The Dissertation Committee for Cristian Ioan Tiu
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ON THE MERTON PROBLEM IN INCOMPLETE MARKETS

Committee:

Thaleia Zariphopoulou, Supervisor

Klaus Bichteler

Patrick R. Brockett

Patrick Jaillet

Stathis Tompaidis

Jack Xin

ON THE MERTON PROBLEM IN INCOMPLETE MARKETS

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Cristian Ioan Tiu, B.S.

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Cristian Ioan Tiu, Ph.D.
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Supervisor: Thaleia Zariphopoulou

Utility maximization problems occupy an important role in Mathematical Finance and since Merton's (1969, 1971) papers, they have been dubbed as Merton Problems. Combining asset and utility models yields various such problems, three of which are treated in this dissertation. In Chapter 2, we analyze the level curves for the classical Merton problem and compare them with the case when trading constraints, and respectively transaction costs are introduced. In Chapter 3, we study the Merton Problem when the agent faces trading constraints and exhibits recursive utility. A representation of the value function is obtained, which has interesting economic explanations. In Chapter 4, we study the Merton Problem when the agent not only faces trading constraints in one asset, but also she is unable to observe its price. Typical examples are illiquid stocks, pre- IPO stocks or pre- IPO stock options. We provide a numerical algorithm. Chapter 5 states few open problems.

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Chapter 1

Introduction

Utility maximization problems occupy an important role in Mathematical Finance and since Merton's (1969, 1971) papers, they have been dubbed as Merton Problems.

The typical model is that of a market consisting of several assets: a risk free asset following a deterministic equation and one or more risky assets, modelled as Ito processes. The time horizon is either $[0, T]$ or $[0, \infty)$. An agent acts on the market by buying and selling the assets and consuming the risk free asset, subject perhaps to various constraints, like interdiction to trade some of the assets, transaction costs or informational limitations. The agent has a certain utility from consumption which, according to her preferences, might take various forms: utility of terminal wealth in the case when the time horizon is finite and the agent only consumes at the end of the horizon; additive utility of consumption across time; recursive utility in the case the agent has preferences on the timing of resolution of uncertainty as well; and a combination of all of the above. Combining the constraints faced by the agent with her specific utility from consumption yields a large class of "Merton Problems", some of which were not studied previously.

In the present paper we look at three such problems: in Chapter 2, we analyze the level curves for the classical Merton problem and compare them with the case when trading constraints, and respectively transaction costs are introduced.

In Chapter 3, we study the Merton Problem when the agent faces trading constraints and exhibits recursive utility. One question arising in such a setup is how much more "anxiety" is experienced by an agent with recursive preferences when the model includes an extra stochastic factor – namely, the asset in which the agent is forbidden to trade.

Finally, in Chapter 4, we study the Merton Problem when the agent not only faces trading constraints in one asset, but also she is unable to observe its price. Typical examples are illiquid stocks, pre- IPO stock or pre- IPO stock options. An interesting question that arises here is whether the agent can observe her own informational limitations in the presence of another uninformed agent, or, put in specific terms, whether the Separation Principle holds. Also, a numerical algorithm is presented.

Chapter 2

INDIFFERENCE CURVES FOR THE VALUE FUNCTION

We study the level sets of value functions in expected utility stochastic optimization models. We consider optimal portfolio management models in complete markets with lognormally distributed prices as well as asset prices modelled as diffusion processes with non-linear dynamics. Besides the complete market cases, we analyze models in markets with frictions like correlated non-traded assets and diffusion stochastic volatilities. We derive, for all the above models, equations that their level curves solve and we relate their evolution to power transformations of derivative prices. We also study models with proportional transaction costs in a finite horizon setting and we derive their level curve equation; the latter turns out to be a Variational Inequality with mixed gradient and obstacle constraints.

2.1 INTRODUCTION

In this chapter, we initiate a study of the level sets of the value functions of stochastic optimization problems that arise in utility maximization models. Level sets are sets on which the value function is constant and, as the examples below indicate, they might have a natural connection with derivative prices. The utility maximization models are the cornerstone in both areas of portfolio management and derivative security pricing especially in incomplete markets. In fact in the latter case, such models arise in the hedging of contingent claims (see example 1) as well as in the pricing of claims via utility methods. Even though when perfect replication is feasible the utility formulation is clearly redundant, this method has produced fruitful results in the presence of frictions which prohibit exact replication.

The study of the level curves has always been of central interest in nonlinear evolution problems. Problems of this nature also arise in a variety of mathematical finance models but the level curves of their solutions have not been analyzed yet. Besides studying these curves for their own sake, there is concrete evidence that they may also contain valuable information for asset valuation as the following examples indicate.

Example 1: It is well known that in the presence of transaction costs perfect replication of contingent claim payoffs is not feasible. Thus one needs to relax the notion of exact replication in order to be able to price derivatives with transaction costs. Among the various methodologies proposed – the utility maximization approach, the imperfect hedging technique and the super-replication method – the latter produces, from the practical point of view, the least interesting results. In fact, as Davis and Clark (1994) conjectured and Soner et al (1995) established, the cheapest super-replication strategy is to buy and hold one share of the underline security. This result was subsequently generalized by Leventhal and Skorohod (1997) who showed that if the derivative payoff $g(S_T)$ satisfies $g(S) \sim \ell S$ for large S, then in order to have exact super-replication at expiration, the least expensive strategy is to hold ℓ shares of the underlying security. Because these constraints are rather stringent and produce prices of little practical significance, it is imperative to relax the requirement of exact super-replication by allowing for a "small slippage". In other words, one may replace the almost surely super-replication requirement by the condition that the candidate (super) hedging portfolio dominates the security payoff with probability $\epsilon \in (0,1)$ only.

A convenient way to study such questions is to formulate the problem as a singular stochastic control one and identify its value function with the maximal probability of hedging

$$(1.1) \qquad V(x,y,S,t) = \sup_{(L,M)} E[1_{\left\{x_T + \binom{\alpha}{\beta}(y_T - g(S_T)) \geq 0\right\}} / x_t = x, y_t = y, S_t = S].$$

The constants α and β are related to the proportional transaction costs and the controlled processes x_s, y_s, $t \leq s \leq T$ represent the current size of the bond and the stock accounts. The optimization is over the set of admissible (super) hedging strategies and the value function gives the probability of (super) hedging. It is then immediate that given a slippage threshold corresponding to super-hedging probability $\epsilon \in (0,1)$, we can determine the new price by studying the *ϵ-level sets* of V.

Example 2. The utility maximization approach has been proven to be a powerful method in obtaining the so-called reservation derivative prices in the presence of market frictions. The prices are determined by comparing the maximal utility of the derivative holder/buyer to the value function without the opportunity to trade the derivative (see Hodges and Neuberger (1989), Davis et al (1993), Constantinides and Zariphopoulou (1999)). Generally speaking and with a slight abuse of the notation, for a European type derivative of payoff $g(S_T)$, the buyer's value function is

$$u(x,S,t) = \sup_{\mathcal{A}} \left[E \int_t^T U(C_s)ds + V(x_T + g(S_T), T)/X_t = x, S_t = S \right]$$

where

$$V(x,t) = \sup_{\mathring{A}_0} E \left[\int_t^{T_1} U(C_s)ds + \Phi(X_{T_1})/X_t = x \right].$$

The processes X_s and S_s represent, respectively, the wealth and the primitive asset price, the functions U and Φ are the utility functions of intermediate consumption and terminal wealth – satisfying $U(0) = \Phi(0) = 0$; the trading horizon T_1 is taken to dominate the expiration time T. The sets of admissible policies $\mathring{A}$ and $\mathring{A}_0$ are appropriately defined to guarantee that the necessary non-negativity wealth constraints are met.

In the frictionless case, the price of the derivative is the unique function $h \equiv h(S,t)$ such that for *all* (x, S, t)

$$V(x,t) = u(x - h(S,t), S, t).$$

One may easily show – after some tedious but routine calculations – that $h(S,t)$ solves the Black and Scholes equation and that the *zero-level sets* of u are described by the derivative price.

Example 3: Recently Carr, Tari and Zariphopoulou (1999) showed that in the absence of arbitrage, the so-called absolute volatility function $a(S_s, s), t \leq s \leq T$, of the underlying stock price process S_s, must satisfy the nonlinear parabolic problem

$$\begin{cases} a_t + \frac{1}{2}a^2 a_{yy} + k(t) y a_y = q(t) a \\ a(0,t) = 0, \quad a(y,T) = \psi(y), \quad (y,t) \in R^+ \times [0,T]. \end{cases}$$

The functions $k(t)$ and $q(t)$ depend on the interest rate and the dividends. The terminal condition $\psi(y)$ represents the volatility data for a given "*smile*". As we show in Section 2, the *slope* $f(x,t)$ of the *level curves* of the value function of the classical Merton problem ((1969), (1971)), is given by $f(x,t) = \delta\pi(x,t) + rx$. The coefficients δ and r are positive constants and π solves a problem similar to (1.5) (see equation (2.18)).

Motivated by the examples above, we start herein a systematic, albeit preliminary, study of the level sets that arise in various utility maximization problems. The basic analysis is carried out through the properties of the relevant Hamilton-Jacobi-Bellman (HJB) equation that their value function is expected to solve. We analyze the level curves of the Merton problem for lognormally distributed prices as well as for the case of non-linear price dynamics. In the first case, the slope of the level curves solves a terminal value problem similar to (1.5) and in the second case, under CRRA references, the level curves are expressed directly as powers of a derivative price.

In section 3, we study the portfolio optimization problems with stochastic volatility, when the latter is modelled as a diffusion correlated with the underlying stock price, and with transaction costs.

2.2 MODELS WITH NO FRICTIONS

We study the level curves of the value function of the classical optimal portfolio management model with general preferences. This model was introduced by Merton ((1969), (1971)) for the case of Hyperbolic Absolute Risk Aversion (HARA) utility functions and lognormally distributed stock prices, and subsequently generalized by various authors (see, among others, Karatzas et al (1987), Grossman and Zhou (1993), Cvitanic and Karatzas (1996), Vila and Zariphopoulou (1997) and Karatzas (1997)).

We show that for general preferences, the slope of the level curves is proportional to the optimal feedback portfolio rule. Moreover, we prove that it solves a nonlinear partial differential equation for which we establish uniqueness of solutions. A byproduct of the latter fact is a comparison result for the optimal feedback portfolio policies in terms of the individual's absolute risk aversion coefficient.

(i) *Models with lognormal stock prices.*

We start with a brief review of the Merton model assuming general utility functions and market completeness. To this end, we consider an economy with two securities, a bond and a stock. The bond's price B_s, is deterministic and evolves, for $0 \le t \le s \le T$, according to

$$\text{(2.1)} \qquad \begin{cases} dB_s = rB_s ds \\ B_t = B > 0 \end{cases}$$

with r being the *interest rate*. The stock price is modelled as a diffusion process S_s solving for $0 \le t \le s \le T$, the stochastic differential equation

$$\text{(2.2)} \qquad \begin{cases} dS_s = \mu S_s ds + \sigma S_s dW_s \\ S_t = S > 0. \end{cases}$$

The market parameters μ and σ are respectively the *mean rate of return* and the *volatility*; it is assumed that $\mu > r > 0$ and $\sigma > 0$. The process W_s is a standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

Trading takes place between the bond and the stock accounts continuously in time, in the trading horizon $[0, T]$. The *wealth process* satisfies $X_s = \pi_s^0 + \pi_s$ with π_s^0 and π_s representing the current holdings in the bond and the stock accounts.

Using the price equations (2.1) and (2.2) one may easily derive the equation for the state process

$$\text{(2.3)} \qquad dX_s = rX_s ds + (\mu - r)\pi_s ds + \sigma \pi_s dW_s.$$

The wealth process must also satisfy the state constraint

$$\text{(2.4)} \qquad X_s \ge 0 \quad \text{a.e.} \quad t \le s \le T.$$

The control $\pi_s, t \le s \le T$ is admissible if it is $\mathcal{F}_s$-progressively measurable – with

$\mathcal{F}_s = \sigma(W_u; t \leq u \leq s)$ – it satisfies $E\int_t^T \pi_s^2 ds < +\infty$ and, it is such that the state constraint (2.4) is satisfied. We denote the set of admissible policies by $\mathring{A}$.

The value function is defined as

$$u(x,t) = \sup_{\mathcal{A}} E[U(X_T)/X_t = x], \tag{2.5}$$

where $U : R^+ \to R^+$ is the utility function modelling the individual preferences.

Assumption 2.1: The utility function $U \in (C^1[0,+\infty) \cap C^2(0,+\infty))$ is increasing, concave and satisfies the growth condition $U(x) \leq K(1+x)^\gamma$ for some positive constants K and γ, with $\gamma \in (0,1)$. Moreover, $U(0) = 0$ and $-\dfrac{U'(x)}{U''(x)} = O(x)$ for large x.

The following result was proved in Karatzas et al (1987).

Proposition 2.1: *(i) The value function* $u \in C^{2,1}((0,+\infty),[0,T])$ *is the unique increasing and concave solution of the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} u_t + \max_\pi \left[\frac{1}{2}\sigma^2\pi^2 u_{xx} + (\mu - r)\pi u_x\right] + rxu_x = 0 & (2.6) \\ u(x,T) = U(x) \ \ \textit{and} \ \ u(0,t) = 0, \quad t \in [0,T]. & (2.7) \end{cases}$$

(ii) The optimal policy π_s^*, $t \leq s \leq T$ *is given in the feedback form* $\pi_s^* = \hat{\pi}(X_s^*, s)$ *where* $\hat{\pi} : R^+ \times [0,T] \to R^+$ *is*

$$\hat{\pi}(x,t) = -\frac{\mu - r}{\sigma^2}\frac{u_x(x,t)}{u_{xx}(x,t)} \tag{2.8}$$

and X_s^* is the solution of (2.3) with the policy π_s^* being used. ∎

We now explore the HJB equation (2.6) from a different point of view. First, we evaluate it at the optimum point (2.8) yielding

$$u_t - \frac{\mu - r}{2\sigma^2} \frac{u_x^2}{u_{xx}} + rxu_x = 0.$$

Therefore, one may interpret the HJB equation (2.6) as the *first order wave equation*

$$\begin{cases} u_t + f(x,t)u_x = 0 \\ u(x,T) = U(x) \ \text{ and } \ u(0,t) = 0, \end{cases} \tag{2.9}$$

where

$$f(x,t) = \frac{\mu - r}{2} \widehat{\pi}(x,t) + rx. \tag{2.10}$$

The above equation is known as the *travelling wave equation of first order* (see, for example, Zauderer (1983)). It is well known for this class of equations that the solution u of (2.9) is constant along the *characteristic curves*, denoted herein by $\tilde{x}(s), t \leq s \leq T$. For a given positive constant c, the characteristic curve, say $\tilde{x}^c(s)$, is defined as the set $\tilde{x}^c(s)$ on which the value function remains constant, i.e.

$$u(\tilde{x}^c(s), s) = c. \tag{2.11}$$

It is then immediate, in view of (2.9), that the characteristic curves of (2.6) have slope

$$\frac{d\tilde{x}^c(s)}{ds} = f(\tilde{x}^c(s), s) = \frac{\mu - r}{2} \widehat{\pi}(\tilde{x}^c(s), s) + r\tilde{x}^c(s) \tag{2.12}$$

and satisfy at $t = T$,

$$\tilde{x}^c(T) = U^{-1}(c). \tag{2.13}$$

The goal for the rest of this section is to study the evolution of the level curves $\tilde{x}^c(s)$. We accomplish this by studying an autonomous equation that their slope f solves. To this end, we show that f solves a nonlinear equation, see (2.15), and that, under mild growth and regularity conditions, f is in fact its unique solution.

Proposition 2.1: *The slope of the characteristic curves $f(x,t)$, given in (2.10), satisfies for $x > 0$,*

$$f(x,t) > rx, \tag{2.14}$$

and it solves the nonlinear parabolic problem

$$\begin{cases} f_t + \dfrac{2\sigma^2}{(\mu - r)^2}(f - rx)^2 f_{xx} + rxf_x = rf, & (2.15) \\ f(x,T) = -\dfrac{(\mu - r)^2}{2\sigma^2}\dfrac{U'(x)}{U''(x)} + rx, \quad \forall x \geq 0, & (2.16) \\ f(0,t) = 0, \quad 0 \leq t \leq T. & (2.17) \end{cases}$$

Proof: First, we recall that the value function u is concave and strictly increasing for $x > 0$ (see Karatzas (1987)). Therefore, $\hat{\pi}(x,t) > 0$ which in view of (2.10) yields (2.14). To derive equation (2.15), we first use (see He and Huang (1994), Huang and Zariphopoulou (1999)) that under Assumption 2.1, the optimal portfolio feedback function $\hat{\pi}(x,t)$ solves

$$\hat{\pi}_t + \frac{1}{2}\sigma^2\hat{\pi}^2\hat{\pi}_{xx} + rx\hat{\pi}_x = r\hat{\pi} \tag{2.18}$$

with

$$\hat{\pi}(x,T) = -\frac{\mu - r}{\sigma^2}\frac{U'(x)}{U''(x)} \quad \text{and} \quad \hat{\pi}(0,t) = 0. \tag{2.19}$$

The above equalities follow respectively from (2.8) and (2.9) and, the state constraint (2.4). Equation (2.18) was derived by He and Huang (1994) and it was further studied by Huang and Zariphopoulou (1999). The arguments used for its derivation are rather technical and tedious and we do not present them here; instead, we refer the technically oriented reader to the above references.

Equation (2.15) and the terminal and boundary conditions (2.16) and (2.17) are then a direct consequence of (2.18), (2.19) and the definition of f in (2.10).

∎

The following theorem provides a uniqueness result for the solutions of the fully nonlinear equation (2.15). Similar results have been recently used by Carr, Tari and Zariphopoulou (1999) to establish the unique characterization of volatility surfaces given a specified "volatility smile" at the expiration time of European derivatives.

Theorem 2.1: *Let $f : R^+ \times [0,T] \to R^+$ be a solution of (2.15) – (2.17) satisfying the terminal condition $\phi(x) \equiv f(x,T)$ with $\phi \in C^2[0,+\infty)$ and $\phi(x) \sim O(x)$ for x large. Then f is the unique solution of (2.15)–(2.17) in the class of functions satisfying $f(x,t) \sim O(x)$ for x large and $|(f^2(x,t))_{xx}| \leq C$ for $(x,t) \in R^+ \times [0,T]$ and some given constant C.*

Proof: The uniqueness result will follow once we establish that $\hat{\pi}(x,t)$ is the unique solution of (2.18) and (2.19). To simplify the presentation we assume that all coefficients appearing in (2.15)–(2.17) are equal to one and we denote its solution by $a(x,t)$, i.e., with a slight abuse of notation we define, $a(x,t) = \hat{\pi}(x,t;\sigma=1,\mu-r=1,r=1)$ to be a solution of

$$\begin{cases} a_t + \dfrac{1}{2}a^2 a_{xx} + x a_x = a & (2.20) \\ a(x,T) = -\dfrac{U'(x)}{U''(x)} \ \text{ and } \ a(x,t) = 0 \ . & (2.21) \end{cases}$$

First, we observe that if $\tilde{a}(x,t)$ satisfies (2.21) and solves the nonlinear problem

$$\tilde{a}_t + \frac{1}{2}\tilde{a}^2 \tilde{a}_{xx} = 0, \tag{2.22}$$

then the function

$$a(x,t) = e^{-(T-t)}\tilde{a}(xe^{(T-t)},t)$$

solves (2.20) and (2.21); this can be easily verified by direct differentiation.

Given the above, it suffices to establish uniqueness for the solutions of (2.21) and (2.22). To this end, we define $F : R^+ \times [0,T] \to R^+$ to be

$$F(x,t) = \tilde{a}^2(x,t). \tag{2.23}$$

Direct calculations yield that F solves

$$\begin{cases} F_t(x,t) + \dfrac{1}{2}F(x,t)F_{xx}(x,t) = F_x^2(x,t) & (2.24) \\ F(x,T) = \left(-\dfrac{U'(x)}{U''(x)}\right)^2 \ \text{ and } \ F(0,t) = 0, \ \ 0 \le t \le T. & (2.25) \end{cases}$$

From the assumptions on $f(x,t)$ and therefore on $\widetilde{\pi}(x,t)$ and, in turn, on $\widetilde{a}(x,t)$ we get that $F(x,t) \sim O(x^2)$ for x large and that $F(x,t)_{xx} \leq C$ for $(x,t) \in R^+ \times [0,T)$. Using a variation of the results of I. Fukuda, H. Ishii and M. Tsutsumi (1993) we get that (2.24), (2.25) has a unique solution.

Therefore, if $a_1(x,t)$ and $a_2(x,t)$ are two solutions of (2.22), satisfying also (2.21), the above uniqueness result yields that

$$a_1^2(x,t) = a_2^2(x,t). \tag{2.26}$$

Next, we look at the difference $G(x,t) = a_1(x,t) - a_2(x,t)$. Differentiation and use of (2.21) yield that G solves

$$\begin{cases} G_t(x,t) + \frac{1}{2}a_1^2(x,t)G_{xx}(x,t) = 0 \\ G(0,t) = 0 \quad \text{and} \quad G(x,T) = 0, \;\; 0 \leq t \leq T. \end{cases} \tag{2.27}$$

Working as above for $\widehat{G}(x,t) = a_2(x,t) - a_1(x,t)$ yields that $\widehat{G}$ solves

$$\widehat{G}_t(x,t) + \frac{1}{2}a_2^2(x,t)\widehat{G}_{xx}(x,t) = 0 \tag{2.28}$$

which, in view of (2.26), coincides with (2.27). Moreover, $\widehat{G}(0,T) = 0$ and $\widehat{G}(x,T) = 0$. We can easily verify that equation (2.27) (or (2.28)) admits a comparison principle and we readily conclude that $G(x,t) \equiv 0$ and therefore, $a_1(x,t) = a_2(x,t)$ for $(x,t) \in R^+ \times [0,T]$. ∎

The following result is an interesting consequence of the uniqueness of solutions of the autonomous portfolio equation (2.18). It shows that two investors with absolute risk aversion coefficients, say $R_1(x)$ and $R_2(x)$ satisfying $R_1(x) \leq R_2(x)$, always choose their optimal portfolio policies $\hat{\pi}_1(x,t)$ and $\hat{\pi}_2(x,t)$, such that $\hat{\pi}_1(x,t) \geq \hat{\pi}_2(x,t)$. Therefore, it is only the *terminal* ordering in the optimal portfolios – via the absolute risk aversion coefficient – that determines the dynamic ordering of all trading times. Even though this result follows easily in the case of Constant Relative Risk Aversion (CRRA) and exponential utilities, to our knowledge, this is the first time that this monotonic behavior is established for dynamic trading models with general individual preferences.

Proposition 2.2: *Assume that utilities U_1 and U_2 have absolute risk aversion coefficients R_1 and R_2 satisfying $R_1(x) \leq R_2(x)$, i.e.*

$$-\frac{U_1''(x)}{U_1'(x)} \leq -\frac{U_2''(x)}{U_2'(x)} \tag{2.29}$$

and $U_1(0) = U_2(0) = 0$. Consider the relevant utility maximization problems (2.6) and (2.7) for utilities U_1 and U_2 and denote, respectively, by $\pi_1^*(x,t)$ and $\pi_2^*(x,t)$ their optimal feedback portfolio rules. Assume that π_1^* and π_2^* satisfy the growth and regularity conditions $\pi_i^*(x,t) \sim O(x)$ and $|(\pi_i^*)^2_{xx}| \leq C$, for a large constant C. Then

$$\pi_1^*(x,t) \geq \pi_2^*(x,t), \quad 0 \leq t \leq T. \tag{2.30}$$

∎

ii) Models with non-linear stock dynamics.

We consider the generalization of the Merton model in a market with two securities, a deterministic bond and a stock. We allow for the stock price to follow a diffusion process with non-linear dynamics. In this setting, the portfolio optimization problem becomes two dimensional and closed form solutions are not in general available. The case of CRRA functions was recently studied by Zariphopoulou (1999) who produced the solutions in a reduced form (see Proposition 2.3 below).

We represent the stock price as the solution of

$$dS_s = \mu(S_s)S_s ds + \sigma(S_s)S_s dW_s. \tag{2.31}$$

The process W_s is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and the coefficients μ, σ are given functions of the current stock price. They are assumed to satisfy, respectively, the global Lipschitz and linear growth conditions $|f(y) - f(\overline{y})| \leq k|y - \overline{y}|$ and $f^2(y) \leq k^2(1 + y^2)$ for $y \geq 0$, k being a positive constant and f standing for μ and σ. Moreover there exist positive constants ℓ_1 and ℓ_2 such that for $y \geq 0$, $\sigma(y) \geq \ell_1$ and $\dfrac{(\mu(y) - r)^2}{\sigma^2(y)} \leq \ell_2$.

With the above non-linear stock price dynamics, the wealth state equation becomes

$$\begin{cases} dX_s = rX_s ds + (\mu(S_s) - r)\pi_s ds + \sigma(S_s)\pi_s dW_s \\ X_t = x \geq 0, \quad 0 \leq t \leq s \leq T \end{cases} \tag{2.32}$$

with X_s being the current wealth satisfying the state constraint $X_s \geq 0$ a.s., $t \leq s \leq T$.

The utility functions is of Constant Relative Risk Aversion (CRRA) type

$$U(x) = \frac{1}{\gamma} x^{\gamma} \tag{2.33}$$

with $\gamma \in (0, 1)$.

The value function is

$$u(x, S, t) = \sup_{\mathcal{A}} E[U(X_T)/X_t = x, S_t = S]$$

with $\mathcal{A}$ being the set of admissible portfolios.

The proof of the following result may be found in Zariphopoulou (1999).

Proposition 2.3: *i) The value functions* u *is given by*

$$u(x, S, t) = \frac{x^{\gamma}}{\gamma} V(S, t)^{1-\gamma} \tag{2.34}$$

where $V : R^{+} \times [0, T] \to R^{+}$ *solves the linear parabolic equation*

$$\begin{cases} V_t + \frac{1}{2}\sigma^2(S) S^2 V_{SS} + \left[\mu(S) S + \frac{\gamma(\mu(S) - r) S}{(1-\gamma)}\right] V_S \\ + \frac{\gamma}{1-\gamma}\left[r + \frac{(\mu(S) - r)^2}{2\sigma^2(S)(1-\gamma)}\right] V = 0 \\ V(S, T) = 1 \quad \textit{and} \quad V(0, t) = e^{\frac{r\gamma}{1-\gamma}(T-t)}, \ \ 0 \le t \le T \end{cases} \tag{2.35}$$

ii) The optimal portfolio policy π_s^* *is given in the feedback form* $\pi_s^* = \tilde{\pi}_s(X_s^*, S_s, s)$ *where*

$$\tilde{\pi}(x, S, t) = \left[\frac{S V_S}{V} + \frac{\mu(S) - r}{(1-\gamma)\sigma^2(S)}\right] x.$$

∎

Using the above representation, one may obtain the level sets of u in a simplified form. In fact, given $c > 0$ and $x^c(S,t)$ being such that

$$u(x^c(S,t), S, t) = c$$

the representation (2.34) yields

$$x^c(S,t) = (c\gamma)^{\frac{1}{\gamma}} [V(S,t)]^{\frac{\gamma-1}{\gamma}} \tag{2.36}$$

with V solving the linear equation (2.35).

So we see that in the case of complete markets with stocks modelled as diffusion prices but with non-linear dynamics the level sets are represented as powers of solutions of linear parabolic equations. Since such equations are directly related to prices of European type derivative securities, we observe an interesting connection between level sets and derivative prices.

2.3 MODELS WITH FRICTIONS

In this section we derive the level sets of two fundamental models of optimal portfolio management in markets with frictions.

i) Models with non-traded assets

These models are similar to the ones we studied in the previous section but we allow for a *non-traded asset* in the market environment. This asset affects the returns of the underlying and it is in general correlated with it. A special case is when the volatility is stochastic and it is modelled as a correlated diffusion process. Of course, since the volatility is in general unobservable the model might not be very realistic albeit useful for certain approximations.

We assume that trading takes place between a bond account – with the bond price given by (2.1) – and a stock account with the stock price S_s solving

$$dS_s = \mu S_s ds + \sigma(Y_s) S_s dW_s^1 \tag{3.1}$$

where $\mu > r > 0$ and Y_s is given by

$$dY_s = b(Y_s, s)ds + a(Y_s, s)dW_s^2. \tag{3.2}$$

The processes W_s^1 and W_s^2 are standard Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$ correlated with *correlation coefficient* $\rho \in (-1, 1)$. The coefficients $\sigma : R \to R^+$ and $b, \alpha : R \times [0, T] \to R$ satisfy the global Lipschitz and linear growth conditions $|f(y,t) - f(\overline{y}, t)| \leq K|y - \overline{y}|$ and $f^2(y,t) \leq k^2(1+y^2)$, for every $t \in [0,T]$, $y, \overline{y} \in R$, K being a positive constant and f standing for σ, b and a. Moreover, uniformly in $y \in R$ and $t \in [0,T]$, there is a positive constant ℓ such that for $y \in R$ and $t \in [0,T]$, $\sigma(y) \geq \ell$.

The value function w is

$$w(x,y,t) = \sup_{\mathcal{A}_1} E(\frac{1}{\gamma} X_T^\gamma / X_t = x, Y_t = y). \tag{3.3}$$

Here $\mathcal{A}_1$ is the set of admissible policies π_s which are $\mathcal{F}_s$-progressively measurable processes, with $\mathcal{F}_s = \sigma((W_u^1, W_u^2); t \leq u \leq s)$, satisfy the integrability condition

$$E \int_t^T \sigma(Y_s, s)^2 \pi_s^2 ds < +\infty,$$

and are such that the state wealth X_s satisfies $X_s \geq 0$ a.e., $t \leq s \leq T$.

Using the state equations (2.1), (3.1) and (3.2), one easily derives the stochastic differential equation for X_s, namely

$$dX_s = rX_s ds + (\mu - r)\pi_s ds + \sigma(Y_s)\pi_s dW_s^1. \tag{3.4}$$

This generalization of the Merton problem was recently solved in Zariphopoulou (2001). Using the apparent homogeneity of the problem and a convenient power transformation, one may obtain the value function in a reduced form. For the proof of the following result we refer the reader to Theorem 3.3 of Zariphopoulou (1999a).

Theorem 3.1: *The value function w is given by*

$$w(x,y,t) = \frac{x^\gamma}{\gamma} H(y,t)^{\frac{1-\gamma}{1-\gamma+\rho^2\gamma}} \tag{3.5}$$

where $H : R \times [0,T] \to R^+$ solves the linear parabolic problem

$$\begin{aligned} H_t + \tfrac{1}{2}a^2(y,t)H_{yy} + \left[b(y,t) + \rho\tfrac{\gamma(\mu-r)a(y,t)}{(1-\gamma)\sigma(y)}\right] H_y \\ + \tfrac{\gamma(1-\gamma+\rho^2\gamma)}{1-\gamma}\left[r + \tfrac{(\mu-r)^2}{2\sigma^2(y)(1-\gamma)}\right] H = 0 \end{aligned} \tag{3.6}$$

$$H(y,T) = 1. \tag{3.7}$$

∎

The following result is a direct consequence of the representation formula (3.5) for the value function.

Proposition 3.1: *The curve* $x^c(y,t)$ *on which the value function satisfies* $w(x^c(y,t),y,t) = c$ *is given by*

$$x^c(y,t) = (c\gamma)^{\frac{1}{\gamma}} H(y,t)^{\frac{1-\gamma}{\gamma(1-\gamma+\rho^2\gamma)}} \tag{3.8}$$

with H *solving (3.6) and (3.7).* ∎

ii) *Models with transaction costs.*

Transaction costs have always been present in financial transactions and their role in asset pricing has long been of central interest, especially when the financial assets involved have different liquidity.

The stochastic control problems that arise in models with transaction costs are of singular type and their HJB equation becomes a Variational Inequality with gradient constraints. The majority of existing work on the subject deals with infinite horizon problems of optimal consumption; see, the pioneering paper of Magill and Constantinides (1976) and the seminal paper of Davis and Norman (1990). Given that a considerable number of applications deal with dynamic trading in a finite horizon, it is highly desirable to study the finite horizon case as well. Important optimization problems in which the finiteness of the horizon is crucial arise in models of derivative pricing with transaction costs via the utility maximization approach. These stochastic portfolio optimization problems consider the optimal policies of the writer and/or the buyer of the derivative security which in turn yield useful bounds on the selling and the buying price (see for example, Davis et al (1993), Davis and Zariphopoulou (1995), Barles and Soner (1998), Constantinides and Zariphopoulou (1999, 1999a)).

In the sequel we review briefly the underlying finite horizon model and we proceed with the derivation of the equation of the level curves. To this end, we consider a market with two securities, a bond and a stock whose prices solve (2.1) and (2.2) respectively. Trading takes place between the bond the

the stock accounts and there is no intermediate consumption. The amounts x_s and y_s invested, respectively, in the bond and the stock account, evolve according to the controlled state equations

$$(3.9) \qquad \begin{cases} dx_s = rx_s ds - (1+\lambda)dL_s + (1-\mu)dM_s \\ \\ x_t = x, \end{cases}$$

and

$$(3.10) \qquad \begin{cases} dy_s = \mu y_s ds + \sigma y_s dW_s + dL_s - dM_s \\ \\ y_t = y, \quad 0 \le t \le s \le T. \end{cases}$$

The control processes L_s and M_s represent the cumulative purchases and sales of stock. The pair (L_s, M_s) is admissible if the processes L_s and M_s are $\mathcal{F}_s$-progressively measurable, right continuous with left limits, and the state constraint

$$(3.11) \qquad x_s + \binom{\alpha}{\beta} y_s \ge 0 \quad \text{a.e.} \quad t \le s \le T$$

is satisfied, where

$$(3.12) \qquad \alpha = 1 - \mu \quad \text{and} \quad \beta = 1 + \lambda.$$

For the rest of the chapter, to ease the presentation we adopt the notation

$$(3.13) \qquad \binom{\alpha}{\beta} z = \begin{cases} \alpha z & \text{if } z \ge 0 \\ \beta z & \text{if } z < 0. \end{cases}$$

We denote the set of admissible policies by $\mathcal{A}_2$.

The value function is defined as

$$V(x,y,t) = \sup_{\mathcal{A}_2} E\left[\frac{1}{\gamma}\left(x_T + \binom{\alpha}{\beta} y_T\right)^{\gamma} / x_t = x, y_t = y\right], \tag{3.14}$$

where

$$(x,y) \in \overline{D} = \Big\{(x,y) \in R : x + \binom{\alpha}{\beta} y \geq 0\Big\}.$$

Following arguments similar to the ones used in Constantinides and Zariphopoulou (1999) yields the following result.

Theorem 3.2: *The value function is the unique concave and increasing in x and y, constrained viscosity solution on $\overline{D}$ of the Variational Inequality*

$$\min\Big\{-V_t - \frac{1}{2}\sigma^2 y^2 V_{yy} - \mu y V_y - rxV_x, \beta V_x - V_y, -\alpha V_x + V_y\Big\} = 0 \tag{3.15}$$

satisfying

$$V(x,y,T) = \frac{1}{\gamma}\left(x + \binom{\alpha}{\beta} y\right)^{\gamma}. \tag{3.16}$$

∎

The fact that one needs to relax the notion of solutions to the Hamilton-Jacobi-Bellman equation of stochastic control problems involving models with frictions, is by now well established. For the use of viscosity solutions in models with transaction costs, we refer the technically interested reader to the review article by Zariphopoulou (1999).

We are now ready to derive the equation which the level curves of V satisfy. Note that complete results on the regularity of the value function are not in general available and the calculations below are formal.

To this end, we consider a constant $c > 0$ and we look for the function $g : R \times [0, T] \rightarrow R$ such that

$$V(x, g(x,t), t) = c. \tag{3.17}$$

We recall that V is jointly homogeneous of degree γ which yields

$$xV_x(x, g(x,t), t) + g(x,t)V_y(x, g(x,t), t) = \gamma V(x, g(x,t), t) \tag{3.18}$$

and, in turn, that

$$xV_{xx}(x, g(x,t), t) + g(x,t)V_{xy}(x, g(x,t), t) = (1-\gamma)g_x(x,t)V_y(x, g(x,t), t). \tag{3.19}$$

Differentiating twice (3.17) with respect to x yields

$$V_{xx}(x, g(x,t), t) + \quad 2g_x(x,t)V_{yy}(x, g(x,t), t) + \\ + g_{xx}(x,t)V_y(x, g(x,t), t) + g_x^2(x,t)V_{yy}(x, g(x,t), t) = 0. \tag{3.20}$$

Combining (3.19) and (3.20) gives

$$V_{xy} = \frac{[(1-\gamma)g_x + xg_{xx}]V_y + xg_x^2 V_{yy}}{g - 2xg_x} \tag{3.21}$$

with all the above derivatives of V being evaluated at the point $(x, g(x,t), t)$.

Using again the homogeneity of V implies

$$xV_{xy}(x, g(x,t), t) + g(x,t)V_{yy}(x, g(x,t), t) = -(1-\gamma)V_y(x, g(x,t), t)$$

which together with (3.21) results in

$$\frac{V_{yy}(x,g(x,t),t)}{V_y(x,g(x,t),t)} = -\frac{1-\gamma}{g(x,t)-xg_x(x,t)} - \frac{x^2 g_{xx}(x,t)}{(g(x,t)-xg_x(x,t))^2}. \tag{3.22}$$

Differentiating (3.17) with respect to time and x respectively, implies

$$V_t(x,g(x,t),t) = -g_t(x,t)V_y(x,g(x,t),t) \tag{3.23}$$

and

$$V_x(x,g(x,t),t) = -g_x(x,t)V_y(x,g(x,t),t). \tag{3.24}$$

Combining (3.22), (3.23) and (3.24) yields that the second order operator appearing in (3.15), namely

$$\mathcal{L}V = -\{V_t + \frac{1}{2}\sigma^2 y^2 V_{yy} + \mu y V_y + rxV_x\} \tag{3.25}$$

when evaluated at $(x,g(x,t),t)$ becomes

$$\begin{aligned} \mathcal{L}V(x,g(x,t),t) = V_y(x,g(x,t),t)\Big[g_t(x,t)+ \\ +\tfrac{1}{2}\sigma^2 g^2(x,t)\left(\tfrac{1-\gamma}{g(x,t)-xg_x(x,t)} + \tfrac{x^2 g_{xx}(x,t)}{(g(x,t)-xg_x(x,t))^2}\right) - \mu g(x,t)\Big] \end{aligned}. \tag{3.26}$$

From (3.24) we get that the gradient terms

$$\mathcal{L}_1 V = \beta V_x - V_y \quad \text{and} \quad \mathcal{L}_2 V = -\alpha V_x + V_y$$

evaluated at $(x,g(x,t),t)$ become

$$\mathcal{L}_1 V(x,g(x,t),t) = -V_y(x,g(x,t),t)(\beta g_x(x,t)+1) \tag{3.27}$$

and

$$\mathcal{L}_2 V(x,g(x,t),t) = V_y(x,g(x,t),t)(\alpha g_x(x,t)+1). \tag{3.28}$$

Combining (3.26)–(3.28) and cancelling the common term V_y gives the equation that $g(x,t)$ satisfies. The latter turns out to be the Variational Inequality

$$\min\Big\{g_t+\frac{1}{2}\sigma^2 g^2\Big[\frac{1-\gamma}{g-xg_x}+\frac{x^2g_{xx}}{(g-xg_x)^2}\Big]-\mu g,-(\beta g_x+1),\alpha g_x+1\Big\}=0. \tag{3.29}$$

The terminal condition $g(x,T)$ is recovered easily from (3.16) and it is given by

$$g(x,T)=\begin{cases}\dfrac{c^{\frac{1}{\gamma}}-x}{\beta} & \text{if } x\geq c^{\frac{1}{\gamma}}\\[2ex] \dfrac{c^{\frac{1}{\gamma}}-x}{\alpha} & \text{if } x< c^{\frac{1}{\gamma}}.\end{cases} \tag{3.30}$$

Next we make the following transformations.

Remark 3.1: One may further simplify the second order part in (3.29) using a number of transformations. In fact, if $k: R\times[0,T]\to R$ is such that $k(x,t)=e^{-\mu t}e^{-\frac{x}{\gamma}}g(e^{\frac{x}{\gamma}},t)$,
$0\leq t\leq T$ and $p: R\times[0,T]\to R$ is given by $p(x,t)=k\left(x,\frac{2}{\sigma^2}t\right)$ for $0\leq t\leq\overline{T}$ with $\overline{T}=\frac{\sigma^2 T}{2}$, after lengthy arguments, one can argue that there is a well defined function $q(x,t)$ such that $p(q(x,t),t)=x$. Defining

$$S(x,t)=\exp\Big\{-q(e^x,t)+\frac{x}{2}+\frac{t}{4}\Big\}$$

one gets – after tedious but routine calculations – that S solves

$$\min\Big\{S_t+S_{xx},\ \alpha e^{\frac{2\mu}{\sigma^2}t}\,\Big(-\frac{S_x}{S}+\gamma+\frac{1}{2}\Big)+1,-\beta e^{\frac{2\mu}{\sigma^2}t}\left(-\frac{S_x}{S}+\gamma+\frac{1}{2}\right)-1\Big\}=0$$

with terminal condition

$$S(x,\overline{T}) = e^{-\frac{x}{2}-\frac{\overline{T}}{4}} \left[\frac{\alpha 1_{\{x<0\}} + \beta 1_{\{x\geq 0\}}}{e^{\mu \overline{T}}} e^{x} + 1 \right]^{\gamma}.$$

Chapter 3

INCOMPLETE MARKETS AND RECURSIVE PREFERENCES

We consider and solve the problem of maximizing two types of recursive utilities, namely Standard Additive and Kreps-Porteus, in the Markovian case of a market consisting of a stock,which depends on a stochastic factor, and a bond. The noises driving the stock and the stochastic factor are correlated. Intuitively, we expect that the presence of the stochastic factor brings more anxiety to a trader who differentiates preferences by timing of resolution of uncertainty. By providing a certain variational interpretation of the value function , we prove rigorously that the intuitive affirmation above is true.

3.1 INTRODUCTION

We consider a market with two assets,a bond and a stock. The stock price is modelled as a diffusion whose drift and quadratic variation depend upon a third process, which is itself a diffusion, and is interpreted for instance as an observable but non-traded asset, or as a stochastic volatility. We consider an agent that invests in the assets and also consumes. Our goal is to understand how the presence of a “stochastic factor” influences the agent’s optimal policy.

Such a model has been considered earlier by Zariphopoulou (2001): the agent had only terminal HARA consumption. Zariphopoulou (2001) provided an explicit method to solve the associated control problem. In this chapter, we allow the agent to consume such that a combination between her recursive preferences and terminal consumption is maximized. The recursive utility supposedly brings more insight about how the stochastic factor influences the agent’s behavior hence her attitude toward the timing of the resolution of uncertainty. For example, an agent who cares about the time the uncertainty is revealed experiences intuitively more anxiety if the uncertainty comes not only from the noisy structure of a price (which we give exogeneously), but also from an extra noisy parameter related to the stock price, than if the uncertainty were coming solely from a noisy, but log-normal, price.

To see how the particular type of utility used in this chapter treats resolution of uncertainty, we consider a choice∗ among three hypothetical consumption programs, which are informally defined as follows. Consumption during the interval $[0, 1)$ is fixed at the same level for all three programs. In the first program, c^1, a fair coin is flipped at $t = 1$. If the outcome is head, then consumption is constant at level l for the entire remaining time horizon $[1, T]$. Otherwise, it is constant at $L > l$. For consumption program c^2, $T - 1$ independent fair coins are flipped, one for each integer time $t \in [1, T)$, and all coin tosses are revealed at time $t = 1$. If the tth toss yields a head (tail) then consumption over the interval $[t, t+1)$ is $l(L)$. Finally, c^3 differs from c^1 only in that the tth coin toss is not revealed until time t. From the perspective of an agent with time separable additive utility, $c^1 \sim c^2 \sim c^3$. Diversification reasons would want to make another agent to differentiate between c^1 and c^2 (in the former the consumption across time is positively autocorrelated – therefore undiversified, in the latter the consumption is serially independent). However, such criteria still don't distinguish between c^2 and c^3. It is only the recursive utility which distinguish between them; the distinction is typically that early resolution of uncertainty dominates the early one. Now, in our particular model, the uncertainty "increases" due to the presence of a non-traded asset. This asset is the analogue of the coined flipped in the previous example, and its "outcomes" are not known until the price of the non-traded asset is observed. We would expect that this "extra uncertainty" brings disutility to an agent with recursive preferences. This will become clear in the context.

This is not the first time when such an optimization problem was considered. Schroeder and Skiadas (1999) analyze a Markovian model; if we interpret our volatility as a underlying Markovian state, and think the market as consisting of the bond and tradeable stock only, then our model coincides with theirs. However, when one does such identifications, one must be careful with the information available before and after the identification. To be more precise, it is the filtration used in the definition of the recursive utility which matters; in the present chapter, as we will make precise later, this is not a sensitive issue, but it can become one when dealing with partial information.

We want to warn the reader about it.

The present chapter does two things: on the one hand, it solves the control problem associated with a recursive utility in the market model described above, and on the other hand, provides an economically meaningful representation of the value function. A distortion transformation, corresponding to a change from the actual probability to a coherent measure, plays an important role above, as it simultaneously simplifies the problem technically and allows the practical interpretation of the result.

3.2 THE MODEL

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We are placed in a market with two traded assets, namely, a bond of maturity T with price given by

$$\begin{cases} dB_s = rB_s ds \\ B_t = 1, 0 \leq t \leq s \leq T, \end{cases} \tag{2.1}$$

and a stock modelled as

$$\begin{cases} \dfrac{dS_s}{S_s} = \mu(s, Y_s)ds + \sigma(s, Y_s)dW_s^1 \\ S_t = S \geq 0, 0 \leq t \leq s \leq T. \end{cases} \tag{2.2}$$

The stock price itself depends upon a *diffusion coefficient*, which could be interpreted either as a *stochastic volatility* or as a *non-traded asset*, and whose dynamics is described by

$$\begin{cases} dY_s = b(s, Y_s)ds + a(s, Y_s)dW_s^2 \\ Y_t = y \in R. \end{cases} \tag{2.3}$$

The Brownian motions W^1 and W^2 are defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and they have correlation $\rho \in (-1, 1)$.

An agent consumes at a rate c_s, at time s, and invests the amounts π_s^0 and π_s in the bond and stock, respectively. The current wealth $X_s = \pi_s^0 + \pi_s$ solves the state controlled SDE:

$$\begin{cases} dX_s = [rX_s + (\mu(Y_s, s) - r)\pi_s - c_s]ds + \sigma(Y_s, s)\pi_s dW_s^1 \\ X_t = x \geq 0. \end{cases} \tag{2.4}$$

The aim of the agent is to maximize a *recursive utility* of the above consumption, that is, to maximize V_t, where

$$V_s = E[\int_s^T f(c_l, X_l, V_l)ds + g(X_T)|\ \mathcal{F}_s], \quad t \leq s \leq T. \tag{2.5}$$

This type of utility has been introduced by Duffie and Epstein(1992) to incorporate the timing of resolution of uncertainty, which we expect to be an important factor in the presence of the stochastic coefficient Y. The functional f is the *generator* of the recursive utility, and it takes several important forms that will be described later on.

The utility function is

$$\mathcal{J}(t, x, y) = \sup_{\{\pi, c\}} V_t. \tag{2.6}$$

The Hamilton-Jacobi-Bellman equation associated with the above optimization problem could be written (see Duffie and Epstein (1992)) as:

$$\begin{cases} \sup\limits_{\pi, c} \mathcal{L}^{\pi, c}\mathcal{J}(t, x, y) + f(c, x, \mathcal{J}(t, x, y)) = 0 \\ \mathcal{J}(T, x, y) = g(x), \end{cases} \tag{2.7}$$

where

$$\mathcal{L}^{\pi,c} = \mathcal{J}_t + \frac{1}{2}a^2(t,y)\mathcal{J}_{yy} + b(t,y)\mathcal{J}_y + rx\mathcal{J}_x$$

$$+\frac{1}{2}\sigma^2(t,y)\pi^2\mathcal{J}_{xx} + \rho a(t,y)\sigma(t,y)\pi\mathcal{J}_{xy} + [\mu(t,y) - r]\pi\mathcal{J}_x - c\mathcal{J}_x. \tag{2.8}$$

We consider several cases for the generator f, known for their financial significance. For a review of the most meaningful generators (complete list of references to come), and for various forms in which the recursive utility could be represented, we refer the reader to El Karoui, Peng, Quenez (1997) or Duffie and Epstein (1992). We enlist below the cases we analyze:

•*Standard additive utility with Hyperbolic Relative Risk Aversion*

$$f(c,V) = \frac{c^\gamma}{\gamma} - \beta V, \gamma \in (-\infty, 1) - \{0\}$$

$$f(c,V) = \log c - \beta V, \gamma = 0 \tag{2.9a}$$

•*Kreps-Porteus isoelastic, normalized*

$$f(c,V) = \frac{\beta}{\alpha}\frac{c^\gamma - (\gamma V)^{\alpha/\gamma}}{(\gamma V)^{\alpha/\gamma - 1}}, \alpha, \gamma < 1, \alpha, \gamma \neq 0 \tag{2.9b}$$

Please remark that as compared to the existing literature, we used the letter γ for the *intertemporal rate of substitution* in the expression above. This is a parameter which measures the attitude toward the timing of resolution of uncertainty: when $\alpha > 0$, then a bigger γ means more tolerance for later resolution of uncertainty, while when $\alpha < 0$ the role of γ is reversed. We refer the reader to the paper of Schroeder and Skiadas (1999), for more insightful comments on the parameters α and γ. As one could see in (2.10) below, we have chosen the HARA exponent of the utility of terminal wealth to equal the intertemporal rate of substitution. This makes the problem homogeneous, and allows us to manipulate the HJB equation efficiently.

- *Uzawa utility*

$$f(c, V) = u(c) - \beta(c)V. \tag{2.9c}$$

We will not analyze the case of the Uzawa generator (the functions u and β are too general to allow explicit formulae),but we will use it in a later section; the term u in its expression is to be interpreted as a *utility* (recall the properties of a risk aversion utilities: increasing with respect to consumption and concave), while β is a *discount factor*(see Appendix D for an explanation of this terminology). The Uzawa utility appears as a generalization of the Standard Additive utility, that is, the discount factor of the Uzawa generator depends also on consumption. We slightly extend the terminology of "Uzawa generators" by allowing β to be stochastic.

Along this chapter we let the *terminal criterion* to be of HARA type,

$$g(x) = \frac{x^\gamma}{\gamma}, \quad \gamma \neq 0 \tag{2.10}$$

We make the reader aware that the significance of γ differs from the Standard Additive case to the Kreps-Porteus case; we used the same γ to enable us to obtain a closed form solution to our problem and represent it in a meaningful way. While in the Standard Additive case γ relates to the risk aversion of the agent, in the Kreps-Porteus case γ is an indicator of the preference for the timing of the resolution of uncertainty.

3.3 THE OPTIMIZATION PROBLEM

Using a distortion transformation introduced by Zariphopoulou (2001), we simplify the HJB equation (2.7) and prove existence and uniqueness results, for a fairly large class of parameters. We are also able to produce explicit bounds on the value function, depending on given bounds on the coefficients driving the diffusions (2.2) and (2.3). We do the computations separately for the two most significant types of utilities, namely the Standard Additive and the recursive Kreps-Porteus utility. Once again we warn the reader about the different significance of the parameter γ in the two cases. We also mention that the Kreps-Porteus utility degenerates into one similar with the Standard Additive utility when $\alpha = \gamma$.

3.3.1 The Standard Additive case

In this case the HJB equation (1.6) becomes:

$$(3.1.1)\quad \begin{cases} \mathcal{J}_t + \frac{1}{2}a(t,y)^2\mathcal{J}_{yy} + b(t,y)\mathcal{J}_y + rx\mathcal{J}_x \\ \quad + \max_{\pi}\{\frac{1}{2}\sigma(t,y)^2\mathcal{J}_{xx}\pi^2 + \rho\sigma(t,y)a(t,y)\mathcal{J}_{xy}\pi + [\mu(t,y)-r]\mathcal{J}_x\pi\} \\ \quad + \max_{c}\{-c\mathcal{J}_x + \frac{c^\gamma}{\gamma} - \beta\mathcal{J}\} = 0 \\ \mathcal{J}(T,x,y) = \frac{x^\gamma}{\gamma} \end{cases}$$

The maximum with respect to c is

$$(3.1.2)\qquad \frac{1-\gamma}{\gamma}\mathcal{J}_x^{\frac{\gamma}{\gamma-1}} - \beta\mathcal{J}$$

obtained at

$$(3.1.3)\qquad \begin{aligned} c^*(t,x,y) &= \Big(\mathcal{J}_x(t,x,y)\Big)^{\frac{1}{\gamma-1}} \\ \pi^*(t,x,y) &= -\frac{\rho\sigma(t,y)a(t,y)\mathcal{J}_{xy}(t,x,y) + [\mu(t,y)-r]\mathcal{J}_x(t,x,y)}{\sigma(t,y)^2\mathcal{J}_{xx}(t,x,y)}. \end{aligned}$$

The form of the preference functionals suggests that the value function decomposes as

$$(3.1.4)\qquad \mathcal{J}(t,x,y) = \frac{x^\gamma}{\gamma}\tilde{V}(t,y).$$

We will look for a candidate solution of (3.1.1) of the above form. Once a candidate solution if found, its identification with the value function will come from uniqueness results of viscosity solutions in the relevant class. To this end, using $\tilde{V}$ in (3.1.1) yields

(3.1.5)

$$\begin{cases} \frac{1}{\gamma}[\tilde{V}_t + \frac{1}{2}a(t,y)^2\tilde{V}_{yy} + b(t,y)\tilde{V}_y] + r\tilde{V} \\ \quad + \max_{\tilde{\pi}}\{\frac{1}{2}(\gamma-1)\sigma(t,y)^2\tilde{V}\tilde{\pi}^2 + \rho\sigma(t,y)a(t,y)\tilde{V}_y\tilde{\pi} + [\mu(t,y)-r]\tilde{V}\tilde{\pi}\} \\ \quad + \frac{1-\gamma}{\gamma}\tilde{V}^{\frac{\gamma}{\gamma-1}} - \frac{\beta}{\gamma}\tilde{V} = 0 \\ \tilde{V}(T,y) = 1 \end{cases}$$

We inserted an x into the old control π to get the new control $\tilde{\pi} = \pi/x$.

If we solve the maximization problem above, (3.1.5) becomes:

$$\begin{cases} \tilde{V}_t + \frac{1}{2}a(t,y)^2\tilde{V}_{yy} + [b(t,y) + \rho\frac{\gamma}{1-\gamma}a(t,y)\frac{(\mu(t,y)-r)}{\sigma(t,y)}]\tilde{V}_y \\ \quad + [r\gamma + \frac{\gamma}{1-\gamma}\frac{(\mu(t,y)-r)^2}{2\sigma(t,y)^2} - \beta]\tilde{V} \\ \quad + (1-\gamma)\tilde{V}^{\frac{\gamma}{\gamma-1}} + \rho^2\frac{\gamma}{1-\gamma}\frac{a(t,y)^2}{2}\frac{\tilde{V}_y^2}{\tilde{V}} = 0 \\ \tilde{V}(T,y) = 1 \end{cases} \tag{3.1.6}$$

The optimal investment strategy is given (in terms of $\tilde{V}$) by

$$\pi^*(s,x,y) = \frac{\rho\sigma(s,y)a(s,y)\tilde{V}_y(s,y) + [\mu(s,y)-r]\tilde{V}(s,y)}{(1-\gamma)\sigma^2(s,y)\tilde{V}(s,y)}x \tag{3.1.7}$$

We make now the transformation

$$\tilde{V}(t,y) = v(t,y)^\delta. \tag{3.1.8}$$

Calculating the derivatives yields

$$\tilde{V}_t = \delta v_t v^{\delta-1}; \tilde{V}_y = \delta v_y v^{\delta-1}; \tilde{V}_{yy} = \delta v_{yy} v^{\delta-1} + \delta(\delta-1)v_y^2 v^{\delta-2} \tag{3.1.9}$$

Therefore equation (3.1.6) becomes:

$$(3.1.10)\qquad \begin{cases} \delta v_t v^{\delta-1} + \frac{1}{2}a(y,t)^2\delta v_{yy}v^{\delta-1} + \frac{1}{2}a(y,t)^2\delta(\delta-1)v_y^2 v^{\delta-2} \\ \quad +[b(y,t) + \rho\frac{\gamma}{1-\gamma}a(t,y)\frac{(\mu(t,y)-r)}{\sigma(t,y)}]\delta v_y v^{\delta-1} \\ \quad +[r\gamma + \frac{\gamma}{\gamma}\frac{(\mu(y,t)-r)^2}{2\sigma(t,y)^2} - \beta]v^\delta + (1-\gamma)v^{\frac{\delta\gamma}{\gamma-1}} \\ \quad +\frac{1}{2}\rho^2\frac{\gamma}{1-\gamma}a(t,y)^2\frac{\delta^2 v_y^2 v^{2(\delta-1)}}{v^\delta} = 0 \\ v(T,y) = 1 \end{cases}$$

Dividing by $\delta v^{\delta-1}$ yields:

$$(3.1.11)\qquad \begin{cases} v_t + \frac{1}{2}a(t,y)^2 v_{yy} + [b(t,y) + \rho\frac{\gamma}{1-\gamma}a(t,y)\frac{(\mu(t,y)-r)}{\sigma(t,y)}]v_y \\ \quad +\frac{1}{\delta}[r\delta + \frac{\gamma}{1-\gamma}\frac{(\mu(t,y)-r)^2}{2\sigma(t,y)^2} - \beta]v \\ \quad +\frac{a(t,y)^2}{2}\frac{v_y^2}{v}[\delta - 1 + \rho^2\frac{\gamma}{1-\gamma}\delta] + \frac{1-\gamma}{\delta}v^{-\delta\frac{\gamma}{1-\gamma}-(\delta-1)} = 0 \\ v(T,y) = 1 \end{cases}$$

Choosing

$$(3.1.12)\qquad \delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$$

we get the simpler equation:

$$(3.1.13)\qquad \begin{cases} v_t + \frac{1}{2}a(t,y)^2 v_{yy} + [b(t,y) + \rho\frac{\gamma}{1-\gamma}a(t,y)\frac{(\mu(t,y)-r)}{\sigma(t,y)}]v_y \\ \quad +\frac{\gamma}{\delta}[r\delta + \frac{\gamma}{1-\gamma}\frac{(\mu(t,y)-r)^2}{2\sigma(t,y)^2} - \frac{\beta}{\gamma}]v + \frac{1-\gamma}{\delta}v^{\frac{-\gamma(1-\rho^2)}{1-\gamma+\rho^2\gamma}} = 0 \\ v(T,y) = 1 \end{cases}$$

Remark that if the Brownian Motions W^1, W^2 are perfectly correlated, the semilinear term in the equation above is removed. We send the reader to examples in Zariphopoulou (1999), related to the situation $\rho = 1$.

3.3.2 The Kreps-Porteus case

In this case, the HJB equation (2.7) becomes:

$$(3.2.1)\quad \begin{cases} \mathcal{J}_t + \frac{1}{2}a(t,y)^2\mathcal{J}_{yy} + b(t,y)\mathcal{J}_y + rx\mathcal{J}_x \\ \quad + \max_{\pi}\{\frac{1}{2}\sigma(t,y)^2\mathcal{J}_{xx}\pi^2 + \rho\sigma(t,y)a(t,y)\mathcal{J}_{xy}\pi + [\mu(t,y)-r]\mathcal{J}_x\pi\} \\ \quad + \max_{c}\{-c\mathcal{J}_x + \frac{\beta}{\alpha}\frac{c^\alpha - (\gamma\mathcal{J})^{\alpha/\gamma}}{(\gamma\mathcal{J})^{\alpha/\gamma-1}}\} = 0 \\ \mathcal{J}(T,x,y) = \frac{x^\gamma}{\gamma} \end{cases}$$

We proceed exactly as in the previous case, that is, we solve first the maximization problem with respect to c, and see that the optimal value is taken for

$$(3.2.2)\qquad c^*(s,x,y) = [\frac{1}{\beta}\mathcal{J}_x(s,x,y)(\gamma\mathcal{J}(s,x,y))^{\alpha/\gamma-1}]^{\frac{1}{\alpha-1}}$$

then we make the transformation (3.1.4) to transform the equation (3.2.1) into:

(3.2.3)

$$\begin{cases} \frac{1}{\gamma}[\tilde{V}_t + \frac{1}{2}a(t,y)^2\tilde{V}_{yy} + b(t,y)\tilde{V}_y] + r\tilde{V} \\ \quad + \max_{\tilde{\pi}}\{\frac{1}{2}(\gamma-1)\sigma(t,y)^2\tilde{V}\tilde{\pi}^2 + \rho\sigma(t,y)a(t,y)\tilde{V}_y\tilde{\pi} + [\mu(t,y)-r]\tilde{V}\tilde{\pi}\} \\ \quad + \beta^{-\frac{1}{\alpha-1}}(\frac{1}{\alpha}-1)\tilde{V}^{1+\frac{\alpha}{(\alpha-1)\gamma}} - \frac{\beta}{\alpha}\tilde{V} = 0 \\ \tilde{V}(T,y) = 1 \end{cases}$$

We solve the optimization problem (out of which we get π^* as in (3.1.7)), then we make again the transformation (3.1.7) and for

$$(3.2.4)\qquad \delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}\quad ,$$

we get

$$
(3.2.5) \qquad \begin{cases} v_t + \frac{1}{2}a(t,y)^2 v_{yy} + [b(t,y) + \rho\frac{\gamma}{1-\gamma}a(t,y)\frac{(\mu(t,y)-r)}{\sigma(t,y)}]v_y \\ \quad +\frac{1}{\delta}[r\gamma + \frac{\gamma}{1-\gamma}\frac{(\mu(t,y)-r)^2}{2\sigma(t,y)^2} - \frac{\beta}{\alpha}]v \\ \quad +\frac{\gamma}{\delta\beta^{\frac{1}{\alpha-1}}}\frac{1-\alpha}{\alpha}v^{1+\frac{\delta\alpha}{(\alpha-1)\gamma}} = 0 \\ v(T,y) = 1. \end{cases}
$$

In the next section, we will summarize our computations.

3.3.3 Summary of computations

We observe that the equations (3.1.13) and (3.2.5) have the same form, that is, they are *reaction-diffusion* equations. For an extensive study of such equations, we recommend the monograph of Rothe (1984), or the book of Smoller (1994).The difficulty herein comes from the fact that the reaction-diffusion equation is not a standard one (ie, has non-Lipschitz nonlinearity), so we need to provide a complete proof for existence and uniqueness. This is done in the Appendix A. The common form for both (3.1.13) and (3.2.5) is:

$$
(3.3.1) \qquad \begin{cases} v_t + \tilde{\mathcal{L}}v + A(t,y)v + Bv^p = 0 \\ v(T,y) = 1 \end{cases}
$$

whose parameters A, B, p are described in (3.3.2) and (3.3.3) below. The parameters are therefore:

- for *Standard Additive Case*:

$$
\begin{array}{lrcl}
(3.3.2) & A(t,y) & = & \frac{1}{\delta}[r\gamma + \frac{\gamma}{1-\gamma}\frac{(\mu(t,y)-r)^2}{2\sigma(t,y)^2} - \beta] \\
 & B & = & \frac{1-\gamma}{\delta} \\
 & p & = & \frac{-\gamma(1-\rho^2)}{1-\gamma+\rho^2\gamma} < 1
\end{array}.
$$

- for *Kreps-Porteus*:

$$
\begin{array}{lrcl}
(3.3.3) & A(t,y) & = & \frac{1}{\delta}[r\gamma + \frac{\gamma}{1-\gamma}\frac{(\mu(t,y)-r)^2}{2\sigma(t,y)^2} - \frac{\beta}{\alpha}] \\
 & B & = & \frac{\gamma}{\delta\beta^{\frac{1}{\alpha-1}}}\frac{1-\alpha}{\alpha} \\
 & p & = & 1 - \frac{\delta\alpha}{(1-\alpha)\gamma}.
\end{array}
$$

In both cases, $\tilde{\mathcal{L}}$ is a second order linear differential operator given by:

$$
(3.3.4) \quad (\tilde{\mathcal{L}}v)(t,y) = \frac{1}{2}a(t,y)^2 v_{yy}(t,y) + [b(t,y) + \rho\frac{\gamma}{1-\gamma}a(t,y)\frac{(\mu(t,y)-r)}{\sigma(t,y)}]v_y
$$

and

$$
(3.3.5) \qquad \delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma} > 0.
$$

3.3.4 Assumptions

The assumptions below are made to insure, on one hand, strong existence and uniqueness for the diffusions S, Y, and on the other hand, to allow us to prove existence and uniqueness for the reaction diffusion equation (3.3.1).

Assumption A.1: We assume that σ is strictly positive and also invertible with respect to its spatial argument; that is, there is a constant $C_0 > 0$ such that:

$$\sigma(s, y) \geq C_0, t \leq s \leq T, y \in R \tag{3.4.1a}$$

and also

$$\sigma^{-1}(s, \cdot) \text{ exists } , t \leq s \leq T \tag{3.4.1b}$$

Assumption A.2: We assume that any coefficient q which appears in the equations describing any diffusion we encounter is Lipschitz, uniformly in time, that is, we assume that there is a constant $C_1 > 0$ such that:

$$|q(t_1, y_1) - q(t_2, y_2)| \leq C_1 |y_1 - y_2| \tag{3.4.2}$$

Assumption A.3: We assume that the differential operator $\tilde{\mathcal{L}}$ is uniformly parabolic, that is:

$$a(s, y) \geq C_2, t \leq s \leq T, y \in R \tag{3.4.3}$$

for some constant $C_2 > 0$.

Assumption A.4: This assumption is automatically satisfied in the Standard Additive case if $\gamma < 0$, or in the Kreps-Porteus case if $\alpha\gamma < 0$.

a) for the Standard Additive case:

there is a constant $C_3 < 0$ such that:

$$r + \frac{(\mu(s, y) - r)^2}{2(1 - \gamma)\sigma^2(s, y)} - \frac{\beta}{\gamma} \leq C_3 \tag{3.4.4a}$$

b) for the Kreps-Porteus case:

there is a constant $C_3 < 0$ such that:

$$r + \frac{(\mu(s,y) - r)^2}{2(1-\gamma)\sigma^2(s,y)} - \frac{\beta}{\gamma\alpha} \leq C_3 \tag{3.4.4b}$$

Assumption A.5: Any q which is either a coefficient of the diffusions S or Y, or is equal to A from (3.3.1) satisfies:

$$q \in b((t,T); W^{1,\infty}(R)) \tag{3.4.5}$$

Assumption A.1 is *not* only a technical condition, it is exactly the ingredient that insures the observability of the process Y. This was used in a crucial way, for the reason the HJB equation is deterministic is exactly the fact that not only S, but also Y is measurable with the filtration used in the definition of the recursive utility. The motivation is, very shortly, the following: if we observe S, we could observe its quadratic variation, that is, formally, $\sigma^2(s, Y_s)S_s^2 ds$. That basically means that we could recover $\sigma^2(s, Y_s)$ from our observations, and since σ is positive and properly invertible, we recover Y_s. That proves that Y_s is measurable with respect to $\mathcal{F}_s$. We do not work therefore with partial information, and the results of the chapter will prove that *even in the case of full information, incorporating timing of resolution of uncertainty in the utility brings more anxiety to the trader.* For details in the case of partial information, see the next chapter.

Assumptions A.4 and A.5 are technical conditions, sufficient to make our analysis of the pde (3.3.1) work. They are not *necessary* conditions for the solvability of the entire optimization problem, and could be relaxed or modified in various ways.

3.4 THE SOLUTION OF OUR PROBLEM

Even though the reaction-diffusion equation (3.3.1) is not of a standard type (i.e., non-Lipschitz and with non-monotonic nonlinearities), it turns out that the coefficients p and B, which could be responsible for a blow up of the solution, "match their values" in such a way that when one is "bad" for the equation, the other has the proper sign to repair the "bad" effect.

A detailed analysis of the equation, along with constructive, elementary existence proofs that also allows us to write bounds on the value function, is given in the Appendix A. We demonstrate later that the solution constructed is suitable for practical applications.

Among parameters, we distinguish four different cases:

Case I: $\gamma \in (-\infty, 0), \alpha \in (-\infty, 0)$

In this case, $A \geq 0, B > 0, p < 1$.

Case II: $\gamma \in (0, 1), \alpha \in (-\infty, 0)$

In this case, $A \geq 0, B < 0, p > 1$.

Case III: $\gamma \in (-\infty, 0), \alpha \in (0, 1)$

In this case $A \leq 0, B < 0, p > 1$.

Case IV: $\gamma \in (0, 1), \alpha \in (0, 1)$

In this case, $A \leq 0, B > 0, p < 1$.

With a Kreps-Porteus utility, we might encounter any of the four cases above; the Standard Additive Utility belongs to Case IV.

To recover the intuition of the proof, a look at the various cases shows that when the parameter p is bigger than 1, which may cause explosion of the solution, the parameter B is negative. So actually the solution is "slowed down", so that we are able to get existence results. The uniqueness in the problem (3.3.1) should be understood as uniqueness of positive solution. Even though the $v \to v^p$ is not Lipschitz, so we cannot establish uniqueness right away, comparison shows that if solutions exist, then they are "properly" bounded (away from zero), so actually $v \to v^p$ *is* Lipschitz on the domain of interest and that yields the uniqueness. The technically oriented reader is referred once again to the Appendix for a full proof of existence and uniqueness of positive solution for (3.3.1).

We are therefore able to obtain the following result:

Theorem 4.1 : *If the Assumptions A.1-A.5 are satisfied, then the control problem (1.5) has a unique solution of the form*

$$\mathcal{J}(t,x,y) = \frac{x^\gamma}{\gamma} v(t,y)^\delta, \tag{4.1}$$

where

$$\delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma} > 0. \tag{4.2}$$

Also, there is a positive constant $C = C(C_0, C_1, C_2, C_3)$ (and whose functional form is different according to Cases I-IV), such that

- *in case I:*

$$1 \le v(t,y) \le 1 + e^{C(T-t)} \tag{4.3}$$

- *in cases II and III:*

$$0 < \tilde{w}(T) \le \tilde{w}(T-t) \le v(t,y) \le e^{C(T-t)} \tag{4.4}$$

where $\tilde{w}$ depends only upon t and

$$\begin{cases} \tilde{w}_t = -C\tilde{w} - Be^{(p-1)Ct}\tilde{w}^p \\ \tilde{w}(0) = 1 \end{cases} \tag{4.5}$$

- *in Case IV:*

$$e^{-C(T-t)} \le v(t,y) \le 1 + e^{-C(T-t)} \tag{4.6}$$

The optimal consumption and investment strategies can be obtained from (3.1.3), (3.2.2) and (3.1.7) respectively, and they are given in the following

Theorem 4.2: *The solution v of (3.3.1) is of class $C^{1,2}((0,T)\times R)$ and the optimal consumption-investment policies of the optimization problem (2.6) are given by:*

- *For Standard Additive case:*

$$c_s^* = (X_s^*)^{\frac{\gamma}{\gamma-1}} v(s,Y_s)^{\frac{\delta}{\gamma-1}}; \tag{4.7}$$

- *For the Kreps-Porteus case:*

$$c_s^* = (\frac{1}{\beta})^{\frac{1}{\alpha-1}} X_s^* v(s,Y_s)^{\frac{\delta\alpha}{\gamma(\alpha-1)}}. \tag{4.8}$$

The optimal investment strategy is given in both cases by

$$\pi_s^* = \Big[\frac{1}{1-\gamma}\frac{\mu(s,Y_s)-r}{\sigma(s,Y_s)^2} + \frac{\rho\delta}{1-\gamma}\frac{a(s,Y_s)}{\sigma(s,Y_s)}\frac{v(s,Y_s)}{v(s,Y_s)}\Big]X_s^*. \tag{4.9}$$

The process of optimal wealth X_s^* is the unique solution of the stochastic differential equation

$$\begin{cases} dX_s^* = [rX_s^* + (\mu(s,Y_s)-r)\pi^*(s,X_s^*,Y_s) - c^*(s,X_s^*,Y_s)]ds \\ \quad +\sigma(s,Y_s)\pi^*(s,X_s^*,Y_s)dW_s^1 \\ X_t^* = x. \end{cases} \tag{4.10}$$

where the deterministic functions π^* and c^* are the ones from (3.1.3), (3.1.6) and (3.2.2). ∎

In words, *the value function is a HARA utility of the initial wealth times the distorted solution of a reaction-diffusion equation*. A full interpretation of this result will be obtained in the next section.

3.5 INTERPRETATION OF RESULTS AND VARIATIONAL FORMULA FOR v

In addition to the theoretical importance of Theorem 4.1 (establishing existence and uniqueness for the optimization problem), it offers insight about the attitude of the agent toward the presence of a stochastic factor when timing of resolution of uncertainty *does* matter. Albeit the initial value of the stochastic factor influences the value function, we actually proved that it is impossible that the value function vanishes or explodes as a consequence of the presence of the stochastic factor. To rephrase, no “extreme” situations appear because of the presence of Y (this is technically reflected by the fact that we have bounds on v). We will also prove that the value function might separated into two factors: the HARA utility of the wealth and another, distorted value function associated with the problem of *maximizing* a Uzawa-type *disutility*.

We continue by analyzing rigorously the way the stochastic factor influences the value function. To do that, we will provide a variational formula for the solution v of (3.3.1); using comparison principles for stochastic backward differential equations (see the Appendix B for a brief on the theory), we will represent v as the value function of a stochastic problem with a recursive criterion. The power transformation used earlier is going to prove its usefulness again, as the generator of the aforementioned recursive criterion turns out to be linear in V, that is, of Uzawa type.

We begin first by considering the equation (3.3.1): using the generalized Feynman-Kac formula presented in the Appendix B, we could interpret its solution as the initial value of the solution of a Backward-Forward Stochastic Differential Equation(FBSDE).To do that, we introduce first the forward process:

$$\begin{cases} d\tilde{Y}_s = [b(\tilde{Y}_s, s) + \rho \frac{\gamma}{1-\gamma} a(\tilde{Y}_s, s) \frac{(\mu(\tilde{Y}_s, s) - r)}{\sigma(\tilde{Y}_s, s)}]ds + a(\tilde{Y}_s, s) d\tilde{W}_s \\ \tilde{Y}_t = y \in R \end{cases} \tag{5.1}$$

The infinitesimal generator of the diffusion $\tilde{Y}$ is $\tilde{\mathcal{L}}$, given by (3.3.4).

The backward component of the FBSDE is:

$$U_s = E[\int_s^T g(\omega, l, U_l)dl + 1| \mathcal{F}_s] \tag{5.2}$$

where

$$g(\omega, s, U) = A(s, \tilde{Y}_s(\omega))U + BU^p \tag{5.3}$$

and hence the solution v of (3.3.1) is given by

$$v(t, y) = U_t \tag{5.4}$$

We want to warn the reader that it is not obvious that the above equation has a unique positive solution; arguments for existence and uniqueness of positive solution are made in the Appendix C.

We will continue with the interpretation of v as a value function. Looking once more at the four possible cases for our parameters as presented in Section 4, we could readily observe that:

$$\begin{cases} \text{if } p \in (-\infty, 0), Bv^p = \max_{c>0}\{B(1-p)c^p + Bpvc^{p-1}\} \\ \text{if } p \in \Big((0, \infty) - \{1\}\Big), Bv^p = \min_{c>0}\{B(1-p)c^p + Bpvc^{p-1}\} \end{cases} \tag{5.5}$$

We define the generators:

$$g(\omega, c, s, U) = B(1-p)c^p + [Bpc^{p-1} + A(s, \tilde{Y}_s(\omega))]U \tag{5.6}$$

We make the observation that g is linear in U, that is, it has the typical form of the Uzawa utility generator $g = u(c) + \theta(c)U$, in which we allowed the *discount factor* θ to be stochastic.

We also introduce recursive criteria $U^{(c_i)_i}$, given by

$$U_s^{(c_i)_i} = E[\int_s^T g(l, c_l, l, U_l^{(c_i)_i})dl + 1|\mathcal{F}_s] \tag{5.7}$$

Applying Proposition (2.1) in the Appendix B, we get the following

Theorem 5.1 *Let $U^{(c_s)_s}$ defined in (5.7). Then*

- *if* $p < 0$ *we have:*

$$v(t, y) = \max_{c.} U^{(c_t)_t} \tag{5.8}$$

- *if* $p > 0, p \neq 1$, *we have:*

$$v(t, y) = \max_{c.} \{-U^{(c_t)_t}\} \tag{5.9}$$

∎

The first observation we make is that the distortion introduced earlier is useful in the sense that v could be seen as the solution of a stochastic control problem with a *linear* and recursive criterion, that is, one of the form:

$$g(c, s, U) = u(c) + \theta(c)U \tag{5.10}$$

for

$$(5.11) \qquad \begin{cases} u(c) = B(1-p)c^p \\ \theta(c) = [Bpc^{p-1} + A(s, \tilde{Y}_s)] \end{cases}$$

In the above expression, u could be interpreted as a *disutility function*, and θ as a *discount factor*. In our case, we observe that the discount rate is stochastic, as a consequence of the presence of the stochastic factor. We refer the reader to the Appendix D for an explanation of the notions of *utility from consumption* and *discount factor* used for Uzawa utility.

We now look at the specific formula we have gotten in the representation of v. In the case of $p > 0, p \neq 1$, we remark that u is a utility function (that is, it is increasing with respect to consumption; it is not concave, so if $p > 1$ it is not risk averse). In this case, v is the value function of a *maximization of a disutility* problem; if we interpret $U^{(c_s)_s}$ as a Uzawa utility, then v is obtained by *maximizing* $-U^{(c_s)_s}$, so the presence of the stochastic factor is a negative trait, as it could be seen as a factor which enforces the agent to *maximize* a Uzawa-type disutility (then distort the value function obtained and multiply it by the HARA utility of the terminal wealth), in order to obtain the value function of (2.6).

In the case of $p > 0, p \neq 1$, u is decreasing with respect to consumption, hence it may be regarded as a *disutility* function again. The recursive criterion, of Uzawa type, could therefore be regarded as a disutilty function as well, and the control problem we face is to *maximize* this criterion. We see therefore again the negative trait of the presence of the stochastic factor, as we *maximize* a disutility in order to obtain the term v that gives the (distorted) contribution of the stochastic factor to the value function of (2.6).

Since the distortion is a monotonic transform, we see how the "bad character" of the presence of Y inherits from v to $\mathcal{J}$ from (2.6).

Another feature of Theorem 5.1 is that it incorporates the stochastic factor into the criterion to be maximized, and we know exactly how, namely, the stochastic factor comes into the expression of the discount factor of the Uzawa criterion.

To see another situation to which this type of representation applies, we refer to Zariphopoulou (1999).

Chapter 4

MARKETS INFORMATIONALLY INCOMPLETE

Suppose that a company which is not publicly traded yet (but plans to become public) is trying to hire an employee. Due to lack of financial slack – but having very good growth opportunities – the company may promise the future employee a fraction of its equity instead of a bigger salary. If the company does well and goes public, the employee may cash her stock. If the employee lacks strong confidence in the company,then stock options, rather than plain stock, may be awarded initially to the employee. Given that there is no stock price to observe, one cannot use a Black-Scholes formula to value the stock options. How can we value then the package given to the employee? How can we hedge the stock options?

The answer is that initially , the employee knows something about the company, and implicitly about the value of its equity. For example, the value of the company, as known to the future employee, is between \$ 10 and \$ 25. This is the same as saying that the employee believes initially that the value of the company has some distribution over the interval $\$[10, 25]$. Moreover, the company is not isolated; by observing its environment, we can infer even more about it. Suppose for example that our company hosts online auctions. Then most of the people who buy things in auctions use some electronic service to pay, which is, say, offered through a big Internet portal. The success of the online bidding company has a certain impact on the value of the Internet portal, and this success can be publicly observed, and quantified to some extent, if the portal is a public company.

Mathematically, this amounts to filter out the ”price” of our company from observations on the portal. This will become very clear in this chapter. We present a model with two entangled prices where one is not observable. We derive the corresponding HJB equation and we transform it into a linear equation, which has good regularity properties. This linearization is also useful for numerics.

Suppose now that instead of the incompleteness generated by the unobservability of the ”price” of our company, we can observe the price, only we cannot trade in its stock. In this new setting we expect an optimal strategy that is dependent on the stock price that formerly went unobserved. The question arises, is the optimal strategy conditionally the same as the one in the partially observed market ? The answer is yes for logarithmic utility, and no for the other types analyzed. The importance of this question comes from market microstructure models. In such a typical model, different agents have specific information and the prices form in a Rational Expectations Equilibrium. Price itself conveys information. These models rely on the existence of non-informed, or noisy traders, which are agents who do not have private information. The question that arises is whether by observing the price and the relative success (for example in terms of wealth) of the other market participants, the uninformed agents will actually discover their own informational disadvantage and withdraw from the market. An interesting question, remained unsolved in the present work, is the reciprocal of the affirmations above: precisely, if the Separation Principle described above holds, can we infer the functional form of the agent's utility?

This is not the only story behind such models. In fact we provide one example in which the solution can be computed in closed form, where the health of the CEO of a company is the unobservable factor and the stock price of the company is public information. This model is inspired by the anecdotal evidence that sudden shocks to the health of the CEO produce shocks in the stock price, in the opposite if the shock to the health is unexpected. For example, the stock prices go up, on average, following a sudden death of the CEO.

Technically, the present work is the first in which the HJB equation arising is numerically tractable; while this is possible due to the linearization of the HJB equation, we hope it is possible in a general setting (see Chapter 5 "Open Problems"). The structure of this chapter is as follows: Section 1 describes the model. Section 2 analyzes the corresponding HJB equation and states a regularity result for its solution. Section 3 provides an example where the infinite dimensional HJB equation could be solved in closed form. Section 4 studies whether a separation principle holds for the optimal strategy. Section

5 describes a numerical algorithm to solve the problem.

4.1 THE MODEL

We restrict to a finite time horizon $[0, T]$ and we work with a market model hosted by $(\Omega, \mathcal{F}, P)$ consisting of a bond

$$(1.1) \qquad \begin{cases} dB_s = rB_s ds \\ B_t = 1, \qquad t \le s \le T \end{cases}$$

a stock

$$(1.2) \qquad \begin{cases} \frac{dS_s}{S_s} = \mu(s, Y_s)ds + \sigma(s)dW_s^1 \\ S_t = S, \;\; T \le s \le T, \;\; \zeta_s := \sigma(B_u, S_u; t \le u \le s) \end{cases}$$

and a "stochastic factor" Y,

$$(1.3) \qquad dY_s = b(s, Y_s)ds + a(s, Y_s)[\rho dW_s^1 + \sqrt{1-\rho^2}dW_s^2]$$

with initial distribution (at time t). We assume ν_t has a density p_t. W^1 and W^2 are independent Brownian Motions and $\rho \in [-1, 1]$. We assume that an agent on the market has access to observations of S and B, but is unable to observe Y. For future reference, we take the processes

$$(1.4) \qquad \theta_s = \frac{\mu(s, Y_s)}{\sigma(s)}$$

$$(1.5) \qquad L_s = \exp\Big\{ -\int_t^s \theta_u dY_u - \frac{1}{2}\int_t^s \theta_u^2 du \Big\}$$

and the differential operators

$$\mathcal{L} = \frac{1}{2}a^2(s,\cdot)\partial_2 + b(s,\cdot)\partial \tag{1.6}$$

$$\mathcal{B} = \frac{\mu(\int,\cdot)}{\sigma(\int)}\infty + \rho\dashv(\int,\cdot)\partial. \tag{1.7}$$

The agent invests an amount π_s of his wealth in stock, for each instant s, so that his wealth $X_s^{\pi,x,p}$ satisfies:

$$\begin{cases} dX_s^{\pi,x,p} = [rX_s^{\pi,x,p} + (\mu(s,Y_s) - r)\pi_s]ds + \sigma(s)\pi_s dW_s^1 \\ X_t^{\pi,x,p} = x. \end{cases} \tag{1.8}$$

We assume that the strategy π_s is measurable with respect to the filtration generated by S and B but not of that generated by Y. We also ask that $\pi \in \mathcal{A} = \{(K_s)_s : K_s \text{ is } \sigma(S_u; t \le u \le s)\text{-adapted}, E\int |K_s|^2 ds < \infty\}$.

The objective of the agent is to choose π such that the following (terminal) criterion is maximized:

$$E[U(X_T^{\pi,x,p})\tilde{U}(Y_T)] \tag{1.9}$$

The value function associated with the optimization problem above is

$$\mathcal{J}(t,x,p) = \sup_{\pi\in\mathcal{A}} E[U(X_T^{\pi,x,p})\tilde{U}(Y_T)] \tag{1.10}$$

We have an optimization problem with *partial information*. We continue by listing our assumptions on the parameters:

- (H1) $\exists C > 0$ such that $\sigma(\cdot) \ge C$

- (H2) $a^2(s,\cdot)$, $\frac{\mu(s,\cdot)}{\sigma(s)}$, $b(s,\cdot)$ have derivatives up to order two, bounded uniformly with respect to s, $t \leq s \leq T$.

- (H3) $p_t \in H^2(R)$ (p is the density of Y_t).

- (H4) U satisfies: $\frac{U''U}{(U')^2} = k = \text{const.}$ or $U(x) = \log x$

To illustrate the (unusual) choice of the utility, we will give few examples.

Examples of utilities satisfying (H4)

- a)$U(x) = x^\gamma/\gamma, \gamma \in (-\infty, 1), \gamma \neq 0;$ (here $k = -\frac{1-\gamma}{\gamma}$)

- $\tilde{U}(y) = 1$

  This is the Merton problem with partial information and HARA utility of terminal wealth (see Zariphopoulou(2000) for the full information solution).

- b)$U(x) = -e^{-\gamma x}, \gamma > 0;$ (here $k = 1$)

- $\tilde{U}(y) = 1$

  This is the Merton problem with partial information and exponential utility of terminal wealth.

- c)$U(x) = -e^{-\gamma x}, \gamma > 0;$ (here $k = 1$)

- $\tilde{U}(y) = e^{\gamma g(y)}$

In this case the criterion to be maximized is

$$E[-e^{-\gamma(X_T^{x,\pi,p}-g(Y_T))}]$$

We observe that the optimal strategy for the above criterion is also optimal for

$$E[1-e^{-\gamma(X_T^{x,\pi,p}-g(Y_T))}]$$

and this represents the exponential utility of the writer of an European option on the unobserved asset Y, with the payoff $g(Y_T)$ (see Davis, Panas and Zariphopoulou (1993) for details). The optimal strategy will also be the hedging strategy of such an agent. We will obtain an explicit form of this strategy in Example 1, for the case when S and Y are not correlated.

In what follows, it is convenient to change the probability such that the (observable) stock S becomes a Brownian integral. This technique is standard for partial information problems (see Lakner (1998), Lasry and Lions (1999) or Pham and Quenez (2000)). The new probability P^0 is given by

$$\frac{dP^0}{dP} = L_T. \tag{1.11}$$

The Bayes rule gives (see Karatzas and Shreve (1992))

$$E[f(Y_s) \mid \zeta_s] = \frac{E^0[f(Y_s)L_s^{-1} \mid \zeta_s]}{E^0[L_s^{-1} \mid \zeta_s]} \tag{1.12}$$

and

$$E^0[f(Y_s)L_s^{-1} \mid \zeta_s] = \int_R f(y)p_s(y)dy \tag{1.13}$$

where p_s is the unnormalized conditional density of Y_s given ζ_s.

Under P^0, the process

$$\widetilde{W}_s^1 = W_s^1 + \int_t^s \theta_u du \tag{1.14}$$

is a Brownian Motion, independent of W^2.

4.2 THE HJB EQUATION

Firstly, classical results in filtering theory (see Zakaï (1969) for derivation of the filtering equation) give the following

Proposition 2.1. (Krylov and Rozovskii (1982)). *The unnormalized conditional density p_t satisfies the stochastic pde*

$$\begin{cases} dp_s = \mathcal{L}^* p_s ds + \mathcal{B}^* p_s d\widetilde{W}_s^1 \\ p_t \quad \textit{given} \end{cases} \tag{2.1}$$

Also, $p_s \in H^2(R)$, $t \leq s \leq T$ and $\exists C > 0$ such that

$$E^0\Big[\sup_{t\leq s\leq T} \|p_s\|^2_{H^2(R)}\Big] \leq \|p_t\|^2_{H^2(R)} e^{c(T-t)}. \tag{2.2}$$

∎

Knowing that $p_s \in H^2(R)$, $t \leq s \leq T$, we could now look for a value function

$$\mathcal{J} : [t,T] \times (0,\infty) \times (H^2(R) \cap \text{ densitites}) \to (0,\infty). \tag{2.3}$$

We translate the optimization problem (1.10) in the language of P^0:

$$
\begin{aligned}
\mathcal{J}(t,x,p) &= \sup_{\pi\in\mathcal{A}} E[U(X_T^{\pi,x,p})\tilde{U}(Y_t)] \\
&= \sup_{\pi\in\mathcal{A}} E^0[U(X_T^{\pi,x,p})\tilde{U}(Y_T)L_T^{-1}] \\
&= \sup_{\pi\in\mathcal{A}} E^0[E^0[U(X_T^{\pi,x,p})\tilde{U}(Y_T)L_T^{-1} \mid \zeta_T]] \\
&= \sup_{\pi\in\mathcal{A}} E^0\Big[U(X_T^{\pi,x,p}) \textstyle\int_R \tilde{U}(y)p_T(y)dy\Big]
\end{aligned}
\tag{2.4}
$$

(we used (1.13) to infer the last equality.)

In what follows, we need an Itô formula in infinite dimensions. By approximating $p \in H^2(R)$ with finite dimensional functions ($H^2(R)$ is separable Hilbert space) and using finite dimensional formula, it is not difficult to prove the following:

Proposition 2.2. *Let H be a separable Hilbert space, $(P_s)_{s\in[t,T]}$ a H-valued diffusion such that*

$$dP_s = A(P_s)ds + B(P_s)dW_s$$

where $A, B : H \to H$ and let $F : [t,T]\times H \to R$, $F \in C^{P1,2}((t,T)\times H)$ (differentiation with respect to the variable in H is understood in Fréchet sense). Then

$$
\begin{aligned}
dF(s,P_s) &= F_t(s,P_s)ds + F_p(s,P_s)(A(P_s)ds + B(P_s)dW_s) \\
&\quad + \tfrac{1}{2}F_{pp}(s,P_s)(A(P_s)ds + B(P_s)dW_s, A(P_s)ds + B(P_s)dW_s).
\end{aligned}
\tag{2.5}
$$

In the formula above,

$$F_p(s,P_s) \in H', \quad F_{pp}(s,P_s) \in (H\times H)'.$$

Taking $H = (0, \infty) \times H^2(R)$, we are able to derive formally the HJB equation satisfied by the value function. Assuming enough regularity in $\mathcal{J}$, we see that formally (formally because we did not prove any regularity results on $\mathcal{J}$):

$$
\begin{aligned}
d\mathcal{J} \quad = \quad & \Big\{\mathcal{J}_t + (rX_s^{\pi,x,p} - r\pi_s)\mathcal{J}_x + \mathcal{J}_p(\mathcal{L}^* p_s) + \\
& +\tfrac{1}{2}\pi_s^2\sigma^2(s)\mathcal{J}_{xx} \\
& +\tfrac{1}{2}\mathcal{J}_{pp}(\mathcal{B}^* p_s, \mathcal{B}^* p_s) + \\
& +\pi_s\sigma(s)\mathcal{J}_{xp}(\mathcal{B}^* p_s)\Big\}ds \\
& +Q_s d\widetilde{W}_s^1,
\end{aligned}
\tag{2.6}
$$

where Q_s is a predictable square integrable process. Above, any $\mathcal{J}_{uv}$, $u, v \in \{t, x, p\}$ is shorthand for $\mathcal{J}_{uv}(t, x, p)$, for example

$\mathcal{J}_p(\mathcal{L}^* p_s)$ means $\mathcal{J}_p(t, x, p)(\mathcal{L}^* p_s)$, etc.

In order to have a HJB equation on the Hilbert space $H^2(R)$, we need to extend the definition (2.4) to any $p \in H^2(R)$; we will then solve the optimization problem for all $p \in H^2(R)$, and if $\widetilde{\mathcal{J}}$ is the solution of the problem on $H^2(R)$, namely,

$$\widetilde{\mathcal{J}} : [t, T] \times (0, \infty) \times H^2(R) \to (0, \infty).$$

We could obtain the value function (2.3) by

$$\mathcal{J} = \widetilde{\mathcal{J}}\big|_{[t,T]\times(0,\infty)\times(H^2(R)\cap\,\mathcal{P})}. \tag{2.7}$$

Above, $\mathcal{P} = \{p : R \to R^+ \text{ Borel measurable} : \int_R p(y)dy = 1\}$ are the probability densities. We will use only $\mathcal{J}$ alone, and we understand it as defined on $H^2(R)$.

Remark 1: We have restricted our data to $H^2(R)$. To see why this choice has been made, see Ishii (1993) for a discussion of how to formulate correctly infinite dimensional pde's.

Formally, $\mathcal{J}$ is "the solution" of

$$\text{(2.8)} \qquad \begin{cases} \mathcal{J}_t + rx\mathcal{J}_x + \mathcal{J}_p(\mathcal{L}^*p) + \\ \qquad + \frac{1}{2}\mathcal{J}_{pp}(\mathcal{B}^*p, \mathcal{B}^*p) \\ \qquad + \max_{\pi} \Big\{ \frac{1}{2}\pi^2\sigma^2(t)\mathcal{J}_{xx} + \pi\sigma(t)\mathcal{J}_{xp}(\mathcal{B}^*p) \\ \qquad\qquad - r\pi\mathcal{J}_x \Big\} = 0 \\ \\ \mathcal{J}(T, x, p) = U(x) \int_R \tilde{U}(y)p(y)dy. \end{cases}$$

Remark 2: $\int_R p(y)dy$ is not 1, as we extended $\mathcal{J}$ to $H^2(R)$ (the value function is not defined solely on densities now) The aforementioned integral equals 1 if p is also a probability density. We first look for a solution of the form

$$\text{(2.9)} \qquad \mathcal{J}(t, x, p) = K(t, e^{r(T-t)}x, p)$$

and we obtain that K satisfies

$$
(2.10)\qquad \begin{cases} K_t + K_p(\mathcal{L}^* p) + \dfrac{1}{2} K_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + \\ \qquad + \max\limits_{\hat{\pi}} \Big\{ \dfrac{1}{2}\sigma^2(t) K_{xx}\hat{\pi}^2 + \sigma(t) K_{xp}(\mathcal{B}^* p)\hat{\pi} - r\hat{\pi}K_x \Big\} = 0 \\ K(T,x,p) = U(x) \displaystyle\int_R \tilde{U}(y)p(y)dy \end{cases}
$$

We replaced the control π with the new control $\hat{\pi} = \pi e^{r(T-t)}$.

Now, we can simplify further the HJB equation, but this will be done differently for the logarithmic utility than for the other type we are analyzing. We will begin with the logarithmic utility first, as this is a case when the feedback control takes an explicit form.

4.2.a LOGARITHMIC UTILITY

If $U(x) = \log x$, then (2.10) becomes

$$
(2.10a)\qquad \begin{cases} K_t + \dfrac{1}{2} K_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + K_p(\mathcal{L}^* p) + \\ \qquad + \max\limits_{\hat{\pi}} \Big\{ \dfrac{1}{2}\sigma^2(t) K_{xx}\hat{\pi}^2 + \sigma(t) K_{xp}(\mathcal{B}^* p)\hat{\pi} - rK_x\hat{\pi} \Big\} = 0 \\ K(T,x,p) = \log x \displaystyle\int_R \tilde{U}(y)p(y)dy \end{cases}
$$

As we do for the Merton problem with logarithmic utility, we look for

$$
(2.11a)\qquad K(t,x,p) = \log x \int_R \tilde{U}(y)p(y)dy + V(t,p)
$$

and where V solves

$$
(2.12a) \quad \begin{cases} V_t + \frac{1}{2} V_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + V_p(\mathcal{L}^* p) + \\ \quad + \max_{\tilde{\pi}} \Big\{ -\frac{1}{2}\sigma^2(t) \int_R \tilde{U}(y) p(y) dy \tilde{\pi}^2 + \sigma(t) \int_R (\mathcal{B}\tilde{U})(y) p(y) dy \tilde{\pi} \\ \quad -r \int_R \tilde{U}(y) p(y) dy \tilde{\pi} \Big\} = 0 \\ V(T,p) = 0 \end{cases}
$$

Above, $\tilde{\pi} = \frac{\hat{\pi}}{x}$ is a new control. Solving the maximization of the quadratic,

$$
(2.13a) \quad \begin{cases} V_t + \frac{1}{2} V_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + V_p(\mathcal{L}^* p) + \\ \quad + \frac{\Big[\sigma(t) \int_R (\mathcal{B}\tilde{U})(y) p(y) dy - r \int_R \tilde{U}(y) p(y) dy\Big]^2}{2\sigma^2(t) \int_R \tilde{U}(y) p(y) dy} = 0 \\ V(T,p) = 0 \end{cases},
$$

which is a (infinitely dimensional) *linear* pde.

The optimal policy π is given by

$$
(2.14a) \qquad \pi(t,x,p) = \frac{1}{\sigma^2(t)} \Big[\frac{\sigma(t) \int_R (B\tilde{U})(y) p(y) dy}{\int_R \tilde{U}(y) p(y) dy} - r \Big] x
$$

We continue solving the problem for the other utilities presented in (H4), the ones not logarithmic.

4.2.bc HARA AND EXPONENTIAL UTILITIES

We observe that K separates as

$$
(2.11) \qquad K(t,x,p) = U(x) V(t,p)
$$

with V satisfying

$$(2.12) \qquad \begin{cases} V_t + V_p(\mathcal{L}^* p) + \dfrac{1}{2} V_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + \\ \quad + \max_{\tilde{\pi}} \Big\{ \dfrac{1}{2} \sigma^2(t) k V \tilde{\pi}^2 + \sigma(t) V_p(\mathcal{B}^* p) \tilde{\pi} - r \tilde{\pi} V \Big\} = 0 \\ V(T, p) = \displaystyle\int_R \tilde{U}(y) p(y) dy \end{cases}$$

We replaced again the former control $\hat{\pi}$ with the new control $\tilde{\pi} = \hat{\pi} \frac{U'}{U}$.

Solving the maximization of the quadratic, we obtain

$$(2.13) \qquad \begin{cases} V_t + V_p(\mathcal{L}^* p) + \dfrac{1}{2} V_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + \\ \quad - \dfrac{(\sigma(t) V_p(\mathcal{B}^* p) - r V)^2}{2 k V \sigma^2(t)} = 0 \\ V(T, p) = \displaystyle\int_R \tilde{U}(y) p(y) dy \end{cases}$$

The way we continue depends now on the value of k. If $k \neq 1$ then we look for

$$(2.14b) \qquad V(t, p) = v(t, p)^{\delta}$$

and if

$$(2.15b) \qquad \delta = \frac{k}{k-1} (= 1 - \gamma \text{ if } U(x) = x^{\gamma}/\gamma)$$

we obtain

$$
(2.16b) \qquad \begin{cases} v_t + \frac{1}{2} v_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + v_p \Big\{ (\mathcal{L} + \frac{r}{k\sigma(t)} \mathcal{B})^* p \Big\} - \\ \quad - \frac{r^2}{2k\sigma^2(t)} v = 0 \\ v(T,p) = \Big\{ \int_R \tilde{U}(y) p(y) dy \Big\}^{1/\delta} \end{cases}
$$

This is a *linear* pde.

If $k = 1$, then the nonlinear term $\frac{V_p^2(\mathcal{B}^* p)}{V}$ in (2.13) cannot be removed by a substitution of the form (2.14b), and we seek instead a solution of the form

$$
(2.14c) \qquad V(t,p) = e^{v(t,p)}
$$

We obtain that v satisfies

$$
(2.15c) \qquad \begin{cases} v_t + \frac{1}{2} v_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + v_p \Big[(\mathcal{L} + \frac{r}{\sigma(t)} \mathcal{B})^* p \Big] - \\ \quad - \frac{r^2}{2\sigma^2(t)} = 0 \\ v(T,p) = \log \Big(\int_R \tilde{U}(y) p(y) dy \Big) \end{cases}
$$

The optimal policies are given respectively by

$$
(2.16) \qquad \pi(t,x,p) = \begin{cases} \frac{1}{R(xe^{r(T-t)})} \frac{\delta\sigma(t) v_p(\mathcal{B}^* p) - rv}{\sigma^2(t) v} x e^{r(T-t)} & \text{if } k \neq 1 \\ \frac{1}{R(xe^{r(T-t)})} \frac{\sigma(t) v_p(\mathcal{B}^* p) - r}{\sigma^2(t)} & \text{if } k = 1 \end{cases}
$$

where

$$R(x) = -\frac{U'(x)}{xU''(x)} \tag{2.17}$$

is a the risk aversion of the utility U.

The Main Theorem

We can summarize our computations in the following statement:

Main Theorem *The value function* $\mathcal{J} \in C^{1,2,2}\Big([t,T] \times R \times H^2(R)\Big)$. *The value function and the optimal control* π *are given by:*

If $U(x) = \log x$ *then:*

$$\mathcal{J}(t,x,p) = \Big[\log x + r(T-t)\Big] \int_R \tilde{U}(y)p(y)dy + V(t,p)$$

$$\pi(t,x,p) = \tfrac{1}{\sigma^2(t)}\Big[\tfrac{\sigma(t)\int_R (B\tilde{U})(y)p(y)dy}{\int_R \tilde{U}(y)p(y)dy} - r\Big]x$$

where V *satisfies (2.13a).*

If $\frac{U''U}{(U')^2} = k \neq 1$ *then:*

$$\mathcal{J}(t,x,p) = U(xe^{r(T-t)})v(t,p)^{\frac{k}{k-1}}$$

$$\pi(t,x,p) = \tfrac{1}{R(xe^{r(T-t)})}\tfrac{\delta\sigma(t)v_p(\mathcal{B}^*p)-rv}{\sigma^2(t)v}xe^{r(T-t)}$$

where v *satisfies (2.16b) and* R *is given by (2.17).*

If $\frac{U''U}{(U')^2} = k = 1$ *then:*

$$\mathcal{J}(t,x,p) = U(xe^{r(T-t)})e^{v(t,p)}$$

$$\pi(t,x,p) = \tfrac{1}{R(xe^{r(T-t)})}\tfrac{\sigma(t)v_p(\mathcal{B}^*p)-r}{\sigma^2(t)}xe^{r(T-t)}$$

where v satisfies (2.15c) and R is given by (2.17). ∎

Remark 4: In either case, we reduced the difficulty of solving the optimization problem into solving a *linear*, though infinite dimensional pde. In some cases (and we will give an example) this infinite dimensional equation is solvable explicitely.

The proof of the Theorem is a simple exercise for the reader familiar with the DP principle and stochastic partial differential equations. Existence and uniqueness results for (2.16) appear in the literature; the reader may choose between viscosity solutions techniques(as in Ishii (1993), Lions (1984)) or classical treatment (as in Zabczyk(1999)). The Main Theorem is a simple verification theorem, once the existence and uniqueness results mentioned above are available. The fact that we were able to linearize the HJB equation had also its own technical importance because, as we mention in the introduction, results concerning non linear pde's are quite scarce in the literature, while the study of the linear parabolic pde's in Hilbert spaces has quite a long history, beginning with Daleckii (1964).

Despite the infinite dimensionality, the linear equations to which our problem has been reduced aren't always unsolvable. We continue by an example of a situation where such equations admit in fact closed form solution.

4.3 EXAMPLES

In this section we provide two applications of the theory developed in the previous section: one example in which despite the infinite dimensionality, the HJB equation admits closed form solution, and a second example in which we show how to compute prices of European Options with the underlying Y.

4.3.1 An Example of Closed Form Solution

We are investigating a model in which Y follows a pure diffusion process with bounded volatility (see equation (3.1.1) below for the formal expression), and the mean rate of return of S is linear with respect to Y.

This is a model of the influence the health of the CEO of a company has on the stock price. There is anecdotal evidence that the sudden death of the CEO induces a fall in the stock price. In what follows, we assume that Y models the shocks that the health of the CEO produces on returns, and that S is the stock price.

Precisely,

$$dY_s = \alpha(Y_s - \ell)(1 - u^{-1}Y_s)dW_s^2 \tag{3.1.1}$$

$$\frac{dS_s}{S_s} = \lambda(m - Y_s)ds + \sigma dW_s^1. \tag{3.1.2}$$

Y is a pure diffusion process. Suppose a positive shock dW_t^2 is applied to Y; what is the change in Y? Recall that Y is not the health of the CEO *per se*, but the shock that changes in the CEO's health induces on the stock returns. If Y is low, so that its diffusion coefficient is negative, the positive shock dW_t^2, combined with a negative diffusion coefficient, send Y up. That is, if the CEO was previously very sick and his or her health improves, the stock return improves. If the CEO's health decreases further, the stock returns are negatively influenced. If the diffusion coefficient of Y is positive (this happens if the the variable Y is "in the middle" – so we don't know that the CEO is very sick nor that the CEO is extremely healthy) , then a negative shock to the health makes the returns to move up, consistent with the anecdotal evidence that *unexpected* changes in the CEO's health affects the stock prices negatively. When Y is big enough so that it's diffusion coefficient is negative, then positive shocks in health yield positive shocks in returns (similarly with the situation when the CEO is very sick).

The partial information Merton problem is in this context a portfolio management problem in which we can control for the state of health of the CEO.

The utility is such that $UU''/(U')^2 = k$ and $\tilde{U} \equiv 1$. Therefore, with the notations of Section 1, we have

$$
(3.1.3)\qquad
\begin{array}{rcl}
\mu(t,y) & = & \lambda(m-y) \\
\sigma(t) & \equiv & \sigma \\
b(t,y) & \equiv & 0 \\
a(t,y) & = & \alpha(y-\ell)(1-u^{-1}y) \\
(\mathcal{B}f)(y) & = & \frac{\lambda}{\sigma}(m-y)f(y)+\rho\alpha(y-\ell)(1-u^{-1}y)f'(y) \\
(\mathcal{L}f)(y) & = & \frac{1}{2}(y-\ell)^2(1-u^{-1}y)^2 f''(y)
\end{array}
$$

Reducing the degrees of freedom of our model, we also assume that

$$
(3.1.4)\qquad \frac{\lambda}{\sigma}+\frac{\rho\alpha}{u}=0
$$

This assumption, although very restrictive, was made in order to enable us to produce closed form solutions. With the above assumption, we look for a solution of the form:

$$
(3.1.5)\qquad v(t,p)=f(t,\int_R p(y)dy,\int_R yp(y)dy),
$$

where $f:[0,T]\times R^2\Rightarrow R$. To substitute the Fréchet derivatives of v, we have, for example for $q\in H^2(R)$,

$$
\begin{aligned}
v_p(t,p)(q) = f_1(t,\int_R p(y)dy,\int_R yp(y)dy)\int_R q(y)dy+ \\
+f_2(t,\int_R p(y)dy,\int_R yp(y)dy)\int_R yq(y)dy,
\end{aligned}
$$

where $f_i(t,x_1,x_2)=\frac{\partial f}{\partial x_i}(t,x_1,x_2)$.

If we denote

$$(3.1.6) \qquad \begin{array}{ll} c(x_1, x_2) & = \frac{\lambda}{\sigma}(mx_1 - x_2) \\ d(x_1, x_2) & = \Big(\frac{m\lambda}{\sigma} + \rho\alpha + \frac{\rho\alpha\ell}{u}\Big)x_2 - \rho\alpha\ell x_1, \end{array}$$

then f satisfies the following 2-dimensional, linear, parabolic pde:

If $k \neq 1$*:*

$$(3.1.7b) \qquad \begin{cases} f_t + \dfrac{1}{2}(c^2 f_{11} + 2cdf_{12} + d^2 f_{22}) + \\ \quad + \dfrac{r}{k\sigma}(cf_1 + df_2) - \dfrac{r^2}{2k\delta\sigma^2} f = 0 \\ f(T, x_1, x_2) = x_1^{1/\delta} \end{cases}$$

If $k = 1$*:*

$$(3.1.7c) \qquad \begin{cases} f_t + \dfrac{1}{2}(c^2 f_{11} + 2cdf_{12} + d^2 f_{22}) + \\ \quad + \dfrac{r}{\sigma}(cf_1 + df_2) - \dfrac{r^2}{2\sigma^2} = 0 \\ f(T, x_1, x_2) = \log(x_1); x_1 > 0 \end{cases}$$

So up to the solution of the above 2-dimensional linear parabolic equation, the value function and the feedback control are given by:

If $k \neq 1$*:*

$$
(3.1.8b)\quad
\begin{aligned}
\mathcal{J}(t,x,p) &= U(xe^{r(T-t)})f(t,1,M_1(p))^{1/\delta} \\
\pi(t,x,p) &= \tfrac{xR(t,x)}{\sigma^2 f(t,1,M_1(p))}\times \\
&\quad \times\Big(\delta\sigma[c(1,M_1(p))f_1(t,1,M_1(p)) \\
&\quad +d(1,M_1(p))f_2(t,1,M_1(p))] - rf(t,1,M_1(p))\Big)
\end{aligned}
$$

If $k=1$*:*

$$
(3.1.8c)\quad
\begin{aligned}
\mathcal{J}(t,x,p) &= U(xe^{r(T-t)})e^{f(t,1,M_1(p))} \\
\pi(t,x,p) &= \tfrac{xR(t,x)}{\sigma^2}\times \\
&\quad \times\Big(\sigma[c(1,M_1(p))f_1(t,1,M_1(p)) + d(1,M_1(p))f_2(t,1,M_1(p))] \\
&\quad -r\Big)
\end{aligned}
$$

Remark 5: We need to make an observation needed in Section 4. It could be verified by direct inspection that for $m,\lambda,\rho\neq 0$, the solution of the equation (3.1.8b) is not of the form $e^{u(t,x)+v(t,x)y}$ and the solution of the equation (3.1.8c) is not of the form $u(t,x)+v(t,x)y$.

4.3.2 Pricing European Options

Although the object of this presentation is not how to compute option prices, we will present it as an example. We will solely solve the problem for the seller of an European Option; the buyer's problem is similar.

The typical example is of an option written on a pre-IPO stock. The method requires an initial estimation of the probability density of the pre-IPO stock, which should be made available from the underwriter of the stock offering (the so-called prospectus if the IPO offering was filed already with SEC) .

We use a technique presented in Davis, Panas and Zariphopoulou (1993) to compute the seller's price of European Options within the incomplete information framework. Their method yields the indifference price – the unique price P of the option that makes the seller indifferent between between being given extra P dollars initially but having to honor the option's payoff in the end, and not having to deal with the option at all.

We assume that the agent has an exponential utility

$$\hat{U}(x) = 1 - e^{-\gamma x} \tag{3.2.1}$$

and that he *sells* some European Options with underlying asset Y, expiration T and payoff g. Initially, the agent has x dollars and he made δ dollars more by selling the options. If the price of each options was C, the agent has shorted δ/C options, and he will have to shell out $\frac{\delta}{C}g(Y_T)$ dollars at the end of the time horizon to cover these options. His objective is to maximize the expected utility of his terminal wealth, trading in S meanwhile. The criterion to be maximized is therefore

$$E\Big[\hat{U}(X_T^{t,x+\delta,\pi} - \frac{\delta}{C}g(Y_T))\Big] \tag{3.2.2}$$

and we denote the associated value function by

$$I(t,x,p,\delta) = \sup_{\pi} E\Big[\hat{U}(X_T^{t,x+\delta,\pi} - \frac{\delta}{C} g(Y_T))\Big] \tag{3.2.3}$$

Remark that I has the form

$$I(t,x,p,\delta) = 1 + E[U(X_T^{t,x+\delta,\pi})\tilde{U}(\frac{\delta}{C} g(Y_T))] \tag{3.2.4}$$

where U and $\tilde{U}$ are like in the Example c) of Section 1, and I is related to the value function $\mathcal{J}$ of the Main Theorem by

$$I(t,x,p,\delta) = 1 + \mathcal{J}(t,x+\delta,p) \tag{3.2.5}$$

According to the pricing method described in Davis, Panas and Zariphopoulou (1993), the price of the option is given in case exists and it's unique, by the unique solution of the equation

$$\frac{\partial I}{\partial \delta}(t,x,p,\delta=0) = 0 \tag{3.2.6}$$

Using the Main Theorem, (3.2.5) and (3.2.6) we obtain that the price C of the option is given by

$$C(t,p) = e^{-r(T-t)} C^o(t,p) \tag{3.2.6}$$

where C^o is the solution of the partial differential equation

$$(3.2.7) \qquad \begin{cases} C^o_t + \frac{1}{2}C_{pp}(\mathcal{B}^*p, \mathcal{B}^*p) + C^o_p\Big((\mathcal{L} + \frac{r}{\sigma(t)}\mathcal{B})^*p\Big) = 0 \\ C^o(T,p) = \frac{\int_R g(y)p(y)dy}{\int_R p(y)dy} \end{cases}$$

The price C is the *seller's* price of the option.

4.4 THE SEPARATION PRINCIPLE

This section studies how different is the hedging strategy of an agent acting in the full information compared to that of an agent acting in an incomplete information environment, *from the perspective of the agent placed in the partial information market.* In other words, conditional on the common information, are the hedging strategies the same?

The question arises when an uninformed agent observes the market in order to decide whether or not to participate. We assume that the uninformed agent has access to the (conditional) trading strategy of the informed agent and to the stock price S. Based on the information regarding the stock price and her own risk aversion, the uninformed agent opts for her (uninformed) trading strategy. Then this strategy is compared to the one of the informed agent; if they coincide, the uninformed agent may enter the market, while if they are different, the uninformed agent realizes her informational limitations.

The separation holds for log-utility:

Proposition 4.1 *Let $\pi^{inf}(t, X_t, Y_t)$ and $\pi^{uninf}(t, X_t, p_t)$ be the optimal trading strategies of an informed, respectively uninformed agent. Then*

$$E[\pi^{inf}(t, X_t, Y_t)|\zeta_t] = \pi^{uninf}(t, X_t, p_t).$$

Proof: In the case when $\tilde{U} \equiv 1$, that is, when we solve the Merton problem with incomplete information and logarithmic utility, the optimal policy is (from the Main Theorem)

$$\begin{aligned}
\pi^{uninf}(t, x, p) &= \frac{1}{\sigma^2(t)}\Big[\frac{\int_R \mu(t,y)(y)p(y)dy}{\int_R p(y)dy} - r\Big]x \\
&= \frac{1}{\sigma^2(t)}\Big[E[\mu(t, Y_t)|\ \zeta_t] - r\Big]x \\
&= E[\pi^{inf}(t, X_t, Y_t)|\zeta_t]
\end{aligned}$$

so we recover the "separation principle" (4.5) from Pham and Quenez (2000).

∎

If the utility is not logarithmic then the two policies are conditionally different. One agent with non-logarithmic utility may – using only *publicly* available information – to infer that a competitor has private information. This casts doubt on market microstructure or equilibrium models based on the existence of uninformed traders, because uninformed traders are able to realize their exclusive role as liquidity providers and therefore withdraw from the market because they anticipate losses.

To show that the Separation Principle does not hold for power or exponential utilities we present now a counterexample, based on the Example already built in Section 3.1.

Proposition 4.2. *The Separation Principle does not hold in general for HARA or exponential utilities.*

Proof: The counterexample consists of the model presented in Section 3.1, where $m, \lambda, \rho, \alpha \neq 0$.

Suppose by absurd that the Separation Principle holds; then

$$\begin{aligned} \pi^{uninf}(t, X_t, p_t) &= E[\pi^{inf}(t, X_t, Y_t)|\zeta_t] \\ &= \textstyle\int_R g(t, x, y)p(y)dy, \end{aligned} \tag{4.1}$$

for some smooth function $g : [0, T] \times [0, \infty) \times R \to R$ and *any* $H^2(R)$ function p.

From the Example 3.1.(formulae 3.1.8b) and (3.1.8c)) we have that

$$\pi^{uninf}(t, X_t, p_t) = xR(t, x)h(t, \int_R yp(y)dy), \tag{4.2}$$

where h is some smooth function $h : [0, T] \times R \to R$.

Combining (4.1) and (4.2) there should be a function $i(t, y) = g(t, x, y)/(xR(t, x))$ such that

$$h(t, \int_R yp(y)dy) = \int_R i(t, y)p(y)dy, \forall p.$$

Differentiating with respect to p along the direction of some $q \in H^2(R)$, we obtain that

$$h_2(t, \int_R yp(y)dy) \int_R yq(y)dy = \int_R i(t,y)q(y)dy, \forall p, q.$$

But the above relationship is impossible unless $h_2(t, z)$ does not depend on z, so h must be affine with respect to its second argument.

Given the expression for π^{uninf} from (3.1.8b) and (3.1.8c), this happens if and only if

$$f(t,x,y) = \begin{cases} e^{u(t,x)+v(t,x)y}, k \neq 1 \\ u(t,x) + v(t,x)y, k = 1. \end{cases}$$

But the Remark 5 closing Section 3.1. asserts that this is impossible (at least for nontrivial parameters).

Therefore the Separation Principle cannot hold. ∎

4.5 NUMERICAL ALGORITHM

This section provides numerical results for the HJB equations developed in Section 2. It exploits heavily the linear structure created in Section 2, even though the infinite dimensionality remains; it turns out that the problem is numerically tractable with tools already available in the literature: simulations of Zakai equations coupled with Monte Carlo methods for solving partial differential equations. We will present below the main ingredients of the theory and the numerical algorithm.

We present first a Feynman-Kac formula for infinite dimensional parabolic partial differential equations.

Proposition 5.1. *(Feynman-Kac) Let $\mathcal{B}, \mathcal{L}$ as in (1.6), (1.7) and $p_t \in H^2(R)$ the solution of the stochastic partial differential equation*

$$\begin{cases} dp_s = \mathcal{L}^* p_s ds + \mathcal{B}^* p_s d\widetilde{W}_s^1 \\ p_t = p \end{cases} \tag{5.1}$$

and let $v : [0,T] \times H^2(R) \to R$ the solution of the infinite dimensional partial differential equation

$$\begin{cases} v_t + \frac{1}{2} v_{pp}(\mathcal{B}^* p, \mathcal{B}^* p) + v_p(\mathcal{L}^* p) - k(t)v + g(p) = 0 \\ v(T,p) = f(p). \end{cases} \tag{5.2}$$

Then

$$v(t,p) = E[f(p_T) e^{-\int_t^T k(s)ds} + \int_t^T g(p_s) e^{-\int_t^T k(u)du} ds]. \tag{5.3}$$

∎

The proof is similar to the finite dimensional case, based on applying Ito's rule to $f(p_s)e^{-\int_t^s k(u)du}$, except that we have to use the infinite dimensional Ito rule given in Proposition 2 of Section 2.

We present now an algorithm which computes numerically the solution of (5.1), for given Brownian Motion paths $(W_t)_t$. It is presented in Kushner and Dupuis (1992).

Suppose $p \in H^2(R)$ is given. We would like to simulate a path of the solution $p_s, t \leq s \leq T$ of equation (5.1). As p is infinite dimensional, we must approximate p with the values it takes on a grid. We chose the space grid as $\Delta X = \{x_0, x_1, x_2, ..., x_n\} \subset R, x_k = x_0 + kh/n$. $h > 0$ is the space grid mesh. The time is discretized as $\Delta T = \{t_0 = t, t_1, t_2, ..., t_m = T\}, t_k = t + k(T-t)/m$. Remark that we consider the values of the meshes given.

Therefore, p_s is approximated by the vector $\tilde{p}_s \in R^n$,

$$\tilde{p}_s = (p_s(x_0), ..., p_s(x_n))'.$$

The first and second derivative are formally approximated by

$$f'(x_k) \approx \begin{cases} \dfrac{f(x_{k+1}) - f(x_k)}{h}, k < n \\ \dfrac{-f(x_n)}{h}, k = n. \end{cases} \tag{5.4}$$

$$f''(x_k) \approx \begin{cases} \dfrac{f(x_1) - 2f(x_0)}{h^2}, k = 0 \\ \dfrac{f(x_{k+1}) - 2f(x_k) + f(x_{k-1})}{h^2}, 1 \leq k < n \\ \dfrac{-2f(x_n) + f(x_{n-1})}{h^2}, k = n \end{cases} \tag{5.5}$$

Following (5.4) and (5.5), the operators $\mathcal{B}, \mathcal{L}$ of (1.6) and (1.7) are approximated by the matrices $\tilde{\mathcal{B}}_s^n, \tilde{\mathcal{L}}_s^n \in R^{(n+1)\times(n+1)}$,

$$(5.6) \qquad \tilde{\mathcal{L}}_s^n = \begin{pmatrix} l_{02}^n(s) & l_{03}^n(s) & 0 & 0 & ... & 0 & 0 \\ & ..... & & & & & \\ ... & ... & l_{k1}^n(s) & l_{k2}^n(s) & l_{k3}^n(s) & ... & ... \\ & ..... & & & & & \\ 0 & 0 & ... & ... & 0 & l_{n1}^n(s) & l_{n2}(s) \end{pmatrix},$$

where

$$l_{ki}^n(s) = \begin{cases} \frac{a(s,x_k)^2}{h^2}, i = 1 \\ -\frac{2a^2(s,x_k)}{h^2} - \frac{b(s,x_k)}{h}, i = 2 \\ \frac{b(s,x_k)}{h}, i = 3; \end{cases}$$

$$(5.7) \qquad \tilde{\mathcal{B}}_s^n = \begin{pmatrix} b_{01}^n(s) & b_{02}^n(s) & 0 & ... & 0 & 0 \\ & ..... & & & & \\ ... & ... & b_{k1}^n(s) & b_{k2}^n(s) & ... & ... \\ & ..... & & & & \\ 0 & 0 & ... & ... & 0 & b_{n1}^n(s) \end{pmatrix},$$

where

$$b_{ki}^n(s) = \begin{cases} -\frac{\rho a(s,x_k)}{h} + \frac{\mu(s,x_k)}{\sigma(s)}, i = 1 \\ \frac{\rho a(s,x_k)}{h}, i = 2. \end{cases}$$

An approximation to the solution of equation (5.1) is then built as follows:

- start with $p_t^n = (p_t(x_0), ..., p_t(x_n))'$;
- for $j = 1$ to m:

$$p_{t+j(T-t)/m}^n = \Big[1 + \frac{(T-t)}{m}\tilde{\mathcal{L}}_s^n + \sqrt{\frac{(T-t)}{m}}\epsilon_j^n\tilde{\mathcal{B}}_s^n\Big]^* p_{t+(j-1)(T-t)/m}^n,$$

where ϵ_j^n are independent random normal variables.

Denote the simulated sequence of p's by $(p_t^n)_1$ and repeat the above algorithm Q times, obtaining a $(p_t^n)_q$ at each step. Then, by the law of large numbers and Proposition 5.1, an approximation of $v(t,p)$ is given by Monte Carlo:

$$v(t,p) \approx \frac{1}{M}\sum_{q=1}^{Q}\Big[f((p_T^n)_q)e^{-\int_t^T k(s)ds} + \sum_{j=1}^{m}\frac{T-t}{m}g((p_{t+j(T-t)/m}^n)_q)e^{-\int_t^T k(u)du}\Big].$$

Chapter 5

OPEN PROBLEMS

Although we have solved the problems we started with, there are still problems that stem from the proposed models that we do not solve here. We present here a short list.

In Chapter 3, we have proved that a Merton Problem in incomplete markets, with agents exhibiting recursive preferences, could be reduced to a (simpler) Merton Problem in complete markets, where the agent exhibits a Uzawa utility with stochastic discount factor. The question is how general this result is; also, a more profound economic explanation is due.

In Chapter 4, the infinite dimensional HJB equation admitted an easy treatment: we were able to linearize it, so regularity results followed from literature already available. Also, the linear structure made Monte Carlo analysis very easy. It is important to analyze the very features of the model that made these results possible. One aspect is that the process Y does not depend on S. This would be perhaps unrealistic for long time behavior of pre -IPO stocks: as time increases, the big internet portal will definitely influence the value of Y. Therefore, the coefficients of Y should depend on S, but this complicates the structure as another state variable is to be introduced. Another aspect is the dimensionality of the problem: both S and Y are one dimensional. To generalize the result in several dimensions will destroy the linearity. However, from a numerical point of view, the hope is not completely lost: the HJB equation cannot be linearized, but we can use a Feynman - Kac formula for Forward Backward Stochastic Differential Equations, coupled with Monte Carlo schemes for the same FBSDEs. Monte Carlo simulations for such equations are becoming increasingly feasible, thanks to the use of Malliavin Calculus in the simulations of conditional expectations. One direction of research would be to extend the results here (numerics as well as theory) to the case when S and Y are multidimensional. The first step is to reconstruct a theory of FBSDEs whose forward component is a SPDE (like Zakaï 's equation); the second step is to build the approximation scheme; the last step is to establish the convergence of the Monte Carlo simulations.

Another problem that arises in this model is what exactly makes the Separation Principle (presented in Section 4.4) to hold. For example, is it true that if the Separation Principle holds and the agent maximizes utility from terminal wealth, then does it follow that the utility is logarithmic?

Appendices

Appendix A

PROOF OF THEOREM 4.1 OF CHAPTER 3

We split the problem in four smaller ones, according to the values which α, γ take. Cases refer to Kreps-Porteus utility, with the exception of Case IV which also include the Standard Additive utility.

Case I $\gamma \in (-\infty, 0), \alpha \in (-\infty, 0)$

Case II $\gamma \in (0, 1), \alpha \in (-\infty, 0)$

In this case, $A \geq 0, B < 0, p > 1$.

Case III $\gamma \in (-\infty, 0), \alpha \in (0, 1)$

In this case $A \leq 0, B < 0, p > 1$.

Case IV $\gamma \in (0,1), \alpha \in (0,1)$

In this case, $A \leq 0, B > 0, p < 1$. Also, Standard Additive Utility is included here, as the coefficients A, B, p satisfy in this case the inequalities above.

We inquire about existence and uniqueness of *positive* solutions of the pde for v. We continue by proving such results; we stress the fact that cases corresponding to $p > 1$ are treated differently than cases with $p < 1$. We therefore begin with cases I and IV, then we do II and III.

Case I

Uniqueness of positive solution Let v be a positive solution of the studied pde. Then $v_t + \tilde{\mathcal{L}}v \leq 0$ since A, v, B are positive. Also, $v(T, y) = 1$. By comparison principle for regular parabolic pdes, $v \geq v_1$, where v_1 is the solution of

$$v_{1t} + \tilde{\mathcal{L}}v_1 = 0$$

$$v_1(T, y) = 1$$

that is, $v(t, y) = 1$. So $v \geq 1$. Note that the argument above relies on the fact that v is bounded below (by some $C > 0$).

We observe that the only thing that impeaches us in getting uniqueness of positive solution in the pde for v is the fact that $v \to v^p$ is not Lipschitz around 0. But positive solutions have been seen to be greater than one, so uniqueness follows now from standard arguments.

Existence of positive solution Existence for (5) for any time horizon $[0, T]$ is equivalent with global existence for

$$\begin{cases} u_t = \tilde{\mathcal{L}}u + Au + Bu^p, y \in R, t > 0 \\ u(0, y) = 1 \end{cases}$$

Let $l(t - \tau, y - z)$ be the kernel associated with $\tilde{\mathcal{L}}$, that is, the solution of

$$\begin{cases} w_t = \tilde{\mathcal{L}}w + g(t, y) \\ w(0, y) = 1 \end{cases}$$

is given by

$$w(t, y) = 1 + \int_0^t \int_{-\infty}^{+\infty} l(t - \tau, y - z)g(\tau, z)dzd\tau$$

Let

$$u_0(t, y) = 1, \forall t \in [0, T], \forall y \in R$$

and for any $n \geq 0$ let

$$u_{n+1}(t, y) = 1 + \int_0^t \int_{-\infty}^{+\infty} l(t - \tau, y - z)[A(\tau, z)u_n(\tau, z) + Bu_n(\tau, z)^p]dzd\tau$$

Clearly any $u_n \geq 1$, so the above definition is correct. We also have

$$|u_1(t, y) - u_0(t, y)| \leq \int_0^t \int_{-\infty}^{+\infty} l(t - \tau, y - z)[A(\tau, z) + B]dzd\tau$$

$$\leq Ct, \forall t \in [0,T]$$

since A is bounded below. C is a constant such that $|A| + B \leq C$. Assume now that

$$|u_n(t,y) - u_{n-1}(t,y)| \leq \frac{(Ct)^n}{n!}, \forall t \in [0,T] \qquad (*)$$

Then, using that $|a^p - b^p| \leq |a-b|, \forall a,b > 1, \forall p < 1$, we have that

$$|u_{n+1}(t,y) - u_n(t,y)|$$

$$\leq \int_0^t \int_{-\infty}^{+\infty} l(t-\tau, y-z)[A(\tau,z)|u_n(\tau,z) - u_{n-1}(\tau,z)| + B|u_n(\tau,z) - u_{n-1}(\tau,z)|\,]dzd\tau$$

and by (*) and the choice of C,

$$|u_{n+1}(t,y) - u_n(t,y)| \leq C\int_0^t \frac{(C\tau)^n}{n!}d\tau = \frac{(Ct)^{n+1}}{(n+1)!}, \forall t \in [0,T]$$

We therefore proved that

$$|u_{n+1}(t,y) - u_n(t,y)| \leq \frac{(Ct)^{n+1}}{(n+1)!}, \forall t \in [0,T], \forall n \geq 0$$

This yields that $(u_n)_{n \geq 0}$ converges to a function u uniformly on $[0,T]$. Every u_n being clearly continuous, so it is u. Also

$$|u(t,y)| \leq 1 + e^{Ct}, \forall t \in [0,T], \forall y \in R$$

By passing to the limit in the recurrence relationship, we get that

$$u(t,y) = 1 + \int_0^t \int_{-\infty}^{+\infty} l(t-\tau, y-z)[A(\tau,z)u(\tau,z) + Bu(\tau,z)^p]dzd\tau$$

Since the integral with respect to z is a convolution integral, u is also C^∞ in y. Because u is continuous in t it is also C^∞ is t. By the choice of the recurrence relationship, $u(T-t,y)$ is also solution of (3.3.1). It satisfies (3.3.1) in classical sense.

Remark 1: We also proved that $1 \leq v(t,y) \leq 1 + e^{C(T-t)}$

Remark 2: Only the fact that $V(0,\cdot) > m > 0$ was used in the above proof.

Remark 3: We used an approximation sequence in the proof above; this is computationally convenient too.

Case II and III

Uniqueness of positive solution: We rewrite (3.3.1) as

$$\begin{cases} u_t = \tilde{\mathcal{L}}u + A(t,y)u + Bu^p \\ u(0,y) = 1 \end{cases}$$

and we inquire about *global* solutions. Then, in the case when $A > 0$, we choose $M > \sup A \geq 0$ and rewrite the above equation in u as

$$\begin{cases} u_t - Mu = \tilde{\mathcal{L}}u + (A(t,y) - M)u + Bu^p \\ u(0,y) = 1 \end{cases}$$

or, if we denote $w(t,y) = u(t,y)e^{-Mt}$,

$$(1.1) \qquad \begin{cases} w_t = \tilde{\mathcal{L}}w + \tilde{A}(t,y)w + \tilde{B}(t)w^p \\ w(0,y) = 1 \end{cases}$$

where

$$\tilde{A}(t,y) = A(t,y) - M < 0$$

$$\tilde{B}(t) = Be^{(p-1)M} < 0$$

Let now w be a positive solution of (1.1). We will show that $w(t,y) < 2$, $\forall t > 0, \forall y \in R$. If not, then there is (t_0, y_0) such that

$$w(t_0, y_0) = 2$$

and

$$w(t,y) < 2, \forall t < t_0, \forall y \in R$$

Note that it is possible that $y_o \in \{-\infty, +\infty\}$. We suppose that it belongs to R and refer the reader to Proposition(15.4.3) from Taylor (1995). The idea of the proof is the same, it just gets tedious when $y_0 \notin R$. We observe now that $w_t(t_0, y_0) \geq 0$. Also, the function $\phi(y) = w(t_0, y)$ has a maximum at y_0, and since w satisfies (1.1),

$$0 \leq w_t(t_0, y_0) = \tilde{\mathcal{L}}w(t_0, y_0) + \tilde{A}(t_0, y_0)w(t_0, y_0) + \tilde{B}(t_0)w(t_0, y_0)^p < 0$$

therefore a contradiction. That proves that w is bounded above and since it is positive standard arguments invoquing the lipschitzianity close the argument for uniqueness of positive solution of (1.1), and therefore of (3.3.1).

Existence of positive solution: We prove existence of positive solution for (1.1). Take $T_0 < \sup\{|\tilde{A}| + |\tilde{B}|\}$ and define

$$w_0(t,y) = 1$$

and for any $n \geq 0$

$$w_{n+1}(t,y) = 1 + \int_0^t \int_{-\infty}^{+\infty} l(t-\tau, y-z)[\tilde{A}(\tau,z)w_n(\tau,z) + \tilde{B}(\tau)w_n(\tau,z)^p]dzd\tau$$

It follows easily by induction that

$$0 \leq w_n(t,y) \leq 1, \forall t \in [0,T_0], \forall y \in R$$

Using now the fact that

$$w \to w^p$$

is Lipschitz on $[0,1]$ (recall that now $p > 1$), that there is a $C > 0$ such that

$$|w_{n+1}(t,y) - w_n(t,y)| \leq \frac{(Ct)^{n+1}}{(n+1)!}, \forall t \in [0,T_0], \forall y \in R$$

which implies that w_n converges to a solution of (1.1) on $[0,T_0]$. Obviously $0 \leq w(T_0,\cdot) \leq 1$. We therefore got a local solution on $[0,T_0)$. Let T_0 as above be maximal and suppose it's not $+\infty$. We'll show how to extend the solution beyond T_0 in this case, obtaining therefore a contradiction. Remark that above we used only the fact that the initial data was bounded away from 0 uniformly in order to get local existence. If we prove that the solution on $[0,T_0)$ satisfies $w(T_0-,\cdot) \geq C > 0$, the same proof could be used to extend the solution beyond T_0. Now, if we take the solution w on $[0,T_0)$, and if we take

y_t be a minimum point of $w(t,\cdot)$, we have $\tilde{\mathcal{L}}w(t,y_t) \geq 0$ (please remark that $w(t,\cdot)$ is bounded), and therefore

$$w_t \geq -Mw - \tilde{B}(t)w^p$$

where M is an upper bound on $\tilde{A}(t,y)$. It follows easily from the inequality above (with initial condition $w(0,y)=1$) that w is uniformly bounded away from zero, therefore it could be extended.

Remark 4: We constructed a solution v such that $0 < \tilde{w}(T) \leq \tilde{w}(T-t) \leq v(t,y) \leq e^{M(T-t)}$, where $\tilde{w}$ is the solution of

$$\begin{cases} \tilde{w}_t = -M\tilde{w} - \tilde{B}(t)\tilde{w}^p \\ \tilde{w}(0) = 1 \end{cases}$$

Case IV

This time we'll do the existence first.

Existence of positive solution: We would like to reproduce the arguments of *case I*; however, since $a \leq 0$ in this case we cannot infer that every u_n defined as in *Case I* is bigger than 1, so the inequality used there cannot be used here. But we'll rewrite the equation

$$\begin{cases} u_t = \tilde{\mathcal{L}}u + Au + Bu^p \\ u(0,y) = 1 \end{cases} \tag{1.2}$$

in a more convenient form. First, let $M > 0$ such that

$$|A(t, y)| < M, \forall 1 t \in [0, T], \forall y \in R$$

We could rewrite (1.2) as

$$\begin{cases} u_t + Mu = \tilde{\mathcal{L}}u + [A + M]u + Bu^p \\ u(0, y) = 1 \end{cases}$$

or, substituting $w(t, y) = e^{Mt}u(t, y)$,

$$\begin{cases} w_t = \tilde{\mathcal{L}}w + \tilde{A}w + \tilde{B}w^p \\ W(0, y) = 1 \end{cases}$$

with

$$\tilde{A}(t, y) = A(t, y) + M > 0, \tilde{B}(t) = Be^{Mt(1-p)}$$

We obviously continue exactly as in *Case I*, letting first

$$w_0(t, y) = 1$$

then, for $n \geq 0$

$$w_{n+1}(t, y) = 1 + \int_0^t \int_{-\infty}^{+\infty} l(t - \tau, y - z)[\tilde{A}(\tau, z)w_n(\tau, z) + \tilde{B}(\tau)w_n(\tau, z)^p]dzd\tau$$

We take $C > 0$ such that $|\tilde{A}| + |\tilde{B}| \leq C$ (such a C exists because if A, B were bounded, so are $\tilde{A}, \tilde{B}$), and we prove that

$$|w_{n+1}(t, y) - w_n(t, y)| \leq \frac{(Ct)^{n+1}}{(n + 1)!}, \forall t \in [0, T], \forall y \in R$$

which grants existence of a positive solution w and therefore of a positive solution v of (3.3.1).

Remark 5 We also proved that $e^{-M(T-t)} \leq v(t,y) \leq e^{-M(T-t)}(1 + e^{C(T-t)})$.

Uniqueness of positive solution: As in the existence proof, we begin by bringing the equation to the form

$$w_t = \tilde{\mathcal{L}}w + \tilde{A}w + \tilde{B}w^p \geq \tilde{\mathcal{L}}w$$

If solution of (3.3.1) is positive, so is w, so we were able to write the inequality above and we also get that

$$w(t,y) \geq 1, \forall t \in [0,T]$$

Now $v(t,y) \geq e^{-M(T-t)}$, so by standard lipshitzianity arguments we get uniqueness for (3.3.1).

Appendix B

BSDEs

We will follow El Karoui, Peng, Quenez (1997) here and hence we'll be brief. For full details, mathematical proofs and applications of the concept we refer the reader to the aforementioned paper.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let W be a Brownian Motion adapted to the (enlarged) $(\mathcal{F}_s)_{t \leq s \leq T}$. Let

$$H_T^2(R) = \{\zeta.\text{predictable}, E[\int_0^T \zeta_s^2 ds] < \infty\}$$

A *Backward Stochastic Differential Equation* (BSDE), with solution (U, Z) is a stochastic equation of the form:

$$U_s = \xi + \int_s^T g(\omega, l, U_l, Z_l)dl - \int_s^T Z_l dW_l$$

The function g, called *generator*, is considered to be *standard*, that is,

$$\begin{cases} g \text{ adapted to} (\mathcal{F}_s)_s \\ g(\cdot, \cdot, 0, 0) \in H_T^2(R) \\ g \text{ Lipschitz with respect to} U, Z \end{cases}$$

The random variable ξ is the *terminal condition*, and it satisfies

$$\xi \in L^2(\mathcal{F}_T)$$

This conditions being satisfied, it is proven in El Karoui, Peng, Quenez (1997) that the BSDE has a unique solution $(U, Z) \in H_T^2(R)$.

The BSDE could be written in an equivalent form:

$$U_s = E[\xi + \int_s^T g(\cdot, l, U_l, Z_l)dl|\mathcal{F}_s]$$

so that recursive utilities are solutions of BSDEs. The following results are proven in El Karoui, Peng, Quenez (1997):

Proposition 2.1 *We consider a family of BSDEs with standard generators g_α, g and square integrable terminal conditions ξ_α, ξ such that:*

$$\begin{cases} g = \min_\alpha g_\alpha(\max_\alpha f_\alpha) \\ \xi = \min_\alpha \xi_\alpha(\max_\alpha \xi_\alpha) \end{cases}$$

(min, or max, are taken for some values of α the same value for generator as for the terminal condition)

Let $(U^\alpha, Z^\alpha), (U, Z)$ be the solutions of these BSDEs. Then:

$$U_t = \min_\alpha U_t^\alpha (\max_\alpha U_t^\alpha)$$

∎

The following result is not stated in its full generality:

Proposition 2.2 *(Generalized Feynman-Kac formula) Let v a classical bounded solution (that is, of class $\mathcal{C}^{1,2}$) of*

$$\begin{cases} v_t + \tilde{\mathcal{L}}v + g(t, y, v) = 0 \\ \\ v(T, y) = v_T(y) \end{cases}$$

where $\tilde{\mathcal{L}}$ is a second order elliptic operator, and g is standard in the sense described above.

Let $\tilde{Y}$ be a diffusion with infinitesimal generator $\tilde{\mathcal{L}}$, driven by the Brownian Motion W, and let (U, Z) be the solution of the following BSDE (with the generator constructed using g and Y):

$$U_s = v_T(Y_T) + \int_s^T g(l, Y_l, U_l)dl - \int_s^T Z_l dW_l, t \leq s \leq T$$

Then

$$v(t, y) = U_t$$

The proof of this result is simple, consisting only in applying Ito's formula to $v(s, Y_s)$ and using uniqueness properties for the BSDE above. ∎

Appendix C

EXISTENCE AND UNIQUENESS FOR (4.4) OF CHAPTER 3

In our case, U is in $H_T^2(R)$ since it's been proven that v is bounded, and if we rewrite

$$v(t,y) = 1 + \int_0^t \int_{-\infty}^{+\infty} l(t-\tau, y-z)[A(\tau,z)v(\tau,z) + Bv^p(\tau,z)]dzd\tau$$

and differentiate once with respect to y, we could easily see that v_y has linear growth in y. On the other hand, since $\tilde{Y}$ is a "nice" diffusion (ie, the solution of a SDE with regular coefficients) we have that $\tilde{Y} \in H_T^2(R)$ (see for example Revuz and Yor (1994), Ch. 9), and therefore $Z_{\cdot} = a(\cdot, \tilde{Y}_{\cdot})v_y(\cdot, \tilde{Y}_{\cdot}) \in H_T^2(R)$ (a is also bounded from assumptions).

Uniqueness is a bit more complicated. If we try to prove uniqueness showing that the generator is Lipschitz, as in El Karoui, Peng, Quenez (1997) , we obviously fail (g is not Lipschitz). However, we know we have a solution (U, Z) such that, in fact, U is between two positive constants,C_*, C^*, a.s. The two constants are not universal, they depend on the parameters of the problem. Since g doesn't depend on Z, we could look only for solutions bounded by the aforementioned *afortiori* constants, and in such a case $g : [C_1, C_2] \to R$ is

Lipschitz, therefore uniqueness of a solution with U a.s. bounded and bounded away from zero is insured, by classical results contained in El Karoui, Peng, Quenez (1997).

Appendix D

UZAWA UTILITY

We would like to provide a clearer formula for a recursive criterion having the Uzawa generator of the form

$$g(c, V) = u(c) - \beta(c)V$$

To do that in an easy-to-follow way, we start with a recursive criterion U which satisfies

$(Ap.4.1)U_s = E[\int_s^T u(c_l)e^{-\int_s^l \beta(c_\tau)d\tau}dl + U_T|\mathcal{F}_s], t \leq s \leq T$

It's easy to see that U satisfies

$$U_s = e^{\int_s^l \beta(c_\tau)d\tau}\{E[\int_t^T u(c_l)e^{-\int_s^l \beta(c_\tau)d\tau}dl|\mathcal{F}_s] - \int_t^s u(c_l)e^{-\int_s^l \beta(c_\tau)d\tau}dl\}$$

As the process $E[\int_t^T u(c_l)e^{-\int_s^l \beta(c_\tau)d\tau}dl|\mathcal{F}_s]$ is a Brownian martingale, the Martingale Representation Theorem ensures the existence of a square integrable process Z such that

$$U_s = e^{\int_s^l \beta(c_\tau)d\tau}\{\int_t^s Z_l d\tilde{W}_l - \int_t^s u(c_l)e^{-\int_s^l \beta(c_\tau)d\tau}dl\}$$

Writing the above relationship using Ito differentials produces

$$-dU_s = \{u(c_s) - \beta(c_s)U_s\}ds + e^{\int_s^l \beta(c_\tau)d\tau}Z_s d\tilde{W}_s$$

and if we integrate above and write conditional expectation with respect to $\mathcal{F}_s$ we get that

$$U_s = E[\int_s^T g(c_l, U_l)dl + U_T|\mathcal{F}_s]$$

Obviously all the computations could be done in the reverse way, and now (Ap.4.1) justifies the names of *utility* for u and *discount factor* for θ in the formula of the generator of the Uzawa utility.

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Vita

Cristian Ioan Tiu was born in Giurgiu, Romania on 22 January 1975, the son of Ionel Tiu and Elena Tiu. He received the Bachelor of Science degree in Mathematics from University of Bucharest in 1997 and started graduate studies in Mathematics at University of Wisconsin at Madison the same year. He moved to University of Texas at Austin in 2000. He is married and has a son.

Permanent address: 7 Izvorul Oltului, Bl. III/25, Apt. 41
Bucharest 4
Romania

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