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Global Well-posedness and Scattering for the Defocusing Energy-Supercritical Cubic Nonlinear Wave Equation

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Benimle birlikte bu yolda yürüdükleri için,

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Global Well-posedness and Scattering for the Defocusing Energy-Supercritical Cubic Nonlinear Wave Equation

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We study the initial value problem for the defocusing nonlinear wave equation with cubic nonlinearity $F(u) = |u|^2u$ in the energy-supercritical regime, that is dimensions $d \geq 5$. We prove that solutions to this equation satisfying an a priori bound in the critical homogeneous Sobolev space exist globally in time and scatter in the case of spatial dimensions $d \geq 6$ with general (possibly non-radial) initial data, and in the case of spatial dimension $d = 5$ with radial initial data.

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Chapter 1

Introduction

In this thesis, we study the initial value problem for the defocusing nonlinear wave equation with cubic nonlinearity $F(u) = |u|^2 u$ in the energy-supercritical regime, that is dimensions $d \geq 5$. More precisely, we consider

$$\text{(NLW)} \quad \begin{cases} u_{tt} - \Delta u + F(u) &= 0 \\ (u, u_t)|_{t=0} &= (u_0, u_1) \in \dot{H}_x^{s_c}(\mathbb{R}^d) \times \dot{H}_x^{s_c - 1}(\mathbb{R}^d), \end{cases}$$

where $s_c = \frac{d-2}{2}$, $u : I \times \mathbb{R}^d \to \mathbb{R}$, and $0 \in I \subset \mathbb{R}$ is a time interval.

There is a natural scaling associated to this initial value problem: if we set $u_\lambda(t, x) = \lambda u(\lambda t, \lambda x)$, $\lambda > 0$, then the map $u \mapsto u_\lambda$ carries the set of solutions of (NLW) to itself and, moreover, we have

$$\|(u_\lambda, \partial_t u_\lambda)|_{t=0}\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1}} = \|(u_0, u_1)\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1}}. \tag{1.1}$$

The invariance of this norm is closely connected to the existence of a suitable local well-posedness theory for (NLW), and this leads to s_c being referred to as the *critical regularity* for the problem.

Before proceeding to the main results of this thesis, we recall that solutions to (NLW) conserve the *energy*,

$$E(u(t), u_t(t)) = \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t)|^2 + \frac{1}{2}|u_t(t)|^2 + \frac{1}{4}|u(t)|^4 dx \in [0, +\infty],$$

in the sense that if u is a solution to (NLW), then

$$E(u(t), u_t(t)) = E(u(0), u_t(0)) \quad \text{for all} \quad t \in I.$$

One immediately observes that in the case $s_c = 1$, that is $d = 4$, solutions to (NLW) have finite energy and the scaling $u \mapsto u_\lambda$ leaves the energy invariant. For this reason, the problem is referred to as *energy-critical* when $s_c = 1$, while in the setting $s_c > 1$, equivalently $d \geq 5$, the problem is called *energy-supercritical.*

We will work with the following notion of solution for (NLW):

Definition 1.0.1. We say that $u : I \times \mathbb{R}^d \to \mathbb{R}$, $d \geq 5$ with $0 \in I \subset \mathbb{R}$, is a *solution* to (NLW) if (u, u_t) belongs to $C_t(K; \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x)$, u belongs to $L^{d+1}_{t,x}(K \times \mathbb{R}^d)$ for every compact $K \subset I$, and u satisfies the *Duhamel formula*

$$u(t) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|}u_1 + \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|}F(u(t'))dt'$$

for every $t \in I$.

Using the usual convention, we refer to I as the *interval of existence* of u, and we say that I is the *maximal interval of existence* if u cannot be extended to any larger time interval. We say that u is a *global solution* if $I = \mathbb{R}$, and that u is a *blow-up solution* if $\|u\|_{L^{d+1}_{t,x}(I \times \mathbb{R}^d)} = \infty$. Moreover, we say that u *scatters* as $t \to \pm\infty$ if there exist unique $(u_0^\pm, u_1^\pm) \in \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x$ such that

$$\lim_{t \to \pm\infty} \|(u(t), u_t(t)) - (\mathcal{W}(t)(u_0^\pm, u_1^\pm), \partial_t \mathcal{W}(t)(u_0^\pm, u_1^\pm))\|_{\dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x} = 0,$$

where

$$\mathcal{W}(t)(f,g) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|}g \tag{1.2}$$

is the solution to the linear wave equation with Cauchy data (f,g). The Strichartz estimates then imply that u scatters as $t \to \pm\infty$ if $\|u\|_{L^{d+1}_{t,x}} < \infty$.

In this thesis, we study the questions of global well-posedness and scattering for (NLW) in the energy-supercritical regime $s_c > 1$. In particular, we address the following conjecture, which asserts that solutions to (NLW) which remain bounded in the critical space exist globally in time and scatter as $t \to \pm\infty$.

Conjecture 1.0.1. *Let $d \geq 5$ and $s_c = \frac{d-2}{2}$. Assume $u : I \times \mathbb{R}^d \to \mathbb{R}$ is a solution to (NLW) with maximal interval of existence $I \subset \mathbb{R}$ which satisfies*

$$(u, u_t) \in L^\infty_t(I; \dot{H}^{s_c}_x(\mathbb{R}^d) \times \dot{H}^{s_c-1}_x(\mathbb{R}^d)). \tag{1.3}$$

Then u is global, and

$$\|u\|_{L^{d+1}_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \leq C$$

for some constant $C = C(\|(u,u_t)\|_{L^\infty_t(I;\dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x)})$. In particular, u scatters as $t \to \pm\infty$.

We prove the conjecture in the case of dimensions $d \geq 6$ with no radial assumption on the initial data as well as in the case of dimension $d = 5$ when the initial data is assumed to have radial symmetry.

The study of Conjecture 1.0.1 in the case of a general energy-supercritical nonlinearity $F(u) = |u|^p u$ was initiated by Kenig and Merle in [18]. In that work, the authors established the result for radial initial data in dimension $d = 3$ and $p > 4$.[1] Subsequently, Killip and Visan established the conjecture for the energy-supercritical nonlinear Schrödinger equation in dimensions $d \geq 5$ [29]. Returning to (NLW), Killip and Visan established Conjecture 1.0.1 for dimension $d = 3$ with nonlinearity $F(u) = |u|^p u$ for even integers $p > 4$ with no radial assumption in [26], and for radial initial data in dimensions $3 \leq d \leq 6$ with nonlinearity $F(u) = |u|^p u$ and $\frac{4}{d-2} < p < \frac{4}{d-3}$ and in dimensions $d \geq 7$ for a certain range of p in [28].

We note that the results obtained in this thesis have not been previously treated in the literature. In particular, Theorem 1.1.1 below is the first global well-posedness result treating non-radial initial data in dimensions $d \geq 5$. Likewise, Theorem 1.1.2 resolves the cubic radial case in dimension $d = 5$ for the first time. Indeed, in the case of the cubic nonlinearity in the energy-supercritical regime, we present the first results obtained regardless of dimension, since the restrictions on p imposed in [28] exclude the cubic case for any d.

In the present work, consideration of the cubic nonlinearity mainly serves to simplify our discussion of the local theory. In particular, the algebraic nature of this nonlinearity allows us to obtain estimates using the fractional

[1] Note that when the nonlinearity is taken as $F(u) = |u|^p u$, the critical regularity s_c becomes $\frac{d}{2} - \frac{2}{p}$.

product rule. We remark that our arguments also apply to the case of a general energy-supercritical nonlinearity $F(u) = |u|^p u$ by replacing the product rule with the fractional chain rule as appropriate, along with suitable changes in numerology. However, we restrict our presentation to the cubic case to simplify the exposition.

We now briefly discuss the role of (1.3) in Conjecture 1.0.1, which we will often refer to in the sequel as the *a priori bound*. When the cubic nonlinearity in (NLW) is replaced by the energy-critical nonlinearity $F(u) = |u|^{\frac{4}{d-2}}u$ for any $d \geq 3$ (so that $s_c = 1$), global well-posedness was obtained in a series of works [1, 12–14, 33–36, 38, 40, 42]. In particular, Struwe [40] obtained the global well-posedness for energy critical (NLW) with radial initial data in $d = 3$, while Grillakis [12] removed the radial assumption in this dimension. The global well-posedness and persistence of regularity was shown for $3 \leq d \leq 5$ by Grillakis [13], and for $d \geq 3$ by Shatah and Struwe [36–38] and Kapitanski [14].

In all of these works, the key property in obtaining global well-posedness results for the energy critical (NLW) is an immediate uniform control in time of the critical norm $\dot{H}^1_x \times L^2_x$ by virtue of the conservation of energy. In the energy supercritical regime, when $s_c > 1$, the global behavior of solutions to (NLW) is a more delicate matter, since in this context we do not have instantaneous access to any conservation law at the critical regularity. In view of the energy critical theory, it is then natural to impose an a priori uniform in time control of the critical norm to compensate for the lack of such a conservation law.

This is the role of (1.3) in Conjecture 1.0.1.

We are now ready to state the main results of this thesis.

1.1 Main results

In this thesis, we prove Conjecture 1.0.1 in two cases, the first of which covers dimensions $d \geq 6$, with general (not necessarily radial) initial data.

Theorem 1.1.1 (Bulut, [3]). *Let $d \geq 6$ and $s_c = \frac{d-2}{2}$. Assume $u : I \times \mathbb{R}^d \to \mathbb{R}$ is a solution to (NLW) with maximal interval of existence $I \subset \mathbb{R}$ which satisfies*

$$(u, u_t) \in L_t^\infty(I; \dot{H}_x^{s_c}(\mathbb{R}^d) \times \dot{H}_x^{s_c-1}(\mathbb{R}^d)).$$

Then u is global, and

$$\|u\|_{L_{t,x}^{d+1}(\mathbb{R}\times\mathbb{R}^d)} \leq C$$

for some constant $C = C(\|(u, u_t)\|_{L_t^\infty(I;\dot{H}_x^{s_c}\times\dot{H}_x^{s_c-1})})$.

The contribution of Theorem 1.1.1 to the study of NLW in the energy-supercritical regime is to treat the case of general (possibly nonradial) initial data in dimensions $d \geq 6$. Prior to this work, the only nonradial result in the energy-supercritical setting is contained in [26], which dealt with the three dimensional case.

Concerning the remaining dimension, $d = 5$, we obtain the following theorem, in which we prove the conjecture in the case of radial initial data:

Theorem 1.1.2 (Bulut, [4]). *Assume $u : I \times \mathbb{R}^5 \to \mathbb{R}$ is a radial solution to (NLW) with maximal interval of existence $I \subset \mathbb{R}$ which satisfies*

$$(u, u_t) \in L_t^\infty(I; \dot{H}_x^{3/2}(\mathbb{R}^d) \times \dot{H}_x^{1/2}(\mathbb{R}^d)).$$

Then u is global, and

$$\|u\|_{L^6_{t,x}(\mathbb{R}\times\mathbb{R}^5)} \leq C$$

for some constant $C = C(\|(u, u_t)\|_{L_t^\infty(I;\dot{H}_x^{3/2}\times\dot{H}_x^{1/2})})$.

Recall that the works [18] and [28] also consider Conjecture 1.0.1 in the case of radial initial data in dimension $d = 3$ and dimensions $d \geq 3$, respectively. However, the restrictions on the nonlinearity $F(u) = |u|^p u$ in [28] lead that result to not apply to the cubic case we consider in Theorem 1.1.2. We also recently learned that Kenig and Merle have treated the defocusing energy-supercritical NLW with the quintic nonlinearity and radial data in all odd dimensions [19].

We prove Theorem 1.1.1 and Theorem 1.1.2 in Chapter 3 and Chapter 4 of this thesis, respectively. In order to discuss our approach to the proofs of these theorems, we begin with the observation that in the energy-critical works that we mentioned above, a monotonicity formula known as the Morawetz identity played an important role, giving an estimate of the form

$$\int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|} dxdt \lesssim E(u_0, u_1) \tag{1.4}$$

for solutions u of (NLW). Throughout our exposition, we will refer to (1.4) as the *Morawetz estimate*. Although the right hand side of this estimate is finite in the energy-critical setting, such a bound is not immediately accessible in the energy-supercritical regime. To overcome this, one must proceed in a different manner and make use of different tools than in the energy critical case.

1.2 Outline of our approach

We now briefly describe the approach that we follow to prove Theorems 1.1.1 and 1.1.2. For a detailed discussion, we refer the reader to Sections 3.1 and 4.1. Our proofs of the theorems make use of the concentration compactness approach introduced by Kenig and Merle in their study of the focusing energy-critical NLS and NLW [20, 21], which has recently been applied to a wide variety of problems, including the energy-supercritical NLW [18, 26, 28] and energy-supercritical NLS [29].

In particular, our proofs proceed by the use of a contradiction argument: assuming that the theorem fails, one constructs a minimal blow-up solution using the concentration compactness/rigidity approach of Kenig and Merle.

An important ingredient in this construction is a concentration compactness result in the form of a profile decomposition theorem for solutions of the linear wave equation. In a broad sense, this result asserts that any bounded sequence of initial data in the critical space $\dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x$ can be decomposed up to a subsequence as the sum of a superposition of profiles and an error term. The profiles are asymptotically orthogonal and the remainder term is small in a Strichartz norm. The idea behind this decomposition is to compensate for the lack of compactness of the linear wave propagator $\mathcal{W}(t)$ as a map from the space $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ to the Strichartz space. In the present context, the higher dimensional version of the profile decomposition with initial data lying in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ reads as follows:

Theorem 1.2.1 (Profile decomposition, Bulut [5]). *Let $(u_{0,n}, u_{1,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}_x^{s_c}(\mathbb{R}^d) \times \dot{H}_x^{s_c-1}(\mathbb{R}^d)$ with $s_c \geq 1$. Then there exists a subsequence of $(u_{0,n}, u_{1,n})$ (still denoted $(u_{0,n}, u_{1,n})$), a sequence of profiles $(V_0^j, V_1^j)_{j\in\mathbb{N}} \subset \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}(\mathbb{R}^d)$, and a sequence of triples $(\epsilon_n^j, x_n^j, t_n^j) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}$, which are orthogonal in the sense that for every $j \neq j'$,*

$$\frac{\epsilon_n^j}{\epsilon_n^{j'}} + \frac{\epsilon_n^{j'}}{\epsilon_n^j} + \frac{|t_n^j - t_n^{j'}|}{\epsilon_n^j} + \frac{|x_n^j - x_n^{j'}|}{\epsilon_n^j} \underset{n\to\infty}{\longrightarrow} \infty,$$

and for every $\ell \geq 1$, if

$$V^j = \mathcal{W}(t)(V_0^j, V_1^j) \quad \text{and} \quad V_n^j(t,x) = \frac{1}{(\epsilon_n^j)} V^j\left(\frac{t - t_n^j}{\epsilon_n^j}, \frac{x - x_n^j}{\epsilon_n^j}\right),$$

where $\mathcal{W}(t)(f,g)$ is as in (1.2), then

$$(u_{0,n}(x), u_{1,n}(x)) = \sum_{j=1}^{\ell} (V_n^j(0,x), \partial_t V_n^j(0,x)) + (w_{0,n}^\ell(x), w_{1,n}^\ell(x))$$

with

$$\limsup_{n\to\infty} \|\mathcal{W}(t)(w_{0,n}^\ell, w_{1,n}^\ell)\|_{L_t^q L_x^r} \underset{\ell\to\infty}{\longrightarrow} 0$$

for every (q,r) an $\dot{H}_x^{s_c}$-wave admissible pair with $q, r \in (2,\infty)$. For all $\ell \geq 1$, we also have,

$$\|u_{0,n}\|_{\dot{H}_x^{s_c}}^2 + \|u_{1,n}\|_{\dot{H}_x^{s_c-1}}^2 = \sum_{j=1}^{\ell} \left[\|V_0^j\|_{\dot{H}_x^{s_c}}^2 + \|V_1^j\|_{\dot{H}_x^{s_c-1}}^2\right] + \|w_{0,n}^\ell\|_{\dot{H}_x^{s_c}}^2 + \|w_{1,n}^\ell\|_{\dot{H}_x^{s_c-1}}^2$$

$$+\, o(1), \quad n \to \infty.$$

Here, for the definition of $\dot{H}^s_x$-wave admissible pair, we refer the reader to Section 2.1*.*

For initial data in $\dot{H}^1_x \times L^2_x$, the profile decomposition for the wave equation was established by Bahouri and Gerard [1] in dimension 3 and was extended to dimensions $d \geq 3$ by Bulut in [5]. Roughly speaking, the proof of Theorem 1.2.1 is obtained by observing that for any sequence of initial data $\{(u_{0,n}, u_{1,n})\} \subset \dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x$, the sequence $\{(|\nabla|^{s_c-1}u_{0,n}, |\nabla|^{s_c-1}u_{1,n})\}$ lies in the energy space $\dot{H}^1_x \times L^2_x$. Applying the energy-critical profile decomposition to this new sequence, the result then follows from an application of the Sobolev embedding. For more details, we refer the reader to [1, 5].

1.2.1 Existence of minimal blow-up solutions

With the profile decomposition in hand, the first part in the "concentration compactness + rigidity" approach of Kenig and Merle consists of reducing the argument to the study of minimal blow-up solutions to (NLW). Informally speaking, this reduction is a consequence of the observation that if either Theorem 1.1.1 or Theorem 1.1.2 fails, the above profile decomposition can be applied to study a minimizing sequence of blow-up solutions to (NLW) with respect to the $L^\infty_t(\dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x)$ norm. Through this analysis, one extracts a minimal blow-up solution which is then shown to possess an additional compactness property up to the symmetries of the equation. More precisely, we recall the following result from [18].

Theorem 1.2.2. *[18] Suppose that either Theorem* 1.1.1 *or Theorem* 1.1.2 *fails. Then there exists a solution* $u : I \times \mathbb{R}^d \to \mathbb{R}$ *to (NLW) with maximal interval of existence* I*,*

$$(u, u_t) \in L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}), \quad \textit{and} \quad \|u\|_{L_{t,x}^{d+1}(I \times \mathbb{R}^d)} = \infty$$

such that u *is a minimal blow-up solution in the following sense: for any solution* v *with maximal interval of existence* J *such that* $\|v\|_{L_{t,x}^{d+1}(J \times \mathbb{R}^d)} = \infty$*, we have*

$$\sup_{t \in I} \|(u(t), u_t(t))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \le \sup_{t \in J} \|(v(t), v_t(t))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}}.$$

Moreover, there exist $N : I \to \mathbb{R}^+$ *and* $x : I \to \mathbb{R}^d$ *such that the set*

$$K = \{(\frac{1}{N(t)} u(t, x(t) + \frac{x}{N(t)}), \frac{1}{N(t)^2} u_t(t, x(t) + \frac{x}{N(t)})) : t \in I\}, \tag{1.5}$$

has compact closure in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}(\mathbb{R}^d)$*. In the case that Theorem* 1.1.2 *fails, we have* $x(t) \equiv 0$*.*

The above theorem was proved by Kenig and Merle in [18] in three dimensions with radial initial data. However, as pointed out in [16, 17], when a satisfactory local theory is present the proof is independent of the dimension and the assumption of radial symmetry. We briefly summarize the main steps of the argument. First, by means of the profile decomposition along with the local theory (local well-posedness and stability) discussed in Section 2.2 below, a minimal blow-up solution is extracted. Then, the remainder of the proof consists of showing the compactness property (1.5), which is a consequence of the minimality. For a detailed treatment, we refer the reader to the works [18, 21].

We now recall from [26, 28] an equivalent formulation of (1.5) which will be an essential tool for our analysis of blow-up solutions.

Definition 1.2.1. A solution u to (NLW) with time interval I is said to be *almost periodic modulo symmetries* or, for the sake of brevity, *almost periodic*, if $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$ and there exist functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^d$ and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $t \in I$ and $\eta > 0$,

$$\int_{|x-x(t)|\geq C(\eta)/N(t)} ||\nabla|^{s_c} u(t,x)|^2 + ||\nabla|^{s_c-1} u_t(t,x)|^2 dx \leq \eta,$$

and

$$\int_{|\xi|\geq C(\eta)N(t)} |\xi|^{2s_c} |\hat{u}(t,\xi)|^2 + |\xi|^{2(s_c-1)} |\hat{u}_t(t,\xi)|^2 d\xi \leq \eta.$$

An important tool in analysing almost periodic solutions to (NLW) is the following Duhamel formula, which states that if u is an almost periodic solution, the linear components of the evolutions u and u_t vanish as the time t approaches the endpoints of I. In the context of the mass critical NLS, this formula was introduced in [45] (see also [27] for further discussion). We recall the version that we use here from [26].

Lemma 1.2.3. *[26, 45] Let $u : I \times \mathbb{R}^d \to \mathbb{R}$ be a solution to (NLW) with maximal interval of existence I which is almost periodic modulo symmetries. Then for all $t \in I$,*

$$\left(\int_t^T \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(u(t'))dt', \int_t^T \cos((t-t')|\nabla|) F(u(t'))dt' \right)$$

$$\underset{T \to \sup I}{\rightharpoonup} (u(t), u_t(t)), \tag{1.6}$$

and

$$\left(- \int_T^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} F(u(t'))dt', - \int_T^t \cos((t-t')|\nabla|) F(u(t'))dt' \right)$$
$$\underset{T \to \inf I}{\rightharpoonup} (u(t), u_t(t)). \tag{1.7}$$

weakly in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1}$.

With this machinery in place, we are now ready to discuss the remainder of the proofs of Theorem 1.1.1 and Theorem 1.1.2. In both settings, our goal will be to show that the minimal blow-up solution constructed in Theorem 1.2.2 cannot exist, which gives the desired contradiction, proving the theorems. We note that the tools and techniques we use will differ significantly for the proof of each theorem.

1.2.2 Approach for dimensions $d \geq 6$

In order to motivate our approach to complete the proof of Theorem 1.1.1, we recall that, as we mentioned above, the Morawetz estimate (1.4) is an important tool in the global well-posedness theory for the energy-critical NLW. However, in the energy-supercritical setting it is not immediately obvious how one can exploit the control given by this estimate, since solutions do not necessarily possess finite energy. Nevertheless, a similar difficulty in which the relevant monotonicity formula has a different scaling than the known conservation laws, also appears in study of the nonlinear Schrödinger equation.

Our approach to completing the proof of Theorem 1.1.1 is therefore based on the ideas introduced in the NLS setting, and in particular we make use of a technique developed by Killip and Visan to treat the energy-critical and energy-supercritical NLS [29, 30].

The idea behind our approach is two-fold: starting from Theorem 1.2.2, we first use a further reduction due to Killip-Tao-Visan [24] and Killip-Visan [26, 29, 30] to conclude that the failure of Theorem 1.1.1 implies the existence of a solution falling into one of three possible scenarios: the finite time blow-up solution, the soliton-like solution, and the low-to-high frequency cascade solution. We then prove that such solutions have finite energy, so that the conservation of energy and, in the case of the soliton-like solution, the Morawetz estimate can be applied.

In particular, to rule out the finite time blow-up solution, we show that the spatial support of this solution is contained in a ball for which the radius shrinks to 0 as t approaches the blow-up time, by virtue of the finite speed of propagation. This is then shown to be incompatible with the conservation of energy. To handle the remaining two scenarios, we use a double Duhamel technique introduced in [8, 43], which is used for the same purpose in [29, 30], and which allows us to show the finiteness of energy for these scenarios. We remark that this technique is both the source of our ability to treat non-radial initial data as well as the restriction in Theorem 1.1.1 to dimensions $d \geq 6$, due to the need to prove the convergence of a double integral coming from the use of Lemma 1.2.3 both forward and backward in time. We refer the reader to Section 3.1 for a more detailed account.

1.2.3 Approach for dimension $d = 5$

Since the double Duhamel technique cannot be immediately applied in dimension $d = 5$, we prove Theorem 1.1.2 by taking a complementary approach. Rather than, as in the proof of Theorem 1.1.1, proving an additional regularity property for a special class of solutions in order to have access to the Morawetz estimate (1.4), we make use of ideas recently introduced for the mass-critical and energy-critical nonlinear Schrödinger equation [9, 29, 46] to localize this estimate in frequency so that it can be applied to our minimal blow-up solutions.

As in the proof of Theorem 1.1.1, we invoke a preliminary reduction to show that the failure of Theorem 1.1.2 implies the existence of a solution belonging to one of two scenarios: the finite time and infinite time blow-up solutions. We remark that these scenarios are related to, but not identical with, the scenarios identified in the proof of Theorem 1.1.1. From here, the finite time blow-up scenario is ruled out in analogy with the argument used in the setting of Theorem 1.1.1. On the other hand, to rule out the infinite time blow-up scenario we obtain a frequency localized form of the Morawetz estimate. In particular, this estimate, along with the assumption of radial symmetry, negates the need to prove the finiteness of energy.

Chapter 2

Preliminaries

In this section, we introduce the notation and some basic estimates that we use throughout the thesis. For any time interval $I \subseteq \mathbb{R}$, we write $L_t^q L_x^r(I \times \mathbb{R}^d)$ to denote the spacetime norm

$$\|u\|_{L_t^q L_x^r} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(t,x)|^r dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}}$$

with the standard definitions when q or r is equal to infinity. In the case $q = r$, we shorten the notation $L_t^q L_x^r$ and write $L_{t,x}^q$. We write $X \lesssim Y$ to indicate that there exists a constant $C > 0$ such that $X \leq CY$. The constant C may change from line to line, and its dependence will be indicated by subscripts, i.e. $X \lesssim_u Y$ to mean $X \leq C(u)Y$ with $C(u)$ depending on u. For convenience, we will at times also use the explicit constant $C(u)$. We use the symbol ∇ for the derivative operator in the space variable.

In what follows, we define the Fourier transform on $\mathbb{R}^d$ by

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) dx.$$

We also define the homogeneous Sobolev space $\dot{H}_x^s(\mathbb{R}^d)$, $s \in \mathbb{R}$ via the norm

$$\|f\|_{\dot{H}_x^s} := \||\nabla|^s f\|_{L_x^2}$$

where the fractional differentiation operator is given by

$$\widehat{|\nabla|^s f}(\xi) := |\xi|^s \hat{f}(\xi).$$

We next recall some basic facts from Littlewood-Paley theory that will be used frequently in the sequel (see for instance Chapter A of [41]). Let $\phi(\xi)$ be a real valued radially symmetric bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 2\}$ which equals 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For any dyadic number $N = 2^k$, $k \in \mathbb{Z}$, we define the following Littlewood-Paley operators:

$$\begin{aligned}
\widehat{P_{\leq N} f}(\xi) &= \phi(\xi/N)\hat{f}(\xi), \\
\widehat{P_{>N} f}(\xi) &= (1 - \phi(\xi/N)\hat{f}(\xi), \\
\widehat{P_N f}(\xi) &= (\phi(\xi/N) - \phi(2\xi/N))\hat{f}(\xi).
\end{aligned}$$

Similarly, we define $P_{<N}$ and $P_{\geq N}$ with

$$P_{<N} = P_{\leq N} - P_N, \quad P_{\geq N} = P_{>N} + P_N,$$

and also

$$P_{M<\cdot\leq N} := P_{\leq N} - P_{\leq M} = \sum_{M<N_1\leq N} P_{N_1}$$

whenever $M \leq N$.

These operators commute with one another, with derivative operators and with the wave propagator $\mathcal{W}(t)(f,g)$. Moreover, they are bounded on L^p_x for $1 \leq p \leq \infty$ and obey the following *Bernstein inequalities*,

$$\||\nabla|^s P_{\leq N} f\|_{L^p_x} \lesssim N^s \|P_{\leq N} f\|_{L^p_x},$$

$$\begin{aligned}\|P_{>N}f\|_{L^p_x} &\lesssim N^{-s}\|P_{>N}|\nabla|^s f\|_{L^p_x},\\ \||\nabla|^{\pm s}P_N f\|_{L^p_x} &\sim N^{\pm s}\|P_N f\|_{L^p_x},\\ \|P_{\leq N}f\|_{L^q_x} &\lesssim N^{\frac{d}{p}-\frac{d}{q}}\|P_{\leq N}f\|_{L^p_x}\\ \|P_N f\|_{L^q_x} &\lesssim N^{\frac{d}{p}-\frac{d}{q}}\|P_N f\|_{L^p_x},\end{aligned}$$

with $s \geq 0$ and $1 \leq p \leq q \leq \infty$.

We end this section by noting some basic facts concerning the fractional derivative operator.

Remark 2.0.1. Suppose $\phi \in C_0^\infty(\mathbb{R}^d)$, where C_0^∞ denotes the space of smooth functions having compact support. Then for all nonnegative integers s and all $p \geq 1$ we have $|\nabla|^s\phi \in L^p_x$, while for all $s > 0$ and all $p \in [2, d)$, we have $|\nabla|^s\phi \in L^p_x$.

We also note a (simple) version of the chain rule which allows us to compute the fractional derivative of a composition with a linear function.

Remark 2.0.2. For all $s > 0$, $|\nabla|^s\left[u(\alpha\cdot)\right](x) = \alpha^s(|\nabla|^s u)(\alpha x)$.

2.1 The linear and nonlinear wave equation

In this section we recall some classical properties of the linear and nonlinear wave equation. We use $\mathcal{W}(t)$ to denote the linear wave propagator associated to (NLW). This operator is given by (1.2), which is written equivalently in frequency space as

$$\widehat{\mathcal{W}(t)}(f,g)(\xi) = \cos(t|\xi|)\hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|}\hat{g}(\xi).$$

In particular, in terms of the explicit form of the propagator, we recall the following standard dispersive estimate.

Proposition 2.1.1 (Dispersive estimate, [38]). *For any $d \geq 2$, $2 \leq p < \infty$ and $t \neq 0$ we have*

$$\left\| \frac{e^{it|\nabla|}}{|\nabla|} f \right\|_{L_x^p} \lesssim |t|^{-\frac{d-1}{2}\left(1-\frac{2}{p}\right)} \| |\nabla|^{\frac{d-1}{2}-\frac{d+1}{p}} f \|_{L_x^{p'}}. \tag{2.1}$$

In particular,

$$\left\| \frac{\sin(t|\nabla|)}{|\nabla|} f \right\|_{L_x^p(\mathbb{R}^d)} \lesssim |t|^{-\frac{(d-1)}{2}\left(1-\frac{2}{p}\right)} \| |\nabla|^{\frac{d-1}{2}-\frac{d+1}{p}} f \|_{L_x^{p'}(\mathbb{R}^d)} \tag{2.2}$$

and

$$\left\| \frac{\cos(t|\nabla|)}{|\nabla|^2} g \right\|_{L_x^p(\mathbb{R}^d)} \lesssim |t|^{-\frac{(d-1)}{2}\left(1-\frac{2}{p}\right)} \| |\nabla|^{\frac{d-3}{2}-\frac{d+1}{p}} g \|_{L_x^{p'}(\mathbb{R}^d)},$$

for all $f, g \in \mathcal{S}(\mathbb{R}^d)$, where $\frac{1}{p'} + \frac{1}{p} = 1$.

For $s \geq 0$, we say that a pair of exponents (q, r) is $\dot{H}_x^s$-*wave admissible* if $q, r \geq 2$, $r < \infty$ and it satisfies

$$\frac{1}{q} + \frac{d-1}{2r} \leq \frac{d-1}{4},$$
$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s.$$

The *Strichartz estimates* then read as follows; for a proof, see [11, 15, 39]. Assume $u : I \times \mathbb{R}^d \to \mathbb{R}$ with time interval $0 \in I \subset \mathbb{R}$ is a solution to the nonlinear wave equation

$$\begin{cases} u_{tt} - \Delta u + F &= 0 \\ (u, u_t)|_{t=0} &= (u_0, u_1) \in \dot{H}_x^{\mu} \times \dot{H}_x^{\mu-1}(\mathbb{R}^d), \quad \mu \in \mathbb{R}. \end{cases}$$

Then

$$\||\nabla|^s u\|_{L_t^q L_x^r} + \||\nabla|^{s-1} u_t\|_{L_t^q L_x^r} + \||\nabla|^{\mu} u\|_{L_t^\infty L_x^2} + \||\nabla|^{\mu-1} u_t\|_{L_t^\infty L_x^2} \tag{2.3}$$
$$\lesssim \|(u_0, u_1)\|_{\dot{H}_x^{\mu} \times \dot{H}_x^{\mu-1}} + \||\nabla|^{\tilde{s}} F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

for $s \geq 0$, where the pair (q, r) is $\dot{H}_x^{\mu-s}$-wave admissible and the pair $(\tilde{q}, \tilde{r})$ is $\dot{H}_x^{1+\tilde{s}-\mu}$-wave admissible.

We also define the following *Strichartz norms.* For each $I \subset \mathbb{R}$ and $s \geq 0$, we set

$$\|u\|_{S_s(I)} = \sup_{(q,r)\ \dot{H}_x^s-\text{wave admissible}} \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)},$$
$$\|u\|_{N_s(I)} = \inf_{(q,r)\ \dot{H}_x^s-\text{wave admissible}} \|u\|_{L_t^{q'} L_x^{r'}(I \times \mathbb{R}^d)}.$$

Taking the supremum over (q, r) $\dot{H}_x^{\mu-s}$-wave admissible and the infimum over $(\tilde{q}, \tilde{r})$ $\dot{H}_x^{1+\tilde{s}-\mu}$-wave admissible pairs in (2.3), we also have,

$$\||\nabla|^s u\|_{S_{\mu-s}(I)} + \||\nabla|^{s-1} u_t\|_{S_{\mu-s}(I)} \lesssim \|(u_0, u_1)\|_{\dot{H}_x^{\mu} \times \dot{H}_x^{\mu-1}} + \||\nabla|^{\tilde{s}} F\|_{N_{1+\tilde{s}-\mu}(I)}. \tag{2.4}$$

We also recall the following Morawetz estimate for the wave equation.

Theorem 2.1.2 (Morawetz estimate [31, 32])**.** *Assume $u : I \times \mathbb{R}^d \to \mathbb{R}$ is a solution to (NLW). Then we have*

$$\int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|} dx dt \leq C E(u, u_t).$$

We also note that our assumption $u \in L_{t,x}^6(K \times \mathbb{R}^5)$ on the solution in Definition 1.0.1, combined with the local theory and the Strichartz estimates, implies

$$\||\nabla|^s u\|_{S_{\frac{3}{2}-s}(K)} < \infty$$

for $s \in [0, \frac{3}{2}]$ and $K \subset I$ compact.

Moreover, for every nonzero almost periodic solution u to (NLW) there exists $C(u) > 0$ such that for every compact $K \subset I$

$$\frac{1}{C(u)} \int_K N(t)dt \leq \|u\|^6_{L^6_t(K;L^6_x)} \leq C(u)\Big(1 + \int_K N(t)dt\Big), \tag{2.5}$$

together with the bound

$$\frac{1}{C(u)} \int_K N(t)dt \leq \||\nabla|^{3/4}u\|^2_{L^2_t(K;L^4_x)} \leq C_1(u)\Big(1 + \int_K N(t)dt\Big). \tag{2.6}$$

The above bounds are consequences of almost periodicity and the Strichartz estimates (2.4). In the NLS setting, we refer to the analogous estimates in [27, Lemma 5.21] and [46, Lemma 1.7], while for solutions to (NLW) these bounds are obtained in a similar manner, after accounting for the difference in scaling between the equations.

We also record two consequences of almost periodicity from [26, 29].

Remark 2.1.1. If u is an almost periodic solution modulo symmetries, then for each $\eta > 0$ there exist constants $c_1(\eta), c_2(\eta) > 0$ such that for all $t \in I$,

$$\int_{|x-x(t)|\geq c_1(\eta)/N(t)} |u(t,x)|^d dx + \int_{|x-x(t)|\geq c_1(\eta)/N(t)} |u_t(t,x)|^{\frac{d}{2}} dx \leq \eta,$$

$$\int_{|x-x(t)|\geq c_1(\eta)/N(t)} |\nabla u(t,x)|^{\frac{d}{2}} dx + \int_{|x-x(t)|\geq c_1(\eta)/N(t)} |u_t(t,x)|^{\frac{d}{2}} dx \leq \eta,$$

and also

$$\int_{|\xi|\leq c_2(\eta)N(t)} |\xi|^{2s_c}|\hat{u}(t,\xi)|^2 + |\xi|^{2(s_c-1)}|\hat{u}_t(t,\xi)|^2 d\xi \leq \eta. \tag{2.7}$$

2.2 Review of the local theory for (NLW)

In this section, we review the standard local theory for (NLW) : local well-posedness and stability theorems. The versions that we present here are in the spirit of [18, 21, 28, 29, 44], and for the clarity of exposition we restrict ourselves to dimensions $d \geq 6$ (the setting of Chapter 3 below). We remark that analogous results hold in dimension $d = 5$ after a suitable change in the numerology.

We note that the product structure of the cubic nonlinearity $F(u) = |u|^2 u$ gives access to estimates coming from the following product rule for fractional derivatives. In the case of general nonlinearities $F(u) = |u|^p u$, similar arguments to those presented in this section carry through with the fractional product rule replaced by the fractional chain rule.

Lemma 2.2.1. *[7, 23] For all $s \geq 0$ we have*

$$\||\nabla|^s (fg)\|_{L_x^p} \leq \||\nabla|^s f\|_{L_x^{p_1}} \|g\|_{L_x^{p_2}} + \|f\|_{L_x^{p_3}} \||\nabla|^s g\|_{L_x^{p_4}},$$

where $1 < p_1, p_4 < \infty$ and $1 < p, p_2, p_3 \leq \infty$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

In the following two lemmas, using Lemma 2.2.1, we obtain the estimates that will help us control the nonlinear term in establishing the local well-posedness and stability results.

Lemma 2.2.2. *Let $d \geq 6$ be given. Then the following estimate holds:*

$$\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} (fg)\|_{N_{\frac{d-3}{2(d-1)}}} \lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{S_{\frac{d+1}{2(d-1)}}} \|g\|_{L_{t,x}^{\frac{d+1}{2}}}$$

$$+ \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{S_{\frac{d+1}{2(d-1)}}} \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} g\|_{L_t^{\frac{d+1}{2}} L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}}$$

Proof. We begin by noting that $(\frac{2(d+1)}{d-3}, \frac{2(d^2-1)}{d^2-2d+5})$ is an $\dot{H}_x^{\frac{d-3}{2(d-1)}}$ wave admissible pair. Applying Lemma 2.2.1 followed by Sobolev's inequality, we obtain,

$$\begin{aligned}
\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}}(fg)\|_{N_{\frac{d-3}{2(d-1)}}} &\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}}(fg)\|_{L_t^{\frac{2(d+1)}{d+5}} L_x^{\frac{2(d^2-1)}{d^2+2d-7}}} \\
&\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{L_t^2 L_x^{\frac{2(d-1)}{d-3}}} \|g\|_{L_{t,x}^{\frac{d+1}{2}}} \\
&\quad + \|f\|_{L_t^2 L_x^{2d}} \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} g\|_{L_t^{\frac{d+1}{2}} L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}} \\
&\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{L_t^2 L_x^{\frac{2(d-1)}{d-3}}} \|g\|_{L_{t,x}^{\frac{d+1}{2}}} \\
&\quad + \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{L_t^2 L_x^{\frac{2(d-1)}{d-3}}} \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} g\|_{L_t^{\frac{d+1}{2}} L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}}.
\end{aligned}$$

We conclude the proof by noting that $(2, \frac{2(d-1)}{d-3})$ is an $\dot{H}_x^{\frac{d+1}{2(d-1)}}$ admissible pair, which gives the right hand side of the desired inequality. □

We will also need the following estimate, which is a variant of the fractional chain rule for the cubic nonlinearity.

Lemma 2.2.3. *Let $d \geq 6$ be given. Then we have,*

$$\begin{aligned}
\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}}(|f|^2 f)\|_{N_{\frac{d-3}{2(d-1)}}} &\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{L_t^2 L_x^{\frac{2(d-1)}{d-3}}} \|f\|_{L_{t,x}^{d+1}}^2 \\
&\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{S_{\frac{d+1}{2(d-1)}}} \|f\|_{L_{t,x}^{d+1}}^2.
\end{aligned}$$

Proof. We note that, proceeding as in the proof of Lemma 2.2.2,

$$\begin{aligned}
\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}}(|f|^2 f)\|_{N_{\frac{d-3}{2(d-1)}}} &\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}}(|f|^2 f)\|_{L_t^{\frac{2(d+1)}{d+5}} L_x^{\frac{2(d^2-1)}{d^2+2d-7}}} \\
&\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{L_t^2 L_x^{\frac{2(d-1)}{d-3}}} \|f^2\|_{L_{t,x}^{\frac{d+1}{2}}} \\
&\quad + \|f\|_{L_{t,x}^{d+1}} \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}}(f^2)\|_{L_t^{\frac{2(d+1)}{d+3}} L_x^{\frac{2(d^2-1)}{d^2-5}}} \\
&\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{L_t^2 L_x^{\frac{2(d-1)}{d-3}}} \|f\|^2_{L_{t,x}^{d+1}} \\
&\quad + \|f\|_{L_{t,x}^{d+1}} \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{L_t^2 L_x^{\frac{2(d-1)}{d-3}}} \|f\|_{L_{t,x}^{d+1}} \\
&\lesssim \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} f\|_{S_{\frac{d+1}{2(d-1)}}} \|f\|^2_{L_{t,x}^{d+1}}, \qquad (2.8)
\end{aligned}$$

where in the third inequality we use Lemma 2.2.1 and we note that $(2, \frac{2(d-1)}{d-3})$ is an $\dot{H}_x^{\frac{d+1}{2(d-1)}}$-wave admissible pair to obtain the desired estimate. □

2.2.1 Local well-posedness

We now review of the standard local well-posedness theorem for (NLW). The version that we present here is in the spirit of the related results in the works of [21, 29]. For similar results see also [6, 10, 20, 34, 37, 44].

Theorem 2.2.4. *Let $d \geq 6$ and $s_c = \frac{d-2}{2}$. Then for all $A > 0$, there exists $\delta_0 = \delta_0(d, A) > 0$ such that for every $0 < \delta \leq \delta_0$, $0 \in I \subset \mathbb{R}$, and $(u_0, u_1) \in \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}(\mathbb{R}^d)$ with*

$$\|(u_0, u_1)\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \leq A, \qquad (2.9)$$

the condition

$$\|\mathcal{W}(t)(u_0, u_1)\|_{L_{t,x}^{d+1}(I\times\mathbb{R}^d)} \leq \delta,$$

implies that there exists a unique solution u to (NLW) on $I \times \mathbb{R}^d$ with

$$\|u\|_{L^{d+1}_{t,x}} \leq 2\delta,$$

and

$$\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} u\|_{S_{\frac{d+1}{2(d-1)}}(I)} + \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}-1} u_t\|_{S_{\frac{d+1}{2(d-1)}}(I)} < \infty.$$

Proof. We use a contraction mapping argument. Fix $\alpha = \frac{d^2-4d+1}{2(d-1)}$ and note that by the Duhamel representation for the solution to (NLW), we have

$$u(t) = \mathcal{W}(t)(u_0, u_1) + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}(|u(s)|^2 u(s))ds.$$

For all $a, b > 0$, we define the contraction space

$$\begin{aligned} B_{a,b} := \{v : \|v\|_{L^{d+1}_{t,x}} \leq a, \\ \||\nabla|^{\alpha} v\|_{S_{\frac{d+1}{2(d-1)}}} + \||\nabla|^{\alpha-1} v_t\|_{S_{\frac{d+1}{2(d-1)}}} \leq b\}, \end{aligned}$$

and the map

$$\Phi(v)(t) := \mathcal{W}(t)(u_0, u_1) + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}(|v(s)|^2 v(s))ds.$$

We would like to show that for suitably chosen a and b, we have the inclusion $\Phi(B_{a,b}) \subset B_{a,b}$ and the mapping $\Phi : B_{a,b} \to B_{a,b}$ is a contraction.

We first note that using Minkowski's inequality followed by the assumption (2.9) and the Strichartz inequality, we obtain for $v \in B_{a,b}$,

$$\||\nabla|^{\alpha} \Phi(v)\|_{S_{s_c - \alpha}} + \||\nabla|^{\alpha-1} \partial_t \Phi(v)\|_{S_{s_c - \alpha}}$$

$$
\begin{aligned}
&\le \||\nabla|^{\alpha}\mathcal{W}(t)(u_0,u_1)\|_{S_{s_c-\alpha}} + \left\||\nabla|^{\alpha}\int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}(|v(s)|^2 v(s))ds\right\|_{S_{s_c-\alpha}} \\
&\quad + \||\nabla|^{\alpha}\partial_t\mathcal{W}(t)(u_0,,u_1)\|_{S_{s_c-\alpha}} \\
&\quad + \left\||\nabla|^{\alpha-1}\int_0^t \cos((t-s)|\nabla|)(|v(s)|^2 v(s))ds\right\|_{S_{s_c-\alpha}} \\
&\lesssim \|(u_0,u_1)\|_{\dot{H}_x^{s_c}\times\dot{H}_x^{s_c-1}} + \||\nabla|^{\alpha}(|v|^2 v)\|_{N_{1+\alpha-s_c}} \\
&\le CA + C'\||\nabla|^{\alpha}v\|_{S_{s_c-\alpha}}\|v\|^2_{L^{d+1}_{t,x}} \qquad (2.10) \\
&\le CA + Ca^2 b,
\end{aligned}
$$

where we used Lemma 2.2.3 to obtain (2.10).

Similarly, using Minkowski's inequality together with the assumption (2.9), we estimate

$$
\begin{aligned}
\|\Phi(v)\|_{L^{d+1}_{t,x}} &\le \|\mathcal{W}(t)(u_0,u_1)\|_{L^{d+1}_{t,x}} + \left\|\int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|}(|u(s)|^2 u(s))ds\right\|_{L^{d+1}_{t,x}} \\
&\le \delta + C\||\nabla|^{\alpha}(|u|^2 u)\|_{N_{1+\alpha-s_c}} \\
&\le \delta + C\||\nabla|^{\alpha}u\|_{S_{s_c-\alpha}}\|u\|^2_{L^{d+1}_{t,x}} \\
&\le \delta + Ca^2 b.
\end{aligned}
$$

Choosing $b = 2AC$ and a such that $Ca^2 \le \frac{1}{2}$, we obtain

$$
\||\nabla|^{\alpha}\Phi(v)\|_{S_{s_c-\alpha}} \le b. \qquad (2.11)
$$

If we also fix $\delta = \frac{a}{2}$ and a small enough such that $Ca^2 b \le \frac{a}{2}$, we have

$$
\|\Phi(v)\|_{L^{d+1}_{t,x}} \le a. \qquad (2.12)
$$

Combining (2.11) and (2.12) with the above choices of a, b and δ, we have the desired inclusion $\Phi(B_{a,b}) \subset B_{a,b}$.

We now show that the mapping Φ is a contraction for suitable a, b and δ. Let a, b and δ be as chosen above. Note that by the Strichartz inequality and Lemma 2.2.2 along with Minkowski's inequality we have,

$$
\begin{aligned}
&\||\nabla|^{\alpha}[\Phi(u)-\Phi(v)]\|_{S_{s_c-\alpha}} + \||\nabla|^{\alpha-1}\partial_t[\Phi(u)-\Phi(v)]\|_{S_{s_c-\alpha}} + \|\Phi(u)-\Phi(v)\|_{L^{d+1}_{t,x}} \\
&\quad \lesssim \||\nabla|^{\alpha}[(|v|^2 v)-(|u|^2 u)]\|_{N_{1+\alpha-s_c}} \\
&\quad = \||\nabla|^{\alpha}[(v-u)\{v^2+uv+u^2\}]\|_{N_{\frac{d-3}{2(d-1)}}} \\
&\quad \le \||\nabla|^{\alpha}(v-u)\|_{S_{\frac{d+1}{2(d-1)}}} \\
&\qquad \Big[\|\{v^2+uv+u^2\}\|_{L^{\frac{d+1}{2}}_{t,x}} + \||\nabla|^{\alpha}\{v^2+uv+u^2\}\|_{L_t^{\frac{d+1}{2}} L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}}\Big] \\
&\quad \le \||\nabla|^{\alpha}(v-u)\|_{S_{\frac{d+1}{2(d-1)}}} \\
&\qquad \Big[\|v^2\|_{L^{\frac{d+1}{2}}_{t,x}} + \|uv\|_{L^{\frac{d+1}{2}}_{t,x}} + \|u^2\|_{L^{\frac{d+1}{2}}_{t,x}} + \||\nabla|^{\alpha}(v^2)\|_{L_t^{\frac{d+1}{2}} L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}} \\
&\qquad + \||\nabla|^{\alpha}(uv)\|_{L_t^{\frac{d+1}{2}} L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}} + \||\nabla|^{\alpha}(u^2)\|_{L_t^{\frac{d+1}{2}} L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}}\Big] \\
&\quad \lesssim \|v-u\|_{B_{a,b}} \qquad\qquad (2.13) \\
&\qquad \Big[\|v\|^2_{L^{d+1}_{t,x}} + \|u\|_{L^{d+1}_{t,x}}\|v\|_{L^{d+1}_{t,x}} + \|u\|^2_{L^{d+1}_{t,x}} \\
&\qquad + \||\nabla|^{\alpha}v\|_{L_t^{d+1} L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}}\|v\|_{L^{d+1}_{t,x}} + \||\nabla|^{\alpha}u\|_{L_t^{d+1} L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}}\|v\|_{L^{d+1}_{t,x}} \\
&\qquad + \||\nabla|^{\alpha}v\|_{L^{d+1}_{t,x}}\|v\|_{L^{d+1}_{t,x}} + \||\nabla|^{\alpha}u\|_{L_t^{d+1} L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}}\|u\|_{L^{d+1}_{t,x}}\Big] \\
&\quad \lesssim \|u-v\|_{B_{a,b}}(a^2+ab),
\end{aligned}
$$

where we use Hölder's inequality and Lemma 2.2.1 to obtain (2.13). Thus, if a is chosen such that $C(a^2+ab) < 1$ we conclude that Φ is a contraction as desired. □

Remark 2.2.1. Note that if $u^{(1)}$ and $u^{(2)}$ are two solutions to (NLW) as stated in

Definition 1.0.1 with maximal interval of existence I such that $(u^{(1)}(0), u_t^{(1)}(0)) = (u^{(2)}(0), u_t^{(2)}(0))$, then

$$u^{(1)}(t) = u^{(2)}(t) \quad \text{for all} \quad t \in I.$$

This result follows from standard arguments; see for instance [39, §IV.3].

2.2.2 Stability

In this section, we prove a stability result for (NLW). As in the local well-posedness theorem, the argument that we present follows a standard approach. In particular, the argument that we present here is in the spirit of the related works [18, 29]. For similar treatments, see also [6, 17, 28, 44].

Theorem 2.2.5. *Let $d \geq 6$ and $s_c = \frac{d-2}{2}$. Assume $0 \in I \subset \mathbb{R}$ is a compact time interval and $\tilde{u} : I \times \mathbb{R}^d \to \mathbb{R}$ is a solution of the equation*

$$\tilde{u}_{tt} - \Delta\tilde{u} + |\tilde{u}|^2\tilde{u} = e,$$

for some e.

Then for every $E, L > 0$, there exists $\epsilon_1 = \epsilon_1(E, L) > 0$ such that for each $0 < \epsilon < \epsilon_1$, the conditions

$$\sup_{t\in I} \|(\tilde{u}(t), \tilde{u}_t(t))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}(\mathbb{R}^d)} \leq E,$$

$$\|(u_0 - \tilde{u}(0), u_1 - \tilde{u}_t(0))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}(\mathbb{R}^d)} \leq \epsilon,$$

$$\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} e\|_{N_{\frac{d-3}{2(d-1)}}(I)} \leq \epsilon, \quad \text{and}$$

$$\|\tilde{u}\|_{L^{d+1}_{t,x}} \leq L$$

imply that there exists a unique solution $u : I \times \mathbb{R}^d \to \mathbb{R}$ to (NLW) with initial data (u_0, u_1) such that

$$\|\tilde{u} - u\|_{L^{d+1}_{t,x}} \leq C(E, L)\epsilon, \tag{2.14}$$

$$\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}}(u - \tilde{u})\|_{S_{\frac{d+1}{2(d-1)}}(I)} \leq C(E, L)\epsilon, \tag{2.15}$$

$$\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} u\|_{S_{\frac{d+1}{2(d-1)}}(I)} \leq C(E, L). \tag{2.16}$$

Proof. Fix $\alpha = \frac{d^2-4d+1}{2(d-1)}$. We begin by obtaining a bound on

$$\||\nabla|^{\alpha}\tilde{u}\|_{S_{s_c-\alpha}(I)}.$$

To do so, we fix $\epsilon_1, \eta > 0$ (to be determined later in the argument) and partition I into $J_0 = J_0(L, \eta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that for each $j = 1, \cdots, J_0$,

$$\|\tilde{u}\|_{L^{d+1}_{t,x}(I_j \times \mathbb{R}^d)} \leq \eta.$$

Applying the Strichartz inequality followed by Lemma 2.2.3, we obtain

$$\begin{aligned}
\||\nabla|^{\alpha}\tilde{u}\|_{S_{s_c-\alpha}(I_j)} &\lesssim \|(\tilde{u}(t_j), \tilde{u}_t(t_j))\|_{\dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x} \\
&\quad + \||\nabla|^{\alpha} e\|_{N_{1+\alpha-s_c}(I_j)} + \||\nabla|^{\alpha} F(\tilde{u}(s))\|_{N_{1+\alpha-s_c}(I_j)} \\
&\lesssim E + \epsilon + \|\tilde{u}\|^2_{L^{d+1}_{t,x}} \||\nabla|^{\alpha}\tilde{u}\|_{S_{s_c-\alpha}(I_j)} \\
&\lesssim E + \epsilon_1 + \eta^2 \||\nabla|^{\alpha}\tilde{u}\|_{S_{s_c-\alpha}(I_j)}
\end{aligned}$$

for each $\epsilon < \epsilon_1$. Choosing $\eta > 0$ sufficiently small and $\epsilon_1 < E$, we obtain

$$\||\nabla|^\alpha \tilde{u}\|_{S_{s_c-\alpha}(I_j)} \lesssim E.$$

Summing the contributions of the subintervals, we conclude

$$\||\nabla|^\alpha \tilde{u}\|_{S_{s_c-\alpha}(I)} \lesssim C(E, L). \tag{2.17}$$

as desired.

To continue, fixing $\epsilon_1 \leq E$ and $\delta > 0$ (to be determined later in the argument), we note that $(d+1, \frac{2d(d^2-1)}{d^3-d^2-5d+1})$ is an $\dot{H}_x^{\frac{d+1}{2(d-1)}}$-wave admissible pair. Then by virtue of (2.17), we may divide I into $J_1 = J_1(E, L, \delta)$ subintervals $I_j = [t_j, t_{j+1}]$ such that for each $j = 1, \cdots, J_1$, we have

$$\||\nabla|^\alpha \tilde{u}\|_{L_t^{d+1} L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}} \leq \delta.$$

Let $w = u - \tilde{u}$, and define, for $t \in I$ and $j = 1, \cdots, J_1$,

$$\gamma_j(t) := \||\nabla|^\alpha [F(\tilde{u} + w) - F(\tilde{u})]\|_{N_{1+\alpha-s_c}([t_j,t])}.$$

Let $j \in \{1, \cdots, J_1\}$ be given. We now obtain an estimate on $\gamma_j(t)$. We begin by writing

$$F(x) - F(y) = (x-y)[(x-y)^2 + 3xy].$$

Invoking Lemma 2.2.2, followed by Minkowski's and Hölder's inequalities, we obtain

$$\gamma_j(t) \leq \||\nabla|^\alpha w\|_{S_{s_c-\alpha}}$$

$$
\begin{aligned}
&\Big[\|w^2+3(\tilde u+w)\tilde u\|_{L^{\frac{d+1}{2}}_{t,x}}+\||\nabla|^\alpha[w^2+3(\tilde u+w)\tilde u]\|_{L_t^{\frac{d+1}{2}}L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}}\Big]\\
&\lesssim \||\nabla|^\alpha w\|_{S_{s_c-\alpha}}\Big[\|w^2\|_{L^{\frac{d+1}{2}}_{t,x}}+\|\tilde u^2\|_{L^{\frac{d+1}{2}}_{t,x}}+\|w\tilde u\|_{L^{\frac{d+1}{2}}_{t,x}}\\
&\quad+\||\nabla|^\alpha[w^2]\|_{L_t^{\frac{d+1}{2}}L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}}+\||\nabla|^\alpha[\tilde u^2]\|_{L_t^{\frac{d+1}{2}}L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}}\\
&\quad+\||\nabla|^\alpha[w\tilde u]\|_{L_t^{\frac{d+1}{2}}L_x^{\frac{2d(d^2-1)}{d^3+d^2-7d+1}}}\Big]\\
&\lesssim \||\nabla|^\alpha w\|_{S_{s_c-\alpha}}\Big[\|w\|^2_{L^{d+1}_{t,x}}+\|\tilde u\|^2_{L^{d+1}_{t,x}}+\|\tilde u\|_{L^{d+1}_{t,x}}\|w\|_{L^{d+1}_{t,x}}\\
&\quad+\||\nabla|^\alpha w\|_{L_t^{d+1}L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}}\|w\|_{L^{d+1}_{t,x}}+\||\nabla|^\alpha \tilde u\|_{L_t^{d+1}L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}}\|\tilde u\|_{L^{d+1}_{t,x}}\\
&\quad+\||\nabla|^\alpha w\|_{L_t^{d+1}L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}}\|\tilde u\|_{L^{d+1}_{t,x}}+\|w\|_{L^{d+1}_{t,x}}\||\nabla|^\alpha \tilde u\|_{L_t^{d+1}L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}}\Big]\\
&\lesssim \||\nabla|^\alpha w\|^3_{S_{s_c-\alpha}(I_j)}+\delta\||\nabla|^\alpha w\|^2_{S_{s_c-\alpha}(I_j)}+\delta^2\||\nabla|^\alpha w\|_{S_{s_c-\alpha}(I_j)}. \qquad (2.18)
\end{aligned}
$$

where we have used Lemma 2.2.1 along with Sobolev's inequality in obtaining the last inequality.

Having obtained the bound (2.18) on $\gamma_j(t)$ for all $j\in\{1,\cdots,J_1\}$, we next show by induction that for every $j=1,\cdots,J_1$, there exists a constant $C(j,d)>0$ such that

$$
\gamma_j(t)\le C(j,d)\epsilon. \qquad (2.19)
$$

In the remainder of the argument, we let $\epsilon\in\mathbb{R}$ be arbitrary such that $\epsilon<\epsilon_1$ and we note that without loss of generality we may assume $t_1=0$.

To obtain (2.19) we argue as follows: we first observe that when $j=1$, the Strichartz inequality gives, for every $t\in I_1$,

$$
\||\nabla|^\alpha w\|_{S_{s_c-\alpha}([t_1,t])}\lesssim \|(w(t_1),w_t(t_1))\|_{\dot H^{s_c}_x\times\dot H^{s_c-1}_x}
$$

$$
\begin{aligned}
&+ \||\nabla|^\alpha [F(\tilde{u}) - F(u)]\|_{N_{1+\alpha-s_c}([t_1,t])} + \||\nabla|^\alpha e\|_{N_{1+\alpha-s_c}(I_1)} \\
&\lesssim \|(w(0), w_t(0))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} + \gamma_1(t) + \epsilon \\
&\lesssim \epsilon + \gamma_1(t) + \epsilon. \qquad (2.20)
\end{aligned}
$$

Putting (2.18) and (2.20) together, we obtain

$$
\gamma_1(t) \lesssim (\gamma_1(t) + \epsilon)^3 + \delta(\gamma_1(t) + \epsilon)^2 + \delta^2(\gamma_1(t) + \epsilon).
$$

A bootstrap argument then implies that for δ and ϵ sufficiently small, $\gamma_1(t) \lesssim \epsilon$ for all $t \in I_1$.

For the induction step, we now assume that for all $j \leq j_0$ there exists $C(j, d, \delta) > 0$ such that $\gamma_j(t) \leq C(j, d)\epsilon$ for all $t \in I_j$. We then prove the validity of (2.19) for $j = j_0 + 1$.

Note that for every $t \in I_{j_0+1}$, two successive applications of the Strichartz inequality give

$$
\begin{aligned}
\||\nabla|^\alpha w\|_{S_{s_c-\alpha}([t_{j_0+1},t])} &\lesssim \|(w(t_{j_0+1}), w_t(t_{j_0+1}))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \\
&\quad + \||\nabla|^\alpha [F(\tilde{u}) - F(u)]\|_{N_{1+\alpha-s_c}([t_{j_0+1},t])} + \||\nabla|^\alpha e\|_{N_{1+\alpha-s_c}(I_{j_0+1})} \\
&\lesssim \|(w(t_{j_0+1}), w_t(t_{j_0+1}))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} + \gamma_{j_0+1}(t) + \epsilon \\
&\lesssim \|(w(0), w_t(0))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} + \||\nabla|^\alpha [F(\tilde{u}) - F(u)]\|_{N_{1+\alpha-s_c}([0,t_{j_0+1}])} \\
&\quad + \||\nabla|^\alpha e\|_{N_{1+\alpha-s_c}([0,t_{j_0+1}])} + \gamma_{j_0+1}(t) + \epsilon \\
&\lesssim 3\epsilon + \gamma_{j_0+1}(t) + \sum_{k=1}^{j_0} \gamma_k(t_{k+1}) \\
&\lesssim \left(3 + \sum_{k=1}^{j_0} C(k, d)\right)\epsilon + \gamma_{j_0+1}(t) \qquad (2.21)
\end{aligned}
$$

where we used the induction assumption in obtaining the last inequality. Noting $\sum_{k=1}^{j_0} C(k,d) \lesssim C(j_0,d)$ and combining (2.18) and (2.21), we obtain

$$\gamma_{j_0+1}(t) \lesssim (\gamma_{j_0+1}(t)+\epsilon)^3 + \delta(\gamma_{j_0+1}(t)+\epsilon)^2 + \delta^2(\gamma_{j_0+1}(t)+\epsilon).$$

A bootstrap argument then implies that for δ and ϵ_1 sufficiently small, $\gamma_{j_0+1}(t) \lesssim \epsilon$ for all $t \in I_{j_0+1}$. This immediately establishes the inductive step $j_0 \to j_0+1$.

Combining the estimates (2.19) that we have obtained on $\gamma_j(t)$ for $j = 1, \cdots, J_1$, we obtain

$$\||\nabla|^\alpha [F(u) - F(\tilde{u})]\|_{N_{1+\alpha-s_c}(I)} \lesssim \sum_{j=1}^{J_1} \gamma_j(t_{j+1}) \lesssim C(E,L)\epsilon \tag{2.22}$$

where we note that $J_1 = J_1(E,L)$.

We now conclude the proof by showing the desired bounds (2.14)-(2.16). For (2.14), we note that by the Sobolev embedding and the definition of the $S_{s_c-\alpha}$ norm, we have

$$\begin{aligned}\|\tilde{u} - u\|_{L^{d+1}_{t,x}} &\lesssim \||\nabla|^\alpha(\tilde{u}-u)\|_{L_t^{d+1} L_x^{\frac{2d(d^2-1)}{d^3-d^2-5d+1}}} \\ &\lesssim \||\nabla|^\alpha(\tilde{u}-u)\|_{S_{s_c-\alpha}}.\end{aligned}$$

On the other hand, for (2.16), Minkowski's inequality and (2.17) imply

$$\begin{aligned}\||\nabla|^\alpha u\|_{S_{s_c-\alpha}} &\leq \||\nabla|^\alpha(u-\tilde{u})\|_{S_{s_c-\alpha}} + \||\nabla|^\alpha \tilde{u}\|_{S_{s_c-\alpha}} \\ &\lesssim \||\nabla|^\alpha(\tilde{u}-u)\|_{S_{s_c-\alpha}} + C(E,L).\end{aligned}$$

Thus, both (2.14) and (2.16) follow from (2.15), which is proved as follows: by the Strichartz inequality and (2.22), we have

$$\||\nabla|^\alpha(\tilde{u}-u)\|_{S_{s_c-\alpha}} \lesssim \epsilon + \||\nabla|^\alpha F(\tilde{u}) - F(u)\|_{N_{1+\alpha-s_c}}$$

$$\lesssim C(E, L)\epsilon.$$

□

Chapter 3

The Defocusing Energy-Supercritical Cubic NLW in Dimensions Six and Higher

In this chapter, we prove Theorem 1.1.1. In particular, in Section 3.1, we give a detailed overview of the proof of the theorem, starting with a refinement of Theorem 1.2.2 which shows that the failure of Theorem 1.1.1 implies the existence of one of three special blow-up scenarios: the finite time blow-up solution, the soliton-like solution, and the low-to-high frequency cascade solution. In Section 3.2, we then state and prove a lemma which arises as a consequence of the finite speed of propagation and which will be used in Sections 3.3 and 3.5.

In Section 3.3, we rule out the finite time blow-up scenario. In Section 3.4, we prove an additional decay result for the soliton-like and low-to-high frequency cascade scenarios. This result is then used to rule out these two cases in Sections 3.5 and 3.6 respectively.

We conclude the chapter with a brief discussion in Section 3.7, in which we provide the details of some arguments used in the main body of the proof.

3.1 Overview of the proof of Theorem 1.1.1

We now give a brief outline of the proof of Theorem 1.1.1. The approach we pursue here follows the methods introduced by Kenig and Merle [20, 21] and Killip, Tao, and Visan [24], and developed in the works [18, 22, 26, 29, 30]. As we have mentioned in the introduction, the proof of Theorem 1.1.1 is an argument by contradiction and consists of the following components:

3.1.1 Existence of a minimal blow-up solution and three blow-up scenarios

Supposing that Theorem 1.1.1 fails, Theorem 1.2.2 implies that there exists a minimal blow-up solution with the compactness property (1.5). To obtain the desired contradiction, the next step in the argument is to show that no such blow-up solution can exist. The following theorem now shows that failure of Theorem 1.1.1, in addition to implying the existence of a minimal blow-up solution, also implies the existence of an almost periodic solution which belongs to one of three particular classes for which the associated function $N(t)$ is specified further. Thus in order to prove Theorem 1.1.1, it will suffice to show that such solutions cannot exist.

Theorem 3.1.1. *[26] Suppose that Theorem* 1.1.1 *fails. Then there exists a solution* $u : I \times \mathbb{R}^d \to \mathbb{R}$ *to (NLW) with maximal interval of existence* I *such that* u *is almost periodic modulo symmetries,*

$$(u, u_t) \in L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}), \quad \textit{and} \quad \|u\|_{L_{t,x}^{d+1}(I\times\mathbb{R}^d)} = \infty,$$

and u satisfies one of the following:

- *(finite time blow-up solution) either* $\sup I < \infty$ *or* $\inf I > -\infty$.
- *(soliton-like solution)* $I = \mathbb{R}$ *and* $N(t) = 1$ *for all* $t \in \mathbb{R}$.
- *(low-to-high frequency cascade solution)* $I = \mathbb{R}$,

$$\inf_{t\in\mathbb{R}} N(t) \geq 1, \quad \textit{and} \quad \limsup_{t\to\infty} N(t) = \infty.$$

In the context of the mass critical nonlinear Schrödinger equation, a more refined version of this theorem was proved by Killip, Tao and Visan in [24]. The version that we use here was obtained by Killip and Visan in [30]. As remarked in [26], the argument applies equally to the present NLW setting.

3.1.2 The contradiction

We conclude our proof of Theorem 1.1.1 by showing that each of the scenarios identified in Theorem 3.1.1 cannot occur.

The key ingredient that we use to rule out each of these scenarios is the conservation of energy. However, we note that in our current setting we do not have immediate access to the finiteness of energy, since the energy has scaling below the critical regularity. Nevertheless, in our analysis of each scenario, this obstruction is overcome with an observation that the solutions in that case do indeed have finite energy, due to the particular properties they possess. We then exploit the conservation of energy in a manner well-suited to each scenario to obtain the desired contradiction.

We now briefly describe how we exclude each possible scenario in Theorem 3.1.1:

We first consider the finite time blow-up solution. In this case, our arguments are in the spirit of related results in [18, 21]. We also note that a similar approach is taken in [26]. The key observation here is that when the maximal interval of existence of a solution u is finite, the finite speed of propagation forces the supports of u and u_t to be localized to a ball which shrinks to 0 as one approaches the blow-up time (see Lemma 3.3.2). We then show that the energy $E(u(t), u_t(t))$ tends to 0 as t tends to the blow-up time, contradicting the construction of u as a blow-up solution.

We next study the remaining two scenarios, the soliton-like solution and the low-to-high frequency cascade. In these cases, as in [26, 29], we prove that the solutions possess an additional decay property: for almost periodic solutions with the function $N(t)$ bounded away from zero, the a priori bound $(u, u_t) \in L_t^\infty(\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$ allows us to obtain the bound $(u, u_t) \in L_t^\infty(\dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon})$ for some $\epsilon > 0$ (see Theorem 3.4.1 for further details). In the NLS context the corresponding result was obtained in [29, 30], while for the energy-supercritical NLW in $d = 3$, see [26].

Arguing as in [26, 29], we prove the additional decay property as follows:

- (Lemma 3.4.2) We first refine the bound $u \in L_t^\infty L_x^d$ (which is immediate from the Sobolev embedding and the a priori assumption $u \in L_t^\infty \dot{H}_x^{s_c}$) to $L_t^\infty L_x^p$ for some $p < d$. In particular, we use a bootstrap argument to bound the low frequencies of u via Lemma 1.2.3, while the high frequencies are bounded by the a priori bound. We note that this argument imposes the restriction $p > 2(d-1)/(d-3)$.

- (Lemma 3.4.3) We next use this $L_t^\infty L_x^p$ bound to improve bounds of the form $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^s \times \dot{H}_x^{s-1})$ to $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s-s_0} \times \dot{H}_x^{s-1-s_0})$ for some $s_0 > 0$. This is accomplished by using the double Duhamel technique [8, 43]. More precisely, we consider the inner product of the forward-in-time Duhamel formula with its backward-in-time counterpart given in Lemma 1.2.3, and use the dispersive estimate. When p is such that the resulting integrals are convergent, this gives the desired improvement. We note that this argument imposes the restriction $p < d - 1$.

- (Theorem 3.4.1) Once we obtain the second step, we iterate the argument, starting with the a priori bound $(u, u_t) \in L_t^\infty(\dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1})$, to obtain the desired decay $L_t^\infty(\dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon})$ for some $\epsilon > 0$. In particular, we obtain that the energy is finite.

We remark that the balance between the bounds provided by Lemma 3.4.2 and the bound required by Lemma 3.4.3 is the source of our restriction to dimensions $d \geq 6$. As we noted above, Lemma 3.4.2 provides the $L_t^\infty L_x^p$

bounds for $p > 2(d-1)/(d-3)$, while Lemma 3.4.3 requires this bound with $p < d-1$. These conditions on p impose the restriction $d \geq 6$.

We now return to the study of the two remaining blow-up scenarios: the soliton-like solution and the low-to-high frequency cascade solution.

To preclude the soliton-like solution, we note that the finite speed of propagation implies a bound on the growth of $x(t)$ (see Lemma 3.5.2), while the almost periodicity gives a uniform bound from below on the $L_t^4([s, s+1]; L_x^4(\mathbb{R}^d))$ norm (see Lemma 3.5.1). The latter bound is closely related to a similar bound in [26]. However, we point out that in [26] the bound is based on the L_x^d norm, while our estimate is obtained via the $L_x^{2d/(d-2)}$ norm. This allows us to use the dispersive estimate to control the linear propagator, rather than using the Strichartz estimate and a bootstrap argument. Arguing as in [26], we then obtain a contradiction via the Morawetz identity by combining the bound on $x(t)$ with the $L_{t,x}^4$ bound and the finiteness of energy.

To conclude, as in the soliton-like solution, our preclusion of the low-to-high frequency cascade scenario is also based on the additional decay result. We argue in a similar spirit as in [29] to show that the energy tends to 0 as $N(t)$ approaches infinity. Since the energy is conserved, this contradicts our construction of u as a blow-up solution.

3.2 Finite speed of propagation

A key property of NLW which is not present in the NLS setting is the finite speed of propagation. Using this property, we next give the following lemma which will facilitate our arguments in the proofs of Lemma 3.3.2 and Lemma 3.5.2.

Let ψ be a smooth radial function such that $0 \leq \psi \leq 1$ and

$$\psi(x) = \begin{cases} 0, & |x| < 1, \\ 1, & |x| \geq 2. \end{cases}$$

For all $R > 0$, define $\psi_R \in C^\infty(\mathbb{R}^d)$ by

$$\psi_R(x) = \psi(\frac{x}{R}), \quad x \in \mathbb{R}^d.$$

Lemma 3.2.1. *Suppose that $u : I \times \mathbb{R}^d \to \mathbb{R}$ is an almost periodic solution to (NLW) with maximal interval of existence I and $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$.*

Then for each $\epsilon > 0$ there exists $R > 0$ such that for every $t \in I$, if $(v_0^{(t)}, v_1^{(t)})$ is defined by

$$(v_0^{(t)}, v_1^{(t)}) := (\tfrac{1}{N(t)} u(t, x(t) + \tfrac{x}{N(t)}),\ \tfrac{1}{N(t)^2} u_t(t, x(t) + \tfrac{x}{N(t)})) \in \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$$

and $v_R^{(t)}$ is the solution to (NLW) with initial data $(\psi_R v_0^{(t)}, \psi_R v_1^{(t)})$ given by Theorem 2.2.4, then $v_R^{(t)}$ is global, satisfies the bound

$$\|(v_R^{(t)}(\tau), \partial_t v_R^{(t)}(\tau))\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})} < \epsilon, \tag{3.1}$$

and for $r \in I - t = \{s - t : s \in I\}$, and $x \in \{x \in \mathbb{R}^d : |x| \geq 2R + rN(t)\}$ we have

$$v^{(t)}(rN(t), x) = v_R^{(t)}(rN(t), x) \tag{3.2}$$

where $v^{(t)}(\tau,x) = \frac{1}{N(t)}u(t+\frac{\tau}{N(t)}, x(t)+\frac{x}{N(t)})$ *is the solution to (NLW) with initial data* $(v_0^{(t)}, v_1^{(t)})$.

Proof. We argue as in [18]. Fix $R>0$ to be determined later in the argument and let $t\in I$ be arbitrary. Our first goal is to obtain the global solution $v_R^{(t)}$ to (NLW) via the local well-posedness result, Theorem 2.2.4.

We begin by showing that there exists a constant $A>0$ (independent of R and t) such that

$$\|(\psi_R v_0^{(t)}, \psi_R v_1^{(t)})\|_{\dot H_x^{s_c}\times \dot H_x^{s_c-1}} \leq A. \tag{3.3}$$

Using Lemma 2.2.1 followed by the Sobolev embedding and Remark 2.0.2, we argue as follows:

$$\begin{aligned}
&\|(\psi_R v_0^{(t)}, \psi_R v_1^{(t)})\|_{\dot H_x^{s_c}\times \dot H_x^{s_c-1}} \\
&\quad\leq \|(\psi_R-1)v_0^{(t)}\|_{\dot H_x^{s_c}} + \|v_0^{(t)}\|_{\dot H_x^{s_c}} + \|(\psi_R-1)v_1^{(t)}\|_{\dot H_x^{s_c-1}} + \|v_1^{(t)}\|_{\dot H_x^{s_c-1}} \\
&\quad\lesssim \||\nabla|^{s_c}(\psi_R-1)\|_{L_x^{\frac{2d}{d-2}}}\|v_0^{(t)}\|_{L_x^d} + \|\psi_R-1\|_{L_x^\infty}\||\nabla|^{s_c}v_0^{(t)}\|_{L_x^2} + \|v_0^{(t)}\|_{\dot H_x^{s_c}} \\
&\qquad + \||\nabla|^{s_c-1}(\psi_R-1)\|_{L_x^{\frac{2d}{d-4}}}\|v_1^{(t)}\|_{L_x^{\frac{d}{2}}} + \|\psi_R-1\|_{L_x^\infty}\||\nabla|^{s_c-1}v_1^{(t)}\|_{L_x^2} \\
&\qquad + \|v_1^{(t)}\|_{\dot H_x^{s_c-1}} \\
&\quad\lesssim \left[\|\frac{1}{R^{s_c}}|\nabla|^{s_c}(\psi-1)(\frac{x}{R})\|_{L_x^{\frac{2d}{d-2}}} + \|\psi_R-1\|_{L_x^\infty}+1\right]\|v_0^{(t)}\|_{\dot H_x^{s_c}} \\
&\qquad + \left[\|\frac{1}{R^{s_c-1}}|\nabla|^{s_c-1}(\psi-1)(\frac{x}{R})\|_{L_x^{\frac{2d}{d-4}}} + \|\psi_R-1\|_{L_x^\infty}+1\right]\|v_1^{(t)}\|_{\dot H_x^{s_c-1}} \\
&\quad= \left[\||\nabla|^{s_c}(\psi-1)\|_{L_x^{\frac{2d}{d-2}}} + \|\psi_R-1\|_{L_x^\infty}+1\right]\|v_0^{(t)}\|_{\dot H_x^{s_c}} \\
&\qquad + \left[\||\nabla|^{s_c-1}(\psi-1)\|_{L_x^{\frac{2d}{d-4}}} + \|\psi_R-1\|_{L_x^\infty}+1\right]\|v_1^{(t)}\|_{\dot H_x^{s_c-1}}
\end{aligned}$$

$$\lesssim \|(v_0^{(t)}, v_1^{(t)})\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}}. \tag{3.4}$$

where in the last inequality we note that by Remark 2.0.1, $\psi - 1 \in C_0^\infty$ gives the finiteness of $\||\nabla|^{s_c}(\psi - 1)\|_{L_x^{\frac{2d}{d-2}}}$ and $\||\nabla|^{s_c-1}(\psi - 1)\|_{L_x^{\frac{2d}{d-4}}}$, with $s_c = 2$ for $d = 6$ and $\frac{2d}{d-2}, \frac{2d}{d-4} \in [2, d)$ for $d \geq 7$.

Hence, by the scaling invariance (1.1),

$$\begin{aligned}
\|(\psi_R v_0^{(t)}, \psi_R v_1^{(t)})\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} &\lesssim \|(v_0^{(t)}, v_1^{(t)})\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \\
&\lesssim \|(u(t), u_t(t))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \\
&\lesssim \|(u, u_t)\|_{L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})},
\end{aligned}$$

and we set $A = C\|(u, u_t)\|_{L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})}$ to get the desired bound.

Let us now choose $\delta_0 > 0$ as in Theorem 2.2.4. We next show that for every $0 < \delta < \delta_0$ we may choose R independent of t such that

$$\|\mathcal{W}(\tau)(\psi_R v_0^{(t)}, \psi_R v_1^{(t)})\|_{L_{\tau,x}^{d+1}} < \delta. \tag{3.5}$$

To do so, using the Strichartz inequality we see that it suffices to prove

$$\|(\psi_R v_0^{(t)}, \psi_R v_1^{(t)})\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} < \frac{\delta}{C}, \tag{3.6}$$

where C is the constant from the Strichartz inequality. Suppose for contradiction that the claim (3.6) failed. We may then choose $\delta_0' > 0$ together with sequences $R_n \to \infty$ and $t_n \in I$ such that for each $n \in \mathbb{N}$

$$\|(\psi_{R_n} v_0^{(t_n)}, \psi_{R_n} v_1^{(t_n)})\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} > \delta_0', \tag{3.7}$$

where $(v_0^{(t_n)}, v_1^{(t_n)})$ is the pair defined in the statement of the theorem. Since u is almost periodic, we may then choose $(f,g) \in \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ such that $(v_0^{(t_n)}, v_1^{(t_n)})$ converges to (f,g) in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$. Moreover, the density of $C_0^\infty \times C_0^\infty$ in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ allows us to to choose $(f_m, g_m) \in C_0^\infty \times C_0^\infty(\mathbb{R}^d)$ with (f_m, g_m) converging to (f,g) in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$.

Thus, invoking (3.4) and using Minkowski's inequality, we obtain

$$\begin{aligned}
&\|(\psi_{R_n} v_0^{(t_n)}, \psi_{R_n} v_1^{(t_n)})\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \\
&\quad \lesssim \|(\psi_{R_n}(v_0^{(t_n)} - f), \psi_{R_n}(v_1^{(t_n)} - g))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \\
&\qquad + \|(\psi_{R_n}(f - f_m), \psi_{R_n}(g - g_m))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \\
&\qquad + \|(\psi_{R_n} f_m, \psi_{R_n} g_m)\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \\
&\quad \lesssim \|(v_0^{(t_n)} - f, v_1^{(t_n)} - g)\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} + \|(f - f_m, g - g_m)\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \\
&\qquad + \|(\psi_{R_n} f_m, \psi_{R_n} g_m)\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}}.
\end{aligned} \tag{3.8}$$

where we note that (3.4) holds for any $(v_0, v_1) \in \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$. As $(f_m, g_m) \in C_0^\infty \times C_0^\infty$ and $\operatorname{supp} \psi_{R_n} \subset \{x : |x| > R_n\}$, we have

$$\psi_{R_n} f_m \equiv \psi_{R_n} g_m \equiv 0.$$

for n sufficiently large. Thus, taking the limit $n \to \infty$ in (3.8) followed by the limit $m \to \infty$ yields

$$\|(\psi_{R_n} v_0^{(t_n)}, \psi_{R_n} v_1^{(t_n)})\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}} \underset{n\to\infty}{\longrightarrow} 0.$$

But this contradicts (3.7), proving that the desired estimate (3.6) holds.

Collecting (3.3) and (3.5), Theorem 2.2.4 now implies that there exists a global solution $v_R^{(t)}$ with the bounds

$$\|v_R^{(t)}\|_{L_{t,x}^{d+1}} \lesssim \epsilon_1, \tag{3.9}$$

$$\||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} v_R^{(t)}\|_{S_{\frac{d+1}{2(d-1)}}(\mathbb{R})} + \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}-1}\partial_t v_R^{(t)}\|_{S_{\frac{d+1}{2(d-1)}}(\mathbb{R})} < \infty. \tag{3.10}$$

Moreover, using the Stricharz inequality and Lemma 2.2.3 followed by the bounds (3.7), (3.9) and (3.10), we obtain

$$\begin{aligned}
&\|(v_R^{(t)}, \partial_t v_R^{(t)})\|_{L_t^\infty(\mathbb{R};\dot{H}_x^{s_c}\times\dot{H}_x^{s_c-1})} \\
&\quad\lesssim \|(\psi_R v_0^{(t)}, \psi_R v_1^{(t)})\|_{\dot{H}_x^{s_c}\times\dot{H}_x^{s_c-1}} + \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}}(v_R^{(t)})^3\|_{N_{\frac{d-3}{2(d-1)}}(\mathbb{R})} \\
&\quad\lesssim \delta + \||\nabla|^{\frac{d^2-4d+1}{2(d-1)}} v_R^{(t)}\|_{S_{\frac{d+1}{2(d-1)}}(\mathbb{R})}\|v_R^{(t)}\|_{L_{t,x}^{d+1}}^2 \\
&\quad\lesssim \delta + \delta^2.
\end{aligned}$$

Thus, choosing δ small enough such that $C(\delta + \delta^2) < \epsilon$ gives the bound (3.1) as desired.

Finally, we now address (3.2). Given $t \in I$ and $r \in I - t \cap [0, \infty)$ we note that

$$v_R^{(t)}(0,x) = v^{(t)}(0,x) \quad \text{and} \quad \partial_t v_R^{(t)}(0,x) = \partial_t v^{(t)}(0,x)$$

on $|x| > 2R$. Then, the finite speed of propagation implies

$$v_R^{(t)}(rN(t),x) = v^{(t)}(rN(t),x)$$

on $|x| > 2R + rN(t)$ as desired. $\square$

3.3 Finite time blow-up solution

In this section, we show that the finite time blow-up solution described in Theorem 3.1.1 cannot exist. Arguing as in [18, 26], we prove that the solution must have zero energy, contradicting the fact that the solution blows up. We note that without loss of generality we may assume $\sup I = 1$.

The first step is to note that the function $N(t)$ tends to infinity as t approaches the blow-up time. In the context of the nonlinear Schrödinger equation this property is given in [24, 27], while for the nonlinear wave equation, see e.g. Lemma 4.14 in [18] and the proof of Theorem 10.1 in [26].

Lemma 3.3.1. *Let $u : I \times \mathbb{R}^d \to \mathbb{R}$ be an almost periodic solution to (NLW) with maximal interval of existence I, $\sup I = 1$. Then there exist $\epsilon > 0$ and $C > 0$ such that for all $t \in (1-\epsilon, 1)$,*

$$N(t) \geq \frac{C}{1-t}.$$

Proof. Suppose for contradiction that the claim failed, and let us choose a sequence $t_n \to 1$ such that for all $n \in \mathbb{N}$, $N(t_n)(1-t_n) < \frac{1}{n}$. For all $n \in \mathbb{N}$, we set

$$(v_{0,n}, v_{1,n}) = \left(\tfrac{1}{N(t_n)} u(t_n, x(t_n) + \tfrac{x}{N(t_n)}), \tfrac{1}{N(t_n)^2} u_t(t_n, x(t_n) + \tfrac{x}{N(t_n)})\right)$$

and let v_n denote the solution to (NLW) with Cauchy data $(v_{0,n}, v_{1,n})$, with maximal interval of existence I_n. Then for all $n \in \mathbb{N}$, the scaling and space translation symmetries imply that we have $\sup I_n = N(t_n)(1-t_n)$.

Note that since u is almost periodic, we may choose $(f,g) \in \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ such that $(v_{0,n}, v_{1,n}) \to (f,g)$ in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ as $n \to \infty$.

Let $\delta_0(d, \|(u,u_t)\|_{L_t^\infty(I;\dot{H}_x^{s_c}\times\dot{H}_x^{s_c-1})}) > 0$ as in Theorem 2.2.4. Then there exists an open interval $0 \in J \subset \mathbb{R}$ small enough so that

$$\|\mathcal{W}(t)(f,g)\|_{L_{t,x}^{d+1}(J\times\mathbb{R}^d)} < \frac{\delta_0}{3}.$$

On the other hand the Strichartz inequality gives

$$\|\mathcal{W}(t)(f,g) - \mathcal{W}(t)(v_{0,n}, v_{1,n})\|_{L_{t,x}^{d+1}(J\times\mathbb{R}^d)} \to 0$$

as $n \to \infty$, so that we may choose N large enough such that for every $n \geq N$, $\|\mathcal{W}(t)(v_{0,n}, v_{1,n})\|_{L_{t,x}^{d+1}(J\times\mathbb{R}^d)} \leq \frac{2\delta_0}{3}$. Thus for all $n \geq N$, Theorem 2.2.4 implies that $J \subset I_n$, and thus $\frac{1}{2}\sup J \in I_n$. However, this contradicts the limit $\sup I_n \to 0$ as $n \to \infty$. Thus, the desired claim holds. $\square$

A second ingredient that is necessary to rule out the finite time blow-up solution is to control its support.

Lemma 3.3.2. *Let $u : I \times \mathbb{R}^d \to \mathbb{R}$ be an almost periodic solution to (NLW) with maximal interval of existence I, $\sup I = 1$ and $(u,u_t) \in L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$.*

Then there exists $y \in \mathbb{R}^d$ such that for each $0 < s < 1$, we have

$$\operatorname{supp} u(s,\cdot), \quad \operatorname{supp} u_t(s,\cdot) \subset \overline{B(y, 1-s)}$$

Proof. We argue as in Lemma 4.8 of [21] and Lemma 4.15 of [18]. Fix $\epsilon > 0$ and $0 < s < 1$. Let $R, v_0^{(t)}, v_1^{(t)}, v_R^{(t)}$ be as stated in Lemma 3.2.1.

We first show

$$\limsup_{t\to 1} \int_{|x-x(t)|\geq \frac{2R}{N(t)}+t-s} |\nabla u(s,x)|^{\frac{d}{2}} + |u_t(s,x)|^{\frac{d}{2}} dx \leq C\epsilon. \tag{3.11}$$

Indeed, for $t \in I$,

$$\begin{aligned}
&\int_{|x-x(t)|\geq \frac{2R}{N(t)}+t-s} |(\nabla u)(s,x)|^{\frac{d}{2}} dx \\
&\quad = \int_{|x|\geq 2R+(t-s)N(t)} \left|(\nabla u)\left(s, x(t) + \tfrac{x}{N(t)}\right)\right|^{\frac{d}{2}} \tfrac{1}{N(t)^d} dx \\
&\quad \leq \int_{\mathbb{R}^d} |\nabla v_R^{(t)}((s-t)N(t), x)|^{\frac{d}{2}} dx \\
&\quad \lesssim \|v_R^{(t)}((s-t)N(t), x)\|_{\dot{H}_x^{s_c}}^{\frac{d}{2}} \\
&\quad \lesssim \epsilon
\end{aligned}$$

where to obtain the last two inequalities, we used Sobolev's inequality combined with Lemma 3.2.1. A similar argument also shows the corresponding inequality with $\nabla u(s,x)$ replaced by $u_s(s,x)$. As $t \in I$ is arbitrary, this proves the desired inequality (3.11).

We next show that there exists $\epsilon' > 0$ and $A > 0$ such that for all $1 - \epsilon' < t < 1$, we have

$$|x(t)| < A. \tag{3.12}$$

To see this, suppose for a contradiction that the claim failed. Then there exists a sequence of times $\{t_n\}$ such that $t_n \in (1 - \frac{1}{n}, 1)$ and $|x(t_n)| > n$ for all

$n \in \mathbb{N}$. Then given $M > 0$, $|x| < M$ implies $|x - x(t_n)| \geq n - M$. Moreover, by Lemma 3.3.1, $N(t_n) \to \infty$ as $t_n \to 1$ which yields $\frac{2R}{N(t_n)} \to 0$ as $n \to \infty$, so that for n large enough, $\frac{2R}{N(t_n)} \leq 1$. Noting that for all $n \in \mathbb{N}$, $t_n \leq 1$, we deduce that for n large enough,

$$\{x : |x| < M\} \subset \{x : |x - x(t_n)| \geq \tfrac{2R}{N(t_n)} + t_n\}.$$

Using this embedding to expand the domain of integration in (3.11), we obtain

$$\int_{|x|<M} |\nabla u(0,x)|^{\frac{d}{2}} + |u_t(0,x)|^{\frac{d}{2}} dx \leq 2C\epsilon.$$

Letting $\epsilon \to 0$ followed by $M \to \infty$, we derive $\int_{\mathbb{R}^d} |\nabla u(0,x)|^{\frac{d}{2}} + |u_t(0,x)|^{\frac{d}{2}} dx = 0$, and hence $u \equiv 0$. This contradicts the fact that u is a blow-up solution, and thus the desired claim (3.12) holds.

With the bound (3.12) in hand, we are now ready to conclude the proof of the lemma. Let us choose a time sequence $t_n \in (1 - \epsilon', 1)$ such that $t_n \to 1$ as $n \to \infty$. Then by (3.12), $|x(t_n)| < A$ for all n, so that we may choose a subsequence (still labeled t_n) such that $x(t_n) \to y$ as $n \to \infty$.

We now claim that for $\eta > 0$ fixed and for n large enough (depending on η),

$$\{x : |x - y| \geq 1 - s + \eta\} \subset \{x : |x - x(t_n)| \geq \tfrac{2R}{N(t_n)} + t_n - s\}. \tag{3.13}$$

To observe this inclusion, by the convergence of $x(t_n)$ let us choose $N_0 \in \mathbb{N}$ such that for all $n > N_0$, $|x(t_n) - y| < \frac{\eta}{2}$. Then for $n > N_0$ and $|x - y| \geq 1 - s + \eta$, we have

$$|x - x(t_n)| \geq 1 - s + \tfrac{\eta}{2}. \tag{3.14}$$

Moreover, by Lemma 3.3.1 $N(t_n) \to \infty$ as $t_n \to 1$, so that we may choose $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$\tfrac{2R}{N(t_n)} < \tfrac{\eta}{2}. \tag{3.15}$$

Putting together (3.14) and (3.15) and recalling $t_n < 1$, we obtain that for $n > \max\{N_0, N_1\}$,

$$|x - x(t_n)| \geq \tfrac{2R}{N(t_n)} + t_n - s.$$

Returning back to (3.11) and invoking (3.13) followed by letting $n \to \infty$, we get

$$\int_{|x-y| \geq 1-s+\eta} |\nabla u(s,x)|^{\frac{d}{2}} + |u_t(s,x)|^{\frac{d}{2}} dx \leq C\epsilon. \tag{3.16}$$

Letting $\eta \to 0$ and using the monotone convergence theorem together with $\epsilon \to 0$, we deduce

$$\int_{|x-y| \geq 1-s} |\nabla u(s,x)|^{\frac{d}{2}} + |u_t(s,x)|^{\frac{d}{2}} dx = 0.$$

This immediately implies $\operatorname{supp} u_t(s) \subset \overline{B(y, 1-s)}$.

To conclude, we note that (3.16) also implies that $u(s)$ is constant on $\{|x-y| > 1-s\}$. Then $u \in L_t^\infty \dot{H}_x^{s_c}$ gives $u \in L_t^\infty L_x^d$ via the Sobolev embedding. This in turn forces $u = 0$ on $\{|x-y| > 1-s\}$, and thus $\operatorname{supp} u \subset \overline{B(y, 1-s)}$ as desired. □

Arguing as in [18], we can now rule out the finite time blow-up solution:

Proposition 3.3.3. *There is no solution $u : I \times \mathbb{R}^d \to \mathbb{R}$ to (NLW) with maximal interval of existence I satisfying the properties of a finite time blow-up solution in the sense of Theorem 3.1.1.*

Proof. Let us suppose for a contradiction that there is such a solution u. By the time-reversal and scaling symmetries we may assume that $\sup I = 1$. Using Lemma 3.3.2 and the space-translation symmetry, we may further assume that $\operatorname{supp} u(t), \operatorname{supp} u_t(t) \subset \overline{B(0, 1-t)}$. Then for all $t \in (0,1)$, we have

$$\begin{aligned}
E(u(t), u_t(t)) &= \int_{|x| \le 1-t} \tfrac{1}{2}|\nabla u(t)|^2 + \tfrac{1}{2}|u_t(t)|^2 + \tfrac{1}{4}|u(t)|^4 dx \\
&\lesssim (1-t)^{d-4}[\|\nabla u(t)\|^2_{L_x^{\frac{d}{2}}(\mathbb{R}^d)} + \|u_t(t)\|^2_{L_x^{\frac{d}{2}}(\mathbb{R}^d)} + \|u(t)\|^4_{L_x^d(\mathbb{R}^d)}] \\
&\lesssim (1-t)^{d-4}[\|u(t)\|^2_{\dot{H}_x^{s_c}} + \|u_t(t)\|^2_{\dot{H}_x^{s_c-1}} + \|u(t)\|^4_{\dot{H}_x^{s_c}}] \\
&\lesssim (1-t)^{d-4}
\end{aligned}$$

where we have used the fact that $u \in L_t^\infty(I; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$.

Letting $t \nearrow 1$ and using the conservation of energy,

$$E(u(0), u_t(0)) = \lim_{t \to 1} E(u(t), u_t(t)) = 0.$$

This implies $u \equiv 0$ which contradicts the assumption that u is a finite time blow-up solution. Thus such a solution cannot exist. ☐

3.4 Additional decay

In this section, we prove that the soliton-like and frequency cascade solutions identified in Theorem 3.1.1 satisfy an additional decay property.

More precisely, for $d \geq 6$ we show that $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon})$ for some $\epsilon = \epsilon(d) > 0$. In particular, we obtain that such solutions belong to $L_t^\infty(\dot{H}_x^1 \times L_x^2)$. Our approach follows that of Killip and Visan in [26, 29, 30].

The main result of this section is the following:

Theorem 3.4.1. *Assume $d \geq 6$ and that $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is an almost periodic solution to (NLW) with $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$ and*

$$\inf_{t \in I} N(t) \geq 1.$$

Then we have

$$(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon}) \tag{3.17}$$

for some $\epsilon = \epsilon(d) > 0$. In particular, $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^1 \times L_x^2)$.

Arguing as in [26, 29, 30], we obtain Theorem 3.4.1 in two steps. The first step is to prove that the solution u belongs to $L_t^\infty L_x^{q_0}$ for all $q_0 \in (\frac{2(d-1)}{d-3}, d]$. The second step is to perform a double Duhamel technique [8, 43] to improve this decay to $(u, u_t) \in L_t^\infty(\dot{H}_x^{s_c - s_0} \times \dot{H}_x^{s_c-1-s_0})$ for some $s_0 = s_0(d, q_0) > 0$. Iterating the second step finitely many times, we obtain Theorem 3.4.1.

More precisely, Theorem 3.4.1 will follow once we establish the following two lemmas:

Lemma 3.4.2. *Suppose $d \geq 6$ and that $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is an almost periodic solution to (NLW) with $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$ and*

$$\inf_{t \in I} N(t) \geq 1. \tag{3.18}$$

Then for every $q_0 \in (\frac{2(d-1)}{d-3}, d]$ *we have* $u \in L_t^\infty L_x^{q_0}$.

Lemma 3.4.3. *Suppose* $d \geq 6$ *and that* $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ *is an almost periodic solution to (NLW) with* $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1})$ *and*

$$\inf_{t \in I} N(t) \geq 1.$$

Moreover, assume that there exists $4 < q_1 < d - 1$ *and* $s \in [1, s_c]$ *such that* $u \in L_t^\infty L_x^{q_1}$ *and* $|\nabla|^s u \in L_t^\infty L_x^2$. *Then*

$$(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s - s_0} \times \dot{H}_x^{s - 1 - s_0}). \tag{3.19}$$

for some $s_0 = s_0(d, q_1) > 0$.

We will discuss the proofs of Lemma 3.4.2 and Lemma 3.4.3 in detail in the rest of this section; however, with these two lemmas in hand, we immediately complete the proof of the main theorem of this section.

Proof of Theorem 3.4.1. We begin by choosing a suitable exponent to be able to apply Lemma 3.4.2 and Lemma 3.4.3. To this end, we define

$$q(d) := \tfrac{d^2 - d - 2}{2(d-3)}$$

and note that $d \geq 6$ implies $q(d) \in (\frac{2(d-1)}{d-3}, d)$ and $4 < q(d) < d - 1$.

Fix $s_0 = s_0(d, q(d))$ as in Lemma 3.4.3. By induction, we now prove that for each $k \in \mathbb{N}$ with $s_c - (k-1)s_0 \geq 1$, we have $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c - ks_0} \times \dot{H}_x^{s_c - 1 - ks_0})$. We first note that for $k = 0$ the result follows from the hypothesis

$(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$. For the induction step, we assume that the result holds for some $k-1 \in \mathbb{N}$ with $s_c - (k-2)s_0 \geq 1$. We then have $u \in L_t^\infty \dot{H}_x^{s_c-(k-1)s_0}$, so that if k also satisfies $s_c - (k-1)s_0 \geq 1$, then an immediate application of Lemma 3.4.3 gives

$$(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c - ks_0} \times \dot{H}_x^{s_c-1-ks_0})$$

establishing the induction step.

Note that taking $k \in \mathbb{N}$ as the largest integer such that $s_c - (k-1)s_0 \geq 1$ we obtain the desired result (3.17) with $\epsilon = 1 - (s_c - ks_0)$. $\square$

We now turn our attention to the proofs of Lemma 3.4.2 and Lemma 3.4.3. The rest of this section is devoted to proving these two lemmas. We start with,

3.4.1 Proof of Lemma 3.4.2

Let $\eta > 0$ be a small constant to be chosen later. Assume u is a solution to (NLW) as stated in Lemma 3.4.2. Then almost periodicity together with the condition (3.18) imply that we may find a dyadic number N_0 such that

$$\| |\nabla|^{s_c} u_{\leq N_0} \|_{L_t^\infty L_x^2} \leq \eta. \tag{3.20}$$

Let us now fix $R \in (\frac{2(d-1)}{d-3}, \min\{\frac{2d}{d-4}, \frac{3d}{d-1}\})$ and define

$$\mathcal{S}(N) = N^{\frac{d}{R}-1} \|u_N\|_{L_t^\infty L_x^R}$$

for each dyadic number $N \in \{2^n : n \in \mathbb{Z}\}$.

To prove Lemma 3.4.2, it is enough to show $\|u_N\|_{L_t^\infty L_x^R} \lesssim N^\gamma$ for some $\gamma > 0$ and N sufficiently small depending on u, d and R (see the argument at the end of this section). This bound will follow from the following decay estimate, which uses a Gronwall type inequality as stated in [26].

Lemma 3.4.4 (Decay estimate)**.** *For all dyadic numbers $N \leq 8N_0$, we have*

$$\mathcal{S}(N) \lesssim (\tfrac{N}{N_0})^{d-\frac{d}{R}-3} + \eta \sum_{N_1=\frac{2N}{8}}^{N_0} \left[\left(\tfrac{N}{N_1}\right)^{d-\frac{d}{R}-3} \mathcal{S}(N_1) \right] + \eta \sum_{N_1 \leq \frac{N}{8}} \left[\left(\tfrac{N_1}{N}\right)^{\frac{d}{R}-\frac{d}{2}+2} \mathcal{S}(N_1) \right]. \tag{3.21}$$

In particular,

$$\mathcal{S}(N) \lesssim N^{\frac{d-4}{2}} \tag{3.22}$$

for every $N \leq 8N_0$

Proof. We argue as in [29, 30]. Let $N \leq 8N_0$. We first observe that by Bernstein's inequality together with the Sobolev embedding and $u \in L_t^\infty \dot{H}_x^{s_c}$,

$$\mathcal{S}(N) \lesssim N^{\frac{d}{2}-1} \|u_N\|_{L_t^\infty L_x^2} \lesssim \||\nabla|^{s_c} u_N\|_{L_t^\infty L_x^2} < \infty,$$

We now turn our attention to (3.21). We first note that using the time translation symmetry, it suffices to prove the result when $t = 0$. Then, by using the Duhamel formula (1.6) combined with Minkowski's inequality, we obtain

$$N^{\frac{d}{R}-1} \|u_N(0)\|_{L_x^R}$$

$$\lesssim N^{\frac{d}{R}-1}\Big(\int_0^{N^{-1}} \|\tfrac{\sin(-t'|\nabla|)}{|\nabla|} P_N F(u(t'))\|_{L_x^R} dt' + \int_{N^{-1}}^{\infty} \|\tfrac{\sin(-t'|\nabla|)}{|\nabla|} P_N F(u(t'))\|_{L_x^R} dt'\Big). \tag{3.23}$$

We then use Bernstein's inequality on the first term and the dispersive inequality (2.2) on the second term to obtain

$$\begin{aligned}
(3.23) &\lesssim N^{\frac{d}{R}-1}\Big(\int_0^{N^{-1}} N^{\frac{d}{2}-\frac{d}{R}} \|\tfrac{\sin(-t'|\nabla|)}{|\nabla|} P_N F(u(t'))\|_{L_x^2} dt' \\
&\qquad + \int_{N^{-1}}^{\infty} |t'|^{-(d-1)(\frac{1}{2}-\frac{1}{R})} \||\nabla|^{\frac{d-1}{2}-\frac{d+1}{R}} P_N F(u(t'))\|_{L_x^{R'}} dt'\Big) \\
&\lesssim N^{\frac{d}{R}-1}\Big(\int_0^{N^{-1}} N^{\frac{d}{2}-\frac{d}{R}} \||\nabla|^{-1} P_N F(u(t'))\|_{L_x^2} dt' \\
&\qquad + \int_{N^{-1}}^{\infty} |t'|^{-(d-1)(\frac{1}{2}-\frac{1}{R})} \||\nabla|^{\frac{d-1}{2}-\frac{d+1}{R}} P_N F(u(t'))\|_{L_x^{R'}} dt'\Big) \\
&\lesssim N^{\frac{d}{2}-3} \|P_N F(u)\|_{L_t^\infty L_x^2} + N^{d-\frac{d}{R}-3} \|P_N F(u)\|_{L_t^\infty L_x^{R'}} \\
&\lesssim N^{d-\frac{d}{R}-3} \|P_N F(u)\|_{L_t^\infty L_x^{R'}}
\end{aligned} \tag{3.24}$$

where in passing from the the first line to the third we use (2.2) once more and in passing from the fourth line to the fifth line, we used the fact that $(d-1)(\frac{1}{2}-\frac{1}{R}) > 1$ to observe the finiteness of the integral.

Collecting (3.23) and (3.24), we obtain

$$N^{\frac{d}{R}-1} \|u_N(0)\|_{L_x^R} \lesssim N^{d-\frac{d}{R}-3} \|P_N F(u)\|_{L_t^\infty L_x^{R'}}.$$

Now to establish (3.21), it remains to estimate the term $\|P_N F(u)\|_{L_t^\infty L_x^{R'}}$. We start by decomposing u as

$$u = u_{\leq \frac{N}{8}} + u_{\frac{N}{8} < \cdot \leq N_0} + u_{> N_0}$$

$$=: u_1 + u_2 + u_3.$$

Note that this decomposition gives

$$\begin{aligned}
\|P_N(u^3)\|_{L_t^\infty L_x^{R'}} &= \|P_N\Big(\sum_{i=1}^{3} u_i\Big)^3\|_{L_t^\infty L_x^{R'}} \\
&= \|\sum_{i,j,k=1}^{3} P_N(u_i u_j u_k)\|_{L_t^\infty L_x^{R'}} \\
&\lesssim \|P_N(u_1^3)\|_{L_t^\infty L_x^{R'}} + \|P_N(u_2^3)\|_{L_t^\infty L_x^{R'}} \\
&\quad + \sum_{i,j=1}^{3} \|P_N(u_3 u_i u_j)\|_{L_t^\infty L_x^{R'}} + \sum_{i=1}^{2} \|P_N(u_2 u_1 u_i)\|_{L_t^\infty L_x^{R'}},
\end{aligned}$$

where we have grouped some terms.

Using this inequality combined with the boundedness of P_N, we obtain

$$\begin{aligned}
N^{\frac{d}{R}-1}\|u_N(0)\|_{L_x^R} &\lesssim N^{d-\frac{d}{R}-3}\|P_N F(u)\|_{L_t^\infty L_x^{R'}} \\
&\leq N^{d-\frac{d}{R}-3}\Big(\|P_N u_1^3\|_{L_t^\infty L_x^{R'}} + \|u_2^3\|_{L_t^\infty L_x^{R'}} \\
&\quad + \sum_{i,j=1}^{3} \|u_3 u_i u_j\|_{L_t^\infty L_x^{R'}} + \sum_{i=1}^{2} \|u_1 u_2 u_i\|_{L_t^\infty L_x^{R'}}\Big) \\
&= N^{d-\frac{d}{R}-3}\Big((I) + (II) + (III)_{i,j} + (IV)_i\Big) \qquad (3.25)
\end{aligned}$$

We now estimate each of the above terms (I), (II), $(III)_{i,j}$ and $(IV)_i$ separately.

Term (I): By the support of the Fourier transform of $u_{\leq \frac{N}{8}}(t)^3$, we have

$$P_N[u_{\leq \frac{N}{8}}(t)^3] \equiv 0, \qquad (3.26)$$

so that $(I) = 0$.

Term (II): Using Hölder's inequality, the Sobolev embedding, and the boundedness of $P_{>\frac{N}{8}}$ together with Bernstein's inequality, we obtain

$$\|u_2^3\|_{L_t^\infty L_x^{R'}} \leq \|u_2\|_{L_t^\infty L_x^d} \|u_2\|^2_{L_t^\infty L_x^{\frac{2Rd}{Rd-d-R}}}$$

$$\lesssim \||\nabla|^{s_c} u_{\leq N_0}\|_{L_t^\infty L_x^2} \Bigg[\sum_{\frac{2N}{8} \leq N_1 \leq N_2 \leq N_0} \|u_{N_1}\|_{L_t^\infty L_x^{\frac{2Rd}{Rd-d-R}}} \|u_{N_2}\|_{L_t^\infty L_x^{\frac{2Rd}{Rd-d-R}}} \Bigg]$$

$$\lesssim \||\nabla|^{s_c} u_{\leq N_0}\|_{L_t^\infty L_x^2}$$

$$\Bigg[\sum_{\frac{2N}{8} \leq N_1 \leq N_2 \leq N_0} N_1^{\frac{3d-Rd+R}{2R}} \|u_{N_1}\|_{L_t^\infty L_x^R} N_2^{-\frac{Rd-d-3R}{2R}} \||\nabla|^{\frac{Rd-d-3R}{2R}} u_{N_2}\|_{L_t^\infty L_x^{\frac{2Rd}{Rd-d-R}}} \Bigg]$$

$$\lesssim \||\nabla|^{s_c} u_{\leq N_0}\|_{L_t^\infty L_x^2}$$

$$\Bigg[\sum_{N_1=\frac{2N}{8}}^{N_0} \Bigg\{ N_1^{\frac{3d-Rd+R}{2R}} \|u_{N_1}\|_{L_t^\infty L_x^R} \Bigg(\sum_{N_2=N_1}^{N_0} N_2^{-\frac{Rd-d-3R}{2R}} \||\nabla|^{s_c} u_{N_2}\|_{L_t^\infty L_x^2} \Bigg) \Bigg\} \Bigg]$$

$$\lesssim \||\nabla|^{s_c} u_{\leq N_0}\|_{L_t^\infty L_x^2}$$

$$\Bigg[\sum_{N_1=\frac{2N}{8}}^{N_0} \Bigg\{ N_1^{\frac{3d-Rd+R}{2R}} \|u_{N_1}\|_{L_t^\infty L_x^R} \Bigg(N_1^{-\frac{Rd-d-3R}{2R}} \|(u, u_t)\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})} \Bigg) \Bigg\} \Bigg].$$

where to obtain the third inequality we note that $R < \frac{3d}{d-1}$.

Thus, using (3.20) in the last inequality above, we obtain

$$(II) \lesssim \eta \sum_{N_1=\frac{2N}{8}}^{N_0} N_1^{\frac{2d}{R}-d+2} \|u_{N_1}\|_{L_t^\infty L_x^R}$$

$$= \eta \sum_{N_1=\frac{2N}{8}}^{N_0} N_1^{\frac{d}{R}-d+3} \mathcal{S}(N_1). \qquad (3.27)$$

Term $(III)_{i,j}$: Fix $i, j \in \{1,2,3\}$. Using Hölder's inequality followed by the Bernstein and Sobolev inequalities, we get

$$\begin{aligned}
\|u_{>N_0} u_i u_j\|_{L_t^\infty L_x^{R'}} &\leq \|u_{>N_0}\|_{L_t^\infty L_x^{\frac{dR'}{d-2R'}}} \|u_i\|_{L_t^\infty L_x^d} \|u_j\|_{L_t^\infty L_x^d} \\
&\lesssim N_0^{3-\frac{d}{R'}} \||\nabla|^{\frac{d}{R'}-3} u_{>N_0}\|_{L_t^\infty L_x^{\frac{dR'}{d-2R'}}} \|u\|_{L_t^\infty L_x^d}^2 \\
&\lesssim N_0^{3-\frac{d}{R'}} \||\nabla|^{s_c} u_{>N_0}\|_{L_t^\infty L_x^2} \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2}^2 \\
&\lesssim N_0^{3+\frac{d}{R}-d} \qquad (3.28)
\end{aligned}$$

where in passing from the second line to the third line, we use $R < \frac{2d}{d-4}$, and in the last inequality we observed that

$$\||\nabla|^{s_c} u_{>N_0}\|_{L_t^\infty L_x^2} \leq \|(u, u_t)\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c - 1})}.$$

Term $(IV)_i$: Fix $i \in \{1,2\}$. By Hölder's inequality, together with the Sobolev and Bernstein inequalities, we have

$$\begin{aligned}
&\|u_{\frac{N}{8} < \cdot \leq N_0} u_{\leq \frac{N}{8}} u_i\|_{L_t^\infty L_x^{R'}} \\
&\quad\leq \|u_{\frac{N}{8} < \cdot \leq N_0}\|_{L_t^\infty L_x^2} \|u_{\leq \frac{N}{8}}\|_{L_t^\infty L_x^{\frac{2Rd}{(d-2)R-2d}}} \|u_i\|_{L_t^\infty L_x^d} \\
&\quad\lesssim \|P_{>\frac{N}{8}} P_{\leq N_0} u\|_{L_t^\infty L_x^2} \|u_{\leq \frac{N}{8}}\|_{L_t^\infty L_x^{\frac{2Rd}{(d-2)R-2d}}} \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2} \\
&\quad\lesssim \left(\tfrac{N}{8}\right)^{-s_c} \||\nabla|^{s_c} u_{\frac{N}{8} < \cdot \leq N_0}\|_{L_t^\infty L_x^2} \sum_{N_1 \leq \frac{N}{8}} \|u_{N_1}\|_{L_t^\infty L_x^{\frac{2Rd}{(d-2)R-2d}}} \\
&\quad\lesssim N^{1-\frac{d}{2}} \eta \sum_{N_1 \leq \frac{N}{8}} \|u_{N_1}\|_{L_t^\infty L_x^{\frac{2Rd}{(d-2)R-2d}}} \qquad (3.29) \\
&\quad\lesssim N^{1-\frac{d}{2}} \eta \sum_{N_1 \leq \frac{N}{8}} N_1^{\frac{d}{R} - \frac{(d-2)R-2d}{2R}} \|u_{N_1}\|_{L_t^\infty L_x^R} \qquad (3.30)
\end{aligned}$$

$$\lesssim N^{1-\frac{d}{2}}\eta \sum_{N_1 \le \frac{N}{8}} N_1^{\frac{2d}{R}-\frac{d}{2}+1} N_1^{1-\frac{d}{R}} \mathcal{S}(N_1)$$

$$= N^{\frac{d}{R}-d+3}\eta \sum_{N_1 \le \frac{N}{8}} \left(\tfrac{N_1}{N}\right)^{\frac{d}{R}-\frac{d}{2}+2}. \tag{3.31}$$

where to obtain (3.29) we note that $N \le 8N_0$ and to obtain (3.30) we used $R < \frac{3d}{d-1}$.

Collecting the estimates (3.25), (3.26), (3.27), (3.28) and (3.31), we obtain the desired inequality (3.21).

To obtain (3.22), we invoke Lemma 3.7.1 from Section 3.7. This is a version of Gronwall's inequality which we recall from [26]. In particular, we define $x_k = \mathcal{S}(2^{-k}N_0)$, $k \in \mathbb{N}$ and note that (3.21) combined with Lemma 3.7.1 gives the bound

$$x_k \lesssim 2^{-k\rho} \tag{3.32}$$

for each $\rho \in (0, d - \frac{d}{R} - 3)$. For the details in obtaining the bound (3.32) we refer the reader to Section 3.7. Thus, for each $N = 2^{-k}N_0 \le 8N_0$ we obtain

$$\mathcal{S}(N) = \mathcal{S}(2^{-k}N_0) \lesssim (2^{-k})^{\rho} \sim N^{\rho}.$$

Taking $\rho = \frac{d-4}{2}$ gives the desired bound (3.22). □

With this lemma in hand, we are now ready to prove Lemma 3.4.2:

Proof of Lemma 3.4.2. Recalling the definition of $\mathcal{S}(N)$, (3.22) shows that for all $N \le 8N_0$,

$$\|u_N\|_{L_t^\infty L_x^R} \lesssim N^{\frac{d}{2}-\frac{d}{R}-1}. \tag{3.33}$$

Then, using (3.33) along with the Bernstein inequalities, we obtain

$$\begin{aligned}\|u\|_{L_t^\infty L_x^R} &\leq \|u_{\leq N_0}\|_{L_t^\infty L_x^R} + \|u_{>N_0}\|_{L_t^\infty L_x^R} \\ &\lesssim \sum_{N\leq N_0} \|u_N\|_{L_t^\infty L_x^R} + \sum_{N>N_0} N^{\frac{d}{2}-\frac{d}{R}} \|u_N\|_{L_t^\infty L_x^2} \\ &\lesssim \sum_{N\leq N_0} N^{\frac{d}{2}-\frac{d}{R}-1} + \sum_{N>N_0} N^{1-\frac{d}{R}} \||\nabla|^{s_c} u\|_{L_t^\infty L_x^2} \\ &\lesssim 1,\end{aligned}$$

where we note that our hypotheses on d and R ensure that $\frac{d}{2} - \frac{d}{R} - 1 > 0$ and $1 - \frac{d}{R} < 0$. Since R is arbitrary, we obtain the lemma for every $q_0 \in (\frac{2(d-1)}{d-3}, \min\{\frac{2d}{d-4}, \frac{3d}{d-1}\})$.

We note that the lemma then follows for every $q_0 \in (\frac{2(d-1)}{d-3}, d]$ by using interpolation with the $L_t^\infty L_x^d$ bound which results from combining the a priori bound $u \in L_t^\infty \dot{H}_x^{s_c}$ with the Sobolev embedding. $\square$

3.4.2 Proof of Lemma 3.4.3

Let u, q_1 and s be given as stated in the lemma and choose $s_0 \in (0, \frac{2(d-q_1)}{q_1})$. Applying the Bernstein inequalities, we argue as follows:

$$\begin{aligned}&\||\nabla|^{s-s_0} u\|_{L_t^\infty L_x^2} + \||\nabla|^{s-1-s_0} u_t\|_{L_t^\infty L_x^2} \\ &\quad\leq \sum_{N\leq 1} \||\nabla|^{s-s_0} u_N\|_{L_t^\infty L_x^2} + \||\nabla|^{s-1-s_0} \partial_t u_N\|_{L_t^\infty L_x^2} \\ &\qquad + \sum_{N>1} \||\nabla|^{s-s_0} u_N\|_{L_t^\infty L_x^2} + \||\nabla|^{s-1-s_0} \partial_t u_N\|_{L_t^\infty L_x^2} \\ &\quad\lesssim \sum_{N\leq 1} N^{-s_0} \Big[\||\nabla|^{s} u_N\|_{L_t^\infty L_x^2} + \||\nabla|^{s-1} \partial_t u_N\|_{L_t^\infty L_x^2} \Big]\end{aligned}$$

$$
\begin{aligned}
&+\sum_{N>1} N^{s-s_0-s_c}\Big[\||\nabla|^{s_c}u_N\|_{L_t^\infty L_x^2}+\||\nabla|^{s_c-1}\partial_t u_N\|_{L_t^\infty L_x^2}\Big]\\
\lesssim&\sum_{N\le 1} N^{-s_0}\Big[\||\nabla|^{s}u_N\|_{L_t^\infty L_x^2}+\||\nabla|^{s-1}\partial_t u_N\|_{L_t^\infty L_x^2}\Big]+\sum_{N>1} N^{s-s_0-s_c}\\
\lesssim&\sum_{N\le 1} N^{-s_0}\Big[\||\nabla|^{s}u_N\|_{L_t^\infty L_x^2}+\||\nabla|^{s-1}\partial_t u_N\|_{L_t^\infty L_x^2}\Big]+1 \qquad (3.34)
\end{aligned}
$$

where we note $\|(u,u_t)\|_{L_t^\infty(\dot H_x^{s_c}\times \dot H_x^{s_c-1})}\le C$ to obtain the third inequality followed by $\sum_{N>1} N^{s-s_0-s_c}<\infty$ for $s-s_0-s_c<0$ to obtain the fourth inequality.

To obtain (3.19), it thus remains to estimate the term $\||\nabla|^s u_N\|_{L_t^\infty L_x^2}+\||\nabla|^{s-1}\partial_t u_N\|_{L_t^\infty L_x^2}$ in (3.34). We begin by noting that the unitary property of the linear propagator $\mathcal{W}(\cdot)$ implies that for every $t_1,t_2\in\mathbb{R}$ and $g,h\in L^2$,

$$
\begin{aligned}
\langle|\nabla|\tfrac{\sin(t_1|\nabla|)}{|\nabla|}g,-|\nabla|\tfrac{\sin(t_2|\nabla|)}{|\nabla|}h\rangle+\langle\cos(t_1|\nabla|)g,-\cos(t_2|\nabla|)h\rangle&\\
=\langle g,-\cos((t_1-t_2)|\nabla|)h\rangle,&
\end{aligned}
$$

Next, without loss of generality we take $t=0$, and note that by using the above observation and Lemma 1.2.3 we write

$$
\begin{aligned}
&\||\nabla|^s u_N(0)\|_{L_x^2}^2+\||\nabla|^{s-1}\partial_t u_N(0)\|_{L_x^2}^2\\
&\quad=\lim_{T\to\infty}\lim_{T'\to-\infty}\langle|\nabla|\int_0^T \tfrac{\sin(-t'|\nabla|)}{|\nabla|}P_N|\nabla|^{s-1}F(u(t'))dt',\\
&\qquad\qquad -|\nabla|\int_{T'}^0 \tfrac{\sin(-\tau'|\nabla|)}{|\nabla|}P_N|\nabla|^{s-1}F(u(\tau'))d\tau'\rangle\\
&\qquad+\langle\int_0^T\cos(-t'|\nabla|)P_N|\nabla|^{s-1}F(u(t'))dt',\\
&\qquad\qquad -\int_{T'}^0\cos(-\tau'|\nabla|)P_N|\nabla|^{s-1}F(u(\tau'))d\tau'\rangle
\end{aligned}
$$

$$\leq \int_0^\infty \int_{-\infty}^0 \Big| \langle P_N|\nabla|^{s-1}F(u(t')), -\cos((t'-\tau')|\nabla|)P_N|\nabla|^{s-1}F(u(\tau'))\rangle \Big| d\tau' dt' \tag{3.35}$$

Setting $r = \frac{2q_1}{q_1+4}$ and using Hölder's inequality followed by Proposition 2.1.1 and Bernstein's inequalities, we obtain

$$\begin{aligned}
&\big|\langle P_N|\nabla|^s F(u(t')), \tfrac{\cos((t'-\tau')|\nabla|)}{|\nabla|^2} P_N|\nabla|^s F(u(\tau'))\rangle\big| \\
&\qquad \lesssim \|P_N|\nabla|^s F(u(t'))\|_{L_x^r} \|\tfrac{\cos((t'-\tau')|\nabla|)}{|\nabla|^2} P_N|\nabla|^s F(u(\tau'))\|_{L_x^{r'}} \\
&\qquad \lesssim \frac{1}{|t'-\tau'|^{(d-1)\left(\frac{1}{2}-\frac{1}{r'}\right)}} \|P_N|\nabla|^s F(u(t'))\|_{L_x^r} \||\nabla|^{\frac{d-3}{2}-\frac{d+1}{r'}} P_N|\nabla|^s F(u(\tau'))\|_{L_x^r} \\
&\qquad \lesssim \frac{N^{\frac{d-3}{2}-\frac{d+1}{r'}}}{|t'-\tau'|^{(d-1)\left(\frac{1}{2}-\frac{1}{r'}\right)}} \|P_N|\nabla|^s F(u(t'))\|_{L_t^\infty L_x^r}^2
\end{aligned} \tag{3.36}$$

On the other hand, using the Cauchy-Schwarz inequality followed by Proposition 2.1.1 (with $p = 2$) and Bernstein's inequality, we obtain

$$\begin{aligned}
&\big|\langle P_N|\nabla|^s F(u(t')), \tfrac{\cos((t'-\tau')|\nabla|)}{|\nabla|^2} P_N|\nabla|^s F(u(\tau'))\rangle\big| \\
&\qquad \lesssim \|P_N|\nabla|^s F(u(t'))\|_{L_x^2} \|\tfrac{\cos((t'-\tau')|\nabla|)}{|\nabla|^2} P_N|\nabla|^s F(u(\tau'))\|_{L_x^2} \\
&\qquad \lesssim \|P_N|\nabla|^s F(u)\|_{L_x^2} \||\nabla|^{-2} P_N|\nabla|^s F(u)\|_{L_x^2} \\
&\qquad \lesssim N^{-2} \|P_N|\nabla|^s F(u)\|_{L_x^2}^2 \\
&\qquad \lesssim N^{-2+\frac{2d}{r}-d} \||\nabla|^s F(u)\|_{L_t^\infty L_x^r}^2,
\end{aligned} \tag{3.37}$$

where we recall that $r < \frac{2d}{d+4} < 2$.

Invoking the bounds (3.36) and (3.37) in (3.35) and using Lemma 2.2.1, we obtain

$$\||\nabla|^s u_N(0)\|_{L_x^2}^2 + \||\nabla|^{s-1}\partial_t u_N(0)\|_{L_x^2}^2$$

$$\leq \||\nabla|^s F(u)\|^2_{L^\infty_t L^r_x} \int_0^\infty \int_{-\infty}^0 \min\{\frac{N^{\frac{d-3}{2}-\frac{d+1}{r'}}}{|t'-\tau'|^{(d-1)(\frac{1}{2}-\frac{1}{r'})}}, N^{-2+d-\frac{2d}{r'}}\} dt' d\tau'$$
$$\leq \||\nabla|^s u\|^2_{L^\infty_t L^2_x} \|u\|^4_{L^\infty_t L^{q_1}_x} \int_0^\infty \int_{-\infty}^0 \min\{\frac{N^{\frac{d-3}{2}-\frac{d+1}{r'}}}{|t'-\tau'|^{(d-1)(\frac{1}{2}-\frac{1}{r'})}}, N^{-2+d-\frac{2d}{r'}}\} dt' d\tau'$$
$$= N^{-2+d-\frac{2d}{r'}} \||\nabla|^s u\|^2_{L^\infty_t L^2_x} \|u\|^4_{L^\infty_t L^{q_1}_x} \int_0^\infty \int_{-\infty}^0 \min\{\frac{N^{-(d-1)}}{|t'-\tau'|^{d-1}}, 1\}^{\frac{1}{2}-\frac{1}{r'}} dt' d\tau'$$

We conclude the proof by estimating the above integral. To this end, we use the bound

$$\int_0^\infty \int_{-\infty}^0 \min\{\frac{N^{-(d-1)}}{|t'-\tau'|^{d-1}}, 1\}^{\frac{1}{2}-\frac{1}{r'}} dt' d\tau' \lesssim N^{-2}, \tag{3.38}$$

which follows from the assumption $q_1 < d-1$ and a straightforward computation.

Invoking this bound in (3.34) and using the hypotheses $u \in L^\infty_t L^{q_1}_x$ and $|\nabla|^s u \in L^\infty_t L^2_x$, we get

$$\begin{aligned} &\||\nabla|^{s-s_0} u\|_{L^\infty_t L^2_x} + \||\nabla|^{s-s_0-1} u_t\|_{L^\infty_t L^2_x} \\ &\qquad \lesssim \sum_{N \leq 1} N^{-s_0} N^{-2+\frac{d}{2}-\frac{d}{r'}} + 1 \\ &\qquad = \sum_{N \leq 1} N^{\frac{2d}{q}-2-s_0} + 1 \end{aligned}$$

Note that by our choice of s_0, we have $\frac{2d}{q_1} - 2 - s_0 > 0$, so that the desired bound (3.19) holds. □

3.5 Soliton-like solution

In this section, we rule out the second blow-up scenario identified in Theorem 3.1.1, the soliton-like solution.

As in [26, 29], our approach to obtain the desired contradiction is to get an upper and lower bound on the quantity

$$\int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|} dxdt, \tag{3.39}$$

with a time interval $I \subset \mathbb{R}$. Indeed, the Morawetz estimate (Theorem 2.1.2) and the additional decay property given in Theorem 3.4.1 immediately imply that (3.39) is bounded from above independent of I. The contradiction will then follow once we obtain a lower bound on (3.39) which grows to infinity as $|I| \to \infty$.

We obtain the lower bound in two steps: the first step is to get an estimate on the growth of $x(t)$ via the finite speed of propagation in the form of Lemma 3.2.1. The second step is then to show that the $L^4_{t,x}$ norm of u over unit time intervals and localized in space near $x(t)$ is bounded away from zero.

The key ingredient used to control $x(t)$ in Step 1 is to obtain a bound from below in a suitable space for all times. This requires the additional decay result, Theorem 3.4.1.

Lemma 3.5.1. *Suppose that $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a solution to (NLW) which satisfies the properties of a soliton-like solution stated in Theorem* 3.1.1*. Then there exists $\eta > 0$ such that for all $t \in \mathbb{R}$,*

$$\int_{\mathbb{R}^d} |u(t,x)|^d + |u_t(t,x)|^{\frac{d}{2}} dx \geq \eta.$$

Proof. Suppose to the contrary that the claim failed. Then there exists a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$(u(t_n), u_t(t_n)) \to (0,0) \quad \text{in} \quad L_x^d \times L_x^{\frac{d}{2}} \tag{3.40}$$

as $n \to \infty$. Since u is a soliton-like solution, $\{(u(t_n, x(t_n)+\cdot), u_t(t_n, x(t_n)+\cdot)) : n \in \mathbb{N}\}$ has compact closure in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$.

Note that by the precompactness of $\{u(t_n, x(t_n) + \cdot), u_t(t_n, x(t_n) + \cdot) : n \in \mathbb{N}\}$ there exists a subsequence (still indexed by n) such that $(u(t_n, x(t_n) + \cdot), u_t(t_n, x(t_n)+\cdot)) \to (u_0^*, u_1^*)$ in $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$. However, (3.40) and the change of variable $x \mapsto x(t_n) + x$ imply $(u(t_n, x(t_n) + \cdot), u_t(t_n, x(t_n) + \cdot)) \to (0,0)$ in $L_x^d \times L_x^{\frac{d}{2}}$, so that the continuous embedding $\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1} \hookrightarrow L_x^d \times L_x^{\frac{d}{2}}$ and the uniqueness of limits give $(u_0^*, u_1^*) = (0,0)$. Thus by the change of variable $x \mapsto -x(t_n) + x$, we have

$$(u(t_n), u_t(t_n)) \to (0,0) \quad \text{in} \quad \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}. \tag{3.41}$$

We now note that for all $n \in \mathbb{N}$, if ϵ is as in Theorem 3.4.1, then there exist $\theta_1, \theta_2, \theta_3 \in (0,1)$ such that

$$\begin{aligned}
E(u_0, u_1) &= E(u(t_n), u_t(t_n)) \\
&= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u(t_n)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^d} |u_t(t_n)|^2 dx + \frac{1}{4}\int_{\mathbb{R}^d} |u(t_n)|^4 dx \\
&\lesssim \|u(t_n)\|_{\dot{H}_x^{s_c}}^{\theta_1} \|(u, u_t)\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon})}^{1-\theta_1} \\
&\quad + \|u_t(t_n)\|_{\dot{H}_x^{s_c-1}}^{\theta_2} \|(u, u_t)\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon})}^{1-\theta_2} \\
&\quad + \|u(t_n)\|_{\dot{H}_x^{s_c}}^{4\theta_3} \|(u, u_t)\|_{L_t^\infty(\mathbb{R}; \dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon})}^{4(1-\theta_3)}.
\end{aligned}$$

where in obtaining the inequality we used $\|u(t_n)\|_{L^4_x} \lesssim \|u(t_n)\|_{\dot{H}^{\frac{d}{4}}_x}$ and interpolation.

Letting $n \to \infty$ and applying (3.41) followed by the conservation of energy, we obtain

$$E(u_0, u_1) = 0.$$

Thus $u \equiv 0$, contradicting our assumption that $\|u\|_{L^{d+1}_{t,x}} = \infty$. □

Based on the previous lemma and the finite speed of propagation in the sense of Lemma 3.2.1, we now prove the following estimate for $x(t)$:

Lemma 3.5.2. *Suppose that $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a solution to (NLW) which satisfies the properties of a soliton-like solution stated in Theorem* 3.1.1*. Then there exists $C > 0$ such that for every $t \geq 0$ we have,*

$$|x(t) - x(0)| \leq C + t.$$

Proof. We argue in a similar spirit to [26]. Fix $\eta > 0$ to be determined later in the argument. Let us first note that by Remark 2.1.1 there exists $c(\eta) > 0$ such that

$$\int_{|x-x(t)|>c(\eta)} |u(t,x)|^d dx + \int_{|x-x(t)|>c(\eta)} |u_t(t,x)|^{\frac{d}{2}} dx \leq \eta \tag{3.42}$$

for all $t \in \mathbb{R}$.

Next, applying Lemma 3.2.1 with $\epsilon = \eta$ and $t = 0$, we choose $R > 0$ such that for all $r \in \mathbb{R}$ and $x \in \{x \in \mathbb{R}^d : |x| \geq 2R + r\}$ we have

$$v^{(0)}(r,x) = v^{(0)}_R(r,x)$$

where $v^{(t)}$ and $v_R^{(t)}$ are defined as in Lemma 3.2.1.

Then, for all $t \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\int_{|x-x(0)|>2R+t} |u(t,x)|^d dx + \int_{|x-x(0)|>2R+t} |u_t(t,x)|^{\frac{d}{2}} dx \\
&\quad = \int_{|x|>2R+t} |u(t,x+x(0))|^d dx + \int_{|x|>2R+t} |u_t(t,x+x(0))|^{\frac{d}{2}} dx \\
&\quad = \int_{|x|>2R+t} |v^{(0)}(t,x)|^d dx + \int_{|x|>2R+t} |\partial_t v^{(0)}(t,x)|^{\frac{d}{2}} dx \\
&\quad = \int_{|x|>2R+t} |v_R^{(0)}(t,x)|^d dx + \int_{|x|>2R+t} |\partial_t v_R^{(0)}(t,x)|^{\frac{d}{2}} dx \\
&\quad \le \int_{\mathbb{R}^d} |v_R^{(0)}(t,x)|^d dx + \int_{\mathbb{R}^d} |\partial_t v_R^{(0)}(t,x)|^{\frac{d}{2}} dx \\
&\quad \le \left(\int_{\mathbb{R}^d} ||\nabla|^{s_c} v_R^{(0)}(t,x)|^2 dx\right)^{d/2} + \left(\int_{\mathbb{R}^d} ||\nabla|^{s_c-1} \partial_t v_R^{(0)}(t,x)|^2 dx\right)^{d/4} \\
&\quad \le (C\eta)^d + (C\eta)^{d/2} \\
&\quad \le C,\eta
\end{aligned} \tag{3.43}$$

where in the second to last inequality we used the smallness given by (3.1) in Lemma 3.2.1.

Combining the bounds (3.42) and (3.43), we obtain

$$\begin{aligned}
&\int_{\{x:|x-x(t)|\ge c(\eta)\}\cup\{x:|x-x(0)|\ge 2R+t\}} |u(t,x)|^d + |u_t(t,x)|^{\frac{d}{2}} dx \\
&\quad \le \int_{|x-x(t)|\ge c(\eta)} |u(t,x)|^d + |u_t(t,x)|^{\frac{d}{2}} dx + \int_{|x-x(0)|\ge 2R+t} |u(t,x)|^d + |u_t(t,x)|^{\frac{d}{2}} dx \\
&\quad \le (1+C)\eta.
\end{aligned} \tag{3.44}$$

for all $t \geq 0$. We now determine η. Note that by Lemma 3.5.1 together with the assumption $(u, u_t) \in L_t^\infty(\mathbb{R}; \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1})$, we have

$$0 < \inf_{t \in \mathbb{R}} \left(\|u(t)\|_{L_x^d}^d + \|u_t(t)\|_{L_x^{\frac{d}{2}}}^{\frac{d}{2}} \right) < \infty,$$

so that we may choose $\eta > 0$ such that

$$\eta < \frac{1}{4(1+C)} \inf_{t \in \mathbb{R}} \left(\|u(t)\|_{L_x^d}^d + \|u_t(t)\|_{L_x^{\frac{d}{2}}}^{\frac{d}{2}} \right).$$

Thus invoking this choice of η in (3.44), we have for all $t \geq 0$,

$$\begin{aligned}
&\int_{\{x:|x-x(t)|<c(\eta)\}\cap\{x:|x-x(0)|<2R+t\}} |u(t,x)|^d + |u_t(t,x)|^{\frac{d}{2}} dx \\
&\quad = \int_{\mathbb{R}^d} |u(t,x)|^d + |u_t(t,x)|^{\frac{d}{2}} dx \\
&\qquad - \int_{\{x:|x-x(t)|\geq c(\eta)\}\cup\{x:|x-x(0)|\geq 2R+t\}} |u(t,x)|^d + |u_t(t,x)|^{\frac{d}{2}} dx \\
&\quad \geq \inf_{t \in \mathbb{R}} \left(\|u(t)\|_{L_x^d}^d + \|u_t(t)\|_{L_x^{\frac{d}{2}}}^{\frac{d}{2}} \right) - (1+C)\eta \\
&\quad \geq \left(1 - \frac{1}{4}\right) \inf_{t \in \mathbb{R}} \left(\|u(t)\|_{L_x^d}^d + \|u_t(t)\|_{L_x^{\frac{d}{2}}}^{\frac{d}{2}} \right) \\
&\quad > 0.
\end{aligned}$$

Thus, we conclude that for all $t \geq 0$, the set

$$X(t) = \{x : |x - x(t)| < c(\eta)\} \cap \{x : |x - x(0)| < 2R + t\} \neq \emptyset.$$

We may then choose $x \in X(t)$, $t \geq 0$, so that

$$|x(t) - x(0)| \leq |x(t) - x| + |x - x(0)| \leq c(\eta) + 2R + t.$$

Noting that η and R are independent of t, we conclude that there exists $C > 0$

such that for all $t \geq 0$ we have

$$|x(t) - x(0)| \leq C + t$$

as desired. $\square$

The second step in obtaining the lower bound on (3.39) is the following lemma which employs the almost periodicity as well as the dispersive estimate.

Lemma 3.5.3. *Suppose that $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is a solution to (NLW) which satisfies the properties of a soliton-like solution stated in Theorem* 3.1.1. *Then there exists $R > 0$ and $c > 0$ such that for every $s \in \mathbb{R}$,*

$$\int_s^{s+1} \int_{|x-x(t)|\leq R} |u(t,x)|^4 dx dt \geq c. \tag{3.45}$$

Proof. We argue in a similar manner as in [26]. As a first step, we claim that there exists $C_1 > 0$ such that for every $s \in \mathbb{R}$,

$$\left| \left\{ t \in [s, s+1] : \int_{\mathbb{R}^d} |u(t)|^{\frac{2d}{d-2}} dx \geq C_1 \right\} \right| \geq C_1. \tag{3.46}$$

To this end, suppose to the contrary that the claim failed. Then there exists a sequence of times $\{s_n\} \subset \mathbb{R}$ such that for every $n \in \mathbb{N}$,

$$\left| \left\{ \tau \in [0,1] : \int_{\mathbb{R}^d} |u(s_n + \tau)|^{\frac{2d}{d-2}} dx \geq \frac{1}{n} \right\} \right| < \frac{1}{n}.$$

This in turn implies that the sequence $g_n : [0,1] \to \mathbb{R}$ defined by

$$g_n(\tau) = \int_{\mathbb{R}^d} |u(s_n + \tau)|^{\frac{2d}{d-2}} dx$$

converges to zero in measure as $n \to \infty$. We next extract a subsequence (still labeled s_n) such that

$$\int_{\mathbb{R}^d} |u(s_n+\tau)|^{\frac{2d}{d-2}} dx \to 0 \quad \text{for a.e.} \quad \tau \in [0,1] \quad \text{as} \quad n \to \infty. \tag{3.47}$$

To continue, using the hypothesis that u is a soliton-like solution together with the almost periodicity of u, we choose a further subsequence (still labeled s_n) and a pair $(f,g) \in \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}$ such that

$$(u(s_n, x(s_n)+\cdot), u_t(s_n, x(s_n)+\cdot) \to (f,g) \quad \text{in} \quad \dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}. \tag{3.48}$$

Moreover, using the additional decay property (Theorem 3.4.1) we observe that the sequence $\{(u(s_n, x(s_n)+\cdot), u_t(s_n, x(s_n)+\cdot))\}$ is bounded in $\dot{H}_x^1 \times L_x^2$, and we therefore pass to another subsequence to find $(f', g') \in \dot{H}_x^1 \times L_x^2$ such that

$$(u(s_n, x(s_n)+\cdot), u_t(s_n, x(s_n)+\cdot)) \rightharpoonup (f', g') \quad \text{weakly in} \quad \dot{H}_x^1 \times L_x^2. \tag{3.49}$$

Next, we show that we have $(f'(x), g'(x)) = (0,0)$ for a.e. $x \in \mathbb{R}^d$. To prove this, we begin by noting that it suffices to show

$$\mathcal{W}(\tau)(f', g')(x) = 0, \quad \text{for a.e.} \quad \tau \in [0,1] \quad \text{and a.e.} \quad x \in \mathbb{R}^d. \tag{3.50}$$

Indeed, if we assume (3.50), then in view of $\mathcal{W}(\tau)(f', g') \in C_\tau^0(\dot{H}_x^1) \cap C_\tau^1(L_x^2)$, we obtain

$$\|f'\|_{L_x^{\frac{2d}{d-2}}} \lesssim \|f'\|_{\dot{H}_x^1} = \lim_{\tau \to 0} \|\mathcal{W}(\tau)(f', g')\|_{\dot{H}_x^1} = 0,$$

as well as

$$\|g'\|_{L^2_x} = \lim_{\tau\to 0}\|\partial_\tau \mathcal{W}(\tau)(f',g')\|_{L^2_x} = \lim_{\tau\to 0}\lim_{h\to 0}\|\frac{1}{h}[\mathcal{W}(\tau+h)(f',g') - \mathcal{W}(\tau)(f',g')]\|_{L^2_x} = 0.$$

We now turn to verifying the assertion (3.50). We first note that (3.49) yields $\mathcal{W}(\tau)(u(s_n), x(s_n)+\cdot), u_t(s_n, x(s_n)+\cdot)) \rightharpoonup \mathcal{W}(\tau)(f',g')$ weakly in $L_x^{\frac{2d}{d-2}}$ for every $\tau \in \mathbb{R}$ (for a justification of this claim, we refer to Proposition 3.7.2 in Section 3.7). The weak lower semicontinuity of the norm then yields

$$\|\mathcal{W}(\tau)(f',g')\|_{L_x^{\frac{2d}{d-2}}} \leq \lim_{n\to\infty}\|\mathcal{W}(\tau)(u(s_n), u_t(s_n))\|_{L_x^{\frac{2d}{d-2}}} \tag{3.51}$$

for every $\tau \in \mathbb{R}$.

Fix $\tau \in [0,1]$. Using the Duhamel formula, the dispersive estimate followed by Lemma 2.2.1 twice, and the Sobolev embedding, we obtain for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\|\mathcal{W}(\tau)(u(s_n), u_t(s_n))\|_{L_x^{\frac{2d}{d-2}}} \\
&\quad\leq \|u(s_n+\tau)\|_{L_x^{\frac{2d}{d-2}}} + \int_{s_n}^{s_n+\tau} \|\frac{\sin((s_n+\tau-\tau')|\nabla|)}{|\nabla|}[u(\tau')]^3\|_{L_x^{\frac{2d}{d-2}}}\, d\tau' \\
&\quad\lesssim \|u(s_n+\tau)\|_{L_x^{\frac{2d}{d-2}}} + \int_{s_n}^{s_n+\tau} |s_n+\tau-\tau'|^{-\frac{d-1}{d}} \||\nabla|^{\frac{1}{d}}[u(\tau')^3]\|_{L_x^{\frac{2d}{d+2}}}\, d\tau' \\
&\quad\lesssim \|u(s_n+\tau)\|_{L_x^{\frac{2d}{d-2}}} + \int_{s_n}^{s_n+\tau} |s_n+\tau-\tau'|^{-\frac{d-1}{d}} \|u(\tau')^2\|_{L_x^{\frac{2d^2}{d^2-2}}} \||\nabla|^{\frac{1}{d}} u(\tau')\|_{L_x^{\frac{d^2}{d+1}}}\, d\tau' \\
&\quad\lesssim \|u(s_n+\tau)\|_{L_x^{\frac{2d}{d-2}}} + \int_{s_n}^{s_n+\tau} |s_n+\tau-\tau'|^{-\frac{d-1}{d}} \|u(\tau')\|^2_{L_x^{\frac{4d^2}{d^2-2}}} \|u(\tau')\|_{\dot{H}_x^{s_c}}\, d\tau' \\
&\quad\lesssim \|u(s_n+\tau)\|_{L_x^{\frac{2d}{d-2}}} + \int_{s_n}^{s_n+\tau} |s_n+\tau-\tau'|^{-\frac{d-1}{d}} \|u(\tau')\|^2_{L_x^{\frac{4d^2}{d^2-2}}}\, d\tau'.
\end{aligned} \tag{3.52}$$

We estimate the above integral as follows: Using interpolation, we deduce

$$\int_{s_n}^{s_n+\tau} |s_n+\tau-\tau'|^{-\frac{d-1}{d}} \|u(\tau')\|^2_{L_x^{\frac{4d^2}{d^2-2}}}\, d\tau'$$

$$
\begin{aligned}
&= \int_0^\tau |\tau-\tau'|^{-\frac{d-1}{d}} \|u(s_n+\tau')\|^2_{L_x^{\frac{4d^2}{d^2-2}}}\, d\tau' \\
&\lesssim \int_0^\tau |\tau-\tau'|^{-\frac{d-1}{d}} \|u(s_n+\tau')\|^{2\theta}_{L_x^{\frac{2d}{d-2}}} \|u(s_n+\tau')\|^{2(1-\theta)}_{L_x^d}\, d\tau' \\
&\lesssim \int_0^\tau |\tau-\tau'|^{-\frac{d-1}{d}} \|u(s_n+\tau')\|^{2\theta}_{L_x^{\frac{2d}{d-2}}} \|u\|^{2(1-\theta)}_{L_t^\infty \dot{H}_x^{s_c}}\, d\tau' \\
&\lesssim \int_0^\tau |\tau-\tau'|^{-\frac{d-1}{d}} \|u(s_n+\tau')\|^{2\theta}_{L_x^{\frac{2d}{d-2}}}\, d\tau' \qquad (3.53)
\end{aligned}
$$

for some $\theta \in (0,1)$. Then, by virtue of Theorem 3.4.1 and (3.47), the dominated convergence theorem yields

$$
\int_0^\tau |\tau-\tau'|^{-\frac{d-1}{d}} \|u(s_n+\tau')\|^{2\theta}_{L_x^{\frac{2d}{d-2}}}\, d\tau' \to 0. \qquad (3.54)
$$

Thus appealing to (3.47) once again, together with (3.54), we use (3.52) to obtain

$$
\|\mathcal{W}(s)(u(s_n), u_t(s_n))\|_{L_x^{\frac{2d}{d-2}}} \to 0
$$

which in turn gives the claim (3.50) so that $f'(x) = g'(x) = 0$ a.e. as claimed.

Now, note that by combining (3.48) and (3.49) with the Sobolev embedding and uniqueness of weak limits in L_x^p spaces, we obtain $(f(x), g(x)) = (f'(x), g'(x))$ for a.e. $x \in \mathbb{R}^d$. Thus, using (3.48) with $f(x) = g(x) = 0$ for a.e. $x \in \mathbb{R}^d$, we may choose n so that $\|(u(s_n, x(s_n)+\cdot), u_t(s_n, x(s_n)+\cdot))\|_{\dot{H}_x^{s_c} \times \dot{H}_x^{s_c-1}}$ is arbitrarily small. The local theory then gives $\|u\|_{L_{t,x}^{d+1}} < \infty$, contradicting our hypothesis that u is a blow-up solution. Thus (3.46) holds as desired.

Our second step is to adjust the domain of integration in (3.46). To this end, let C_1 be as in (3.46). Fix $\eta > 0$ to be determined later in the argument and let $s \in \mathbb{R}$ be given. Then, by the almost periodicity of u, we may choose $C_2(\eta) > 0$ such that

$$\|u(t)\|_{L^d_x(|x-x(t)|\geq C_2(\eta))} \leq \eta^{\frac{1}{d}}.$$

Let $\epsilon > 0$ be as in Theorem 3.4.1. Using interpolation followed by the Sobolev embedding, we have

$$\begin{aligned}\|u(t)\|_{L_x^{\frac{2d}{d-2}}(|x-x(t)|\geq C_2(\eta))} &\leq \|u(t)\|^{\gamma}_{L^d_x(|x-x(t)|\geq C_2(\eta))}\|u(t)\|^{1-\gamma}_{L_x^{\frac{2d}{d-2(1-\epsilon)}}(\mathbb{R}^d)} \\ &\leq C\|u(t)\|^{\gamma}_{L^d_x(|x-x(t)|\geq C_2(\eta))}\|u(t)\|^{1-\gamma}_{\dot{H}^{1-\epsilon}_x(\mathbb{R}^d)} \\ &\leq C\eta^{\frac{\gamma}{d}} \end{aligned} \tag{3.55}$$

for some $\gamma \in (0,1)$, where we note that $d \geq 6$ yields $\frac{2d}{d-2(1-\epsilon)} < \frac{2d}{d-2} < d$.

Choose η small enough so that $(C\eta^{\frac{\gamma}{d}})^{\frac{2d}{d-2}} < \frac{C_1}{2}$. Then for all $t \in [s, s+1]$, $\int_{\mathbb{R}^d} |u(t,x)|^{\frac{2d}{d-2}} \geq C_1$ implies

$$\begin{aligned}\int_{|x-x(t)|\leq C_2(\eta)} |u(t,x)|^{\frac{2d}{d-2}}dx &= \int_{\mathbb{R}^d} |u(t,x)|^{\frac{2d}{d-2}}dx - \int_{|x-x(t)|\geq C_2(\eta)} |u(t,x)|^{\frac{2d}{d-2}}dx \\ &\geq \frac{C_1}{2}.\end{aligned}$$

Thus, we obtain from (3.46) that for all $s \in \mathbb{R}$

$$\left|\left\{t \in [s,s+1] : \int_{|x-x(t)|\leq C_2(\eta)} |u(t,x)|^{\frac{2d}{d-2}}dx \geq \frac{C_1}{2}\right\}\right| \geq C_1. \tag{3.56}$$

from which we settle the second step.

To conclude the proof, we use (3.56) to obtain the desired estimate (3.45). Arguing similarly as in (3.55), we obtain

$$\|u(t)\|_{L_x^{\frac{2d}{d-2}}(|x-x(t)|\leq C_2(\eta))} \leq \|u(t)\|^{\theta}_{L^4_x(|x-x(t)|\leq C_2(\eta))}\|u(t)\|^{1-\theta}_{L_x^{\frac{2d}{d-2(1-\epsilon)}}(|x-x(t)|\leq C_2(\eta))}$$

$$
\begin{aligned}
&\leq \|u(t)\|^{\theta}_{L^4_x(|x-x(t)|\leq C_2(\eta))}\|u(t)\|^{1-\theta}_{L_x^{\frac{2d}{d-2(1-\epsilon)}}(\mathbb{R}^d)} \\
&\leq C\|u(t)\|^{\theta}_{L^4_x(|x-x(t)|\leq C_2(\eta))}\|u(t)\|^{1-\theta}_{\dot{H}^{1-\epsilon}_x(\mathbb{R}^d)} \\
&\leq C\|u(t)\|^{\theta}_{L^4_x(|x-x(t)|\leq C_2(\eta))}
\end{aligned}
$$

for some $\theta \in (0,1)$.

Then for all $s \in \mathbb{R}$ we have

$$
\begin{aligned}
\int_s^{s+1}\int_{|x-x(t)|\leq C_2(\eta)} |u(t,x)|^4 dxdt &= \int_s^{s+1} \|u(t)\|^4_{L^4_x(|x-x(t)|\leq C_2(\eta)}dt \\
&\geq \int_s^{s+1} C^{-4/\theta}\|u(t)\|^{4/\theta}_{L_x^{\frac{2d}{d-2}}(|x-x(t)|\leq C_2(\eta))} dt \\
&\geq C_1 \cdot C^{-4/\theta}\left(\frac{C_1}{2}\right)^{\frac{4(d-2)}{2d\theta}}
\end{aligned}
$$

where we used (3.56) to obtain the last inequality. Since C_1, C_2 and C are independent of s, this yields the desired estimate (3.45). □

Having shown the two steps we outlined above, we are now ready to address the proof of the main proposition of this section, which precludes the soliton-like scenario.

Proposition 3.5.4. *Assume $d \geq 6$. Then there is no $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that u solves (NLW) and satisfies the properties of a soliton-like solution in the sense of Theorem* 3.1.1*.*

Proof. We argue as in [26]. Suppose for a contradiction that such a solution u existed. Fix $T > 0$ and choose C as in Lemma 3.5.2 and R, c as in Lemma

3.5.3. We then write,

$$\int_0^T \int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|} dxdt \geq \sum_{i=0}^{\lfloor T \rfloor -1} \int_i^{i+1} \int_{|x-x(t)|\leq R} \frac{|u(t,x)|^4}{|x|} dxdt. \tag{3.57}$$

Note that for all $i \in \{0, \cdots, \lfloor T \rfloor - 1\}$ the conditions $t \in [i, i+1)$ and $x \in \{x \in \mathbb{R}^d : |x - x(t)| \leq R\}$ yield

$$|x| \leq |x - x(t)| + |x(t) - x(0)| + |x(0)| \leq R + C + t + |x(0)| \leq C' + i.$$

Using this bound,

$$\begin{aligned}(3.57) &\geq \sum_{i=0}^{\lfloor T \rfloor -1} \frac{1}{C'+i} \int_i^{i+1} \int_{|x-x(t)|\leq R} |u(t,x)|^4 dxdt \\ &\geq c \sum_{i=0}^{\lfloor T \rfloor -1} \frac{1}{C'+i} \\ &\geq c \int_0^{\lfloor T \rfloor} \frac{1}{C'+t} dt. \end{aligned} \tag{3.58}$$

Combining (3.57) with (3.58) and invoking Theorem 2.1.2, we obtain

$$c \log\big(\frac{C' + \lfloor T \rfloor}{C'}\big) \leq \int_0^T \int_{\mathbb{R}^d} \frac{|u(t,x)|^4}{|x|} dxdt \leq CE(u_0, u_1).$$

Since u is a soliton-like solution, by Theorem 3.4.1 we have $E(u_0, u_1) < \infty$. Noting that $T > 0$ is arbitrary and the constants C, R and c are independent of T, letting T tend to infinity, we derive a contradiction. This completes the proof of the proposition. □

3.6 Low-to-high frequency cascade solution

In this section, we rule out the low-to-high frequency cascade scenario identified in Theorem 3.1.1.

Proposition 3.6.1. *There is no $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that u solves (NLW), and satisfies the properties of a low-to-high frequency cascade solution in the sense of Theorem* 3.1.1.

Proof. We proceed in a similar manner as in [29]. Assume to the contrary that there exists such a solution u. Since u is a low-to-high frequency cascade solution, we may choose a sequence $\{t_n\} \subset \mathbb{R}$ with $t_n \to \infty$ such that $N(t_n) \to \infty$ as $n \to \infty$.

Using (2.7) followed by Hölder's inequality with $u \in L_t^\infty(\dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon})$ for some $\epsilon > 0$ (Theorem 3.4.1) we have, for all $n \in \mathbb{N}$ and $\eta > 0$,

$$
\begin{aligned}
&\int_{|\xi| \le c(\eta) N(t_n)} |\xi|^2 |\hat{u}(t_n,\xi)|^2 + |\hat{u}_t(t_n,\xi)|^2 d\xi \\
&\lesssim \left(\int_{|\xi| \le c(\eta) N(t_n)} |\xi|^{2s_c} |\hat{u}(t_n,\xi)|^2 d\xi \right)^{\frac{\epsilon}{\epsilon + s_c - 1}} \left(\int_{|\xi| \le c(\eta) N(t_n)} |\xi|^{2(1-\epsilon)} |\hat{u}(t_n,\xi)|^2 d\xi \right)^{\frac{s_c-1}{\epsilon + s_c - 1}} \\
&+ \left(\int_{|\xi| \le c(\eta) N(t_n)} |\xi|^{2(s_c-1)} |\hat{u}_t(t_n,\xi)|^2 d\xi \right)^{\frac{\epsilon}{\epsilon + s_c - 1}} \left(\int_{|\xi| \le c(\eta) N(t_n)} |\xi|^{-2\epsilon} |\hat{u}_t(t_n,\xi)|^2 d\xi \right)^{\frac{s_c-1}{\epsilon + s_c - 1}} \\
&\lesssim \eta^{\frac{\epsilon}{\epsilon + s_c - 1}} \|(u,u_t)\|_{L^\infty(\mathbb{R}; \dot{H}_x^{1-\epsilon} \times \dot{H}_x^{-\epsilon})}^{\frac{2(s_c-1)}{\epsilon + s_c - 1}} \\
&\lesssim \eta^{\frac{\epsilon}{\epsilon + s_c - 1}}. \qquad (3.59)
\end{aligned}
$$

On the other hand, by Chebyshev's inequality

$$
\int_{|\xi| \ge c(\eta) N(t_n)} |\xi|^2 |\hat{u}(t_n,\xi)|^2 + |\hat{u}_t(t_n,\xi)|^2 d\xi
$$

$$\leq [c(\eta)N(t)]^{-2(s_c-1)} \int_{\mathbb{R}^d} |\xi|^{2s_c}|\hat{u}(t_n,\xi)|^2 + |\xi|^{2(s_c-1)}|\hat{u}_t(t_n,\xi)|^2 d\xi$$
$$\lesssim [c(\eta)N(t_n)]^{-2(s_c-1)} \|(u,u_t)\|^2_{L^\infty(\mathbb{R};\dot{H}^{s_c}_x \times \dot{H}^{s_c-1}_x)}$$
$$\lesssim [c(\eta)N(t_n)]^{-2(s_c-1)}. \tag{3.60}$$

for all $\eta > 0$ and $n \in \mathbb{N}$.

To continue, we now estimate the nonlinear term in the energy. Note that using Sobolev's inequality followed by interpolation with $u \in L^\infty_t \dot{H}^{s_c}_x$,

$$\|u(t_n)\|_{L^4_x} \lesssim \||\nabla|^{\frac{d}{4}} u(t_n)\|_{L^2_x} \lesssim \|\nabla u(t_n)\|^{\frac{1}{2}}_{L^2_x} \|u\|^{\frac{1}{2}}_{L^\infty_t \dot{H}^{s_c}_x} \lesssim \|\nabla u(t_n)\|^{\frac{1}{2}}_{L^2_x}. \tag{3.61}$$

Combining (3.59), (3.60) and invoking Plancherel's theorem in (3.61), we estimate the energy as

$$\begin{aligned} E(u(t_n), u_t(t_n)) &\lesssim \int_{\mathbb{R}^d} |\xi|^2 |\hat{u}(t_n)|^2 d\xi + \int_{\mathbb{R}^d} |\hat{u}_t(t_n)|^2 d\xi + \left(\int_{\mathbb{R}^d} |\xi|^2 |\hat{u}(t_n)|^2 d\xi \right)^2 \\ &\lesssim \int_{|\xi| \leq c(\eta)N(t_n)} |\xi|^2 |\hat{u}(t_n)|^2 d\xi + \int_{|\xi| \geq c(\eta)N(t_n)} |\xi|^2 \hat{u}(t_n)|^2 d\xi \\ &+ \int_{|\xi| \leq c(\eta)N(t_n)} |\hat{u}_t(t_n)|^2 d\xi + \int_{|\xi| \geq c(\eta)N(t_n)} |\hat{u}_t(t_n)|^2 d\xi \\ &+ \left[\int_{|\xi| \leq c(\eta)N(t_n)} |\xi|^2 |\hat{u}(t_n)|^2 d\xi + \int_{|\xi| \geq c(\eta)N(t_n)} |\xi|^2 \hat{u}(t_n)|^2 d\xi \right]^2 \\ &\lesssim \eta^{\frac{\epsilon}{\epsilon + s_c - 1}} + [c(\eta)N(t_n)]^{-2(s_c-1)} + \eta^{\frac{2\epsilon}{\epsilon + s_c - 1}} + [c(\eta)N(t_n)]^{-4(s_c-1)}, \end{aligned} \tag{3.62}$$

for all $\eta > 0$ and $n \in \mathbb{N}$.

Letting $n \to \infty$ in (3.62) and using the conservation of energy, now $N(t_n) \to \infty$ yields for all $\eta > 0$,

$$E(u(0), u_t(0)) \lesssim \eta^{\frac{\epsilon}{\epsilon + s_c - 1}} + \eta^{\frac{2\epsilon}{\epsilon + s_c - 1}}.$$

Taking $\eta \to 0$, we obtain $E(u(0), u_t(0)) = 0$. Thus $u \equiv 0$ contradicting our assumption that u is a blow-up solution. □

3.7 Some auxiliary results

In this section, we present the detailed proofs of some observations that we used in the discussion above. More precisely,

3.7.1 The bound (3.22)

Here, we present the argument used in obtaining the bound (3.32) from the decay estimate (3.21) in the proof of Lemma 3.4.4. We begin by recalling the following Gronwall inequality from [26].

Lemma 3.7.1. *Let $\gamma, \gamma', C, \eta > 0$ and $\rho \in (0, \gamma)$ be given such that*

$$\eta \leq \frac{1}{4} \min\{1 - 2^{-\gamma}, 1 - 2^{-\gamma'}, 1 - 2^{\rho-\gamma}\}.$$

Then for every bounded sequence $\{x_k\} \subset \mathbb{R}^+$ satisfying

$$x_k \leq C2^{-\gamma k} + \eta \sum_{l=0}^{k-1} 2^{-\gamma(k-l)} x_l + \eta \sum_{l=k}^{\infty} 2^{-\gamma'|k-l|} x_l,$$

we have

$$x_k \leq (4C + \|x\|_{l^\infty}) 2^{-\rho k}.$$

We now turn our attention to the proof of the bound (3.32).

Fix $\gamma = d - \frac{d}{R} - 3$, $\gamma' = \frac{d}{R} - \frac{d}{2} + 2$, $C = 1$ and $\rho \in (0, \gamma)$. Let C' be the constant in the inequality given in (3.21) (note that this constant comes from the combinatorial considerations, as well as the constants in each application of the Sobolev and Bernstein inequalities, and thus may be chosen independent of η and N_0).

We now choose $\eta > 0$ such that

$$\eta' := C'\eta \leq \left(\frac{1}{4}\min\{1-2^{-\gamma}, 1-2^{-\gamma'}, 1-2^{\rho-\gamma}\}\right)^2$$

and

$$\eta' \leq 2^{-4(\gamma+\gamma')}. \tag{3.63}$$

Having chosen η, we may use our hypothesis on u (in the context of the proof of Lemma 3.4.2) to choose $N_0 \in \mathbb{N}$ such that

$$\||\nabla|^{s_c} u_{\leq N_0}\|_{L^\infty L^2} < \eta.$$

For all $k \in \mathbb{N}$, we define $x_k = \mathcal{S}(2^{-k}N_0)$. Then, applying (3.21) for all $k \geq 0$, we have

$$\begin{aligned}
x_k &= \mathcal{S}(2^{-k}N_0) \\
&\leq C'\left(\frac{2^{-k}N_0}{N_0}\right)^{\gamma} + C'\eta\sum_{i=0}^{k+2}\left(\frac{2^{-k}N_0}{2^{-i}N_0}\right)^{\gamma} x_i + C'\eta\sum_{i=k+3}^{\infty}\left(\frac{2^{-i}N_0}{2^{-k}N_0}\right)^{\gamma'} x_i \\
&= C'2^{-k\gamma} + \eta'\sum_{i=0}^{k+2} 2^{(i-k)\gamma}x_i + \eta'\sum_{i=k+3}^{\infty} 2^{(k-i)\gamma}x_i \\
&= C'2^{-k\gamma} + \eta'\sum_{i=0}^{k-1} 2^{-\gamma|k-i|}x_i + \eta' x_k + \eta' 2^{[(k+1)-k]\gamma}x_{k+1} \\
&\quad + \eta' 2^{[(k+2)-k]\gamma}x_{k+2} + \eta'\sum_{i=k+3}^{\infty} 2^{-\gamma'|k-i|}x_i
\end{aligned}$$

$$\leq C'2^{-k\gamma} + (\eta')^{\frac{1}{2}} \sum_{i=0}^{k} 2^{-\gamma|k-i|} x_i + (\eta')^{\frac{1}{2}} x_k + (\eta')^{\frac{1}{2}} 2^{-\gamma'} x_{k+1}$$

$$+ (\eta')^{\frac{1}{2}} 2^{-2\gamma'} x_{k+2} + (\eta')^{\frac{1}{2}} \sum_{i=k+3}^{\infty} 2^{-\gamma'|k-i|} x_i$$

$$\leq C'2^{-k\gamma} + (\eta')^{\frac{1}{2}} \sum_{i=0}^{k-1} 2^{-\gamma|k-i|} x_i + (\eta')^{\frac{1}{2}} \sum_{i=k}^{\infty} 2^{-\gamma'|k-i|} x_i \tag{3.64}$$

where we have used (3.63) and noted that $\eta' < 1$ and $2^{-\gamma|k-k|} = 2^{-\gamma'|k-k|} = 2^0$.

Applying the estimate (3.64) and invoking Lemma 3.7.1, we obtain the bound

$$x_k \lesssim 2^{-k\rho}.$$

Thus, for all $N = 2^{-k}N_0 \leq 8N_0$, we have

$$\mathcal{S}(N) = \mathcal{S}(2^{-k}N_0) \lesssim (2^{-k})^\rho = N^\rho$$

where $\rho \in (0, d - \frac{d}{R} - 3)$. This gives the desired inequality (3.32).

3.7.2 Weak continuity of the wave propagator

We now recall that the wave propagator $\mathcal{W}(t)$ is weakly continuous for all $t \in \mathbb{R}$, which was used to obtain the inequality (3.51) in the proof of Proposition 3.5.3.

Proposition 3.7.2. *Suppose $\{(f_n, g_n)\} \subset \dot{H}^1_x \times L^2_x$ is a sequence such that for some $(f, g) \in \dot{H}^1_x \times L^2_x$, we have*

$$(f_n, g_n) \rightharpoonup (f, g) \quad \textit{weakly in} \quad \dot{H}^1_x \times L^2_x. \tag{3.65}$$

Then for every $\tau \in \mathbb{R}$,

$$\mathcal{W}(\tau)(f_n, g_n) \rightharpoonup \mathcal{W}(\tau)(f, g) \quad \textit{weakly in} \quad L_x^{\frac{2d}{d-2}}$$

Proof. Fix $\tau > 0$ and note that by the Strichartz inequality the operators $A : \dot{H}_x^1 \to L_x^{\frac{2d}{d-2}}$ defined by $Af = \mathcal{W}(\tau)(f, 0)$ and $B : L_x^2 \to L_x^{\frac{2d}{d-2}}$ defined by $Bg = \mathcal{W}(\tau)(0, g)$ are bounded and linear. Thus, they are weakly continuous and the hypothesis (3.65) implies that

$$\mathcal{W}(\tau)(f_n - f, 0) \rightharpoonup 0 \quad \text{and} \quad \mathcal{W}(\tau)(0, g_n - g) \rightharpoonup 0 \tag{3.66}$$

weakly in $L_x^{\frac{2d}{d-2}}$.

Next, by the linearity of the propagator $\mathcal{W}(\tau)$, we have

$$\mathcal{W}(\tau)(f_n, g_n) = \mathcal{W}(\tau)(f_n - f, 0) + \mathcal{W}(\tau)(0, g_n - g) + \mathcal{W}(\tau)(f, g). \tag{3.67}$$

Invoking the weak limits (3.66) in (3.67), we obtain the desired weak convergence. $\square$

Chapter 4

The Radial Defocusing Energy-Supercritical Cubic NLW in Dimension Five

In this chapter, we prove Theorem 1.1.2. In particular, and in analogy with Chapter 3, we give a detailed overview of the proof of the theorem in Section 4.1. As in the previous chapter, the key component of this outline is a reduction to two special blow-up scenarios, which we label the finite time blow-up solution and the infinite time blow-up solution. As mentioned in the introduction, our strategy to complete the proof of the theorem is based on a frequency localized form of the Morawetz estimate. In order to obtain this estimate, we first obtain a frequency localized form of the Strichartz estimate in Section 4.2. Section 4.3 is then devoted to the frequency localized Morawetz estimate. The finite and infinite time blow-up solutions are then ruled out in Sections 4.4 and 4.5, respectively.

4.1 Outline of the proof of Theorem 1.1.2

In this section, we give an outline of our proof of Theorem 1.1.2. As in the previous chapter, we proceed by contradiction following the concentration compactness approach of Kenig and Merle. We first recall that Theorem

1.2.2 shows that the failure of Theorem 1.1.2 gives the existence of a minimal counterexample which belongs to the class of almost periodic solutions.

4.1.1 Existence of a minimal blow-up solution and two blow-up scenarios

We will use the following refinement of Theorem 1.2.2, which shows that the almost periodic solution u and associated function $N(t)$ can be chosen so that $N(t)$ is piecewise constant on $I^+ := I \cap [0, \infty)$ and $N(t) \geq 1$ for all t in this set.

Theorem 4.1.1. *Suppose that Theorem* 1.1.2 *failed. Then there exists a radial solution* $u : I \times \mathbb{R}^5 \to \mathbb{R}$ *to (NLW) with maximal interval of existence* I *such that* u *is almost periodic modulo symmetries,* $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{3/2} \times \dot{H}_x^{1/2})$, $\|u\|_{L^6_{t,x}(I\times\mathbb{R}^5)} = \infty$*, and there exists* $\delta > 0$ *and a family of disjoint intervals* $\{J_k\}_{k\geq 1}$ *with* $I^+ = \cup J_k$,

$$N(t) = N_k \geq 1 \text{ for } t \in J_k, \quad \text{and} \quad |J_k| = \delta N_k^{-1}.$$

Moreover, either

$$|I^+| < \infty \quad \text{or} \quad |I^+| = \infty.$$

This theorem is proved by applying a rescaling argument to the function obtained in Theorem 1.2.2 to find another almost periodic solution with $N(t) \geq 1$ for $t \in I^+$ (see Theorem 7.1 in [21]). One then observes that the function $N(t)$ obeys $N(s) \sim_u N(t)$ for $|s - t| \leq \delta N(t)^{-1}$ and δ suitably chosen, as a consequence of the scaling symmetry and local theory for (NLW).

This property is proved in the NLS setting in [25, Corollary 3.6]; however, the arguments apply equally to (NLW). After a suitable modification of $N(t)$ and $C(t)$, the desired result is obtained.

In Theorem 4.1.1 we divide the solutions of (NLW) into two classes depending on the control granted by the frequency localized Morawetz estimate, Lemma 4.3.3. This is inspired by recent works in the mass and energy critical NLS settings [9, 46]. In the present context, this corresponds to distinguishing the cases $|I^+| < \infty$ and $|I^+| = \infty$; we also note that this distinction is also present in [21].

We next give a quick remark concerning the decay of norms of the Littlewood-Paley projections of u.

Remark 4.1.1. Suppose that u is as in Theorem 4.1.1. The property $\inf_{t\in I^+} N(t) = \inf_k N_k \geq 1$ along with the definition of almost periodicity implies

$$\lim_{N\to 0}[\|u_{\leq N}\|_{L_t^\infty(I^+;\dot{H}_x^{3/2})} + \|\partial_t u_{\leq N}\|_{L_t^\infty(I^+;\dot{H}_x^{1/2})}] = 0.$$

4.1.2 The contradiction

The proof of Theorem 1.1.2 is therefore reduced to the task of showing that solutions satisfying the properties given in Theorem 4.1.1 cannot occur. This is accomplished in Sections 4.4 and 4.5 below, corresponding to the cases $|I^+| < \infty$ and $|I^+| = \infty$, respectively.

To handle the case $|I^+| < \infty$, we show that the solution at time t must be supported in space inside a ball centered at the origin with radius shrinking to 0 as t approaches the blow-up time. This is then shown to be incompatible with the conservation of energy. We remark that this is essentially the same argument as in Section 3.3, adapted to the radial setting under consideration.

On the other hand, the case $|I^+| = \infty$ requires significantly more analysis. For this case, we will observe that, given $\eta > 0$, the frequency localized Morawetz estimate obtained in Section 4.3 implies the bound

$$\int_{I_0}\int_{\mathbb{R}^5} \frac{|u_{\geq N}(t,x)|^4}{|x|} dxdt \lesssim_u \eta(N^{-1} + |I_0|)$$

for N sufficiently small and all $I_0 \subset I^+$ compact. We then obtain a bound from below on the left hand side of this inequality by a multiple of $|I_0|$. Choosing η sufficiently small then gives the desired contradiction.

4.2 Frequency localized Strichartz estimate

We now obtain a frequency localized version of the Strichartz estimates that we will use as a main ingredient in proving the frequency localized Morawetz estimate in Section 4.3. The proof of this result is inspired by analogous results for the mass and energy critical nonlinear Schrödinger equation due to Dodson [9] and Visan [46].

Theorem 4.2.1 (Frequency localized Strichartz estimate.)**.** *Suppose that u is an almost periodic solution to (NLW) with maximal interval of existence I, $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{3/2} \times \dot{H}_x^{1/2})$, and such that there exist disjoint intervals $\{J_k\}_{k\geq 1}$ with $I^+ = \cup J_k$ and for every k, $N(t) = N_k \in [1, \infty)$ on J_k, $|J_k| = \delta N_k^{-1}$.*

Then there exists $C = C(u) > 0$ such that for all dyadic N and compact intervals $I_0 = \cup J_k \subset I^+$ we have

$$\||\nabla|^{3/4} u_{\leq N}\|_{L_t^2(I_0; L_x^4)} \leq C(u)(1 + (N|I_0|)^{1/2}) \tag{4.1}$$

Moreover, for every $\eta > 0$ there exists $N_0 > 0$ such that for $N < N_0$ we have

$$\||\nabla|^{3/4} u_{\leq N}\|_{L_t^2(I_0; L_x^4)} \leq C(u)\eta(1 + (N|I_0|)^{1/2}). \tag{4.2}$$

Before we proceed with the proof of the theorem, we record the following related estimates, derived by interpolating (4.1) and (4.2) with the a priori bound on $L_t^\infty(I; \dot{H}_x^{3/2} \times \dot{H}_x^{1/2})$.

Corollary 4.2.2. *Let u be as in Theorem 4.2.1. Then there exists $C(u) > 0$ such that*

- *for each dyadic $N > 0$ and compact interval $I_0 = \cup J_k \subset I^+$ we have*

$$\|u_{>N}\|_{L_t^3(I_0; L_x^{30/7})} \leq C(u) N^{-1/2}(1 + N|I_0|)^{1/3},$$
$$\|u_{>N}\|_{L_t^4(I_0; L_x^{20/7})} \leq C(u) N^{-1}(1 + N|I_0|)^{1/4},$$

- *and for each $\eta > 0$ there exists $N_0 > 0$ such that for $N < N_0$ we have*

$$\|\nabla u_{\leq N}\|_{L_t^3(I_0; L_x^3)} \leq C(u)\eta(1 + N|I_0|)^{1/3},$$
$$\|\nabla u_{\leq N}\|_{L_t^4(I_0; L_x^{20/7})} \leq C(u)\eta(1 + N|I_0|)^{1/4}.$$

Proof of Theorem 4.2.1. We begin by showing (4.1). Let $I_0 \subset I^+$ be given as stated and observe that the bound (2.6) implies that (4.1) holds with $C_1(u)$

for all

$$N \geq \frac{\int_{I_0} N(t)dt}{|I_0|}.$$

For general dyadic numbers N, we proceed by induction. Fix

$$C(u) > \max\{C_1(u), 1\}$$

to be determined, and suppose that (4.1) holds for all N larger than some N_0. Our goal is to show that (4.1) holds for $N = N_1 := N_0/2$ (with $C(u)$ unchanged). Toward this end, we apply the Strichartz inequality to obtain

$$\begin{aligned} &\| |\nabla|^{3/4} u_{\leq N_1} \|_{L_t^2(I_0; L_x^4)} \\ &\quad \lesssim \inf_{t \in I_0} \| (u_{\leq N_1}(t), \partial_t u_{\leq N_1}(t)) \|_{\dot{H}_x^{3/2} \times \dot{H}_x^{1/2}} + \| |\nabla|^{5/4} P_{\leq N_1}[u^3] \|_{L_t^2(I_0; L_x^{\frac{4}{3}})}. \end{aligned} \tag{4.3}$$

In the remainder of the proof, all space-time norms will be over the set $I_0 \times \mathbb{R}^5$, unless otherwise indicated.

To estimate the nonlinear term in (4.3), we fix $0 < \eta_0 \leq \frac{1}{2}$ (to be determined later in the argument) and use the almost periodicity of u to choose $c_0 = c_0(\eta_0)$ such that

$$\| |\nabla|^{3/2} u_{\leq c_0 N(t)} \|_{L_t^\infty L_x^2} + \| |\nabla|^{1/2} \partial_t u_{\leq c_0 N(t)} \|_{L_t^\infty L_x^2} \leq \eta_0. \tag{4.4}$$

Then, writing

$$u(t) = u_{\leq N_1/\eta_0}(t) + u_{> N_1/\eta_0}(t)$$

and using the identity

$$(u_{> N_1/\eta_0}(t) + u_{\leq N_1/\eta_0}(t))^3$$

$$= u_{>N_1/\eta_0}(t)^3 + 3u_{>N_1/\eta_0}(t)u_{\leq N_1/\eta_0}(t)u(t) + u_{\leq N_1/\eta_0}(t)^3,$$

we obtain

$$\begin{aligned}
\||\nabla|^{3/4} u_{\leq N_1}\|_{L_t^2 L_x^4} \lesssim \inf_{t\in I_0} &\|(u_{\leq N_1}(t), \partial_t u_{\leq N_1}(t))\|_{\dot H_x^{3/2}\times \dot H_x^{1/2}} \\
&+ \||\nabla|^{5/4} P_{\leq N_1}[u^3_{>N_1/\eta_0}]\|_{L_t^2 L_x^{\frac{4}{3}}} && (4.5)\\
&+ \||\nabla|^{5/4} P_{\leq N_1}[u_{>N_1/\eta_0} u_{\leq N_1/\eta_0} u]\|_{L_t^2 L_x^{\frac{4}{3}}} && (4.6)\\
&+ \||\nabla|^{5/4} P_{\leq N_1}[u^3_{\leq N_1/\eta_0}]\|_{L_t^2 L_x^{\frac{4}{3}}}.
\end{aligned}$$

Furthermore, we bound the last term by a multiple of

$$\begin{aligned}
&\Big\||\nabla|^{5/4} P_{\leq N_1}\big[u_{\leq N_1/\eta_0}[P_{\leq} u_{<c_0 N(t)}]^2\big]\Big\|_{L_t^2 L_x^{\frac{4}{3}}} && (4.7)\\
&\quad+ \Big\||\nabla|^{5/4} P_{\leq N_1}\big[u_{\leq N_1/\eta_0}[P_{\leq} u_{\leq c_0 N(t)}][P_{\leq} u_{\geq c_0 N(t)}]\big]\Big\|_{L_t^2 L_x^{\frac{4}{3}}} && (4.8)\\
&\quad+ \Big\||\nabla|^{5/4} P_{\leq N_1}\big[u_{\leq N_1/\eta_0}[P_{\leq} u_{\geq c_0 N(t)}]^2\big]\Big\|_{L_t^2 L_x^{\frac{4}{3}}}, && (4.9)
\end{aligned}$$

where we have set $P_{\leq} = P_{\leq N_1/\eta_0}$ and used the decomposition

$$P_{\leq} u(t) = P_{\leq} u_{\leq c_0 N(t)}(t) + P_{\leq} u_{>c_0 N(t)}(t),$$

and where c_0 is chosen in (4.4).

Thus, it suffices to bound (4.5) through (4.9). Before estimating each of these terms, we will need the following estimate, which is obtained via Hölder's inequality in time and interpolation: for each dyadic $M > 0$,

$$\begin{aligned}
\||\nabla|^{5/4} u_{\leq M}\|_{L_t^2 L_x^{20/7}} &\leq (M|I_0|)^{1/4} \|\nabla u_{\leq M}\|_{L_t^4 L_x^{20/7}} \\
&\lesssim (M|I_0|)^{1/2} + \|\nabla u_{\leq M}\|^2_{L_t^4 L_x^{20/7}}
\end{aligned}$$

$$\lesssim (M|I_0|)^{1/2} + \||\nabla|^{3/4} u_{\le M}\|_{L_t^2 L_x^4} \||\nabla|^{5/4} u_{\le M}\|_{L_t^\infty L_x^{20/9}}$$
$$\lesssim_u (M|I_0|)^{1/2} + \||\nabla|^{3/4} u_{\le M}\|_{L_t^2 L_x^4}. \tag{4.10}$$

where in obtaining the last inequality, we have used the a priori bound $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{3/2} \times \dot{H}_x^{1/2})$.

With this bound in hand, we are now ready to estimate the above terms. For (4.5), we note that an application of Bernstein's inequality gives

$$\begin{aligned}
(4.5) &\lesssim N_1^{5/4} \|u^3_{>N_1/\eta_0}\|_{L_t^2 L_x^{4/3}} \\
&\lesssim N_1^{5/4} \sum_{M > N_1/\eta_0} \|u_M\|_{L_t^2 L_x^{20/7}} \|u_{>N_1/\eta_0}\|^2_{L_t^\infty L_x^5} \\
&\lesssim_u N_1^{5/4} \sum_{M > N_1/\eta_0} M^{-5/4} \||\nabla|^{5/4} u_M\|_{L_t^2 L_x^{20/7}} \\
&\le \eta_0^{3/4} C_2(u) C(u) (N_1|I_0|)^{\frac{1}{2}} + \eta_0^{5/4} C_2(u) C(u),
\end{aligned}$$

where to obtain the last line we have used (4.10) followed by the induction hypothesis. We may use the same argument to estimate (4.6), obtaining

$$(4.6) \le \eta_0^{3/4} C_2(u) C(u) (N_1|I_0|)^{1/2} + \eta_0^{5/4} C_2(u) C(u).$$

On the other hand, to estimate (4.7), we apply the fractional product rule [7, 23] to obtain

$$\begin{aligned}
(4.7) \le& \||\nabla|^{5/4} u_{\le N_1/\eta_0}\|_{L_t^2 L_x^{20/7}} \|[P_\le u_{\le c_0 N(t)}]^2\|_{L_t^\infty L_x^{5/2}} \\
&+ \|u_{\le N_1/\eta_0}\|_{L_t^2 L_x^{10}} \||\nabla|^{5/4} [P_\le u_{\le c_0 N(t)}]^2\|_{L_t^\infty L_x^{20/13}} \\
\lesssim_u& \ \eta_0^2 \left[\||\nabla|^{5/4} u_{\le N_1/\eta_0}\|_{L_t^2 L_x^{20/7}} + \||\nabla|^{3/4} u_{\le N_1/\eta_0}\|_{L_t^2 L_x^4} \right],
\end{aligned}$$

where to obtain the second inequality we have used (4.4) to estimate the first term and the Sobolev embedding, fractional product rule, and (4.4) to estimate the second term. Then, using (4.10) and the induction hypothesis once again, we get

$$(4.7) \leq \eta_0^2 C_3(u)(\eta_0^{-1/2} C(u)(N_1|I_0|)^{1/2} + C(u)).$$

We now turn our attention to the two remaining terms. In what follows, we will use the notation $v(t)$ to refer to either of the functions $P_{\leq N_1/\eta_0} u_{\leq c_0 N(t)}(t)$ and $P_{\leq N_1/\eta_0} u_{>c_0 N(t)}(t)$. In particular, using Bernstein's inequalities combined with the fractional product rule, we obtain the preliminary bound

$$\begin{aligned}
&\| |\nabla|^{5/4} P_{\leq N_1} \big[u_{\leq N_1/\eta_0} v P_{\leq} u_{>c_0 N(t)} \big] \|_{L_t^2(J_k; L_x^{\frac{4}{3}})} \\
&\quad \lesssim N_1^{1/2} \| |\nabla|^{3/4} \big[u_{\leq N_1/\eta_0} v \big] \|_{L_t^\infty L_x^{20/11}} \| P_{\leq} u_{>c_0 N(t)} \|_{L_t^2(J_k; L_x^5)} \\
&\qquad + N_1^{1/2} \| u_{\leq N_1/\eta_0} \|_{L_t^6(J_k; L_x^6)} \| u \|_{L_t^6(J_k; L_x^6)} \| |\nabla|^{3/4} P_{\leq} u_{>c_0 N(t)} \|_{L_t^6(J_k; L_x^{12/5})}. \qquad (4.11)
\end{aligned}$$

We then use the fractional product rule again combined with the a priori bound $(u, u_t) \in L_t^\infty(I; \dot{H}_x^{3/2} \times \dot{H}_x^{1/2})$ to bound the factor

$$\begin{aligned}
\| |\nabla|^{3/4} [u_{\leq N_1/\eta_0} v] \|_{L_t^\infty L_x^{20/11}} &\lesssim \| |\nabla|^{3/4} u_{\leq N_1/\eta_0} \|_{L_t^\infty L_x^{20/7}} \| v \|_{L_t^\infty L_x^5} \\
&\quad + \| u_{\leq N_1/\eta_0} \|_{L_t^\infty L_x^5} \| |\nabla|^{3/4} v \|_{L_t^\infty L_x^{20/7}} \qquad (4.12) \\
&\lesssim_u 1.
\end{aligned}$$

Invoking this bound in (4.11), we obtain

$$\max\{(4.8), (4.9)\} \leq \left(\sum_{J_k \subset I_0} \| |\nabla|^{5/4} P_{\leq N_1} \big[u_{\leq N_1/\eta_0} v P_{\leq} u_{>c_0 N(t)} \big] \|_{L_t^2(J_k; L_x^{4/3})}^2 \right)^{1/2}$$

$$\lesssim_u N_1^{1/2}\left(\sum_{J_K\subset I_0}\left\{\|u_{>c_0N_k}\|^2_{L^2_t(J_k;L^5_x)}+\||\nabla|^{3/4}u_{>c_0N_k}\|^2_{L^6_t(J_k;L^{12/5}_x)}\right\}\right)^{1/2} \tag{4.13}$$

where in the second term of (4.11) we use the bound (2.5) in the form

$$\|u\|_{L^6_t(J_k;L^6_x)}\le C(u)(1+\delta)\lesssim_u 1. \tag{4.14}$$

Moreover, using Bernstein's inequalities and the bounds $\||\nabla|^{1/2}u\|_{L^2_t(J_k;L^5_x)}\lesssim_u 1$ and $\||\nabla|^{5/4}u\|_{L^6_t(J_k;L^{12/5}_x)}\lesssim_u 1$ (these bounds are obtained via an argument identical to that used to prove (2.6)),

$$\begin{aligned}(4.13)&\lesssim_u N_1^{1/2}\left(\sum_{J_k\subset I_0}\frac{1}{c_0N_k}\left\{\||\nabla|^{1/2}u_{>c_0N_k}\|^2_{L^2_t(J_k;L^5_x)}+\||\nabla|^{5/4}u_{>c_0N_k}\|^2_{L^6_t(J_k;L^{12/5}_x)}\right\}\right)^{1/2}\\ &\lesssim_u N_1^{1/2}\left(\sum_{J_k\subset I_0}\frac{1}{c_0N_k}\right)^{1/2}\\ &\le\frac{C_4(u)}{c_0^{1/2}}(N_1|I_0|)^{1/2}.\end{aligned}$$

Combining the estimates of (4.5) through (4.9), we obtain

$$\begin{aligned}&\||\nabla|^{3/4}u_{\le N_1}\|_{L^2_tL^4_x}\\ &\quad\le C_0\inf_{t\in I_0}\|(u_{\le N_1}(t),\partial_tu_{\le N_1}(t))\|_{\dot H^{3/2}_x\times\dot H^{1/2}_x}\\ &\qquad+2\eta_0^{3/4}C_2(u)C(u)((N_1|I_0|)^{1/2}+\eta_0^{1/2})\\ &\qquad+\eta_0^2C_3(u)C(u)(\eta_0^{-1/2}(N_1|I_0|)^{1/2}+1)+\frac{C_4(u)}{c_0^{1/2}}(N_1|I_0|)^{1/2}.\end{aligned} \tag{4.15}$$

We now choose η_0 sufficiently small (depending on $C_2(u)$ and $C_3(u)$) to ensure that

$$\||\nabla|^{3/4}u_{\le N_1}\|_{L^2_tL^4_x}\le C_0\inf_{t\in I_0}\|(u_{\le N_1}(t),\partial_tu_{\le N_1}(t))\|_{\dot H^{3/2}_x\times\dot H^{1/2}_x}+\frac{2C(u)}{3}(N_1|I_0|)^{1/2}$$

$$+\frac{2C(u)}{3}+\frac{C_4(u)}{c_0^{1/2}}(N_1|I_0|)^{1/2}.$$

We now choose $C(u)$ large enough so that

$$C(u)>\max\{\frac{3C_4(u)}{c_0(\eta_0)^{1/2}},3C_0\|(u,u_t)\|_{L_t^\infty(I;\dot H_x^{3/2}\times\dot H_x^{1/2})}\}$$

With such a choice of $C(u)$ we obtain

$$\||\nabla|^{3/4}u_{\le N_1}\|_{L_t^2L_x^4}\le C(u)(1+(N_1|I_0|)^{1/2}), \tag{4.16}$$

completing the induction.

We now turn to (4.2). Let $\eta>0$ be given and fix $N_0=N_0(\eta)>0$ to be determined later in the argument. Let $N\le N_0$ be given and recall that (4.1) is satisfied for all $N>0$. As a consequence, (4.15) is satisfied for any $\eta_0\in(0,\frac{1}{2}]$ with N_1 replaced by N. More precisely, after setting

$$f(N)=\|(u_{\le N},\partial_t u_{\le N})\|_{L_t^\infty(\dot H_x^{3/2}\times\dot H_x^{1/2})}+\sup_{J_k\subset I}\|u_{\le N}\|_{L_t^6(J_k;L_x^6)},$$

we have

$$\begin{aligned}\||\nabla|^{3/4}u_{\le N}\|_{L_t^2L_x^4}\lesssim_u\ & f(N)+\eta_0^{3/4}((N|I_0|)^{1/2}+\eta_0^{1/2})\\ &+\eta_0^2(\eta_0^{-1/2}(N|I_0|)^{1/2}+1)+\frac{f(N/\eta_0)}{c_0^{1/2}}(N|I_0|)^{1/2}\end{aligned} \tag{4.17}$$

for any $\eta_0\in(0,\frac{1}{2}]$, where we have replaced $C_4(u)$ in (4.15) by $f(N/\eta_0)$ in view of (4.11) and (4.12). We next show that $f(N)\to 0$ as $N\to 0$. Indeed, invoking the Strichartz inequality (2.4) and using the decomposition $u=u_{\le N^{1/2}}+u_{>N^{1/2}}$, we obtain

$$f(N)\lesssim\|(u_{\le N},\partial_t u_{\le N})\|_{L_t^\infty(\dot H_x^{3/2}\times\dot H_x^{1/2})}+\sup_{J_k\subset I}\||\nabla|^{5/4}P_{\le N}[u^3]\|_{L_t^2(J_k;L_x^{4/3})}$$

$$\begin{aligned}
&\lesssim \|(u_{\le N}, \partial_t u_{\le N})\|_{L_t^\infty(\dot H_x^{3/2}\times \dot H_x^{1/2})} \\
&\quad + \sup_{J_k\subset I}\Big[\||\nabla|^{5/4}P_{\le N}[u^3_{>N^{1/2}}]\|_{L_t^2(J_k;L_x^{4/3})} \\
&\qquad\qquad + \||\nabla|^{5/4}P_{\le N}[u_{>N^{1/2}}u_{\le N^{1/2}}u]\|_{L_t^2(J_k;L_x^{4/3})} \\
&\qquad\qquad + \||\nabla|^{5/4}P_{\le N}[u^3_{\le N^{1/2}}]\|_{L_t^2(J_k;L_x^{4/3})}\Big] \qquad (4.18)\\
&\lesssim \|(u_{\le N}, \partial_t u_{\le N})\|_{L_t^\infty(\dot H_x^{3/2}\times \dot H_x^{1/2})} \\
&\quad + \sup_{J_k\subset I}\Big[N^{5/4}\|u_{>N^{1/2}}\|_{L_t^6(J_k;L_x^{12/5})}\|u\|^2_{L_t^6(J_k;L_x^6)} \\
&\qquad\qquad + \|u\|^2_{L_t^4(J_k;L_x^{20/3})}\||\nabla|^{5/4}u_{<N^{1/2}}\|_{L_t^\infty L_x^{20/9}}\Big] \qquad (4.19)
\end{aligned}$$

for any $N > 0$, where we have used the Bernstein inequalities followed by the Hölder inequality for the second and third terms of (4.18) and the fractional product rule for the fourth term of (4.18). We then bound the second and third terms in (4.19) by using the Bernstein inequalities followed by (4.14) along with the bounds $\||\nabla|^{5/4}u\|_{L_t^6(J_k;L_x^{12/5})} \lesssim_u 1$ and $\|u\|_{L_t^4(J_k;L_x^{20/3})} \lesssim_u 1$ (as before, these bounds are obtained through an argument identical to that used for (2.6)) to obtain, for $N < 1$,

$$f(N) \lesssim_u \|(u_{\le N^{1/2}}, \partial_t u_{\le N^{1/2}})\|_{L_t^\infty(\dot H_x^{3/2}\times \dot H_x^{1/2})} + N^{5/8},$$

which tends to 0 as $N \to 0$ as a consequence of Remark 4.1.1. With this limit in hand, we choose η_0 small enough to ensure $\eta_0^{3/4} < \eta$ and N_0 small enough to guarantee that $N < N_0$ implies $f(N) < \eta$ and $f(N/\eta_0) < \eta c_0(\eta_0)^{1/2}$. The inequality (4.17) then gives

$$\||\nabla|^{3/4}u_{\le N}\|_{L_t^2 L_x^4} \lesssim_u \eta(1 + (N|I_0|)^{1/2})$$

as desired. ∎

Proof of Corollary 4.2.2. We note that interpolation gives

$$\|u_{>N}\|_{L_t^3 L_x^{30/7}} \lesssim \|u_{>N}\|_{L_t^\infty L_x^5}^{1/3} \|u_{>N}\|_{L_t^2 L_x^4}^{2/3} \tag{4.20}$$

and

$$\|u_{>N}\|_{L_t^4 L_x^{20/7}} \lesssim \||\nabla|^{5/4} u_{>N}\|_{L_t^\infty L_x^{20/9}}^{1/2} \||\nabla|^{-5/4} u_{>N}\|_{L_t^2 L_x^4}^{1/2}. \tag{4.21}$$

The Sobolev inequality followed by the boundedness of the Littlewood-Paley projection then yields

$$\||\nabla|^{5/4} u_{>N}\|_{L_t^\infty L_x^{20/9}} \lesssim \|(u, u_t)\|_{L_t^\infty (\dot{H}_x^{3/2} \times \dot{H}_x^{1/2})}.$$

On the other hand, the Bernstein inequalities along with Lemma 4.2.1 give the bounds

$$\begin{aligned} \|u_{>N}\|_{L_t^2 L_x^4} &\leq \sum_{M>N} M^{-3/4} \||\nabla|^{3/4} u_M\|_{L_t^2 L_x^4} \\ &\lesssim \sum_{M>N} M^{-3/4} \||\nabla|^{3/4} u_{\leq 2M}\|_{L_t^2 L_x^4} \\ &\lesssim_u \sum_{M>N} M^{-3/4} (1 + (M|I_0|)^{1/2}) \\ &\lesssim_u N^{-3/4} (1 + N|I_0|)^{1/2} \end{aligned}$$

and (by an identical argument)

$$\||\nabla|^{-5/4} u_{>N}\|_{L_t^2 L_x^4} \lesssim_u N^{-2} (1 + N|I_0|)^{1/2}.$$

Thus, we obtain

$$(4.20) \lesssim_u N^{-1/2}(1+N|I_0|)^{1/3}, \qquad (4.21) \lesssim_u N^{-1}(1+N|I_0|)^{1/4}$$

as desired.

The bounds on $\|\nabla u_{\leq N}\|_{L^3_{t,x}}$ and $\|\nabla u_{\leq N}\|_{L^4_t L^{20/7}_x}$ are obtained by interpolating (4.2) with the a priori bound $(u, u_t) \in L^\infty_t(\dot{H}^{3/2}_x \times \dot{H}^{1/2}_x)$. □

4.3 Frequency-localized Morawetz estimate

In this section, we obtain a frequency localized Morawetz estimate. The proof of this result is inspired by the recent work of Visan [46] on the energy critical NLS.

We begin by deriving a general form of the classical Morawetz estimate; for the classical form, see [31, 32]. To obtain this, when u is a solution to $u_{tt} - \Delta u + \mathcal{N} = 0$, we set

$$M(t) = \int_{\mathbb{R}^5} -a_j(x)u_t(t,x)u_j(t,x) - \frac{1}{2}a_{jj}(x)u(t,x)u_t(t,x)dx,$$

where $a : \mathbb{R}^5 \to \mathbb{R}$, subscripts indicate partial derivatives, and we have used the summation convention. A brief calculation then yields the identity

$$\frac{dM}{dt}(t) = \int_{\mathbb{R}^5} a_{jk}(x)u_j(t,x)u_k(t,x) + \frac{1}{2}a_j(x)\{\mathcal{N}, u\}_j - \frac{1}{4}a_{jjkk}(x)u(t,x)^2dx,$$

with $\{f, g\} := f\nabla g - g\nabla f$, where the subscript on $\{\mathcal{N}, u\}$ denotes the jth component. Taking $a(x) = |x|$, integrating in time, and using the fundamental theorem of Calculus, we then have

$$\int_I \int_{\mathbb{R}^5} \left(\frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3} \right) u_j(t,x) u_k(t,x) + \frac{x_j \{\mathcal{N}, u\}_j}{2|x|} + \frac{8}{|x|^3} u(t,x)^2 dxdt \lesssim \sup_{t \in I} |M(t)| \tag{4.22}$$

for every $I \subset \mathbb{R}$. Moreover, the triangle inequality followed by the Cauchy-Schwartz and Hardy inequalities give

$$|M(t)| \lesssim \|u_t\|_{L_t^\infty L_x^2} \|\nabla u\|_{L_t^\infty L_x^2} \tag{4.23}$$

for all $t \in I$. Combining (4.22) with (4.23), observing that the first term on the left hand side of (4.22) is non-negative and invoking an approximation argument, we obtain

Lemma 4.3.1 (Morawetz estimate)**.** *Suppose $u : I \times \mathbb{R}^5 \to \mathbb{R}$ solves $u_{tt} - \Delta u + \mathcal{N} = 0$. Then,*

$$\int_I \int_{\mathbb{R}^5} \frac{x \cdot \{\mathcal{N}(t,x), u(t,x)\}}{|x|} dxdt \lesssim \|u_t\|_{L_t^\infty L_x^2} \|\nabla u\|_{L_t^\infty L_x^2}. \tag{4.24}$$

We also recall the following Hardy-type bound, which will be used to estimate the error terms resulting from the frequency localization.

Proposition 4.3.2 (Hardy-type bound, [2])**.** *Fix $1 < p < \infty$, and $0 \leq \alpha < 5$. Then there exists $C = C(\alpha, p) > 0$ such that for every $g \in \mathcal{S}(\mathbb{R}^5)$,*

$$\| |x|^{-\alpha/p} g(x) \|_{L_x^p(\mathbb{R}^5)} \leq C(\alpha, p) \| |\nabla|^{\alpha/p} g(x) \|_{L_x^p(\mathbb{R}^5)}. \tag{4.25}$$

In particular, we prove the following:

Lemma 4.3.3 (Frequency localized Morawetz estimate)**.** *If* $u : I \times \mathbb{R}^5 \to \mathbb{R}$ *is an almost periodic solution to (NLW) on* $I^+ = \cup J_k \subset \mathbb{R}$ *with* $N(t) = N_k \geq 1$ *on each* J_k *and* $(u, u_t) \in L_t^\infty(\dot{H}_x^{3/2} \times \dot{H}_x^{1/2})$*, then for any* $\eta > 0$ *there exists* $N_0 = N_0(\eta) > 0$ *such that for all* $N \leq N_0$ *one has*

$$\int_{I_0} \int_{\mathbb{R}^5} \frac{|u_{\geq N}(t,x)|^4}{|x|} dxdt \leq \eta C(u)(N^{-1} + |I_0|)$$

on any compact interval $I_0 = \cup J_k$*.*

Proof. Fix a compact time interval $I_0 = \cup J_k \subset I^+$. In what follows, all spacetime norms will be taken over $I_0 \times \mathbb{R}^5$, unless otherwise indicated. Let $\eta > 0$ be given, and fix $N_0 > 0$ to be determined later in the argument. Let $N \leq N_0$ be given. We begin by observing that the Morawetz estimate (4.24) applied to $u_{\geq N}$ yields

$$\int_{I_0} \int_{\mathbb{R}^5} \frac{x \cdot \{P_{\geq N}[u(t,x)^3], u_{\geq N}(t,x)\}}{|x|} dxdt \lesssim \|\partial_t u_{\geq N}\|_{L_t^\infty L_x^2} \|\nabla u_{\geq N}\|_{L_t^\infty L_x^2} \tag{4.26}$$

Note that by Remark 4.1.1, we may choose $N_1 > 0$ so that $N \leq N_1$ implies

$$\|(u_{\leq N}, \partial_t u_{\leq N})\|_{L_t^\infty(I; \dot{H}_x^{3/2} \times \dot{H}_x^{1/2})} < \eta^{1/2}.$$

Now, by choosing N_0 small enough so that $N_0 < \eta N_1$, we may estimate the right hand side of (4.26) by

$$\begin{aligned}
&(\|\partial_t u_{N \leq \cdot < N_1}\|_{L_t^\infty L_x^2} + \|\partial_t u_{\geq N_1}\|_{L_t^\infty L_x^2}) \cdot (\|\nabla u_{N \leq \cdot < N_1}\|_{L_t^\infty L_x^2} + \|\nabla u_{\geq N_1}\|_{L_t^\infty L_x^2}) \\
&\quad \lesssim (N^{-1/2} \|\partial_t u_{< N_1}\|_{L_t^\infty \dot{H}_x^{1/2}} + N_1^{-1/2} \|\partial_t u_{\geq N_1}\|_{L_t^\infty \dot{H}_x^{1/2}}) \\
&\quad\quad \cdot (N^{-1/2} \|u_{< N_1}\|_{L_t^\infty \dot{H}_x^{3/2}} + N_1^{-1/2} \|u\|_{L_t^\infty \dot{H}_x^{3/2}})
\end{aligned}$$

$$\lesssim_u \eta^2 N^{-1}. \tag{4.27}$$

We now estimate the left hand side of (4.26). For this, we use the identity

$$\begin{aligned}\{P_{\geq N}[u(t)^3], u_{\geq N}(t)\} = \{u(t)^3, u(t)\} &- \{u_{<N}(t)^3, u_{<N}(t)\} \\ &- \{u(t)^3 - u_{<N}(t)^3, u_{<N}(t)\} \\ &- \{P_{<N}[u(t)^3], u_{\geq N}(t)\}\end{aligned}$$

to obtain

$$\begin{aligned}&\int_{I_0}\int_{\mathbb{R}^5} \frac{x\cdot\{P_{\geq N}[u(t,x)^3], u_{\geq N}(t,x)\}}{|x|} dxdt \\ &\quad= \int_{I_0}\int_{\mathbb{R}^5} \frac{x\cdot\{u(t,x)^3, u(t,x)\}}{|x|} - \frac{x\cdot\{u_{<N}(t,x)^3, u_{<N}(t,x)\}}{|x|} dxdt \\ &\quad\quad - \int_{I_0}\int_{\mathbb{R}^5} \frac{x\cdot\{u(t,x)^3 - u_{<N}(t,x)^3, u_{<N}(t,x)\}}{|x|} dxdt \\ &\quad\quad - \int_{I_0}\int_{\mathbb{R}^5} \frac{x\cdot\{P_{<N}[u(t,x)^3], u_{\geq N}(t,x)\}}{|x|} dxdt.\end{aligned} \tag{4.28}$$

A simple calculation then shows $\{f^3, f\} = -\frac{1}{2}\nabla[f^4]$, so that integrating the first two terms in (4.28) by parts gives

$$\begin{aligned}&\int_{I_0}\int_{\mathbb{R}^5} \frac{x\cdot\{P_{\geq N}[u(t,x)^3], u_{\geq N}(t,x)\}}{|x|} dxdt \\ &\quad= \int_{I_0}\int_{\mathbb{R}^5} \frac{2(|u(t,x)|^4 - |u_{<N}(t,x)|^4)}{|x|} dxdt \\ &\quad\quad - \int_{I_0}\int_{\mathbb{R}^5} \frac{x\cdot\{u(t,x)^3 - u_{<N}(t,x)^3, u_{<N}(t,x)\}}{|x|} dxdt \\ &\quad\quad - \int_{I_0}\int_{\mathbb{R}^5} \frac{x\cdot\{P_{<N}[u(t,x)^3], u_{\geq N}(t,x)\}}{|x|} dxdt.\end{aligned} \tag{4.29}$$

On the other hand, applying the decomposition $u = u_{<N} + u_{\geq N}$ gives

$$\int_{I_0}\int_{\mathbb{R}^5} \frac{|u_{\geq N}(t,x)|^4}{|x|} dxdt \lesssim \int_{I_0}\int_{\mathbb{R}^5} \frac{|u(t,x)|^4 - |u_{<N}(t,x)|^4}{|x|} dxdt + \sum_{i=1}^{3} \int_{I_0}\int_{\mathbb{R}^5} \frac{|u_{<N}(t,x)|^{4-i}|u_{\geq N}(t,x)|^i}{|x|} dxdt.$$

In view of (4.26) and (4.29), we therefore obtain the bound

$$\int_{I_0}\int_{\mathbb{R}^5} \frac{|u_{\geq N}(t,x)|^4}{|x|} dxdt \lesssim \eta N^{-1} + \sum_{i=1}^{3} (I)_i + (II) + (III),$$

where we have set

$$\begin{aligned} (I)_i &= \int_{I_0}\int_{\mathbb{R}^5} \frac{|u_{<N}(t,x)|^{4-i}|u_{\geq N}(t,x)|^i}{|x|} dxdt, \quad i = 1, \cdots, 3, \\ (II) &= \left| \int_{I_0}\int_{\mathbb{R}^5} \frac{x \cdot \{u(t,x)^3 - u_{<N}(t,x)^3, u_{<N}(t,x)\}}{|x|} dxdt \right|, \quad \text{and} \\ (III) &= \left| \int_{I_0}\int_{\mathbb{R}^5} \frac{x \cdot \{P_{<N}[u(t,x)^3], u_{\geq N}(t,x)\}}{|x|} dxdt \right|. \end{aligned}$$

We estimate each of these terms individually. For $(I)_i$, we use the Hölder inequality with the Hardy-type bound (4.25), along with the Sobolev embedding and Corollary 4.2.2 (after choosing N_0 sufficiently small) to obtain 4.2.1 to obtain the bounds

$$\begin{aligned} (I)_1 &\lesssim \|u_{\geq N}\|_{L_t^4 L_x^{20/7}} \left\| \frac{u_{<N}}{|x|^{1/3}} \right\|^3_{L_t^4 L_x^{60/13}} \\ &\lesssim \|u_{\geq N}\|_{L_t^4 L_x^{20/7}} \||\nabla|^{1/3} u_{<N}\|^3_{L_t^4 L_x^{\frac{60}{13}}} \\ &\lesssim \|u_{\geq N}\|_{L_t^4 L_x^{20/7}} \|\nabla u_{<N}\|^3_{L_t^4 L_x^{20/7}} \\ &\lesssim_u \eta^3 (N^{-1} + |I_0|). \end{aligned}$$

For the term $(I)_2$, we write

$$\begin{aligned}(I)_2 &\lesssim \int_{I_0}\int_{\mathbb{R}^5} \frac{|u_{<N}(t,x)|\,|u_{\geq N}(t,x)|}{|x|}(|u_{<N}(t,x)|^2 + |u_{\geq N}(t,x)|^2)dxdt \\ &\lesssim (I)_1 + (I)_3,\end{aligned}$$

while for the term $(I)_3$, we note that for each $\epsilon > 0$,

$$\begin{aligned}(I)_3 &\lesssim \int_{I_0}\int_{\{x:|u_{<N}(t,x)|\leq\epsilon|u_{\geq N}(t,x)|\}} \frac{|u_{<N}(t,x)|\,|u_{\geq N}(t,x)|^3}{|x|}dxdt \\ &\quad + \int_{I_0}\int_{\{x:|u_{<N}(t,x)|>\epsilon|u_{\geq N}(t,x)|\}} \frac{|u_{<N}(t,x)|\,|u_{\geq N}(t,x)|^3}{|x|}dxdt \\ &\leq \epsilon\int_{I_0}\int_{\mathbb{R}^5} \frac{|u_{\geq N}(t,x)|^4}{|x|}dxdt + \frac{1}{\epsilon}(I)_1.\end{aligned}$$

We now estimate term (II). Using the identity

$$\{u^3 - u^3_{<N}, u_{<N}\} = 2(u^3 - u^3_{<N})\nabla u_{<N} - \nabla((u^3 - u^3_{<N})u_{<N}),$$

we apply the triangle inequality and integrate by parts in the second term of the resulting integral to obtain

$$\begin{aligned}(II) &\lesssim \int_{I_0}\int_{\mathbb{R}^5} |(u(t,x)^3 - u_{<N}(t,x)^3)|\,|\nabla u_{<N}(t,x)|dxdt \\ &\quad + \int_{I_0}\int_{\mathbb{R}^5} \frac{|(u(t,x)^3 - u_{<N}(t,x)^3)|\,|u_{<N}(t,x)|}{|x|}dxdt \\ &\lesssim \sum_{i=1}^{3}\int_{I_0}\int_{\mathbb{R}^5} |u_{\geq N}(t,x)|^i|u_{<N}(t,x)|^{3-i}|\nabla u_{<N}(t,x)|dxdt + \sum_{i=1}^{3}(I)_i\end{aligned}$$

We now use the Hölder inequality, Sobolev embedding, and Corollary 4.2.2 to estimate the first term,

$$\|u_{\geq N}u^2_{<N}\nabla u_{<N}\|_{L^1_{t,x}} \leq \|u_{\geq N}\|_{L^4_tL^{20/7}_x}\|u_{<N}\|_{L^2_tL^{10}_x}\|u_{<N}\|_{L^\infty_tL^5_x}\|\nabla u_{<N}\|_{L^4_tL^{20/7}_x}$$

$$\lesssim_u \|u_{\geq N}\|_{L_t^4 L_x^{20/7}} \||\nabla|^{3/4} u_{<N}\|_{L_t^2 L_x^4} \|\nabla u_{<N}\|_{L_t^4 L_x^{20/7}}$$
$$\lesssim_u \eta N^{-1}(1 + N|I_0|),$$

the second term,

$$\begin{aligned}\|u_{\geq N}^2 u_{<N} \nabla u_{<N}\|_{L_{t,x}^1} &\leq \|u_{\geq N}\|_{L_t^4 L_x^{20/7}} \|u_{\geq N}\|_{L_t^\infty L_x^5} \|u_{<N}\|_{L_t^2 L_x^{10}} \|\nabla u_{<N}\|_{L_t^4 L_x^{20/7}} \\ &\lesssim_u \|u_{\geq N}\|_{L_t^4 L_x^{20/7}} \||\nabla|^{3/4} u_{<N}\|_{L_t^2 L_x^4} \|\nabla u_{<N}\|_{L_t^4 L_x^{20/7}} \\ &\lesssim_u \eta N^{-1}(1 + N|I_0|),\end{aligned}$$

and the third term,

$$\begin{aligned}\|u_{\geq N}^3 \nabla u_{<N}\|_{L_{t,x}^1} &\leq \|u_{\geq N}\|_{L_t^3 L_x^{30/7}}^2 \|u_{\geq N}\|_{L_t^\infty L_x^5} \|\nabla u_{<N}\|_{L_{t,x}^3} \\ &\lesssim_u \eta N^{-1}(1 + N|I_0|).\end{aligned}$$

Combining these estimates then gives

$$(II) \lesssim_u \eta N^{-1}(1 + N|I_0|) + \sum_{i=1}^{3} (I)_i.$$

To continue, we estimate the remaining term, (III). In a similar manner as above, we use the identity

$$\{P_{<N}[u(t)^3], u_{\geq N}(t)\} = \nabla(P_{<N}[u(t)^3] u_{\geq N}(t)) - 2u_{\geq N}(t) \nabla P_{<N}[u(t)^3]$$

and integrate by parts in the first term of the resulting integral to obtain

$$\begin{aligned}(III) \lesssim &\int_{I_0} \int_{\mathbb{R}^5} \frac{|P_{<N}[u(t,x)^3] u_{\geq N}(t,x)|}{|x|} dx dt \\ &+ \int_{I_0} \int_{\mathbb{R}^5} |u_{\geq N}(t,x) \nabla P_{<N}[u(t,x)^3]| dx dt\end{aligned}$$

$$\lesssim \sum_{i=0}^{3} \left\| \frac{P_{<N}[u_{<N}^{i} u_{\geq N}^{3-i}] u_{\geq N}}{|x|} \right\|_{L^1_{t,x}} + \| u_{\geq N} \nabla P_{<N}[u_{<N}^{i} u_{\geq N}^{3-i}] \|_{L^1_{t,x}}$$

We estimate the terms containing the gradient and remark that the other terms may then be bounded through the use of the Hardy-type inequality (4.25). In particular, we apply the Hölder, Bernstein, and Sobolev inequalities along with Corollary 4.2.2 to obtain, for the first term (using the bound from (4.27)),

$$\begin{aligned} \| u_{\geq N} \nabla P_{<N}[u_{\geq N}^{3}] \|_{L^1_{t,x}} &\leq \| u_{\geq N} \|_{L_t^\infty L_x^{10/3}} \| \nabla P_{<N}[u_{\geq N}^{3}] \|_{L_t^1 L_x^{10/7}} \\ &\lesssim N \| u_{\geq N} \|_{L_t^\infty \dot{H}_x^1} \| u_{\geq N} \|_{L_t^3 L_x^{30/7}}^3 \\ &\lesssim_u \eta N^{-1} (1 + N|I_0|), \end{aligned}$$

for the second term,

$$\begin{aligned} \| u_{\geq N} \nabla P_{<N}[u_{<N} u_{\geq N}^{2}] \|_{L^1_{t,x}} &\leq \| u_{\geq N} \|_{L_t^\infty L_x^{5}} \| \nabla P_{<N}[u_{<N} u_{\geq N}^{2}] \|_{L_t^1 L_x^{5/4}} \\ &\lesssim_u N \| u_{<N} \|_{L_t^2 L_x^{10}} \| u_{\geq N} \|_{L_t^4 L_x^{20/7}}^2 \\ &\lesssim_u N \| |\nabla|^{3/4} u_{<N} \|_{L_t^2 L_x^{4}} \| u_{\geq N} \|_{L_t^4 L_x^{20/7}}^2 \\ &\lesssim_u \eta N^{-1} (1 + N|I_0|), \end{aligned}$$

for the third term,

$$\begin{aligned} \| u_{\geq N} \nabla P_{<N}[u_{<N}^{2} u_{\geq N}] \|_{L^1_{t,x}} &\leq \| u_{\geq N} \|_{L_t^4 L_x^{20/7}} \| \nabla P_{<N}[u_{<N}^{2} u_{\geq N}] \|_{L_t^{4/3} L_x^{20/13}} \\ &\lesssim N \| u_{\geq N} \|_{L_t^4 L_x^{20/7}}^2 \| u_{<N} \|_{L_t^4 L_x^{20/3}}^2 \\ &\lesssim N \| u_{\geq N} \|_{L_t^4 L_x^{20/7}}^2 \| \nabla u_{<N} \|_{L_t^4 L_x^{20/7}}^2 \\ &\lesssim_u \eta N^{-1} (1 + N|I_0|), \end{aligned}$$

and for the fourth term,

$$\begin{aligned}\|u_{\geq N}\nabla P_{<N}[u_{<N}^3]\|_{L^1_{t,x}} &\leq \|u_{\geq N}\|_{L_t^4 L_x^{20/7}}\|\nabla P_{<N}[u_{<N}^3]\|_{L_t^{4/3}L_x^{20/13}}\\ &= \|u_{\geq N}\|_{L_t^4 L_x^{20/7}}\|u_{<N}^2\nabla u_{<N}\|_{L_t^{4/3}L_x^{20/13}}\\ &\leq \|u_{\geq N}\|_{L_t^4 L_x^{20/7}}\|u_{<N}\|_{L_t^\infty L_x^5}\|u_{<N}\|_{L_t^2 L_x^{10}}\|\nabla u_{<N}\|_{L_t^4 L_x^{20/7}}\\ &\lesssim_u \eta N^{-1}(1+N|I_0|).\end{aligned}$$

Combining these estimates, we obtain

$$\int_{I_0}\int_{\mathbb{R}^5}\frac{|u_{\geq N}(t,x)|^4}{|x|}dxdt \lesssim_u \eta(N^{-1}+|I_0|)+C_\epsilon(I)_1+\epsilon\int_{I_0}\int_{\mathbb{R}^5}\frac{|u_{\geq N}(t,x)|^4}{|x|}dxdt.$$

Choosing ϵ sufficiently small, we obtain

$$\int_{I_0}\int_{\mathbb{R}^5}\frac{|u_{\geq N}(t,x)|^4}{|x|}dxdt \lesssim_u 3\eta(N^{-1}+|I_0|)$$

as desired. □

4.4 Finite time blow-up solution

In this section, we rule out the existence of finite time blow-up solutions satisfying the properties stated in Theorem 4.1.1. Arguing as in [3, 18, 26, 28], this is accomplished by showing that such solutions must have zero energy, which in the defocusing case implies that the solution must be identically zero, contradicting its blow up.

In particular, we have the following theorem:

Theorem 4.4.1. *Suppose that u is an almost periodic solution to (NLW) with maximal interval of existence I, satisfying the properties given in Theorem* 4.1.1*. Then the case $|I^+| < \infty$ cannot occur.*

Proof. Let u be given as stated and suppose to the contrary that $|I^+| < \infty$. By the time reversal and scaling symmetries we may assume that $\sup I = 1$.

We first show that

$$\operatorname{supp} u(t,\cdot), \quad \operatorname{supp} u_t(t,\cdot) \subset \overline{B(0,1-t)}, \quad 0 < t < 1. \tag{4.30}$$

Indeed, the almost periodicity of u in the form of Remark 2.1.1 gives that for all $\epsilon > 0$ there exists $R = R(\epsilon) > 0$ such that for every $0 < s < 1$ we have

$$\int_{|x| \geq \frac{R}{N(s)}} |\nabla u(s,x)|^{5/2} + |u_t(s,x)|^{5/2} dx < \epsilon.$$

An invocation of the finite speed of propagation (see, for instance, [3, Proposition 5.1]) then gives

$$\int_{|x| \geq \frac{R}{N(s)} + s - t} |\nabla u(t,x)|^{5/2} + |u_t(t,x)|^{5/2} dx \leq \epsilon \tag{4.31}$$

whenever $0 < t < s < 1$, yielding

$$\limsup_{s \to 1} \int_{|x| \geq \frac{R}{N(s)} + s - t} |\nabla u(t,x)|^{5/2} + |u_t(t,x)|^{5/2} dx \leq \epsilon$$

for $t \in (0,1)$. On the other hand, recalling $N(t) \to \infty$ as $t \to 1$ (a consequence of the local theory and the almost periodicity), for all $t \in (0,1)$ and $\eta > 0$ we have

$$\{x : |x| \geq 1 - t + \eta\} \subset \{x : |x| \geq \frac{R}{N(s)} + s - t\}.$$

when $s = s(t, \eta)$ is sufficiently close to 1. Combining this inclusion with (4.31) and letting η and ϵ tend to zero, we obtain

$$\int_{|x| \geq 1-t} |\nabla u(t,x)|^{5/2} + |u_t(t,x)|^{5/2} dx = 0,$$

which in turn yields that $(u(t,\cdot)$ is constant a.e. on $\{x : |x| \geq 1-t\}$ as well as $\operatorname{supp} u_t(t,\cdot) \subset \overline{B(0, 1-t)}$. To bound the support of u, we note that u belongs to $L^\infty_x L^d_x$, which gives (4.30).

To continue, by (4.30), we write the energy by

$$\begin{aligned} E(u, u_t) &= \int_{|x| \leq 1-t} \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{2} |u_t(t,x)|^2 + \frac{1}{4} |u(t,x)|^4 dx \\ &\lesssim (1-t)[\|\nabla u(t)\|^2_{L^{5/2}_x(\mathbb{R}^5)} + \|u_t(t)\|^2_{L^{5/2}_x(\mathbb{R}^5)} + \|u(t)\|^4_{L^5_x(\mathbb{R}^5)}] \\ &\lesssim_u 1-t \end{aligned}$$

where we have used the a priori bound $(u, u_t) \in L^\infty_t(\dot{H}^{3/2}_x \times \dot{H}^{1/2}_x)$. Letting $t \to 1$ and using the conservation of energy, we obtain $u \equiv 0$, contradicting its blow-up. $\square$

4.5 Infinite time blow-up solution

In this section, we consider the second class of solutions identified in Theorem 4.1.1, almost periodic solutions to (NLW) which blow up in infinite time. By making use of a frequency localized variant of the concentration of potential energy along with the frequency localized Morawetz estimate obtained in Section 4.3, we obtain a bound on the length of the maximal interval of existence, contradicting the assumption of infinite time blow-up. When combined with the results of the previous section, this completes the proof of Theorem 1.1.2. In particular, we prove the following theorem:

Theorem 4.5.1. *There is no solution u to (NLW) satisfying the properties of Theorem* 4.1.1 *with* $|I^+| = \infty$.

Proof. Suppose to the contrary that such a solution u existed. We begin by showing that there exists $C > 0$ and $N_0 > 0$ such that for all $N \leq N_0$ and every $k \geq 1$,

$$\int_{J_k} \int_{|x| \leq C/N_k} |u_{\geq N}(t,x)|^4 dx dt \gtrsim_u N_k^{-2}. \tag{4.32}$$

To show this claim, we recall that [28, Lemma 2.6] gives the existence of $C > 0$ such that for every $k \geq 1$,

$$\int_{J_k} \int_{|x| \leq C/N_k} |u(t,x)|^4 dx dt \gtrsim_u N_k^{-2}.$$

An application of Minkowski's inequality then gives

$$\begin{aligned}
&\left(\int_{J_k} \int_{|x| \leq C/N_k} |u_{\geq N}(t,x)|^4 dx dt \right)^{1/4} \\
&\quad = \left(\int_{J_k} \int_{|x| \leq C/N_k} |u(t,x) - u_{\leq N/2}(t,x)|^4 dx dt \right)^{1/4} \\
&\quad \geq \left(\int_{J_k} \int_{|x| \leq C/N_k} |u(t,x)|^4 dx dt \right)^{1/4} - \left(\int_{J_k} \int_{|x| \leq C/N_k} |u_{\leq N/2}(t,x)|^4 dx dt \right)^{1/4} \\
&\quad \gtrsim_u N_k^{-1/2} - \left(\int_{J_k} \int_{|x| \leq C/N_k} |u_{\leq N/2}(t,x)|^4 dx dt \right)^{1/4}. \qquad (4.33)
\end{aligned}$$

On the other hand, fixing $\eta_1 > 0$ and applying Hölder's inequality along with Remark 4.1.1, we obtain that for N sufficiently small

$$\int_{|x| \leq C/N_k} |u_{\leq N/2}(t,x)|^4 dx \lesssim_u N_k^{-1} \left(\int_{|x| \leq C/N_k} |u_{\leq N/2}(t,x)|^5 dx \right)^{4/5}$$

$$\lesssim_u N_k^{-1}\|u_{\le N/2}\|^4_{L_t^\infty \dot{H}_x^{3/2}} \lesssim_u \frac{\eta_1^4}{N_k}.$$

This implies the bound

$$\int_{J_k}\int_{|x|\le C/N_k} |u_{\le N/2}(t,x)|^4 dxdt \lesssim \frac{\delta\eta_1^4}{N_k^2},$$

so that, after choosing η_1 sufficiently small and substituting this bound into (4.33), we obtain (4.32).

We now fix $\eta > 0$ to be determined later in the argument and recall that Lemma 4.3.3 implies the existence of $N_1 \in (0, N_0)$ such that for all $N \le N_1$ and $I_0 = \cup J_k \subset I$ compact,

$$\int_{I_0}\int_{\mathbb{R}^5} \frac{|u_{\ge N}(t,x)|^4}{|x|} dxdt \lesssim_u \eta(N^{-1} + |I_0|). \tag{4.34}$$

Combining (4.34) with (4.32) then gives

$$\begin{aligned}
\eta\Big(N^{-1} + |I_0|\Big) &\gtrsim_u \sum_{J_k\subset I_0} \int_{J_k}\int_{|x|\le C/N_k} \frac{|u_{\ge N}(t,x)|^4}{|x|} dxdt \\
&\gtrsim_u \sum_{J_k\subset I_0} N_k \int_{J_k}\int_{|x|\le C/N_k} |u_{\ge N}(t,x)|^4 dxdt \\
&\gtrsim_u \sum_{J_k\subset I_0} N_k^{-1} \\
&\gtrsim_u |I_0|
\end{aligned} \tag{4.35}$$

for all $N \le N_1$. Choosing η sufficiently small (depending on the constant in (4.35)), we obtain the bound

$$|I_0| \lesssim_u N^{-1}$$

for all $N \le N_1$ and all I_0. Fixing N and letting I_0 tend to I^+ then gives the desired contradiction. □

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