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Algorithms and Hardness Results for Resource Allocation and Facility Location Problems

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Algorithms and Hardness Results for Resource Allocation and Facility Location Problems

by

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Algorithms and Hardness Results for Resource Allocation and Facility Location Problems

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Many real-life scenarios involve making decisions based on agent preferences, for example, electing leaders, developing public facilities, and allocating resources in market. In this thesis, we consider two such problems where we want to make decisions based on agent preferences. In our first problem, we study a variant of the classic housing markets model. Each agent is initially assigned a unique house, and the houses form a graph. Each agent has strict preferences over the houses. Two agents can swap houses if the swap is Pareto-improving and the houses are adjacent. We study three reachability questions in the context of various graph structures. In our second problem, we study a variant of the facility location game. In this setting, a central planner has to build a set of heterogeneous facilities. Every agent reports the set of facilities that they consider "obnoxious". The goal is to design strategyproof and group-strategyproof mechanisms that maximize either the total utility of agents or the minimum utility of agents.

Table of Contents

List of Tables

Chapter 1

Introduction

Many real-life scenarios involve making decisions based on agent preferences. For example, in an election each agent may be asked to select their preferred candidates in order to choose a leader. Other examples include deciding where public facilities should be built based on the preferences of the agents living in a locality, and deciding which house should be assigned to each agent based on their preference over the houses. In such problems, besides finding an optimal solution, it is also desirable for the solutions to satisfy certain game-theoretic properties. Below we mention several of the game-theroretic properties to be discussed in this thesis.

1. **Pareto-Efficiency:** A solution is Pareto-efficient if there is no other solution where some agent gets higher utility and no agent gets lower utility.

2. **Strategyproof:** An algorithm (mechanism) is strategyproof if it is a weakly dominant strategy for any agent to report their true preferences. In other words, no agent can get higher utility by misreporting their preferences.

3. **Weakly Group-Strategyproof:** An algorithm (mechanism) is weakly group-strategyproof if no group of agents can misreport such that every agent in the group gets higher utility.

4. **Strongly Group-Strategyproof:** An algorithm (mechanism) is strongly group-strategyproof if no group of agents can misreport such that some agent in the group gets higher utility and no agent in the group gets lower utility.

5. **Efficiency:** A solution is efficient (or utilitarian) if it maximizes the sum of the agent utilities.

6. **Egalitarian:** A solution is egalitarian if it maximizes the minimum utility of any agent.

In this thesis, we study the three problems outlined in the Sections 1.1 through 1.3. The problems described in Sections 1.1 and 1.3 are resource allocation problems, and the problem described in Section 1.2 is a facility location problem.

1.1 Object Allocation Over a Network of Objects

In recent work, Gourvès, Lesca, and Wilczynski propose a variant of the classic housing markets model. In this variant, each agent has strict preferences over the houses and is initially matched to a unique house. The matching between agents and objects evolves through Pareto-improving swaps between pairs of adjacent agents in a social network. To explore the swap dynamics of their model, they pose several basic questions concerning the set of reachable matchings. In their work and other follow-up works, these questions have been studied for various classes of graphs: stars, paths, generalized stars (i.e., trees where at most one vertex has degree greater than two), trees, and cliques. For generalized stars and trees, it remains open whether a Pareto-efficient reachable matching can be found in polynomial time.

We pursue the same set of questions under a natural variant of their model. In our model, the social network is replaced by a network of objects, and a swap is allowed to take place between two agents if it is Pareto-improving and the associated objects are adjacent in the network. In those cases where the question of polynomial-time solvability versus NP-hardness has been resolved for the social network model, we are able to show that the same result holds for the network-of-objects model. In addition, for our model, we present a polynomial-time algorithm for computing a Pareto-efficient reachable matching in generalized star networks. Moreover, the object reachability algorithm that we present for path networks is significantly faster than the known polynomial-time algorithms for the same question in the social network model.

1.2 Obnoxious Facility Location with Dichotomous Preferences

We consider a facility location game in which n agents reside at known locations on a path, and k heterogeneous facilities are to be constructed on the path. Each agent is adversely affected by some subset of the facilities, and is unaffected by the others. The preferences of each agent specify the set of facilities that they are adversely affected by. We design two classes of mechanisms for choosing the facility locations given the reported agent preferences: utilitarian mechanisms that strive to maximize social welfare (i.e., to be efficient), and egalitarian mechanisms that strive to maximize the minimum welfare. For the utilitarian objective, we present a weakly group-strategyproof efficient mechanism for up to three facilities, we give a strongly group-strategyproof mechanism that guarantees at least half of the optimal social welfare for arbitrary k, and we prove that no strongly group-strategyproof mechanism achieves an approximation ratio of $5/4$ for one facility. For the egalitarian objective, we present a strategyproof egalitarian mechanism for arbitrary k, and we prove that no weakly group-strategyproof mechanism achieves a $o(\sqrt{n})$ approximation ratio for two facilities. We extend these egalitarian results to the case where the agents are located on a cycle, and we extend the first result to the case where the agents are located in the unit square.

1.3 Offline Resource Allocation

Pooling resources helps agents gain more utility when their demands are higher than their individual resources and prevents wastage of resources when their demands are low. In this problem, we consider n agents pooling r resources. Each agent has a demand (preference) for every kind of resource. We propose a max-min mechanism that strives to be fair by maximizing the minimum utility that any agent gets. We show that the max-min mechanism is strategyproof. It also guarantees each agent at least half of the utility they have by not pooling the resources.

1.4 Organization of the Thesis

All of the work presented in this thesis was done in collaboration with Fu Li, a doctoral student at the University of Texas at Austin, and Greg Plaxton, who advised this thesis. We have collaborated together on three projects. In order to minimize duplication, we have partitioned the associated results between this thesis and Fu Li's forthcoming doctoral dissertation. This partitioning is discussed below.

Our first joint work presents various efficient algorithms and hardness results related to the problem of object allocation over a network of objects discussed in Section 1.1. This work has recently been published as an extended abstract at a conference [41]; we have also made a full version available [40]. Chapter 2 presents the main hardness results associated with this work.

Our second joint work addresses the obnoxious facility location game with dichotomous preferences discussed in Section 1.2. Chapter 3 presents all of the results associated with this work. The proofs of Lemma 2.4.1 and Theorem 3.5.4 are primarily due to Fu, and are also expected to appear in his dissertation; these proofs are included here for the sake of completeness.

Our third joint work studies the offline resource allocation problem discussed in Section 1.3. The manuscript for this work is in preparation, and hence is not presented in this thesis. It is expected to appear in Fu Li's doctoral dissertation.

Chapter 4 provides concluding remarks.

Chapter 2

Object Allocation Over a Network of Objects: Mobile Agents with Strict Preferences

2.1 Introduction

Problems related to resource allocation under preferences are widely studied in both computer science and economics. Research in this area seeks to gain mathematical insight into the structure of resource allocation problems, and to exploit this structure to design fast algorithms. In one important class of resource allocation problems, sometimes referred to as one-sided matching problems [44], we seek to allocate indivisible objects to a set of agents, where each agent has preferences over the objects and wants to receive at most one object (unit demand). The allocation should enjoy one or more strong game-theoretic properties, such as Pareto-efficiency.

In a seminal work, Shapley and Scarf [55] introduced the notion of a housing market, which corresponds to the special case of one-sided matching in

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A part of the contents of this chapter appeared in: Fu Li, C. Gregory Plaxton, and Vaibhav B. Sinha. Object allocation over a network of objects: Mobile agents with strict preferences. In Proceedings of the 20th International Conference on Autonomous Agents and Multiagent Systems, pages 1578–1580, 2021. Except for the proof of Lemma 2.4.1, which is due to my co-authors, this chapter presents the results for which I was the primary contributor. The proof of Lemma 2.4.1 is included for the sake of completeness.

which there are an equal number of agents and objects, each agent is initially endowed with a distinct object, and each agent is required to be matched to exactly one object. They present an elegant algorithm (attributed to David Gale) for housing markets called the top trading cycles (TTC) algorithm. The TTC algorithm enjoys a number of strong game-theoretic properties. For example, when agents have strict preferences, the output of the TTC algorithm is the unique matching in the core. The TTC algorithm has subsequently been generalized to handle more complex variants of the original housing market problem (e.g., [7, 19, 23, 50, 53]).

Like many one-sided matching algorithms, the TTC algorithm is centralized: it takes all of the agent preference information as input and computes the output matching. In some resource allocation scenarios of practical interest, it may be difficult or impossible to coordinate such a global recomputation of the matching. Accordingly, researchers have studied decentralized (or distributed) variants of one-sided matching problems in which the initial allocation gradually evolves as "local" trading opportunities arise. In this setting, restrictions are imposed on the sets of agents that are allowed to participate in a single trade. For example, we might only allow (certain) pairs of agents to trade. In addition, all trades are required to be Pareto-improving. Locality-based restrictions on trade are generally enforced through graph-theoretic constraints.

Of particular relevance to the present paper is the line of research initiated by Gourvès et al. [31] on decentralized allocation in housing markets.

They propose a model in which agents have strict preferences and are embedded in an underlying social network. A pair of agents are allowed to swap objects with each other only if (1) they will be better off after the swap, and (2) they are directly connected (socially tied) via the network. The underlying social network is modeled as an undirected graph, and five different graph classes are considered: paths, stars, generalized stars, trees, and general graphs. The swap dynamics of the model are investigated by considering three computational questions. The first question, Reachable Object, asks whether there is a sequence of swaps that results in a given agent being matched to a given target object. The second question, Reachable Matching, asks whether there is a sequence of swaps that results in a given target matching. The third question, Pareto Efficiency, asks how to find a sequence of swaps that results in a Pareto-efficient matching with respect to the set of reachable matchings.

Gourvès et al. [31] studied each of the three questions in the context of the aforementioned graph classes, with the goal of either exhibiting a polynomial-time algorithm or establishing NP-hardness. For some of these problems, it is a relatively straightforward exercise to design a polynomial-time algorithm (even for the search version). In particular, this is the case for all three reachability questions on stars, for Pareto Efficiency on paths, and for Reachable Matching on trees (which subsumes Reachable Matching on generalized stars, and hence also on paths). Gourvès et al. present an elegant reduction from 2P1N-SAT [59] to establish the NP-completeness of Reachable Object on generalized stars (and hence also on trees and general graphs).

They establish the NP-completeness of Reachable Matching on general graphs via a reduction from Reachable Object on trees. The latter reduction has the property that for any given instance of Reachable Object on trees, the target matching in the instance of Reachable Matching on general graphs produced by the transformation matches each agent to its most preferred object. Consequently, the same reduction establishes the NP-hardness of Pareto Efficiency on general graphs. The work of Gourvès et al. left three of these problems open: Reachable Object on paths and Pareto Efficiency on generalized stars and trees. Subsequently, two sets of authors independently presented polynomial-time algorithms for Reachable Object on paths [9, 33]. Both groups obtained an $O(n^4)$-time algorithm by carefully studying the structure of swap dynamics on paths and then reducing the problem to 2-SAT. The complexity of Pareto Efficiency remains open for generalized stars and for trees. Gourvès et al. noted, "It appears interesting to see if Pareto (Efficiency) is polynomial time solvable in a generalized star by a combination of the techniques used to solve the cases of paths and stars."

Bentert et al. [9] established that Reachable Object on cliques is NP-complete, and Müller and Bentert [45] established that Reachable Matching on cliques is NP-complete. It is easy to extend the latter result to show that Pareto Efficiency on cliques is NP-hard. These three hardness results for cliques subsume the corresponding results obtained previously for general graphs by Gourvès et al.

We study a natural variant of the decentralized housing markets model of Gourvès et al. [31]. Instead of enforcing locality constraints on trade via a network where the locations of the agents are fixed (since they correspond to the vertices of the network) and the objects move around (due to swaps), we consider a network where the locations of the objects are fixed and the agents move around. We refer to these two models as the object-moving model and the agent-moving model. Table 2.1 summarizes the current state of the art for the object-moving model.

To motivate the study of the agent-moving model, consider a cloud computing environment with a large number of servers (objects) connected by a network that are available to rent. A set of customers (agents) are each interested in renting one server. The servers vary in CPU capacity, storage capacity, physical security, and rental cost. Varying customer workloads and requirements result in varying customer preferences over the servers. Rather than attempting to globally optimize the entire matching of customers to servers, it might be preferable to allow local swaps between adjacent servers to gradually optimize the matching. Given that customer workloads are likely to vary significantly over time, an optimization strategy based on frequent local updates might outperform a strategy based on less frequent global updates. Alternatively, one can envision a system that performs occasional global updates to optimize the matching, and that relies on local updates to maintain a reasonable matching between successive global updates.

Our Results. We initiate the study of the agent-moving model by revisiting each of the questions associated with Table 2.1 in the context of the

	Reachable Object	Reachable Matching	Pareto Efficiency
Star	poly-time	(poly-time)	poly-time
Path	poly-time	(poly-time)	poly-time
Generalized Star	NP-complete	(poly-time)	open
Tree	(NP-complete)	poly-time	open
Clique	NP-complete	NP-complete	NP-hard

Table 2.1: This table presents known complexity results for various questions related to the object-moving model of Gourvès et al. [31]. The results in parentheses follow directly from other table entries. For the agent-moving model, we obtain the same results, except that we also give a polynomial-time algorithm for Pareto Efficiency on generalized stars.

agent-moving model. We emphasize that the sole difference between the agent-moving model and the object-moving model is that the locality constraint prevents an agent a currently matched to an object b from trading with an agent a' currently matched to an object b' unless objects b and b' (two vertices in a given network of objects) are adjacent, rather than requiring agents a and a' (two vertices in a given network of agents) to be adjacent. Both models also require swaps to be Pareto-improving. The two models have strong similarities. In fact, for all of the questions in Table 1 for which a polynomial-time algorithm or hardness result has been established in the object-moving model, we establish a corresponding result in the agent-moving model. Moreover, for Pareto Efficiency on generalized stars, which is open in the object-moving model, we provide a polynomial-time algorithm in the agent-moving model. In some cases, it is relatively straightforward to adapt known results for the object-moving model to the agent-moving model. Below we highlight our four main technical contributions, which address more challenging cases.

Our first main technical result is an $O(n^2)$ time algorithm for Reachable Object on paths in the agent-moving model, which is much faster than the known $O(n^4)$-time algorithms for Reachable Object on paths in the object-moving model. (Here n denotes the number of agents/objects; the size of the input is quadratic in n since the preference list of each agent is of length n.) The speedup is due to a simpler local characterization of the reachable matchings on a path in the agent-moving model.

In our second main technical result, we obtain the same $O(n^2)$ time bound for Pareto Efficiency on paths. Our algorithms for Reachable Object and Pareto Efficiency are based on an efficient subroutine for solving a certain constrained reachability problem. Roughly speaking, this subroutine determines all of the possible matches for a given agent when certain agent-object pairs are required to be matched to one another. Our implementation involves a trivial $O(n^2)$-time preprocessing phase followed by an $O(n)$-time greedy phase. The preferences of the agents are only examined during the preprocessing phase. The proof of correctness of the greedy phase is somewhat nontrivial. We solve Reachable Object on paths using a single application of the subroutine, yielding an $O(n^2)$ bound. Our polynomial-time algorithm for Pareto Efficiency on paths uses n applications of our algorithm for Reachable Object on paths. Since the preprocessing phase only needs to be performed once, the overall running time remains $O(n^2)$.

In our third main technical result, we present a polynomial-time algorithm for Pareto Efficiency on generalized stars, which remains open in the object-moving model. To tackle this problem, we use the serial dictatorship algorithm with the novel idea of dynamically choosing the dictator sequence. We also leverage our techniques for solving Pareto Efficiency on paths.

The faster time bounds discussed above for the case of paths suggest that the agent-moving model is simpler than the object-moving model, at least from an upper bound perspective. Accordingly, we can expect it to be a bit more challenging to establish the NP-completeness results stated in Table 2.1 for the agent-moving model than for the object-moving model. In our fourth main technical result, we adapt an NP-completeness proof developed by Bentert et al. [9] in the context of the object-moving model to the more challenging setting of the agent-moving model. Specifically, we modify their reduction from 2P1N-SAT to establish that Reachable Object on cliques remains NP-complete in the agent-moving model.

Related work. For the object-moving model, Huang and Xiao [33] study Reachable Object with weak preferences, i.e., where an agent can be indifferent between different objects. Bentert et al. [9] establish NP-hardness for Reachable Object on cliques, and consider the case where the preference lists have bounded length. Saffidine and Wilczynski [52] propose a variant of Reachable Object where we ask whether a given agent is guaranteed to achieve a specified level of satisfaction after any maximal sequence of rational exchanges. Müller and Bentert [45] study Reachable Matching on cliques and cycles. Aspects related to social connectivity are also addressed in recent work on envy-free allocations [10, 13] and on trade-offs between efficiency and fairness [36].

Our agent-moving model can be viewed as a game in which each agent seeks to be matched to an object that is as high as possible on its preference list. If the game reaches a state in which no further swaps can be performed, we say that an equilibrium matching has been reached. Agarwal et al. [3] study a similar game motivated by Schelling's well-known residential segregation model. As in our game, there are an equal number of agents and objects, the objects correspond to the nodes of a graph, a matching is maintained between the agents and the objects, and the matching evolves via Pareto-improving, agent-moving swaps. There are also some significant differences. In our model, each agent has static preferences over the set of objects, and swaps can only occur between adjacent agents (i.e, agents matched to adjacent objects). In the Agarwal et al. game, each agent has a type, the desirability of an object b to an agent a depends on the current fraction of agents in the "neighborhood" of b (i.e., the set of agents matched to an object adjacent to b) with the same type as a, and swaps can occur between any pair of agents. Agarwal et al. study the existence, computational complexity, and quality of equilibrium matchings in such games. Bilò et. al [12] further investigated the influence of the graph structure on the resulting strategic multi-agent system.

Organization of the chapter. The remainder of the chapter is organized as follows. Section 2.2 provides formal definitions. Section 2.3 and 2.4 present our NP-completeness results for Reachable Object and Reachable Matching on cliques, respectively. Section 2.5 presents our NP-hardness results for Pareto Efficiency on cliques. The other results in Table 2.1 are included in our full version [40]. Section 2.6 offers concluding remarks.

2.2 Preliminaries

We define an object allocation framework (OAF) as a 4-tuple $F = (A, B, \succ, E)$ where A is a set of agents, B is a set of objects such that $|A| = |B|$, $\succ$ is a collection of strict linear orderings $\{\succ_a\}_{a \in A}$ over B such that $\succ_a$ specifies the preferences of agent a over B, and E is the edge set of some undirected graph (B, E).

We define a matching μ of given OAF $F = (A, B, \succ, E)$ as a subset of $A \times B$ such that no agent or object belongs to more than one pair in μ. (Put differently, μ is a matching in the complete bipartite graph of agents and objects.) We say that such a matching is perfect if $|\mu| = |A|$. For any matching μ, we define agents(μ) (resp., objects(μ)) as the set of all matched agents (resp., objects) with respect to μ. For any matching μ and any agent a that is matched in μ, we use the shorthand notation $\mu(a)$ to refer to the object matched to agent a. For any matching μ and any object b that is matched in μ, we use the notation $\mu^{-1}(b)$ to refer to the agent matched to object b.

For any OAF $F = (A, B, \succ, E)$, any perfect matching μ of F, and any edge $e = (b, b')$ in E such that $b' \succ_a b$ and $b \succ_{a'} b'$ where $a = \mu^{-1}(b)$ and $a' = \mu^{-1}(b')$, we say that a swap operation is applicable to μ across edge e, and we write $\mu \rightarrow_{F,e} \mu'$ where

$$\mu' = (\mu \setminus \{(a,b),(a',b')\}) \cup \{(a,b'),(a',b)\},$$

is the matching of F that results from applying this operation. We write $\mu \rightarrow_F \mu'$ to denote that $\mu \rightarrow_{F,e} \mu'$ for some edge e. We write $\mu \rightsquigarrow_F \mu'$ if there exists a sequence $\mu = \mu_0, \ldots, \mu_k = \mu'$ of matchings of F such that $\mu_{i-1} \rightarrow_F \mu_i$ for $1 \leq i \leq k$.

We define a configuration as a pair $\chi = (F, \mu)$ where F is an OAF and μ is a perfect matching of F.

For any configuration $\chi = (F, \mu)$ where $F = (A, B, \succ, E)$, any agent a in A, and any object b in B, we define $\chi(a)$ as a shorthand for the object $\mu(a)$, and we define $\chi^{-1}(b)$ as a shorthand for the agent $\mu^{-1}(b)$.

For any configuration $\chi = (F, \mu)$ where $F = (A, B, \succ, E)$, and any matching μ' of F such that $\mu \rightarrow_{F,e} \mu'$ for some edge e in E, we say that a swap is applicable to χ across edge e, and the result of applying this operation is the configuration (F, μ').

For any configuration $\chi = (F, \mu)$, we define reach(χ) as the set of all perfect matchings μ' of F such that $\mu \rightsquigarrow_F \mu'$. For any configuration $\chi = (F, \mu)$ and any matching μ' of F, we define reach(χ, μ') as the set of all matchings μ'' in reach(χ) such that μ'' contains μ'.

We now state the three reachability problems studied in this chapter.

- The reachable matching problem: Given a configuration $\chi = (F, \mu)$ and a perfect matching μ' of F, determine whether μ' belongs to reach(χ).

- The reachable object problem: Given a configuration $\chi = (F, \mu)$ where $F = (A, B, \succ, E)$, an agent a in A, and an object b in B, determine whether there is a matching μ' in reach(χ) such that $\mu'(a) = b$.

- The Pareto-efficient matching problem: Given a configuration χ, find a matching in reach(χ) that is not Pareto-dominated by any other matching in reach(χ).

2.3 NP-Completeness of Reachable Object on Cliques

It is easy to see that the reachable object on cliques problem belongs to NP. In this section, we prove that the problem is NP-hard by presenting a polynomial-time reduction from the known NP-complete problem 2P1N-SAT to reachable object on cliques.

An instance of 2P1N-SAT is a propositional formula f over n variables $x_1, \ldots, x_n$ with the following properties: f is the conjunction of a number of clauses, where each clause is the disjunction of a number of literals, and each literal is either a variable or the negation of a variable; each variable occurs exactly three times in f, once in each of three distinct clauses; each variable x_i occurs twice as a positive literal (i.e., x_i) and once as a negative literal (i.e., $\neg x_i$).

Throughout the remainder of Section 2.3, let f denote a given instance of 2P1N-SAT with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$.

In Section 2.3.1, we describe a polynomial-time procedure for transforming f into an instance I of reachable object on cliques. In Section 2.3.2, we prove that f is a positive instance of 2P1N-SAT if and only if I is a positive instance of reachable object on cliques.

2.3.1 Description of the reduction

We now describe how we transform a 2P1N-SAT instance f into a corresponding instance I of reachable object on cliques. For each variable x_i in f, there are two agents $\hat{x}_i^1$ and $\hat{x}_i^2$ in I. For each clause C_j in f, there are three agents $\hat{u}_j, \hat{v}_j$ and $\hat{w}_j$ in I. Note that the name we use to refer to each agent in I includes a hat symbol. We adopt the convention that if the hat symbol is removed from the name of such an agent, we obtain the name of the initial endowment of that agent. For example, agents $\hat{u}_j$ and $\hat{x}_i^1$ are initially endowed with objects u_j and x_i^1, respectively. For convenience, we define $\hat{U}$ as the set of agents $\{\hat{u}_j \mid j \in [m]\}$, and we define U as the set of objects $\{u_j \mid j \in [m]\}$. We define $\hat{V}$, V, $\hat{W}$, W, $\hat{X}$, and X similarly.

Let A (resp., B) denote the set of all agents (resp., objects) in I. Let N denote $|B|$. Thus $N = 3m + 2n$. Let K_N denote the complete graph with vertex set B, and let E denote the edge set of K_N. Let μ_0 denote the initial matching of agents with objects.

Below we describe the preferences $\succ$ of the agents in A over the objects in B. Let $\chi = (F, \mu_0)$ denote the initial configuration of I, where $F = (A, B, \succ$

, E). Instance I asks the following reachability question: Is there a matching μ in reach(χ) such that $\mu(\hat{w}_m) = u_1$?

Let variable x_i appear in clauses $C_{p_i^1}$ and $C_{p_i^2}$ as a positive literal, where $p_i^1 < p_i^2$, and in clause C_{n_i} as a negative literal. The definition of 2P1N-SAT implies that p_i^1, p_i^2, and n_i are distinct.

Below we list the preferences of each agent in A. In doing so, we specify only the objects that an agent prefers to its initial endowment; the order of the remaining objects is immaterial. The initial endowment is shown in a box. For any i in $[n]$, the preferences of agent $\hat{x}_i^1$ are

$$\hat{x}_i^1 : x_i^2 \succ w_{p_i^1} \succ v_{p_i^1} \succ \boxed{x_i^1}$$

and the preferences of agent $\hat{x}_i^2$ are

$$\hat{x}_i^2 : w_{n_i} \succ v_{n_i} \succ w_{p_i^2} \succ v_{p_i^2} \succ x_i^1 \succ \boxed{x_i^2}$$

if $n_i < p_i^2$, and are

$$\hat{x}_i^2 : w_{p_i^2} \succ v_{p_i^2} \succ w_{n_i} \succ v_{n_i} \succ x_i^1 \succ \boxed{x_i^2}$$

otherwise.

For any j in $[m]$, the preferences of agent $\hat{u}_j$ are

$$\hat{u}_j : v_j \succ \boxed{u_j}.$$

For any j in $[m-1]$, the preferences of agent $\hat{w}_j$ are

$$\hat{w}_j : u_{j+1} \succ \boxed{w_j}.$$

The preferences of agent $\hat{w}_m$ are

$$\hat{w}_m : u_1 \succ v_1 \succ w_1 \succ u_2 \succ v_2 \succ w_2 \succ \ldots \succ u_m \succ v_m \succ \boxed{w_m}.$$

For any j in $[m]$, the preferences of agent $\hat{v}_j$ are

$$\hat{v}_j : \{x_i^1 \mid j \in \{p_i^1, n_i\}\} \cup \{x_i^2 \mid j = p_i^2\} \succ \boxed{v_j},$$

where the set of objects preceding v_j may be ordered arbitrarily.

This completes the description of the reachable object on cliques instance I.

2.3.2 Correctness of the reduction

In this section, we prove that f is a positive instance of 2P1N-SAT if and only if I is a positive instance of reachable object on cliques. Lemma 2.3.4 establishes the only if direction. Lemmas 2.3.11 through 2.3.17 lay the groundwork for Lemma 2.3.18, which establishes the if direction.

For the purposes of our analysis, it is convenient to assign a nonnegative integer rank to each object in B, as follows. For any j in $[m]$, we define $\text{rank}(u_j)$ as $3j - 2$, $\text{rank}(v_j)$ as $3j - 1$, and $\text{rank}(w_j)$ as $3j$. The rank of any object in X is defined to be 0.

Observation 2.3.1 below can be justified by enumerating all those agents who like object x_i^1 at least as well as their initial endowment. Observations 2.3.2 and 2.3.3 can be justified in a similar manner.

Observation 2.3.1. For any i in $[n]$ and any matching μ in reach(χ), agent $\mu^{-1}(x_i^1)$ belongs to $\{\hat{x}_i^1, \hat{x}_i^2, \hat{v}_{p_i^1}, \hat{v}_{n_i}\}$.

Observation 2.3.2. For any i in $[n]$ and any matching μ in reach(χ), agent $\mu^{-1}(x_i^2)$ belongs to $\{\hat{x}_i^1, \hat{x}_i^2, \hat{v}_{p_i^2}\}$.

Observation 2.3.3. For any j in $[m]$ and any matching μ in reach(χ), agents $\mu^{-1}(v_j)$ and $\mu^{-1}(w_j)$ belong to $\{\hat{u}_j, \hat{v}_j, \hat{w}_m\} \cup A_j$ and $\{\hat{w}_j, \hat{w}_m\} \cup A_j$, respectively, where A_j denotes

$$\{\hat{x}_i^1 \mid j = p_i^1\} \cup \{\hat{x}_i^2 \mid j \in \{p_i^2, n_i\}\}.$$

Lemma 2.3.4. Assume that 2P1N-SAT instance f is satisfiable. Then there is a matching μ in reach(χ) such that $\mu(\hat{w}_m) = u_1$ in the reachable object on cliques instance I.

Proof. Let $\sigma : \{x_1, \ldots, x_n\} \to \{0, 1\}$ denote a satisfying assignment for f. We specify a sequence of matchings $\mu_0, \ldots, \mu_{3n+3m-1}$, which depends on σ, such that (1) $\mu_{3n+3m-1}(\hat{w}_m) = u_1$ and (2) $\mu_{k-1} = \mu_k$ or $\mu_{k-1} \to_F \mu_k$ for all k in $[3n + 3m - 1]$. We obtain this sequence in two phases.

In the first phase, we perform the following three steps for each i from 1 to n.

1. If $\sigma(x_i) = 1$ and $\mu_{3i-3}(\hat{v}_{p_i^1}) = v_{p_i^1}$, we set μ_{3i-2} to the matching obtained by swapping $\hat{v}_{p_i^1}$ with $\hat{x}_i^1$ in μ_{3i-3}. Otherwise, we set μ_{3i-2} to μ_{3i-3}.

2. If $\sigma(x_i) = 1$ and $\mu_{3i-2}(\hat{v}_{p_i^2}) = v_{p_i^2}$, we set μ_{3i-1} to the matching obtained by swapping $\hat{v}_{p_i^2}$ with $\hat{x}_i^2$ in μ_{3i-2}. Otherwise, we set μ_{3i-1} to μ_{3i-2}.

3. If $\sigma(x_i) = 0$ and $\mu_{3i-1}(\hat{v}_{n_i}) = v_{n_i}$, we set μ_{3i} to the matching obtained by first swapping $\hat{x}_i^1$ with $\hat{x}_i^2$, and then swapping $\hat{v}_{n_i}$ with $\hat{x}_i^2$, in μ_{3i-1}. Otherwise, we set μ_{3i} to μ_{3i-1}.

It is easy to check that all of the swaps in the first phase are valid.

Let A_j be as defined in the statement of Observation 2.3.3. We claim that at the end of the first phase, agent $\mu_{3n}^{-1}(v_j)$ belongs to A_j for all j in $[m]$. Assume for the sake of contradiction that for some j in $[m]$, agent $\mu_{3n}^{-1}(v_j)$ does not belong to A_j. Since agents $\hat{u}_j$ and $\hat{w}_m$ do not participate in any swap in the first phase, $\mu_{3n}(\hat{u}_j) = u_j$ and $\mu_{3n}(\hat{w}_m) = w_m$. Since $\mu_{3n}^{-1}(v_j)$ does not belong to $\{\hat{u}_j, \hat{w}_m\} \cup A_j$, Observation 2.3.3 implies that $\mu_{3n}(\hat{v}_j) = v_j$. Let variable x_i satisfy clause C_j. The swaps in the ith iteration of first phase imply that $\mu_{3i}^{-1}(v_j) \neq \hat{v}_j$. Since no agent in $\hat{V}$ participates in more than one swap in the first phase, we have $\mu_{3n}^{-1}(v_j) \neq \hat{v}_j$, a contradiction since $\mu_{3n}(\hat{v}_j) = v_j$. This completes the proof of the claim.

Note that there is a unique object of rank k for each k in $[3m]$. Using the claim stated above, together with the fact that no agent in $\hat{U} \cup \hat{W}$ participates in a swap in the first phase, it is easy to verify that for any rank k in $[3m-1]$, there is an agent a such that $\text{rank}(\mu_{3n}(a)) = k$ and a prefers the object of rank $k+1$ to the object of rank k. The preferences of agent $\hat{w}_m$ imply that for any rank k in $[3m-1]$, agent $\hat{w}_m$ prefers the object of rank k to the object of rank $k+1$. Moreover, $\text{rank}(\mu_{3n}(\hat{w}_m)) = 3m$. In the second phase we perform $3m-1$ swaps, each involving agent $\hat{w}_m$. For k running from $3m-1$ down to

1, we set $\mu_{3n+3m-k}$ to the matching obtained by swapping $\hat{w}_m$ with the agent matched to the object of rank k in $\mu_{3n+3m-k-1}$. The foregoing arguments show that all of these $3m-1$ swaps are valid and $\text{rank}(\mu_{3n+3m-1}(\hat{w}_m)) = 1$. Thus $\mu_{3n+3m-1}(\hat{w}_m) = u_1$. □

Observation 2.3.5 below can be justified by using the preferences of agent $\hat{w}_m$ and the fact that if any agent a swaps from object b to b' then $b' \succ_a b$. Observations 2.3.6 through 2.3.10 can be justified similarly.

Observation 2.3.5. For any matchings μ_1 and μ_2 in reach(χ) such that $\mu_1 \to_F \mu_2$, we have $\text{rank}(\mu_2(\hat{w}_m)) \leq \text{rank}(\mu_1(\hat{w}_m))$.

Observation 2.3.6. For any j in $[m-1]$ and any matchings μ_1 and μ_2 in reach(χ) such that $\mu_1 \to_F \mu_2$, we have $\text{rank}(\mu_2(\hat{w}_j)) \leq \text{rank}(\mu_1(\hat{w}_j)) + 1$.

Observation 2.3.7. For any j in $[m]$ and any matchings μ_1 and μ_2 in reach(χ) such that $\mu_1 \to_F \mu_2$, we have $\text{rank}(\mu_2(\hat{u}_j)) \leq \text{rank}(\mu_1(\hat{u}_j)) + 1$.

Observation 2.3.8. For any j in $[m]$ and any matchings μ_1 and μ_2 in reach(χ) such that $\mu_1 \to_F \mu_2$, we have $\text{rank}(\mu_2(\hat{v}_j)) \leq \text{rank}(\mu_1(\hat{v}_j))$.

Observation 2.3.9. For any i in $[n]$ and any matchings μ_1 and μ_2 in reach(χ) such that $\mu_1 \to_F \mu_2$ and $\mu_1(\hat{x}_i^1)$ belongs to $B \setminus X$, we have $\text{rank}(\mu_2(\hat{x}_i^1)) \leq \text{rank}(\mu_1(\hat{x}_i^1)) + 1$.

Observation 2.3.10. For any i in $[n]$ and any matchings μ_1 and μ_2 in reach(χ) such that $\mu_1 \to_F \mu_2$ and $\mu_1(\hat{x}_i^2)$ belongs to $B \setminus X$, we have $\text{rank}(\mu_2(\hat{x}_i^2)) \leq \text{rank}(\mu_1(\hat{x}_i^2)) + 1$.

Lemma 2.3.11 below follows immediately from Observations 2.3.5 through 2.3.10.

Lemma 2.3.11. For any agent a in A and any matchings μ_1 and μ_2 in reach(χ) such that $\mu_1 \to_F \mu_2$ and $\mu_1(a)$ belongs to $B \setminus X$, we have $\text{rank}(\mu_2(a)) \leq \text{rank}(\mu_1(a)) + 1$.

Lemma 2.3.12. Let μ_1 and μ_2 be matchings in reach(χ) such that $\mu_1 \to_F \mu_2$. Then

$$\text{rank}(\mu_2(\hat{w}_m)) \geq \text{rank}(\mu_1(\hat{w}_m)) - 1.$$

Proof. Assume for the sake of contradiction that

$$\text{rank}(\mu_2(\hat{w}_m)) < \text{rank}(\mu_1(\hat{w}_m)) - 1.$$

The preferences of $\hat{w}_m$ imply that $\mu_1(\hat{w}_m)$ and $\mu_2(\hat{w}_m)$ belong to $B \setminus X$. Thus there is an agent a such that $\mu_1(a)$ belongs to $B \setminus X$ and $\text{rank}(\mu_2(a)) > \text{rank}(\mu_1(a)) + 1$, contradicting Lemma 2.3.11. □

Lemma 2.3.13. Let i belong to $[n]$ and j belong to $[m]$. If $\hat{v}_j$ prefers x_i^1 to v_j, then $\hat{v}_j$ prefers v_j to x_i^2. Similarly, if $\hat{v}_j$ prefers x_i^2 to v_j, then $\hat{v}_j$ prefers v_j to x_i^1.

Proof. Assume that $\hat{v}_j$ prefers x_i^1 to v_j. The preferences of $\hat{v}_j$ imply that $j \in \{p_i^1, n_i\}$. Since p_i^1, p_i^2, and n_i are distinct, $j \neq p_i^2$. Hence the preferences of $\hat{v}_j$ imply that $\hat{v}_j$ prefers v_j to x_i^2. We can use a similar argument to prove that if $\hat{v}_j$ prefers x_i^2 to v_j, then $\hat{v}_j$ prefers v_j to x_i^1. □

Lemma 2.3.14. Let μ_1 and μ_2 be matchings in reach(χ) such that $\mu_1 \rightarrow_F \mu_2$ and $\mu_1(\hat{x}_i^2) = x_i^2$. Then $\mu_2(\hat{x}_i^2)$ belongs to $\{x_i^1, x_i^2, v_{p_i^2}\}$.

Proof. By examining the preferences of $\hat{x}_i^2$ we deduce that object $\mu_2(\hat{x}_i^2)$ belongs to

$$\{w_{n_i}, w_{p_i^2}, v_{n_i}, v_{p_i^2}, x_i^1, x_i^2\}.$$

Assume for the sake of contradiction that $\mu_2(\hat{x}_i^2)$ belongs to $\{w_{n_i}, w_{p_i^2}, v_{n_i}\}$. We consider three cases.

Case 1: $\mu_2(\hat{x}_i^2) = w_{n_i}$. Thus $\mu_1^{-1}(w_{n_i}) = \mu_2^{-1}(x_i^2)$. Since $\mu_2^{-1}(x_i^2) \neq \hat{x}_i^2$, Observation 2.3.2 implies that $\mu_2^{-1}(x_i^2)$ belongs to $\{\hat{x}_i^1, \hat{v}_{p_i^2}\}$. We consider two cases.

Case 1.1: $\mu_2^{-1}(x_i^2) = \hat{x}_i^1$. Thus $\mu_1^{-1}(w_{n_i}) = \hat{x}_i^1$. By examining the preferences of $\hat{x}_i^1$, we deduce that $\mu_1(\hat{x}_i^1) \neq w_{n_i}$, a contradiction.

Case 1.2: $\mu_2^{-1}(x_i^2) = \hat{v}_{p_i^2}$. A contradiction follows using a similar argument as in Case 1.1.

Case 2: $\mu_2(\hat{x}_i^2) = w_{p_i^2}$. A contradiction follows using a similar argument as in Case 1.

Case 3: $\mu_2(\hat{x}_i^2) = v_{n_i}$. A contradiction follows using a similar argument as in Case 1.

Thus $\mu_2(\hat{x}_i^2)$ belongs to $\{x_i^1, x_i^2, v_{p_i^2}\}$. □

Throughout the remainder of Section 2.3.2, we say that an agent a is satisfied in a matching μ if $\mu(a)$ is the most preferred object of a. In

Lemmas 2.3.15 through 2.3.18 below, let $\mu_0, \ldots, \mu_t$ be a sequence of matching such that $\mu_{s-1} \rightarrow_F \mu_s$ for all s in $[t]$, and for each i in $[n]$ let $P(i)$, (resp., $Q(i)$ and $R(i)$) denote the predicate that holds if there is an integer s in $[t]$ such that $\mu_s(\hat{x}_i^1) = v_{p_i^1}$ (resp., $\mu_s(\hat{x}_i^2) = v_{p_i^2}$, $\mu_s(\hat{x}_i^2) = v_{n_i}$). Lemmas 2.3.15 and 2.3.16 below present useful properties of these predicates.

Lemma 2.3.15. Let i be an element of $[n]$ such that $R(i)$ holds. Then $Q(i)$ does not hold.

Proof. Let s be an element of $[t]$ such that $\mu_s(\hat{x}_i^2) = v_{n_i}$; such an s exists since $R(i)$ holds. Assume that $Q(i)$ holds. Let s' be an element of $[t]$ such that $\mu_{s'}(\hat{x}_i^2) = v_{p_i^2}$; such an s' exists as $Q(i)$ holds. We consider two cases.

Case 1: $s' > s$. Let s'' be an integer such that $s \leq s'' \leq s'$. The preferences of agent $\hat{x}_i^2$ imply that $p_i^2 < n_i$ and $\mu_{s''}(\hat{x}_i^2)$ belongs to $\{w_{p_i^2}, v_{p_i^2}, w_{n_i}, v_{n_i}\}$. Hence $\text{rank}(\mu_{s''}(\hat{x}_i^2))$ belongs to $\{3p_i^2, 3p_i^2-1, 3n_i, 3n_i-1\}$. Note that $\text{rank}(\mu_s(\hat{x}_i^2))$ $= 3n_i-1$, and $\text{rank}(\mu_{s'}(\hat{x}_i^2)) = 3p_i^2-1$. Hence there is an s''' such that $s \leq s''' < s'$ and $\text{rank}(\mu_{s'''+1}(\hat{x}_i^2)) \leq \text{rank}(\mu_{s'''}(\hat{x}_i^2)) - 2$. It follows that there is an agent a such that $\mu_{s'''}(a)$ belongs to $B \setminus X$ and $\text{rank}(\mu_{s'''+1}(a)) \geq \text{rank}(\mu_{s'''}(a)) + 2$, contradicting Lemma 2.3.11.

Case 2: $s' < s$. We can derive a contradiction using a similar argument as in Case 1. □

Lemma 2.3.16. Let i be an element of $[n]$ such that $R(i)$ holds. Then $P(i)$ does not hold.

Proof. Let s be an element of $[t]$ such that $\mu_s(\hat{x}_i^2) = v_{n_i}$; such an s exists since $R(i)$ holds. We begin by proving the following claim: There is an integer s'' in $[s-1]$ such that $\mu_{s''}(\hat{x}_i^2) = x_i^1$. Assume for the sake of contradiction that there is no s'' in $[s-1]$ such that $\mu_{s''}(\hat{x}_i^2) = x_i^1$. Let s'' be the least index in $[s]$ such that $\mu_{s''}(\hat{x}_i^2) \neq x_i^2$. Since $\mu_{s''}(\hat{x}_i^2)$ does not belong to $\{x_i^1, x_i^2\}$, Lemma 2.3.14 implies that $\mu_{s''}(\hat{x}_i^2) = v_{p_i^2}$. Thus $Q(i)$ holds, contradicting Lemma 2.3.15. This completes the proof of the claim.

Having established the claim, we let s'' denote the least integer in $[s-1]$ such that $\mu_{s''}(\hat{x}_i^2) = x_i^1$. The preferences of agent $\hat{x}_i^2$ imply that $\mu_{s''-1}(\hat{x}_i^2) = x_i^2$. Let a be the agent $\mu_{s''-1}^{-1}(x_i^1)$. Since $a \neq \hat{x}_i^2$, Observation 2.3.1 implies that a belongs to $\{\hat{x}_i^1, \hat{v}_{p_i^1}, \hat{v}_{n_i}\}$. We consider two cases.

Case 1: $a \in \{\hat{v}_{p_i^1}, \hat{v}_{n_i}\}$. Lemma 2.3.13 implies that a does not prefer x_i^2 to their initially endowment. Hence $\mu_{s''}^{-1}(x_i^2) \neq a$, a contradiction.

Case 2: $a = \hat{x}_i^1$. Since $\mu_0(\hat{x}_i^1) = \mu_{s''-1}(\hat{x}_i^1) = x_i^1$, we deduce that $\mu_s(\hat{x}_i^1) = x_i^1 \neq v_{p_i^1}$ for all s' such that $0 \leq s' < s''$. Moreover, $\hat{x}_i^1$ is satisfied in $\mu_{s''}$ and hence $\mu_{s'}(\hat{x}_i^1) = x_i^2 \neq v_{p_i^1}$ for all s' such that $s'' \leq s' \leq t$. Hence $\mu_{s'}(\hat{x}_i^1) \neq v_{p_i^1}$ for all s' such that $0 \leq s' \leq t$. Thus $P(i)$ does not hold. □

Lemma 2.3.17. Let j belong to $[m]$ and assume that $\mu_t(\hat{w}_m) = u_1$. Then there is an s in $[t-1]$ such that $\mu_s(\hat{w}_m) = w_j$ and $\mu_{s+1}(\hat{w}_m) = v_j$.

Proof. The only object with rank $3j$ (resp., $3j-1$) is w_j (resp., v_j). Since $\text{rank}(\mu_0(\hat{w}_m)) = 3m$ and $\text{rank}(\mu_t(\hat{w}_m)) = 1$, Lemma 2.3.12 implies that for every rank k in $[3m-1]$, there is an integer s in $[t-1]$ such that $\text{rank}(\mu_s(\hat{w}_m)) =$

$k+1$ and $\text{rank}(\mu_{s+1}(\hat{w}_m)) = k$. The lemma follows by choosing k to be $3j-1$. □

Lemma 2.3.18. Assume that $\mu_t(\hat{w}_m) = u_1$. Then the 2P1N-SAT instance f is satisfiable.

Proof. We construct an assignment $\sigma : \{x_1, \ldots, x_n\} \to \{0,1\}$ for f as follows: for any i in $[n]$, we set $\sigma(x_i)$ to 1 if $P(i) \vee Q(i)$ holds, and to 0 otherwise.

We now show that σ satisfies f. Let j belong to $[m]$. Let s be an element of $[t-1]$ such that $\mu_s(\hat{w}_m) = w_j$ and $\mu_{s+1}(\hat{w}_m) = v_j$; such an s exists by Lemma 2.3.17. Thus there is an agent a such that $\mu_s(a) = v_j$ and $\mu_{s+1}(a) = w_j$. Since $a \neq \hat{w}_m$, Observation 2.3.3 implies that exactly one of the following three statements holds: (1) $j = p_i^1$ and $\mu_s(\hat{x}_i^1) = v_j$; (2) $j = p_i^2$ and $\mu_s(\hat{x}_i^2) = v_j$; (3) $j = n_i$ and $\mu_s(\hat{x}_i^2) = v_j$. We consider two cases.

Case 1: (1) or (2) holds. Then $P(i)$ or $Q(i)$ holds, respectively. Hence $\sigma(x_i) = 1$. By construction, the variable x_i appears as the positive literal x_i in clause C_j. Thus, C_j is satisfied.

Case 2: (3) holds. Then $R(i)$ holds. Lemmas 2.3.15 and 2.3.16 imply that $P(i)$ and $Q(i)$ do not hold. Hence $\sigma(x_i) = 0$. By construction, the variable x_i appears as the negative literal $\neg x_i$ in clause C_j. Thus, C_j is satisfied.

Since σ satisfies each clause C_j in f, σ satisfies f. □

Theorem 2.3.19. Reachable object on cliques is NP-complete.

Proof. In Section 2.3.1, we described a polynomial-time reduction from 2P1N-SAT instance f to reachable object on cliques instance I. Thus the theorem follows from Lemmas 2.3.4 and 2.3.18. □

2.4 NP-Completeness for Reachable Matching on Cliques

We begin by proving in Lemma 2.4.1 that reachable matching on general graphs is NP-complete. We use Lemma 2.4.1 to establish that reachable matching on cliques is also NP-complete.

Lemma 2.4.1. Reachable matching on general graphs is NP-complete.

Proof. It is easy to see that reachable matching on general graphs belongs to NP. We use a reduction from reachable object on generalized stars to reachable matching on general graphs to establish that reachable matching on general graphs is NP-complete.

Fix an arbitrary reachable object on generalized stars instance I. Without loss of generality, we can assume that the associated configuration $\chi = (F, \mu)$ is such that $F = (A, B, \succ, E)$, $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and $\chi(a_i) = b_i$ for $1 \le i \le n$. We can also assume without loss of generality that our goal is to determine whether there is a matching μ_1 in reach(χ) such that $\mu_1(a_1) = b_n$.

Below we describe how to transform reachable object on generalized stars instance I into a reachable matching on general graphs instance I' such that I is a positive instance of reachable object on generalized stars if and only if I' is a positive instance of reachable matching on general graphs . The reachable matching on general graphs instance I' has two associated configurations $\chi' = (F', \mu')$ and $\chi'' = (F', \mu'')$, where $F' = (A', B', \succ', E')$. The set of agents A' is equal to $A \cup A^*$ where $A^* = \{a_1^*, \ldots, a_n^*\}$. The set of objects B' is equal to $B \cup B^*$ where $B^* = \{b_1^*, \ldots, b_n^*\}$. The perfect matching μ' from A' to B' satisfies $\mu'(a_i) = b_i$ and $\mu'(a_i^*) = b_i^*$ for $1 \leq i \leq n$. The subgraph of (B', E') induced by the set of objects B is equal to (B, E). The subgraph of (B', E') induced by the set of objects B^* is a clique. There are n edges connecting these two subgraphs: there is an edge from object b_i to object b_i^* for $1 \leq i \leq n$. The agent preferences $\succ'$ are defined as follows.

- For any integer i such that $1 \leq i \leq n$, the most preferred object of agent a_i^* is b_i, followed by object b_i^*, followed by the remaining objects in B' in arbitrary order.

- The most preferred object of agent a_1 is b_n^*, followed by the objects in B in the order specified by the preferences of agent a_1 under $\succ$, followed by the objects in $B^* - b_n^*$ in arbitrary order.

- The most preferred objects of agent a_n are $b_1^*, \ldots, b_{n-1}^*$, followed by the objects in B in the order specified by the preferences of agent a_n under $\succ$, followed by object b_n^*.

- For any integer i such that $1 < i < n$, the most preferred objects of a_i are $b_i^*, \ldots, b_{n-1}^*$, followed by $b_{i-1}^*, \ldots, b_1^*$, followed by the objects in B in the order specified by the preferences of agent a_i under $\succ$, followed by object b_n^*.

The perfect matching μ'' associated with configuration χ'' maps each agent in A' to its most preferred object in B'. (Note that μ'' is a perfect matching from A' to B', since no two agents in A' share the same most preferred object.)

It is easy to see that we can construct instance I' in polynomial time in the size of instance I. It remains to argue that instance I is a positive instance of reachable object on generalized stars if and only if I' is a positive instance of reachable matching on general graphs.

We begin by addressing the "only if" direction. Assume that instance I is a positive instance of reachable object on generalized stars. Thus there is a configuration χ_1 in reach(χ) such that $\chi_1(a_1) = b_n$. Our construction of the agent preferences therefore ensures the existence of a configuration χ_1' in reach(χ') such that $\chi_1'(a_i) = \chi_1(a_i)$ and $\chi_1'(a_i^*) = \chi(a_i^*) = b_i^*$ for $1 \leq i \leq n$.

It is easy to check that a swap across edge (b_i, b_i^*) can be applied to configuration χ_1' for $1 \leq i \leq n$. Let χ_2' denote the configuration obtained by applying these n swaps to χ_1'. Thus χ_2' belongs to reach(χ'). Furthermore, it is easy to check that each agent in $A^* + a_1$ is matched in χ_2' to its most preferred object under $\succ'$.

Next, we iteratively construct a sequence of $n-1$ configurations $\chi_3', \ldots, \chi_{n+1}'$ such that configuration χ_k' satisfies the following properties for $3 \leq k \leq n+1$: χ_k' belongs to reach(χ'); every agent in $A \cup \{a_1, \ldots, a_{k-2}\} + a_n$ is matched in χ'_k to its most preferred object in configuration χ'_k. We begin by applying zero or one swaps to configuration χ'_2 to obtain configuration χ'_3. If $\chi'_2(a_n) = b^*_1$, then we define χ'_3 as χ'_2. If not, then $\chi'_2(a_i) = b^*_1$ for some i in $\{2, \ldots, n-1\}$. The preferences of agents a_i and a_n ensure that a swap between these two agents can be applied to configuration χ'_2. We define χ'_3 as the configuration that results from applying this swap. It easy to see that configuration χ'_3 belongs to reach(χ') and that every agent in $A \cup \{a_1, a_n\}$ is matched in χ'_3 to its most preferred object under $\succ'$.

Now fix an integer k such that $4 \leq k \leq n+1$, and inductively assume that we have constructed a configuration χ'_{k-1} in reach(χ') such that every agent in $A \cup \{a_1, \ldots, a_{k-3}\} + a_n$ is matched to its most preferred object under $\succ'$. We apply zero or one swaps to configuration χ'_{k-1} to obtain configuration χ'_k. If $\chi'_{k-1}(a_{k-2}) = b^*_{k-2}$, then we define χ'_k as χ'_{k-1}. If not, then $\chi'_{k-1}(a_i) = b^*_{k-2}$ for some i in $\{k-1, \ldots, n-1\}$. The preferences of agents a_{k-2} and a_i ensure that a swap between these two agents can be applied to configuration χ'_{k-1}. We define χ'_k as the configuration that results from applying this swap. It easy to see that configuration χ'_k belongs to reach(χ') and that every agent in $A \cup \{a_1, \ldots, a_{k-2}\} + a_n$ is matched in χ'_k to its most preferred object under $\succ'$.

Since χ'_{n+1} belongs to reach(χ') and every agent in A' is matched in χ'_{n+1} to its most preferred object under $\succ'$, we conclude that $\chi'_{n+1} = \chi''$ and hence that I' is a positive instance of reachable matching on general graphs.

Now we address the "if" direction. Assume that instance I' is a positive instance of reachable matching on general graphs. Thus χ'' belongs to reach(χ'), and hence there is a sequence of swaps S that transforms configuration χ' into configuration χ''.

By examining the preferences of the agents in A^*, we deduce that each agent in A^* participates in exactly one swap in S, and that the other agent participating in each of these swaps belongs to A. By examining the preferences of the agents in A, we deduce that agent a_1 is the agent that swaps with agent a_n^*, and that once an agent in A becomes matched to an object in B^*, it remains matched to an object in B^* thereafter. It follows that there is a permutation π of $\{1, \ldots, n\}$ such that $\pi(n) = 1$ and a_i^* swaps with $a_{\pi(i)}$ for $1 \leq i \leq n$.

For any integer k such that $0 \leq k \leq |S|$, let χ_k' denote the configuration reached by applying the first k swaps of sequence S to configuration χ'. Thus $\chi' = \chi_0'$, $\chi'' = \chi_{|S|}'$, and χ_k' is of the form (F', μ_k') where μ_k' is a perfect matching from A' to B'.

For any integer k such that $0 \leq k \leq |S|$, we use the perfect matching μ_k' to construct a perfect matching μ_k from A to B, as follows: for each agent a_i^* in A^* such that $\mu_k'(a_i^*)$ belongs to B^*, we define $\mu_k(a_{\pi(i)})$ as $\mu_k'(a_{\pi(i)})$; for each agent a_i^* in A^* such that $\mu_k'(a_i^*)$ belongs to B, we define $\mu_k(a_{\pi(i)})$ as $\mu_k'(a_i^*)$.

For any integer k such that $0 \leq k \leq |S|$, we define χ_k as the configuration (F, μ_k). It is straightforward to prove by induction on k that χ_k belongs to reach(χ) for $0 \leq k \leq |S|$.

Let ℓ denote the least integer such that $\chi_\ell^{-1}(b_n) = a_n^*$. We know that

ℓ exists since $\chi_{|S|}^{-1}(b_n) = a_n^*$, and that ℓ is positive since $\chi_0^{-1}(b_n) = a_n$. As discussed earlier, agent a_1 is the only agent that participates in a swap with agent a_n^*. Hence $\chi_{\ell-1}^{-1}(b_n) = a_1$. Since $\chi_{\ell-1}$ belongs to reach(χ), we conclude that I is a positive instance of reachable object on generalized stars, as required. $\square$

It is easy to see that the reachable matching on cliques problem belongs to NP. We use a reduction from reachable object on cliques to reachable matching on cliques to establish that reachable matching on cliques is NP-complete. This reduction is similar to the one used in the proof of Lemma 2.4.1.

Fix an arbitrary reachable object on cliques instance I. Without loss of generality, we can assume that the associated configuration $\chi = (F, \mu)$ is such that $F = (A, B, \succ, E)$, $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and $\chi(a_i) = b_i$ for $1 \leq i \leq n$. We can also assume without loss of generality that our goal is to determine whether there is a matching μ_1 in reach(χ) such that $\mu_1(a_1) = b_n$.

We now describe how to transform reachable object on cliques instance I into a reachable matching on cliques instance I'. Instance I' has two associated configurations $\chi' = (F', \mu')$ and $\chi'' = (F', \mu'')$, where $F' = (A', B', \succ', E')$. The set of agents A' is equal to $A \cup A^*$ where $A^* = \{a_1^*, \ldots, a_n^*\}$. The set of objects B' is equal to $B \cup B^*$ where $B^* = \{b_1^*, \ldots, b_n^*\}$. Let K_{2n} denote the complete graph with vertex set B', and let E' denote the edge set of K_{2n}. The agent preferences $\succ'$ and the matchings μ' and μ'' are as described in the proof of Lemma 2.4.1.

Let $\hat{E}$ denote the union of three sets of edges: $\{(b_i, b_j) \mid i, j \in [n] \wedge i \neq$

$j\}$; $\{(b_i, b_i^*) \mid i \in [n]\}$; $\{(b_i^*, b_j^*) \mid i, j \in [n] \wedge i \neq j\}$. Lemma 2.4.2 below establishes that if a swap occurs on an edge e in I', then e belongs to $\hat{E}$.

Lemma 2.4.2. Let i and j be elements of $[n]$ such that $i \neq j$. Let μ_1 and μ_2 be matchings in reach(χ') such that $\mu_1 \rightarrow_F \mu_2$. Then $\mu_2^{-1}(b_i) \neq \mu_1^{-1}(b_j^*)$.

Proof. We consider two cases.

Case 1: $\mu_1^{-1}(b_j^*)$ belongs to A^*. By examining the preferences of agents in A^*, we deduce that $\mu_1^{-1}(b_j^*) = a_j^*$. The only object that agent a_j^* prefers to b_j^* is b_j. Hence $\mu_2^{-1}(b_i) \neq a_j^* = \mu_1^{-1}(b_j^*)$.

Case 2: $\mu_1^{-1}(b_j^*)$ belongs to A. By examining the preferences of agents in A, we deduce that $\mu_2(\mu_1^{-1}(b_j^*))$ belongs to B^*. Hence $\mu_2^{-1}(b_i) \neq \mu_1^{-1}(b_j^*)$. □

Using Lemma 2.4.2, along with the same reasoning as in the proof of Lemma 2.4.1, we deduce that I' is a positive instance of reachable matching on cliques if and only if I is a positive instance of reachable object on cliques. Thus Theorem 2.4.3 below holds.

Theorem 2.4.3. Reachable matching on cliques is NP-complete.

2.5 NP-Hardness for Pareto-Efficiency on Cliques

Theorem 2.5.1. Pareto-efficient matching on cliques is NP-hard.

Proof. We use the same reduction as we used in Section 2.4 to establish the NP-completeness of reachable matching on cliques. In analyzing that reduction, we proved that a given instance of reachable object on cliques is positive if and only if every agent gets its most preferred object in the corresponding instance of reachable matching on cliques. Therefore an efficient algorithm for computing a Pareto-efficient matching on cliques yields an efficient algorithm for reachable object on cliques. Since reachable object on cliques is NP-complete, we deduce that Pareto-efficient matching on cliques is NP-hard. □

2.6 Concluding Remarks

In this chapter, we have introduced the agent-moving model, and we have revisited the collection of problems listed in Table 2.1, which were previously considered in the context of the object-moving model. In all cases where a polynomial-time algorithm or hardness result has been established for the object-moving model, we have established a corresponding result for the agent-moving model.

In addition, we have presented a polynomial-time algorithm for Pareto Efficiency on generalized stars in the agent-moving model, a problem that remains open in the object-moving model. It is natural to ask whether our techniques can be extended to address this open problem. Our algorithm relies on the polynomial-time solvability of the Reachable Object problem for the center agent, which allows us to compute an object that is matched to the center agent in some Pareto-efficient matching. In the object-moving model, no polynomial-time algorithm is known to compute an agent that is matched to the center object in some Pareto-efficient matching. (We do know how to compute the agents that can be reached by the center object in polynomial time, but it isn't clear how to use this information to compute a Pareto-efficient matching in polynomial time.) An interesting direction for future research in the agent-moving model is to determine whether our techniques for solving Pareto Efficiency on generalized stars can be extended to trees. It would also be interesting to study strategic aspects of this model.

Chapter 3

The Obnoxious Facility Location Game with Dichotomous Preferences

3.1 Introduction

The facility location game (FLG) was introduced by Procaccia and Tannenholtz [48]. In this setting, a central planner wants to build a facility that serves agents located on a path. The agents report their locations, which are fed to a mechanism that decides where the facility should be built. Procaccia and Tannenholtz studied two different objectives that the planner seeks to minimize: the sum of the distances from the facility to all agents and the maximum distance of any agent to the facility.

Every agent aims to maximize their welfare, which increases as their distance to the facility decreases. An agent or a coalition of agents can misreport their location(s) to try to increase their welfare. So the goal is to design strategyproof (SP) or group-strategyproof (GSP) mechanisms that incentivize truthful reporting. Often such mechanisms cannot simultaneously optimize the planner's objective. In these cases, the mechanisms should approximately optimize the planner's objective.

In real scenarios, an agent might dislike a certain facility, such as a power plant, and want to stay away from it. This variant, called the obnoxious facility location game (OFLG), was introduced by Cheng et al., who studied the problem of building an obnoxious facility on a path [17]. In this chapter, we consider the problem of building multiple obnoxious facilities on a path. With multiple facilities, there are different ways to define the welfare function. For example, in the case of two facilities, the welfare of the agent can be the sum, minimum, or maximum of the distance to the two facilities. In our work, as all the facilities are obnoxious, a natural choice for welfare is the minimum distance to any obnoxious facility: the closest facility to an agent causes them the most annoyance, and if it is far away, then the agent is satisfied.

A facility might not be universally obnoxious. Consider, for example, a school or sports stadium. An agent with no children might consider a school to be obnoxious due to the associated noise and traffic, while an agent with children might not consider it to be obnoxious. Another agent who is not interested in sports might similarly consider a stadium to be obnoxious. We assume that each agent has dichotomous preferences; they dislike some subset of the facilities and are indifferent about the others. Each agent reports a subset of facilities to the planner. As the dislikes are private information, the reported subset might not be the subset of facilities that the agent truly dislikes. On the other hand, we assume that the agent locations are public and cannot be misreported.

In this chapter, we study a variant of FLG, which we call DOFLG (Dichotomous Obnoxious Facility Location Game), that combines the three aspects mentioned above: multiple (heterogeneous) obnoxious facilities, minimum distance as welfare, and dichotomous preferences. We seek to design mechanisms that perform well with respect to either a utilitarian or egalitarian objective. The utilitarian objective is to maximize the social welfare, that is, the total welfare of all agents. A mechanism that maximizes social welfare is said to be efficient. The egalitarian objective is to maximize the minimum welfare of any agent. For both objectives, we seek mechanisms that are SP, or better yet, weakly or strongly group-strategyproof (WGSP / SGSP).

3.1.1 Our contributions

We study DOFLG with n agents. In Section 3.5, we consider the utilitarian objective. We present 2-approximate SGSP mechanisms for any number of facilities when the agents are located on a path, cycle, or square. We obtain the following two additional results for the path setting. In the first main result of the chapter, we obtain a mechanism that is WGSP for any number of facilities and efficient for up to three facilities. To show that this mechanism is WGSP, we relate it to a weighted approval voting mechanism. To prove its efficiency, we identify two crucial properties that the welfare function satisfies, and we use an exchange argument. In our second result for the path setting, we show that no SGSP mechanism can achieve an approximation ratio better than 5/4, even for one facility.

In Section 3.6, we consider the egalitarian objective. We provide SP mechanisms for any number of facilities when the agents are located on a path,

<table>
<tr><th></th><th colspan="2">Utilitarian</th><th colspan="2">Egalitarian</th></tr>
<tr><th></th><th>LB</th><th>UB</th><th>LB</th><th>UB</th></tr>
<tr><td>SP</td><td rowspan="2">1</td><td rowspan="2">1 for $k \leq 3$</td><td>1</td><td>1</td></tr>
<tr><td>WGSP</td><td rowspan="2">$\Omega(\sqrt{n})$</td><td rowspan="2">$O(n)$</td></tr>
<tr><td>SGSP</td><td>5/4</td><td>2</td></tr>
</table>

Table 3.1: Summary of results for DOFLG when the agents are located on a path.

cycle, or square. In the second main result of the chapter, we prove that the approximation ratio achieved by any WGSP mechanism is $\Omega(\sqrt{n})$, even for two facilities. Also, we present a straightforward $O(n)$-approximate WGSP mechanism. Both of the results for WGSP mechanisms hold for DOFLG when the agents are located on a path or cycle.

Table 3.1 summarizes our results. The heading LB (resp., UB) in Table 3.1 stands for lower (resp., upper) bound. The results in the egalitarian column in the table also hold when the agents are located on a cycle. Boldface results in the table hold when the agents are located on a path, cycle, or square.

3.2 Related Work

FLG was introduced by Procaccia and Tannenholtz [48]. Many generalizations and extensions of FLG have been studied [43, 27, 28, 4, 21, 25, 26, 61]. Below, we highlight the works that are most relevant to the present paper.

Cheng *et al.* introduced OFLG and presented a WGSP mechanism to build a single facility on a path [17]. Later they extended the model to cycles and trees [18]. A complete characterization of single-facility SP/WGSP mechanisms for paths has been developed [32, 35]. Duan *et al.* studied the problem of locating two obnoxious facilities at least distance d apart [22]. Other variants of this game have been considered [16, 58, 46, 29].

Agent preferences over the facilities were introduced to FLG in the works of Feigenbaum and Sethuraman [24], and Zou and Li [62]. Serafino and Ventre studied FLG for building two facilities where each agent likes a subset of the facilities [54]. Anastasiadis and Deligkas extended this model to allow the agents to like, dislike, or be indifferent to the facilities [5]. The aforementioned works address linear (sum) welfare function. Yuan *et al.* studied non-linear welfare function (max and min) for building two non-obnoxious facilities [60]; their results have subsequently been strengthened [42, 15]. In the present chapter, we initiate the study of a non-linear welfare function (min) for building multiple obnoxious facilities.

3.3 Preliminaries

There are n agents and a planner. In this chapter, we consider three settings depending on whether the agents are located on a path, cycle, or square. In the path (resp., cycle, square) setting, we assume without loss of generality that the path (resp., cycle, square) is the unit interval (resp., unit-circumference circle, unit square). We map the points on the unit-circumference circle to $[0, 1)$ in the natural manner. Thus, in the path (resp., cycle, square) setting, each agent i is located at x_i in $[0, 1]$ (resp., $[0, 1)$, $[0, 1]^2$). The distance between any two points x and y is denoted $d(x, y)$. In the path and square settings, $d(x, y)$ is defined as the Euclidean distance between x and y. In the cycle setting, $d(x, y)$, is defined as the length of the shorter arc between x and y. In all settings, the vector $\mathbf{x} = (x_1, \ldots, x_n)$ of agent locations is called the *location profile*, and is known to all agents and the planner.

The planner seeks to build k facilities $F_1, \ldots, F_k$ on the path, cycle, or square. We define an *aversion profile* as a vector $(v_1, \ldots, v_n)$ where each v_i is a set of facilities. Let a_i be the set of the facilities that agent i dislikes. The true aversion profile $\mathbf{a} = (a_1, \ldots, a_n)$ is the vector of the facilities that agents dislike.

For given $\mathbf{x}$ and $\mathbf{a}$, the welfare of an agent i when $F_1, \ldots, F_k$ are built at $\mathbf{y} = (y_1, \ldots, y_k)$ is denoted by $w_{\mathbf{x},\mathbf{a}}(i, \mathbf{y})$. If a_i is not empty, we define $w_{\mathbf{x},\mathbf{a}}(i, \mathbf{y})$ as

$$\min_{j:F_j \in a_i} d(x_i, y_j),$$

i.e., the minimum distance from x_i to any facility in a_i. If a_i is empty, we define $w_{\mathbf{x},\mathbf{a}}(i, \mathbf{y})$ as $1/2$ in the cycle setting, $\max(d(x_i, 0), d(x_i, 1))$ in the path setting, and the maximum distance from x_i to a corner in the square setting. (In all settings, this is the distance to the most distant location that agent i can reach without encountering a facility in a_i.)

The *social welfare* (resp., *minimum welfare*) is the sum (resp., minimum) of the individual welfare of agents. When the facilities are built at $\mathbf{y}$, the social welfare and minimum welfare are denoted by $\mathrm{SW}_{\mathbf{x},\mathbf{a}}(\mathbf{y})$ and $\mathrm{MW}_{\mathbf{x},\mathbf{a}}(\mathbf{y})$ respectively. Thus,

$$\mathrm{SW}_{\mathbf{x},\mathbf{a}}(\mathbf{y}) = \sum_{i \in [n]} w_{\mathbf{x},\mathbf{a}}(i, \mathbf{y}),$$
$$\mathrm{MW}_{\mathbf{x},\mathbf{a}}(\mathbf{y}) = \min_{i \in [n]} w_{\mathbf{x},\mathbf{a}}(i, \mathbf{y}).$$

Each agent reports to dislike a set of facilities which is denoted by a'_i. Note that a'_i need not be the same as a_i, that is, the agents can lie (misreport) about the facilities that they dislike. The reported aversion profile is denoted by $\mathbf{a}' = (a'_1, \ldots, a'_n)$. In the dichotomous obnoxious facility location game (DOFLG), the goal is to design a mechanism M that takes as input the number of facilities to build k, the location profile $\mathbf{x}$, and the reported aversion profile $\mathbf{a}'$ and outputs the locations $\mathbf{y} = (y_1, \ldots, y_k)$ where the facilities $F_1, \ldots, F_k$ should be built respectively. To help in our analysis, we define $Z_j = \{i \in [n] \mid F_j \in a'_i\}$ as the set of agents who report to dislike F_j (and possibly some other facilities) and $U_\emptyset = \{i \in [n] \mid a'_i = \emptyset\}$ as the set of agents who report to dislike no facilities. Lastly, we define a special case of DOFLG with $k = 1$ which we refer to as *single-facility* DOFLG.

In this paper, we study five properties that a mechanism can satisfy: SP, WGSP, SGSP, efficiency, and egalitarianism. The SP property says that no agent can increase their welfare by lying about their dislike. Let $\mathbf{a}_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ denote the true aversion profile of all agents except agent i and $(\mathbf{a}_{-i}, a'_i)$ denote the aversion profile where agent i reports a'_i and all of the other agents report truthfully.

Definition 3.3.1. A mechanism M is SP if for all $k, \mathbf{x}, \mathbf{a}, i$ and $\mathbf{a}' = (\mathbf{a}_{-i}, a'_i)$,

$$w_{\mathbf{x},\mathbf{a}}(i, M(k, \mathbf{x}, \mathbf{a})) \geq w_{\mathbf{x},\mathbf{a}}(i, M(k, \mathbf{x}, \mathbf{a}')).$$

The WGSP property says that if a coalition $C \subseteq [n]$ of agents lies then every agent in the coalition cannot increase their welfare. Let $\mathbf{a}_{-C}$ denote the true aversion profile of all agents except the agents in C and let $\mathbf{a}'_C$ denote the reported aversion profile of the agents in C. Define $(\mathbf{a}_{-C}, \mathbf{a}'_C)$ to be the aversion profile where agents in C report $\mathbf{a}'_C$ and all the other agents report truthfully.

Definition 3.3.2. A mechanism M is WGSP if for any $k, \mathbf{x}$, and $\mathbf{a}$ there is no coalition $C \subseteq [n]$ of agents and $\mathbf{a}' = (\mathbf{a}_{-C}, \mathbf{a}'_C)$ such that for all i in C,

$$w_{\mathbf{x},\mathbf{a}}(i, M(k, \mathbf{x}, \mathbf{a})) < w_{\mathbf{x},\mathbf{a}}(i, M(k, \mathbf{x}, \mathbf{a}')).$$

The SGSP property says that if a coalition of agents lies then no agent in the coalition can increase their welfare while the other agents of the coalition have welfare at least as much as when all agents reported truthfully.

Definition 3.3.3. A mechanism M is SGSP if for any $k, \mathbf{x}$, and $\mathbf{a}$ there is no coalition $C \subseteq [n]$ of agents and $\mathbf{a}' = (\mathbf{a}_{-C}, \mathbf{a}'_C)$ such that

$$w_{\mathbf{x},\mathbf{a}}(i, M(k, \mathbf{x}, \mathbf{a})) < w_{\mathbf{x},\mathbf{a}}(i, M(k, \mathbf{x}, \mathbf{a}'))$$

for some agent i in C and

$$w_{\mathbf{x},\mathbf{a}}(i', M(k, \mathbf{x}, \mathbf{a})) \leq w_{\mathbf{x},\mathbf{a}}(i', M(k, \mathbf{x}, \mathbf{a}'))$$

for all agents i' in $C - i$.

Every SGSP mechanism is WGSP and every WGSP mechanism is SP. We now define efficient and egalitarian mechanism. These are defined relative to the reported aversion profile as the true aversion profile is unknown to the mechanism.

Definition 3.3.4. A mechanism M is efficient if for all $k, \mathbf{x}$, and $\mathbf{a}'$,

$$\max_{\mathbf{y}} \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y}) = \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(M(k, \mathbf{x}, \mathbf{a}')).$$

Definition 3.3.5. A mechanism M is egalitarian if for all $k, \mathbf{x}$, and $\mathbf{a}'$,

$$\max_{\mathbf{y}} \mathrm{MW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y}) = \mathrm{MW}_{\mathbf{x},\mathbf{a}'}(M(k, \mathbf{x}, \mathbf{a}')).$$

A mechanism that is efficient or egalitarian might not be SP, WGSP, or SGSP. Hence to measure the performance of a mechanism we define the approximation ratio α.

Definition 3.3.6. For $\alpha \geq 1$, a mechanism M is α-efficient if for all $k, \mathbf{x}$, and $\mathbf{a}'$, $\max_{\mathbf{y}} \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y}) \leq \alpha \, \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(M(k, \mathbf{x}, \mathbf{a}'))$. Similarly M is α-egalitarian if for all $k, \mathbf{x}$, and $\mathbf{a}'$, $\max_{\mathbf{y}} \mathrm{MW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y}) \leq \alpha \, \mathrm{MW}_{\mathbf{x},\mathbf{a}'}(M(k, \mathbf{x}, \mathbf{a}'))$.

As agents can lie, there is not much value in an efficient or egalitarian mechanism that is not SP. Thus we seek to design SP (or, better yet, WGSP or SGSP) mechanisms that are α-efficient or α-egalitarian.

3.4 Weighted Approval Voting

Before studying efficient mechanisms we study a variant of the Approval Voting mechanism [14]. This voting mechanism is an abstraction of our efficient mechanism.

Consider a scenario where m voters $1, \ldots, m$ have to elect a candidate among the set of candidates $C = \{c_1, \ldots, c_n\}$. Each voter has dichotomous preferences. Voter i prefers some candidates denoted by the set C_i and does not prefer the rest who are denoted by $\overline{C_i} = C \setminus C_i$ and is indifferent among any candidate within C_i (and similarly within $\overline{C_i}$). Each voter i is associated with weights $w_i^+ > w_i^- \geq 0$. We now present our weighted approval voting mechanism[1].

Mechanism 1. Every voter i votes by partitioning C into C_i' and $\overline{C_i'}$. Let the weight function w be such that for voter i and candidate c_j, $w(i, j) = w_i^+$ if c_j is in C_i' and $w(i, j) = w_i^-$ otherwise. For each candidate c_j, compute *approval* $A(j) = \sum_{i \in [m]} w(i, j)$. The candidate j with highest approval $A(j)$ is declared the winner. In case of a tie, candidates are chosen in a pre-decided order which is without loss of generality: $c_1, \ldots, c_n$.

Note that the standard Approval Voting mechanism can be obtained from our mechanism by setting weights $w_i^+ = 1$ and $w_i^- = 0$ for each voter i.

We extend the notion of SP, WGSP, and SGSP for the voting setting, similar in spirit to Section 3.3. Suppose, when all voters votes truthfully the winner c is in $\overline{C_i}$. A mechanism is SP if by lying voter i (while the other voters vote truthfully) cannot get a candidate c' from C_i elected.

[1]Our mechanism differs from the homonymous mechanism by Massó *et al.* who have weights for candidates instead of voters [57].

Theorem 3.4.1. Mechanism 1 is SP.

Proof. A lie can benefit a voter only if on voting their true preference $C_i, \overline{C_i}$, the winning candidate c is in $\overline{C_i}$. Suppose that they lie and vote $C_i', \overline{C_i'}$. For any candidate d, let $A(d)$ denote the approval when voter i reported truthfully and $A'(d)$ when they lied. For d in $C_i \setminus C_i'$ we have $A'(d) = A(d) - w_i^+ + w_i^- < A(d)$ and for d in $C_i \cap C_i'$ we have $A'(d) = A(d) - w_i^+ + w_i^+ = A(d)$. Thus for d in C_i, $A'(d) \leq A(d)$. For c, if c is in $\overline{C_i} \cap \overline{C_i'}$ we have $A'(c) = A(c) - w_i^- + w_i^- = A(c)$ and if c is in $\overline{C_i} \setminus \overline{C_i'}$, we have $A'(c) = A(c) - w_i^- + w_i^+ > A(c)$. Thus we have $A'(c) \geq A(c)$. As c won the election when voter i reported truthfully, for all candidates d in C_i either (1) $A(d) < A(c)$ or (2) $A(d) = A(c)$ and c is higher than d in the tie-breaking order. As after lying we have $A'(c) \geq A(c)$ and $A'(d) \leq A(d)$, no d in C_i wins the election and voter i does not benefit. □

Now we extend the definition of WGSP to this setting. Let the winner c be in $\overline{C_i}$ for all i in a coalition $V \subseteq [m]$ when all the voters vote truthfully. Then a mechanism is WGSP if by lying V (while the other voters vote truthfully) cannot get c' elected where c' is in C_i for all i in V. Like Approval Voting, weighted approval voting is WGSP as shown in Theorem 3.4.2 below.

Theorem 3.4.2. Mechanism 1 is WGSP.

Proof. The proof is similar to that of Theorem 3.4.1. For any candidate j let $A(j)$ denote the approval when V reported truthfully and $A'(j)$ when they lied. Consider any d in $\bigcap_{i \in V} C_i$. Denote the lied vote of voter i from V by

$C_i', \overline{C_i'}$. Then,

$$A'(d) = A(d) + \sum_{i \in V, d \in C_i'} (w_i^+ - w_i^+) + \sum_{i \in V, d \in \overline{C_i'}} (w_i^- - w_i^+).$$

As $w_i^+ > w_i^-$, we have $A'(d) \leq A(d)$. Assume that candidate c had won the election when V had voted truthfully. For all voters in V to benefit, c is in $\bigcap_{i \in V} \overline{C_i}$. We have,

$$A'(c) = A(c) + \sum_{i \in V, c \in C_i'} (w_i^+ - w_i^-) + \sum_{i \in V, c \in \overline{C_i'}} (w_i^- - w_i^-).$$

Thus we get that $A'(c) \geq A(c)$. Now using the same arguments as in proof of Theorem 3.4.1 we get that d will not be elected. □

Lastly, we consider the extension of SGSP to this setting. Let $V \subseteq [m]$ be some coalition of voters. Let the winner be c when all of the voters vote truthfully. The mechanism is SGSP if when all the voters in V lie (while others vote truthfully), they cannot get c' elected where c' is in C_i and c is in $\overline{C_i}$ for at least one i in V (voter i gets benefited) and for the other voters i' in V, if c is in $C_{i'}$ then c' is also in $C_{i'}$ (no other voter is worse off).

Theorem 3.4.3. Mechanism 1 is not SGSP.

The above theorem can be shown by adapting the instance shown in Section 3.5 to prove that Mechanism 2 is not SGSP.

3.5 Efficient Mechanisms

We now present our efficient mechanism for DOFLG.

Mechanism 2. Compute social welfare $\mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y})$ for all $\mathbf{y}$ in $\{0,1\}^k$. Output $\mathbf{y}$ for which the social welfare is the highest. In case of a tie, use the lexicographic order over all $\{0,1\}^k$ as the tie-breaking order.

We show that Mechanism 2 is WGSP for any k by reducing it to Mechanism 1.

Theorem 3.5.1. Mechanism 2 is WGSP.

Proof. To show this theorem, we reduce Mechanism 2 to the weighted approval voting mechanism. Then Theorem 3.4.2 implies that Mechanism 2 is WGSP.

Any agent i such that $x_i = 1/2$ cannot influence where the facilities are built and so we assume that they do not exist for the rest of the proof. First, we view each agent $i \in [n]$ as a voter i. Every $\mathbf{y}$ in $\{0,1\}^k$ is considered a candidate. We obtain the preferred candidates of voter i, C_i and their voted candidates C'_i, from a_i and a'_i respectively. Assume that $x_i < 1/2$ (the other agents can be converted similarly). Set $C_i = \{\mathbf{y} = (y_1, \ldots, y_k) \in \{0,1\}^k \mid y_j = 1 \text{ for all } F_j \in a_i\}$ and similarly $C'_i = \{\mathbf{y} = (y_1, \ldots, y_k) \in \{0,1\}^k \mid y_j = 1 \text{ for all } F_j \in a'_i\}$. Also set $w_i^+ = 1 - x_i$ and $w_i^- = x_i$. With this notation, it is easy to see that $A(\mathbf{y}) = \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y})$. Then choosing the $\mathbf{y}$ with the highest social welfare in Mechanism 2 is the same as electing the candidate with the highest approval in Mechanism 1. This completes the reduction. □

We show that Mechanism 2 is efficient for $k = 3$. First, we note a well-known result about the 1-Maxian problem. In this problem, there are n

agents located at $z_1, \ldots, z_n$ in the interval $[a, b]$ and the task is to choose a point in the interval such that the sum of the distances from that point to all the agent locations is maximized.

Lemma 3.5.2 (Optimality of the 1-Maxian Problem). Let $f(z) = \sum_{i\in[n]} |z - z_i|$, where $z_1, \ldots, z_n$ are n arbitrary real numbers in the interval $[a, b]$. Then, either $f(a) = \max_{z\in[a,b]} f(z)$ or $f(b) = \max_{z\in[a,b]} f(z)$.

Before proving the main theorem, we establish an important lemma which also follows from Lemma 3.5.2.

Lemma 3.5.3. For any k, let there be some y in $[0, 1]$ such that it is efficient to build the facilities at $(y, \ldots, y)$. Then there is at least one choice of locations among $(0, \ldots, 0)$ and $(1, \ldots, 1)$ which is also efficient.

Proof. When all the facilities are built at y, then we have,

$$\mathrm{SW}_{\mathbf{x},\mathbf{a}'}(y, \ldots, y) = \sum_{i\in[n]\setminus U_\emptyset} |x_i - y| + \sum_{i\in U_\emptyset} w_{\mathbf{x},\mathbf{a}'}(x_i, y).$$

Since $\sum_{i\in U_\emptyset} w_{\mathbf{x},\mathbf{a}'}(x_i, y)$ does not depend on y, from Lemma 3.5.2 we deduce that $\max(\mathrm{SW}_{\mathbf{x},\mathbf{a}'}(0, \ldots, 0), \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(1, \ldots, 1)) = \max_z \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(z, \ldots, z) = \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(y, \ldots, y)$. Thus if $\mathrm{SW}_{\mathbf{x},\mathbf{a}'}(0, \ldots, 0) \geq \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(1, \ldots, 1)$ then it is efficient to build all the facilities at 0 and in the other case it is efficient to build all of them at 1. □

Theorem 3.5.4. Mechanism 2 is efficient for $k = 3$.

Proof. We begin by proving the following claim: There is a solution $\mathbf{y}^* = (y_1^*, y_2^*, y_3^*)$ that optimizes $\max_{\mathbf{y}} \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y})$ and builds facilities in at most two locations.

Let $\mathbf{y}^* = (y_1^*, y_2^*, y_3^*)$ be an optimal solution such that $y_1^* \leq y_2^* \leq y_3^*$. We establish the claim by proving that either $\mathrm{SW}_{\mathbf{x},\mathbf{a}'}(y_1^*, y_1^*, y_3^*) \geq \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y}^*)$ or $\mathrm{SW}_{\mathbf{x},\mathbf{a}'}(y_1^*, y_3^*, y_3^*) \geq \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y}^*)$.

Consider fixing variables y_1 and y_3 in the social welfare function $\mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y})$. That is, we have

$$\mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y})|_{y_1=y_1^*, y_3=y_3^*} = \sum_{i \in [n]} w_{\mathbf{x},\mathbf{a}'}(i, \mathbf{y})|_{y_1=y_1^*, y_3=y_3^*}.$$

For convenience, let

$$\mathrm{SW}(y_2) = \mathrm{SW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y})|_{y_1=y_1^*, y_3=y_3^*}$$

and let

$$w_i(y_2) = w_{\mathbf{x},\mathbf{a}'}(i, \mathbf{y})|_{y_1=y_1^*, y_3=y_3^*}$$

for each agent i. We start by proving that for each agent i, the welfare function $w_i(y_2)$ with $y_2 \in [y_1^*, y_3^*]$ satisfies at least one of the following two properties:

1. $w_i(y_2) = |y_2 - x_i|$;
2. $w_i(y_1^*) = w_i(y_3^*) = \max_{y \in [y_1^*, y_3^*]} w_i(y)$.

Consider an agent i such that $a'_i = \{F_1, F_2\}$. By definition, we have $w_i(y_2) = \min(|y_1^* - x_i|, |y_2 - x_i|)$. Notice that $w_i(y_1^*) = \min(|y_1^* - x_i|, |y_1^* - x_i|) =$

$|y_1^* - x_i| = \max_{y \in [y_1^*, y_3^*]} w_i(y)$. Moreover, $w_i(y_3^*) = \min(|y_1^* - x_i|, |y_3^* - x_i|)$. We consider two cases.

Case 1: $|y_1^* - x_i| > |y_3^* - x_i|$. Then $w_i(y_3^*) = |y_3^* - x_i|$ and hence $w_i(y_2) = |y_2 - x_i|$ for all y_2 in $[y_1^*, y_3^*]$, that is, $w_i(y_2)$ satisfies property 1.

Case 2: $|y_1^* - x_i| \leq |y_3^* - x_i|$. Then $w_i(y_3^*) = |y_1^* - x_i| = \max_{y \in [y_1^*, y_3^*]} w_i(y) = w_i(y_1^*)$ and hence $w_i(y_2)$ satisfies property 2.

Thus, we conclude that for an agent i with $a_i' = \{F_1, F_2\}$, the function $w_i(y_2)$ with $y_2 \in [y_1^*, y_3^*]$ satisfies either property 1 or 2. For any agent i with another $a_i' \subseteq \{F_1, F_2, F_3\}$, a similar method can be used to deduce that $w_i(y_2)$ with $y_2 \in [y_1^*, y_3^*]$ satisfies either property 1 or property 2.

Therefore, the set of agents $[n]$ can be partitioned into two sets $(S, \overline{S})$ such that $w_i(y_2)$ satisfies property 1 for all i in S, and $w_i(y_2)$ satisfies property 2 for all i in $\overline{S}$. Thus, we have $\mathrm{SW}(y_2) = \sum_{i \in [n]} w_i(y_2) = \sum_{i \in S} w_i(y_2) + \sum_{i \in \overline{S}} w_i(y_2)$. By Lemma 3.5.2, there is a b in $\{y_1^*, y_3^*\}$ such that $\sum_{i \in S} w_i(b) \geq \sum_{i \in S} w_i(y_2)$ for all y_2 in $[y_1^*, y_3^*]$. For any i in $\overline{S}$, we deduce from property 2 that $w_i(b) \geq w_i(y_2)$ for all y_2 in $[y_1^*, y_3^*]$. Therefore, $\mathrm{SW}(b) \geq \mathrm{SW}(y_2)$ for all y_2 in $[y_1^*, y_3^*]$. This completes the proof of the claim.

Having established the claim, we can assume without loss of generality that there is an optimal solution (y_1^*, y_2^*, y_3^*) such that $y_1^* \leq y_2^* = y_3^*$. A similar argument as above can be used to prove that either $(0, y_2^*, y_2^*)$ or (y_2^*, y_2^*, y_2^*) is an optimal solution. Now if $(0, y_2^*, y_2^*)$ is optimal, then one can use a similar argument to prove that either $(0, 0, 0)$ or $(0, 1, 1)$ is optimal. And if (y_2^*, y_2^*, y_2^*)

is optimal, then by applying Lemma 3.5.3 with $k = 3$, we deduce that either $(0, 0, 0)$ or $(1, 1, 1)$ is optimal. Thus, there is a 0-1 optimal solution. The efficiency of Mechanism 2 follows. □

When $k = 2$ (resp., 1), we can add one (resp., two) dummy facilities and use Theorem 3.5.4 to establish that Mechanism 2 is efficient for $k = 2$ (resp., 1).

While Mechanism 2 is WGSP, it is not SGSP. To see this, consider two DOFLG instances I and I' with 7 agents. In both I and I', the location profile and true aversion profile are $\{0, 0, 0, 0, 0, 1, 1\}$ and $\{\{F_1\}, \emptyset, \emptyset, \emptyset, \emptyset, \{F_1\}, \{F_1\}\}$, respectively. All agents in I report truthfully. In I', the first five agents form a coalition and report $\{F_1\}$, and the other two agents report truthfully. Mechanism 2 builds F_1 at 0 (resp., 1) in I (resp., I'). The welfare of agent 1 is higher in I' than in I, and the welfare of the other agents in the coalition is the same in I and I'. Thus Mechanism 2 is not SGSP. Additionally, observe that if M is an SGSP mechanism that builds F_1 at y in I, then M does not build F_1 to the right of y in I'. Using this observation, and generalizing instances I and I' to have $3n/2 + 1$ agents, $n + 1$ of which form the coalition, we establish Theorem 3.5.5 below. Theorem 3.5.5 provides a lower bound on the approximation ratio of any SGSP mechanism.

Theorem 3.5.5. There is no SGSP α-efficient DOFLG mechanism with $\alpha < 5/4$.

Proof. We construct two DOFLG instances I and I' with $\frac{3n}{2} + 1$ agents having the same location profile and true aversion profile, where n is a large even integer. In both instances, agent 1 is located at 0 and dislikes $\{F_1\}$, $n/2$ agents are located at 1 and all of them dislike $\{F_1\}$, and the remaining n agents, which are denoted by set U, are located at 0 dislike $\emptyset$. In I all the agents report truthfully while in I' all the agents in U report $\{F_1\}$ while the others report truthfully.

Let the maximum possible social welfare on these instances be OPT and OPT′ respectively. It is easy to see that $\mathrm{OPT} = 3n/2$ and $\mathrm{OPT}' = n+1$ (obtained by building the facilities at 0 and 1 respectively). Let the social welfare achieved by some SGSP mechanism M on these instances be ALG and ALG′ respectively.

Let M build F_1 at y in I. It follows that $\mathrm{ALG} = y+3n/2-ny/2$. If the agents in U and agent 1 form a coalition in I and agents in U report $\{F_1\}$, then the instance becomes I'. Thus, as M is SGSP, M cannot build F_1 to the right of y in I'. Using this fact it is easy to see that $\mathrm{ALG}' \leq (n+1)y+n/2(1-y) = ny/2+n/2+y$.

Using $\mathrm{OPT} = 3n/2$ and $\mathrm{ALG} = y + 3n/2 - ny/2$, we obtain

$$\alpha \geq \frac{3n/2}{y + 3n/2 - ny/2}. \tag{3.1}$$

Similarly using $\mathrm{OPT}' = n + 1$ and $\mathrm{ALG}' \leq ny/2 + n/2 + y$, we obtain

$$\alpha \geq \frac{n+1}{ny/2 + n/2 + y}. \tag{3.2}$$

Let

$$f(y) = \max\left(\frac{3n/2}{y + 3n/2 - ny/2}, \frac{n+1}{ny/2 + n/2 + y}\right).$$

From (3.1) and (3.2) we have $\alpha \geq f(y)$. It is easy to verify that y^* in $[0, 1]$ that minimizes $f(y)$ gives a value of $f(y^*) = (5n^2 + 4n - 4)/4n(n + 1)$. Thus, $\alpha \geq f(y^*)$. As n approaches infinity, $f(y^*)$ approaches $5/4$. Thus, for any SGSP utilitarian mechanism $\alpha \geq 5/4$. □

A natural question at this point is to ask for an SGSP mechanism with the lowest approximation ratio. Below we present a 2-efficient SGSP mechanism. It remains an interesting open problem to improve the approximation ratio of 2, or to establish a tighter lower bound for the approximation ratio.

Mechanism 3. Build all of the facilities at 0 if

$$\sum_{i \in [n]} x_i \geq \sum_{i \in [n]} (1 - x_i);$$

otherwise, build all of them at 1.

Theorem 3.5.6. Mechanism 3 is SGSP.

Proof. Reported dislikes do not affect the locations at which the facilities are built. Hence the theorem follows. □

Theorem 3.5.7. Mechanism 3 is 2-efficient.

Proof. Fix an arbitrary instance of DOFLG. Let ALG denote the social welfare obtained by Mechanism 3 on this instance, and let OPT denote the maximum possible social welfare on this instance. We need to prove that $2 \cdot \mathrm{ALG} \geq \mathrm{OPT}$.

Assume without loss of generality that Mechanism 3 builds all of the facilities at 0. (A symmetric argument handles the case where all of the facilities are built at 1). Then the welfare of an agent i not in U_ϕ is x_i and an agent i' in U_ϕ is $\max(x_{i'}, 1 - x_{i'}) \geq x_{i'}$. Thus, $\text{ALG} \geq \sum_{i \in [n]} x_i$. As Mechanism 3 builds the facilities at 0 and not 1, we have $\sum_{i \in [n]} x_i \geq \sum_{i \in [n]} (1 - x_i)$ which implies that $\sum_{i \in [n]} x_i \geq n/2$. Combining the above two inequalities we have, $\text{ALG} \geq n/2$. Note that as the welfare of any agent can never exceed 1, $n \geq \text{OPT}$. Thus, $2 \cdot \text{ALG} \geq n \geq \text{OPT}$, as required. □

We now show that the analysis of Theorem 3.5.7 is tight by exhibiting a two-facility instance on which Mechanism 3 achieves only half of the optimal social welfare. Consider an instance with two agents at 0 and 1 who report dislikes $\{F_1\}$ and $\{F_2\}$, respectively. It is easy to verify that optimal social welfare is $\text{SW}_{\mathbf{x},\mathbf{a}'}(1, 0) = 2$, while the social welfare obtained by Mechanism 3 is $\text{SW}_{\mathbf{x},\mathbf{a}'}(0, 0) = 1$.

3.5.1 Efficient mechanisms for the unit square and the unit circumference circle

We now present a simple adaptation of Mechanism 3 to the case where the agents are located on a circle.

Mechanism 8. Build all of the facilities at 0 if

$$\sum_{i \in [n]} d(x_i, 0) \geq \sum_{i \in [n]} d(x_i, 1/2);$$

otherwise, build all of them at 1/2.

As with Mechanism 3, reported dislikes do not affect the locations at which Mechanism 8 builds the facilities. Hence Mechanism 8 is SGSP.

Theorem 3.5.8. Mechanism 8 is SGSP when the agents are located on a circle.

Theorem 3.5.9. Mechanism 8 is 2-efficient when the agents are located on a circle.

Proof. This theorem can be shown using the same arguments in the proof of Theorem 3.5.7. Consider similar definitions of ALG and OPT and assume that Mechanism 8 builds the facilities at 0. Then using similar arguments we have $\mathrm{ALG} \geq \sum_{i\in[n]} d(x_i, 0)$. Also we have $\sum_{i\in[n]} d(x_i, 0) \geq \sum_{i\in[n]} d(x_i, 1/2)$, and for any agent i, $d(x_i, 0) + d(x_i, 1/2) \geq 1/2$ which together imply that $\sum_{i\in[n]} d(x_i, 0) \geq n/4$. Thus we have, $\mathrm{ALG} \geq n/4$. Note that as the welfare of any agent can never exceed 1/2, $n/2 \geq \mathrm{OPT}$. Thus, $2 \cdot \mathrm{ALG} \geq n/2 \geq \mathrm{OPT}$, as required. □

Lastly, we present the adaptation of Mechanism 3 to the case where the agents are located in a unit square.

Mechanism 9. Compute $d_{(0,0)} = \sum_{i\in[n]} d(x_i, (0, 0))$, $d_{(0,1)} = \sum_{i\in[n]} d(x_i, (0, 1))$, $d_{(1,0)} = \sum_{i\in[n]} d(x_i, (1, 0))$, and $d_{(1,1)} = \sum_{i\in[n]} d(x_i, (1, 1))$. Build all of the facilities at point p if d_p is the highest among $d_{(0,0)}, d_{(0,1)}, d_{(1,0)}$, and $d_{(1,1)}$, where p belongs to $\{0, 1\}^2$. Break ties in the following order: $d_{(0,0)}, d_{(0,1)}, d_{(1,0)}, d_{(1,1)}$.

As with Mechanism 3, reported dislikes do not affect the locations at which Mechanism 9 builds the facilities. Hence Mechanism 9 is SGSP.

Theorem 3.5.10. Mechanism 9 is SGSP when the agents are located in the unit square.

Theorem 3.5.11. Mechanism 9 is 2-efficient when the agents are located in the unit square.

Proof. This theorem again can be shown using the same arguments in the proof of Theorem 3.5.7. Consider similar definitions of ALG and OPT and assume that Mechanism 9 builds the facilities at $(0,0)$. Then using similar arguments we have $\text{ALG} \geq \sum_{i\in[n]} d(x_i,(0,0))$. Also we have $\sum_{i\in[n]} d(x_i,(0,0)) \geq \max_{p\in\{(0,1),(1,0),(1,1)\}}(\sum_{i\in[n]} d(x_i,p))$, and for any agent i,

$$d(x_i,(0,0)) + d(x_i,(0,1)) + d(x_i,(1,0)) + d(x_i,(1,1)) \geq 2\sqrt{2}$$

which together imply that $\sum_{i\in[n]} d(x_i,(0,0)) \geq n/\sqrt{2}$. Thus we have, $\text{ALG} \geq n/\sqrt{2}$. Note that as the welfare of any agent can never exceed $\sqrt{2}$, $\sqrt{2}n \geq \text{OPT}$. Thus, $2 \cdot \text{ALG} \geq \sqrt{2}n \geq \text{OPT}$, as required. □

3.6 Egalitarian Mechanisms

We now design egalitarian mechanisms for DOFLG when the agents are located on an interval, circle, or square.

For single-facility DOFLG mechanisms, specifying Z_1 is equivalent to specifying $\mathbf{a}'$. We begin by describing a simple way to convert a single-facility DOFLG mechanism into a DOFLG mechanism.

Definition 3.6.1. For any single-facility DOFLG mechanism M, we define Parallel(M) as the DOFLG mechanism that takes as input $k, \mathbf{x}$, and $\mathbf{a}'$ and outputs $\mathbf{y} = (y_1, \ldots, y_k)$, where y_j is the location where M builds F_j on input $\mathbf{x}$ and Z_j.

Lemmas 3.6.2 and 3.6.3 below reduce the task of designing a SP egalitarian DOFLG mechanism to the single-facility case.

Lemma 3.6.2. Let M be a SP single-facility DOFLG mechanism. Then Parallel(M) is a SP DOFLG mechanism.

Proof. Consider the true aversion profile $\mathbf{a}$ and another aversion profile $\mathbf{a}' = (\mathbf{a}_{-i}, a'_i)$. Suppose when the agents report $\mathbf{a}$, (resp., $\mathbf{a}'$) Parallel(M) builds the facilities at $\mathbf{y} = (y_1, \ldots, y_k)$ (resp., $\mathbf{y}' = (y'_1, \ldots, y'_k)$). Since M is SP, we have, $d(x_i, y_j) \geq d(x_i, y'_j)$ for each facility F_j in a_i. This gives us $w_{\mathbf{x},\mathbf{a}}(i, \mathbf{y}) \geq w_{\mathbf{x},\mathbf{a}}(i, \mathbf{y}')$ which implies that agent i does not benefit by reporting a'_i instead of a_i. □

Lemma 3.6.3. Let M be an egalitarian single-facility DOFLG mechanism. Then Parallel(M) is an egalitarian DOFLG mechanism.

Proof. Fix an arbitrary instance of DOFLG. Let the optimal location be $\mathbf{y}^* = (y^*_1, \ldots, y^*_k)$ and the optimal (maximum) value of the minimum welfare be $\text{OPT} = \text{MW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y}^*)$. Also let Parallel($M$) build the facilities at $\mathbf{y}' = (y'_1, \ldots, y'_k)$ giving a minimum welfare $\text{ALG} = \text{MW}_{\mathbf{x},\mathbf{a}'}(\mathbf{y}')$. If Z_j is empty define $\text{OPT}_j = \text{ALG}_j = \infty$. Otherwise, let the closest distance from y^*_j (resp., y'_j) to an agent in Z_j be OPT_j (resp., ALG_j).

Then we have,

$$\mathrm{OPT} = \min\left(\min_j \mathrm{OPT}_j, \min_{i \in U_\emptyset} w_{\mathbf{x},\mathbf{a}'}(i, \mathbf{y}^*)\right)$$

and

$$\mathrm{ALG} = \min\left(\min_j \mathrm{ALG}_j, \min_{i \in U_\emptyset} w_{\mathbf{x},\mathbf{a}'}(i, \mathbf{y}')\right).$$

Since M is egalitarian we have $\mathrm{OPT}_j = \mathrm{ALG}_j$ for all j. Also the welfare of agents in $U_\emptyset$ does not depend on the location of the facilities. Thus, $\mathrm{ALG} = \mathrm{OPT}$ implying that Parallel(M) is egalitarian. □

3.6.1 Egalitarian mechanisms for the unit interval

We begin by describing a SP egalitarian mechanism for single-facility DOFLG when the agents are located in the unit interval.

Mechanism 4. If Z_1 is empty, build F_1 at 0. Otherwise, let Z_1 contain ℓ agents $z_1, \ldots, z_\ell$ such that $x_{z_1} \leq x_{z_2} \leq \cdots \leq x_{z_\ell}$. Let $d_1 = x_{z_1}$ and $d_3 = 1 - x_{z_\ell}$. If $\ell = 1$, then build F_1 at 0 if $d_1 \geq d_3$, and at 1 otherwise. If $\ell > 1$, let m be the midpoint of the leftmost largest interval between consecutive agents in Z_1. Formally, $m = (x_{z_o} + x_{z_{o+1}})/2$, where o is the smallest number in $[\ell - 1]$ such that $x_{z_{o+1}} - x_{z_o} = \max_{j \in [\ell-1]}(x_{z_{j+1}} - x_{z_j})$. Let $d_2 = m - x_{z_o}$. Then build facility F_1 at 0 if $d_1 \geq d_2$ and $d_1 \geq d_3$, at m if $d_2 \geq d_3$, and at 1 otherwise.

Lemma 3.6.4. Mechanism 4 is SP for single-facility DOFLG.

Proof. Let the true aversion profile be $\mathbf{a}$ and the reported aversion profile be $(\mathbf{a}_{-i}, a'_i)$. If agent i is such that F_1 is not in a_i, then the location of F_1 does not affect the welfare of agent i. Moreover, if F_1 is in $a_i \cap a'_i$, then the location of F_1 does not change by reporting a'_i instead of a_i. So for the remainder of the proof we only consider i such that F_1 is in $a_i \setminus a'_i$.

Let F_1 be built at y when agent i reports truthfully. We assume that $y < x_i$ (the other case can be treated symmetrically). Let Z_1 be the set of agents who report $\{F_1\}$ when all agents report truthfully. Note that the mechanism never builds F_1 exactly at the location of any agent in Z_1, that is, no agent in Z_1 is at y. We consider two cases based on whether there is an agent from Z_1 between y and x_i.

Case 1: No agent in $Z_1 - i$ is located in $[y, x_i]$. We consider two cases based on y.

Case 1.1: $y = 0$. Then $d_1 = x_i$. When agent i reports a'_i, then either (1) d'_1, the new value in Mechanism 4, satisfies $d'_1 > d_1$ or (2) Z_1 is empty. Then the facility is again built at 0 which does not benefit agent i.

Case 1.2: $y \neq 0$. Then $d_2 = x_i - y$, there is some agent i' in Z_1 at $y - d_2$ and there are no agents in Z_1 in $(y - d_2, y + d_2)$. We consider two cases.

Case 1.2.1: No agent in Z_1 is located to the right of x_i. Then $x_i \geq 1 - d_2$ and when agent i reports a'_i, F_1 is built at 1 which again does not benefit agent i.

Case 1.2.2: Let the first agent in Z_1 to the right of agent i be i''. Then $x_{i''} - x_i \leq 2d_2$. Thus when agent i reports a'_i, F_1 is built somewhere in $[y, x_i]$ which does not benefit agent i.

Case 2: There is some agent in $Z_1 - i$ in $[y, x_i]$. Let the first agent to the right of y in $Z_1 - i$ be i'. Let $d = d_1 = x_{i'}$ if $y = 0$ otherwise $d = d_2 = x_{i'} - y$. We also have that the distance of agent i from F_1 is $x_i - y \geq d$. We consider two cases.

Case 2.1: No agent in Z_1 is located to the right of x_i. Then $x_i \geq 1 - d$. It is easy to check that when agent i reports a'_i, F_1 is either built again at y or at 1, neither of which benefits agent i.

Case 2.2: Let agent a and b be the first agent in $Z_1 - i$ to the weak left (at $\leq x_i$) and right of agent i respectively. Then $x_i - x_a \leq 2d$ and $x_b - x_i \leq 2d$. It is easy to check then that when agent i reports a'_i, F_1 is built at y or somewhere in $[x_i - d, x_i + d]$ which again does not benefit agent i.

Thus, agent i never benefits by reporting a'_i. □

Lemma 3.6.5. Mechanism 4 is egalitarian for single-facility DOFLG.

Proof. The welfare of any agent in $[n] \setminus Z_1$ is independent of the location of the facility. Thus, a mechanism is egalitarian if it maximizes the minimum welfare of the agents in Z_1. Mechanism 4 ignores all the agents not in Z_1. Thus it is sufficient to show that Mechanism 4 maximizes the minimum welfare on instances that have all agents in Z_1. Hence for the remainder of the proof, we assume that all the agents are in Z_1. Let y^* denote an optimal location for the facility and let OPT denote $\mathrm{MW}_{\mathbf{x},\mathbf{a}'}(y^*)$. Let y' denote the location where Mechanism 4 builds the facility and let ALG denote $\mathrm{MW}_{\mathbf{x},\mathbf{a}'}(y')$. Below we establish that ALG $\geq$ OPT, which implies that Mechanism 4 is egalitarian.

If Z_1 is empty then trivially Mechanism 4 is egalitarian. For the remainder of the proof, assume that Z_1 is non-empty. We say that an agent in Z_1 is *tight* if it is as close to y^* as any other agent in Z_1. Thus for any tight agent i, $\mathrm{OPT} = |y^* - x_i|$. Similarly , ALG is the distance between y' and a closest agent in Z_1.

If $y^* = 0$, consider any tight agent i. Then no agent in Z_1 is located in $[0, x_i)$. It follows that $d_1 = x_i = \mathrm{OPT}$. As $\mathrm{ALG} \geq d_1$, we have $\mathrm{ALG} \geq \mathrm{OPT}$. A symmetric argument can be made for the case $y^* = 1$.

It remains to consider the case where y^* does not belong to $\{0, 1\}$. Assume that $x_i < y^*$ where i is a tight agent (the other case can be treated symmetrically). We have $\mathrm{OPT} = y^* - x_i$. Thus no agent in Z_1 is located in $(x_i = y^* - \mathrm{OPT}, y^* + \mathrm{OPT})$. We consider two cases.

Case 1: There is no agent i' to the right of y^*. Thus $d_3 \geq \mathrm{OPT}$. Since $\mathrm{ALG} \geq d_3$, we have $\mathrm{ALG} \geq \mathrm{OPT}$.

Case 2: There is an agent in Z_1 to the right of y^*. Consider the leftmost such agent i'. Then as $x_{i'} \geq y^* + \mathrm{OPT}$, we have $d_2 \geq \mathrm{OPT}$. Since $\mathrm{ALG} \geq d_2$, we have $\mathrm{ALG} \geq \mathrm{OPT}$. □

We define Mechanism 5 as the DOFLG mechanism Parallel(M), where M denotes Mechanism 4. Using Lemmas 3.6.2 through 3.6.5, we immediately obtain Theorem 3.6.6 below.

Theorem 3.6.6. Mechanism 5 is SP and egalitarian.

Below we provide a lower bound on the approximation ratio of any WGSP egalitarian mechanism. Remark: Theorem 3.6.7 implies that Mechanism 5 is not WGSP.

Theorem 3.6.7. Let M be a WGSP α-egalitarian mechanism. Then α is $\Omega(\sqrt{n})$, where n is the number of agents.

Proof. Let q be a large even integer and let p denote $q^2 + 1$. We construct two DOFLG instances I and I' with $p + 3$ agents having the same location profile, and true aversion profile, and different reported aversion profiles. In both instances, there is an agent located at i/q^2 (called agent i) for each $0 < i < q^2/2$ and $q^2/2 < i < q^2$ and there are two agents each at 0, 1/2, and 1. In both I and I', agent i for $0 < i < q^2/2$ dislikes $\{F_2\}$ and agent i for $q^2/2 < i < q^2$ dislikes $\{F_1\}$, and one agent at 0 (resp., 1/2, 1) dislikes $\{F_1\}$ and the other dislikes $\{F_2\}$. All the agents in I report truthfully. In I', agents i for $q \leq i < q^2/2$ have alternating reports: agent q reports $\{F_1\}$, agent $q + 1$ reports $\{F_2\}$, agent $q + 2$ reports $\{F_1\}$, and so on. Symmetrically, the agents i for $q^2/2 < i \leq q^2 - q$ have alternating reports: agent $q^2 - q$ reports $\{F_2\}$, agent $q^2 - q - 1$ reports $\{F_1\}$, agent $q^2 - q - 2$ reports $\{F_2\}$, and so on. All the other agents in I' report truthfully.

Let the optimal minimum welfare for instance I (resp., I') be OPT (resp., OPT$'$). It is easy to see that OPT $= 1/4$ and OPT$' = \frac{1}{2q}$ (obtained by building the facilities at $(1/4, 3/4)$ and $(\frac{1}{2q}, 1 - \frac{1}{2q})$, respectively). Let ALG (resp., ALG$'$) denote the minimum welfare achieved by M on instance I (resp.,

I'). Below we prove that either $\text{OPT}/\text{ALG} \geq \frac{q}{4}$ or $\text{OPT}'/\text{ALG}' \geq q/2$.

Let M build facilities at (y_1, y_2) (resp., (y_1', y_2')) on instance I (resp., I'). We consider two cases.

Case 1: $0 \leq y_1' < 1/q$ and $1 - 1/q < y_2' \leq 1$. Let S denote the set of agents who lie in I'. If $y_1' < y_1$ and $y_2' > y_2$, then all agents in S benefit by lying. Hence for M to be WGSP either $y_1' \geq y_1$ or $y_2' \leq y_2$. Let us assume that $y_1' \geq y_1$; the other case can be treated symmetrically. Since $y_1' < 1/q$, we have $y_1 < 1/q$. Note that there is an agent at 0 who reported $\{F_1\}$. Thus $\text{ALG} \leq y_1 < 1/q$. Hence $\text{OPT}/\text{ALG} \geq \frac{q}{4}$.

Case 2: $y_1' \geq 1/q$ or $y_2' \leq 1 - 1/q$. If $y_1' \geq 1/q$, then at least one agent within distance $1/q^2$ of y_1' reported $\{F_1\}$ in I'. A similar observation holds for the case $y_2' \leq 1 - 1/q$. Thus $\text{ALG}' \leq 1/q^2$. Hence $\text{OPT}'/\text{ALG}' \geq q/2$.

The preceding case analysis shows that $\alpha \geq q/4$. Since $q = \sqrt{p-1} = \sqrt{n-4}$, we obtain the desired result. □

The following variant of Mechanism 5 is SGSP. In this variant, we first replace the reported dislikes of all agents with $\{F_1\}$ and use Mechanism 4 to determine where to build F_1. Then we build all of the remaining facilities at the same location as F_1. This mechanism is SGSP because it disregards the reported aversion profile. We claim that this mechanism is $2(n+1)$-egalitarian where n denotes the number of agents. To prove this claim, we first observe that when Mechanism 4 is run as a subroutine within this mechanism, we have $\max(d_1, 2d_2, d_3) \geq 1/(n+1)$. Thus the minimum welfare achieved by the mechanism is at least $1/(2(n+1))$. Since the optimal minimum welfare is at most 1, the claim holds.

3.6.2 Egalitarian mechanisms for the unit circumference cycle

In this section, we extend the results from Section 3.6.1 to the case where the agents are located on a cycle. We denote the point antipodal to u on C by $\widehat{u}$. We now consider the natural extension of Mechanism 4 to a circle.

Mechanism 10. If Z_1 is empty build facility F_1 at 0. If Z_1 has only one agent j, then build F_1 at $\widehat{x_j}$. Otherwise build the facility at the midpoint of the largest gap between any two consecutive agents from Z_1. Formally, let Z_1 have ℓ agents $z_0, \ldots, z_{\ell-1}$ such that $x_{z_0} \leq x_{z_1} \leq \cdots \leq x_{z_{\ell-1}}$. Let $\oplus$ mean addition modulo ℓ. Define m to be the midpoint of x_{z_o} and $x_{z_{o\oplus 1}}$ where o is the smallest number in $\{0, \ldots \ell-1\}$ such that $d(x_{z_{o\oplus 1}}, x_{z_o}) = \max_{j \in \{0, \ldots \ell-1\}} d(x_{z_{j\oplus 1}}, x_{z_j})$. Build F_1 at m.

Lemma 3.6.8. Mechanism 10 is SP for single-facility DOFLG.

Proof. Let the true aversion profile be $\mathbf{a}$ and the reported aversion profile be $(\mathbf{a}_{-i}, a'_i)$. Due to same arguments as in the proof of Lemma 3.6.4 we only consider agent i such that F_1 is in $a_i \setminus a'_i$.

Let F_1 be built at y when agent i reports truthfully. Let Z_1 be the set of agents who report $\{F_1\}$ when all agents report truthfully. Note that the mechanism never builds F_1 exactly at the location of any agent in Z_1. Let the arc of C that goes from y in the clockwise direction to x_i be r_1 and the arc of

C that goes from y in the anti-clockwise direction to x_i be r_2. Both arcs r_1 and r_2 include the end-points y and x_i. We consider three cases.

Case 1: No agent in $Z_1 - i$ is located on r_1 or r_2. Thus $Z_1 = \{i\}$. Hence $y = \widehat{x_i}$ and $d(x_i, y) = 1/2$. Also when agent i reports a_i', F_1 is built at 0. Since $d(x_i, 0) \leq 1/2$, this does not benefit the agent.

Case 2: There are agents in $Z_1 - i$ located on either r_1 or r_2 but not both. Without loss of generality, we assume that the agent is on r_2 (and there are no agents in $Z_1 - i$ on r_1). Let the closest agent from y in $Z_1 - i$ on r_2 be i'. Let $d' = d(y, x_{i'})$. By the design of the algorithm, y is the midpoint of $x_{i'}$ and x_i. Hence $d' = d(x_i, y)$. Let the closest agent in $Z_1 - i$ in the clockwise direction from x_i be i''. Since the mechanism builds the facility at the midpoint of the largest gap, $d(x_{i''}, x_i) \leq 2d'$. It follows that when agent i reports a_i', F_1 is built between y and x_i (that is, in the arc going from y in the clockwise direction to x_i) which does not benefit agent i.

Case 3: There are agents in $Z_1 - i$ on r_1 and r_2. Let the closest agent from y in $Z_1 - i$ on r_2 and r_1 be a and b. Let $d' = d(x_a, y) = d(y, x_b)$. Note that $d(x_i, y) \geq d'$. Also let the first agent from $Z_1 - i$ in the anti-clockwise and clockwise direction from x_i be i' and i''. Since the gap between agents a and b is the largest, $d(x_i, x_{i'}) \leq 2d'$ and $d(x_i, x_{i''}) \leq 2d'$. So when agent i reports a_i', either F_1 is built again at y or it is built somewhere within a distance of d' from x_i which does not benefit agent i.

Thus, agent i never benefits by reporting a_i'. $\square$

Lemma 3.6.9. Mechanism 10 is egalitarian for single-facility DOFLG.

Proof. Following the same arguments as in the proof of Lemma 3.6.5 we assume that all the agents are in Z_1. Let y^* denote an optimal location for the facility and let OPT denote $\mathrm{MW}_{\mathbf{x},\mathbf{a}'}(y^*)$. Let y' denote the location where Mechanism 10 builds the facility and let ALG denote $\mathrm{MW}_{\mathbf{x},\mathbf{a}'}(y')$. Below we establish that ALG $\geq$ OPT, which implies that Mechanism 10 is egalitarian.

If Z_1 is empty then trivially Mechanism 10 is egalitarian. For the remainder of the proof, assume that Z_1 is non-empty. We say that an agent in Z_1 is *tight* if it is as close to y^* as any other agent in Z_1. Thus for any tight agent i, $\mathrm{OPT} = d(y^*, x_i)$.

Consider any tight agent i. Assume that in the shorter arc between x_i and y^*, x_i is on the anti-clockwise side of y^* (the other case can be treated similarly). Thus $\mathrm{OPT} = d(x_i, y^*)$. Let i' be the closest agent in Z_1 in the clockwise direction from y^*. Since agent i the closest to y^*, we have $d(x_{i'}, y^*) \geq \mathrm{OPT}$. Thus $d(x_i, x_{i'}) \geq 2 \cdot \mathrm{OPT}$. Since $d(x_i, x_{i'})$ is the gap between two consecutive agents from Z_1, and Mechanism 10 builds the facility at the midpoint of the largest gap, we deduce that ALG $\geq$ OPT. □

We define Mechanism 11 as the DOFLG mechanism Parallel(M), where M denotes Mechanism 10. Using Lemmas 3.6.2, 3.6.3, 3.6.8, and 3.6.9, we immediately obtain Theorem 3.6.10 below.

Theorem 3.6.10. Mechanism 11 is SP and egalitarian.

Next we extend Theorem 3.6.7. Remark: Theorem 3.6.11 implies that Mechanism 11 is not WGSP.

Theorem 3.6.11. Let M be a WGSP α-egalitarian mechanism. Then α is $\Omega(\sqrt{n})$, where n is the number of agents.

Proof. It is easy to verify that the construction from the proof of Theorem 3.6.7 also works for circle and establishes the same lower bound. (We identify the point 1 with the point 0.) □

The following variant of Mechanism 11 is SGSP. As in the SGSP mechanism for the case when the agents are located on the unit interval, in this variant, we first replace the reported dislikes of all agents with $\{F_1\}$ and use Mechanism 10 to determine where to build F_1. Then we build all of the remaining facilities at the same location as F_1. This mechanism is SGSP because it disregards the reported aversion profile. We claim that this mechanism is n-egalitarian where n denotes the number of agents. To prove this claim, we first observe that the largest gap between two consecutive agents in Z_1 is at least $1/n$. Thus the minimum welfare achieved by the mechanism is at least $1/(2n)$. Since the optimal minimum welfare is at most $1/2$, the claim holds.

3.6.3 An egalitarian mechanism for the unit square

We begin by extending Mechanism 4 to a SP egalitarian mechanism for single-facility DOFLG when the agents are located in the unit square. Let S denote $[0,1]^2$ and let B denote the boundary of S. Agent i is located at $x_i = (a_i, b_i)$. For convenience, we assume that all agents are located at distinct points: the results below are easy to generalize to the case where this assumption does not hold.

Mechanism 6. If Z_1 is empty, build F_1 at $(0, 0)$. Otherwise, construct the Voronoi diagram D based on the locations of the agents in Z_1. Let V be the union of the following three sets of vertices: the vertices of D in the interior of S; the points of intersection between B and D; the four vertices of S. For each v in V, compute the minimum distance d_v from v to any agent in Z_1. Build F_1 at a vertex v maximizing d_v, breaking ties by x-coordinate and then by y-coordinate.

Toussaint presents an efficient $O(n \log n)$ algorithm to find the optimal v in Mechanism 6 [56]. The following lemma establishes that Mechanism 6 is egalitarian. The lemma is shown using Theorem 2 of [56] for the largest empty circle with location constraints problem.

Lemma 3.6.12. Mechanism 6 is egalitarian for single-facility DOFLG.

Proof. Using the same arguments as in the proof of Lemma 3.6.5, we assume that all the agents are in Z_1. Let the optimal location be y^* and corresponding minimum welfare be $\mathrm{OPT} = \mathrm{MW}_{\mathbf{x},\mathbf{a}'}(y^*)$. Also let Mechanism 6 build F_1 at y' giving a minimum welfare $\mathrm{ALG} = \mathrm{MW}_{\mathbf{x},\mathbf{a}'}(y')$. We show that $\mathrm{ALG} = \mathrm{OPT}$ implying that Mechanism 6 is egalitarian.

If Z_1 is empty then clearly $\mathrm{ALG} = \mathrm{OPT}$. Otherwise, the problem of finding the optimal location to build facility F_1 is equivalent to finding the circle with the largest radius having center in the interior or on the boundary of S such that the interior of the circle has no points from $\mathbf{x}$. This problem is known as the Largest Empty Circle with Location Constraint in literature [56]. Toussaint (Theorem 2 [56]) [2] shows that the optimal center for the circle is either a vertex of the Voronoi diagram in S, a point of intersection of D with B or a vertex of S. This result implies that ALG = OPT. □

Lemma 3.6.13. Mechanism 6 is SP for single-facility DOFLG.

Proof. Let the true aversion profile be $\mathbf{a}$ and the reported aversion profile be $(\mathbf{a}_{-i}, a'_i)$. Using the same arguments as in the proof of Lemma 3.6.4, we only consider agent i such that F_1 is in $a_i \setminus a'_i$.

Let Mechanism 6 build F_1 at y when all the agents report truthfully. Note that the mechanism never builds F_1 exactly at the location of any agent in Z_1. The Voronoi diagram within S consists of $|Z_1|$ non-overlapping polygons with each polygon having one agent. Let the polygon containing agent i when they report truthfully be P. When agent i reports a'_i, the only change to the Voronoi diagram is in the interior and on the boundary of P. So when agent i reports a'_i, F_1 is built either at the same location as when they reported truthfully or inside P (boundary inclusive). Clearly, if it is built at the same location, agent i does not benefit. Thus for the remainder of the proof, we consider the case when y' is inside P.

[2]Toussiant assumes that no three points are collinear and no four are cocircular but the theorem can be proven without these assumptions too.

Let OPT (resp., OPT$'$) be the closest distance of any agent from Z_1 (resp., $Z_1' = Z_1 - i$) to the point y (resp., y') when agent i reports a_i (resp., a_i'). Let $d = d(x_i, y)$ and $d' = d(x_i, y')$. Hence $d \geq \text{OPT}$. Also $\text{OPT}' \geq \text{OPT}$ since y is at a distance of at least OPT from any agent in Z_1'.

Assume for contradiction that Mechanism 6 is not SP and $d' > d$. We show that $\text{OPT} = \text{OPT}'$. Suppose this was not true. Then $\text{OPT}' > \text{OPT}$. Consider the situation when F is built at y' when agent i reports truthfully. We know $d' > d$ and the agent from $Z_1 - i$ closest to y' is at a distance of OPT$'$, which we know is greater than OPT. This shows that y' is a strictly better location to build F_1 as compared to y when everyone reports truthfully. But this leads to contradiction as we know that Mechanism 6 is egalitarian from Lemma 3.6.12. Thus $\text{OPT} = \text{OPT}'$.

Recollect that y' is either in the interior of P or on its boundary. Hence the closest agent in Z_1 to y' is agent i. Thus $d' \leq \text{OPT}'$. But we have established $\text{OPT} \leq d$ and assumed $d < d'$. Combining these facts with $d' \leq \text{OPT}'$, we get $\text{OPT} < \text{OPT}'$, which contradicts the result $\text{OPT} = \text{OPT}'$. So $d' \not> d$ and agent i does not benefit by reporting a_i'. □

We define Mechanism 7 as the DOFLG mechanism Parallel(M), where M denotes Mechanism 6. Using Lemmas 3.6.2, 3.6.3, 3.6.12, and 3.6.13, we immediately obtain Theorem 3.6.14 below.

Theorem 3.6.14. Mechanism 7 is SP and egalitarian.

3.7 Concluding Remarks

In this chapter, we studied the obnoxious facility location game with dichotomous preferences. This game generalizes the obnoxious facility location game to more realistic scenarios. Our results are summarized in Table 3.1.

All the mechanisms presented in this chapter run in polynomial time, except that the running time of Mechanism 2 has exponential dependence on k (and polynomial dependence on n). We can extend the results of Section 3.6.3 to obtain an analogue of Theorem 3.6.14 that holds for arbitrary convex polygon. We showed that Mechanism 2 is WGSP for all k and is efficient for $k \leq 3$. Properties 1 and 2 in the proof of the associated theorem, Theorem 3.5.4, do not hold for $k > 3$. Nevertheless, we conjecture that Mechanism 2 is efficient for all k. It remains an interesting open problem to reduce the gap between the $\Omega(\sqrt{n})$ and $O(n)$ bounds on the approximation ratio α of WGSP α-egalitarian mechanisms.

Chapter 4

Concluding Remarks

In this thesis, we presented results for two game-theoretic models. The first, object allocation over a network of objects, is a variant of the classic housing markets model. In Chapter 2, we presented hardness results for solving three reachability problems on cliques in this setting.

In Chapter 3, we proposed the obnoxious facility location game with dichotomous preferences. We designed mechanisms that are strategyproof and group-strategyproof for the utilitarian and egalitarian objectives. We also provided upper and lower bounds for the approximation ratios of these mechanisms.

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