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Differential fppf descent obstructions

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Differential fppf descent obstructions

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Dedicated to the cats in my life, Mulder, Scully and Shark, although they cannot appreciate it on an abstract level.

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Differential fppf descent obstructions

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In this dissertation, we consider the category of schemes equipped with a derivation and investigate a differential analogue of the fppf site on a differential scheme. We interpret the obstruction to the existence of integral points in affine varieties given by certain differential equations introduced by Voloch as the descent obstruction associated with torsors under a certain sheaf for our differential fppf topology. We also consider a multiplicative analogue of those differential descent obstructions and show that it is the only obstruction to the existence of integral points in affine varieties over function fields. Finally, we describe the obstruction set in the case of smooth projective isotrivial curves of genus $g \geq 2$ over a function field, extending a result of Voloch.

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Chapter 1

Introduction

1.1 Conventions

Let R be a (commutative) ring. By a derivation of R, we mean a $\mathbf{Z}$-linear map $\delta : R \to R$ satisfying the Leibniz rule

$$\delta(ab) = \delta(a)b + a\delta(b), \qquad a, b \in R.$$

We will write

$$R^\delta = \{r \in R : \delta(r) = 0\}$$

for the ring of differential constants of R. When R is a field, so is R^δ. Note that, if R has characteristic $p > 0$, then

$$R^p = \{r^p : r \in R\}$$

is a subset of R.

Given a field k, we fix a separable closure $\overline{k}$ of k. Any derivation of k extends uniquely to a derivation of $\overline{k}$, which will again be denoted by δ. When k has characteristic zero, $\overline{k}$ is also an algebraic closure. However, in positive characteristic, algebraic closures have no interesting differential properties: if k is algebraically closed of characteristic $p > 0$, then any derivation $\delta : k \to k$

must be identically zero. Indeed, every element of such k is a p-th power, hence a differential constant.

Let K be a function field in one variable over a field k. Fix a separating element t of K/k, that is, $t \in K$ such that K is a finite and separable extension of the rational function field $k(t)$. We write δ for the derivation d/dt of $k(t)$. Since K is a separable extension of $k(t)$, δ extends uniquely to a derivation of K, which is again denoted by $\delta = d/dt$.

Let v be a place of K. We write K_v for the completion of K with respect to v and $\mathcal{O}_v$ for the ring of integers of K_v. Upon writing

$$R = \{a \in K : v(a) \geq 0\},$$
$$\mathfrak{m} = \{a \in K : v(a) > 0\},$$

we have

$$\mathcal{O}_v = \varprojlim_i R/\mathfrak{m}^i.$$

Therefore, δ extends to a canonical derivation of $\mathcal{O}_v$ (hence of K_v), which is again denoted by δ.

Let S be a (finite) set of "bad" places of our function field K. We write

$$\mathcal{O}_{K,S} = \{a \in K : v(a) \geq 0, \text{ for all } v \notin S\}$$

for the ring of S-integers of K and

$$\mathbf{A}_{K,S} = \prod_{v \notin S} \mathcal{O}_v \times \prod_{v \in S} K_v$$

for the ring of S-adeles.

Let X be a scheme. We write $\Gamma(X, \mathcal{O}_X)$ for the ring of global sections of X. Given an fppf sheaf G of abelian groups on X, we write $H^i(X, G)$ for the i-th derived functor of the global sections functor $\Gamma(\cdot, G)$. For convenience, when X is the spectrum of a ring R, we write $H^i(R, G)$ instead of $H^i(\operatorname{Spec} R, G)$. We will also omit Spec in the notation of fiber products. For instance, if X is a scheme over a field K and L is a field containing K, we will write

$$X \times_K L$$

for the base-change of X to L.

1.2 Local-global principle and descent obstructions

Given an algebraic variety X over a field K, a central problem in arithmetic geometry is to decide whether the set $X(K)$ of K-rational points of X is non-empty. When K/k is a function field in one variable, then one may first try to solve the "easier" problem of deciding whether $X(K_v) \neq \emptyset$, for each place v of K. Note that, if $X(K_v) = \emptyset$, for some v, then $X(K) = \emptyset$.

One then naturally wonders whether $X(K_v) \neq \emptyset$, for every place v of K, implies $X(K) = \emptyset$. If the answer is positive, then one says that X satisfies the local-global principle. If the answer is negative, the next natural problem is to understand why the local-global principle fails.

Let G be a (smooth) group scheme over X. The fppf cohomology group $H^1(X, G)$ classifies (sheaves of) torsors over X under G. Given an (isomorphism class of a) torsor $[Y] \in H^1(X, G)$ and $x \in X(K)$, we look at the fiber of $Y \to X$ above x, which is a torsor $[Y](x) \in H^1(K,G)$ called the evaluation of $[Y]$ at x. Similarly, for every place v of K and $x_v \in X(K_v)$, we may evaluate $[Y]$ at x_v yielding a torsor $[Y](x_v) \in H^1(K_v,G)$. Therefore, for each torsor $[Y] \in H^1(X,G)$ we have a commutative diagram

$$\begin{tikzcd} X(K) \arrow[r] \arrow[d] & X(\mathbf{A}_K) \arrow[d] \\ H^1(K,G) \arrow[r] & \prod_v H^1(K_v,G) \end{tikzcd}$$

where the vertical maps are given by evaluation of $[Y]$ at rational points, and the horizontal maps are the usual diagonal embeddings and

$$X(\mathbf{A}_K) := \prod_v X(K_v),$$

the product being over all places v of K.

Definition 1.2.1. We say that a point $(x_v) \in X(\mathbf{A}_K)$ is unobstructed by a torsor $Y \to X$ under G if the evaluation $([Y](x_v)) \in \prod_v H^1(K_v,G)$ is in the image of $H^1(K,G)$ under the diagonal $H^1(K,G) \to \prod_v H^1(K_v,G)$. We write

$$X(\mathbf{A}_K)^{ob} = \{(x_v) \in X(\mathbf{A}_K) : (x_v) \text{ is unobstructed by every } [Y] \in H^1(X,G)\}$$

and call it the G-descent obstruction (set) of X/K.

The name "obstruction set" is motivated by the fact that

$$X(K) \subset X(\mathbf{A}_K)^{ob} \subset X(\mathbf{A}_K),$$

and thus $X(\mathbf{A}_K)^{ob} = \emptyset$ implies $X(K) = \emptyset$, explaining the failure of the local-global principle for X. It is natural to ask how big $X(\mathbf{A}_K)^{ob}$ is compared to $X(K)$. We say that the G-descent obstruction is the only obstruction to the existence of rational points in X if

$$X(\mathbf{A}_K)^{ob} = X(K).$$

Fix a (finite) set S of places of K/k. Suppose X/K is the generic fiber of a scheme $\mathbf{X}$ of finite type over $\mathcal{O}_{K,S}$. In this case, one can still use X-torsors under G to study the set $\mathbf{X}(\mathcal{O}_{K,S})$ of S-integral points of $\mathbf{X}$ inside the S-adelic space

$$X(\mathbf{A}_{K,S}) := \prod_{v \notin S} \mathbf{X}(\mathcal{O}_v) \times \prod_{v \in S} X(K_v).$$

We say the G-descent obstruction is the only obstruction to the existence of S-integral points in X if

$$X(\mathbf{A}_{K,S})^{ob} = \mathbf{X}(\mathcal{O}_{K,S}).$$

1.3 Voloch's differential obstructions

Let $\mathbf{X}$ be an affine scheme of finite type over $\mathcal{O}_S$ with generic fiber X over the function field K as in the previous section. In [18], Voloch characterizes $\mathbf{X}(\mathcal{O}_S)$ as a certain descent obstruction set. In this section, we recall Voloch's construction.

For $c \in K$, one regards the equation $\delta(z) = c$ as defining a torsor under the group K^δ of differential constants of K. By a result of Kolchin ([10, Corollary 1, page 193]), those torsors are classified by the group

$$H^1(K, \delta) := K/\delta(K).$$

Explicitly, the class of $c \in K$ in the above quotient corresponds to the torsor given by $\delta(z) = c$.

For each F in the coordinate ring $K[X]$ of X, we may consider the commutative diagram

$$\begin{tikzcd} X(K) \arrow[r] \arrow[d, "F"'] & X(\mathbf{A}_{K,S}) \arrow[d, "F"] \\ H^1(K, \delta) \arrow[r] & \prod_v H^1(K_v, \delta) \end{tikzcd}$$

where $H^1(K_v, \delta) := K_v/\delta(K_v)$, the horizontal arrows are the natural diagonal embeddings and the vertical maps are evaluation of the regular function F at a rational point. This is very similar to the commutative diagram that one considers when studying the descent obstruction associated with a torsor over X, as we discussed in the previous section.

In this case, one says that $(x_v) \in X(\mathbf{A}_{K,S})$ is unobstructed by the torsor

$$Y_F : \delta(z) = F$$

if $(F(x_v)+\delta(K_v)) \in \prod_v H^1(K_v, \delta)$ is in the image of $H^1(K, \delta) \to \prod_v H^1(K_v, \delta)$. Explicitly, (x_v) is unobstructed by Y_F if there exist $c \in K$ and $z_v \in K_v$ such that

$$\delta(z_v) = F(x_v) + c,$$

for every place v of K. Voloch's differential descent obstruction set is the set of all adelic points that are unobstructed by the torsor $Y_F : \delta(z) = F$, for every $F \in K[X]$.

The main results in [18] are:

Theorem 1.3.1. *Let* $\mathbf{X}$ *be an affine scheme of finite type over* $\mathcal{O}_{K,S}$ *with generic fiber* X *over the function field* K*. If* $(x_v) \in X(\mathbf{A}_{K,S})$ *is unobstructed by all torsors*

$$Y_F : \delta(z) = F,$$

for $F \in K[X]$*, then* $(x_v) \in \mathbf{X}(\mathcal{O}_{K,S})$*.*

Theorem 1.3.2. *Let* X *be a smooth projective curve of genus* $g \geq 2$ *over* K*. Suppose that the derivation* $\delta = d/dt$ *does not extend to a derivation on* X*. Then,* $X(K)$ *is described by differential descent obstructions.*

The proof of theorem 1.3.1 consists of reducing the problem to the case $\mathbf{X} = \mathbf{A}^1_{\mathcal{O}_{K,S}}$ and then applying Serre's duality and the Riemann-Roch theorem.

The proof of theorem 1.3.2 consists of embedding X as a closed subscheme of an affine scheme and then using theorem 1.3.1. This affine scheme is the first jet scheme of X along δ, which we will discuss in section 1.5.

In section 3.1, we will show how to interpret Voloch's differential obstruction as an obstruction associated with torsors under group schemes. To do that, we will work in the category of differential schemes (section 1.4), then consider a differential analogue of the fppf site on a scheme (section 2.1), and understand the cohomology of sheaves for this differential fppf topology (section 2.2). In addition, in section 3.2, we prove a multiplicative analogue of theorem 1.3.1.

It is natural to wonder what happens if we remove some of the hypothesis in theorem 1.3.2. In chapters 4 and 5, we show that when we allow $\delta = d/dt$ to extend to a derivation on X, the differential descent obstruction set of X is in general bigger than $X(K)$, the additional non-global points being "differential constants".

1.4 Differential schemes

By a differential scheme (X, D) we mean a scheme X together with a global vector field $D \in \mathrm{Der}(\mathcal{O}_X)$. A morphism $(X', D') \to (X, D)$ of differential schemes is a morphism $f : X' \to X$ of schemes such that the diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "f^\sharp"] \arrow[d, "D"'] & f_*\mathcal{O}_{X'} \arrow[d, "D'"] \\ \mathcal{O}_X \arrow[r, "f^\sharp"'] & f_*\mathcal{O}_{X'} \end{tikzcd}$$

commutes.

The category of differential schemes was studied by Buium, for example, in [2]. Although results in [2] were obtained for differential schemes in characteristic 0, the results we need remain true in positive characteristic. For instance, fiber products exist in the category of differential schemes. Explicitly, if (X_1, D_1) and (X_2, D_2) are differential schemes over a scheme S, then we equip $X_1 \times_S X_2$ with the derivation D given by

$$D(s_1 \otimes s_2) = D_1(s_1) \otimes s_2 + s_1 \otimes D_2(s_2),$$

where s_1 and s_2 are local sections of $\mathcal{O}_{X_1}$ and $\mathcal{O}_{X_2}$, respectively.

We now fix a differential scheme (S, δ). An (S, δ)-scheme is a differential scheme (X, D) where X is an S-scheme whose structure map $X \to S$ induces a morphism $(X, D) \to (S, \delta)$ of differential schemes. We define morphisms of (S, δ)-schemes in the natural way and write

$$\mathbf{DiffSch}/(S, \delta)$$

for the category of (S, δ)-differential schemes and their morphisms.

It is natural to wonder if we can equip an S-scheme X with a derivation D turning (X, D) into an (S, δ)-scheme. We assume that the structure map $f : X \to S$ is smooth. In this case, there exists an exact sequence

$$0 \longrightarrow f^*\Omega_S \longrightarrow \Omega_X \longrightarrow \Omega_{X/S} \longrightarrow 0, \tag{1.1}$$

where Ω_S (resp. Ω_X) denotes the cotangent sheaf of S (resp. X) over $\operatorname{Spec} \mathbf{Z}$. It induces an exact sequence

$$\operatorname{Der}(\mathcal{O}_X) \longrightarrow \operatorname{Der}(\mathcal{O}_S, f_*\mathcal{O}_X) \xrightarrow{ks} \operatorname{Ext}(\Omega_{X/S}, \mathcal{O}_X) = H^1(X, TX), \tag{1.2}$$

where $TX = \operatorname{Hom}_{\mathcal{O}_X}(\Omega_{X/S}, \mathcal{O}_X)$ is the tangent bundle of X (over S). The connecting homomorphism

$$ks : \operatorname{Der}(\mathcal{O}_S, f_*\mathcal{O}_X) \longrightarrow H^1(X, TX)$$

will be referred to as the Kodaira-Spencer map associated with $f : X \to S$. For simplicity, when f is understood, we write $ks(\delta)$ instead of $ks(f^\sharp \delta)$.

Proposition 1.4.1. *There exists a derivation $D \in \mathrm{Der}(\mathcal{O}_X)$ turning (X, D) into an (S, δ)-scheme if and only if $ks(\delta) = 0$.*

Proof. A derivation $D \in \mathrm{Der}(\mathcal{O}_X)$ extends δ via f if and only if D maps to $f^\sharp \delta$ under $\mathrm{Der}(\mathcal{O}_X) \to \mathrm{Der}(\mathcal{O}_S, f_*\mathcal{O}_X)$ in the exact sequence (1.2). By exactness, this is equivalent to $f^\sharp \delta$ being in the kernel of the Kodaira-Spencer map.

□

We also remark that:

Proposition 1.4.2. *If $f : X \to S$ is étale, every derivation on S lifts to a unique derivation on X.*

Proof. For an étale S-scheme X, the cotangent sheaf $\Omega_{X/S}$ vanishes. Therefore,

$$\mathrm{Der}(\mathcal{O}_X) \to \mathrm{Der}(\mathcal{O}_S, f_*\mathcal{O}_X)$$

is an isomorphism.

□

We can say more when X is a smooth projective curve and S is the spectrum of a field K of characteristic $p \geq 0$ equipped with a derivation δ. Let

$$K^\delta = \{a \in K : \delta(a) = 0\}$$

be the field of differential constants of K. We say that X is *(infinitesimally) isotrivial* if there exists an algebraic and separable field extension L/K and a scheme X' defined over L^δ such that

$$X \times_K L \cong X' \times_{L^\delta} L.$$

Proposition 1.4.3. *Let X be a smooth projective curve over a differential field (K, δ). There exists $D \in \mathrm{Der}(\mathcal{O}_X)$ extending δ if and only if X is (infinitesimally) isotrivial.*

Proof. Cf. [3] for $p = 0$ and [16] for $p > 0$.

$\square$

Remark 1.4.1.

1. Let K be a function field in one variable over a perfect field k. Let t be a separating element of K/k and consider $\delta = d/dt$. Traditionally, one says that X/K is isotrivial if there exists a finite extension L/k and a scheme X' defined over k such that $X \times_K L \cong X' \times_k L$. In characteristic zero, this notion of isotriviality coincides with that of infinitesimal isotriviality defined above. When the characteristic is $p > 0$, the two notions will not coincide since $K^\delta = K^p$. *In this dissertation, we are only interested in infinitesimal isotriviality. Thus we shall hereafter use the term "isotrivial" to mean "infinitesimally isotrivial".*

2. Every smooth projective curve X of genus 0 over (K,δ) is isotrivial. Indeed, if L/K is a finite extension such that $X(L) \neq \emptyset$, then X is isomorphic to the projective line $\mathbf{P}^1$ over L and $\mathbf{P}^1$ is defined over K^δ.

3. An elliptic curve over K is isotrivial if and only if its j-invariant is a differential constant. For instance, if K has characteristic zero, then the elliptic curve with Weierstrass equation

$$y^2 = x^3 + ax + b, \qquad a, b \in K,$$

has j-invariant

$$j = 1728 \frac{4a^3}{4a^3 + 27b^2},$$

and $\delta(j) = 0$ if and only if

$$3a^2b^2\delta(a) - 2a^3b\delta(b) = 0.$$

This happens if and only there exist differential constants $\lambda_1, \lambda_2 \in \overline{K}^\delta$ and $c \in \overline{K}$ such that

$$a = \lambda_1 c^2 \quad \text{and} \quad b = \lambda_2 c^3.$$

4. In chapter 5, we will obtain an explicit description of the equation defining a smooth isotrivial hyperelliptic curve similar to that of isotrivial elliptic curves as above. In particular, we will have explicit examples of isotrivial and non-isotrivial curves of every genus $g \geq 1$.

1.5 Prolongations

Let X be a scheme over a differential field (K, δ). By proposition 1.4.1, δ extends to a derivation of $\mathcal{O}_X$ if and only if the Kodaira-Spencer class $ks(\delta)$ is trivial. However, even when δ extends to a derivation on X, some relevant data on X may not be differential in nature. For instance, a rational point $\operatorname{Spec} K \to X$ may not define a morphism $(\operatorname{Spec} K, \delta) \to (X, \delta)$ of differential schemes. We can remedy this by embedding X as a closed subscheme in a "best possible" scheme over K equipped with a derivation extending δ, and this construction is possible even when $ks(\delta) \neq 0$. In this section, we outline the construction of such scheme and cite its key properties. The main reference is [4].

Let $\pi_{-1} : X \to \operatorname{Spec} K$ be the structure morphism. Set

$$X^{-1} = \operatorname{Spec} K,$$
$$X^0 = X,$$

and recursively define

$$X^{n+1} = \operatorname{Spec}(S(\Omega_{X^n})/I_n), \qquad n = 0, 1, \ldots,$$

where I_n is the ideal of the symmetric algebra $S(\Omega_{X^n})$ generated by local sections of the form

$$d(\pi_{n-1}^* f) - \delta_{n-1}(f), \qquad f \in \mathcal{O}_{X^{n-1}},$$

where $\pi_{n-1} : X^n \to X^{n-1}$ is the canonical projection and

$$\delta_{n-1} : \mathcal{O}_{X^{n-1}} \to \pi_{n-1 *}\mathcal{O}_{X^n}$$

is a derivation extending δ induced by the universal derivation $d : \mathcal{O}_{X^n} \to \Omega_{X^n}$.

We then have a sequence

$$\ldots \longrightarrow X^2 \longrightarrow X^1 \longrightarrow X$$

known as the prolongation sequence of X. Note that the collection (X^n) forms a projective system whose transition maps π_n are affine and we have derivations $\delta_n : \mathcal{O}_{X^n} \to \pi_{n*}\mathcal{O}_{X^{n+1}}$. Therefore, the inverse limit

$$X^\infty := \varprojlim X^n$$

exists as a scheme. It is a smooth scheme defined over K, affine over X and it comes equipped with a derivation extending δ, which will again be denoted by δ. The scheme X^n will be called the n-th jet scheme (of X along δ) and the differential scheme (X^∞, δ) will be called the infinite jet scheme (of X along δ).

Example 1.5.1. Let X be the affine variety in $\mathbf{A}^n_K$ given by polynomials

$$f_i(x_1, \ldots, x_n) = 0, \qquad i = 1, \ldots, m.$$

Then, X^1 is an affine variety in $\mathbf{A}^{2n}_K$ given by the additional equations

$$\sum_{i=1}^{n} f_i'(x_1, \ldots, x_n)x_i' + f^\delta(x_1, \ldots, x_n), \qquad i = 1, \ldots, m,$$

where f_i' denotes the usual the derivative of f_i as a polynomial, f_i^δ denotes the polynomial obtained by applying δ to the coefficients of f_i, and

$$x_1' = \delta_0(x_1),$$
$$\vdots$$
$$x_n' = \delta_0(x_n)$$

are coordinates of $\mathbf{A}_K^{2n}$ in addition to the coordinates $x_1, \ldots, x_n$ of $\mathbf{A}_K^n$.

Note that the equations defining X^1 in the example above are very similar to the equations

$$\sum_{i=1}^{n} f_i'(x_1, \ldots, x_n) dx_i, \qquad i = 1, \ldots, m,$$

defining the tangent bundle TX of X. For instance, if the coefficients of the equations defining the variety X are differential constants, then X^1 is isomorphic to the tangent bundle TX of X via an isomorphism over X given by

$$dx_i \mapsto x', \qquad i = 1, \ldots, m.$$

This is a special case of the following more general fact:

Proposition 1.5.1. *X^1 is an X-torsor under the tangent bundle TX. Its isomorphism class in the fppf cohomology group $H^1(X, TX)$ is the Kodaira-Spencer class $ks(\delta)$.*

Proof. Cf. [1, Chapter 3, Proposition 2.5].

$\square$

As example 1.5.1 suggests, there exists a closed immersion

$$\nabla_1 : X \to X^1,$$

which is a section of the canonical projection $X^1 \to X$ (cf. [1, Chapter 3, (3.8)]). Fixing local affine coordinates $x_1, \ldots, x_n$ around a point $P \in X(K)$, we have

$$\nabla_1(P) = (x_1(P), \ldots, x_n(P), \delta(x_1(P)), \ldots, \delta(x_n(P))).$$

Therefore, a regular function on X^1 in a sense defines a differential operator on X of order at most 1 (see [1, Chapter 3, Proposition 1.15] for a more precise statement). This same discussion applies to higher jet schemes. We will refer to an element in the coordinate ring $K[X^n]$ as a regular differential function on X of order $\leq n$. An element of

$$K\{X\} := K[X^\infty]$$

will be referred to simply as a regular differential function on X. Note that a regular function on X, that is, an element of the coordinate ring $K[X]$, is in particular a regular differential function on X (of order 0).

Finally, we remark that "forming the infinite jet scheme along δ" defines a functor

$$\mathbf{Sch}/K \longrightarrow \mathbf{DiffSch}/(K, \delta)$$

from the category of schemes over K to the category of differential schemes over (K, δ). This functor is right adjoint to the functor

$$\mathbf{DiffSch}/(K, \delta) \longrightarrow \mathbf{Sch}/K$$

that forgets the derivation of a differential scheme, that is,

$$\mathrm{Hom}_{\mathbf{Sch}}(S, X) \simeq \mathrm{Hom}_{\mathbf{DiffSch}}((S, D), (X^{\infty}, \delta)), \tag{1.3}$$

for every differential scheme (S, D) over (K, δ). In particular:

Proposition 1.5.2. *Every K-rational point $x : \operatorname{Spec} K \to X$ lifts to a unique morphism $(\operatorname{Spec} K, \delta) \to (X^{\infty}, \delta)$ of differential schemes.*

Chapter 2

The differential fppf site

2.1 Differential fppf sheaves

We fix a differential scheme (S, δ) and consider a differential analogue of the fppf site on S.

Definition 2.1.1. Let (X, D) be an (S, δ)-scheme. A differential fppf cover of (X, D) is a family

$$\{(X_i, D_i) \to (X, D)\}_{i \in I}$$

of morphisms of (S, δ)-schemes such that the associated family $\{X_i \to S\}$ of morphisms of S-schemes is an fppf cover of X.

The category of (S, δ)-schemes together with the class of all differential fppf coverings defines a Grothendieck topology as in [12, Chapter II]. This Grothendieck topology will be called the differential fppf topology. The big differential fppf site on (S, δ) will be denoted by S_{dfl}.

Remark 2.1.1. We could also consider differential analogues of the Zariski and étale sites. However, by proposition 1.3.2, the differential Zariski (resp. étale) site on (S, δ) coincides with the usual Zariski (resp. étale) site on S.

Let S_{fl} denote the (big) fppf site on S. Since a cover of (S, δ) with respect to the differential fppf topology is, in particular, a cover of S with respect to the (non-differential) fppf topology, there exists a morphism of sites

$$S_{fl} \to S_{dfl}$$

induced by the identity and forgetting derivations. Every fppf sheaf G on S thus restricts to a differential fppf sheaf on (S, δ), which will again be denoted by G. For instance, the additive sheaf $\mathbf{G}_a$ and the multiplicative sheaf $\mathbf{G}_m$ define differential fppf sheaves on (S, δ).

Let $\mathrm{der} : \mathbf{G}_a \to \mathbf{G}_a$ be the morphism of differential fppf sheaves given by

$$\mathrm{der}(x) = D(x), \quad x \in \Gamma(X, \mathcal{O}_X),$$

for (X, D) an (S, δ)-scheme. Its kernel is a differential fppf sheaf, denoted $\mathbf{G}_a^\delta$.

Theorem 2.1.1. *The sequence*

$$0 \longrightarrow \mathbf{G}_a^\delta \longrightarrow \mathbf{G}_a \xrightarrow{\mathrm{der}} \mathbf{G}_a \longrightarrow 0 \tag{2.1}$$

of differential fppf sheaves is exact. Moreover, over an affine base $(\operatorname{Spec} R, \delta)$, $\mathbf{G}_a^\delta$ *is represented by the differential group scheme*

$$(\operatorname{Spec} R[t], \partial),$$

where ∂ *is the unique derivation of the polynomial ring* $R[t]$ *extending* δ *and such that* $\partial(t) = 0$.

Proof. The only nontrivial fact about the exactness of (2.1) is the surjectivity of der. Let (X, D) be a differential scheme and $x \in \Gamma(X, \mathcal{O}_X)$. We need to show that the equation $\operatorname{der}(z) = x$ has a solution in some differential fppf cover of X. Consider the scheme

$$X' = \operatorname{Spec}_X(\mathcal{O}_X[t]),$$

where t is an indeterminant. For each open affine $U \subset X$, we endow the ring $\mathcal{O}_X(U)[t]$ with the unique derivation D_U extending $D \in \operatorname{Der}(\mathcal{O}_U)$ and such that $D_U(t) = x$. Gluing the derivations D_U for every $U \subset X$ gives $D' \in \operatorname{Der}(\mathcal{O}_{X'})$ satisfying $D'(t) = x$. Clearly, $\{(X', D') \to (X, D)\}$ is a differential fppf cover.

We now prove the representability of $\mathbf{G}_a^\delta$ over an affine base $(\operatorname{Spec} R, \delta)$. By definition, $\mathbf{G}_a^\delta$ is the contravariant functor that sends a differential scheme (X, D) to the set

$$\Gamma(X, \mathcal{O}_X)^D = \{x \in \Gamma(X, \mathcal{O}_X) : D(x) = 0\}$$

and a morphism $(X, D) \to (X', D')$ to the induced map

$$\Gamma(X', \mathcal{O}_{X'})^{D'} \to \Gamma(X, \mathcal{O}_X)^D.$$

A morphism $f : (X, D) \to (\operatorname{Spec} R[t], \partial)$ is uniquely determined by the image of t under the associated map $f^\sharp : R[t] \to \Gamma(X, \mathcal{O}_X)$ and it must satisfy

$$D(f^\sharp(t)) = f^\sharp(\partial(t)) = 0.$$

Therefore, the map that sends a morphism

$$f : (X, D) \to (\operatorname{Spec} R[t], \partial)$$

to

$$f^\sharp(t) \in \Gamma(X, \mathcal{O}_X)^D$$

is a bijection, and the representability of $\mathbf{G}_a^\delta$ follows.

$\square$

Remark 2.1.2.

1. The surjectivity of der : $\mathbf{G}_a \to \mathbf{G}_a$ illustrates an important feature of differential fppf covers: they introduce solutions to differential equations. More precisely, let Λ be an order n differential operator, that is, Λ is a map $\mathbf{G}_a \to \mathbf{G}_a$ over (S, δ) given by a polynomial P in the variables $\delta^0, \delta, \delta^2, \ldots, \delta^n$ and with coefficients in $\mathcal{O}_S$. Suppose that the separant of Λ, that is, the partial derivative $S_\Lambda = \partial\Lambda/\partial\delta^n$, is invertible. Then, the differential equation $\Lambda(z) = x$, $x \in \mathcal{O}_S$, has a solution over the differential scheme

   $$(\operatorname{Spec}_S \mathcal{O}_S[t_0, \ldots, t_n, s]/I, D)$$

   where $t_1, \ldots, t_n$ are variables, I is the ideal of $\mathcal{O}_S[t_1, \ldots, t_n, s]$ generated by

   $$\{P(t_0, \ldots, t_n) - x,\ sS_\Lambda - 1\},$$

   and D is the unique derivation of $\mathcal{O}_S[t_0, \ldots, t_n, s]$ extending δ and such that

$$\begin{aligned}
D(t_i) &= t_{i+1}, \qquad i = 0, \ldots, n-1, \\
D(t_n) &= s\delta(x) - s\sum_{i=0}^{n-1} t_{i+1} P_i(t_0, \ldots, t_n), \\
D(s) &= -s^2 D(S_\lambda),
\end{aligned}$$

where P_i denotes the partial derivative of P with respect to the i-th variable. Note that $\Lambda(t_0) = x$ and, clearly,

$$(\operatorname{Spec}_S \mathcal{O}_S[t_0, \ldots, t_n, s]/I, D) \to (S, \delta)$$

is a differential fppf covering.

2. For a concrete example of the above remark, consider the linear differential operator

$$\Lambda = a_0\delta^0 + a_1\delta + \cdots + a_{n-1}\delta^{n-1} + a_n\delta^n$$

with $a_i \in \mathcal{O}_S$ and $a_n \in \mathcal{O}_S^\times$. Its separant is $\partial\Lambda/\partial\delta^n = a_n$. The morphism of sheaves $\Lambda : \mathbf{G}_a \to \mathbf{G}_a$ given by

$$\Lambda(x) = a_0 x + a_1 D(x) + \ldots + a_n D^n(x), \quad x \in \Gamma(X, \mathcal{O}_X),$$

for (X, D) an (S, δ)-scheme, is surjective since the equation $\Lambda(z) = x$, $x \in \mathcal{O}_X$, has a solution in

$$(\operatorname{Spec}_X \mathcal{O}_X[t_0, t_1, \ldots, t_{n-1}], D'),$$

where D' is the unique derivation extending D and such that $D'(t_i) = t_{i+1}$, for $i = 0, ..., n-2$, and

$$D'(t_{n-1}) = a_n^{-1}x - a_n^{-1}(a_0t_0 + \cdots + a_{n-1}t_{n-1}).$$

Moreover, if we write $\mathbf{G}_a^{\Lambda}$ for the kernel of Λ, we have an exact sequence

$$0 \longrightarrow \mathbf{G}_a^{\Lambda} \longrightarrow \mathbf{G}_a \xrightarrow{\Lambda} \mathbf{G}_a \longrightarrow 0$$

generalizing (2.1).

3. Let $F \in \Gamma(S, \mathcal{O}_S)$. The proof of the representability of $\mathbf{G}_a^{\delta}$ can be adapted to show that the sheaf $\mathcal{F}$ on (S, δ) given by

$$\mathcal{F}(X) = \{x \in \Gamma(X, \mathcal{O}_X) : D(x) = F\},$$

for (X, D) an (S, δ)-scheme, is represented by the differential scheme

$$(\mathbf{A}_S^1, \partial_F),$$

where ∂_F is the unique derivation on $\mathbf{A}_S^1 = \operatorname{Spec}_S \mathcal{O}_S[t]$ extending δ and such that $\partial_F(t) = F$. Note that $(\mathbf{A}_S^1, \partial_F)$ is an (S, δ)-torsor under $\mathbf{G}_a^{\delta}$.

Now, let $\operatorname{dlog} : \mathbf{G}_m \to \mathbf{G}_a$ be the morphism of differential fppf schemes given by

$$\operatorname{dlog}(x) = D(x)/x, \quad x \in \Gamma(X, \mathcal{O}_X)^{\times},$$

for (X, D) an (S, δ)-scheme. Its kernel is a differential fppf sheaf denoted by $\mathbf{G}_m^{\delta}$. One should think of $\mathbf{G}_m^{\delta}$ as a multiplicative analogue of $\mathbf{G}_a^{\delta}$.

Theorem 2.1.2. *The sequence*

$$0 \longrightarrow \mathbf{G}_m^\delta \longrightarrow \mathbf{G}_m \xrightarrow{\text{dlog}} \mathbf{G}_a \longrightarrow 0 \tag{2.2}$$

of differential fppf sheaves is exact. Moreover, over an affine base $(\operatorname{Spec} R, \delta)$, $\mathbf{G}_m^\delta$ *is represented by the differential group scheme*

$$(\operatorname{Spec} R[t, t^{-1}], \partial),$$

where ∂ *is the unique derivation of* $R[t, t^{-1}]$ *extending* δ *and such that* $\partial(t) = 0$.

Proof. We proceed as in the proof of theorem 2.1.1. Let (X, D) be a differential scheme and $x \in \Gamma(X, \mathcal{O}_X)$. Consider the scheme

$$X' = \operatorname{Spec}_X(\mathcal{O}_X[t, t^{-1}]),$$

where t is an indeterminant. For each open affine $U \subset X$, we endow the ring $\mathcal{O}_X(U)[t, t^{-1}]$ with the unique derivation D_U extending $D \in \operatorname{Der}(\mathcal{O}_U)$ and such that $D_U(t) = xt$. Gluing the derivations D_U for every $U \subset X$ gives $D' \in \operatorname{Der}(\mathcal{O}_{X'})$ satisfying $D'(t) = xt$. In other words, $\operatorname{dlog}(t) = x$. Since $\{(X', D') \to (X, D)\}$ is a differential fppf cover, we conclude that dlog is surjective and the exactness of (2.2) is now clear.

To prove the representability of $\mathbf{G}_m^\delta$, notice that $\mathbf{G}_m^\delta$ is the contravariant functor that sends the differential scheme (X, D) to the set

$$\Gamma(X, \mathcal{O}_X^\times)^D = \{x \in \Gamma(X, \mathcal{O}_X^\times) : D(x) = 0\}$$

and a morphism $(X, D) \to (X', D')$ to the induced map

$$\Gamma(X', \mathcal{O}_{X'}^\times)^{D'} \to \Gamma(X, \mathcal{O}_X^\times)^D.$$

A morphism $f : (X, D) \to (\operatorname{Spec} R[t, t^{-1}], \partial)$ is uniquely determined by the image of t under $f^\sharp : R[t, t^{-1}] \to \Gamma(X, \mathcal{O}_X)$, and

$$D(f^\sharp(t)) = f^\sharp(\partial(t)) = 0.$$

Moreover, since t is invertible, its image $f^\sharp(t)$ lies in $\Gamma(X, \mathcal{O}_X^\times)^D$. Therefore, the map that sends

$$f : (X, D) \to (\operatorname{Spec} R[t, t^{-1}], \partial)$$

to

$$f^\sharp(t) \in \Gamma(X, \mathcal{O}_X^\times)^D$$

is a bijection and the representability of $\mathbf{G}_m^\delta$ follows.

$\square$

Remark 2.1.3.

1. As in remark 2.1.2, the surjectivity of dlog can be seen as a special case of the more general fact that we can use differential fppf covers to solve a differential equation $\Lambda(z) = x$ as long as the separant of the differential operator Λ can be made invertible. Note that the separant of the differential operator $\operatorname{dlog}(z) = \delta(z)/z$ is $1/z$.

2. Let $F \in \Gamma(S, \mathcal{O}_S)$. The proof of the representability of $\mathbf{G}_m^\delta$ can be adapted to show that the sheaf on (S, δ) given by

$$\mathcal{F}(X) = \{x \in \Gamma(X, \mathcal{O}_X^\times) : \operatorname{dlog}(x) = F\},$$

for (X, D) an (S, δ)-scheme, is represented by

$$(\mathbf{A}^1_S - \{0\}, \partial_F),$$

where ∂_F is the unique derivation on $\mathbf{A}^1_S - \{0\} = \operatorname{Spec}_S \mathcal{O}_S[t, t^{-1}]$ extending δ and such that $\partial_F(t) = Ft$. The scheme $(\mathbf{A}^1_S - \{0\}, \partial_F)$ is an (S, δ)-torsor under $\mathbf{G}^\delta_m$.

If (X, D) is an (S, δ)-scheme, then D acts on

$$\mathbf{GL}_n(X) = \mathrm{GL}_n(\Gamma(X, \mathcal{O}_X))$$

componentwise. We set

$$\mathbf{GL}^\delta_n(X) := \{A \in \mathbf{GL}_n(X) : D(A) = 0\},$$

defining the differential fppf sheaf $\mathbf{GL}^\delta_n$.

In the non-differential setting, $\mathbf{GL}_n$ is the automorphism group of the free $\mathcal{O}_S$-module of rank n. The sheaf $\mathbf{GL}^\delta_n$ has an analogous description.

Definition 2.1.2. A differential $\mathcal{O}_S$-module is a pair (L, D_L) consisting of an $\mathcal{O}_S$-module L and $D_L \in \mathrm{Der}(L)$ satisfying

$$D_L(af) = \delta(a)f + aD_L(f),$$

for all local sections $a \in \mathcal{O}_S$ and $f \in L$. A morphism $(L, D_L) \to (L', D_{L'})$ is a morphism $\varphi : L \to L'$ of $\mathcal{O}_S$-modules satisfying the differential condition

$$D_{L'}(\varphi(f)) = \varphi(D_L(f)),$$

for every $f \in L$.

The free $\mathcal{O}_S$-module of rank n can be made differential as follows: choose a basis $\{t_1, \ldots, t_n\}$ for $\mathcal{O}_S^n$, choose $A = [a_{ij}]$ in the sheaf $\mathbf{M}_n$ of $n \times n$ matrices on S and define D_A as the unique derivation of $\mathcal{O}_S^n$ extending $\delta \in \mathrm{Der}(\mathcal{O}_S)$ and such that

$$D_A(t_i) = a_{i1}t_1 + \ldots a_{in}t_n,$$

for $i = 1, \ldots, n$.

Proposition 2.1.3. *Let (S, δ) be a differential scheme.*

1. *The automorphism group of $(\mathcal{O}_S^n, D_A)$ is isomorphic to $\mathbf{GL}_n^\delta$, for every A;*

2. *$(\mathcal{O}_S^n, D_A)$ and $(\mathcal{O}_S^n, D_B)$ are isomorphic if and only if*

$$A - B = \mathrm{dlog}(M) := M^{-1}\delta(M),$$

for some $M \in \mathbf{GL}_n$.

Proof.

1. Let σ be a differential automorphism of $(\mathcal{O}_S^n, D_A)$ and let T be the diagonal matrix $\mathrm{diag}(t_1, \ldots, t_n)$. We write $[s_{ij}]$ for the matrix representing σ with respect to the basis $\{t_1, \ldots, t_n\}$. The differential condition

$$D_A(\sigma(T)) = \sigma(D_A(T))$$

can then be rewritten as $D_A([s_{ij}]T) = [s_{ij}]AT$. Since

$$D_A([s_{ij}]T) = [\delta(s_{ij})]T + [s_{ij}]AT,$$

we conclude that

$$[\delta(s_{ij})]T = 0,$$

implying that $\sigma \in \mathbf{GL}_n^\delta$.

2. Let $\sigma : (\mathcal{O}_S^n, D_A) \to (\mathcal{O}_S^n, D_B)$ be an isomorphism, let T be the diagonal matrix $\operatorname{diag}(t_1, \ldots, t_n)$ and write M for the matrix representing σ with respect to the basis $\{t_1, \ldots, t_n\}$. The differential condition

$$\sigma(D_A(T)) = D_B(\sigma(T))$$

can then be rewritten as

$$MAT = D_B(MT) = \delta(M)T + MBT,$$

implying that $A - B = \operatorname{dlog}(M)$.

Reversing the argument above shows that, if there exists $M \in \mathbf{GL}_n$ such that $A - B = \operatorname{dlog}(M)$, then the map $\sigma : (\mathcal{O}_S^n, D_A) \to (\mathcal{O}_S^n, D_B)$ given by $\sigma(T) = MT$ is a differential isomorphism.

□

In the non-differential setting, $\mathbf{PGL}_n$ is the automorphism group of $\mathbf{M}_n$ as an $\mathcal{O}_S$-algebra. We define $\mathbf{PGL}_n^\delta$ as the group of differential automorphisms of $(\mathbf{M}_n, \delta)$. Conjugation by an element of $\mathbf{GL}_n^\delta$ is a differential automorphism of $\mathbf{M}_n$ yielding a morphism

$$\mathbf{GL}_n^\delta \longrightarrow \mathbf{PGL}_n^\delta$$

whose kernel is $\mathbf{G}_m^\delta$.

Theorem 2.1.4 (Differential Skolem-Noether). *Let (R, δ) be a differential commutative local ring. If σ is a differential automorphism of the R-algebra $(\mathbf{M}_n(R), \delta)$, then there exists $u \in \mathbf{GL}_n^{\delta}(R)$ such that*

$$\sigma(a) = uau^{-1},$$

for every $a \in \mathbf{M}_n$. In particular, the sequence

$$0 \longrightarrow \mathbf{G}_m^{\delta} \longrightarrow \mathbf{GL}_n^{\delta} \longrightarrow \mathbf{PGL}_n^{\delta} \longrightarrow 0 \tag{2.3}$$

of differential fppf sheaves is exact.

Proof. Let σ be a differential automorphism of $(\mathbf{M}_n(R), \delta)$. By the non-differential Skolem-Noether theorem (cf. [12, Chapter IV, Corollary 2.4]), there is $u \in \mathbf{GL}_n(R)$ such that

$$\sigma(a) = uau^{-1},$$

for every $a \in \mathbf{M}_n(R)$. By the differential hypothesis on σ,

$$u\delta(a)u^{-1} = \delta(uau^{-1}),$$

for every $a \in \mathbf{M}_n(R)$.

Since $\delta(uau^{-1}) = \delta(u)au^{-1} + u\delta(a)u^{-1} - uau^{-2}\delta(u)$, we see that

$$\delta(u)au^{-1} = uau^{-2}\delta(u),$$

and thus

$$\begin{aligned}
\operatorname{dlog}(u)a &= u^{-1}uau^{-2}\delta(u)u \\
&= au^{-1}(u^{-1}\delta(u)u) \\
&= au^{-1}\delta(u) \\
&= a \operatorname{dlog}(u),
\end{aligned}$$

for every $a \in \mathbf{M}_n(R)$. It follows that there exists $\lambda \in R$ such that

$$\delta(u) = \lambda u.$$

If $\delta(u) = 0$, then $u \in \mathbf{GL}_n^\delta(R)$, and we are done. If $\delta(u) \neq 0$, then the equation $\delta(z) = \lambda z$ has a solution r in $R^\times$ and every solution of $\delta(z) = \lambda z$ in R is of the form cr for some $c \in R^\delta$. Set $\tilde{u} = r^{-1}u \in \mathbf{GL}_n^\delta(R)$ and notice that

$$\tilde{u}a\tilde{u}^{-1} = uau^{-1} = \sigma(a),$$

for every $a \in \mathbf{M}_n(R)$.

$\square$

Remark 2.1.4.

1. Proceeding as in the proof of the representability of $\mathbf{G}_a^\delta$ or $\mathbf{G}_m^\delta$, one can show that the differential fppf sheaves $\mathbf{GL}_n^\delta$ and $\mathbf{PGL}_n^\delta$ on (S, δ) are represented by affine differential group schemes.

2. $\mathbf{GL}_1 = \mathbf{G}_m$, so one wonders to what extent the exact sequence (2.2) generalizes for $n > 1$. Although, for $n > 1$, the cokernel of the inclusion

$$\mathbf{GL}_n^\delta \to \mathbf{GL}_n$$

does not exist, there exists an exact sequence

$$0 \longrightarrow \mathbf{GL}_n^\delta \longrightarrow \mathbf{GL}_n \xrightarrow{\text{dlog}} \mathfrak{gl}_n \longrightarrow 0,$$

where dlog is regarded as a 1-cocycle for the conjugation action of $\mathbf{GL}_n$ on its Lie algebra $\mathfrak{gl}_n$.

2.2 Differential fppf cohomology

Let (S, δ) be a differential scheme and G a differential fppf sheaf of abelian groups on (S, δ). We write $H^i(S_{dfl}, G)$ for the degree i differential fppf sheaf cohomology group of S_{dfl} with coefficients in G. More precisely, $H^i(\cdot, G)$ is the i-th right derived functor of the global sections functor $\Gamma(\cdot, G)$.

We then have the usual Čech-to-derived spectral sequence

$$\check{H}^p(S_{dfl}, \underline{H}^q(G)) \Longrightarrow H^{p+q}(S_{dfl}, G),$$

and, by the same proof as in the non-differential setting, $H^1(S_{dfl}, G)$ is isomorphic to the Čech cohomology group $\check{H}^1(S_{dfl}, G)$ (cf. [12, Chapter III, Propositions 2.7, 2.9 and Corollary 2.10]).

If $\mathcal{F}$ is a differential fppf sheaf of not necessarily abelian groups, then it is convenient to define $H^1(S_{dfl}, G)$ as the degree 1 non-abelian Čech cohomology set of S_{dfl} with coefficients in G. This allows short exact sequences of differential fppf sheaves of not necessarily abelian groups to induce a (not so) long exact sequence of cohomology sets:

Lemma 2.2.1. *Every central extension*

$$1 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow 1$$

of differential fppf sheaves on (S, δ) *induces a long exact sequence*

$$\begin{tikzcd} 1 \arrow[r] & \Gamma(S, G_1) \arrow[r] & \Gamma(S, G_2) \arrow[r] & \Gamma(S, G_3) \arrow[dll, rounded corners] & \\ & H^1(S_{dfl}, G_1) \arrow[r] & H^1(S_{dfl}, G_2) \arrow[r] & H^1(S_{dfl}, G_3) \arrow[r] & H^2(S_{dfl}, G_1). \end{tikzcd}$$

of groups and pointed sets.

Proof. Cf. [8, Remarque 4.2.10].

□

Let S_{et} and S_{fl} be the (small) étale site and the (big) fppf site on S, respectively.

Theorem 2.2.2. *Let* G *be a sheaf of abelian groups on* S_{fl} *(hence, on* S_{dfl} *and on* S_{et}*). If the canonical map* $H^1(S_{et}, G) \to H^1(S_{fl}, G)$ *is an isomorphism, then*

$$H^1(S_{dfl}, G) \cong H^1(S_{et}, G).$$

Proof. We have continuous morphisms of sites

$$S_{fl} \to S_{dfl} \to S_{et}$$

induced by the identity. They induce injective homomorphisms

$$H^1(S_{et}, G) \to H^1(S_{dfl}, G) \to H^1(S_{fl}, G).$$

If $H^1(S_{et}, G) \to H^1(S_{fl}, G)$ is an isomorphism, then $H^1(S_{dfl}, G) \to H^1(S_{fl}, G)$ is surjective, hence an isomorphism. It follows that $H^1(S_{et}, G) \to H^1(S_{dfl}, G)$ is an isomorphism.

□

Remark 2.2.1.

1. Étale and fppf cohomologies agree when G is a smooth, quasi-projective group scheme (cf. [12, Chapter III, Lemma 3.9]]). In particular, theorem 2.2.2 holds when $G = \mathbf{G}_a$, $\mathbf{G}_m$, or an abelian variety.

2. When étale and fppf cohomologies don't agree, $H^1(S_{fl}, G)$ could lie anywhere between $H^1(S_{et}, G)$ and $H^1(S_{fl}, G)$ depending on the choice of δ. For instance, when $S = \operatorname{Spec} K$, K a field of characteristic $p > 0$, and $G = \mu_p$, then

$$\begin{aligned} H^1_{et}(K, \mu_p) &= 0 \\ H^1_{fl}(K, \mu_p) &= K^\times/(K^\times)^p, \\ H^1_{dfl}(K, \mu_p) &= (K^\times)^\delta/(K^\times)^p. \end{aligned}$$

   Note that the differential fppf cohomology and the (non-differential) fppf cohomology agree when δ is the trivial derivation. Étale and differential fppf cohomologies agree when $(K^\times)^\delta = (K^\times)^p$ (e.g. when K is a function field over a perfect field k and $\delta = d/dt$, where t is a separating element of K/k).

It is possible to generalize theorem 2.2.2 to higher degree cohomology groups. However, we only need the following:

Proposition 2.2.3.

1. *(Differential Serre's theorem) If S is affine, then*

$$H^q(S_{dfl}, \mathbf{G}_a) = 0,$$

for every $q \geq 1$.

2. *(Cohomological Brauer group) $H^2(S_{dfl}, \mathbf{G}_m) = H^2(S_{et}, \mathbf{G}_m)$.*

Proof. Consider the continuous morphism of sites $S_{fl} \to S_{dfl}$ induced by the identity and forgetting derivations. Let $G = \mathbf{G}_a$ or $\mathbf{G}_m$. The associated Leray spectral sequence is

$$H^p(S_{dfl}, R^q i_* G) \Longrightarrow H^{p+q}(S_{fl}, G), \tag{2.4}$$

and thus $H^q(S_{dfl}, G) \cong H^q(S_{fl}, G)$ if $R^q i_* G = 0$. In our context, the vanishing of the sheaf $R^q i_* G$ can be phrased as follows: for every (S, δ)-scheme (X, D), there exists a differential fppf covering

$$\{(X_i, D_i) \to (X, D)\}$$

such that the image of $H^q(X_{fl}, G)$ in $H^q((X_i)_{fl}, G)$ is zero.

Given an (S, δ)-scheme (X, D), we choose an étale covering $\{X_i \to X\}$ where X_i is the spectrum of a strictly Henselian ring A_i. By proposition 1.4.2, $\{X_i \to X\}$ defines a differential fppf covering.

1. By Serre's theorem, since each X_i is affine, $H^q((X_i)_{fl}, \mathbf{G}_a) = 0$. Hence, the image of $H^q(X_{fl}, \mathbf{G}_a)$ in $H^q((X_i)_{fl}, \mathbf{G}_a)$ is trivially zero, for every $q \geq 1$, showing that $R^q i_* \mathbf{G}_a = 0$. Therefore, $H^q(S_{dfl}, \mathbf{G}_a) \cong H^q(S_{fl}, \mathbf{G}_a)$, which implies that Serre's theorem holds for the differential fppf topology.

2. Note that

$$H^2((X_i)_{fl}, \mathbf{G}_m) = H^2((X_i)_{et}, \mathbf{G}_m) = \mathrm{Br}(A_i) = 0,$$

since each A_i is a strictly local ring (cf. [12, Chapter IV, Corollary 1.7]). This shows that $R^2 i_* \mathbf{G}_m = 0$, and the Leray spectral sequence (2.4) then gives the desired isomorphism.

$\square$

In what follows, unless otherwise stated, all cohomology groups are with respect to the differential fppf topology and we write S instead of S_{dfl}.

Theorem 2.2.4. *There exist exact sequences*

$$0 \to \Gamma(S, \mathcal{O}_S)/\delta(\Gamma(S, \mathcal{O}_S)) \to H^1(S, \mathbf{G}_a^\delta) \to H^1(S, \mathbf{G}_a) \xrightarrow{\mathrm{der}} H^1(S, \mathbf{G}_a), \quad (2.5)$$

$$0 \to \Gamma(S, \mathcal{O}_S)/\operatorname{dlog}(\Gamma(S, \mathcal{O}_S^\times)) \to H^1(S, \mathbf{G}_m^\delta) \to \mathrm{Pic}(S) \xrightarrow{\mathrm{dlog}} H^1(S, \mathbf{G}_a). \quad (2.6)$$

Proof. The exact sequence (2.1) of differential fppf sheaves on (S, δ) yields the long exact sequence

$$\Gamma(S, \mathcal{O}_S) \xrightarrow{\text{der}} \Gamma(S, \mathcal{O}_S) \to H^1(S, \mathbf{G}_a^\delta) \to H^1(S, \mathbf{G}_a) \xrightarrow{\text{der}} H^1(S, \mathbf{G}_a) \to \cdots,$$

from which (2.5) follows.

Similarly, (2.2) induces the long exact sequence

$$\Gamma(S, \mathcal{O}_S^\times) \xrightarrow{\text{dlog}} \Gamma(S, \mathcal{O}_S) \to H^1(S, \mathbf{G}_m^\delta) \to \operatorname{Pic}(S) \xrightarrow{\text{dlog}} H^1(S, \mathbf{G}_a) \to \cdots,$$

from which (2.6) follows.

$\square$

Remark 2.2.2.

1. Applying theorem 2.2.4 to the differential scheme $(\operatorname{Spec} K, \delta)$ yields

$$H^1(K, \mathbf{G}_a^\delta) \cong K/\delta(K)$$

and

$$H^1(K, \mathbf{G}_m^\delta) \cong K/\operatorname{dlog}(K^\times).$$

Those isomorphisms are consistent with Kolchin's classification of principal homogeneous spaces under differential groups (cf. [10, Chapter VII, Section 6, Corollaries 1 and 2]). Explicitly, let $\lambda \in K$. The K-torsor $\mathcal{F}_{a,\lambda}$ under $\mathbf{G}_a^\delta$ associated with the class of λ in $K/\delta(K)$ is given by

$$\mathcal{F}_{a,\lambda}(X) = \{x \in \Gamma(X, \mathcal{O}_X) : D(x) = \lambda\},$$

for (X, D) a $(\operatorname{Spec} K, \delta)$-scheme. The K-torsor $\mathcal{F}_{m,\lambda}$ under $\mathbf{G}_m^\delta$ associated with the class of λ in $K/\operatorname{dlog}(K^\times)$ is given by

$$\mathcal{F}_{m,\lambda}(X) = \{x \in \Gamma(X, \mathcal{O}_X)^\times : D(x) = \lambda x\},$$

for (X, D) a $(\operatorname{Spec} K, \delta)$-scheme. In both additive and multiplicative cases, those sheaves of torsors are represented by differential schemes (cf. remarks 2.1.2 and 2.1.3).

2. The differential fppf sheaf $\mathbf{G}_a^\delta$ restricts to a sheaf on the étale site. One might wonder if the differential fppf topology captures something the étale topology can't. The answer is that it does. In fact:

Lemma 2.2.5. *Fix a separable closure $\overline{K}$ of K. The derivation δ of K then extends to a unique derivation of $\overline{K}$, which will be denoted by the same letter δ. Given $\lambda \in K$, if*

$$\delta(z) = \lambda$$

has a solution in $\overline{K}$, then it has a solution in K.

Proof. Let $z \in \overline{K}$ such that $\delta(z) = \lambda$, and let

$$m(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0$$

be the minimal polynomial of z over K. Applying δ to $m(z) = 0$ yields

$$0 = \sum_{i=1}^{d} (ia_i\lambda + \delta(a_{i-1}))z^{i-1},$$

where, for convenience, we have set $a_d = 1$. By the minimality of m, we must have

$$ia_i\lambda + \delta(a_{i-1}) = 0, \qquad i = 1, \ldots, d.$$

Since m is separable, there exists a greatest integer $i \in \{1, \ldots, d\}$ such that the characteristic $p \geq 0$ of K does not divide i and $a_i \neq 0$. If $i = d$, then

$$\delta(-a_{d-1})/d) = \lambda,$$

and we are done. If $i < d$, then the relation

$$(i+1)a_{i+1}\lambda + \delta(a_i) = 0$$

together with the definition of i implies that $\delta(a_i) = 0$,. The identity

$$ia_i\lambda + \delta(a_{i-1}) = 0$$

then yields

$$\delta\left(-\frac{a_{i-1}}{ia_i}\right) = \lambda.$$

□

Proposition 2.2.6. $H^1_{et}(K, \mathbf{G}^\delta_a) = 0$.

Proof. By lemma 2.2.5, if the differential equation $\delta(z) = \lambda$ has a solution in some étale cover of K, then it has a solution in K. Therefore, the differential fppf K-torsor $\mathcal{F}_{a,\lambda}$ given by $\delta(z) = \lambda$, $\lambda \in K$, is a K-torsor for the étale topology if and only if it is trivial.

□

Chapter 3

Cohomological differential descent obstructions

3.1 $\mathbf{G}_a^\delta$-descent obstructions

In this section, we restate Voloch's results (theorems 1.3.1 and 1.3.2) using the language of differential schemes. First, we recall the basic setup. Let K be a function field in one variable over an arbitrary field k of characteristic $p \geq 0$. We fix a separating element $t \in K$ and consider the derivation $\delta := d/dt$ of K. Given a scheme X over K the goal is to understand the set $X(K)$ inside the adelic space $X(\mathbf{A}_K)$. Given a (finite) set S of places of K/k, when X is the generic fiber of an affine scheme $\mathbf{X}$ of finite type over $\operatorname{Spec} \mathcal{O}_{K,S}$, we are also interested in understanding the set $\mathbf{X}(\mathcal{O}_{K,S})$ inside

$$X(\mathbf{A}_{K,S}) = \prod_{v \notin S} \mathbf{X}(\mathcal{O}_v) \times \prod_{v \in S} X(K_v).$$

Let G be a differential fppf sheaf of abelian groups. The derivation δ of K does not have to extend to a derivation on X, so we cannot in general talk about a differential fppf cohomology group $H^1(X_{dfl}, G)$. However, we can embed X in the infinite jet scheme X^∞ as discussed in section 1.5. The derivation δ now extends to a derivation on X^∞ and we may consider the differential fppf cohomology group $H^1(X^\infty, G)$. Moreover, by proposition 1.5.2, each rational point $x : \operatorname{Spec} K \to X$ lifts to a unique (differential) rational point $x : (\operatorname{Spec} K, \delta) \to (X^\infty, \delta)$, allowing us to consider, for each (class of a) torsor $[Y] \in H^1(X^\infty, G)$, the diagram

$$\begin{CD}
X(K) @>>> X(\mathbf{A}_K) \\
@VVV @VVV \\
H^1(K, G) @>>> \prod_v H^1(K_v, G)
\end{CD}$$

where all cohomology groups are with respect to the differential fppf topology, the vertical maps are given by evaluation of $[Y]$ at (differential) rational points of X^∞, and the horizontal maps are the usual diagonal embeddings.

Voloch's differential obstructions arise when we consider the additive sheaf $G = \mathbf{G}_a^\delta$ of differential constants. By theorem 2.2.4, the differential fppf cohomology group $H^1(X^\infty, \mathbf{G}_a^\delta)$ classifying X^∞-torsors under $\mathbf{G}_a^\delta$ sits in the exact sequence

$$0 \to \frac{K\{X\}}{\delta(K\{X\})} \to H^1(X^\infty, \mathbf{G}_a^\delta) \to H^1(X^\infty, \mathbf{G}_a)^\delta \to 0, \tag{3.1}$$

where $H^1(X^\infty, \mathbf{G}_a)^\delta = \ker(H^1(X^\infty, \mathbf{G}_a) \xrightarrow{\text{der}} H^1(X^\infty, \mathbf{G}_a))$. Recall that $K\{X\}$ denotes the coordinate ring of X^∞, that is, the set of all differential regular maps on X, and we view $K[X] \subset K\{X\}$.

By theorem 2.1.1, the sheaf $\mathbf{G}_a^\delta$ is represented by an affine differential group scheme. As in the non-differential setting, it follows that all (sheaves of) torsors corresponding to elements in $H^1(X^\infty, \mathbf{G}_a^\delta)$ are represented by differential schemes (cf. [12, Chapter 3, Theorem 4.3]). Moreover, by theorem 2.2.2, we can compute the differential fppf cohomology group $H^1(X^\infty, \mathbf{G}_a)$ using the fppf or étale topologies, and the map

$$H^1(X^\infty, \mathbf{G}_a^\delta) \to H^1(X^\infty, \mathbf{G}_a)$$

can be interpreted as "forgetting derivations". In particular, the underlying scheme of a torsor corresponding to a class in

$$\frac{K\{X\}}{\delta(K\{X\})} \subset H^1(X^\infty, \mathbf{G}_a^\delta)$$

is the trivial $\mathbf{G}_a$-torsor $\mathbf{A}^1_{X^\infty} = \operatorname{Spec} \mathcal{O}_{X^\infty}[t]$. A derivation on $\mathbf{A}^1_{X^\infty}$ compatible with the action of $\mathbf{G}_a^\delta$ must be given by $\partial_F : \mathcal{O}_{X^\infty}[t] \to \mathcal{O}_{X^\infty}[t]$, $\partial_F(t) = F$, for some $F \in K\{X\}$. Such $\mathbf{G}_a^\delta$-torsor $(\mathbf{A}^1_{X^\infty}, \partial_F)$ corresponds to the class of F in

$$\frac{K\{X\}}{\delta(K\{X\})} \subset H^1(X^\infty, \mathbf{G}_a^\delta).$$

The fiber of the torsor $(A^1_{X^\infty}, \partial_F) \to (X^\infty, \delta)$ above $x \in X(K)$ viewed as a differential rational point $x : (\operatorname{Spec} K, \delta) \to (X^\infty, \delta)$ is

$$(\mathbf{A}^1_K, \partial_{F(x)}),$$

which is the torsor over $\operatorname{Spec} K$ corresponding to the class of $F(x) \in K$ in

$$\frac{K}{\delta(K)} \cong H^1(K, \mathbf{G}_a^\delta).$$

Proposition 3.1.1. *A point (x_v) in the adelic space $X(\mathbf{A}_K^S)$ is unobstructed by $(A^1_{X^\infty}, \partial_F)$ if and only if there exist $(z_v) \in \prod_v K_v$ and $c \in K$ such that*

$$\delta(z_v) = F(x_v) + c. \tag{3.2}$$

Proof. Recall definition 1.2.1. We say that (x_v) is unobstructed by $(A^1_{X^\infty}, \partial_F)$ if there exists $c \in K$ such that the evaluation $(\mathbf{A}^1_{K_v}, \partial_{F(x_v)})$ is isomorphic to the K-torsor $(\mathbf{A}^1_K, \partial_c)$, for every place v of K. By our discussion above, this is equivalent to there existing $c \in K$ whose class is mapped to the class of $(F(x_v)) \in \prod_v K_v$ under the diagonal

$$\frac{K}{\delta(K)} \to \prod_v \frac{K_v}{\delta(K_v)},$$

yielding the criterion in the statement.

□

Remark 3.1.1.

1. It is possible to verify directly that, for any differential field (L, δ) and $a, b \in L$, the differential schemes $(\mathbf{A}^1_L, \partial_a)$ and $(\mathbf{A}^1_L, \partial_b)$ are isomorphic as L-torsors under $\mathbf{G}_a^\delta$ if and only if $b - a \in \delta(L)$. First, one shows that a $\mathbf{G}_a^\delta$-equivariant isomorphism $\sigma : L[t] \to L[t]$ must be given by $\sigma(t) = t + \lambda$, $\lambda \in L$. Then, one uses the differential condition $\sigma(\partial_b(t)) = \partial_a(\sigma(t))$ to conclude that $b - a = \delta(\lambda)$.

2. Without the language of differential schemes, Voloch in [18] uses (3.2) as the definition of a point being unobstructed by his differential descent obstructions. Proposition 3.1.1 thus shows that, with the language of differential schemes, Voloch's differential descent obstructions are more

specifically the descent obstructions coming from (differential fppf) torsors in

$$K[X] \hookrightarrow \frac{K\{X\}}{\delta(K\{X\})} \subset H^1(X^\infty, \mathbf{G}_a^\delta).$$

A rational point $(\operatorname{Spec} K, \delta) \to (\mathbf{A}^1_{X^\infty}, \partial_F)$ corresponds to a maximal ideal of the polynomial ring $\mathcal{O}_X[t]$ stable under the action of ∂_F and whose residue field is K. It is easy to check that such maximal ideal must be generated by $t-z$, where z is a solution to $\delta(z) = F$. Therefore, from now on, to simplify, we will refer to the X-torsor $(\mathbf{A}^1_{X^\infty}, \partial_F)$ as the torsor given by $\delta(z) = F$.

We are ready to restate theorems 1.3.1 and 1.3.2 using the language of differential schemes and differential fppf descent.

Theorem 3.1.2. *Let $\mathbf{X}$ be an affine $\mathcal{O}_{K,S}$-scheme of finite type with generic fiber X. If $(x_v) \in \{(\mathbf{A}_{K,S})$ is unobstructed by all X-torsors corresponding to elements in*

$$K[X] \subset \frac{K\{X\}}{\delta(K\{X\})} \subset H^1(X^\infty, \mathbf{G}_a^\delta),$$

then $(x_v) \in \mathbf{X}(\mathcal{O}_{K,S})$.

Theorem 3.1.3. *Let X be a smooth non-isotrivial curve of genus $g > 1$ over K. If $(x_v) \in \prod_v X(K_v)$ is unobstructed by all $\mathbf{G}_a^\delta$-torsors given by first order regular differential maps on X, that is, corresponding to elements in*

$$\Gamma(X^1, \mathcal{O}_{X^1}) \to \frac{K\{X\}}{\delta(K\{X\})} \subset H^1(X^\infty, \mathbf{G}_a^\delta),$$

then $(x_v) \in X(K)$.

Proof. We sketch the proof from [18] since the statement of theorem 1.3.2 is less specific. First, one considers the embedding

$$\nabla_1 : X \to X^1,$$

where X^1 is the first jet scheme of X along δ discussed in section 1.5. When X is a smooth non-isotrivial curve of genus $g > 1$, the first jet scheme X^1 is affine (more specifically, an affine surface), as proven in [5] and [6]. The argument in [18] then shows that

$$\nabla_1(X(K)) \subset X^1(\mathbf{A}_{K,S}),$$

for a suitable (finite) set S of places of K/k. Since X^1 is affine, we can now use theorem 3.1.2 to see that (x_v) is global if and only if (x_v) is unobstructed by all X^1-torsors corresponding to regular functions on X^1. The result as in the statement of theorem 3.1.2 then follows from interpreting a regular map on X^1 as an element in $K\{X\}$, more specifically an order 1 differential regular function on X, and noticing that $(X^1)^\infty \cong X^\infty$ (which implies that differential torsors over X^1 are differential torsors over X).

□

3.2 $\mathbf{G}_m^\delta$-descent obstructions

Let K, δ, $\mathbf{X}$ and X be as in the previous section. In this section, we study the descent obstruction to integral points associated with X-torsors under the affine differential scheme $\mathbf{G}_m^\delta$.

By theorem 2.2.4, the differential fppf cohomology group $H^1(X^\infty, \mathbf{G}_m^\delta)$ sits in the exact sequence

$$0 \to \frac{K\{X\}}{\mathrm{dlog}(K\{X\}^\times)} \to H^1(X^\infty, \mathbf{G}_m^\delta) \to \mathrm{Pic}(X^\infty)^\delta \to 0, \tag{3.3}$$

where $\mathrm{Pic}(X^\infty)^\delta = \ker(\mathrm{Pic}(X^\infty) \xrightarrow{\mathrm{dlog}} H^1(X^\infty, \mathbf{G}_a))$.

As in the case of $\mathbf{G}_a^\delta$, every torsor under $\mathbf{G}_m^\delta$ is represented by a differential scheme. By theorem 2.2.2, we can compute the differential fppf cohomology group $H^1(X^\infty, \mathbf{G}_m) = \mathrm{Pic}(X^\infty)$ using the étale topology, and the map

$$H^1(X^\infty, \mathbf{G}_m^\delta) \to \mathrm{Pic}(X)$$

can thus be interpreted as "forgetting derivations". In particular, the underlying scheme of a torsor corresponding to a class in

$$\frac{K\{X\}}{\mathrm{dlog}(K\{X\}^\times)} \subset H^1(X^\infty, \mathbf{G}_m^\delta)$$

is the trivial $\mathbf{G}_m$-torsor

$$\mathbf{A}^1_{X^\infty} - \{0\} = \mathrm{Spec}\, \mathcal{O}_{X^\infty}[t, t^{-1}].$$

A derivation on $\mathbf{A}^1_{X^\infty} - \{0\}$ compatible with the action of $\mathbf{G}_m^\delta$ must be given by $\partial_F : \mathcal{O}_{X^\infty}[t, t^{-1}] \to \mathcal{O}_{X^\infty}[t, t^{-1}]$, $\partial_F(t) = Ft$, for some $F \in K\{X\}$. Such $\mathbf{G}_m^\delta$-torsor $(\mathbf{A}^1_{X^\infty} - \{0\}, \partial_F)$ corresponds to the class of F in

$$\frac{K\{X\}}{\mathrm{dlog}(K\{X\}^\times)} \subset H^1(X^\infty, \mathbf{G}_m^\delta).$$

We will refer to the torsor $(\mathbf{A}^1_{X^\infty} - \{0\}, \partial_F) \to (X^\infty, \delta)$ as the torsor given by $\mathrm{dlog}(z) = F$. Its fiber above $x \in X(K)$ viewed as a differential rational point $x : (\operatorname{Spec} K, \delta) \to (X^\infty, \delta)$ is the K-torsor under $\mathbf{G}_m^\delta$ given by $\operatorname{dlog}(z) = F(x)$, that is, $(\mathbf{A}_K^1, \partial_{F(x)})$. This K-torsor corresponds to the class of $F(x) \in K$ in

$$\frac{K}{\operatorname{dlog}(K^\times)} \cong H^1(K, \mathbf{G}_m^\delta).$$

Proposition 3.2.1. *A point (x_v) in the adelic space $X(\mathbf{A}_K^S)$ is unobstructed by the torsor given by $\operatorname{dlog}(z) = F$ if and only if there exist $(z_v) \in \prod_v K_v^\times$ and $c \in K$ such that*

$$\operatorname{dlog}(z_v) = F(x_v) + c. \tag{3.4}$$

Proof. Recall definition 1.2.1. We say that (x_v) is unobstructed by the torsor $\operatorname{dlog}(z) = F$ if there exists $c \in K$ such that the fiber $\operatorname{dlog}(z) = F(x_v)$ is isomorphic to the torsor given by $\operatorname{dlog}(z) = c$, for every place v of K. This is equivalent to there existing $c \in K$ whose class is mapped to the class of $(F(x_v)) \in \prod_v K_v$ under the diagonal

$$\frac{K}{\operatorname{dlog}(K^\times)} \to \prod_v \frac{K_v}{\operatorname{dlog}(K_v^\times)},$$

yielding the criterion in the statement.

□

We now prove a multiplicative analogue of theorem 3.1.2.

Lemma 3.2.2. *Let K be a function field in one variable over a finite field k of characteristic $p > 0$ and let $a \in K$. If the Artin-Schreier polynomial*

$$f(x) = x^p - x + b$$

has a root in K_v, for every place v of K, then f has a root in K.

Proof. Fix an algebraic closure $\overline{K}$ of K and an algebraic closure $\overline{k}$ of k inside $\overline{K}$. If f remains irreducible over the compositum $K\overline{k}$, then f remains irreducible over K_v, for some place v of K (in fact, infinitely many places), by Chebotarev's density theorem. However, we assumed that f has a root in K_v, for all v.

So, f must be reducible over $K' = KL$, for some finite extension L/k. Hasse-Weil's bound for the number of places of degree n of K shows that, for n large, there exist places v_1 and v_2 of K of degree n and $n+1$. Therefore, there exists a place v of degree coprime with $[L : K]$. This guarantees that v is inert in the function field extension K'/K (cf. [15, Theorem 3.6.3]). In this case, the tensor product $K' \otimes_K K_v$ is a field (namely, the completion of K' with respect to v), implying that K'/K and K_v/K don't have K-isomorphic finite subextensions. In particular, f cannot be irreducible over K. Since an Artin-Schreier polynomial is reducible over K if and only if it contains a root in K, we conclude that f has a root in K.

□

Theorem 3.2.3. *Let $\mathbf{X}$ be an affine scheme of finite type over $\mathcal{O}_{K,S}$ with generic fiber X over K, where S is a finite set of places of K. If $(x_v) \in \mathbf{X}(\mathbf{A}_{K,S})$ is unobstructed by all X-torsors in*

$$\frac{K\{X\}}{\mathrm{dlog}(K\{X\}^\times)} \subset H^1(X^\infty, \mathbf{G}_m^\delta),$$

then $(x_v) \in \mathbf{X}(\mathcal{O}_{K,S})$.

Proof. Since we may embed $\mathbf{X}$ in the affine space $\mathbf{A}^n_{\mathcal{O}_{K,S}}$, for some positive integer n, it suffices to consider the case $\mathbf{X} = \mathbf{A}^1_{\mathcal{O}_{K,S}}$.

Let x be a coordinate in $\mathbf{A}^1$ and $f \in K$. By proposition 3.2.1, (x_v) being unobstructed by the torsor $\operatorname{dlog}(z) = fx$ means that there exist $c \in K$ and $z_v \in \prod_v K_v^\times$ such that $\operatorname{dlog}(z_v) = fx_v + c$, for every place v of K, or, equivalently, in the language of 1-forms,

$$\frac{dz_v}{z_v} = x_v f dt + cdt.$$

Taking residues yields

$$v(z_v)\deg(v) = \operatorname{Res}_v \frac{dz_v}{z_v} = \operatorname{Res}_v x_v f dt + \operatorname{Res}_v cdt,$$

where $\deg(v)$ is the degree of the place v. By the residue theorem, we have

$$\sum_v \operatorname{Res}_v x_v f dt = \sum_v v(z_v)\deg(v),$$

which converges since the set S of "bad places" is finite and $f dt$ has finitely many poles. This being true for every $f \in K$ implies that the residue map

$$\langle (x_v), \cdot \rangle : \Omega_{K/k} \to k$$

given by

$$\langle (x_v), \omega \rangle = \sum_v \operatorname{Res}_v x_v \omega$$

has image contained in $\mathbf{Z}/p\mathbf{Z}$, where $p \geq 0$ is the characteristic of K. If k is not the field with p elements, since $\langle (x_v), \cdot \rangle$ is k-linear, we must have $\langle (x_v), \cdot \rangle$ identically zero. By Serre's duality and the Riemann-Roch theorem, it follows that (x_v) is global.

We now consider the case $p > 0$ and $k = \mathbf{Z}/p\mathbf{Z}$. For each $i = 0, \ldots, p-1$, assume that (x_v) is unobstructed by $\mathrm{dlog}(z) = t^i x$. Then, there exist $c_i \in K$ and $z_{iv} \in \prod_v K_v^\times$ such that $\mathrm{dlog}(z_{iv}) = t^i x_v + c_i$, for every place v of K. Suppose $\mathrm{dlog}(z_{iv}) \neq 0$. Applying the Cartier operator to the 1-form dz_{iv}/z_{iv} (or by lemma 1.3 in [17]), it follows that

$$\delta^{p-1}(t^i x_v + c_i) = -(t^i x_v + c_i)^p.$$

Note that, if $\mathrm{dlog}(z_{iv}) = 0$, the above equality is trivially true.

Therefore, when (x_v) is unobstructed by $\mathrm{dlog}(z) = t^i x$, for $i = 0, \ldots, p-1$, we have a system of p equations

$$\left\{ \sum_{k=0}^{i} \binom{p-1}{k} \frac{i!}{(i-k)!} t^{i-k} \delta^{p-1-k}(x_v) = -\delta^{p-1}(c_i) - t^{ip} x_v^p - c_i^p : i = 0, \ldots, p-1 \right\},$$

or, equivalently, a matrix equation

$$A \begin{pmatrix} x_v \\ \delta(x_v) \\ \vdots \\ \delta^{p-1}(x_v) \end{pmatrix} = \begin{pmatrix} -\delta^{p-1}(c_0) - x_v^p - c_0^p \\ -\delta^{p-1}(c_1) - t^p x_v^p - c_1^p \\ \vdots \\ -\delta^{p-1}(c_{p-1}) - t^{(p-1)p} x_v^p - c_{p-1}^p \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 0 & \ldots & 0 & \binom{p-1}{0} \\ 0 & 0 & \ldots & \binom{p-1}{1}\delta(t) & \binom{p-1}{0} t \\ \vdots & \vdots & & \vdots & \vdots \\ \binom{p-1}{p-1}(p-1)! & \binom{p-1}{p-2}\delta^{p-2}(t^{p-1}) & \ldots & \binom{p-1}{1}\delta(t^{p-1}) & \binom{p-1}{0} t^{p-1} \end{pmatrix}.$$

Since

$$\det A = (-1)^{\lfloor n/2 \rfloor} \prod_{i=0}^{p-1} \binom{p-1}{i} i! \neq 0,$$

we may consider the inverse of A and write

$$\begin{pmatrix} x_v \\ \delta(x_v) \\ \vdots \\ \delta^{p-1}(x_v) \end{pmatrix} = A^{-1} \begin{pmatrix} -\delta^{p-1}(c_0) - x_v^p - c_0^p \\ -\delta^{p-1}(c_1) - t^p x_v^p - c_1^p \\ \vdots \\ -\delta^{p-1}(c_{p-1}) - t^{(p-1)p} x_v^p - c_{p-1}^p \end{pmatrix}. \tag{3.5}$$

In order to solve for x_v, we determine the first row of A^{-1} explicitly by computing the cofactors C_{i1} of the first column of A. It is easy to see that $C_{p1} = -\det A$. In order to compute the other first column cofactors, write $A = (a_{ij}t^{i+j-p-1})$ with $a_{ij} \in \mathbf{Z}/p\mathbf{Z}$ and note that, for each $j = 2, \ldots, p$,

$$\begin{aligned} \sum_{i=1}^{p} a_{ij} &= \sum_{i=p+1-j}^{p} a_{ij} \\ &= \sum_{i=p+1-j}^{p} \binom{p-1}{p-j} \frac{(i-1)!}{(i+j-p-1)!} \\ &= \binom{p-1}{p-j} \sum_{i=p+1-j}^{p} \binom{i-1}{p-j} (p-j)! \\ &= \binom{p-1}{p-j} \binom{p}{p-j+1} (p-j)! \\ &= 0. \end{aligned}$$

In the computation of C_{i1}, for $i = 1, ..., p-1$, we may thus perform the row operations

$$\begin{aligned} L_{p-1} &\longrightarrow L_{p-1} + L_p t^{-1} \\ L_{p-2} &\longrightarrow L_{p-2} + L_{p-1} t^{-1} \\ &\vdots \\ L_i &\longrightarrow L_i + L_{i+1} t^{-1} \end{aligned}$$

to get

$$C_{i1} = (-1)^{i+1} \det \begin{pmatrix} & & \text{\huge 0} & & & & \binom{p-1}{0}0! \\ & & & & & \iddots & \\ & & & & \binom{p-1}{i}i! & & \\ & & & -\binom{p-1}{i-1}(i-1)!t & & & \\ & & & & & * & \\ & & \iddots & & & & \\ -\binom{p-1}{p-2}(p-2)!t & & & & & & \end{pmatrix}$$

$$= -t^{p-i} \det A.$$

This shows that the first row of A^{-1} is

$$\begin{pmatrix} -t^{p-1} & \dots & -t & -1 \end{pmatrix},$$

and, by (3.5), we see that x_v satisfies

$$(t^{p-1} + t^{2(p-1)} + \dots + t^{p(p-1)})x_v^p - x_v + \sum_{i=0}^{p-1}(\delta^{p-1}(c_i) + c_i^p)t^i = 0.$$

Let $y_v = (t^{p-1} - 1)x_v$. Then, (y_v) satisfies

$$y_v^p - y_v + a = 0,$$

for some $a \in K$. By lemma 3.2.2, it follows that $y_v \in K$ and thus $x_v \in K$, for every v.

Fix a place v_0 of K and consider

$$\widetilde{x_v} := x_v - x_{v_0},$$

for each place v of K. If (x_v) is unobstructed by the torsor $\operatorname{dlog}(z) = f^p x$, $f \in K$, then so is $(\widetilde{x_v})$, which means that there exist $c \in K$ and $(z_v) \in \prod_v K_v^\times$ such that

$$\operatorname{dlog}(z_v) = \begin{cases} f^p \widetilde{x_v} + c, & \text{if } v \neq v_0 \\ c, & \text{if } v = v_0. \end{cases}$$

Applying the Cartier operator to the 1-form dz_v/z_v (or by lemma 1.3 in [17]), we conclude that

$$\delta^{p-1}(c) = -c^p$$

and

$$f^p \delta^{p-1}(\widetilde{x_v}) + \delta^{p-1}(c) = -f^{p^2} \widetilde{x_v}^p - c^p,$$

implying that

$$\delta^{p-1}(\widetilde{x_v}) = -f^{p^2-p} \widetilde{x_v}^p,$$

if $f \neq 0$. Letting f vary in $K^\times$, we conclude that $\widetilde{x_v} = 0$, hence $x_v = x_{v_0} \in K$, for every place v of K, that is, (x_v) is global.

□

We finish this section stating the multiplicative analogue of theorem 3.1.3. The proof is identical replacing $\mathbf{G}_a^\delta$ with $\mathbf{G}_m^\delta$ and using theorem 3.2.3 instead of theorem 3.1.2.

Theorem 3.2.4. *Let X be a smooth non-isotrivial curve of genus $g > 1$ over K. If $(x_v) \in \prod_v X(K_v)$ is unobstructed by all $\mathbf{G}_m^\delta$-torsors given by first order regular differential maps on X, that is, corresponding to elements in*

$$\Gamma(X^1, \mathcal{O}_{X^1}) \to \frac{K\{X\}}{\operatorname{dlog}(K\{X\}^\times)} \subset H^1(X^\infty, \mathbf{G}_m^\delta),$$

then $(x_v) \in X(K)$.

3.3 An example

We consider the same example from [18] but here we use $\mathbf{G}_m^\delta$-descent obstructions:

Proposition 3.3.1. *Let* $k = \mathbf{C}$ *and* $K = \mathbf{C}(t)$. *Write* K_λ *for the completion of* $K = \mathbf{C}(t)$ *with respect to the place* v_λ *corresponding to* $\lambda \in \mathbf{C} \cup \{\infty\}$ *and write* $\mathcal{O}_\lambda$ *for the ring of integers of* K_λ. *If* $\mathbf{X}$ *is the affine scheme over* $\mathcal{O}_S = \mathbf{C}[t]$ *given by*

$$x^2 + (t^3 + t^2)y^2 = t + 1,$$

then:

1. $\mathbf{X}(\mathbf{A}_{K,S}) \neq \emptyset$;
2. *if* $P_\infty = (x_\infty, y_\infty) \in X(K_\infty)$, *then* $v_\infty(y_\infty) = 1$;
3. *every adelic point*

$$(P_\lambda) = ((x_\lambda, y_\lambda)) \in X(K_\infty) \times \prod_{\lambda \in \mathbf{C}} \mathbf{X}(\mathcal{O}_\lambda)$$

is obstructed by the $\mathbf{G}_m^\delta$*-torsor*

$$\operatorname{dlog}(z) = ay,$$

where a *is half the reciprocal of the coefficient of* t^{-1} *in the expression of* y_∞ *(as a power series in* t^{-1}*).*

In particular, $\mathbf{X}(\mathbf{C}[t]) = \emptyset$.

Proof.

1. When $\lambda \notin \{-1, \infty\}$, we have a local solution of the form $P_\lambda = (x_\lambda, 0)$ with $x_\lambda^2 = t+1$. When $\lambda = -1$, we have a local solution of the form $P_{-1} = (x_{-1}, 1)$ with $x_{-1}^2 = (t+1)^2(1-t)$. When $\lambda = \infty$, we have the solution $P_\infty = (0, t^{-1})$.

2. By the equation defining $\mathbf{X}$, since $v_\infty((t^3+t^2)y_\infty^2)$ and $v_\infty(x_\infty^2)$ have different parities, we must have $v_\infty((t^3+t^2)y_\infty^2) = -1$, and thus $v_\infty(y_\infty) = 1$.

3. By the previous item, we may write

$$y_\infty = \frac{a_1}{t} + \frac{a_2}{t^2} + \dots,$$

where $a_i \in \mathbf{C}$ and $a_1 \neq 0$. Assume for a contradiction that (P_λ) is unobstructed by $\operatorname{dlog}(z) = ay$, where $a = 1/(2a_1)$ Then, there exist $(z_\lambda) \in \prod_v K_\lambda^\times$ and $c \in K$ such that

$$\operatorname{dlog}(z_\lambda) = ay_\lambda + c$$

or, in terms of 1-forms,

$$\frac{dz_\lambda}{z_\lambda} = ay_\lambda dt + cdt,$$

for every $\lambda \in \mathbf{C} \cup \{\infty\}$. We take residues. For $\lambda \in \mathbf{C}$, since y_λ is integral, we have

$$\operatorname{Res}_\lambda cdt = \operatorname{Res}_\lambda \frac{dz_\lambda}{z_\lambda} - \operatorname{Res}_\lambda ay_\lambda dt = v(z_\lambda) \in \mathbf{Z}.$$

Therefore, on one hand, for the place at infinity, we have

$$\operatorname{Res}_\infty cdt = -\sum_{\lambda \in \mathbf{C}} \operatorname{Res}_\lambda cdt \in \mathbf{Z}$$

However, on the other hand, we have

$$\begin{aligned}
\operatorname{Res}_\infty cdt &= \operatorname{Res}_\infty \frac{dz_\infty}{z_\infty} - \operatorname{Res}_\infty ay_\infty dt \\
&= v_\infty(z_\infty) - \operatorname{Res}_\infty \frac{1}{2a_1}\left(\frac{a_1}{t} + \frac{a_2}{t^2} + \dots\right) dt \\
&= v_\infty(z_\infty) - \frac{1}{2} \notin \mathbf{Z},
\end{aligned}$$

a contradiction.

$\square$

Chapter 4

$\mathbf{G}_a^\delta$-descent for isotrivial canonical curves

4.1 Background

Let K/k be our function field with the derivation $\delta = d/dt$, where t is a separating element, as in chapter 3. In this chapter and in the next, we investigate what happens to theorem 3.1.3 when we drop the non-isotrivial hypothesis.

Let C be a smooth isotrivial curve of genus g over K. By proposition 1.5.1, the first jet scheme C^1 is a torsor under the tangent bundle

$$TC = \operatorname{Spec}_C S(\Omega_{C/K})$$

of C corresponding to the Kodaira-Spencer class $ks(\delta) \in H^1(C, TC)$. Therefore, in the isotrivial case, $C^1 \cong TC$ is not affine and the argument used in the proof of theorem 3.1.3 has to be adapted.

Our adaptation will rely on whether C^1 has enough non-constant global sections, despite C^1 not being affine. For instance, when $g = 0$, $C^1 \cong TC$ has no non-constant global section and we cannot hope to obtain a local-global principle using $\mathbf{G}_a^\delta$ (or $\mathbf{G}_m^\delta$) descent obstructions. However, when $g > 0$, the ring of global sections of the tangent bundle of C, i.e., the canonical ring

$$\bigoplus_{n\geq 0} H^0(C, \Omega^n_{C/K}),$$

has non-constant functions.

We will investigate the case when C is hyperelliptic in chapter 5. For the rest of this chapter, we assume that C is a smooth non-hyperelliptic isotrivial curve of genus $g \geq 3$ over K. In this case, we have the canonical map

$$\varphi : C \to \operatorname{Proj} S(H^0(X, \Omega_{X/K})) = \mathbf{P}^{g-1}$$

induced by the linear system associated with the (very ample) canonical divisor on C. The canonical map is a closed immersion, and the canonical ring

$$\bigoplus_{n\geq 0} H^0(C, \Omega^n_{C/K}) \cong \Gamma(C^1, \mathcal{O}_{C^1})$$

identifies with the homogeneous coordinate ring of the canonical curve $\varphi(C)$. It is generated in degree 1 by Max Noether's theorem.

We fix a basis $\{s_1, \ldots, s_n\}$ of $H^0(C, \Omega_{C/K})$ so that

$$\varphi = (s_1 : \cdots : s_g),$$

and we write $\omega_i \in \Gamma(C^1, \mathcal{O}_{C^1})$ for the first order regular differential function on C corresponding to $s_i \in H^0(C, \Omega_{C/K})$ under a fixed isomorphism $C^1 \cong TC$. Given $P \in C(K)$, note that, if $\omega_i(P) \neq 0$, for some $i \in \{1, \ldots, g\}$, we may "dehomogenize" $\mathbf{P}^{g-1}_K = \operatorname{Proj} S(H^0(C, \Omega_{C/K}))$ with respect to s_i to obtain the expression

$$\varphi(P) = \left(\frac{\omega_1(P)}{\omega_i(P)} : \cdots : \frac{\omega_{i-1}(P)}{\omega_i(P)} : 1 : \frac{\omega_{i+1}(P)}{\omega_i(P)} : \cdots : \frac{\omega_g(P)}{\omega_i(P)}\right). \tag{4.1}$$

4.2 Main result

With the notations and conventions from section 4.1, set

$$C^{\delta}(\mathbf{A}_K) := \{(P_v) \in \prod_v C(K_v) : \omega_i(P_v) = 0 \text{ for all } i \text{ and } v\}.$$

We refer to a point in $C^{\delta}(\mathbf{A}_K)$ as a differential constant.

Theorem 4.2.1. *If $(P_v) \in \prod_v C(K_v)$ is unobstructed by the $\mathbf{G}_a^{\delta}$-torsor*

$$\delta(z) = f\omega_i,$$

for every $f \in K$ and $\omega_i \in \Gamma(C^1, \mathcal{O}_{C^1})$, $i = 1, \dots, g$, then

$$(P_v) \in C(K) \cup C^{\delta}(\mathbf{A}_K).$$

Proof. Suppose (P_v) is unobstructed by the $\mathbf{G}_a^{\delta}$-torsor $\delta(z) = f\omega_i$, for every $f \in K$ and $i = 1, \dots, g$. This means that there exist $c \in K$ and $(z_v) \in \prod_v K_v$ such that

$$\delta(z_v) = f\omega_i(P_v) + c,$$

for every place v of K. In other words, the point $(\omega_i(P_v)) \in \prod \mathbf{A}_K^1(K_v)$ is unobstructed by the $\mathbf{G}_a^{\delta}$-torsor over $\mathbf{A}_K^1$ given by

$$\delta(z) = fx, \quad x \in \mathbf{A}_K^1,$$

for every $f \in K$. By theorem 3.1.2, $\omega_i(P_v)$ is global, for every $i = 1, \dots, g$.

We embed C in $\mathbf{P}_K^{g-1}$ via the canonical map

$$\varphi : C \to \operatorname{Proj} S(H^0(C, \Omega_{C/K})).$$

Suppose that $\omega_i(P_v) \neq 0$, for some $i \in \{1, \dots, g\}$. We can then use (4.1) to express $\varphi(P_v)$ in terms of $\omega_1(P_v), \dots, \omega_g(P_v)$. Since $(\omega_j(P_v))$ is global for every $j = 1, \dots, g$, we conclude that $(\varphi(P_v))$ is global. Since φ is a closed immersion, it follows that (P_v) is global.

$\square$

Remark 4.2.1.

The argument used in this proof is similar to the following construction from [1]. For C a smooth isotrivial non-hyperelliptic curve of genus $g \geq 3$, one considers a sequence of maps

$$C \hookrightarrow C^1 \to \operatorname{Spec} \Gamma(C^1, \mathcal{O}_{C^1}) \cong \operatorname{Spec} \Gamma(TC, \mathcal{O}_{TC}) \to \operatorname{Spec} S(H^0(C, \Omega_{C/K}))$$

where, under the identification $C^1 \cong TC$, the map $C^1 \to \operatorname{Spec} \Gamma(C^1, \mathcal{O}_{C^1})$ is the contraction of the zero section, and

$$\operatorname{Spec} \Gamma(TC, \mathcal{O}_{TC}) = \operatorname{Spec} \bigoplus_{n \geq 0} H^0(C, \Omega^n_{C/K}) \to \operatorname{Spec} S(H^0(C, \Omega_{X/K}))$$

follows from Max Noether's theorem that states that the canonical ring

$$\bigoplus_{n \geq 0} H^0(C, \Omega^n_{C/K})$$

is generated in degree 1. This map $C \to \mathbf{A}^g$ is called the δ-Lagrangian map in [1]. The δ-Lagrangian map is constant on C^δ and is an injective local immersion on $C \backslash C^\delta$, the image of C being the affine cone over the canonical curve $\varphi(C)$ (cf. [1, Chapter 6, Theorem 2.4]).

4.3 An example

Let $K = \mathbf{C}(t)$ with $\delta = d/dt$ and consider the smooth isotrivial projective curve $C \subset \mathbf{P}^2_K$ given by

$$Y^4 + tX^4 = Z^4.$$

Since C is given by a nonsingular polynomial of degree $d = 4$, C has genus $\binom{d-1}{2} = 3$. We consider the affine model of C given by

$$y^4 + tx^4 = 1,$$

where $x = X/Z$ and $y = Y/Z$. When $P \in C(K)$ is a point on this affine portion of C, we write $P = (x, y)$. We also use the suggestive symbols x, y, dx, dy (resp. x, y, x', y') for the corresponding local system of affine coordinates on the tangent bundle TC (resp. the first jet scheme C^1). Explicitly, TC is given by the additional equation

$$y^3 dy + tx^3 dx = 0,$$

whereas C^1 is given by the additional equation

$$4y^3y' + 4tx^3x' + x^4 = 0.$$

It is not hard to see that the equation defining C^1 can be rewritten as

$$4y^3y' + 4tx^3\left(x' + \frac{x}{4t}\right) = 0$$

providing an explicit isomorphism $TC \cong C^1$ over C, namely, the one given by

$$\begin{aligned} dx &\mapsto x' + \frac{x}{4t} \\ dy &\mapsto y'. \end{aligned}$$

The regular 1-forms

$$s_1 = \frac{dx}{y^3},\ s_2 = x\frac{dx}{y^3},\ s_3 = \frac{dx}{y^2}$$

form a basis of $H^0(C, \Omega_{C/K})$ over K. Under the isomorphism $TC \cong C^1$ above, the 1-forms s_1, s_2, s_3 correspond to first order regular differential maps on C given by

$$\omega_1 = \frac{4t\delta(x) + x}{4ty^3},\ \omega_2 = x\frac{4t\delta(x) + x}{4ty^3},\ \omega_3 = y\frac{4t\delta(x) + x}{4ty^3},$$

which generate $\Gamma(C^1, \mathcal{O}_{C^1})$ as a K-algebra. Since $dx/y^3 = -dy/(tx^3)$, the isomorphism $TC \cong C^1$ provides the alternative expressions

$$\omega_1 = -\frac{\delta(y)}{tx^3}, \omega_2 = -x\frac{\delta(y)}{tx^3}, \omega_3 = -y\frac{\delta(y)}{tx^3}.$$

Proposition 4.3.1. *Let C, K and ω_i be as above. Write K_λ for the completion of $K = \mathbf{C}(t)$ with respect to the place v_λ corresponding to $\lambda \in \mathbf{C} \cup \{\infty\}$. Then:*

1. $C(K) \cap C^\delta(K) = \{(0, \pm 1), (0, \pm i)\}$;
2. *there exists $P_\lambda \in C(K_\lambda)$ such that $\omega_1(P_\lambda) \neq 0$, for every $\lambda \in \mathbf{C} \cup \{\infty\}$;*
3. *$t^k\omega_1(P_\lambda) \in \mathcal{O}_\lambda$, for every positive integer k and $\lambda \neq \infty$.*
4. *every adelic point*

$$(P_\lambda) = ((x_\lambda, y_\lambda)) \in \prod_{\lambda \in \mathbf{C} \cup \{\infty\}} C(K_\lambda)$$

not in $C^\delta(K)$ is obstructed by the $\mathbf{G}_a^\delta$-torsor

$$\delta(z) = t^n \omega_1,$$

where $n := v_\infty(x_\infty)$.

In particular, $C(K) = \{(0, \pm 1), (0, \pm i)\}$.

Proof. It is clear that the curve C has no K-rational point at infinity, so we restrict ourselves to the affine portion of C given by $y^4 + tx^4 = 1$.

1. For each $\lambda \in \mathbf{C} \cup \{\infty\}$, a K-rational point $P = (x, y)$ is a differential constant if and only if $-4\delta(x) = t^{-1}x$, that is,

   $$x^4 = ct^{-1},$$

   for some differential constant $c \in K^\delta = \mathbf{C}$ satisfying $y^4 = 1 - c$. This is only possible when $c = 0$, in which case $x = 0$ and $y^4 = 1$.

2. When $\lambda \notin \{1, \infty\}$, we have a local solution of the form $P_\lambda = (1, y_\lambda)$ with $y_\lambda^4 = 1 - t$, and

   $$\omega_1(P_\lambda) = t^{-1}y_\lambda^{-3}/4 \neq 0.$$

   When $\lambda = 1$, we have a local solution of the form $P_1 = (2, y_1)$ with $y_1^4 = 1 - 16t$, and

   $$\omega_1(P_1) = t^{-1}y_1^{-3}/2 \neq 0.$$

   When $\lambda = \infty$, we have a local solution of the form $P_\infty = (t^{-1}, y_\infty)$ with $y_\infty^4 = 1 - t^{-3}$, and

   $$\omega_1(P_\infty) = -3t^{-2}y_\infty^{-3}/4 \neq 0.$$

3. By the equation $y^4 + tx^4 = 1$ defining C, we have

   $$v_0(y_0) = 0 \quad \text{and} \quad v_0(x_0) = m,$$

for some integer $m \geq 0$. Writing $x_0 = u_0 t^m$ with $u_0 \in \mathcal{O}_0^\times$, we see that

$$\omega_1(P_0) = \frac{\delta(u_0)}{y_0^3} t^m + \frac{(4m+1)u_0}{4y_0^3} t^{m-1}$$

and thus $t^k \omega_1(P_0)$ is integral, for every positive integer k.

For $\lambda \neq 0, \infty$, again by the equation definition C, we see that one of the following must happen:

- $v_\lambda(x_\lambda) = v_\lambda(y_\lambda) \leq 0$; or
- $v_\lambda(x_\lambda) = 0$ and $v_\lambda(y_\lambda) \geq 0$; or
- $v_\lambda(y_\lambda) = 0$ and $v_\lambda(x_\lambda) \geq 0$;

In all cases, using the expressions

$$\frac{4t\delta(x_\lambda) + x}{4ty_\lambda^3} = -\frac{\delta(y_\lambda)}{tx_\lambda^3}$$

for $\omega_1(P_\lambda)$, we conclude that $t^k \omega_1(P_\lambda)$ is integral, for every integer k.

4. For $\lambda = \infty$, the equation defining C tells us that $v_\infty(y_\infty) = 0$ and $v_\infty(x_\infty)$ is a positive integer. Set $n := v_\infty(x_\infty)$, and assume for a contradiction that P is unobstructed by $\delta(z) = t^n \omega_1$. This means that there exist $c \in K$ and $(z_\lambda) \in \prod_\lambda K_\lambda$ such that

$$\delta(z_v) = t^n \omega_1(P_\lambda) + c,$$

for every $\lambda \in \mathbf{C} \cup \{\infty\}$. By what we just proved in the previous item, $t^n \omega_1(P_\lambda)$ is integral for every $\lambda \neq \infty$. It follows that the polar parts of

c at the places corresponding to $\lambda \neq \infty$ are integrable. We may then change z_λ if necessary and assume that c is a polynomial.

Writing $x_\infty = u_\infty t^{-n}$ with $u_\infty \in \mathcal{O}_\infty^\times$, we see that

$$\omega_1(P_\infty) = \frac{\delta(u_\infty)}{y_\infty^3} t^{-n} - \frac{(4n-1)u_\infty}{4y_\infty^3} t^{-n-1}.$$

Note that $v_\infty(\delta(u_\infty)) \geq 2$. Therefore, the coefficient of t^{-1} in

$$\delta(z_v) = t^n \omega_1(P_\infty) + c$$

is non-zero, a contradiction.

$\square$

Chapter 5

$\mathbf{G}_a^\delta$-descent for isotrivial hyperelliptic curves

5.1 Background

We now investigate the case when the smooth projective curve C is hyperelliptic of genus $g \geq 2$ over our function field K. In this case, the canonical map

$$\varphi : C \to \operatorname{Proj} S(H^0(C, \Omega_{C/K})) = \mathbf{P}^{g-1}$$

is no longer a closed immersion. It is instead a degree 2 morphism with $\varphi(C)$ a rational normal curve.

We first assume that the characteristic p of K is not 2 and, at the end of this chapter, in section 5.5, we discuss the case $p = 2$. In what follows, let x be an element in the function field $K(C)$ of C such that the function field of the canonical image $\varphi(C)$ identifies with $K(x)$ via φ.

Lemma 5.1.1. *$K(x)$ is the only rational subfield of $K(C)$ with*

$$[K(C) : K(x)] = 2.$$

There exist $y \in K(C)$ and a separable polynomial f in one variable of degree $2g + 1$ or $2g + 2$ over K such that

$$y^2 = f(x)$$

and $K(C) = K(x, y)$*. Moreover, if* C *has a* K*-rational Weierstrass point, i.e.,* $P \in C(K)$ *such that*

$$\dim_K H^0(C, \mathcal{O}_C(2P)) > 1,$$

then we may choose f *monic and separable of degree* $2g+1$*. In this case, the affine equation* $y^2 = f(x)$ *defining* C *is unique up to a change of coordinates of the form*

$$x = \alpha^2 \tilde{x} + \beta, \quad y = \alpha^{2g+1} \tilde{y},$$

where $\alpha \in K^\times$ *and* $\beta \in K$*.*

Proof. Cf. [15, Chapter 6, section 3] and [11, Proposition 1.2].

$\square$

Remark 5.1.1. Explicitly, C is the gluing of affine curves given by

$$y^2 = f(x)$$

and

$$v^2 = u^{2g+2} f(1/u)$$

via

$$u = \frac{1}{x} \quad \text{and} \quad v = \frac{y}{x^{g+1}}.$$

However, for our purposes, when C has a K-rational Weierstrass point P, it will be more convenient to think of C as the union of the affine curve given by $y^2 = f(x)$ and a single distinguished point at infinity corresponding to P.

For the rest of this chapter, we will assume that our field K is large enough so that C has a K-rational Weierstrass point. By lemma 5.1.1, C is then given by an equation

$$y = f(x) = x^{2g+1} + a_{2g}x^{2g} + \cdots + a_0,$$

with $a_i \in K$. We will assume that this equation is in *reduced form*, that is:

($\star$) if $m \geq 0$ is the smallest integer such that the characteristic $p \geq 0$ of K does not divide $2g+1-m$ and $a_{2g+1-m} \neq 0$, then we assume

$$a_{2g-m} = 0, \qquad \text{if } m \not\equiv -1 \pmod{p}$$

and

$$\delta(a_{2g-m}) = 0, \qquad \text{if } m \equiv -1 \pmod{p}.$$

Every isotrivial hyperelliptic curve is isomorphic to a hyperelliptic curve given by an equation in reduced form (cf. remark 5.2.1). When $p = 0$, we have $m = 0$, hence being in reduced form means that $a_{2g} = 0$. When $m \equiv -1 \pmod{p}$, we must have $2g+1$ divisible by p and thus p divides $2g - m$.

5.2 Differential functions on a hyperelliptic curve

Recall that, by proposition 1.5.1, when C isotrivial, the first jet scheme C^1 identifies with the tangent bundle TC. Let $\omega_1, \ldots, \omega_g \in \Gamma(C^1, \mathcal{O}_{C^1})$ be the first order regular differential functions on C corresponding to the regular 1-forms

$$s_i := x^{i-1}\frac{dx}{y} \in H^0(C, \Omega_{C/K}), \quad i = 1, \dots, g,$$

under an isomorphism $C^1 \cong TC$ over C. We will give explicit formulae for ω_i.

If the coefficients $a_0, \dots, a_{2g}$ of $f(x)$ in the equation defining C are differential constants, then the first jet scheme is trivially isomorphic to the tangent bundle TC and we have the expression

$$\omega_i = x^{i-1}\frac{\delta(x)}{y} = x^{i-1}\frac{2\delta(y)}{f'(x)}.$$

Note that, although we write ω_i as a rational function in order to simplify computations, ω_i is really a polynomial in the variables x, y, $\delta(x)$ and $\delta(y)$. Indeed, since $f(x)$ is separable, there exist polynomials g and h in the variable x such that

$$1 = gf + hf',$$

and thus

$$\omega_i = x^{i-1}g(x)y\delta(x) + 2x^{i-1}h(x)\delta(y).$$

If one of the coefficients $a_0, \dots, a_{2g}$ is not a differential constant, then an isomorphism $C^1 \cong TC$ is no longer trivial and

$$\frac{\delta(x)}{y}$$

need not be regular on C. For instance, let C be the isotrivial hyperelliptic curve over $\mathbf{Q}(t)$ given by $y^2 = x^{2g+1} - t^{2g+1}$ and let $\delta = d/dt$. Then, $\delta(x)/y$ is not regular at the rational point $(t, 0)$.

Theorem 5.2.1. *Let m be as in $(\star)$ and let n be the smallest positive integer greater than m such that p does not divide $n(2g+1-n)$ and $a_{2g+1-n} \neq 0$, where, for convenience, $a_k := 1$, for $k \notin \{0, \dots, 2g\}$. Then,*

$$\omega_i = x^{i-1} \frac{n\delta(x) - \mathrm{dlog}(a_{2g+1-n})x}{ny}, \qquad i = 1, \dots, g,$$

defines a differential regular map on C.

Proof. Being isotrivial means that there exist a field extension F/K and a hyperelliptic curve C' over K^δ such that $C \times_K F$ is isomorphic to $C' \times_{K^\delta} F$. Note that C' need not have a K^δ-rational Weierstrass point but $C' \times_{K^\delta} F$ has an F-rational Weierstrass point. Since C' is defined over K^δ, such F-rational Weierstrass point must be a differential constant. By lemma 5.1.1, $C' \times_{K^\delta} F$ is given by an equation

$$y^2 = g(x),$$

where g is a monic and separable polynomial of degree $2g+1$ with coefficients in F^δ. Moreover, once again by lemma 5.1.1, there exist $\alpha \in F^\times$, $\beta \in F$ such that the coefficients of

$$g(x) = \frac{f(\alpha^2 x + \beta)}{\alpha^{4g+2}}$$

are differential constants. Explicitly, if we set

$$A_{2g+1-i} = a_{2g+1-i}\alpha^{-2i},$$
$$z = \beta\alpha^{-2},$$

then the coefficient $\lambda_{2g+1-j} \in F^\delta$ of x^{2g+1-j} in $g(x)$ is given by

$$\lambda_{2g+1-j} = \sum_{i=0}^{j} \binom{2g+1-i}{2g+1-j} A_{2g+1-i} z^{j-i}, \qquad j = 0, \dots, 2g. \tag{5.1}$$

Let n be as in the statement of the theorem. We note some important consequences of (5.1):

1. For $0 \leq l < m$, setting $j = l$ in (5.1) yields
$$\lambda_{2g+1-l} = A_{2g+1-l} + \sum_{i<l} \binom{2g+1-i}{2g+1-l} A_{2g+1-i} z^{l-i}.$$
By the definition of m, a summand in the sum over $i < l$ is nonzero only if both $2g+1-i$ and $2g+1-l$ are divisible by p, in which case, $l-i$ is divisible by p, and z^{l-i} is a differential constant. Therefore, by the above expression for λ_{2g+1-l} together with induction on l, we see that A_{2g+1-l} is a differential constant, for $0 \leq l < m$.

2. Setting $j = m$ in (5.1) yields
$$\lambda_{2g+1-m} = A_{2g+1-m} + \sum_{l<m} \binom{2g+1-l}{2g+1-m} A_{2g+1-l} z^{m-l}. \qquad (5.2)$$
For $l < m$, by the definition of m, when p does not divide $2g+1-l$, we have $a_{2g+1-l} = 0$. When p divides $2g+1-l$, the binomial coefficient $\binom{2g+1-l}{2g+1-m}$ is 0 modulo p. Therefore, (5.2) reduces to
$$\lambda_{2g+1-m} = A_{2g+1-m},$$
showing that A_{2g+1-m} is a differential constant.

3. Setting $j = m+1$ in (5.1) yields
$$\begin{aligned}\lambda_{2g-m} = A_{2g-m} &+ \binom{2g+1-m}{2g-m} A_{2g+1-m} z \\ &+ \sum_{l<m} \binom{2g+1-l}{2g-m} A_{2g+1-l} z^{m-l+1}.\end{aligned}$$

By the definition of m, a summand in the sum over $l < m$ is nonzero only if both $2g+1-l$ and $2g-m$ are divisible by p, in which case, $m-l+1$ is divisible by p, and z^{m-l+1} is a differential constant. Moreover, by the discussion in (1) and (2), A_{2g+1-l} is a differential constant, for every $l \leq m$. Note that, by the definition of m, we have $A_{2g+1-m} \neq 0$. Applying δ to the expression for λ_{2g-m} above then yields

$$(2g+1-m)\delta(z) = -A_{2g+1-m}^{-1}\delta(A_{2g-m}).$$

We claim that

$$\delta(z) = 0. \tag{5.3}$$

When $m \not\equiv -1 \pmod{p}$, recall that we are assuming $a_{2g-m} = 0$, which immediately gives $\delta(z) = 0$. When $m \equiv -1 \pmod{p}$, we are assuming $\delta(a_{2g-m}) = 0$. Moreover, in this case, α^{-2m-2} is a differential constant, and thus

$$\delta(z) = (2g+1-m)\delta(z) = -A_{2g+1-m}^{-1}\alpha^{-2m-2}\delta(a_{2g-m}) = 0.$$

4. Let r be the smallest positive integer greater than m such that p does not divide $2g+1-r$ and $a_{2g+1-r} \neq 0$. In general, $m < r \leq n$. By the same argument used to reduce (5.2), setting $j = r$ in (5.1) yields

$$\begin{aligned}\lambda_{2g+1-r} = A_{2g+1-r} &+ \binom{2g-m}{2g+1-r} A_{2g-m} z^{r-m-1} \\ &+ \binom{2g+1-m}{2g+1-r} A_{2g+1-m} z^{r-m}.\end{aligned}$$

Note that the term with A_{2g-m} is always zero. Indeed, when $r \not\equiv -1 \pmod p$, we have $A_{2g-m} = 0$, and, when $m \equiv -1 \pmod p$, we have $2g-m$ divisible by p, in which case the corresponding binomial coefficient vanishes modulo p. So,

$$\lambda_{2g+1-r} = A_{2g+1-r} + \binom{2g+1-m}{2g+1-r} A_{2g+1-m} z^{r-m}.$$

By the discussion in (2) and (3), both A_{2g+1-m} and z are differential constants. It follows that $A_{2g+1-r} = a_{2g+1-r}\alpha^{-2r}$ is a differential constant and thus

$$2r\,\mathrm{dlog}(\alpha) = \mathrm{dlog}(a_{2g+1-r}).$$

Notice that if $r \neq n$, that is, if p divides r, then a_{2g+1-r} is a differential constant. More generally, by induction, we have $\delta(a_{2g+1-l}) = 0$ for all $m < l < n$ such that p does not divide $2g+1-l$ and $a_{2g+1-l} \neq 0$.

5. If $n \geq 2g+1$, then (1), (2) and (4) show that C is given by an equation with constant coefficients. In this case, $\omega_i = x^{i-1}\delta(x)/y$ guaranteed in the statement is the regular differential function on C corresponding to $s_i = x^{i-1}dx/y$ under the trivial isomorphism $TC \cong C^1$ as we discussed earlier in this section.

6. In what follows, suppose $n \leq 2g+1$. As in (4), setting $j = n$ in (5.1) yields

$$\begin{aligned}\lambda_{2g+1-n} = \quad & A_{2g+1-n} + \binom{2g+1-m}{2g+1-n} A_{2g+1-m} z^{n-m} \\ & + \sum_{m<l<n} \binom{2g+1-l}{2g+1-n} A_{2g+1-l} z^{n-l}.\end{aligned}$$

By the discussion in (2), (3) and (4), we see that $A_{2g+1-n} = a_{2g+1-n}\alpha^{-2n}$ is a differential constant and thus

$$2\,\mathrm{dlog}(\alpha) = \frac{1}{n}\,\mathrm{dlog}(a_{2g+1-n}). \tag{5.4}$$

We construct a regular differential function on $C \times_K F$ as the composition of the change of coordinates $C \times_K F \to C' \times_{K^\delta} F$ with the regular differential map $\delta(x)/y$ on C'. Explicitly, this regular differential function on $C \times_K F$ is given by

$$w_1 = \frac{\delta(\alpha^{-2}x - \beta\alpha^{-2})}{\alpha^{-2g-1}y}.$$

By the discussion in (3) above, $z = \beta\alpha^{-2}$ is a differential constant. By (5.4), it follows that

$$\begin{aligned}
\omega_1 &:= \alpha^{-2g+1}w_1 \\
&= \frac{\alpha^2\delta(\alpha^{-2}x)}{y} \\
&= \frac{\delta(x) - 2\,\mathrm{dlog}(\alpha)x}{y} \\
&= \frac{n\delta(x) - \mathrm{dlog}(a_{2g+1-n})x}{ny}.
\end{aligned}$$

By construction, this differential function is regular on C and it is clearly defined over K.

Finally, for $i = 1, \ldots, g$, let w_i be the composition of the change of coordinates $C \times_K F \to C' \times_{K^\delta} F$ with the regular differential function $x^{i-1}\delta(x)/y$ on C'. Note that

$$(x - \beta)^{i-1}\omega_1 = \alpha^{-2g+3-2i}w_i$$

is regular on $C \times_K F$. Induction on i then shows that

$$\omega_i := x^{i-1}\omega_1 = (x-\beta)^{i-1}\omega_1 - \sum_{j=0}^{i-1}(-1)^{i-1-j}\binom{i-1}{j}\beta^{i-1-j}x^j\omega_1$$

is a regular differential map on C and it is defined over K.

$\square$

Remark 5.2.1.

1. The hypothesis that $y^2 = f(x)$ is in reduced form was crucial to obtain (5.3). However, we point out that this hypothesis was only introduced to simplify our expression for ω_i. Indeed, we can always change coordinates (over K) to assume that an isotrivial hyperelliptic curve is given by an equation in reduced form. More precisely, by the expression for λ_{2g-m} in the proof above, this can be done through the change of coordinates

$$x = \tilde{x} - \frac{a_{2g-m}}{(2g+1-m)a_{2g+1-m}},$$
$$y = \tilde{y}.$$

2. We revisit the example where C is given by $y^2 = x^{2g+1} - t^{2g+1}$ over $K = \mathbf{Q}(t)$. By theorem 5.2.1,

$$\omega_1 = \frac{t\delta(x) - x}{ty} = \frac{2t\delta(y) - (2g+1)y}{(2g+1)tx^{2g}}$$

defines a regular differential map on C. We may now evaluate ω_1 at the rational point $(t, 0)$. We have $\omega_1(t, 0) = 0$.

3. As an example in positive characteristic, consider the isotrivial hyperelliptic curve given by $y^2 = x^5 + t^4x^3 + t^{10}$ over $K = \mathbf{F}_3(t)$. In this case, $m = 0$, $n = 5$ and thus
$$\omega_1 = \frac{t\delta(x) - 2x}{ty}$$
defines a regular differential map on C. Perhaps, the reader will notice that we could have used $n = 2$ to obtain the correct expression for ω_i despite $2g + 1 - n = 3$ being divisible by $p = 3$. This is explained in remark 5.2.2.2.

We can use theorem 5.2.1 to decide when an equation $y^2 = f(x)$ over K defines an isotrivial hyperelliptic curve:

Corollary 5.2.2. *Let $\overline{K}$ be a separable closure of K. The equation*

$$y^2 = x^{2g+1} + a_{2g}x^{2g} + a_{2g-1}x^{2g-1} + \cdots + a_0$$

defines an isotrivial hyperelliptic curve in reduced form over K if and only if there exist $a \in \overline{K}$ and differential constants $\lambda_1, \ldots, \lambda_{2g-1} \in \overline{K}$ such that

$$a_{2g+1-i} = \lambda_i a^i.$$

Proof. Let L be the splitting field of $f(x)$ in $\overline{K}$. Since L/K is separable, δ extends to a unique derivation of L, which we again denote by δ. By theorem 5.2.1,
$$\omega_1 = \frac{n\delta(x) - \mathrm{dlog}(a_{2g+1-n})x}{ny}$$

is regular on C and thus on $C \times_K L$. Therefore, for $(\alpha, 0) \in C(L)$, we must have

$$n\delta(\alpha) - \mathrm{dlog}(a_{2g+1-n})\alpha = 0,$$

implying that

$$\alpha^n = \lambda a_{2g+1-n},$$

for some differential constant $\lambda \in \overline{K}$. If a is an n-th root of a_{2g+1-n} in $\overline{K}$, then the zeros of $f(x)$ are

$$c_1 a, \ldots, c_{2g+1} a,$$

where $c_1, \ldots, c_{2g+1} \in \overline{K}$ are differential constants, and the result follows.

□

Remark 5.2.2.

1. We have not used the hypothesis $g \geq 2$ yet. So, theorem 5.2.1 and corollary 5.2.2 are true for elliptic curves. In the elliptic case, one could alternatively prove corollary 5.2.2 using the fact that an elliptic curve is isotrivial if and only if its j-invariant is a differential constant (cf. remark 1.4.1.3)

2. With a as in the statement of corollary 5.2.2, we have the formula

$$\omega_i = x^{i-1}\frac{\delta(x) - \mathrm{dlog}(a)x}{y},$$

which is independent of n and m.

3. Corollary 5.2.2. also allows us to write

$$2y\left(\delta(y) - \frac{2g+1}{2}\,\mathrm{dlog}(a)y\right) = f'(x)(\delta(x) - \mathrm{dlog}(a)x)$$

providing an explicit isomorphism $C^1 \cong TC$ over C and the alternative formula

$$\omega_i = x^{i-1}\frac{2\delta(y) - (2g+1)\,\mathrm{dlog}(a)y}{f'(x)}.$$

We point out that, as in the constant coefficient case discussed earlier in this section, although we represent ω_i as a rational function, ω_i is actually a polynomial in the variables x, y, $\delta(x)$ and $\delta(y)$. To write ω_i as a polynomial, let g and h be polynomials in the variable x such that

$$1 = gf + hf'.$$

Then,

$$\omega_i = x^{i-1}g(x)(\delta(x) - \mathrm{dlog}(a)x)y + x^{i-1}h(x)(2\delta(y) - (2g+1)\,\mathrm{dlog}(a)y).$$

Using the coordinates $u = 1/x$ and $v = y/x^{g+1}$, it is easy to check that $\omega_1(P) = 0$, where P is the point at infinity corresponding to $(u,v) = (0,0)$.

5.3 Main result

As in chapter 4, we set

$$C^\delta(K) = \{(P_v) \in \prod_v C(K) : \omega_1(P_v) = \cdots = \omega_g(P_v) = 0\}.$$

We prove that:

Theorem 5.3.1. *If $(P_v) \in \prod_v C(K_v)$ is unobstructed by the $\mathbf{G}_a^\delta$-torsor given by*

$$\delta(z) = a\omega_i,$$

for every $a \in K$ and $i = 1, \dots, g$, then

$$(P_v) \in C(K) \cup C^\delta(K).$$

Proof. Suppose (P_v) is unobstructed by the $\mathbf{G}_a^\delta$-torsor $\delta(z) = a\omega_i$, for every $a \in K$ and $i = 1, \dots, g$. Then, there exist $c \in K$ and $(z_v) \in \prod_v K_v$ such that

$$\delta(z_v) = f\omega_i(P_v) + c,$$

for every place v of K. In other words, the point $(\omega_i(P_v)) \in \prod \mathbf{A}_K^1(K_v)$ is unobstructed by the $\mathbf{G}_a^\delta$-torsor over $\mathbf{A}_K^1$ given by

$$\delta(z) = ax, \quad x \in \mathbf{A}_K^1,$$

for every $a \in K$. By theorem 3.1.2, it follows that $\omega_i(P_v)$ is global, for every i.

Assume $(P_v) \notin C^\delta(K)$, that is, $\omega_i(P_v) \neq 0$, for some $i \in \{1, \dots, g\}$. Then, P_v is in the affine portion of C given by the equation $y^2 = f(x)$ and we write $P_v = (x_v, y_v)$. Since the field $K(x)$ is generated by quotients of the regular 1-forms on C and $(\omega_i(P_v))$ is global for every i, we conclude that (x_v) is global.

By theorem 5.2.1,

$$\omega_1(P_v) = \frac{n\delta(x_v) - \mathrm{dlog}(a_{2g+1-n})x_v}{ny_v}.$$

Since $(\omega_1(P_v))$ and (x_v) are global, we have (y_v) global, implying that (P_v) is global.

$\square$

Remark 5.3.1. The hypothesis $g > 1$ is important in this theorem. More specifically, we used it when we said that $K(x)$ is generated by quotients of regular 1-forms. This is not true for genus one curves, since in this case a quotient of two regular 1-forms is a constant function. In fact, in the elliptic case, by the same argument used to show the invariance of the 1-form dx/y under translations [14, Chapter III, Proposition 5.1], we have

$$\omega_1(P + R_v) = \omega_1(P),$$

for every $P \in C(K)$ and $(R_v) \in C^\delta(K)$. If $\omega_1(P) \neq 0$, we then have many non-constant non-global adelic points of C that are unobstructed by all $\mathbf{G}_a^\delta$-torsors. For a concrete example, consider the function field $K = \mathbf{C}(s,t)$ with $s^2 = t^4 - 1$, $\delta = d/dt$, and let C be the elliptic curve over K given by

$$y^2 = 4x^3 + 6x^2 + 4x + 1.$$

Let

$$P = \left(\frac{1}{t-1}, \frac{s}{(t-1)^2}\right) \quad \text{and} \quad Q = \left(-\frac{1}{t+1}, -\frac{s}{(t+1)^2}\right).$$

Fix a place v_0 of K and set

$$P_v = \begin{cases} P, & \text{if } v \neq v_0 \\ Q, & \text{if } v = v_0. \end{cases}$$

Then,

$$\omega_1(P_v) = s^{-1} \in K^\times,$$

whence (P_v) is unobstructed by all torsors coming from $K[\omega_1] = \Gamma(C^1, \mathcal{O}_{C^1})$ but clearly $(P_v) \notin C(K) \cup C^\delta(K)$. However, note that $P - Q = (1/2, 0)$ is a differential constant.

5.4 An example

We illustrate theorem 5.3.1 with an example:

Proposition 5.4.1. *Let C be the smooth hyperelliptic curve given*

$$y^2 = x^5 + t$$

over $K = \mathbf{C}(t)$. Let P be the distinguished point of C "at infinity". Write K_λ for the completion of $K = \mathbf{C}(t)$ with respect to the place v_λ corresponding to $\lambda \in \mathbf{C} \cup \{\infty\}$. Then:

1. *$C(K) \cap C^\delta(K) = \{P\}$;*
2. *there exists $P_\lambda \in C(K_\lambda)$ such that $\omega_1(P_\lambda) \neq 0$, for every $\lambda \in \mathbf{C} \cup \{\infty\}$;*
3. *$t^k \omega_1(P_\lambda) \in \mathcal{O}_\lambda$, for every positive integer k and $\lambda \neq \infty$.*
4. *every adelic point*

$$(P_\lambda) = ((x_\lambda, y_\lambda)) \in \prod_{\lambda \in \mathbf{C} \cup \{\infty\}} C(K_\lambda)$$

not in $C^\delta(K)$ is obstructed by the $\mathbf{G}_a^\delta$-torsor

$$\delta(z) = t^{3n}\omega_1,$$

where $n := v_\infty(x_\infty)/2$.

In particular, $C(K) = \{P\}$.

Proof. By corollary 5.2.2, C is isotrivial with $a = t^{1/5}$ (up to multiplication by a differential constant). Theorem 5.2.1 then tells us that the differential regular function ω_1 is given by

$$\omega_1 = \frac{5t\delta(x) - x}{5ty} = \frac{2t\delta(y) - y}{10tx^4}$$

on the affine portion of C given by $y^2 = x^5 + t$, and by

$$\omega_1 = u\frac{5t\delta(u) - u}{5tv} = \frac{2t\delta(v)u - 6tv\delta(u) - vu}{10t},$$

on the affine portion of C given by $v^2 = tu^6 + u$, where $x = 1/u$ and $y = v/u^3$. In particular, we have $\omega_1(P) = 0$, where $P = (u, v) = (0, 0)$ is the distinguished point "at infinity".

1. We have already seen that $\omega_1(P) = 0$. A K-rational point $P = (x, y)$ on the affine portion of C given by $y^2 = x^5 + t$ is a differential constant if and only if $5\delta(x) = t^{-1}x$, that is,

$$x^5 = ct,$$

for some differential constant $c \in K^\delta = \mathbf{C}$ satisfying $y^2 = (c+1)t$. Note that $x^5 = ct$ can only happen if $c = 0$. However, in this case, we would need $y^2 = t$, which is not possible over $K = \mathbf{C}(t)$.

2. When $\lambda \notin \{-1, \infty\}$, we have a local solution of the form $P_\lambda = (1, y_\lambda)$ with $y_\lambda^2 = 1 + t$, and

$$\omega_1(P_\lambda) = -\frac{1}{5ty_\lambda} \neq 0.$$

When $\lambda = -1$, we have a local solution of the form $P_1 = (2, y_{-1})$ with $y_{-1}^2 = 32 + t$, and

$$\omega_1(P_{-1}) = -\frac{2}{5ty_{-1}} \neq 0.$$

When $\lambda = \infty$, we have a local solution of the form $P_\infty = (t^2, t^5 b)$ with $b^2 = 1 + t^{-9}$, and

$$\omega_1(P_\infty) = \frac{9}{5t^4 b} \neq 0.$$

3. By the equation $y^2 = x^5 + t$ defining C, we must have

$$2v_0(y_0) = 5v_0(x_0) \leq 1$$

and thus there exists an integer $m \geq 0$ such that $v_0(x_0) = -2m$ and $v_0(y_0) = -5m$. Writing $x_0 = u_0 t^{-2m}$ and $y_0 = w_0 t^{-5m}$ with $u_0, w_0 \in \mathcal{O}_0^\times$, we see that

$$\omega_1(P_0) = \frac{\delta(u_0)}{w_0} t^{3m} - \frac{(10m+1)u_0}{5w_0} t^{3m-1}.$$

Hence, $t^k \omega_1(P_0)$ is integral, for every positive integer k.

For $\lambda \neq 0, \infty$, again by the equation definition C, we see that one of the following must happen:

- $5v_\lambda(x_\lambda) = 2v_\lambda(y_\lambda) \leq 0$; or
- $v_\lambda(x_\lambda) = 0$ and $v_\lambda(y_\lambda) \geq 0$; or

- $v_\lambda(y_\lambda) = 0$ and $v_\lambda(x_\lambda) \geq 0$;

In all cases, using the expressions

$$\frac{5t\delta(x_\lambda) - x}{5ty_\lambda} = -\frac{2t\delta(y_\lambda) - y}{10tx_\lambda^4}$$

for $\omega_1(P_\lambda)$, we conclude that $t^k\omega_1(P_\lambda)$ is integral, for every integer k.

4. For $\lambda = \infty$, the equation defining C tells us that

$$2v_\infty(y_\infty) = 5v_\infty(x_\infty) \leq -1$$

and thus there exists a positive integer n such that $v_\infty(x_\infty) = -2n$ and $v_\infty(y_\infty) = -5n$. Assume for a contradiction that P is unobstructed by $\delta(z) = t^{3n}\omega_1$. This means that there exist $c \in K$ and $(z_\lambda) \in \prod_\lambda K_\lambda$ such that

$$\delta(z_v) = t^{3n}\omega_1(P_\lambda) + c,$$

for every $\lambda \in \mathbf{C} \cup \{\infty\}$. Since $t^{3n}\omega_1(P_\lambda)$ is integral for every $\lambda \neq \infty$, the polar parts of c at the places corresponding to $\lambda \neq \infty$ can be integrated. Changing z_λ if necessary, we may thus assume that c is a polynomial.

Writing $x_\infty = u_\infty t^{2n}$ and $y_\infty = w_\infty t^{5n}$ with $u_\infty, w_\infty \in \mathcal{O}_\infty^\times$, we see that

$$\omega_1(P_\infty) = \frac{\delta(u_\infty)}{w_\infty}t^{-3n} + \frac{(10n-1)u_\infty}{5w_\infty}t^{-3n-1}.$$

Note that $v_\infty(\delta(u_\infty)) \geq 2$. Therefore, the coefficient of t^{-1} in

$$\delta(z_v) = t^{3n}\omega_1(P_\infty) + c$$

is non-zero, a contradiction.

$\square$

5.5 Isotrivial hyperelliptic curves in characteristic 2

So far, we have assumed $p \neq 2$. When $p = 2$, the conclusion in theorem 5.3.1 must be weakened:

Theorem 5.5.1. *Let K be a function field in one variable over a finite field k of characteristic 2. Fix a separating element t of K/k and let $\delta = d/dt$. Let C be an isotrivial hyperelliptic curve of genus $g \geq 2$ over K given by an equation*

$$y^2 - h(x)y = f(x),$$

where $h(x)$ is a polynomial of degree at most g over K and $f(x)$ is a monic and separable polynomial of degree $2g+1$ over K. If $(P_v) \in \prod_v C(K_v) - C^\delta(K)$ is unobstructed by all $\mathbf{G}_a^\delta$-torsors coming from

$$\Gamma(C^1, \mathcal{O}_{C^1}) \to \frac{K\{C\}}{\delta(K\{C\})} \hookrightarrow H^1(C^\infty, \mathbf{G}_a^\delta),$$

then, upon writing $P_v = (x_v, y_v)$, we have (x_v) global and $y_v \in K$, for every place v of K.

Proof. Suppose (P_v) is unobstructed by the $\mathbf{G}_a^\delta$-torsor $\delta(z) = a\omega_i$, for every $a \in K$ and $i = 1, \dots, g$. Then, there exist $c \in K$ and $(z_v) \in \prod_v K_v$ such that

$$\delta(z_v) = a\omega_i(P_v) + c,$$

for every place v of K. In other words, the point $(\omega_i(P_v)) \in \prod \mathbf{A}^1_K(K_v)$ is unobstructed by the $\mathbf{G}_a^\delta$-torsor over $\mathbf{A}^1_K$ given by

$$\delta(z) = ax, \quad x \in \mathbf{A}^1_K,$$

for every $a \in K$. By theorem 3.1.2, it follows that $\omega_i(P_v)$ is global, for every i.

Assume $(P_v) \notin C^\delta(K)$, that is, $\omega_i(P_v) \neq 0$, for some $i \in \{1, \ldots, g\}$. Then, P_v is in the affine portion of C given by the equation $y^2 - h(x)y = f(x)$ and we write $P_v = (x_v, y_v)$. Since the field $K(x)$ is generated by quotients of the regular 1-forms on C and $(\omega_i(P_v))$ is global for every i, we conclude that (x_v) is global.

This implies that y_v is a root of the polynomial

$$y^2 - ay + b$$

with $a = h(x_v) \in K$ and $b = f(x_v) \in K$, for every place v of K. If $a = 0$, then b is a square in every completion of K, implying that K is a square in K and thus $y_v \in K$. If $a \neq 0$, then the Artin-Schreier polynomial

$$z^2 - z + ba^{-2}$$

has a root in every completion of K, namely, $z_v = y_v a^{-1}$. Lemma 3.2.2 then guarantees that z_v (hence, y_v) is in K, for every place v.

$\square$

Example 5.5.1. Consider the function field $K = \mathbf{F}_2(t, s)$ with $s^2 + s = t^5 + 1$ and the isotrivial hyperelliptic curve C of genus 2 over K given by

$$y^2 + y = x^5 + 1.$$

Let v_0 be a place of K and set

$$P_v = \begin{cases} (t, s), & \text{if } v \neq v_0 \\ (t, s+1), & \text{if } v = v_0. \end{cases}$$

Since C has constant coefficients, the regular differential maps on C corresponding to the basis $\{dx, xdx\}$ of $H^0(C, \Omega_{C/K})$ are

$$\omega_1 = \delta(x) \quad \text{and} \quad \omega_2 = x\delta(x).$$

We have $\omega_1(P_v) = 1$ and $\omega_2(P_v) = t$, showing that (P_v) is unobstructed by all torsors coming from $\Gamma(C^1, \mathcal{O}_{C^1}) = K[\omega_1, \omega_2]$. Clearly $(P_v) \notin C(K) \cup C^\delta(K)$, but as theorem 5.5.1 guarantees, we have $(x_v) = (t)$ global and $y_v \in K$, for every place v of K.

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Vita

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