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Infinitesimal Symmetries of Dixmier-Douady Gerbes

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Infinitesimal Symmetries of Dixmier-Douady Gerbes

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Infinitesimal Symmetries of Dixmier-Douady Gerbes

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This thesis introduces the infinitesimal symmetries of Dixmier-Douady gerbes over smooth manifolds. The collection of these symmetries are the counterpart for gerbes of the Lie algebra of circle invariant vector fields on principal circle bundles, and are intimately related to connective structures and curvings. We prove that these symmetries possess a Lie 2-algebra structure, and relate them to equivariant gerbes via a "differentiation functor". We also explain the relationship between the infinitesimal symmetries of gerbes and other mathematical structures including Courant algebroids and the String Lie 2-algebra.

Table of Contents

Chapter 1

Introduction

1.1 Overview

This thesis develops some new aspects of the geometry of Dixmier-Douady (DD) gerbes in the setting of smooth manifolds. A gerbe, roughly speaking, is a type of principal bundle, except that, unlike an ordinary bundle, the fibers of a gerbe are not *sets* but *groupoids*; DD gerbes correspond to the case of structure group $G = \mathbb{T} := U(1)$. The study of gerbes thus lies at the intersection of geometry and category theory, involving an interplay between techniques used in both subjects. In particular, many geometric structures which play a role in the study of principal bundles, such as connections, have more categorical counterparts in the world of gerbes. The main goal of this thesis is to introduce and study the *infinitesimal symmetries* of DD gerbes, which are the counterpart for gerbes of the Lie algebra of circle invariant vector fields on a principal circle bundle. Aside from their intrinsic interest, the study of these symmetries unifies a number of geometric and algebraic structures associated to gerbes, and will hopefully play a role in future applications.

Gerbes arise naturally in many areas of mathematics and physics. As such they have been studied from a number of different points of view, especially since the appearance of the influential book [Br1] by Brylinski in 1993. Much of this interest can be traced to the role gerbes play in string theory, where they appear in the guise of the physicists "B-field". They also appear in areas of pure mathematics related to and inspired by physical concepts, such as twisted K-theory and generalized complex geometry. In many of these applications the symmetries of gerbes play a crucial role, providing some of the motivation for the present work.

1.2 Gerbes, cohomology, and category number

Gerbes belong to a hierarchy of objects indexed by the natural numbers. For each $n \in \mathbb{N}$, an *n-gerbe* over a manifold M is a geometric representative of an element in $H^n(M;\mathbb{Z})$: equivalence classes of n-gerbes are in one-to-one correspondence with elements of $H^n(M;\mathbb{Z})$. For example, a 1-gerbe is simply a smooth map g from M to the circle group $\mathbb{T}$, and a 2-gerbe is a principal $\mathbb{T}$-bundle P over M. The principal focus of this thesis is the $n = 3$ case, which are simply called gerbes (without any modification). Roughly speaking, an n-gerbe is a sort of principal bundle whose fibers are elements of an $(n-1)$-category (more precisely, an $(n-1)$-groupoid); for example, the fibers of a circle bundles are sets, i.e. elements of the 1-category of sets. Similarly, with the convention that any specific set forms a 0-category, a function $g : M \to \mathbb{T}$ defines a bundle whose fibers take values in the 0-category $\mathbb{T}$. On the one hand, the objects associated to different values of n are qualitatively quite different; for example, a principal $\mathbb{T}$-bundle has much more structure than a simple $\mathbb{T}$-

valued function, and a gerbe is not even a manifold. On the other hand, there are some structures which make sense for any n, and understanding how they specialize for different cases leads to surprising insights. Indeed, it seems likely that many of the new structures introduced in this thesis in the $n = 3$ case continue to make sense for arbitrary n. The present work thus fits into the larger project of understanding higher categories from a geometric perspective.

1.3 Symmetries of circle bundles and gerbes

As mentioned above, equivalence classes of gerbes over a manifold M correspond to elements of $H^3(M;\mathbb{Z})$. One advantage of working with the gerbes themselves, as opposed to their equivalence classes, is that geometric objects have symmetries. As a prelude to discussing the symmetries of gerbes, let us first consider the case of a principal circle bundle P over M. A symmetry of P is a diffeomorphism $\Phi : P \to P$ commuting with the action of the circle group $\mathbb{T}$ on P. Such a diffeomorphism necessarily covers a diffeomorphism φ of M. The group of such symmetries fits into a group extension

$$1 \longrightarrow C^\infty_M(\mathbb{T}) \longrightarrow \mathrm{Diff}^{\mathbb{T}}(P) \longrightarrow \mathrm{Diff}_P(M) \longrightarrow 1\ , \tag{1.3.1}$$

where $\mathrm{Diff}_P(M)$ is the subgroup of the diffeomorphisms of M which preserve the isomorphism class of P. The infinitesimal version of this extension

$$0 \longrightarrow C^\infty_M(i\mathbb{R}) \longrightarrow C^\infty(TP)^{\mathbb{T}} \xrightarrow{\ \pi\ } C^\infty(TM) \longrightarrow 0 \tag{1.3.2}$$

is obtained by replacing $\mathrm{Diff}_P(M)$ and $\mathrm{Diff}^{\mathbb{T}}(P)$ by the Lie algebras of vector fields on M and $\mathbb{T}$-invariant vector fields on P, respectively; here we identify

$C^\infty_M(i\mathbb{R})$ with the set of $\mathbb{T}$-invariant vertical vector fields on P (with trivial Lie algebra structure). In particular, for each vector field ξ on M, the set $\mathcal{L}_P(\xi) = \pi^{-1}(\xi)$ of "lifts" of ξ to P is a torsor for the group $C^\infty(i\mathbb{R})$: the set of such lifts is non-empty, and any two differ by a unique vertical vector field. We also note that the extension (1.3.2) can be refined to an extension of sheaves over M, which may be viewed as sections of the Atiyah sequence of vector bundles

$$0 \longrightarrow M \times i\mathbb{R} \longrightarrow TP/\mathbb{T} \longrightarrow TM \longrightarrow 0. \tag{1.3.3}$$

In particular, for each vector field ξ on M, we have a principal $i\mathbb{R}$-bundle $\underline{\mathcal{L}}_P(\xi)$ whose set of global sections is $\mathcal{L}_P(\xi)$.

Similarly, given a gerbe $\mathcal{G}$ over M, in this paper we consider symmetries of $\mathcal{G}$ that cover symmetries of M. For example, given a diffeomorphism φ of M, a symmetry of $\mathcal{G}$ lifting φ is an isomorphism of gerbes

$$\hat{\varphi} : \mathcal{G} \xrightarrow{\cong} \varphi^*\mathcal{G}, \tag{1.3.4}$$

where $\varphi^*\mathcal{G}$ is the pull-back of $\mathcal{G}$ via φ. The collection of all such lifts comprise the objects of a category $\mathcal{L}_{\mathcal{G}}(\varphi)$, which when non-empty is (non-canonically) isomorphic to the category of principal circle bundles over M. If $\mathcal{G}$ is equipped with a connective structure $\mathcal{A}$ in the sense of Byrlinski [Br1], then we can pull-back the pair $(\mathcal{G}, \mathcal{A})$ and again define a lift to be an isomorphism $(\mathcal{G}, \mathcal{A}) \xrightarrow{\cong} \varphi^*(\mathcal{G}, \mathcal{A})$. In this case, the category $\mathcal{L}_{(\mathcal{G},\mathcal{A})}(\varphi)$ of all such lifts for a fixed φ (again, assuming it is non-empty) is isomorphic to the category of principal circle bundles over M with connection.

The first new structure we introduce in this paper is a gerbe $\underline{\mathcal{L}}_{\mathcal{G}}(\xi)$ (with band $\underline{i\mathbb{R}}$) associated to each vector field ξ on M, which we propose as an infinitesimal version of the categories of symmetries discussed in the previous paragraph: we call an object of $\underline{\mathcal{L}}_{\mathcal{G}}$ a *lift* of ξ to an infinitesimal symmetry of $\mathcal{G}$. We denote by $\underline{\mathcal{L}}_{\mathcal{G}}$ the sheaf of groupoids which is the disjoint union of the sheaves $\underline{\mathcal{L}}_{\mathcal{G}}(\xi)$ as ξ ranges over $C^{\infty}(TM)$. For simplicity, we generally work with the categories $\mathcal{L}_{\mathcal{G}}(\xi)$ and $\mathcal{L}_{\mathcal{G}}$ which are the global sections of these sheaves. In particular, for each $\xi \in C^{\infty}(TM)$, the category $\mathcal{L}_{\mathcal{G}}(\xi)$ is isomorphic to the category of principal $i\mathbb{R}$-bundles on M, and this isomorphism is canonical in the case $\xi = 0$. In the case that $\mathcal{G}$ is equipped with a connective structure $\mathcal{A}$, we also introduce categories $\mathcal{L}_{(\mathcal{G},\mathcal{A})}(\xi)$ and $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ of infinitesimal symmetries of the pair $(\mathcal{G},\mathcal{A})$ (which again can be refined to sheaves of groupoids). In this case, for each vector field ξ on M the category $\mathcal{L}_{(\mathcal{G},\mathcal{A})}(\xi)$ is isomorphic to the category of principal $i\mathbb{R}$-bundles with connection over M.

Similarly, when $\mathcal{G}$ is equipped with both a connective structure and a curving (see §3.6), there is a subcategory $\mathcal{L}_{(\mathcal{G},\mathcal{A},K)}$ of $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ consisting of infinitesimal symmetries of $(\mathcal{G},\mathcal{A})$ which preserve K.

1.4 Properties of the categories $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$

We attempt to motivate the definitions of the categories of symmetries $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ from a number of different points of view, in particular emphasizing the analogy to the circle bundle case. The bulk of the thesis is then devoted to elucidating the underlying geometric content of these symmetries.

For example, we try to answer the following questions:

(1) What is the relationship of the categories $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ to categories of non-infinitesimal symmetries of $\mathcal{G}$? For example, given a 1-parameter family of symmetries of $\mathcal{G}$, is there a process analogous to differentiation that produces an infinitesimal symmetry of $\mathcal{G}$?

(2) In the case of a circle bundle P, there is an intimate connection between invariant vector fields on P and the geometry of connections on P. What is the relationship between the infinitesimal symmetries of $\mathcal{G}$ and connective structures and curvings on $\mathcal{G}$?

(3) The infinitesimal symmetries of a circle bundle form a Lie algebra. What type of algebraic structures do $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ possess?

(4) It has been suggested in a number of places that the infinitesimal symmetries of a gerbe with connective structure are encoded in a certain Courant algebroid. How is this Courant algebroid related to our definition of the infinitesimal symmetries of $(\mathcal{G},\mathcal{A})$?

Let us briefly explain our answers to these questions, starting with the case of the category $\mathcal{L}_{\mathcal{G}}$. To explain our answer to question (1), suppose we are given a vector field ξ on M generating a 1-parameter family of diffeomorphisms of M

$$\Phi : M \times \mathbb{R} \to M. \tag{1.4.1}$$

Then there is a category $\mathcal{L}_{\mathcal{G}}(\Phi)$ of lifts of Φ to $\mathcal{G}$ [1], and we construct a "differentiation" functor

$$D : \mathcal{L}_{\mathcal{G}}(\Phi) \to \mathcal{L}_{\mathcal{G}}(\xi). \tag{1.4.2}$$

Furthermore, we prove that D gives an equivalence of categories. This is a categorical version of the statement in differential geometry that every vector field on a manifold can locally be integrated to a unique flow, and shows that each 1-parameter symmetry $\hat{\Phi}$ of $\mathcal{G}$ lifting Φ may be reconstructed up to canonical isomorphism from the corresponding infinitesimal symmetry $D(\hat{\Phi}) \in \mathcal{L}_{\mathcal{G}}(\xi)$.

As for questions (2) and (3), recall that in the circle bundle case a connection Θ on P is equivalent to a linear splitting of the Atiyah sequence (1.3.3) of vector bundles, and in particular gives a splitting of the exact sequence (1.3.2). Similarly, regarding $C^\infty(TM)$ as a category with only identity morphisms, we have a functor $\pi : \mathcal{L}_{\mathcal{G}} \to C^\infty(TM)$, and as explained in §3.6 a connective structure $\mathcal{A}$ on $\mathcal{G}$ produces a splitting of π, i.e. a way to construct a "horizontal lift" of each vector field on M to $\mathcal{G}$.

Returning to the circle bundle case, note that the extension (1.3.2) is an exact sequence of Lie algebras. A connection Θ produces a linear splitting of this extension, and the curvature 2-form Θ acts as a cocycle measuring the failure of this splitting to be a homomorphism of Lie algebras. Correspondingly,

[1] Actually, we do not assume that the flow Φ is defined for all $x \in M$ and $t \in \mathbb{R}$. An object of the category $\mathcal{L}_{\mathcal{G}}(\Phi)$ we define in §3.4 may be thought of as a local version of an $\mathbb{R}$-equivariant gerbe over M whose underlying gerbe is $\mathcal{G}$.

in §3.5 we construct operations on $\mathcal{L}_{\mathcal{G}}$ giving it the structure of a categorified Lie algebra, in the same way that a monoidal category is a categorified version of a monoid. This structure is most concretely described in a Cech-type picture relative to a collection of local trivializations for $\mathcal{G}$, and we therefore introduce a Cech version $\mathcal{L}_{g_{ijk}}$ of the category of infinitesimal symmetries. The category $\mathcal{L}_{g_{ijk}}$ then obtains the structure of a Lie 2-algebra, or alternatively a 2-term L_∞-algebra [BCr]. We can also regard $C^\infty(TM)$ as a Lie 2-algebra, and from this point of view we have a strict extension of Lie 2-algebras (as defined in [Ro2])

$$\mathcal{L}_{g_{ijk}}(0) \longrightarrow \mathcal{L}_{g_{ijk}} \longrightarrow C^\infty(TM), \tag{1.4.3}$$

where $\mathcal{L}_{g_{ijk}}(0)$ is the set of lifts of the zero vector field in the Cech picture and is isomorphic to the category of principal $i\mathbb{R}$-bundles on M. As mentioned above, a connective structure on $\mathcal{G}$ is essentially equivalent to a linear splitting of the extension (1.4.3), sending each vector field ξ on M to its horizontal lift $\hat{\xi}^h$. On the other hand, given a pair of vector fields $\xi, \eta \in C^\infty(TM)$, without additional structure there is no natural way to compare $[\hat{\xi}^h, \hat{\eta}^h]$ to $[\hat{\xi}, \hat{\eta}]^h$. On the other hand, if $\mathcal{G}$ is equipped with a curving K for the connective structure $\mathcal{A}$, then we may construct a natural isomorphism $[\hat{\xi}^h, \hat{\eta}^h] \xrightarrow{\cong} [\hat{\xi}, \hat{\eta}]^h$. The curvature 3-form of K then acts as a cocycle whose cohomology class is the obstruction to splitting the sequence (1.4.3) on the level of Lie 2-algebras.

The above discussion gives one interpretation of connective structures and curvings on $\mathcal{G}$ in terms of the infinitesimal symmetries of $\mathcal{G}$. As mentioned previously, however, we may also fix a particular connective structure $\mathcal{A}$ on $\mathcal{G}$ and consider the infinitesimal symmetries $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ of the pair $(\mathcal{G},\mathcal{A})$. Most of the constructions and results discussed above involving $\mathcal{L}_{\mathcal{G}}$ can then be generalized to this case. Furthermore, it is the symmetries of $(\mathcal{G},\mathcal{A})$ which are related to the Courant algebroid associated to $(\mathcal{G},\mathcal{A})$.

1.5 Infinitesimal connective symmetries and Courant algebroids

To explain our construction of the Courant algebroid (at the level of global sections), we begin by noting that there is a functor $\pi : \mathcal{L}_{(\mathcal{G},\mathcal{A})} \to \mathcal{L}_{\mathcal{G}}$ which forgets the connective structure. We prove that, for each $\hat{\xi} \in \mathcal{L}_{\mathcal{G}}$ the set of objects $\pi^{-1}(\hat{\xi}) \subset \mathcal{L}_{(\mathcal{G},\mathcal{A})}$ exending $\hat{\xi}$ is a torsor for the group of 1-forms on M. On the other hand, as mentioned above, for each vector field ξ on M the connective structure $\mathcal{A}$ determines a horizontal lift

$$\hat{\xi}^h \in \mathcal{L}_{\mathcal{G}}. \tag{1.5.1}$$

Thus, for each vector field ξ we obtain a torsor for Ω^1_M given by $\pi^{-1}(\hat{\xi}^h)$, which we denote by $E_{\hat{\xi}}$. Taking the disjoint union of the sets E_{ξ} as ξ ranges over all vector fields on M, we obtain a C^∞_M-module[2] $E_{(\mathcal{G},\mathcal{A})}$ fitting into an exact sequence of C^∞_M-modules

$$0 \longrightarrow C^\infty(T^*M) \longrightarrow E_{(\mathcal{G},\mathcal{A})} \longrightarrow C^\infty(TM) \longrightarrow 0. \tag{1.5.2}$$

[2]By a straightforward generalization we could refine this to a sheaf of modules.

As we explain below, $E_{(\mathcal{G},\mathcal{A})}$ may be identified with the global sections of a Courant algebroid.

We can also explain the relationship between the other structures possessed by a Courant algebroid and the algebraic structures on $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$. Given a pair of vector fields ξ, η, and a pair of connective lifts $\check{\xi}, \check{\eta} \in E$ extending $\hat{\xi}^h, \hat{\eta}^h$, their bracket $[\check{\xi}, \check{\eta}]$ in the category $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ is *not* in general itself a connective extension of $\widehat{[\xi,\eta]}^h$, i.e. is not an element of $E_{[\xi,\eta]}$. On the other hand, $[\check{\xi}, \check{\eta}]$ is *naturally isomorphic* to an element $[\check{\xi}, \check{\eta}]_E \in E_{[\xi,\eta]}$, where the bracket $[\cdot,\cdot]_E$ corresponds to the Courant bracket. We also construct a non-degenerate symmetric bilinear pairing

$$\langle \cdot, \cdot \rangle : E \times E \to C^\infty_M \tag{1.5.3}$$

which is suitably compatible with the bracket $[\cdot,\cdot]_E$ and the projection to $C^\infty(TM)$.

The construction of the Courant algebroid in [Hi] is given in terms of Cech data $\{g_{ijk}, A_{ij}\}$ for $(\mathcal{G},\mathcal{A})$, so to compare this construction with ours we work with the Cech version $\mathcal{L}_{(g_{ijk},A_{ij})}$ of our category of infinitesimal symmetries. After explaining the relationship between the two constructions, we construct an isomorphism of Lie 2-algebras

$$L_{E_{(g_{ijk},A_{ij})}} \xrightarrow{\cong} \mathcal{L}_{(g_{ijk},A_{ij})}, \tag{1.5.4}$$

where $L_{E_{(g_{ijk},A_{ij})}}$ is the 2-term L_∞-algebra constructed from $E_{(g_{ijk},A_{ij})}$ using the results of [RW] and [Ro1]. Since the proof that $\mathcal{L}_{(g_{ijk},A_{ij})}$ is an L_∞-algebra follows almost immediately from its construction, the isomorphism (1.5.4) gives some intuition as to the origin of the L_∞-structure on $L_{E_{(g_{ijk}, A_{ij})}}$.

1.6 Outline of paper

The first few sections of the paper are devoted to reviewing well-known material and introducing convenient notation and terminology. In §2.1 we review the geometry of vector fields on principal $\mathbb{T}$-bundles; this material serves as a useful point of reference for the remainder of the paper and motivates many of our later constructions involving gerbes. In §2.2 we explain some constructions involving torsors for sheaves of abelian groups. In §2.3 we review basic definitions and results involving gerbes, in particular emphasizing maps between gerbes with different band, as well as the the 2-categorical structure possessed by the collection of gerbes over a fixed manifold M.

We begin our study of the infinitesimal symmetries of gerbes in §3. Motivated ideas from the circle bundle case, as well as some ideas from algebraic geometry, we introduce the sheaf of infinitesimal symmetries $\underline{\mathcal{L}}_{\mathcal{G}}$ in Definition 3.2.4. In §3.3 we explain how infinitesimal symmetries appear in the Cech picture. In §3.4 we explain the local relationship between 1-parameter symmetries of $\mathcal{G}$ and infinitesimal symmetries. In §3.5 we examine the algebraic structure of the category of infinitesimal gerbe symetries, in particular introducing operations of addition, scalar multiplication and the Lie bracket. We use these operations to give $\mathcal{L}_{g_{ijk}}$ the structure of a 2-term L_∞-algebra. In §3.6 we introduce connective structures and curvings, and in §4 introduce infinitesimal connective symmetries. In §4.3 give a description of the infinitesimal symmetries of a gerbe in terms of so-called "transition circle torsors". We then use this description in §4.4 to explore our a wide class of examples provided by gerbes over certain compact Lie groups. In §4.5, we generalize the discussion in §3.4 to the connective case. Finally, in §4.6 we discuss the relationship between connective symmetries and Courant algebroids. There are two appendices which contain some of the more technical arguments and definitions. Appendix A goes into more detail about the relationship between 1-parameter families of gerbe symmetries and infinitesimal symmetries. In appendix B we recall the precise definitions related to the 2-category of gerbes. We use this material to prove some results from the main text.

1.7 Remark on models of gerbes

There exist in the literature various models of gerbes. In this paper we adopt the perspective of Brylinski [Br1], where a Dixmier-Douady gerbe is a sheaf of categories equipped with some extra structure. We have found it to be easier to understand certain conceptual questions about the symmetries of gerbes using this model. On the other hand, because the categorical model keeps track of so much information (isomorphic objects are never identified), constructions in this picture can become quite intricate. As a result, we have found it useful to recast many of the main definitions and constructions in a more concrete Cech-type model for gerbes, in which a gerbe is represented by a $\mathbb{T}$-valued Cech 2-cocycle on M. Most constructions can be done equally well in either approach, although our construction of the Lie 2-algebra structures on $\mathcal{L}_{\mathcal{G}}$ and $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ are done only in the Cech picture. One could similarly phrase our ideas in terms of other models of gerbes, for example bundle gerbes or presentations of gerbes as groupoids.

Chapter 2

Preliminaries

2.1 Infinitesimal Symmetries of Circle Bundles

We begin by reviewing the basic geometry of vector fields on circle bundles and their relationship to 1-parameter groups of symmetries. In particular, we formulate the basic definitions and structures in sheaf-theoretic terms. The discussion in this section lays the groundwork for our discussion of infinitesimal symmetries of Dixmier-Douady gerbes, and we will continually refer back to the circle bundle case throughout the rest of the paper as a point of reference.

Let $\pi : E \to M$ be a principal $\mathbb{T}$-bundle over a smooth manifold M. Thus E is a smooth manifold with a free action of the circle group $\mathbb{T}$ whose orbits are the fibers of the projection map π. For every $x \in E$, the kernel of $\pi_* : T_x E \to T_{\pi(x)} M$ is the one dimensional subspace consisting of *vertical vectors*. The $\mathbb{T}$-action induces a canonical isomorphism of this subspace with the Lie algebra $\mathrm{Lie}(\mathbb{T})$, which we identify with the $\mathbb{R}$-vector space $i\mathbb{R}$ consisting of purely imaginary complex numbers.

A smooth 1-parameter group of diffeomorphisms of M is a collection of diffeomorphisms $\{\varphi_t : M \to M\}$ such that $\varphi'_t \circ \varphi_t = \varphi_{t+t'}$ for each $t, t' \in \mathbb{R}$, and such that the corresponding map

$$\Phi : M \times \mathbb{R} \to M \tag{2.1.1}$$

is smooth.

Definition 2.1.2. *Let Φ be a be a smooth 1-parameter group of diffeomorphisms of M. A 1-parameter group of symmetries of E lifting Φ is a 1-parameter group of diffeomorphism*

$$\hat{\Phi} : E \times \mathbb{R} \to E \tag{2.1.3}$$

such that for each $t \in \mathbb{R}$, the map $\Phi(\cdot, t) := \varphi_t$ commutes with the $\mathbb{T}$-action and satisfies $\varphi_t \pi = \pi \hat{\varphi}_t$.

Given such a pair $(\Phi, \hat{\Phi})$, we obtain vector fields $\xi \in C^\infty(TM)$ and $\hat{\xi} \in C^\infty(TE)$ by differentiating at $t = 0$. Definition 2.1.2 implies that $\hat{\xi}$ is $\mathbb{T}$-invariant and projects to ξ; we call such a vector field a *lift* of ξ to E.

Let us describe a lift $\hat{\xi}$ in terms of the sheaf of sections $\underline{E}$ of $E \to M$. Let σ be a local section of E. Then $\hat{\xi} - \sigma_* \xi$ is a vertical vector at each point in the image of σ, and therefore the section σ determines a local $i\mathbb{R}$-valued function f_σ according to the formula

$$\hat{\xi}_{\sigma(x)} = \sigma_* \xi_x + f_\sigma(x). \tag{2.1.4}$$

Because $\hat{\xi}$ is $\mathbb{T}$-invariant, it is completely determined locally by f_σ. Moreover, it is easy to see that the assignment $\sigma \mapsto f_\sigma$ is compatible with restrictions to smaller open sets, so that we obtain a sheaf homomorphism

$$\begin{aligned} F_{\hat{\xi}} : \underline{E} &\to \underline{i\mathbb{R}}_M \\ \sigma &\mapsto f_\sigma, \end{aligned} \tag{2.1.5}$$

where $\underline{i\mathbb{R}}_M$ denotes the sheaf of smooth $i\mathbb{R}$-valued functions on M.

The sheaf of sections $\underline{E}$ is naturally a torsor for the sheaf $\underline{\mathbb{T}}_M$ of smooth $\mathbb{T}$-valued functions on M, as discussed in detail in the next section. Given a local section σ of E and a local $\mathbb{T}$-valued function g, equation (2.1.4) implies that

$$F_{\hat{\xi}}(\sigma \cdot g) = F_{\hat{\xi}}(\sigma) - \iota_\xi d \log(g), \tag{2.1.6}$$

where $d\log(g)$ denotes $g^{-1}dg$. Conversely, given a sheaf homomorphism $\underline{E} \to \underline{i\mathbb{R}}_M$ satisfying (2.1.6) we obtain a unique lift of ξ to E.

To set the stage for our discussion in §3.4 and appendix A, let us derive (2.1.6) by considering the relationship between the vector field $\hat{\xi}$ and the 1-parameter family $\hat{\Phi}$.

Notation 2.1.7. We will denote $M \times \mathbb{R}$ by M^1. We will denote the projection

$$\pi_1 : M^1 \to M \tag{2.1.8}$$

by p_1 and

$$\Phi : M^1 \to M \tag{2.1.9}$$

by p_0. Furthermore, for each open set U of M, we define the open set

$$\nu(U) = \{(x,t) \in U \times \mathbb{R} : \varphi_t(x) \in U\} \subset M^1. \tag{2.1.10}$$

Note that there is a natural inclusion $U \hookrightarrow \nu^1(U)$ given by $x \mapsto (x, 0)$.

For each section σ of E over U define a function $g_\sigma : \nu(U) \to \mathbb{T}$ by

$$\hat{\varphi}_t(\sigma(x)) = \sigma(\varphi_t(x)) g_\sigma(x,t). \tag{2.1.11}$$

The function $f_\sigma : U \to i\mathbb{R}$ is then related to g_σ by

$$f_\sigma(x) = \iota_{\frac{d}{dt}} d \log(g_\sigma(x,t))|_U. \tag{2.1.12}$$

Given another local section σ', there is a unique function $h : U \to \mathbb{T}$ such that $\sigma' = \sigma \cdot h$. Equation (2.1.11) then implies that

$$g_{\sigma'}(x,t) = g_\sigma(x,t) h(x) h^{-1}(\varphi_t(x)). \tag{2.1.13}$$

Combining this with (2.1.12) we obtain (2.1.6).

Next, suppose E has a connection Θ. We view Θ as a map A_Θ from the sheaf of sections of E to the sheaf $\underline{i\Omega^1_M}$ of imaginary 1-forms on M by defining

$$A_\Theta(\sigma) = \sigma^* \Theta. \tag{2.1.14}$$

The behavior of A_Θ under gauge transformations is

$$A_\Theta(\sigma \cdot g) = A_\Theta(\sigma) + g^{-1} dg. \tag{2.1.15}$$

Given a vector field ξ, there is a unique *horizontal lift* $\hat{\xi}^h$ of ξ to P, characterized by the equation

$$f_{\hat{\xi}^h}(\sigma) = -\iota_\xi A_\Theta(\sigma). \tag{2.1.16}$$

On the other hand, if we define $\hat{\xi}^h$ by equation (2.1.16), it is easy to see from equation (2.1.15) that $f_{\hat{\xi}^h}$ satisfies equation (2.1.6).

Suppose that $\hat{\xi}$ and $\hat{\eta}$ are a pair of vector fields on E lifting vector fields ξ and η on M. Then the bracket $[\hat{\xi}, \hat{\eta}]$ is a lift of $[\xi, \eta]$ satisfying

$$f_{[\hat{\xi},\hat{\eta}]} = \xi(f_{\hat{\eta}}) - \eta(f_{\hat{\xi}}). \tag{2.1.17}$$

In particular, if $\hat{\xi} = \hat{\xi}^h$ and $\hat{\eta} = \hat{\eta}^h$ are the horizontal lifts, then

$$\begin{aligned} & f_{[\xi,\eta]^h} - f_{[\xi^h,\eta^h]} \\ &= \iota_{[\xi,\eta]} A_\Theta - \iota_\xi d \iota_\eta A_\Theta + \iota_\eta d \iota_\xi A_\Theta \\ &= -\iota_\eta \iota_\xi K(\Theta), \end{aligned} \tag{2.1.18}$$

where $K(\Theta) \in i\Omega^2(M)$ is the curvature form of Θ and satisfies $K(\Theta) = dA_\Theta(\sigma)$ for any local section σ.

2.2 Torsors

Let $\mathrm{Bund}_M(\mathbb{T})$ denote the category of principal $\mathbb{T}$-bundles over a manifold M. The starting point for this section is the observation that $\mathrm{Bund}_M(\mathbb{T})$ possesses many structures analogous to those of an abelian group, i.e. it is a Picard category. For example, given $E, E' \in \mathrm{Bund}_M(\mathbb{T})$ we can form their tensor product $E \otimes E'$; this operation is commutative and associative up to natural isomorphism, and the trivial $\mathbb{T}$-bundle $M \times \mathbb{T}$ acts as a multiplicative unit. Furthermore, for any bundle E the dual bundle $E^\vee$ plays the role of an inverse for E. As explained below, there are also operations corresponding to taking Lie derivatives and the exterior derivative. These examples can be conveniently described as special cases of the *associated torsor* construction.

After introducing this construction and examining its properties, we use it to rephrase some of the discussion in the last section in a way which will easily generalize to the gerbe setting.

Definition 2.2.1. *Let A be an abelian group.*

1. *An A-*torsor *is a set S with a simply transitive action of A on S:*

$$(s, a) \mapsto s + a, \text{ for } s \in S \text{ and } a \in A. \tag{2.2.2}$$

2. *Given A-torsors S and T, a* homomorphism of A-torsors *is a map $f : S \to T$ such that $f(s+a) = f(s) + a$ for all $s \in S$ and $a \in A$.*

Notation 2.2.3. We will denote the category of all A-torsors by $\mathbf{Tor}_A$. We write the action of A on a torsor additively unless multiplication in the group A is conventionally written multiplicatively (e.g. $\mathbb{T}$). Note that because A is abelian we do not distinguish between left and right A-torsors.

We now introduce the associated torsor construction.

Definition 2.2.4. *Let $\varphi : A \to B$ be a homomorphism of abelian groups. Given $T \in \mathbf{Tor}_A$ let $\sim$ denote the equivalence relation on $T \times B$ given by*

$$(t + a, b) \sim (t, b + \varphi(a)) \text{ for all } a \in A. \tag{2.2.5}$$

The associated B-torsor *$\varphi[T]$ is the quotient*

$$T \times B / \sim, \tag{2.2.6}$$

with elements of B acting on the second factor.

Example 2.2.7. Let $A = \mathbb{T} \times \mathbb{T}$, $B = \mathbb{T}$; because $\mathbb{T}$ is abelian group multiplication defines a homomorphism $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$. Given $S, T \in \mathbf{Tor}_{\mathbb{T}}$, $S \times T$ is naturally a $\mathbb{T} \times \mathbb{T}$-torsor; the associated torsor $\mu[S \times T]$ is the tensor product $S \otimes T$. In the case where A is written addititively (e.g. $i\mathbb{R}$), if $\varphi : A \times A \to A$ is the addition homomorphism we will use the notation $\varphi[S \times T] := S \boxplus T$.

Again, because A is abelian, the inverse map $a \mapsto -a$ defines a group homomorphism, and in this case the associated torsor construction takes a torsor T to its dual $T^{\vee}$. Given $S, T \in \mathbf{Tor}_A$, we will sometimes use the notation $T \boxminus S$ to denote the A-torsor $\mathrm{Hom}(S, T)$; this can be identified with the torsor associated to $S \times T$ via the homomorphism from $A \times A \to A$ taking $(a, a') \mapsto a' - a$.

In the case that $A = i\mathbb{R}$ (or more generally any vector space), given $T \in \mathbf{Tor}_{i\mathbb{R}}$ and $\lambda \in \mathbb{R}$ regarded as a homomorphism $i\mathbb{R} \to i\mathbb{R}$, we will denote the associated torsor $\lambda[T \times i\mathbb{R}]$ by $\lambda \odot T$.

Definition 2.2.8. *Given $T \in \boldsymbol{Tor}_A$, $R \in \boldsymbol{Tor}_B$ and $\varphi : A \to B$, a map $\psi : T \to R$* intertwines *φ if $\psi(t + a) = \psi(t) + \varphi(a)$ for all $t \in T$ and $a \in A$.*

Remark 2.2.9. Note that, by construction, there is a natural map $\varphi[\cdot] : T \to \varphi[T]$ intertwining φ; the associated torsor $\varphi[T]$ can in fact be characterized by the existence of such a map.

We also note without proof the following easily verified facts about the associated torsor construction:

Proposition 2.2.10. *1. Let $\varphi : A \to B$ be a homomorphism of abelian groups. Then there is a functor*

$$\varphi[\cdot] : \boldsymbol{Tor}_A \to \boldsymbol{Tor}_B \tag{2.2.11}$$

sending $T \to \varphi[T]$, and a homomorphism $\psi : T \to T'$ of A-torsors to the unique homomorphism $\varphi[\psi] : \varphi[T] \to \varphi[T']$ satisfying $\varphi[\psi](\varphi[t]) = \varphi[\psi(t)]$ for all $t \in T$.

2. There is a canonical isomorphism

$$\varphi[A] \cong B, \tag{2.2.12}$$

where we consider A as an A-torsor with the obvious action on itself.

3. Given another abelian group C and a homomorphism $\psi : B \to C$, there is a natural isomorphism

$$(\alpha_{\psi,\varphi})_T : \psi[\varphi[T]] \cong (\psi\varphi)[T]. \tag{2.2.13}$$

We now generalize to the context of sheaves over a manifold M. For example, given a principal A-bundle E for A a Lie group, the sheaf of sections of E carries an action of the sheaf of smooth A-valued functions.

Definition 2.2.14. *Let M be a manifold, and let A be a sheaf of abelian groups over M. Then a* sheaf of A-torsors *over M is a sheaf T of sets over M together with a homomorphism of sheaves $\alpha : T \times A \to T$ such that*

1. *For every open subset $U \subset M$ such that $T(U)$ is non-empty, $(T(U), \alpha(U))$ is an $A(U)$-torsor,*

2. *Every $x \in M$ is contained in a neighborhood $U \subset M$ such that $T(U)$ is non-empty.*

Remark 2.2.15. Given a sheaf of abelian groups A, one defines a morphism of A-torsors in the obvious way. We will denote the resulting category by $\mathbf{Tor}_A$.

Example 2.2.16. As mentioned above, given any principal A-bundle E, its sheaf of sections $\underline{E}$ is naturally an $\underline{A}_M$-torsor, where $\underline{A}_M$ denotes the sheaf of smooth A-valued functions on M. Moreover, the assignment $E \mapsto \underline{E}$ is functorial and induces an equivalence of categories.

Example 2.2.17. We will also be interested in sheaves of groups which are not of the form $\underline{A}_M$ for some fixed Lie group A. For example, given a principal circle bundle E over M the set of connections for E is torsor for the group of 1-forms on M. More generally, we have a sheaf of connections which is a torsor for the sheaf of (imaginary) 1-forms on M.

Definition 2.2.18. *Given a homomorphism $\varphi : A \to B$ of sheaves of groups over M and torsors $S \in \boldsymbol{Tor}_A$, $T \in \boldsymbol{Tor}_B$, a sheaf homomorphism $\psi : S \to T$* intertwines φ *if for each open set $U \subset M$ such that $S(U)$ is non-empty, $T(U)$ is also non-empty and the map of torsors*

$$\psi_U : S(U) \to T(U) \tag{2.2.19}$$

intertwines

$$\varphi_U : A(U) \to B(U). \tag{2.2.20}$$

We may extend Definition 2.2.4 of the associated torsor construction to the sheaf setting. For brevity we give a definition in terms of a universal property, and then sketch a construction.

Definition 2.2.21. *Let $\varphi : A \to B$ be a homomorphism of sheaves of groups over M, and let S be an A-torsor. Than a* torsor associated to S via φ *is a B-torsor $\varphi[S]$ together with a homomorphism of sheaves*

$$\varphi[\cdot] : S \to \varphi[S] \tag{2.2.22}$$

intertwining φ.

One easily shows from the definition that the associated torsor $\varphi[S]$ is unique up to unique isomorphism, so that for will speak of *the* associated torsor. For a specific construction, we may first take

$$\varphi[S](U) = \varphi_U[S(U)]. \tag{2.2.23}$$

Given an inclusion of open sets $i : V \hookrightarrow U$, the restriction map is then characterized by the equation

$$i^*(\varphi_U[s]) = \varphi_V[i^* s] \tag{2.2.24}$$

for each $s \in S(U)$. In general this procedure only defines a presheaf, and we must take its sheafification to finish the construction of $\varphi[S]$; for example, if $A = \underline{\mathbb{T}}_M$ and $B = \underline{i\mathbb{R}}_M$ then S may have no global sections, whereas any $\underline{i\mathbb{R}}_M$-torsor does have a global section. It is then easily checked that the appropriate generalization of Proposition 2.2.10 holds in the sheaf setting.

Remark 2.2.25. Let us consider how the associated torsor construction looks in terms of local trivializations and transition functions. Given a sheaf A of abelian groups over M, let $E \in \mathbf{Tor}_A$. Given an open cover $\{U_i\}$ of M and local sections $\sigma_i \in E(U_i)$, we obtain "transition functions" $a_{ij} \in A(U_{ij} = U_i \cap U_j)$ given by

$$\sigma_j|_{U_{ij}} = \sigma_i|_{U_{ij}} + a_{ij}. \tag{2.2.26}$$

It is easily verified that on triple overlaps $U_i \cap U_j \cap U_k := U_{ijk}$ we have

$$a_{jk} - a_{ik} + a_{ij} = 0, \tag{2.2.27}$$

where each term in equation (2.2.27) is implicitly restricted to U_{ijk}. Let B be another sheaf of abelian groups over M and $\varphi : A \to B$ a sheaf homomorphism. By the universal property of the associated torsor construction, we obtain local sections $\varphi[\sigma_i]$ of $\varphi[E]$ over U_i. Since the map taking $\sigma_i \mapsto \varphi[\sigma_i]$ intertwines φ, it follows that the transition functions for $\varphi[E]$ with respect to the local sections $\varphi[\sigma_i]$ are given by $\varphi(a_{ij})$.

In the next example we introduce an operation on $\mathbb{T}$-bundles analogous to taking the Lie derivative with respect to a vector field on M.

Example 2.2.28. Given a manifold M, note that a vector field $\xi \in C^\infty(TM)$ determines a sheaf homomorphism $\underline{\mathbb{T}}_M \to \underline{i\mathbb{R}}_M$ given by

$$\iota_\xi \mathrm{dlog} : g \mapsto \iota_\xi g^{-1} dg. \tag{2.2.29}$$

Let E is a principal $\mathbb{T}$-bundle over M and $\underline{E}$ its sheaf of sections. By the discussion in section §2.1, a lift of ξ to E is equivalent to a sheaf homomorphism

$f_{\hat{\xi}} : \underline{E} \to \underline{i\mathbb{R}}$ intertwining $-\iota_\xi d\log$. We may therefore identify the sheaf of such lifts with[1]

$$\underline{\mathrm{Hom}}(-\iota_\xi d\log[\underline{E}], \underline{i\mathbb{R}}_M) \cong \underline{\mathrm{Hom}}(\underline{i\mathbb{R}}_M, \iota_\xi d\log[\underline{E}]) \cong \iota_\xi d\log[\underline{E}]. \tag{2.2.30}$$

To see this isomorphism concretely on the level of global sections, suppose that we are given a global section S of $\iota_\xi d\log[\underline{E}]$. Given a local section $\sigma \in \underline{E}(U)$, there is a unique function $f_S(\sigma) : U \to i\mathbb{R}$ such that

$$S|_U = \iota_\xi d\log[\sigma] + f_S(\sigma). \tag{2.2.31}$$

A short calculation shows that given $g : U \to \mathbb{T}$ we must have $f_S(\sigma \cdot g) = f_S(\sigma) - \iota_\xi d\log(g)$, so that $\sigma \mapsto f_S(\sigma)$ is a homomorphism from $\underline{E}$ to $\underline{i\mathbb{R}}_M$ intertwining $-\iota_\xi d\log$. Conversely, given such a homomorphism $f_{\hat{\xi}}$, we can define a global section $\iota_\xi d\log[\underline{E}]$ which is given locally by the formula (2.2.31).

Example 2.2.32. Let Θ be a connection on the principal $\mathbb{T}$-bundle E. Then equation (2.1.15) says that the sheaf homomorphism $A_\Theta : \underline{E} \to \underline{\Omega}^1_M(i\mathbb{R})$ defined by equation (2.1.14) is consistent with the homomorphism

$$\begin{aligned} \mathrm{dlog} : \underline{\mathbb{T}}_M &\to \underline{\Omega}^1_M \\ g &\mapsto d\log(g). \end{aligned} \tag{2.2.33}$$

Proceeding as in the last example, we see that the $i\Omega^1_M$-torsor of connections on E can be identified with the global sections of $-d\log[\underline{E}]$. Explicitly,

[1] Given sheaves A and B, we denote by $\underline{\mathrm{Hom}}(A, B)$ the *sheaf* of homomorphisms from A to B

given a connection (viewed as a sheaf homomorphism $A_\Theta : \underline{E} \to \underline{i\Omega^1_M}$) the corresponding section Θ of $-d\log[\underline{E}]$ is given locally by

$$\Theta|_U = -d\log[\sigma] + A_\Theta(\sigma) \tag{2.2.34}$$

for σ a local section of E. We remark also that, given such a section Θ, we obtain a section

$$\iota_{-\xi}[\Theta] \tag{2.2.35}$$

of

$$\iota_{-\xi}[-d\log[\underline{E}]] \cong \iota_\xi d\log[\underline{E}]. \tag{2.2.36}$$

As explained in the last example, global sections of $\iota_\xi d\log[\underline{E}]$ can be identified with lifts of ξ to E; in this case, the section (2.2.35) is the horizontal lift determined by Θ as described in §2.1.

2.3 Gerbes

In this section we recall some of the definitions and constructions related to gerbes which we will need in the remainder of the paper. More detailed definitions can be found in Appendix B. The standard reference for most of this material is [Br1], and much of our notation and terminology follows this source. In addition to this standard material, we introduce some terminology which generalizes the material in §2.2 to the context of gerbes. We will also find it convenient to describe the family of all gerbes over a given manifold using the language of (strict) two-categories. Our reference for 2-categories is [Bo].

Following the approach in [Br1], we use the model of a gerbe $\mathcal{G}$ over a manifold M as a sheaf of groupoids equipped with extra structure. Briefly, a presheaf of groupoids $\mathcal{G}$ consists of the following data: for each open set U of M we have a groupoid $\mathcal{G}(U)$, and for each inclusion of open sets $i : V \hookrightarrow U$ we a restriction functor $i^* : \mathcal{G}(U) \to \mathcal{G}(V)$. Given another inclusion of open sets $j : W \hookrightarrow V$, there is a specified natural isomorphism $\alpha_{i,j} : j^* i^* \xrightarrow{\cong} (ij)^*$. These natural isomorphisms are themselves required to satisfy a coherence condition with respect to chains of inclusions of the form

$$T \hookrightarrow W \hookrightarrow V \hookrightarrow U. \tag{2.3.1}$$

It is a consequence of the above discussion (as summarized more carefully in the Appendix B in Definition .2.1), that for every open set $U \subset M$ and every pair of objects $P, Q \in \mathcal{G}(U)$ there is a presheaf $\underline{\text{Hom}}(P, Q)$ over U. $\mathcal{G}$ is called a *prestack* if $\underline{\text{Hom}}(P, Q)$ is actually a sheaf for each open set U and each pair P, Q. A prestack $\mathcal{G}$ is a *stack* (or sheaf of groupoids) if in addition it satisfies a gluing property, as outlined in the discussion following Definition .2.1.

A gerbe is a stack equipped with additional structure analogous to that possessed by a principal bundle.

Definition 2.3.2. *Let A be a sheaf of abelian groups over M. A* gerbe with band *A over M is a stack $\mathcal{G}$ equipped with a family of isomorphisms*

$$\alpha_P : \underline{Aut}(P) \xrightarrow{\cong} A|_U \tag{2.3.3}$$

for each open set $U \subset M$ and each object $P \in \mathcal{G}(U)$, such that for each object $Q \in \mathcal{G}(U)$ and each isomorphism $\psi : P \to Q$, α_P and α_Q are compatible with the induced isomorphism $\underline{Aut}(P) \to \underline{Aut}(Q)$. In addition, we require that

1. *Every point $x \in M$ is contained in a neighborhood $U \subset M$ such that $\mathcal{G}(U)$ is non-empty.*

2. *Any two $P, Q \in \mathcal{G}(U)$ are* locally isomorphic*; that is, for each $x \in U$ there exists a neighborhood V of x in U such that the restrictions of P and Q to V are isomorphic.*

Terminology 2.3.4. A gerbe with band $\underline{\mathbb{T}}$ is called a *Dixmier-Douady gerbe.*

Remark 2.3.5. Note that for any two local sections $P, Q \in \mathcal{G}(U)$, the sheaf $\underline{\mathrm{Hom}}(P, Q)$ has the structure of a $\underline{\mathbb{T}}_U$-torsor: given $\psi \in \underline{\mathrm{Hom}}(P, Q)(V)$ for $V \subset U$ and $g : V \to \mathbb{T}$, we define

$$\psi \cdot g = \alpha_P(g) \circ \psi. \tag{2.3.6}$$

Example 2.3.7. Let A be a sheaf of abelian groups over M. Then the *trivial gerbe* $\mathrm{B}(A)$ with band A associates to each open set $U \subset M$ the category of $A|_U$-torsors.

Example 2.3.8. The following example closely follows the discussion in §7.2 of [Br1]. Let $M = S^3$, viewed as the one-point compactification of $\mathbb{R}^3$, i.e. $M = \mathbb{R}^3 \cup \{\infty\}$. Let $U_a = M\backslash\{0\}$ and $U_b = M\backslash\{\infty\}$. Then there is a deformation retraction of $U_{ab} = U_a \cap U_b$ onto the unit 2-sphere $S^2 \in \mathbb{R}^3$. Let x be the generator of $H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$ corresponding to the standard orientation of $S^2 \subset \mathbb{R}^3$, and let $\tilde{x}$ be the corresponding generator of $U_{ab} \cong S^2$. Choose a principal $\mathbb{T}$-bundle $E \to U_{ab}$ whose isomorphism class corresponds to $\tilde{x}$.

We construct a gerbe $\mathcal{G}$ from this structure as follows. For each open set $W \subset S^3$, an object of $\mathcal{G}(W)$ consists of a principal circle bundle P_a over $W \cap U_a$, a principal circle bundle P_b over $W \cap U_b$, and an isomorphism

$$\phi : (P_a)|_{W \cap U_{ab}} \otimes (P_b)^*|_{W \cap U_{ab}} \xrightarrow{\cong} E|_{W \cap U_{ab}}$$

Given another object $(P'_a, P'_b, \phi') \in \mathcal{G}(W)$, an isomorphism from (P_a, P_b, ϕ) to (P'_a, P'_b, ϕ') consists of a pair of isomorphisms $\psi_a : P_a \xrightarrow{\cong} P'_a$ and $\psi_b : P_b \to P'_b$ suitably compatible with ϕ, ϕ'. The restriction functors and other data needed to fully define the gerbe $\mathcal{G}$ are constructed in the obvious way; for a proof that the above recipe does in fact define a gerbe, see Definition/Proposition 7.2.1 in [Br1]. Furthermore, a variation of the argument used in the proof of Definition/Proposition 7.2.4 in [Br1] shows that the class of this gerbe in $H^3(S^3, \mathbb{Z})$ is the generator corresponding to the standard orientation.

Next, let us discuss morphisms and 2-morphisms between presheaves of groupoids; for precise definitions see Definitions .2.11 and .2.16. Given presheaves of groupoids $\mathcal{G}$ and $\mathcal{G}'$, a morphism of $\Phi : \mathcal{G} \to \mathcal{G}'$ consists first of all of a collection of functors

$$\Phi_U : \mathcal{G}(U) \to \mathcal{G}'(U). \tag{2.3.9}$$

In addition, for each inclusion $i : V \hookrightarrow U$ the data of Φ includes coherent natural transformations $\Phi_i : i^*_{\mathcal{G}'}\Phi_U \Rightarrow \Phi_V i^*_{\mathcal{G}}$. Given a pair of morphisms $\Phi, \Psi :$

$\mathcal{G} \to \mathcal{G}'$, a 2-morphism $\tau : \Phi \Rightarrow \Psi$ consists of natural transformations $\tau_U : \Phi_U \Rightarrow \Psi_U$ which are suitably compatible with the structure of Φ and Ψ.

The following is a consequence of proposition (7.5.4) in [Bo].

Proposition 2.3.10. *There is a 2-category whose objects are presheaves of groupoids over M, whose 1-morphisms are pseudo-natural transformations, and whose 2-morphisms are modifications.*

In particular, we have a well-defined composition of 1-morphisms, as well as vertical and horizontal composition of 2-morphisms.

In the case that $\mathcal{G}$ and $\mathcal{G}'$ are gerbes, we will be interested in morphisms which interact well with the additional structure specified in Definition 2.3.2.

Definition 2.3.11. *Let $\varphi : A \to B$ be a homomorphism of sheaves of abelian groups over M. Given a gerbe $\mathcal{G}$ over M with band A and a gerbe $\mathcal{G}'$ with band B, a 1-morphism $\Phi : \mathcal{G} \to \mathcal{G}'$ between their underlying sheaves of groupoids is said to* intertwine φ *if for each object $P \in \mathcal{G}(U)$, the diagram*

$$\begin{array}{ccc} \underline{Aut}(P) & \xrightarrow{\Phi_U} & \underline{Aut}(\Phi_U(P)) \\ {\scriptstyle \alpha_P}\big\downarrow & & \big\downarrow{\scriptstyle \alpha'_{\Phi_U(P)}} \\ A|_U & \xrightarrow[\varphi|_U]{} & B|_U \end{array} \tag{2.3.12}$$

commutes.

Remark 2.3.13. Concretely, for each $P, Q \in \mathcal{G}(U)$, each local section $\psi \in \underline{\mathrm{Hom}}(P, Q)(V)$ and each $g : V \to \mathbb{T}$, the diagram (2.3.12) implies that

$$\Phi(\psi \cdot g) = \Phi(\psi) \cdot \varphi(g). \tag{2.3.14}$$

Remark 2.3.15. By proposition (2.3.10), given a gerbe $\mathcal{G}$ with band A and $\mathcal{G}'$ with band B, there is a category of 1-morphisms between the presheaves underlying $\mathcal{G}$ and $\mathcal{G}'$. Given $\varphi : A \to B$, we then have a subcategory of 1-morphisms from $\mathcal{G}$ to $\mathcal{G}'$ that intertwine φ, which we will denote

$$\mathrm{Hom}_{\varphi}(\mathcal{G}, \mathcal{G}'). \tag{2.3.16}$$

The following proposition says that the associated torsor construction can be extended to a 1-morphism of gerbes. The proof, which is omitted, is a straightforward application of the definitions.

Proposition 2.3.17. *1. Let $\varphi : A \to C$ be a homomorphism of sheaves of abelian groups over M. Then there is a 1-morphism $\varphi[\cdot] : B(A) \to B(C)$ intertwining φ defined on each open set $U \subset M$ by the functor $\varphi|_U[\cdot] : \boldsymbol{Tor}_{A|_U} \to \boldsymbol{Tor}_{B|_U}$ given in definition (2.2.21).*

2. Given another sheaf of abelian groups D and a homomorphism $\psi : C \to D$, there exists a 2-morphism

$$\psi[\varphi[\cdot]] \Rightarrow (\psi\varphi)[\cdot]. \tag{2.3.18}$$

We now recall several operations on gerbes which we will need. First, given a gerbe $\mathcal{G}$ over M with band A, we may construct in an obvious way the opposite gerbe $\mathcal{G}^{op}$, which assigns to each open set $U \subset M$ the category $\mathcal{G}(U)^{op}$. Since A is abelian, $\mathcal{G}^{op}$ canonically has the structure of a gerbe with band A. There is a canonical 1-morphism of gerbes

$$\mathcal{G} \to \mathcal{G}^{op} \tag{2.3.19}$$

intertwining the homomorphism

$$g \mapsto g^{-1}. \tag{2.3.20}$$

This morphism is a bijection on the level of both objects and morphisms: on the set of objects it is the identity, and on the level of morphisms it maps

$$\psi : P \to Q \tag{2.3.21}$$

to

$$\psi^{-1} : Q \to P. \tag{2.3.22}$$

The next construction involves a homomorphism $\varphi : A \to B$ of sheaves of abelian groups over M. On page 199 of [Br1] Brylinski describes the construction of an associated gerbe with band B (Brylinski calls this gerbe $\mathcal{G} \times^A B$, but following our earlier notation for torsors we will call it $\varphi[\mathcal{G}]$). Furthermore, from the construction of $\varphi[\mathcal{G}]$ one obtains a 1-morphism

$$\varphi[\cdot] : \mathcal{G} \to \varphi[\mathcal{G}] \tag{2.3.23}$$

intertwining φ. As discussed in Example 2.2.7 for torsors, one can use the associated gerbe construction to construct the tensor product of gerbes, as well as the dual of a gerbe.

Next, let $f : M \to M'$ be a smooth map of manifolds. Given a sheaf of groupoids $\mathcal{G}$ over M with band A, one constructs the direct image $f_*\mathcal{G}$ in the obvious way: for example, for each open set $U \subset N$ we have

$$f_*(\mathcal{G})(U) = \mathcal{G}(f^{-1}(U)). \tag{2.3.24}$$

In general, if $\mathcal{G}$ has the structure of a gerbe with band A, $f_*\mathcal{G}$ will not necessarily have the structure of a gerbe with band f_*A; on the other hand, Proposition 5.2.7 from [Br1] shows that this will hold under certain assumptions about the map f and the sheaf of groups A. We also point out that, given another smooth map $g : M' \to M''$, we have a natural identification of $g_*(f_*\mathcal{G})$ with $(gf)_*\mathcal{G}$.

Given a gerbe $\mathcal{G}$ over M' with band A, we may also form the inverse image $f^*\mathcal{G}$, which is a gerbe with band f^*A over M. If $\mathcal{G}$ is a Dixmier-Douady gerbe (i.e. $A = \underline{\mathbb{T}}_{M'}$), then the inverse image sheaf $f^*(\underline{\mathbb{T}}_{M'})$ will generally not be equal to $\underline{\mathbb{T}}_M$, so that $f^*\mathcal{G}$ is not a DD gerbe. On the other hand, we have a natural sheaf homomorphism

$$\varphi : f^*(\underline{\mathbb{T}}_{M'}) \to \underline{\mathbb{T}}_M, \tag{2.3.25}$$

and we can form the associated gerbe $\varphi[f^*\mathcal{G}]$, which is a DD gerbe over M; from now on we will use the notation $f^*\mathcal{G}$ denote this gerbe over M. We remark that, as in the case of the associated gerbe, the inverse image $f^*\mathcal{G}$ of a DD gerbe is characterized by the following universal property. There exists a morphism of gerbes

$$f^* : \mathcal{G} \to f_*(f^*\mathcal{G}) \tag{2.3.26}$$

intertwining the canonical homomorphism

$$f^* : \underline{\mathbb{T}}_{M'} \to f_*\underline{\mathbb{T}}_M. \tag{2.3.27}$$

Chapter 3

Infinitesimal Symmetries of Gerbes

In this section, we explain how to generalize the discussion of §2.1 to define the infinitesimal symmetries of a Dixmier Douady gerbe $\mathcal{G}$ over a manifold M. Since $\mathcal{G}$ is not itself a manifold, we cannot directly import structures from differential geometry such as vector fields or flows. We instead proceed by analogy to the circle bundle case. To make the analogy clearer, we recall a concept from algebraic geometry. The basic idea, inspired by the discussion in [Ra], is to use a ringed space I_1, sometimes called the "dual numbers", which can be thought of as a curve of infinitesimal length. In [Ra], the authors study maps from I_1 into a stack X to motivate the definition of the tangent stack to X. Similarly, given a gerbe $\mathcal{G}$ over M, we will consider the pullback of $\mathcal{G}$ to I_1 via a map $I_1 \to M$. This somewhat informal discussion motivates our Definition 3.2.4. In §3.4 (and in appendix A), we provide another (more geometric) perspective on $\mathcal{L}_{\mathcal{G}}$ by relating infinitesimal symmetries to families of (non-infinitesimal) symmetries of $\mathcal{G}$ through a process analogous to differentiation.

3.1 Gerbes over the formal interval

Consider the ring

$$R = \mathbb{R}[\epsilon]/(\epsilon)^2. \tag{3.1.1}$$

We can form a ringed space

$$I_1 = \mathrm{Spec}(R), \tag{3.1.2}$$

which has a single underlying point, and whose ring of functions is

$$\mathcal{O}(\{*\}) = R. \tag{3.1.3}$$

Let X be a smooth manifold, considered as a ringed space with its sheaf of smooth real valued-functions. Then the set of maps from I_1 to X in the category of ringed spaces is identified with the set of ring homomorphisms

$$\mathrm{Hom}(C^\infty(X), R). \tag{3.1.4}$$

Note in particular that we have an inclusion

$$\iota : \{*\} \hookrightarrow I_1 \tag{3.1.5}$$

corresponding to the homomorphism from $R \to \mathbb{R}$ sending $a + b\epsilon \mapsto a$, as well as a retraction

$$r : I_1 \to \{*\} \tag{3.1.6}$$

corresponding to the inclusion $\mathbb{R} \hookrightarrow R$. Given a map ϕ from I_1 to a smooth manifold X, let $\phi^* : C^\infty(X) \to R$ be the corresponding ring homomorphism, and write

$$\phi^* f = a(f) + b(f)\epsilon \tag{3.1.7}$$

for each $f \in C^\infty(X)$. Let $x = \phi \circ \iota(*) \in M$ be the image of the underlying point of I_1 in M, then

$$a(f) = (\phi \circ \iota)^* f = f(x), \tag{3.1.8}$$

i.e. $a : C^\infty(X) \to \mathbb{R}$ is the homomorphism which evaluates each function at the point x. The condition that ϕ^* be a ring homomorphism then implies that for every pair of functions $f, g \in C^\infty(X)$ we have

$$b(fg) = f(x)b(g) + g(x)b(f), \tag{3.1.9}$$

so that there is a unique tangent vector $\xi \in T_x X$ such that

$$b(f) = \xi(f). \tag{3.1.10}$$

Conversely, any tangent vector $\xi \in TX$ determines a map

$$\phi_\xi : I_1 \to M. \tag{3.1.11}$$

Thus $\mathrm{Map}(I_1, X)$ is naturally identified with the tangent space of X. Heuristically, we think of I_1 as a "formal interval" and the map ϕ_ξ as an infinitesimal curve in X in the direction ξ.

Next, let E be a principal $\mathbb{T}$-bundle over M, and let $\underline{E}$ be its sheaf of sections. The restriction of $\underline{E}$ to the point x in the category of sheaves is a torsor for the group of germs of smooth $\mathbb{T}$-valued functions at x. On the other hand, there is a homomorphsim

$$\rho_x : St_x(\underline{\mathbb{T}}_M) \to \underline{\mathbb{T}}_{\{x\}} \tag{3.1.12}$$

which evaluates each germ at the point x . We define the restriction $(\underline{E})_x$ of $\underline{E}$ to x (in the category of $\underline{\mathbb{T}}$-torsors) to be the associated torsor.

Given a Lie group G, the set of maps $\mathrm{Map}(I_1, G)$ also has the structure of a group which we identify with the product $G \times \mathfrak{g} \cong TG$. In particular, we may define a principal $\mathbb{T}$-bundle over I_1 to be a $\mathbb{T} \times i\mathbb{R}$-torsor, and we can similarly define a DD gerbe over I_1 to be a gerbe with band $\mathbb{T} \times i\mathbb{R}$. Given a tangent vector $\xi \in T_xM$ thought of as a map $\phi_\xi : I_1 \to M$, we can then construct the inverse image torsor $\phi_\xi^*\underline{E}$ over I_1. There is a restriction map

$$\underline{E} \to (\phi_\xi)_*(\phi_\xi)^*\underline{E} \tag{3.1.13}$$

intertwining the map

$$g \mapsto (g(x), \iota_{\xi_x} d\log(g)). \tag{3.1.14}$$

The maps $\iota : \{*\} \to I_1$ and $r : I_1 \to \{*\}$ yield maps

$$\iota^* : \phi_\xi^*\underline{E} \to \underline{E}_x \tag{3.1.15}$$

and

$$r^* : \underline{E}_x \to \phi_\xi^*\underline{E} \tag{3.1.16}$$

such that $\iota^* \circ r^* = \mathrm{id}_{\underline{E}_x}$. The kernel of (3.1.15) is isomorphic to $\iota_\xi d\log[\underline{E}]_x$, and we therefore obtain an isomorphism

$$\phi_\xi^*\underline{E} \cong \underline{E}_x \times \iota_\xi d\log[\underline{E}]_x. \tag{3.1.17}$$

Recall from Example 2.2.28 that the second factor can be identified with the set of $\mathbb{T}$-invariant lifts of ξ_x to E. Thus, we can encode the infinitesimal symmetries of E point-wise by pulling back $\underline{E}$ to I_1; informally this amounts to restricting E to an infinitesimal curve in M.

Recall from the previous section that the inverse image and associated torsor construction also make sense for gerbes. Thus, given a tangent vector $\xi \in T_xM$ and a DD gerbe $\mathcal{G}$ over M we can form a gerbe $\phi_\xi^*\mathcal{G}$ over I_1 with band $\mathbb{T} \times i\mathbb{R}$. This gerbe is naturally isomorphic to a product

$$\mathcal{G}_x \times \iota_\xi dlog[\mathcal{G}]_x, \tag{3.1.18}$$

where the first factor should be thought of as the fiber of $\mathcal{G}$ at x. By analogy with the circle bundle case, it is natural to identify the infinitesimal symmetries of $\mathcal{G}$ lifting ξ with the second factor $\iota_\xi d\log[\mathcal{G}]$.

3.2 The definition

Rather than working with the associated gerbe $\iota_\xi d\log[\mathcal{G}]$, we will instead use an alternative definition of the infinitesimal symmetries of $\mathcal{G}$. Recall from §2.3 that there is a morphism of gerbes $\mathcal{G} \to \iota_\xi d\log[\mathcal{G}]$ intertwining $\iota_\xi d\log$. In particular, for each local section Q of $\mathcal{G}$ we obtain a local section $\iota_\xi d\log[Q]$ of $\iota_\xi d\log[\mathcal{G}]$. Suppose we are given a global section S of $\iota_\xi d\log[\mathcal{G}]$. Then for each open set $U \subset M$ and each $Q \in \mathcal{G}(U)$, we can form the $\underline{i\mathbb{R}}_U$-torsor $\underline{\mathrm{Hom}}(\iota_\xi d\log[Q], S|_U)$. More formally, the section S defines a morphism of gerbes from $\mathcal{G}^{op}$ to $\mathrm{B}(\underline{i\mathbb{R}}_M)$ intertwining $\iota_\xi d\log$. Moreover one can check that the assignment

$$S \mapsto \underline{\mathrm{Hom}}(\iota_\xi d\log[\cdot], S) \tag{3.2.1}$$

defines an isomorphism from the category of global sections of $\iota_\xi d\log[\mathcal{G}]$ to the category

$$\mathrm{Hom}_{\iota_\xi d\log}(\mathcal{G}^{op}, \mathrm{B}(\underline{i\mathbb{R}}_M)) \tag{3.2.2}$$

of 1-morphisms from $\mathcal{G}^{op}$ to $\mathrm{B}(\underline{i\mathbb{R}}_M)$ intertwining $\iota_\xi d\log$. Using the canonical 1-morphism from $\mathcal{G}$ to $\mathcal{G}^{op}$ intertwining $g \mapsto g^{-1}$, we have a canonical equivalence (which is actually a bijection on the level of objects and morphisms)

$$\mathrm{Hom}_{\iota_\xi d\log}(\mathcal{G}^{op}, \mathrm{B}(\underline{i\mathbb{R}}_M)) \xrightarrow{\cong} \mathrm{Hom}_{-\iota_\xi d\log}(\mathcal{G}, \mathrm{B}(\underline{i\mathbb{R}}_M)). \tag{3.2.3}$$

Definition 3.2.4. *Let $\mathcal{G}$ be a DD gerbe over a manifold M, and let ξ be a vector field on M. Then the category of infinitesimal symmetries of $\mathcal{G}$ lifting ξ is*

$$\mathcal{L}_\mathcal{G}(\xi) = Hom_{-\iota_\xi dlog}(\mathcal{G}, B(i\mathbb{R}_M)). \tag{3.2.5}$$

Notation 3.2.6. We will often use the notation $\hat{\xi}$ to denote an element of $\mathcal{L}_\mathcal{G}(\xi)$; we call $\hat{\xi}$ a *lift* of ξ to $\mathcal{G}$. We also introduce the following notational convention, which in the present context is vacuous but will prove notationally useful when we consider connective lifts in §4: given a lift $\hat{\xi} \in \mathcal{L}_\mathcal{G}(\xi)$, we will say $\hat{\xi}$ *determines* a 1-morphism

$$F_{\hat{\xi}} : \mathcal{G} \to \mathrm{B}(\underline{i\mathbb{R}}_M). \tag{3.2.7}$$

Of course, by definition an element of $\mathcal{L}_\mathcal{G}(\xi)$ is a 1-morphisms, so in the present case $\xi = F_{\hat{\xi}}$.

Remark 3.2.8. Although for simplicity we have defined a *category* of symmetries of $\mathcal{G}$ lifting ξ, we clearly could have defined a gerbe of symmetries

$\underline{\mathrm{Hom}}_{-\iota_\xi dlog}(\mathcal{G}, \mathrm{B}(\underline{i\mathbb{R}_M}))$ assigning to each open set $U \subset M$ the category of morphisms from $\mathcal{G}|_U$ to $\mathrm{B}(\underline{i\mathbb{R}_U})$ intertwining $-\iota_{\xi|_U} d\log$. In other contexts (for example in a holomorphic setting) it may be that the gerbe analogous to $\underline{\mathrm{Hom}}_{-\iota_\xi dlog}(\mathcal{G}, \mathrm{B}(i\mathbb{R}_M))$ has no global sections, in which case it would be crucial to work with the gerbe of symmetries itself.

Example 3.2.9. Suppose that $\mathcal{G} = \mathrm{B}(\underline{\mathbb{T}}_M)$ is the trivial Dixmier-Douady gerbe over M.

Definition 3.2.10. *The* trivial lift *of a vector field ξ to $B(\underline{\mathbb{T}}_M)$ is the 1-morphism*

$$\hat{\xi}_0 = -\iota_\xi d\log[\cdot] : B(\underline{\mathbb{T}}_M) \to B(\underline{i\mathbb{R}_M}) \tag{3.2.11}$$

sending each $\underline{\mathbb{T}}_U$-torsor to the associated $\underline{i\mathbb{R}_U}$-torsor described in Proposition 2.3.17.

To understand the geometric origin of the trivial lift, consider by analogy the trivial $\underline{\mathbb{T}}_M$-torsor $\underline{\mathbb{T}}_M$. A section of this torsor is simply a function f, and since we can compare the value of f at distinct points on M we can differentiate f. Put differently, given a diffeomorphism $\varphi : M \to M$, we can pull-back a principal bundle P to obtain a different principal bundle, but if P is the trivial bundle then we can pull-back *sections* of P via φ to obtain sections of the same bundle. Similarly, given any gerbe $\mathcal{G}$ over M, we can pull back via φ to obtain another gerbe over M, but if $\mathcal{G}$ is the trivial gerbe then we can pull-back sections of $\mathcal{G}$, i.e. principal bundles, and $-\iota_\xi d\log[\cdot]$ is (minus) the corresponding directional derivative operation.

This analogy can be made more precise by considering the relationship between 1-parameter families of symmetries of gerbes (in particular, $\mathbb{R}$-equivariant gerbes) and infinitesimal symmetries, as discussed in appendix A. Given any 1-parameter family Φ of diffeomorphisms of M, there is a canonical lift of Φ to the trivial gerbe over M. Applying the differentiation functor in Definition.1.24, one obtains a lift of the vector field generating Φ which is naturally isomorphic to the one defined in Definition 3.2.10.

More generally, let G be a Lie group acting on M. If $\mathcal{G}$ is given the structure of a G-equivariant gerbe, then for every 1-parameter subgroup of G we obtain an infinitesimal symmetry of $\mathcal{G}$ by applying the differentiation functor.

3.3 The Cech picture

In this section we introduce a more concrete "Cech" version of the category $\mathcal{L}_{\mathcal{G}}$. In this picture one works not with sheaves of categories but with differential forms. We could similarly develop a version of $\mathcal{L}_{\mathcal{G}}$ in terms of other models of gerbes, such as bundle gerbes.

Definition 3.3.1. *Let $\mathcal{G}$ be a DD gerbe over a manifold M. A* collection of local trivializations *of $\mathcal{G}$ consists of a triple $\{\{U_i\}, \{Q_i\}, \{s_{ij}\}\}$ where*

1. *$\{U_i\}$ is an open cover of M,*

2. *Q_i is a section of $\mathcal{G}(U_i)$, and*

3. $s_{ij} : Q_i|_{U_{ij}} \to Q_j|_{U_{ij}}$*, where* $U_{ij} = U_i \cap U_j$*.*

Remark 3.3.2. For the remainder of this paper, we will restrict ourselves to the case where $\{U_i\}$ is a *good cover*, meaning that each open set U_k as well as each (non-empty) k-fold intersection $U_{i_1} \cap U_{i_2} \cap \cdots U_{i+k}$ is contractible.

Remark 3.3.3. Such a collection of local trivializations gives rise to *Cech data* for $\mathcal{G}$, which is a collection of functions $\{g_{ijk}\}$ from triple intersections $U_{ijk} := U_i \cap U_j \cap U_k$ to $\mathbb{T}$. In terms of the discussion of the descent property for gerbes in appendix B, the functions g_{ijk} are are a cocycle whose cohomology class is the obstruction to gluing the local data $\{\{Q_i\}, \{s_{ij}\}\}$ to form a global section of $\mathcal{G}$. Explicitly, we define

$$s_{jk} \circ s_{ij} = s_{ik} \cdot g_{ijk}. \tag{3.3.4}$$

Because of the associativity of the composition of morphisms, it follows that on 4-fold overlaps U_{ijkl} the cocycle condition

$$g_{jkl} g_{ikl}^{-1} g_{ijl} g_{ijk}^{-1} = 1 \tag{3.3.5}$$

is satisfied.

Notation 3.3.6. If $\mathcal{F}$ is any sheaf over an open set $U \subset M$, we will use the notation $x \in \mathcal{F}$ to mean that x is a global section of $\mathcal{F}$ over U.

Definition 3.3.7. *Let* $\hat{\xi}$ *be a lift of a vector field* ξ *to* $\mathcal{G}$*. A* collection of local trivializations *for* $\hat{\xi}$ *relative to* $\{\{U_i\}, \{Q_i\}, \{s_{ij}\}\}$ *is a choice of sections* $r_i \in F_{\hat{\xi}}(Q_i)(U_i)$ *for each* i*.*

Given such a collection of local trivializations of $\hat{\xi}$, we can define $i\mathbb{R}$-valued functions f_{ij} on double intersections according to the formula

$$r_j = F_{\hat{\xi}}(s_{ij})(r_i) + f_{ij}. \tag{3.3.8}$$

We will refer to these functions as *Cech data* for the lift $\hat{\xi}$. Definition 3.2.4 together with equation (3.3.4) imply that on triple intersections we have

$$f_{jk} - f_{ik} + f_{ij} = \iota_\xi d\log(g_{ijk}), \tag{3.3.9}$$

where $\{g_{ijk}\}$ is the Cech data corresponding to the local trivializations of $\mathcal{G}$.

Given another lift $\hat{\xi}'$ of ξ to $\mathcal{G}$ with sections r_i' of $F_{\hat{\xi}'}(Q_i)$, and given an equivalence $T : F_{\hat{\xi}} \to F_{\hat{\xi}'}$, define functions $u_i : U_i \to i\mathbb{R}$ by

$$r_i' = T_{Q_i}(r_i) + u_i. \tag{3.3.10}$$

The naturality of T implies that on double intersections we have

$$f_{ij} - f_{ij}' = u_j - u_i. \tag{3.3.11}$$

These considerations motivate the following definition.

Definition 3.3.12. *Let $\{g_{ijk} : U_{ijk} \to \mathbb{T}\}$ be a Cech cocycle on a manifold M with respect to a good open cover $\{U_i\}$. The category $\mathcal{L}_{g_{ijk}}$ has as objects the set of pairs $(\xi, \{f_{ij}\})$, where ξ is a vector field on M and $\{f_{ij} : U_{ij} \to i\mathbb{R}\}$ is a collection of functions satisfying equation (3.3.9) on triple intersections. A morphism from $(\xi, \{f_{ij}\})$ to $(\xi, \{f_{ij}'\})$ is a collection of functions $\{u_i : U_i \to i\mathbb{R}\}$ satisfying equation (3.3.11).*

Notation 3.3.13. Given a fixed vector field ξ, let $\mathcal{L}_{g_{ijk}}(\xi)$ denote the subcategory of $\mathcal{L}_{g_{ijk}}$ with objects of the form $(\xi, \{f_{ij}\})$ for some collection of functions $\{f_{ij}\}$.

Theorem 3.3.14. *For each vector field ξ on M*

(1) The set of isomorphism classes of objects $\pi_0(\mathcal{L}_{g_{ijk}})$ has exactly one element.

(2) For any object $\hat{\xi} = (\xi, \{f_{ij}\})$, the map which to a smooth function $h : M \to i\mathbb{R}$ associates the automorphism of $\hat{\xi}$ given by $\{u_i = h|_{U_i}\}$ determines an isomorphism of groups

$$C^\infty_M(i\mathbb{R}) \xrightarrow{\cong} Aut(\hat{\xi}). \tag{3.3.15}$$

Proof. Consider $h_{ijk} = d\log(g_{ijk}) := g_{ijk}^{-1} dg_{ijk}$; this is a Cech cocycle with values in the sheaf $\underline{i\mathbb{R}}_M$. By definition an element of $\mathcal{L}_{g_{ijk}}(\xi)$ is a trivialization of this cocycle, i.e. a Cech cochain $\{f_{ij}\}$ such that

$$(\delta f)_{ijk} = h_{ijk}. \tag{3.3.16}$$

Since $\underline{i\mathbb{R}}_M$ is a fine sheaf (i.e. admits partitions of unity), the cohomology class $[h_{ijk}] \in H^3(M, \underline{i\mathbb{R}})$ is necessarily zero, and since $\{U_i\}$ is a good open cover we can find such a trivialization $\{f_{ij}\}$. Thus, for each vector field ξ, the category $L_{g_{ijk}}(\xi)$ is non-empty. On the other hand, given another lift $\{f'_{ij}\}$, it follows that $\{f_{ij} - f'_{ij}\}$ is closed, and therefore also exact. Thus we can find a Cech cochain $\{u_i\}$ such that for each i, j we have

$$f_{ij} - f'_{ij} = u_j - u_i. \tag{3.3.17}$$

Thus any two lifts are isomorphic.

Given an element $\hat{\xi} = (\xi, \{f_{ij}\}) \in \mathcal{L}(\xi)$, if $\{u_i\}$ is an automorphism of $\hat{\xi}$ than by equation (3.3.11) we must have $u_j = u_i$ on $U_i \cap U_j$, and therefore there is a global function f with $u_i = f|_{U_i}$; conversely any such function gives an automorphism of $\hat{\xi}$. □

We saw above that every element of $\mathcal{L}_{\mathcal{G}}$ (non-canonically) gives rise to an element of $\mathcal{L}_{g_{ijk}}$. On the other hand, given an element of $\mathcal{L}_{g_{ijk}}$ one can glue to obtain an element of $\mathcal{L}_{\mathcal{G}}$. We may also glue equivalences of lifts, obtaining a functor $\mathcal{L}_{g_{ijk}} \to \mathcal{L}_{\mathcal{G}}$. Our next result is that this leads to an equivalence of categories.

Proposition 3.3.18. *Let $\mathcal{G}$ be a DD gerbe over a manifold M, and let $\{\{U_i\}, \{Q_i\}, \{s_{ij}\}\}$ be a collection of local trivializations for $\mathcal{G}$ with associated Cech cocycle g_{ijk} relative to a good open cover. Then there is an equivalence of categories $\mathcal{L}_{g_{ijk}} \to \mathcal{L}_{\mathcal{G}}$.*

We defer the proof to Appendix B. To see a similar argument to that used there, we direct the reader to the proof of Proposition 5.3.2 in [Br1].

Corollary 3.3.19. *For each vector field ξ on M*

1. *The set of isomorphism classes of objects $\pi_0(L_{\mathcal{G}})$ has exactly one element.*

2. *For any object $\hat{\xi} \in \mathcal{L}_{\mathcal{G}}(\xi)$, the group of automorphisms of $\hat{\xi}$ is isomorphic to $C^{\infty}_M(i\mathbb{R})$. Explicitly, given $f \in C^{\infty}_M(i\mathbb{R})$, the associated automorphism*

τ of $\hat{\xi}$ is given by

$$(\tau)_Q = \alpha^{-1}_{F_{\hat{\xi}}(Q)}(f|_U) : F_{\hat{\xi}}(Q) \to F_{\hat{\xi}}(Q) \tag{3.3.20}$$

for each $Q \in \mathcal{G}(U)$, where $\alpha_{F_{\hat{\xi}}(Q)} : \mathbb{T}_U \to \underline{Aut}(Q)$ is as in definition (2.3.2) (see also definition (.2.16).

3.4 Local flows and infinitesimal symmetries

Let Φ be a 1-parameter group of diffeomorphisms of M generated by a vector field ξ. In appendix A we describe the category $\mathcal{L}_{\mathcal{G}}(\Phi)$ of lifts of Φ to $\mathcal{G}$ and construct a "differentiation" functor

$$\mathcal{L}_{\mathcal{G}}(\Phi) \to \mathcal{L}_{\mathcal{G}}(\xi). \tag{3.4.1}$$

Roughly speaking, an element of $\mathcal{L}_{\mathcal{G}}(\Phi)$ may be thought of as a flow on $\mathcal{G}$, and the corresponding infinitesimal symmetry as a vector field generated by the flow. In the case of manifolds, every vector field can be integrated locally to obtain a unique flow, and this establishes a bijective correspondence. Similarly, in this section we establish an equivalence between local versions of the categories $\mathcal{L}_{\mathcal{G}}(\Phi)$ and $\mathcal{L}_{\mathcal{G}}(\xi)$.

To describe the local version of the category $\mathcal{L}_{\mathcal{G}}(\Phi)$, it will be convenient to use employ the language of simplicial manifolds. Given a vector field ξ on M, for each x in M there exists an open set U containing x, a positive number

$\tilde{\epsilon}$, and a smooth map

$$\Phi : U \times (-\tilde{\epsilon}, \tilde{\epsilon}) \to M \tag{3.4.2}$$

$$x, t \mapsto \varphi_t(x),$$

such that for each t, φ_t is a diffeomorphism from U to $\varphi_t(U)$, and such that, for each $(x,t) \in U \times (-\tilde{\epsilon}, \tilde{\epsilon})$, we have

$$\frac{d}{ds}|_{s=t}\varphi_s(x) = \xi_{\varphi_t(x)}. \tag{3.4.3}$$

Next, we can choose a smaller open set $V \subset U$ and a smaller positive number $\epsilon < \tilde{\epsilon}$ such that, for each $t \in I = (-\epsilon, \epsilon)$ and each $x \in V$ we have $\varphi_t(x) \in U$. For each $k \geq 0$, consider the subset of $V \times \mathbb{R}^k$ given by

$$U^k = \{(\varphi_{t_0}(x), t_1, \cdots, t_k) : x \in V, \sum_{i=0}^{k} |t_i| < \epsilon\}. \tag{3.4.4}$$

Note that U^k is open since it is a union of open sets of the form $\varphi_t(V) \times W_t$, where for each $t \in I$ we define

$$W_t = \{(t_1, \cdots t_k) : \sum_{i=1}^{k} |t_i| < \epsilon - |t|\}. \tag{3.4.5}$$

The open sets $U^\bullet$ fit together into a simplicial manifold. The boundary maps $p_j : U^k \to U^{k-1}$ for $j = 0, 1 \cdots k$ are given by

$$p_0(x, t_1, \cdots, t_k) = (\varphi_{t_1}(x), t_2, \cdots, t_k), \tag{3.4.6}$$

$$p_i(x, t_1, \cdots, t_k) = (x, \cdots, t_i + t_{i+1}, \cdots, t_k) \tag{3.4.7}$$

for $i = 1, \cdots k-1$, and

$$p_k(x, t_1, \cdots, t_k) = (x, t_1 \cdots, t_{k-1}). \tag{3.4.8}$$

The degeneracy maps $s_i : U^k \to U^{k+1}$ for $i = 0, \cdots k$ are given by

$$s_0(x, t_1, \cdots, t_k) = (x, 0, t_1, \cdots, t_k) \tag{3.4.9}$$

and

$$s_i(x, t_1, \cdots, t_k) = (x, t_1, \cdots, t_i, 0, \cdots, t_k) \tag{3.4.10}$$

for $i = 1, \cdots k$.

It is easily checked that these satisfy the correct relations to define a simplicial manifold.

Notation 3.4.11. Given a $\underline{\mathbb{T}}_{U^k}$-torsor S, we define a $\underline{\mathbb{T}}_{U^{k+1}}$-torsor

$$\delta S = p_0^* S \otimes p_1^* S^\vee \otimes \cdots \otimes p_{k+1}^* S^{(\vee)}. \tag{3.4.12}$$

Similarly, given a function $g : U^k \to \mathbb{T}$, define $\delta g : U^{k+1} \to \mathbb{T}$ by

$$(p_0^* g)(p_1^* g^{-1}) \cdots (p_{k+1}^* g). \tag{3.4.13}$$

It them follows from the simplicial relations that, for any such g we have

$$\delta^2 g = 1; \tag{3.4.14}$$

Similarly, for any $ul\mathbb{T}_{U^k}$-torsor S there is a natural isomorphism of $\delta^2 S$ with the trivial torsor over U^{k+2}. There are also natural isomorphisms

$$s_0^* \delta S \cong p_0^* s_0^* S \tag{3.4.15}$$

and

$$s_1^* \delta S \cong p_1^* s_0^* S \tag{3.4.16}$$

over U^1.

Any DD gerbe is locally isomorphic to the trivial gerbe with band $\underline{\mathbb{T}}$, and therefore without loss of generality we will restrict ourselves to the case that $\mathcal{G}$ is the trivial gerbe in the following. The following definition is a local version of an equivariant gerbe (for the group $\mathbb{R}$)–see for example [Me], [G2], [Br2].

Definition 3.4.17. *Let $U^\bullet$ be the simplicial manifold described above. A local lift of Φ to the trivial gerbe consists of a $\underline{\mathbb{T}}_{U^1}$-torsor S over U^1, together with a section e of s_0^*S over U^0 and a section σ of δE over U^2 satisfying the following conditions:*

(i) $s_0^*\sigma = p_0^*e$ *and* $s_1^*\sigma = p_1^*e$,

(ii) $\delta\sigma$ *is equal to the canonical section of* $\delta^2 S$ *over* U^3.

Given a pair of local lifts $\hat{\Phi} = (S, e, \sigma)$ and $\hat{\Phi}' = (S', e', \sigma')$, an isomorphism from $\hat{\Phi}$ to $\hat{\Phi}'$ is an isomorphism of torsors $\Psi : S \xrightarrow{\cong} S'$ compatible with the sections e, e', σ, σ'. We will denote the corresponding groupoid by $\mathcal{L}(\Phi)$.

Proposition 3.4.18. $\mathcal{L}(\Phi)$ *has a single isomorphism class of objects.*

Proof. Note that there is a distinguished element $\hat{\Phi}_0 \in \mathcal{L}(\Phi)$, which we call the *trivial lift*. Namely, we take S to be the trivial $\underline{\mathbb{T}}_{U^1}$-torsor, and e and σ to be the trivial sections. We will show that an arbitrary element $(S, e, \sigma) \in \mathcal{L}(\Phi)$ is isomorphic to $\hat{\Phi}_0$. Note that s_0^*S can be identified with the restriction of S to $U^0 \times \{0\} \subset U^1$. We therefore begin by extending the section e to a smooth section τ of S. We can then define a smooth function $g : U^2 \to \mathbb{T}$ by the formula

$$\sigma = \delta\tau \cdot g. \tag{3.4.19}$$

Actually, it will be more convenient to work with logarithm of g; this is well-defined since U^2 deformation retracts onto $U^0 \times \{(0,0)\}$ and since by condition (i) in Definition 3.4.17 we have $g(x,0,0) = 1$. Thus, we let $f : U^2 \to i\mathbb{R}$ be the unique function such that $f(x,0,0) = 0$ and $g = e^f$. Condition (i) in definition (3.4.17) is equivalent to the condition that, for all $(x,t) \in U^1$ we have

$$f(x,t,0) = f(x,0,t) = 0, \tag{3.4.20}$$

and by condition (ii) we have $\delta f(x,t,t',t'') =$

$$f(\varphi_t(x),t',t'') - f(x,t+t',t'') + f(x,t,t'+t'') - f(x,t,t') = 0. \tag{3.4.21}$$

Furthermore, it is easy to see that specifying an isomorphism from $\hat{\Phi}_0$ to (S,e,σ) is equivalent to giving a smooth function $h : U^1 \to i\mathbb{R}$ such that $\delta h = f$, i.e. for each $(x,t,t') \in U^2$, h satisfies

$$h(\varphi_t(x),t') - h(x,t+t') + h(x,t) = f(x,t,t'), \tag{3.4.22}$$

and such that $h(x,0) = 0$ for all x. Define a function $k : U^1 \to i\mathbb{R}$ by the equation

$$k(x,t) = \frac{\partial}{\partial u}|_{u=0} f(x,t,u). \tag{3.4.23}$$

Differentiating equation (3.4.21) with respect to t'' at $t'' = 0$ we obtain the relation

$$k(x,t+t') - k(\varphi_t(x),t') = \frac{\partial}{\partial u}|_{u=t'} f(x,t,u). \tag{3.4.24}$$

If we then define

$$h(x,t) = -\int_0^t k(x,s)ds, \tag{3.4.25}$$

we have

$$\begin{aligned}
&h(\varphi_t(x), t') - h(x, t+t') + h(x,t) \\
=&-\int_0^{t'} k(\varphi_t(x), s)ds + \int_0^{t+t'} k(x,s)ds - \int_0^t k(x,s)ds \\
=&-\int_0^{t'} k(\varphi_t(x), s)ds + \int_t^{t+t'} k(x,s)ds \\
=&\int_0^{t'} [k(x, t+s) - k(\varphi_t(x), s)]ds \\
=&\int_0^{t'} \frac{\partial}{\partial u}|_{u=s} f(x,t,u)ds \\
=&f(x,t,t') - f(x,t,0) = f(x,t,t').
\end{aligned} \tag{3.4.26}$$

□

We now define a concrete local version of the differentiation functor (.1.24) considered in Appendix A. Given a vector field ξ on U^0, let $\mathcal{L}(\xi)$ denote $\mathcal{L}_{\mathrm{B}(\underline{\mathbb{T}}_{U^0})}(\xi)$. Then we will construct a functor

$$D : \mathcal{L}(\Phi) \to \mathcal{L}(\xi). \tag{3.4.27}$$

It follows from the results of §3.3 that every lift $\hat{\xi}$ of ξ to $\mathrm{B}(\underline{\mathbb{T}}_{U^0})$ is determined up to canonical isomorphism by its action on the trivial $\underline{\mathbb{T}}_{U^0}$-torsor. Put differently, there is a functor

$$\begin{aligned}
\mathcal{L}(\xi) &\to \mathbf{Tor}_{i\underline{\mathbb{R}}_{U^0}} \\
\hat{\xi} &\mapsto F_{\hat{\xi}}(\underline{\mathbb{T}}_{U^0}),
\end{aligned} \tag{3.4.28}$$

and it follows from Proposition 3.3.19 that it is an equivalence of categories. For example, under this isomorphism the trivial lift discussed in Example 3.2.9 is sent to $\iota_\xi d\log[\underline{\mathbb{T}}_{U^0}]$, which is canonically isomorphic to the trivial $\underline{i\mathbb{R}}_{U^0}$-torsor. For simplicity we will therefore take the codomain of the differentiation functor to be the category $\mathbf{Tor}_{\underline{i\mathbb{R}}_{U^0}}$.

Definition 3.4.29.

$$D : \mathcal{L}(\Phi) \to \boldsymbol{Tor}_{\underline{i\mathbb{R}}_{U^0}} \tag{3.4.30}$$

is the functor taking (S, e, σ) to the $\underline{i\mathbb{R}}_{U^0}$-torsor

$$s_0^*(\iota_{\frac{d}{dt}} d\log[S]). \tag{3.4.31}$$

Theorem 3.4.32. *D is an equivalence of categories.*

Proof. We proceed to by showing that D is both essentially surjective and fully faithful. Since $\mathbf{Tor}_{\underline{i\mathbb{R}}_{U^0}}$ has a single isomorphism class of objects, D is trivially essentially surjective. Furthermore, since $\mathcal{L}(\Phi)$ has a single isomorphism class of objects, to check that D is fully faithful it is sufficient to check that D induces an isomorphism of groups

$$\mathrm{Aut}(\hat{\Phi}_0) \to \mathrm{Aut}(D(\hat{\Phi}_0)), \tag{3.4.33}$$

where $\hat{\Phi}_0$ is the trivial lift.

Define the set

$$Z_{\mathbb{T}} := \{g : U^1 \to \mathbb{T} : \delta g = 1, s_0^* g = 1\}. \tag{3.4.34}$$

There is an isomorphism

$$Z_{\mathbb{T}} \xrightarrow{\cong} \mathrm{Aut}(\hat{\Phi}_0) \tag{3.4.35}$$

sending each function g to the bundle automorphism of $\underline{\mathbb{T}}_{U^1}$ given by right multiplication by g. Define

$$D_{\mathbb{T}} : Z_{\mathbb{T}} \to C^\infty(U^0; i\mathbb{R}) \tag{3.4.36}$$

by

$$g \mapsto \iota_{\frac{d}{dt}} d\log(g)|_{U^0}. \tag{3.4.37}$$

Then the definition of the functor D implies that there is a commutative diagram

$$\begin{array}{ccc} Z_{\mathbb{T}} & \xrightarrow{\cong} & \mathrm{Aut}(\hat{\Phi}_0) \\ \downarrow{\scriptstyle D_{\mathbb{T}}} & & \downarrow{\scriptstyle D} \\ C^\infty(U^0; i\mathbb{R}) & \xrightarrow{\cong} & \mathrm{Aut}(D(\hat{\Phi}_0)) \end{array} \tag{3.4.38}$$

Since both horizontal arrows are isomorphisms, to finish the proof we must show that $D_{\mathbb{T}}$ is an isomorphism. To see this, note that for any $g \in Z$, the condition $s^*g = 1$ can be written

$$g(x, 0) = 1 \tag{3.4.39}$$

for every $x \in U^0$. Therefore there exists a unique $f : U^1 \to i\mathbb{R}$ such that

$$g(x, t) = e^{f(x,t)} \tag{3.4.40}$$

and

$$f(x, 0) = 0. \tag{3.4.41}$$

The condition $\delta g = 1$ then implies that for each $(x, t, t') \in U^1$ we have

$$f(\varphi_t(x), t') - f(x, t + t') + f(x, t) = 0. \tag{3.4.42}$$

Differentiating with respect to t' we obtain

$$\frac{\partial f}{\partial s}|_{s=0} f(\varphi_t(x), s) = \frac{\partial f}{\partial s}|_{s=t} f(x, s). \tag{3.4.43}$$

Define

$$h(x) = D_{\mathbb{T}}(g)(x) = \frac{\partial f}{\partial s}|_{s=0} f(\varphi_t(x), s). \tag{3.4.44}$$

By the fundamental theorem of calculus we have

$$f(x, t) = \int_0^t h(\varphi_s(x)) ds, \tag{3.4.45}$$

so that g is completely determined by $D_{\mathbb{T}}(g) = h$. Conversely, given an arbitrary function $h : U^0 \to i\mathbb{R}$, the function on U^1 defined by

$$g(x, t) = e^{\int_0^t h(\varphi_s(x)) ds} \tag{3.4.46}$$

is in $Z_{\mathbb{T}}$, and is mapped by $D_{\mathbb{T}}$ to h.

□

3.5 Operations on lifts

Given diffeomorphisms $\varphi, \psi : M \to M$, suppose that $\tilde{\varphi}$ $\tilde{\psi}$ are lifts of these symmetries to $\mathcal{G}$. Then we can compose $\tilde{\varphi}$ and $\tilde{\psi}$ to obtain a symmetry of $\mathcal{G}$ covering $\varphi\psi : M \to M$. In this way the category of symmetries of $\mathcal{G}$ obtains a structure which is a categorical version of a group. Infinitesimally, this structure gives rise to a Lie bracket-like operation on the infinitesimal symmetries of $\mathcal{G}$. In this section we give a direct definition of this bracket, together with other structures on $\mathcal{L}_{\mathcal{G}}$ analogous to those possessed by a Lie algebra.

Proposition 3.5.1. *(I) There exists a functor $\boxplus : \mathcal{L}_{\mathcal{G}} \times \mathcal{L}_{\mathcal{G}} \to \mathcal{L}_{\mathcal{G}}$ such that, if $\hat{\xi}, \hat{\eta}$ are lifts of vector fields ξ, η to $\mathcal{G}$, then $\hat{\xi} \boxplus \hat{\eta}$ is a lift of $\xi + \eta$ satisfying*

$$F_{\hat{\xi} \boxplus \hat{\eta}}(Q) = F_{\hat{\xi}}(Q) \boxplus F_{\hat{\eta}}(Q) \tag{3.5.2}$$

for each object $Q \in \mathcal{G}(U)$.

(II) For each real number $\lambda \in \mathbb{R}$ there is a functor $\lambda \odot : \mathcal{L}_{\mathcal{G}} \to \mathcal{L}_{\mathcal{G}}$ such that, if $\hat{\xi}$ is a lift of ξ, then $\lambda[\xi]$ is a lift of $\lambda\xi$ satisfying

$$F_{\lambda \odot \hat{\xi}}(Q) = \lambda \odot (F_{\hat{\xi}}(Q)). \tag{3.5.3}$$

(III) There is a functor $[\cdot, \cdot] : \mathcal{L}_{\mathcal{G}} \times \mathcal{L}_{\mathcal{G}} \to \mathcal{L}_{\mathcal{G}}$ such that, if $\hat{\xi}, \hat{\eta}$ are lifts of ξ, η, then $[\hat{\xi}, \hat{\eta}]$ is a lift of $[\xi, \eta]$ satisfying

$$F_{[\hat{\xi}, \hat{\eta}]}(Q) = \xi[F_{\hat{\eta}}(Q)] \boxminus \eta[F_{\hat{\xi}}(Q)]. \tag{3.5.4}$$

(IV) There exists a lift ζ of the zero vector field to $\mathcal{G}$ such that for each open set $U \subset M$, $F_{\zeta, U} : \mathcal{G}(U) \to B(\underline{i\mathbb{R}}_M)(U)$ is the constant functor sending every object to the sheaf of groups $\underline{i\mathbb{R}}_U$. We call this the zero lift.

Proof. Let $+ : \underline{i\mathbb{R}}_M \times \underline{i\mathbb{R}}_M \to \underline{i\mathbb{R}}_M$ denote the addition homomorphism, and recall from proposition (2.3.17) that there exists a canonical 1-morphism of gerbes $\cdot \boxplus \cdot : \mathrm{B}(\underline{i\mathbb{R}_M}) \times \mathrm{B}(\underline{i\mathbb{R}_M}) \to \mathrm{B}(\underline{i\mathbb{R}_M})$ intertwining $+$ and sending $R, S \in \mathbf{Tor}_{\underline{i\mathbb{R}}(U)}$ to $R \boxplus S$ for each open set $U \subset M$. Given lifts $\hat{\xi}, \hat{\eta}$ of vector fields ξ, η to $\mathcal{G}$, we then define $\hat{\xi} \boxplus \hat{\eta}$ to be the composition

$$\mathcal{G} \xrightarrow{\hat{\xi} \times \hat{\eta}} \mathrm{B}(\underline{i\mathbb{R}_M}) \times \mathrm{B}(\underline{i\mathbb{R}_M}) \xrightarrow{\cdot \boxplus \cdot} \mathrm{B}(\underline{i\mathbb{R}_M}). \tag{3.5.5}$$

If $\psi : \hat{\xi} \Rightarrow \hat{\xi}'$ and $\varphi : \hat{\eta} \Rightarrow \hat{\eta}'$ are equivalences of lifts, we define $\psi \boxplus \varphi : \hat{\xi} \boxplus \hat{\eta} \to \hat{\xi}' \boxplus \hat{\eta}'$ to be the horizontal composition of the identity 2-morphism $\cdot \boxplus \cdot \Rightarrow \cdot \boxplus \cdot$ with the 2-morphism $\psi \times \varphi : \hat{\xi} \times \hat{\eta} \Rightarrow \hat{\xi}' \times \hat{\eta}'$. The functors $\lambda[\cdot]$ and $[\cdot, \cdot]$ are defined similarly.

We omit the proof of (IV), which is straightforward. □

One can further show that the operations above satisfy the axioms of a Lie algebra up to natural isomorphism. For example, given lifts $\hat{\xi}, \hat{\eta}$, and $\check{\tau}$ there is a natural isomorphism

$$(\hat{\xi} \boxplus \hat{\eta}) \boxplus \hat{\tau} \xrightarrow{\cong} \hat{\xi} \boxplus (\hat{\eta} \boxplus \hat{\tau}). \tag{3.5.6}$$

Furthermore, these natural isomorphisms themselves satisfy various coherence conditions, similar to (but much more elaborate than) those satisfied by the associator in a monoidal category. Roughly speaking, the category $\mathcal{L}_{\mathcal{G}}$ has the structure of a "Lie algebra object in the 2-category of categories." Rather than give a precise definition of such an algebraic structure (which so far as we know does not exist in the literature), we will instead work in the Cech picture, where the relevant algebraic structure can be described in the language of L_∞-algebras.

Thus, let us choose a collection of local trivializations $\{\{U_i\}, \{Q_i\}, \{s_{ij}\}\}$ for $\mathcal{G}$ with corresponding Cech data $\{g_{ijk}\}$. Given lifts $\hat{\xi}, \hat{\eta}$ of vector fields ξ, η to $\mathcal{G}$, let $\{r_i^{\hat{\xi}}\}, \{r_i^{\hat{\eta}}\}$ be local sections for these lifts, and let $\{f_{ij}^{\hat{\xi}}\}, \{f_{ij}^{\hat{\eta}}\}$ be the corresponding Cech data defined in equation (3.3.8). Then we obtain local sections

$$r_i^{\lambda \odot \hat{\xi}} = \lambda \odot r_i^{\hat{\xi}}, \tag{3.5.7}$$

$$r_i^{\hat{\xi} \boxplus \hat{\eta}} = r_i^{\hat{\xi}} \boxplus r_i^{\hat{\eta}}, \tag{3.5.8}$$

and

$$r_i^{[\hat{\xi}, \hat{\eta}]} = \xi[r_i^{\hat{\eta}}] \boxminus \eta[r^{\hat{\xi}}] \tag{3.5.9}$$

of the lifts $\lambda \odot \hat{\xi}$, $\hat{\xi} \boxplus \hat{\eta}$, and $[\hat{\xi}, \hat{\eta}]$, respectively. The corresponding Cech data is given by

$$f_{ij}^{\lambda \odot \hat{\xi}} = \lambda f_{ij}^{\hat{\xi}}, \tag{3.5.10}$$

$$f_{ij}^{\hat{\xi} \boxplus \hat{\eta}} = f_{ij}^{\hat{\xi}} + f_{ij}^{\hat{\eta}}, \tag{3.5.11}$$

and

$$f_{ij}^{[\hat{\xi}, \hat{\eta}]} = \xi(f_{ij}^{\hat{\eta}}) - \eta(f_{ij}^{\hat{\xi}}). \tag{3.5.12}$$

We now show that these operations give the category $\mathcal{L}_{g_{ijk}}$ the structure of a (strict) Lie 2-algebra, or equivalently of a 2-term L_∞-algebra. For a discussion of Lie 2-algebras and 2-term L_∞-algebras, we direct the reader to [BCr].

Definition 3.5.13. *A* 2-term L_∞ algebra *is a 2-term chain complex of vector spaces* $V_1 \xrightarrow{d} V_0$ *equipped with:*

1. *an antisymmetric chain map* $[\cdot,\cdot] : V \otimes V \to V$,

2. *an antisymmetric chain homotopy* $J : V \otimes V \otimes V \to V$ *from the chain map*

$$\begin{aligned} V \otimes V \otimes V &\to V \\ x \otimes y \otimes z &\mapsto [x,[y,z]] \end{aligned} \tag{3.5.14}$$

to the chain map

$$\begin{aligned} V \otimes V \otimes V &\to V \\ x \otimes y \otimes z &\mapsto [[x,y],z] + [y,[x,z]], \end{aligned} \tag{3.5.15}$$

such that the following equation holds for each $x, y, z, w \in V$*:*

$$\begin{aligned} [x, J(y,z,w)] + J(x,[y,z],w) + J(x,z,[y,w]) + [J(x,y,z),w] \\ +[z, J(x,y,w)] = J(x,y,[z,w]) + J([x,y],z,w) \\ +[y, J(x,z,w)] + J(y,[x,z],w) + J(y,z,[x,w]). \end{aligned} \tag{3.5.16}$$

Remark 3.5.17. In the language of Lie 2-algebras, the equation (3.5.16) is the *Jacobiator identity*, which is an analogue of the pentagon identity for monoidal categories.

Theorem 3.5.18. *Let* $V_0 = Obj(\mathcal{L}_{g_{ijk}})$*, with vector space structure given by*

1. $\lambda(\xi, \{f_{ij}^{\hat{\xi}}\}) = (\lambda\xi, \{\lambda f_{ij}^{\hat{\xi}}\})$ *for each* $\lambda \in \mathbb{R}$ *and* $(\xi, \{f_{ij}^{\hat{\xi}}\}) \in \mathcal{L}_{g_{ijk}}$*, and*

2. $(\xi, \{f_{ij}^{\hat{\xi}}\}) + (\eta, \{f_{ij}^{\hat{\eta}}\}) = (\xi + \eta, \{f_{ij}^{\hat{\xi}} + f_{ij}^{\hat{\eta}}\})$ *for each* $(\xi, \{f_{ij}^{\hat{\xi}}\}), (\eta, \{f_{ij}^{\hat{\eta}}\}) \in \mathcal{L}_{g_{ijk}}$.

Let $V_1 = \{\{u_i : U_i \to i\mathbb{R}\}\}$ *with vector space structure given by addition and scalar multiplication of functions. Then* $V = V_0 \oplus V_1$ *has the structure of a 2-term* L_∞*-algebra with*

1. $d : V_1 \to V_0$ *given by* $\{u_i\} \mapsto (0, \{u_i - u_j\})$ *for each* $\{u_i\} \in V_1$

2. $[\cdot,\cdot] : V \otimes V \to V$ *given by*

   (a) $[(\xi, \{f^{\hat{\xi}}_{ij}\}), (\eta, \{f^{\hat{\eta}}_{ij}\})] = ([\xi,\eta], \{\xi(f^{\hat{\eta}}_{ij}) - \eta(f^{\hat{\xi}}_{ij})\})$ *for each* $(\xi, \{f^{\hat{\xi}}_{ij}\}), (\hat{\eta}, \{f^{\eta}_{ij}\}) \in V_0$

   (b) $[(\xi, \{f^{\hat{\xi}}_{ij}\}), \{u_i\}] = \{\xi(u_i)\} = -[\{u_i\}, (\xi\{f^{\hat{\xi}}_{ij}\})]$ *for each* $(\xi\{f^{\hat{\xi}}_{ij}\}) \in V_0$, $\{u_i\} \in V_1$

   (c) $[\{u_i\}, \{v_i\}] = 0$ *for each* $\{u_i\}, \{v_i\} \in V_1$.

3. $J = 0$.

Proof. To check that $[\cdot,\cdot]$ is a chain map, it is sufficient to check that for each $v_0 = (\xi_i, \{f^{\hat{\xi}}_{ij}\}) \in V_0$ and $v_1 = \{u_i\}$ and $v_1' = \{u_i'\}$ in V_1 that

$$d[v_0, v_1] = [v_0, dv_1], \tag{3.5.19}$$

and

$$[dv_1, v_1'] = [v_1, dv_1']. \tag{3.5.20}$$

To verify (3.5.19), note that

$$d[(\xi, f_{ij}), \{u_i\}] = d\{\xi(u_i)\} = (0, \{\xi(u_i - u_j)\}), \tag{3.5.21}$$

whereas

$$[(\xi,\{f_{ij}\}),d\{u_i\}] = [(\xi,\{f_{ij}\},(0,\{u_i-u_j\})] = (0,\{\xi(u_i-u_j)\}). \qquad (3.5.22)$$

On the other hand, by inspection both the left and right-hand sides of equation (3.5.20) are zero.

To verify that J is a chain homotopy from $[[x,y],z] \to [[x,y],z] + [y,[x,z]]$, there are two conditions to verify. First, for each $u_0, v_0, w_0 \in L_0$, we must check that

$$dJ(u_0,v_0,w_0) = -[[u_0,v_0],w_0] + [[u_0,w_0],v_0] + [u_0,[v_0,w_0]]. \qquad (3.5.23)$$

The left-hand side is 0 by definition, whereas it is easy to verify that the right-hand side vanishes using the Jacobi identity for vector fields on M. The second condition is that for $v_0, w_0 \in V_0$ and $v_1 \in V_1$, we have

$$J(dv_1,v_0,w_0) = -[[v_0,w_0],v_1] + [[v_0,v_1],w_0] + [v_0,[w_0,v_1]]. \qquad (3.5.24)$$

Again, the left-hand side is zero, and a simple computation shows that the right-hand side vanishes as well.

Finally, because $J = 0$, the condition (3.5.16) trivial. □

Remark 3.5.25. Each of the properties verified in the proof of (3.5.18) (with the exception of the Jacobiator identity) follows from the functoriality of operations in Definition 3.5.1.

3.6 Connective structures and curvings

Given a Dixmier-Douady gerbe $\mathcal{G}$ over a manifold M, Brylinski [Br1] introduced the notions of a connective structure $\mathcal{A}$ on $\mathcal{G}$ and a curving K for $\mathcal{A}$. We now explain how these structures emerge very naturally from the point of view of the infinitesimal symmetries of $\mathcal{G}$ introduced in the previous sections. For simplicity we begin by working in the Cech picture.

Let $\{\{U_i\},\{Q_i\},\{s_{ij}\}\}$ be such a collection of local trivializations with corresponding Cech data $\{g_{ijk}\}$, and let $\mathcal{L}_{g_{ijk}}$ be the corresonding category of lifts described in Definition 3.3.12; note that there is a natural projection π from $\mathcal{L}_{g_{ijk}}$ to $C^\infty(TM)$. Since by definition the category $C^\infty(TM)$ has only trivial morphisms, we take a linear splitting of the projection π to be a linear splitting of the sequence of vector spaces

$$0 \longrightarrow \mathrm{Ker}(\pi) \longrightarrow \mathrm{Obj}(\mathcal{L}_{g_{ijk}}) \xrightarrow{\pi} C^\infty(TM) \longrightarrow 0. \qquad (3.6.1)$$

Such a splitting can be obtained from a collection of 1-forms $\{A_{ij} \in T^*U_{ij}\}$ by setting $\{f_{ij}(\xi) = \iota_\xi A_{ij}\}$; the condition (3.3.9) is equivalent to the condition

$$A_{jk} - A_{ik} + A_{ij} = d\log(g_{ijk}). \qquad (3.6.2)$$

Such a collection of 1-forms is precisely the data needed to specify a connective structure on $\mathcal{G}$ in the Cech picture [Hi]. Thus, given such a collection of 1-forms, we may define the *horizontal lift* $\hat{\xi}^h$ of a vector field ξ to be given by the Cech data $\{f_{ij}^{\xi^h} = \iota_\xi A_{ij}\}$.

Next, let ξ, η be a pair of vector fields on M. Note that in general we have no natural way to compare the lifts $\widehat{[\xi,\eta]}^h$ and $[\hat{\xi}^h, \hat{\eta}^h]$, whose Cech data are given respectively by

$$f_{ij}^{\widehat{[\xi,\eta]}^h} = \iota_{[\xi,\eta]} A_{ij} \tag{3.6.3}$$

and

$$f_{ij}^{[\hat{\xi}^h, \hat{\eta}^h]} = \xi \cdot (\iota_\eta A_{ij}) - \eta \cdot (\iota_\xi A_{ij}). \tag{3.6.4}$$

On the other hand, suppose that we are given the additional structure of a *curving*: in the Cech picture this is a collection of 2-forms $\{B_i \in i\Omega^2(U_i)\}$ such that on overlaps we have

$$B_j - B_i = dA_{ij}. \tag{3.6.5}$$

A simple calculation shows that

$$f_{ij}^{[\hat{\xi}^h, \hat{\eta}^h]} - f_{ij}^{\widehat{[\xi,\eta]}^h} = \iota_\eta \iota_\xi dA_{ij}, \tag{3.6.6}$$

and it therefore follows that

$$\{u_i = \iota_\eta \iota_\xi B_i\} \tag{3.6.7}$$

defines a morphism in the Cech picture between the two lifts (3.6.3) and (3.6.4).

With the above discussion as motivation, we recall the definition of a connective structure on $\mathcal{G}$ in the language of sheaves of categories. The following is a restatement of definition (5.31) from [Br1] in the language of §2.3.

Definition 3.6.8. *A* connective structure *on $\mathcal{G}$ is a morphism of gerbes*

$$\mathcal{A} : \mathcal{G} \to B(\underline{i\Omega^1_M}) \tag{3.6.9}$$

intertwining the homomorphism

$$-d\log : \underline{\mathbb{T}}_M \to \underline{i\Omega^1_M}. \tag{3.6.10}$$

Notation 3.6.11. For each object Q of $\mathcal{G}$ over U, $\mathcal{A}(Q)$ is a sheaf over U. We often write $\mu \in A(Q)$ to denote that μ is a global section of $\mathcal{A}(Q)$; we call such a μ a *connection* on Q.

Notation 3.6.12. Given an inclusion of open sets $i : V \hookrightarrow U$, $P \in \mathcal{G}(U)$ and $\mu \in \mathcal{A}(P)$, we obtain an element of $\mathcal{A}(i^*P)$ in two steps (see Definition .2.11): first we restrict μ to V to obtain a section of the $\underline{i\Omega^1_V}$ torsor $i^*\mathcal{A}(P)$. Then using the natural isomorphism

$$i^*\mathcal{A}(P) \xrightarrow{\cong} \mathcal{A}(i^*P) \tag{3.6.13}$$

(which is given as part of the data of the 1-morphism $\mathcal{A}$), we obtain a section of $\mathcal{A}(i^*P)$. Abusing notation slightly, we will denote this element either by $\mu|_V$ or $i^*\mu$.

Remark 3.6.14. Definition 3.6.8 can easily be generalized to any gerbe with band $\underline{A}_M$ for A an abelian Lie group; the sheaf $\underline{i\Omega^1_M}$ is replaced by the sheaf of Lie(A)-valued 1-forms. We will later use the case $A = i\mathbb{R}$, where we make the identification Lie($i\mathbb{R}$) $:= i\mathbb{R}$.

Example 3.6.15. Let $\mathrm{B}(\underline{\mathbb{T}}_M)$ denote the trivial DD gerbe over M. whose objects are $\underline{\mathbb{T}}_U$-torsors for open sets $U \subset M$. We define the *trivial connective structure* $\mathcal{A}^0_{\underline{\mathbb{T}}_M}$ on $\mathrm{B}(\underline{\mathbb{T}}_M)$ as the 1-morphism which assigns to each torsor its sheaf of connections. We can rephrase this definition using the discussion of Example 2.2.32 as well as Proposition 2.3.17. Namely, the trivial connective structure on $\mathrm{B}(\underline{\mathbb{T}}_M)$ can be define as the 1-morphism of gerbes

$$-d\log[\cdot] : \mathrm{B}(\underline{\mathbb{T}}_M) \to \mathrm{B}(\underline{i\Omega}^1_M). \tag{3.6.16}$$

Example 3.6.17. Let us construct a connective structure on the gerbe $\mathcal{G}$ on S^3 described in Example 2.3.8. Again, we follow the discussion in §7.2 of [Br1]. Recall that $\mathcal{G}$ was defined using an open cover of S^3 by two open sets U_a and U_b, together with a principal circle bundle E on their intersection $U_{ab} = U_a \cap U_b \simeq S^2$. To define a connective structure $\mathcal{A}$ on $\mathcal{G}$, we choose a connection Θ on E. Given an open set $W \subset S^3$, recall that an object of $\mathcal{G}(W)$ consists of a triple $Q = (P_a, P_b, \phi)$, where P_a is a principal circle bundle on $W \cap U_a$, P_b is a principal circle bundle on $W \cap U_b$, and ϕ is an isomorphism

$$P_a \otimes P_b^* \xrightarrow{\cong} E,$$

where each bundle is implicitly restricted to $W \cap U_{ab}$. For each such object Q, we need to define an $\underline{i\Omega}^1_W$-torsor $\mathcal{A}(Q)$. Given an open set $V \subset W$, $\mathcal{A}(Q)(V)$ consists of a pair of connections $\mu = (\Theta_a, \Theta_b)$ on (the restrictions to V of) P_a and P_b, such that the isomorphism ϕ maps $\Theta_a - \Theta_b$ to the connection Θ. The action of $\alpha \in i\Omega^1(V)$ on $\mathcal{A}(Q)(V)$ is given by $(\Theta_a, \Theta_b) \mapsto (\Theta_a + \alpha, \Theta_b + \alpha)$.

We have the following definition of the horizontal lift in sheaf language.

Definition 3.6.18. *For each vector field* ξ *on* M*, the* horizontal lift $\hat{\xi}^h$ *of* ξ *to* $\mathcal{G}$ *is the composition of 1-morphisms*

$$\mathcal{G} \xrightarrow{\mathcal{A}} B(\underline{i\Omega^1_M}) \xrightarrow{(\iota_\xi)[\cdot]} B(\underline{i\mathbb{R}_M}) \tag{3.6.19}$$

$$P \longmapsto \mathcal{A}(P) \longmapsto \iota_\xi[\mathcal{A}(P)].$$

In order to explain the connection between this definition and the definition of the horizontal lift in the Cech picture, let us return to the situation that we have a collection of local trivializations for $\mathcal{G}$. To express $\mathcal{A}$ in terms of Cech data, let us choose connections $\mu_i \in \mathcal{A}(Q_i)$. If we define 1-forms A_{ij} on overlaps by the equation

$$\mu_j = (s_{ij})_* \mu_i + A_{ij}, \tag{3.6.20}$$

then (3.6.8) implies that on triple overlaps these 1-forms satisfy equation (3.6.2). Given a vector field ξ on M, let $\hat{\xi}^h = \iota_\xi d[\mathcal{A}]$ be the horizontal lift given in Definition 3.6.18. Then we obtain sections

$$r_i = \iota_\xi[\mu_i] \in F_{\hat{\xi}^h}(Q_i), \tag{3.6.21}$$

and a simple calculation shows that the corresponding Cech data is given by $\{\iota_\xi A_{ij}\}$.

Let us also recall Brylinski's definition of a *curving* for $\mathcal{A}$.

Definition 3.6.22. ([Br1], Def. 5.3.7) *Let $\mathcal{G}$ be a DD gerbe over a manifold M equipped with a connective structure $\mathcal{A}$. A* curving *of the connective structure is a function which assigns to each object $P \in \mathcal{G}(U)$ and each local section $\mu \in \mathcal{A}(P)(V)$ (for $V \subset U$ an open subset) a 2-form $K(\mu) \in i\Omega^2(V)$, such that the following properties are satisfied:*

(1) For each inclusion of open sets $i : W \hookrightarrow V$, each $P \in \mathcal{G}(U)$ and $\mu \in \mathcal{A}(P)(V)$, we have

$$K(i^*\mu) = K(\mu)|_W \tag{3.6.23}$$

(2) For each pair of objects $P, Q \in \mathcal{G}(U)$, each morphism $\psi \in \underline{Hom}(P, Q)(V)$, and each $\mu \in \mathcal{A}(P)(V)$, we have

$$K(\psi_*\mu) = K(\mu) \tag{3.6.24}$$

(3) For each object $P \in \mathcal{G}(U)$, each $\mu \in \mathcal{A}(P)(V)$ and each 1-form $\alpha \in i\Omega^1(V)$, we have

$$K(\mu + \alpha) = K(\mu) + d\alpha \tag{3.6.25}$$

Remark 3.6.26. Given an $\underline{i\Omega}^1_U$-torsor A, a map from A to $\underline{i\Omega}^2_U$ intertwining $d : \underline{i\Omega}^1_U \to \underline{i\Omega}^2_U$ is equivalent to a isomorphism of $\underline{i\Omega}^2_U$-torsors $d[A] \xrightarrow{\cong} \underline{i\Omega}^2_U$. Unravelling Definition .2.16, it follows that specifying a curving of$\mathcal{A}$ is equivalent to specifying a 2-morphism

$$d[\mathcal{A}] \xrightarrow{\cong} \underline{i\Omega}^2_M, \tag{3.6.27}$$

where $\underline{i\Omega}^2_M$ denotes the trivial 1-morphism from $\mathcal{G}$ to $\mathrm{B}(\underline{i\Omega}^2_M)$ sending every object $Q \in \mathcal{G}(U)$ to the trivial $\underline{i\Omega}^2_U$-torsor $\underline{i\Omega}^2_U$.

Example 3.6.28. Let $\mathcal{A}^0_{\mathbb{T}}$ denote the trivial connective structure on the trivial DD gerbe $\underline{\mathrm{B}\mathbb{T}}_M$. Then the *trivial curving* for $\mathcal{A}^0_{\mathbb{T}}$ assigns to each connection its curvature 2-form.

Example 3.6.29. Continuing with Examples 2.3.8 and 3.6.17, let $\omega \in i\Omega^2(U_{ab})$ denote the curvature of E. Let $\{\rho_a, \rho_b\}$ be a partition of unity subordinate to $\{U_a, U_b\}$; then we define 2-forms $B_{a,b} \in i\Omega^2(U_{a,b})$ such that restricted to U_{ab} we have $B_a = -\rho_b\omega$ and $B_b = \rho_a\omega$. Note that on U_{ab} we have $B_b - B_a = \omega$. Given $Q \in \mathcal{G}(W)$ and $\mu = (\Theta_a, \Theta_b) \in \mathcal{A}(Q)$, define $K(\mu) \in i\Omega^2(W)$ by $K(\mu) = \mathrm{curv}(\Theta_a) + B_a$ on $W \cap U_a$ and $K(\mu) = \mathrm{curv}(\Theta_b) + B_b$. Using the fact that $\mathrm{curv}(\Theta_a) - \mathrm{curv}(\Theta_b) = \omega$ on $W \cap U_{ab}$, it follows $K(\mu)$ is well-defined on W. One easily checks that this defines a curving for $\mathcal{A}$.

To see how a curving appears in the Cech picture, define 2-forms $B_i \in \underline{i\Omega}^2(U_i)$ by

$$B_i = K(\mu_i). \tag{3.6.30}$$

It then follows from Definition 3.6.22 that on overlaps these satisfy equation (3.6.5).

Next, we recall the definition of the curvature 3-form associated to a curving.

Definition 3.6.31. *Given a connective structure $\mathcal{A}$ on a gerbe $\mathcal{G}$ and curving K on a DD gerbe over a manifold M, the* curvature *of $(\mathcal{G}, \mathcal{A}, K)$ is the 3-form C on M defined locally by $C = dK(\mu)$, where μ is a section of $\mathcal{A}(Q)$ for $Q \in \mathcal{G}(U)$.*

Recall that if Θ is a connection on a principal $\mathbb{T}$-bundle $E \to M$, then the curvature 2-form of Θ measures the failure of the splitting of the sequence (1.3.2) determined by Θ to be a homomorphism of Lie algebras. To understand the analogous situation for gerbes, we need the notion of a homomorphism of 2-term L_∞-algebras.

Definition 3.6.32. ([BCr] def. 34) *Let V and V' be 2-term L_∞-algebras. An L_∞-*homomorphism *$\phi : V \to V'$ consists of*

1. *a degree 0 chain map $\phi : V \to V'$,*

2. *an antisymmetric degree 1 chain map $\phi_2 : V \otimes V \to V'$*

such that the following equations hold:

(i) $d(\phi_2(x, y)) = \phi[x, y] - [\phi(x), \phi(y)]$ *for all $x, y \in V$, and*

(ii) $[\phi_2(x, y), \phi(z)] + \phi_2([x, y], z) + \phi(J(x, y, z)) = J(\phi(x), \phi(y), \phi(z)) + [\phi(x), \phi_2(y, z)] + [\phi_2(x, z), \phi(y)] + \phi_2(x, [y, z]) + \phi_2([x, z], y)$ *for all $x, y, z \in V_0$.*

We make $C^\infty(TM)$ into a 2-term L_∞-algebra by setting $V_0 = C^\infty(TM)$ and $V_1 = 0$, with bracket given by the Lie bracket. We then can extend the linear map from $\mathrm{Obj}(\mathcal{L}_{g_{ijk}}) \to C^\infty(TM)$ in an obvious way to an L_∞-homomorphism. In the language of Definition 9.3 from [Ro2], we obtain a *strict central extension of L_∞-algebras*

$$\mathcal{L}_{\{g_{ijk}\}}(0) \to \mathcal{L}_{\{g_{ijk}\}} \to C^\infty(TM). \tag{3.6.33}$$

One should compare this extension to (1.3.2), noting that the category $\mathcal{L}_{\{g_{ijk}\}}(0)$ is naturally equivalent to the category of principal $i\mathbb{R}$-bundles over M.

Let us try to construct a splitting of the extension (3.6.33), i.e. an L_∞-homomorphism $\phi : C^\infty(TM) \to \mathcal{L}_{g_{ijk}}$ such that $\pi \circ \phi$ is the identity on $C^\infty(TM)$.[1] A degree zero chain map $\phi : C^\infty \to \mathcal{L}_{g_{ijk}}$ is simply a linear splitting of the projection $\mathrm{Obj}(\mathcal{L}_{g_{ijk}}) \to C^\infty(TM)$, which we have already seen is essentially equivalent to a connective structure $\{A_{ij}\}$ on $\mathcal{G}$ in the Cech picture. Similarly, it follows from the discussion at the beginning of this section that an antisymmetric degree 1 chain map $\phi_2 : C^\infty(TM) \otimes C^\infty(TM) \to \mathcal{L}_{g_{ijk}}$ is essentially equivalent to a curving $\{B_i\}$.

Finally, since the Jacobiator maps J vanish for both $C^\infty(M)$ and L, it follows from Definition 3.6.32 that (ϕ, ϕ_2) defines an L_∞-homomorphism if and only if for each $\xi, \eta, \tau \in C^\infty(M)$ we have

$$\begin{aligned}
&[\phi_2(\xi,\eta), \phi(\tau)] + \phi_2([\xi,\eta],\tau) - [\phi(\xi), \phi_2(\eta,\tau)] \\
&- [\phi_2(\xi,\tau), \phi(\eta)] - \phi_2(\xi, [\eta,\tau]) = 0.
\end{aligned} \tag{3.6.34}$$

However, the left hand side of this equation is the element $\{u_i\} \in L^1$ given by

$$\begin{aligned}
u_i &= \tau \cdot B_i(\xi,\eta) - B_i([\xi,\eta],\tau) + \xi \cdot B_i(\eta,\tau) \\
&\quad - \eta \cdot B_i(\xi,\tau) - B_i([\eta,\tau],\xi) + B([\xi,\tau],\eta) \\
&\quad = C(\xi,\eta,\tau),
\end{aligned} \tag{3.6.35}$$

[1] More generally we would ask only for a chain homotopy from $\pi \circ \phi$ to the identity map.

where $C = dB_i$ is the curvature 3-form. We therefore see that the curvature (or rather its de Rham cohomology class) is an obstruction to the existence of a splitting of (3.6.33) as Lie 2-algebras.

Chapter 4

Infinitesimal symmetries of gerbes with connective structures and curvings

4.1 Symmetries of gerbes with connective structures

In this section we will study symmetries of a gerbe $\mathcal{G}$ equipped with a connective structure $\mathcal{A}$. We begin by extending Definition 3.2.4 to take the connective structure into account. The motivation for this definition can best be understood by studying the relationship to families of symmetries of gerbes with connective structure, as discussed in Appendix A.

If (P, Θ) and (P', Θ') are principal $\mathbb{T}$-bundles with connection over a manifold M, then given an isomorphism $\psi : P \to P'$ of the underlying principal bundles, we may ask whether ψ is *compatible* with the connections Θ, Θ': concretely, Θ and Θ' are differential forms and we may compare Θ with $\psi^*\Theta'$. By contrast, given Dixmier-Douady gerbes $(\mathcal{G}, A)$, $(\mathcal{G}', \mathcal{A}')$ with connective structures and a 1-morphism $\Psi : \mathcal{G} \to \mathcal{G}'$, it is no longer the right question to ask whether $\mathcal{A}$ is *equal* to $\Psi^*\mathcal{A}$. Rather, we must extend Ψ to a *connective isomorphism* by specifying extra data in the form of an isomorphism $\mathcal{A} \xrightarrow{\cong} \Psi^*\mathcal{A}'$. Recall from definition (3.6.8) that the connective structure $\mathcal{A}'$ is a 1-morphism of gerbes

$$\mathcal{G}' \to \mathrm{B}(\underline{i\Omega^1_M}); \tag{4.1.1}$$

by definition the pull-back $\Psi^*(\mathcal{A}')$ is the composition

$$\mathcal{A}' \circ \Psi : \mathcal{G} \to \mathrm{B}(\underline{i\Omega^1_M}). \tag{4.1.2}$$

In this language an extension of Φ to a connective isomorphism is given by a 2-morphism

$$\mathcal{A} \overset{\cong}{\Longrightarrow} \mathcal{A}' \circ \Psi. \tag{4.1.3}$$

The analogous situation for infinitesimal symmetries is given in the following definition.

Definition 4.1.4. *Let $\mathcal{G}$ be a DD gerbe over a manifold M with connective structure $\mathcal{A}$. Given $\xi \in C^\infty(TM)$, a* connective lift $\check{\xi}$ *of ξ to $\mathcal{G}$ is a pair $\check{\xi} = (F_{\check{\xi}}, \Theta_{\check{\xi}})$, where $F_{\check{\xi}}$ is a lift of ξ to $\mathcal{G}$ in the sense of definition (3.2.4), and $\Theta_{\hat{\xi}} : \pounds_{-\xi}[\mathcal{A}] \overset{\cong}{\Longrightarrow} \mathcal{A}^0_{\underline{i\mathbb{R}}} F_{\hat{\xi}}$ is a 2-morphism.*

More explicitly, for every object $Q \in \mathcal{G}(U)$ and every local section μ of $\mathcal{A}(Q)$, a connective lift determines a local connection $\Theta_{\check{\xi}}(\mu)$ on $F_{\check{\xi}}(Q)$. Furthermore, this assignment is natural in Q in the sense described in (.2.16), and for every 1-form $\alpha \in i\Omega^1(U)$ we have

$$\Theta_{\check{\xi}}(\mu + \alpha) = \Theta_{\check{\xi}}(\mu) - \pounds_\xi \alpha. \tag{4.1.5}$$

Definition 4.1.6. *Let $(\mathcal{G}, \mathcal{A})$ is equipped with a curving K. Then a connective lift $\check{\xi}$ as in Definition 4.1.4 is said to* preserve K *if for each object $Q \in \mathcal{G}(U)$*

and each local section μ *of* $\mathcal{A}(Q)$*, we have*

$$K_0(\Theta_{\check{\xi}}(\mu)) = -\pounds_\xi K(\mu), \tag{4.1.7}$$

where $K_0(\Theta_{\check{\xi}}(\mu))$ *is the curvature of* $\Theta_{\check{\xi}}(\mu)$*.*

Remark 4.1.8. Given $(\mathcal{G}, \mathcal{A}, K)$, let $H \in i\Omega^3(M)$ be the curvature 3-form. Suppose $\check{\xi}$ is a connective lift of a vector field ξ preserving K. Then given an object $\mathcal{G}(U)$ and each section $\mu \in \Gamma(\mathcal{A}(Q))$, let $\nu = \Theta_{\check{\xi}}(\mu)$ denote the corresponding connection on $F_{\check{\xi}}(Q)$, and let $\omega \in i\Omega^2(U)$ denote its curvature. Then

$$\pounds_\xi dK(\mu) = -d\iota_\xi \pounds_\xi \omega = 0 \tag{4.1.9}$$

since $d\omega = 0$. By the definition of H we therefore have $\pounds_\xi H = 0$, which is therefore a necessary condition for the existence of a connective lift of ξ preserving K.

Example 4.1.10. Given a vector field ξ on M, recall from Example 3.2.9 the trivial lift $\hat{\xi}_0$ of $\hat{\xi}$ to the trivial gerbe $\mathrm{B}(\underline{\mathbb{T}}_M)$. This lift assigns to every $\underline{\mathbb{T}}_U$-torsor P the associated $\underline{i\mathbb{R}}_U$-torsor

$$-\iota_\xi d\log[P]. \tag{4.1.11}$$

We wish to extend this to define a trival connective lift of ξ to $\mathrm{B}(\underline{\mathbb{T}}_M)$ equipped with the trivial connective structure $\mathcal{A}^0_{\underline{\mathbb{T}}_M}$. Suppose Θ is a connection on a $\underline{\mathbb{T}}_M$-torsor P, then we will define a connection $-\pounds_\xi(\Theta)$ on $-\iota_\xi d\log[P]$ as follows. Each local section of $-\iota_\xi d\log[P]$ is of the form

$$-\iota_\xi d\log[\sigma] + f, \tag{4.1.12}$$

where σ is a local section of P and f is a local $i\mathbb{R}$-valued function. We then define $-\pounds_\xi(\Theta)$ by the formula

$$A_{-\pounds_\xi(\Theta)}(-\iota_\xi d\log[\sigma] + f) = -\pounds_\xi A_\Theta(\sigma) + df. \tag{4.1.13}$$

Using the Cartan formula $\pounds_\xi = d\iota_\xi + \iota_\xi d$, one easily checks that equation (4.1.13) is well-defined, i.e. independent of the choice of σ.

Definition 4.1.14. *The* trivial connective lift $\check{\xi}_0$ *of* ξ *to* $(B(\underline{\mathbb{T}}_M), \mathcal{A}^0_{\underline{\mathbb{T}}_M})$ *is the pair* $(\hat{\xi}_0, \Theta_{\check{\xi}_0})$*, where* $\hat{\xi}_0$ *is the trivial lift defined in (3.2.9), and for each* $\underline{i\mathbb{R}}_U$*-torsor* P *we have*

$$\begin{aligned}(\Theta_\xi)_P : \pounds_{-\xi}[\mathcal{A}^0_{\underline{\mathbb{T}}_M}(P)] &\to \mathcal{A}^0_{\underline{i\mathbb{R}}_M} \circ \hat{\xi}_0(P) \\ \pounds_{-\xi}[\Theta] &\mapsto -\pounds_\xi(\Theta).\end{aligned} \tag{4.1.15}$$

Note that, if we let K be the trivial connective structure, then the trivial connective lift in the above definition preserves K.

We next wish to define the category of connective lifts.

Definition 4.1.16. *Given connective lifts* $\check{\xi} = (F_{\check{\xi}}, \Theta_{\check{\xi}})$ *and* $\check{\xi}' = (F_{\check{\xi}'}, \Theta_{\check{\xi}'})$ *of a vector field* ξ*, an equivalence* T *between the underlying (non-connective) lifts* $F_{\check{\xi}}, F_{\check{\xi}'}$ *is an* equivalence of connective lifts *if the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{A} & & \\ {\scriptstyle \Theta_{\check{\xi}}}\Downarrow & \overset{\Theta_{\check{\xi}'}}{\searrow} & \\ \mathcal{A}_0 \circ F_{\check{\xi}} & \underset{1_{\mathcal{A}_0} * T}{\Longrightarrow} & \mathcal{A}_0 \circ F_{\check{\xi}'}. \end{array} \tag{4.1.17}$$

Note that if there exists such an equivalence relating $\check{\xi}$ *and* $\check{\xi}'$*, then* $\check{\xi}$ *preserves* K *if and only if* $\check{\xi}'$ *does.*

Remark 4.1.18. In more concrete terms, the condition (4.1.17) says that for each $Q \in \mathcal{G}(U)$, $\mu \in \mathcal{A}(Q)$ and each $\sigma \in F_{\check{\xi}}(Q)$, we have

$$A_{\Theta_{\check{\xi}'(\mu)}}(T_Q(\sigma)) = A_{\Theta_{\check{\xi}(\mu)}}(\sigma). \tag{4.1.19}$$

In particular, suppose $\check{\xi} = \check{\xi}'$, then by Theorem 3.3.19, T is given by

$$\begin{aligned} T_Q : F_{\check{\xi}}(Q) &\to F_{\check{\xi}}(Q) \\ \sigma &\mapsto \sigma + f|_U, \end{aligned} \tag{4.1.20}$$

for each $Q \in \mathcal{G}(U)$, where $f : M \to i\mathbb{R}$ is a globally defined function. Equation (4.1.19) therefore implies that T gives an automorphism of $\check{\xi}$ if and only if f is locally constant.

Definition 4.1.21. $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ *is the category whose objects are connective lifts and whose morphisms are connective equivalences of lifts. If* K *is a curving, then* $\mathcal{L}_{(\mathcal{G},\mathcal{A},K)}$ *is the subcategory of* $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$ *consisting of connective lifts which preserve* K.

Notation 4.1.22. For each vector field ξ on M $\check{L}_{\mathcal{G}}(\xi)$ denotes the subcategory of connective lifts of ξ to $\mathcal{G}$.

Remark 4.1.23. We remark that the construction of the functors $\boxplus, \lambda\odot$ and $[\cdot,\cdot]$ given in proposition (3.5.1) generalizes to the connective setting. In particular, the bracket functor is extended explicitly as follows. Let $\check{\xi} = (F_{\check{\xi}}, \Theta_{\check{\xi}})$ and $\check{\eta} = (F_{\check{\eta}}, \Theta_{\check{\xi}})$ be connective lifts of the vector fields ξ, η, respectively. Then for each local section μ of $\mathcal{A}(P)$, $\Theta_{[\check{\xi},\check{\eta}]}(\mu)$ is the connection on $F_{[\check{\xi},\check{\eta}]}(P)$ given by

$$A_{\Theta_{[\hat{\xi},\hat{\eta}]}} : \xi[\sigma_1] \boxminus \eta[\sigma_2] \mapsto \pounds_\xi A_{\Theta_{\hat{\eta}(\mu)}}(\sigma_1) - \pounds_\eta A_{\Theta_{\hat{\xi}(\mu)}}(\sigma_2) \tag{4.1.24}$$

where σ_1 and σ_2 are local sections of $F_{\check{\xi}}(P)$ and $F_{\check{\eta}}$, respectively. Note that if $\check{\xi}$ and $\check{\eta}$ both preserve K, then so does their bracket. We will explain how the bracket appears in the Cech picture below.

4.2 Connective lifts in the Cech picture

Let us describe the category of connective lifts in the Cech picture. Let

$$\{\{U_i\}, \{Q_i\}, \{s_{ij}\}, \{\mu_i\}\} \tag{4.2.1}$$

be a collection of local trivializations of $(\mathcal{G}, \mathcal{A})$. Recall that if we define 1-forms A_{ij} on U_{ij} by

$$\mu_j = (s_{ij})_*\mu_i + A_{ij}, \tag{4.2.2}$$

then these satisfy

$$A_{jk} - A_{ik} + A_{ij} = d\log(g_{ijk}) \tag{4.2.3}$$

on triple overlaps.

Next, let $\check{\xi} = (F_{\check{\xi}}, \Theta_{\check{\xi}})$ be a connective lift of ξ to $(\mathcal{G}, \mathcal{A})$, and suppose we have chosen local sections $r_i \in F_{\check{\xi}}(Q_i)$. Then for each i we obtain a connection $\nu_i = \Theta_{\check{\xi}}(\mu_i)$ on $F_{\check{\xi}}(Q_i)$, and we can define 1-forms

$$a_i = A_{\nu_i}(r_i) \in i\Omega^1_{U_i}. \tag{4.2.4}$$

Note that if $\check{\xi}$ preserves K then

$$da_i + \pounds_\xi B_i = 0 \tag{4.2.5}$$

on each open set U_i.

Proposition 4.2.6. *On overlaps U_{ij}, we have*

$$a_j - a_i = df_{ij} - \pounds_\xi A_{ij}. \tag{4.2.7}$$

Proof. The naturality of of $\Theta_{\check{\xi}}$ implies that the diagram

$$\begin{array}{ccc} \mathcal{A}(Q_i) & \xrightarrow{(\Theta_{\check{\xi}})_{Q_i}} & \mathcal{A}_0(F_{\check{\xi}}Q_i) \\ {\scriptstyle \mathcal{A}(s_{ij})}\downarrow & & \downarrow{\scriptstyle \mathcal{A}_0(F_{\check{\xi}}(s_{ij}))} \\ \mathcal{A}(Q_j) & \xrightarrow[(\Theta_{\check{\xi}})_{Q_j}]{} & \mathcal{A}_0(F_{\check{\xi}}Q_j) \end{array} \tag{4.2.8}$$

commutes on each overlap U_{ij}; we therefore have

$$\mathcal{A}_0(F_{\check{\xi}}(s_{ij}))\Theta_{\check{\xi}}(\mu_i) = \Theta_{\check{\xi}}(\mathcal{A}(s_{ij})(\mu_i)). \tag{4.2.9}$$

The left-hand side of this equation is equal to

$$F_{\check{\xi}}(s_{ij})_*\nu_i, \tag{4.2.10}$$

whereas the right-hand side is

$$\Theta_{\check{\xi}}(\mu_j - A_{ij}) = \nu_j + \pounds_\xi A_{ij}. \tag{4.2.11}$$

We therefore have

$$\begin{aligned} a_j =& A_{\nu_j}(r_j) \\ =& A_{\nu_j}(F_{\check{\xi}}(s_{ij})r_i + f_{ij}) \\ =& A_{\nu_j}(F_{\check{\xi}}(s_{ij})r_i) + df_{ij} \\ =& A_{F_{\check{\xi}}(s_{ij})_*\nu_i - \pounds_\xi A_{ij}}(F_{\check{\xi}}(s_{ij})r_i) + df_{ij} \\ =& A_{F_{\check{\xi}}(s_{ij})_*\nu_i}(F_{\check{\xi}}(s_{ij})r_i) - \pounds_\xi A_{ij} + df_{ij} \\ =& A_{\nu_i}(r_i) + df_{ij} - \pounds_\xi A_{ij} \\ =& a_i + df_{ij} - \pounds_\xi A_{ij}. \end{aligned} \tag{4.2.12}$$

□

Next, let $\tau : (F_{\check{\xi}}, \Theta_{\check{\xi}}) \to (F_{\check{\xi}'}, \Theta_{\check{\xi}'})$ be an equivalence of connective lifts, and let $\{r_i'\}$ be a collection of local trivializations of $F_{\check{\xi}'}$, with corresponding Cech data $\{a_i'\}$ and $\{f_{ij}'\}$. Recall from §3.3, that if we define local $i\mathbb{R}$-valued functions u_i by

$$r_i' = \tau_{Q_i}(r_i) + u_i, \tag{4.2.13}$$

then these functions satisfy

$$f_{ij}' - f_{ij} = u_i - u_j. \tag{4.2.14}$$

Unravelling definition (4.1.16), we see that

$$\nu_i' = \Theta_{\check{\xi}'}(\mu_i) = (\tau_{Q_i})_* \nu_i. \tag{4.2.15}$$

We therefore have

$$\begin{aligned} a_i' =& A_{\nu_i'}(r_i') && (4.2.16)\\ =& A_{(\tau_{Q_i})_*\nu_i}(\tau_{Q_i}(r_i) + u_i) \\ =& A_{(\tau_{Q_i})_*\nu_i}(\tau_{Q_i}(r_i)) + du_i \\ =& A_{\nu_i}(r_i) + du_i \\ =& a_i + du_i, && (4.2.17) \end{aligned}$$

and therefore $a_i' - a_i = du_i$ on overlaps.

Definition 4.2.18. *Let $\{g_{ijk} : U_{ijk} \to \mathbb{T}\}$ be a Cech cocycle on a manifold M with respect to a good open cover $\{U_i\}$, and let $\{A_{ij} \in i\Omega^1(U_{ij})\}$ be a collection of 1-forms satisfying equation (3.6.2) on triple overlaps. The category*

$\mathcal{L}_{(g_{ijk},A_{ij})}$ *has as objects the set of triples* $(\xi, \{f_{ij}\}, \{a_i\})$*, where the functions* f_{ij} *satisfy equation (3.3.9), and* $a_i \in i\Omega(U_i)$ *is a collection of 1-forms satisfying equation (4.2.7). A morphism from* $(\xi, \{f_{ij}^{\check{\xi}}\}, \{a_i^{\check{\xi}}\}) \to (\xi, \{f_{ij}^{\check{\xi}'}\}, \{a_i^{\check{\xi}'}\})$ *is a collection of local functions* $\{u_i : U_i \to i\mathbb{R}\}$ *such that on each overlap* U_{ij} *we have* $f_{ij}^{\check{\xi}'} - f_{ij}^{\check{\xi}} = u_i - u_j$*, and on each* U_i *we have* $a_i^{\check{\xi}'} - a_i^{\check{\xi}} = du_i$*.*

We can now classify connective lifts via Theorem 3.3.14, augmented by the following result.

Theorem 4.2.19. *Let* $\{g_{ijk}\}$ *and* $\{A_{ij}\}$ *be as in Definition 4.2.18, and let*

$$\pi : \mathcal{L}_{(g_{ijk},A_{ij})} \to \mathcal{L}_{g_{ijk}} \tag{4.2.20}$$

be the obvious forgetful functor. Then for each vector field ξ *on* M *and each* $\{f_{ij}^{\check{\xi}}\} \in \mathcal{L}_{g_{ijk}}(\xi)$*, the set* $\pi^{-1}(\{f_{ij}^{\check{\xi}}\})$ *of connective lifts extending* $\{f_{ij}^{\check{\xi}}\}$ *is a torsor for global 1-forms on* M*, where the action of* $\alpha \in i\Omega^1(M)$ *on* $\check{\xi} = (\xi, \{f_{ij}^{\check{\xi}}\}, \{a_i\})$ *is given by*

$$\check{\xi} + \alpha = (\xi, \{f_{ij}^{\check{\xi}}\}, \{a_i + \alpha|_{U_i}\}). \tag{4.2.21}$$

Furthermore, the automorphism group of each such extension may be identified with the group of locally constant functions $f : M \to i\mathbb{R}$*.*

Proof. Let $\hat{\xi} = (\xi, \{f_{ij}^{\hat{\xi}}\})$ be an object of $\mathcal{L}_{g_{ijk}}(\xi)$, and let

$$\beta_{ij} = df_{ij}^{\hat{\xi}} - \pounds_\xi A_{ij}. \tag{4.2.22}$$

Then the Cech coboundary of $\{\beta_{ij}\}$ is given by

$$\begin{aligned} (\partial\beta)_{ijk} &= d(\partial f)_{ijk} - \pounds_\xi(\partial A)_{ijk} \\ &= d(\iota_\xi d\log(g_{ijk})) - \pounds_\xi(d\log(g_{ijk})) = 0. \end{aligned} \tag{4.2.23}$$

Using partitions of unity we may therefore find $a_i \in i\Omega^1_{U_i}$ such that

$$a_j - a_i = df^{\hat{\xi}}_{ij} - \pounds_\xi A_{ij} \tag{4.2.24}$$

on overlaps. Thus there exists an extension $\check{\xi} = (\xi, \{f^{\hat{\xi}}_{ij}\}, \{a_i\})$ of $(\xi, \{f^{\hat{\xi}}_{ij}\})$ to a connective lift.

Given a global 1-form α, the formula (4.2.21) clearly defines a new connective lift $\check{\xi}+\alpha$. On the other hand, given two connective lifts $(\xi, \{f^{\check{\xi}}_{ij}\}, \{a^{\check{\xi}}_i\})$, $(\xi, \{f^{\check{\xi}'}_{ij}\}, \{a^{\check{\xi}'}_i\})$, if we define $\alpha_i = a^{\check{\xi}'}_i - a^{\check{\xi}}_i$, then we must have $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$, so that there is a global 1-form α with $\alpha_i = \alpha|_{U_i}$. □

We can also generalize Theorem 3.3.18, the proof of which is in Appendix B.

Theorem 4.2.25. *Let $\{\{U_i\}, \{s_{ij}\}, \{\mu_i\}\}$ be a collection of local trivializations of $(\mathcal{G}, \mathcal{A})$ with corresponding Cech data $\{g_{ijk}, A_{ij}\}$. Then there is an equivalence of categories $\mathcal{L}_{(g_{ijk}, A_{ij})} \to \mathcal{L}_{(\mathcal{G},\mathcal{A})}$.*

For each vector field ξ on M, let $\mathcal{L}_{(\mathcal{G},\mathcal{A})}(\xi)$ denote the subcategory consisting of connective lifts of ξ to $\mathcal{G}$. Also, let $\pi : \mathcal{L}_{(\mathcal{G},\mathcal{A})}(\xi) \to \mathcal{L}_{\mathcal{G}}(\xi)$ denote the obvious forgetful functor.

Corollary 4.2.26. *For each vector field ξ on M and each $\hat{\xi} \in \mathcal{L}_{\mathcal{G}}(\xi)$, the set $\pi^{-1}(\hat{\xi})$ of connective lifts extending ξ is a torsor for the group of global 1-forms on M. For each $\check{\xi} \in \check{\mathcal{L}}_{(\mathcal{G},\mathcal{A})}(\xi)$, the automorphism group of $\check{\xi}$ is canonically isomorphic to the group of locally constant $i\mathbb{R}$-valued functions on M.*

To end this section, we will generalize the construction in Theorem 3.5.18 to give $\mathcal{L}_{(g_{ijk},A_{ij})}$ the structure of a 2-term L_∞ algebra. First, one can checks that the bracket functor in the Cech picture is given by

$$\begin{aligned}&[(\xi,\{f_{ij}^{\hat{\xi}}\},\{a_i^{\hat{\xi}}\}),(\eta,\{f_{ij}^{\hat{\eta}}\},\{a_i^{\hat{\eta}}\})]\\=&([\xi,\eta],\{\xi(f_{ij}^{\hat{\eta}})-\eta(f_{ij}^{\hat{\xi}})\},\{\pounds_\xi a_i^{\hat{\eta}}-\pounds_\eta a_i^{\hat{\xi}}\}).\end{aligned} \tag{4.2.27}$$

We then have the following proposition, whose proof is an easy extension of the proof of (3.5.18).

Proposition 4.2.28. *Let $\{U_i\}$ be an open cover of a manifold M, and let $\{g_{ijk}\}$ be a $\mathbb{T}$-valued Cech cocycle. Then there is a 2-term L_∞ algebra $\check{V}$ with $\check{V}_0 = Ob(\mathcal{L}_{(g_{ijk},A_{ij})})$ and $\check{V}_1 = \{\{u_i : U_i \to i\mathbb{R}\}\}$, and such that*

1. *$d : \check{V}_1 \to \check{V}_0$ is given by*

$$\{u_i\} \mapsto (0, \{u_i - u_j\}, \{du_i\}), \tag{4.2.29}$$

   for each $\{u_i\} \in \check{V}_1$.

2. *$[\cdot,\cdot]$ is given on elements of $\check{V}_0$ by equation (4.2.27). Given $\check{\xi} = (\xi, \{f_{ij}^{\check{\xi}}\}, \{a_i^{\check{\xi}}\}) \in \check{V}_0$ and $u = \{u_i\} \in \check{V}_1$, we define*

$$[\check{\xi}, u] = \{\xi(u_i)\} = -[u, \check{\xi}]. \tag{4.2.30}$$

3. *$J = 0$.*

Furthermore, if $\{B_i \in i\Omega^2(U_i)\}$ is a collection of 2-forms satisfying $B_j - B_i = dA_{ij}$ on each overlap U_{ij}, then there is a sub Lie-2 algebra which in degree 0 consists of objects of $\mathcal{L}_{(g_{ijk},A_{ij})}$ satisfying condition (4.2.5).

4.3 Infinitesimal symmetries and transition circle torsors

In this short section we wish to present a variation of the Cech picture which will be useful for studying the examples presented in §4.4. Let $\mathcal{G}$ be a DD gerbe over a manifold M, let $\{U_i\}$ be an open cover of M such that there exist sections $Q_i \in \mathcal{G}(U_i)$. Then on each overlap we have the $\underline{\mathbb{T}}_{U_{ij}}$-torsor

$$E_{ij} = \underline{\mathrm{Hom}}(Q_i, Q_j), \tag{4.3.1}$$

where Q_i and Q_j are implicitly restricted to U_{ij}. The composition of morphisms gives isomorphisms

$$\lambda_{ijk} : E_{ij} \otimes E_{jk} \xrightarrow{\cong} E_{ik}, \tag{4.3.2}$$

on triple overlaps U_{ijk}, and the associativity of composition implies that on each 4-fold intersection U_{ijkl} the following diagram commutes:

$$\begin{array}{ccc} Q_{ij} \otimes Q_{jk} \otimes Q_{kl} & \xrightarrow{id_{Q_{ij}} \otimes \lambda_{jkl}} & Q_{ij} \otimes Q_{jl} \\ {\scriptstyle \lambda_{ijk} \otimes id_{Q_{kl}}} \downarrow & & \downarrow {\scriptstyle \lambda_{ijl}} \\ Q_{ik} \otimes Q_{kl} & \xrightarrow[\lambda_{ikl}]{} & Q_{il}. \end{array} \tag{4.3.3}$$

Next, suppose $\mathcal{G}$ is equipped with a connective structure $\mathcal{A}$. If for each open set U_i we choose a global section μ_i of $\mathcal{A}(Q_i)$, then we obtain connections Θ_{ij} on the torsors E_{ij} as follows. For each open set $W \subset U_{ij}$, recall that $\psi \in E_{ij}(W)$ is an isomorphism $Q_i|_W \xrightarrow{\cong} Q_j|_W$. The connection Θ_{ij} is then specified by the equation

$$\mu_j = \psi_*(\mu_i) + A_{\Theta_{ij}}(\psi). \tag{4.3.4}$$

One easily checks that these connections are compatible with the isomorphisms $\{\lambda_{ijk}\}$ in the sense that λ_{ijk} maps the connection $\Theta_{ij}+\Theta_{jk}$ on $E_{ij}\otimes E_{jk}$ to the connection Θ_{ik} on E_{ik}. Given a curving K, if we define $B_i = K(\mu_i) \in i\Omega^2(U_i)$, then the curvature $\omega_{ij} = K_0(\Theta_{ij})$ is equal to $B_j - B_i$ on U_{ij}.

Let ξ be a vector field on M, and $\hat{\xi}$ a lift of ξ to $\mathcal{G}$. For each i let $F_i = F_{\hat{\xi}}(Q_i)$, and choose global sections $r_i \in \Gamma(F_i)$. Then on each overlap U_{ij}, this data determines a lift $\hat{\xi}_{ij}$ of ξ to E_{ij} as follows. For each open subset $W \subset U_{ij}$ and each local section $\psi \in E_{ij}(W)$, define

$$r_j = F_{\hat{\xi}}(\psi) r_i + f_{\hat{\xi}_{ij}}(\psi). \tag{4.3.5}$$

Then for each $g : W \to \mathbb{T}$ we clearly have $f_{\hat{\xi}_{ij}}(\psi \cdot g) = f_{\hat{\xi}_{ij}}(\psi) - \iota_\xi d\log(g)$, so that this does in fact define a lift. Furthermore, these lifts are compatible with the structure maps λ_{ijk} in the following sense. Given a vector field ξ and $\underline{\mathbb{T}}_W$-torsors E and E', suppose that we have lifts $\hat{\xi}, \hat{\xi}'$ of ξ to E, E'. Then we obtain a lift $\hat{\xi} \otimes \hat{\xi}'$ of ξ to $E \otimes E'$ defined by the formula

$$f_{\xi\otimes\xi'}(\sigma \otimes \sigma') = f_{\hat{\xi}}(\sigma) + f_{\hat{\xi}'}(\sigma') \tag{4.3.6}$$

for each pair of local sections σ, σ' of E, E'. Furthermore, if E'' is another $\underline{\mathbb{T}}_W$-torsor and $\psi : E \xrightarrow{\cong} E''$ is an isomorphism, then the lift $\hat{\xi}$ of ξ to E determines a lift $\psi(\hat{\xi})$ of ξ to E'' in the obvious way.

Next, suppose we have a connective lift $\check{\xi} = (F_{\check{\xi}}, \Theta_{\check{\xi}})$ of ξ to $(\mathcal{G}, \mathcal{A})$ extending the lift above. For each local section σ of E_{ij}, note that the quantity

$$f_{\check{\xi}_{ij}}(\sigma) + A_{\Theta_{ij}}(\sigma) \tag{4.3.7}$$

is independent of the choice of σ. We therefore obtain functions $\zeta_{ij} : U_{ij} \to i\mathbb{R}$ given locally by the expression (4.3.7). Using the connective lift, we obtain connections $\nu_i = \Theta_{\check{\xi}}(\mu_i)$ on the torsors $F_i := F_{\hat{\xi}}(Q_i)$. Define 1-forms $a_i^{\hat{\xi}} = A_{\nu_i}(r_i)$. The proof of the following proposition, which we omit, is similar to the proof of Proposition 4.2.6.

Proposition 4.3.8. *On each overlap U_{ij}, we have*

$$a_i^{\hat{\xi}} - a_j^{\hat{\xi}} = d\zeta_{ij}^{\hat{\xi}} + \iota_\xi \omega_{ij}. \tag{4.3.9}$$

Furthermore, if $(F_{\hat{\xi}}, \Theta_{\check{\xi}})$ preserves the curving K, then

$$da_i^{\hat{\xi}} + \pounds_\xi B_i = 0 \tag{4.3.10}$$

on each U_i.

Next, suppose we have a pair of connective lifts $\check{\xi}, \check{\xi}'$ of ξ to $\mathcal{G}$, and an equivalence of lifts $\Psi : \check{\xi} \xrightarrow{\cong} \check{\xi}'$. Then it is not hard to show that we obtain functions $u_i : U_i \to i\mathbb{R}$ such that on overlaps

$$\hat{\xi}'_{ij} = \hat{\xi}_{ij} + u_j - u_i, \tag{4.3.11}$$

and which also satisfy

$$du_i = a_i^{\check{\xi}'} - a_i^{\check{\xi}}. \tag{4.3.12}$$

These considerations motivate the following definition.

Definition 4.3.13. *Given an open cover $\{U_i\}$ of a manifold M, suppose we have a collection of $\mathbb{T}_{U_{ij}}$ torsors with connection $\{(E_{ij}, \Theta_{ij})\}$ and coherent* *structure maps* $\lambda_{ijk} : E_{ij} \otimes E_{jk} \xrightarrow{\cong} E_{ik}$ *on triple overlaps compatible with the connections. We define a groupoid* $\mathcal{L}_{(E_{ij},\Theta_{ij})}$ *(the structure maps* λ_{ijk} *are suppressed in the notation) as follows. An object of* $\mathcal{L}_{E_{ij}}(\xi)$ *is a triple* $(\xi, \{\hat{\xi}_{ij}\}, \{a_i^{\hat{\xi}}\})$*, where* ξ *is a vector field on* M*, and the* $\hat{\xi}_{ij}'s$ *are lifts of* ξ *to the* $E_{ij}'s$ *which are compatible with the structure maps* $\{\lambda_{ijk}\}$ *as discussed above, and the 1-forms* $\{a_i^{\hat{\xi}}\}$ *satisfy (4.3.9). An equivalence* $(\xi, \{\hat{\xi}_{ij}\}) \xrightarrow{\cong} (\xi, \{\hat{\xi}_{ij}'\})$ *is a collection of functions* $\{u_i : U_i \to i\mathbb{R}\}$ *satisfying conditions (4.3.11) and (4.3.12).*

We also define the subcategory $\mathcal{L}_{(E_{ij},\Theta_{ij},B_i)}$*, which consists of those lifts satisfying condition (4.3.10).*

If $(\{E_{ij}\}, \{\lambda_{ijk}\})$ come from a gerbe as discussed above, then it is possible to construct an equivalence of categories $\mathcal{L}_{(E_{ij},\Theta_{ij})} \xrightarrow{\cong} \mathcal{L}_{(\mathcal{G},\mathcal{A})}$ by glueing. Rather than constructing such an equivalence, however, we will instead explain the relationship between the category of lifts $\mathcal{L}_{(E_{ij},\Theta_{ij})}$ and the category of connective lifts in the Cech picture. Note that, if $\{\tilde{U}_i\}$ is a refinement of the cover $\{U_i\}$, then by restriction we obtain torsors $\tilde{E}_{ij}$ over $\tilde{U}_{ij}$ and structure maps $\tilde{\lambda}_{ijk}$. Since every open cover of M may be refined to a good open cover, without loss of generality we assume that each E_{ij} is trivializable, i.e. admits a global section s_{ij}. We are then in the situation discussed in §4.2, and in particular obtain Cech cocycle $\{g_{ijk}\}$ for the gerbe specified by the equation

$$\lambda_{ijk}(s_{ij} \otimes s_{jk}) = s_{ik} \cdot g_{ijk}. \qquad (4.3.14)$$

Furthermore, if we define 1-forms $A_{ij} = A_{\Theta_{ij}}(s_{ij})$, and 2-forms B_i as above, then these satisfy the conditions (3.6.2) and (3.6.5), respectively.

We now construct an equivalence of categories $\rho : \mathcal{L}_{(E_{ij},\Theta_{ij})} \xrightarrow{\cong} \mathcal{L}_{(g_{ijk},A_{ij})}$, which will actually be a bijection on the level of objects and morphisms. Given $\hat{\xi} = (\xi, \{\hat{\xi}_{ij}\}, \{a_i\}) \in \mathcal{L}_{(E_{ij},\Theta_{ij})}$, define $i\mathbb{R}$-valued functions $f_{ij} = -f_{\hat{\xi}_{ij}}(s_{ij})$. Then it is straightforward to check that $(\xi, \{f_{ij}\}, \{a_i\})$ defines an element of $\mathcal{L}_{(g_{ijk},A_{ij})}(\xi)$, which we denote by $\rho(\hat{\xi})$; actually, this gives a bijection between the objects of the two categories. Furthermore, $\rho(\hat{\xi})$ satisfies (4.2.5) if and only if $\hat{\xi}$ satisfies (4.3.10). Furthermore, if $\tau = \{u_i\}$ is an equivalence of lifts $\hat{\xi}, \hat{\xi}' \in \mathcal{L}_{(E_{ij},\Theta_{ij})}$, then the same collection of functions $\{u_i\}$ defines an equivalence $\rho(\hat{\xi}) \xrightarrow{\cong} \rho(\hat{\xi}')$, which we denote by $\rho(\tau)$. Summing up, we have the following.

Theorem 4.3.15. *$\rho : \mathcal{L}_{(E_{ij},\Theta_{ij})} \to \mathcal{L}_{(g_{ijk},A_{ij})}$ is an equivalence of categories.*

Using this equivalence, we may give $\mathcal{L}_{(E_{ij},\Theta_{ij})}$ the structure of a 2-term L_∞-algebra. We spell out the structure explicitly in the following proposition, whose proof is omitted.

Proposition 4.3.16. *There is a 2-term L_∞-algebra with $V_0 = Obj(\mathcal{L}_{(E_{ij},\Theta_{ij})})$ and $V_1 = \{\{u_i : U_i \to i\mathbb{R}\}\}$, defined as follows:*

1. *The bracket of two elements $\check{\xi} = (\xi, \{\check{\xi}_{ij}\}, \{a_i^{\check{\xi}}\})$ and $\check{\eta} = (\eta, \{\check{\eta}_{ij}\}, \{a_i^{\check{\eta}}\})$ in V_0 is given by*

$$[\check{\xi}, \check{\eta}] = ([\xi, \eta], \{[\check{\xi}_{ij}, \check{\eta}_{ij}\}, \{\pounds_\xi a_i^{\check{\eta}} - \pounds_\eta a_i^{\check{\xi}}\}). \tag{4.3.17}$$

2. *Given* $\check{\xi} = (\xi, \{\check{\xi}_{ij}\}, \{a_i^{\check{\xi}}\}) \in V_0$ *and* $u = \{u_i\} \in V_1$*, their bracket is*

$$[\check{\xi}, u] = -[u, \check{\xi}] = \{\mathcal{L}_\xi u_i\}. \tag{4.3.18}$$

   The bracket of two elements in V_1 *is (necessarily) zero.*

3. $d : V_1 \to V_0$ *maps* $\{u_i\} \in V_1$ *to the lift of the zero vector field* $(0, \{u_j - u_i\}, \{du_i\})$*.*

4. *The Jacobiator is zero.*

With respect to this structure, the equivalence ρ *in Theorem 4.3.15 gives an isomorphism of 2-term* L_∞*-algebras. We also obtain a sub 2-term* L_∞ *algebra which in degree zeros consists of the objects of* $\mathcal{L}_{(E_{ij},\Theta_{ij},B_i)}$*.*

4.4 Example: Gerbes over compact Lie groups

4.4.1 The construction

Let G be a compact, simple, connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Recall that in this case $H^0(G;\mathbb{Z}) = H^1(G;\mathbb{Z}) = H^2(G;\mathbb{Z}) = 0$, whereas $H^3(M;\mathbb{Z}) \cong \mathbb{Z}$. There is a unique G-invariant inner product on $\mathfrak{g}$ (which is necessarily a multiple of the Killing form) such that, for each long root $\alpha \in \mathfrak{g}^*$ we have $\langle \alpha, \alpha \rangle = 2$. With the aid of this inner product, we define a three form $\eta \in i\Omega^3$ by the formula

$$\eta = \frac{1}{12}\langle \theta \wedge [\theta \wedge \theta] \rangle$$

It is well known (see for example [Br1]) that η is bi-invariant, closed, and the deRham cohomology class $[\eta]$ is the image of a generator of $H^3(G;\mathbb{Z})$ under the inclusion $H^3(G;\mathbb{Z}) \hookrightarrow H^3(G;\mathbb{R})$. In particular, η has integral periods, i.e. the integral of η over any 3-dimensional homology class is an integer. Below we will instead work with the imaginary 3-form $H = 2\pi i\eta$.

For each such Lie group G, there is a canonical gerbe with connective structure and curving over G whose curvature form is given by H; moreover, this gerbe is naturally equivariant for the conjugation action of G on itself. There are a number of different (but equivalent) constructions of this gerbe; see for example [Br1] and [Me]. The construction in [Me] is particularly concrete, involving open covers and line bundles. For example, identifying S^3 with the group $SU(2)$, Meinrenken's procedure gives a more systematic construction of the gerbe described in Example 3.6.17.

Let us briefly sketch Meinrenken's construction in the particular case $G = SU(2)$. Recall that the center of $SU(2)$ consists of two points, the identity matrix $e = \mathrm{diag}(1,1)$ and $c = \mathrm{diag}(-1,-1)$. We define open sets $U_0 = SU(2)\backslash\{c\}$ and $U_1 = SU(2)\backslash\{e\}$. Note that the conjugation action of $SU(2)$ on itself restricts to an action on U_0, U_1, and their intersection U_{01}. Furthermore, there are equivariant deformation retractions of U_0 onto $\{e\}$ and U_1 onto $\{c\}$.

Let h_i for $i = 0,1$ be the deRham homotopy operators associated to these deformation retractions. Applying these to H, we obtain 2-forms $B_i \in i\Omega^2(U_i)$ such that $dB_i = H|_{U_i}$. Since H is G-invariant and the deformation retractions are G-equivariant, each B_i is also invariant. On U_{01} we have $d(B_1 - B_0) = 0$; furthermore, by Stoke's theorem for any parameterized 2-dimensional submanifold Σ of U_{01} whose fundamental class generates $H_2(U_{01};\mathbb{Z}) \cong \mathbb{Z}$, we have

$$\int_\Sigma (B_1 - B_0) = \pm 2\pi i, \tag{4.4.1}$$

depending on the orientation of Σ. Meinrenken constructs an equivariant principal $\mathbb{T}$-bundle with invariant connection Θ (actually Hermitian line bundle with unitary connection) whose curvature form ω is equal to $B_1 - B_0$

By setting $(E_{01}, \Theta_{01}) = (E, \Theta)$, $(E_{10}, \Theta_{10}) = (E^*, -\Theta)$, setting (E_{00}, Θ_{00}) and (E_{11}, Θ_{11}) trivial, we obtain the set-up described in §4.3 (with trivial structure maps λ_{ijk}). For each $\xi \in \mathfrak{g}$, let ξ_C be the vector field on G induced by the conjugation action. For each such vector field, we define an element $\check{\xi}_C = (\xi_C, \{(\check{\xi}_C)_{ij}\}, \{a_i^{\check{\xi}_C}\})$ of $\mathcal{L}_{(E_{ij},\Theta_{ij})}$ as follows. For $(i,j) = (0,0)$ or $(1,1)$, $(\check{\xi}_C)_{ij}$ is the trivial lift of ξ_C. For $(i,j) = (0,1)$ we take $(\check{\xi}_C)_{ij}$ to be the lift corresponding to the lift of the conjugation action to E, and for $(i,j) = (1,0)$ we take $(\check{\xi}_C)_{ij}$ to be the corresponding lift of ξ_C to E^*. Also, we set all the 1-forms $a_i^{\check{\xi}_C}$ equal to zero.

Proposition 4.4.2. *For each $\xi \in \mathfrak{g}$, the element $\check{\xi}_C$ described above defines an element of $\mathcal{L}_{(E_{ij},\Theta_{ij},B_i)}$.*

Proof. The fact that the lifts $(\check{\xi}_C)_{ij}$ are compatible with the structure maps follows trivially from the definitions. To show that (4.3.10) holds, we must show that for each (i,j) we have

$$d\zeta_{ij} + \iota(\xi_C)\omega_{ij} = 0. \tag{4.4.3}$$

It is easy to see that the pullback of ζ_{ij} to E_{ij} is the (circle-invariant) function $\iota_{(\check{\xi}_C)_{ij}}\Theta_{ij}$, whereas the pullback of $\iota(\xi_C)\omega_{ij}$ to E_{ij} is equal to $\iota_{(\check{\xi}_C)_{ij}}d\Theta_{ij}$. Therefore, the pullback of the expression (4.4.3) to E_{ij} is

$$d\iota_{(\check{\xi}_C)_{ij}}\Theta_{ij} + \iota_{(\check{\xi}_C)_{ij}}d\Theta_{ij} = \pounds_{(\check{\xi}_C)_{ij}}\Theta_{ij} \tag{4.4.4}$$

which vanishes by construction.

To see that $\check{\xi}_C$ preserves the curving (i.e. is an element of $\mathcal{L}_{(E_{ij},\Theta_{ij},B_i)}$ and not just of $\mathcal{L}_{(E_{ij},\Theta_{ij})}$), note that by construction $\pounds_{\check{\xi}}B_i = 0 = da_i^{\check{\xi}_C}$ for each i. □

If we denote the lift in the above proposition by $\phi(\xi)$, then we easily check that $[\phi(\xi), \phi(\eta)] = \phi([\xi, \eta])$ for each $\xi, \eta \in \mathfrak{g}$. Regarding $\mathfrak{g}$ as a 2-term L_∞ algebra with $V_1 = 0$, we thus obtain an inclusion of Lie 2-algebras $\mathfrak{g} \hookrightarrow \mathcal{L}_{(E_{ij},\Theta_{ij},B_i)}$. In this sense we have a lift of the infinitesimal conjugation action of G on itself to the gerbe.

4.4.2 The string Lie 2-algebra and infinitesimal symmetries

Instead of the conjugation action, we now consider the action of G on itself by left translation. Namely, regarding the Lie algebra $\mathfrak{g}$ as the space of left-invariant vector fields, we will investigate the structure of the Lie 2-algebra $\mathcal{L}_{(E_{ij},\Theta_{ij},B_I)}(\mathfrak{g})$ consisting of connective lifts covering elements of $\mathfrak{g}$ and preserving K. We will show this Lie 2-algebra is quasi-isomorphic to the *string* Lie 2-algebra; in particular, the extension of Lie 2-algebras $\mathcal{L}_{(E_{ij},\Theta_{ij},B_i)}(\mathfrak{g}) \to \mathfrak{g}$

is non-trivial and thus has no splitting, unlike in the case of the conjugation action considered above.

Let G be a compact, connected, simply connected, simple Lie group. Let $\langle \cdot, \cdot \rangle$ be the normalized Killing form described above, and for each positive real number k define $\nu_k \in \Lambda^3 \mathfrak{g}^*$ by

$$\nu_k(\xi, \eta, \tau) = \frac{k}{2} \langle \xi, [\eta, \tau] \rangle$$

for each $\xi, \eta, \tau \in \mathfrak{g}$. Regarding $\mathfrak{g}$ as the space of left-invariant vector fields on G, we have that $kH(\xi, \eta, \tau)$ is the constant function on G with value $\nu_k(\xi, \eta, \tau)$. The following definition is a slight variation on that given in [BR].

Definition 4.4.5. *Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra over $\mathbb{R}$, with $\langle \cdot, \cdot \rangle$ and ν_k as above. Then the* string Lie 2-algebra *(at level k) has*

1. *$V_0 = \mathfrak{g}$,*
2. *$V_1 = i\mathbb{R}$,*
3. *$d : V_1 \to V_0$ is zero,*
4. *The bracket of elements $\xi, \eta \in V_0$ is given by the Lie bracket on $\mathfrak{g}$. For $\xi \in V_0 = \mathfrak{g}$, $a, b \in V_1 = \mathbb{R}$, we have $[\xi, a] = [a, b] = 0$.*
5. *The Jacobiator is given by $2\pi i \nu_k$.*

Remark 4.4.6. It is proven in [BL] that $\mathfrak{g}_k$ and $\mathfrak{g}_{k'}$ are quasi-isomorphic if and only if $k = k'$. In particular, for $k \neq 0$, the extension $\mathfrak{g}_k \to \mathfrak{g}$ is non-trivial.

For our present purposes, we do not need a construction of the gerbe over G as explicit as that given above in the case $SU(2)$. All we need is the existence of a gerbe with connective structure and curving over G whose curvature is the invariant 3-form H. Furthermore, taking tensor powers, for each $k \in \mathbb{Z}$ there exists a gerbe with connective structure and curving whose curvature is equal to kH. For notational simplicity we set $k = 1$; the same arguments given below carry over mutatis mutandis to the case where k is an arbitrary integer. We assume that, as in the example above, we have chosen local trivializations of $(\mathcal{G}, \mathcal{A})$, giving a collection $\{(E_{ij}, \Theta_{ij}, B_i)\}$ and compatible structure maps $\{\lambda_{ijk}\}$ as described in §4.3. Denote the 2-term L_∞-algebra corresponding to the subcategory of $\mathcal{L}_{(E_{ij},\Theta_{ij},B_i)}$ covering $\mathfrak{g}$ (as described in Proposition 4.3.16) by (V_0', V_1'). For k an integer, we now construct a quasi-isomorphism of Lie 2-algebras ϕ from the string Lie 2-algebra $\mathfrak{g}_1 = (V_0, V_1)$ to the Lie 2-algebra (V_0', V_1') corresponding to $\mathcal{L}_{(g_{ijk},\mathcal{A}_{ij},B_i)}(\mathfrak{g})$ The construction is inspired by the paper [BR], where the string Lie 2-algebra is related to the space of *Hamiltonian 1-forms* on G, viewed as a 2-plectic manifold.

To start the construction, for each $\xi \in \mathfrak{g}$ we define a left-invariant 1-form $\alpha^\xi = -\pi i \langle \xi, \cdot \rangle$. Then a simple calculation using the invariant definition of the exterior derivative shows that for each ξ we have

$$d\alpha^\xi = \iota(\xi)H. \tag{4.4.7}$$

Also, we have the relation

$$\pounds_\xi \alpha^\eta = \alpha^{[\xi,\eta]} \tag{4.4.8}$$

for every $\xi, \eta \in \mathfrak{g}$. We first construct a degree 0 map $\phi : V_\bullet \to V'_\bullet$ by

$$\phi(\xi) = (\xi, \{\hat{\xi}^h_{ij}\}, \{-(\iota_\xi B_i + \alpha^\xi|_{U_i})\}) \tag{4.4.9}$$

in degree 0; here $\hat{\xi}^h_{ij}$ is the horizontal lift of ξ to E_{ij} obtained using the connection Θ_{ij}. In degree 1 ϕ maps $c \in \mathbb{R}$ to the collection of constant functions $\{u_i = c\}$. ϕ is easily seen to be a chain map. Note that for the horizontal lift $\hat{\xi}^h_{ij}$ the function $\zeta_{ij} = 0$, so that $d\zeta_{ij} + \iota_\xi \omega_{ij} = \iota_\xi \omega_{ij} = \iota_\xi(B_j - B_i)$. On the other hand, since by definition $a^{\check{\xi}}_i = -\iota_\xi B_i + \alpha^\xi|_{U_i}$ we have $a^{\check{\xi}}_i - a^{\check{\xi}}_j = \iota_\xi(B_j - B_i)$, and therefore the relation (4.3.9) is satisfied, so that we do indeed obtain an element of $\mathcal{L}_{(E_{ij},\Theta_{ij})}$. Furthermore, by construction this element in fact lies in $\mathcal{L}_{(E_{ij}\Theta_{ij},B_i)}$.

Next, we define $\phi_2 : V_0 \to V'_1$ by $\phi_2(\xi, \eta) = \{\iota_\xi \iota_\eta B_i\}$.

Theorem 4.4.10. *$(\phi, \phi_2) : \mathfrak{g}_1 \to (V'_0, V'_1)$ is a quasi-isomorphism of 2-term L_∞ algebras.*

Proof. First, to show that (ϕ, ϕ_2) defines a morphism of 2-term L_∞-algebras, we must check that ϕ_2 satisfies the conditions (i) and (ii) in Definition 3.6.32. On the one hand we have $\phi([\xi, \eta]) - [\phi(\xi), \phi(\eta)]=$

$$(0, \widehat{[\xi,\eta]}^h_{ij} - [\hat{\xi}^h_{ij}, \hat{\eta}^h_{ij}], -\iota_{[\xi,\eta]} B_i - \alpha^{[\xi,\eta]} + \pounds_\xi(\iota_\eta B_i + \alpha^\eta) - \pounds_\eta(\iota_\xi B_i + \alpha^\xi)). \tag{4.4.11}$$

We have

$$\widehat{[\xi,\eta]}^h_{ij} - [\hat{\xi}^h_{ij}, \hat{\eta}^h_{ij}] = \iota_\eta \iota_\xi \omega_{ij} = \iota_\xi \iota_\eta B_i - \iota_\xi \iota_\eta B_j. \tag{4.4.12}$$

On the other hand, using the Cartan formula for the Lie derivative together with the relation (4.4.8), a short calculation shows that

$$-\iota_{[\xi,\eta]}B_i - \alpha^{[\xi,\eta]} + \pounds_\xi(\iota_\eta B_i + \alpha^\eta) - \pounds_\eta(\iota_\xi B_i + \alpha^\xi) = d(\iota_\xi \iota_\eta B_i). \qquad (4.4.13)$$

Thus, we have

$$\phi([\xi,\eta]) - [\phi(\xi), \phi(\eta)] = d\phi_2(\xi,\eta). \qquad (4.4.14)$$

Again using the invariant definition of the exterior derivative, the verification of condition (ii) is given by a quick calculation. Thus (ϕ, ϕ_2) does define a morphism of 2-term L_∞-algebras.

To complete the proof we must show that (ϕ, ϕ_2) is a quasi-isomorphism, i.e. induces an isomorphism on the level of cohomology. It follows from Theorem 4.2.19 that for any fixed element $\check{\xi} \in \mathcal{L}_{(E_{ij},\Theta_{ij})}$, any other element $\check{\xi}' \in \mathcal{L}_{(E_{ij},\Theta_{ij})}$ is cohomologous to one of the form $\check{\xi} + \alpha$ for some 1-form $\alpha \in i\Omega^1(G)$. Now if both $\check{\xi}$ and $\check{\xi}'$ preserve the curving, then α must be closed. Since by assumption G is simply connected, we can find a function $f : G \to i\mathbb{R}$ such that $\alpha = df$. Defining $u_i = f|_{U_i}$ for each i and $u = \{u_i\}$, we see that $\check{\xi} + \alpha = \check{\xi} + du$. Therefore (ϕ, ϕ_0) induces an isomorphism on cohomology in degree 0. On the other hand, given $u = \{u_i\} \in V_1'$ suppose that $du = 0$. Then there exists a locally constant function f on G such that $u_i = f|_{U_i}$. Since by assumption G is connected, f is globally constant, so that the kernel of d consists of the constant functions, which shows that ϕ induces an isomorphism on cohomology in degree 1 also. □

4.5 Local flows and infinitesimal connective symmetries

Given a local flow Φ and vector field ξ on M as described in §3.4, we now wish to generalize Theorem 3.4.32 establishing a local equivalence between the category of connective lifts of ξ to $\mathcal{G}$ and the category of 1-parameter connective lifts of Φ to $\mathcal{G}$. Since every gerbe with connective structure is locally isomorphic to the trivial gerbe with the trivial connective structure, it is enough to consider this case. We work with the simplicial manifold $U^\bullet$ introduced in §3.4. As discussed in Appendix A, we work with sheaves of *relative 1-forms* over the manifolds U^i, with respect to the projections $U^i \to \mathbb{R}^i$. These can be described as follows: for each $i \geq 0$ and each $\vec{t} \in \mathbb{R}^i$, let $U_{\vec{t}} \subset U^i$ be the submanifold $U \times \{\vec{t}\}$. Then $\underline{i\Omega}^1_{U^i,rel}$ is the quotient of the sheaf $\underline{i\Omega}^1_{U^i}$ by the subsheaf of 1-forms whose restriction to $U_{\vec{t}}$ vanishes for each $\vec{t} \in \mathbb{R}^i$.

Terminology 4.5.1. If α is a 1-form whose restriction to each $U_{\vec{t}}$ is zero, we say that α *vanishes in the M-direction.*

Given a principal $\mathbb{T}$-bundle $E \to U^i$, we have the notion of a *relative connection* on E, which we view as a sheaf homomorphism from $\underline{E} \to \underline{i\Omega}^1_{U^i,rel}$ intertwining the homomorphism

$$\underline{\mathbb{T}}_{U^i} \xrightarrow{d\log} \underline{i\Omega}^1_{U^i} \longrightarrow \underline{i\Omega}^1_{U^i,rel}, \tag{4.5.2}$$

where the second map is the quotient. The set $\mathcal{A}_{rel}(E)$ of all such such relative connections is a torsor for the group of relative 1-forms on U^i. Furthermore we have a quotient map $\mathcal{A}(E) \to \mathcal{A}_{rel}(E)$ intertwining the quotient map $\underline{i\Omega}^1_{U^i} \to \underline{i\Omega}^1_{U^i,rel}$.

Definition 4.5.3. *Given $\alpha \in i\Omega^1_{U^i,rel}$, we define*

$$\delta\alpha = p_0^*\alpha - p_1^*\alpha + \cdots (-1)^{i+1}p_{i+1}^*\alpha \in: i\Omega^1_{U^{i+1},rel} \tag{4.5.4}$$

Remark 4.5.5. Given $\alpha \in i\Omega^1_{M^i}$ vanishing in the M-direction, a simple calculation shows that

$$p_0^*\alpha - p_1^*\alpha + \cdots + (-1)^{i+1}p_{i+1}^*\alpha \tag{4.5.6}$$

also vanishes in the M-direction. Therefore (4.5.4) is well-defined.

Remark 4.5.7. Given a $\underline{\mathbb{T}}_{U^i}$-torsor S, recall from §3.4 that we obtain a $\underline{\mathbb{T}}_{U^{i+1}}$-torsor δS. If Θ is a relative connection on S, then we obtain in a natural way a relative connection $\delta\Theta$ on δS.

Remark 4.5.8. Given a 1-form α on U^1 vanishing in the M direction, note that the Lie derivative

$$\pounds_{\frac{d}{dt}}\alpha \tag{4.5.9}$$

also vanishes in the M direction. Therefore we obtain a well-defined Lie derivative

$$\pounds_{\frac{d}{dt}} : i\Omega^1_{U^1,rel} \to i\Omega^1_{U^1,rel}. \tag{4.5.10}$$

Furthermore, given a $\underline{\mathbb{T}}_{U^1}$-torsor E with relative connection Θ, we can generalize the construction in Example 4.1.10 to construct a relative connection $\pounds_{\frac{d}{dt}}(\Theta)$ on the $\underline{i\mathbb{R}}_{U^0}$-torsor $\iota_{\frac{d}{dt}} d\log[E]$ according to the formula

$$A_{\pounds_{\frac{d}{dt}}(\Theta)}(\iota_\xi d\log[\sigma] + f) = \pounds_{\frac{d}{dt}} A_\Theta(\sigma) + df. \tag{4.5.11}$$

Definition 4.5.12. *A* local connective lift *of* Φ *to* $B(\underline{\mathbb{T}}_U)$ *consists of a quadruple* (S, e, σ, Θ)*, where* (S, e, σ) *are as in Definition 3.4.17, and* Θ *is a relative connection on* S *satisfying*

(i) $A_{s_0^*\Theta}(e) = 0$ *and*

(ii) $A_{\delta\Theta}(\sigma) = 0$.

Given another local lift $(S', e', \sigma, \Theta')$ *and isomorphism of lifts from* (S, e, σ, Θ) *to* $(S', e', \sigma', \Theta')$ *is an equivalence* $\Psi : (S, e, \sigma) \to (S', e', \sigma')$ *as in Definition 3.4.17 satisfying* $\Psi^*\Theta' = \Theta$*. We denote the resulting category by* $\check{\mathcal{L}}(\Phi)$*.*

We next extend the differentiation functor D defined in (3.4.29) to a functor $\check{D}$ that takes connections into account. Let $\check{\mathcal{L}}(\xi)$ denote the category of connective lifts of ξ to the trivial gerbe with trivial connective structure over U^0. It follows from (4.2.26) that every connective lift $\check{\xi} = (F_{\check{\xi}}, \Theta_{\check{\xi}}) \in \check{\mathcal{L}}(\xi)$ of ξ is determined by the $i\underline{\mathbb{R}}_{U^0}$-torsor $F_{\check{\xi}}(\underline{\mathbb{T}}_{U^0})$ with the connection

$$\Theta_{\check{\xi}}(\Theta_0), \tag{4.5.13}$$

where Θ_0 is the trivial flat connection on $\underline{\mathbb{T}}_{U^0}$. Put differently, there is an equivalence of categories from $\check{\mathcal{L}}(\xi)$ to the category $\check{\mathbf{Tor}}_{i\underline{\mathbb{R}}_{U^0}}$ whose objects are $i\underline{\mathbb{R}}_{U^0}$-torsors with connection. For simplicity, we will therefore take $\check{\mathbf{Tor}}_{i\underline{\mathbb{R}}_{U^0}}$ to be the codomain of the differentiation functor $\check{D}$.

Given $(S, e, \sigma, \Theta) \in \mathcal{L}(\Phi)$, recall from Remark 4.5.8 that we obtain a relative connection $\pounds_{\frac{d}{dt}}(\Theta)$ on $\iota_{\frac{d}{dt}} d\log[S]$. Furthermore, we can restrict to obtain a connection $D(\Theta) = i^* \pounds_{\frac{d}{dt}}(\Theta)$ on the restriction of $\iota_{\frac{d}{dt}} d\log[S]$ to U^0. One easily checks that the assignment $\Theta \mapsto D(\Theta)$ is functorial.

Definition 4.5.14. *The functor*

$$\check{D} : \check{\mathcal{L}}(\Phi) \to \check{\boldsymbol{Tor}}_{\underline{i\mathbb{R}}_{U^0}} \tag{4.5.15}$$

assigns to each element (S, e, σ, Θ) *the* $\underline{i\mathbb{R}}_{U^0}$*-torsor with connection*

$$(D(S), D(\Theta)), \tag{4.5.16}$$

where $D(S)$ *is defined in (3.4.29).*

Theorem 4.5.17. $\check{D}$ *is an equivalence of categories.*

Proof. Let $\hat{\Phi}_0 \in \mathcal{L}(\Phi)$ be the trivial lift of Φ, and let $\check{\mathcal{L}}_0(\Phi)$ denote the subcategory of $\check{\mathcal{L}}(\Phi)$ whose underlying non-connective lift is $\hat{\Phi}_0$. It follows from Proposition 3.4.18 that the inclusion of $\check{\mathcal{L}}_0(\Phi)$ into $\check{\mathcal{L}}(\Phi)$ is an equivalence of categories. Similarly, let $\check{\mathcal{L}}_0(\xi)$ denote the subcategory of $\check{\mathbf{Tor}}_{\underline{i\mathbb{R}}_{U^0}}$ with underlying torsor $E_0 = D(\hat{\Phi}_0)$; thus an element of $\check{\mathcal{L}}_0(\xi)$ is a connection Θ on E_0 and a morphism from $\Theta \to \Theta'$ is an automorphism of E_0 taking Θ to Θ'. Since every $\underline{i\mathbb{R}}_{U^0}$-torsor is isomorphic to E_0, it follows that the inclusion $\check{\mathcal{L}}_0(\xi) \to \check{\mathbf{Tor}}_{\underline{i\mathbb{R}}_{U^0}}$ is an equivalence of categories. Therefore, it is sufficient to check that the restriction

$$\check{D} : \check{\mathcal{L}}_0(\Phi) \to \check{\mathcal{L}}_0(\xi) \tag{4.5.18}$$

is an equivalence of categories.

Note that we may identify the set of objects of $\check{\mathcal{L}}_0(\Phi)$ with the set of relative 1-forms A on U^1 satisfying conditions (i) and (ii) in Definition 4.5.12. Let us call this set $Z_{i\Omega^1}$. We may view $A \in Z_{i\Omega^1}$ as a section of the pulled-back cotangent bundle of M

$$(x, \vec{t}) \mapsto A_{(x,\vec{t})} := A_x(\vec{t}) \in iT_x^*M. \tag{4.5.19}$$

The conditions (i) and (ii) can then be written

$$\varphi_t^* A_{\varphi_t(x)}(t') - A_x(t+t') + A_x(t) = 0 \tag{4.5.20}$$

for each $x \in V$ and

$$A(0) = 0. \tag{4.5.21}$$

We have a linear differentiation map

$$\Delta_{i\Omega^1} : Z_{i\Omega^1} \to i\Omega^1(U^0) \tag{4.5.22}$$

mapping $A \in Z_{i\Omega^1}$ to

$$\frac{d}{ds}|_{s=0} A_x(s) \tag{4.5.23}$$

Lemma 4.5.24. *$\Delta_{i\Omega^1}$ is an isomorphism of vector spaces.*

Proof. Denote $\Delta_{i\Omega^1} A$ by $\alpha \in i\Omega^1(U^0)$. Equation (4.5.20) implies that

$$\frac{A_x(t+t') - A_x(t)}{t'} = \varphi_t^*(\frac{A(t')}{t'}), \tag{4.5.25}$$

and therefore

$$\frac{d}{ds}|_{s=t} A_x(s) = \varphi_t^* \frac{d}{ds}|_{s=0} A_{\varphi_t(x)}(s) = \varphi_t^* \alpha_{\varphi_t}. \tag{4.5.26}$$

By the fundamental theorem of calculus and condition (4.5.21) we therefore have

$$A_x(t) = \int_0^t \varphi_s^* \alpha_{\varphi_s(x)} ds. \tag{4.5.27}$$

Therefore A is completely determined by α and $\Delta_{i\Omega^1}$ is injective. Conversely, given an arbitrary $\alpha \in i\Omega^1(U^0)$, it is easly verified that the relative 1-form on U^1 defined by equation (4.5.27) is an element of Z, i.e. it satisfies conditions (4.5.20) and (4.5.21).

□

Similarly, we my identify the set of objects of $\check{\mathcal{L}}_0(\xi)$ with the set $i\Omega^1_{U^0}$ of 1-forms on U^0. Moreover, we have a commutative diagram

$$\begin{array}{ccc} Z_{i\Omega^1} & \xrightarrow{\cong} & \mathrm{Ob}(\check{\mathcal{L}}_0(\Phi)) \\ \downarrow{\scriptstyle \Delta_{i\Omega^1}} & & \downarrow{\scriptstyle \check{D}} \\ i\Omega^1_{U^0} & \xrightarrow[\cong]{} & \mathrm{Ob}(\check{\mathcal{L}}_0(\xi)). \end{array} \tag{4.5.28}$$

Thus, Lemma 4.5.24 implies that D is a bijection on the level of objects. Since both categories are groupoids, it only remains to show that for every object $x \in \check{\mathcal{L}}_0(\Phi)$ that D gives an isomorphism from $\mathrm{Aut}(x) \xrightarrow{\cong} \mathrm{Aut}(\check{D}(x))$.

Recall the set $Z_{\mathbb{T}}$ of functions $U^1 \to \mathbb{T}$ defined in the proof of Theorem 3.4.32, as well as the map $\Delta : Z_{\mathbb{T}} \to iC^\infty(U^0)$. One easily checks that an element $g \in Z_{\mathbb{T}}$ defines an automorphism of x if and only if the relative 1-form on U^1 determined by $d \log g$ is equal to zero; call this subset $Z_0 \subset Z_{\mathbb{T}}$. On the other hand, a function $f : U^0 \to i\mathbb{R}$ defines an automorphism of $D(x)$ if and only if f is constant. Equation (3.4.44) then implies that $\Delta_{\mathbb{T}} : Z \to iC^\infty(U^0)$ restricts to a bijection between Z_0 and the constant functions on U^0. $\square$

4.6 The Courant Algebroid Associated to $\check{\mathcal{L}}_{\mathcal{G}}$

In [Hi], starting with Cech data for a DD gerbe with connective structure over a manifold M, Hitchin constructs an extension of vector bundles

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0\,, \tag{4.6.1}$$

which he calls the "generalized tangent bundle".[1] Moreover, the vector bundle E is equipped with a symmetric, non-degenerate pairing and a skew-symmetric bracket, giving E the structure of a *Courant Algebroid* [Gu]. In this section we give an alternative construction of this algebroid in terms of the category of infinitesimal connective symmetries of $(\mathcal{G}, \mathcal{A})$.[2] In particular, our construction does not depend on a choice of local trivializations for $(\mathcal{G}, \mathcal{A})$; given such a choice, however, we explain how to compare our construction with that of Hitchin. It follows from the results of [RW] than any Courant algebroid E gives rise to a 2-term L_∞-algebra L_E. We recall this construction and construct a quasi-isomorphism of L_∞-algebras $L_E \xrightarrow{\cong} \mathcal{L}_{(g_{ijk}, A_{ij})}$. Since the latter is a *strict* 2-term L_∞-algebra (i.e. $J = 0$), this isomorphisms gives a useful perspective on L_E.

[1] For an alternative construction due originally to Severa in terms of "conducting bundles", see [BCh].

[2] More precisely, for simplicity we give a construction of the C^∞_M-module of global sections of E. Since all our constructions in this paper are local, however, we could easily generalize to construct the entire sheaf of sections of E.

4.6.1 The Courant Algebroid associated to $(\{g_{ijk}\}, \{A_{ij}\})$.

On any smooth manifold M, the Courant bracket is defined on sections of $TM \oplus T^*M$ by the formula [Gu]

$$[\xi + a, \eta + b]_c = [\xi, \eta] + \pounds_\xi b - \pounds_\eta a - \frac{1}{2} d\iota_\xi b + \frac{1}{2} d\iota_\eta a. \tag{4.6.2}$$

Here ξ and η are vector fields, a and b are 1-forms, and $[\xi, \eta]$ is the regular Lie bracket of vector fields. This bracket has several interesting features. First, although it is skew symmetric, it does not satisfy the Jacobi identity. However, the defect in the Jacobi identity is easily expressed in terms of the non-degenerate pairing on $C^\infty(TM \oplus T^*M)$ given by

$$\langle \xi + a, \eta + b \rangle = \frac{1}{2}(\iota_\xi b + \iota_\eta a). \tag{4.6.3}$$

If we define

$$\mathrm{Jac}(A, B, C) = [[A, B]_c, C]_c + [[B, C]_c, A]_c + [[C, A]_c, B]_c \tag{4.6.4}$$

for $A, B, C \in C^\infty(TM \oplus T^*M)$, then one can show (see for example [Gu])

$$\mathrm{Jac}(A, B, C) = d(\mathrm{Nij}(A, B, C)), \tag{4.6.5}$$

where the *Nijenhuis tensor* Nij is defined

$$Nij(A, B, C) = \frac{1}{3}(\langle [A, B]_c, C \rangle + \langle [B, C]_c, A \rangle + \langle [C, A]_c, B \rangle). \tag{4.6.6}$$

The properties of the Courant bracket and pairing on $TM \oplus T^*M$ motivate the definition of a *Courant algebroid*[Gu],[BCh].

Another interesting feature of the Courant bracket is that it admits "B-field" transformations as symmetries. Namely, if B is a closed 2-form on M, then a simple computation shows

$$[\xi + a + \iota_\xi B, \eta + b + \iota_\eta B]_c = [\xi + a, \eta + b]_c + \iota_{[\xi,\eta]} B \tag{4.6.7}$$

so that the transformation

$$\xi + a \mapsto \xi + a + \iota_\xi B \tag{4.6.8}$$

is compatible with the bracket. In addition, the skew-symmetry of B implies that the transformation (4.6.8) preserves the pairing (4.6.3). The existence of these symmetries allows one to "twist" the Courant bracket on $TM \oplus T^*M$ to obtain more general Courant algebroids.

In particular, suppose we are given a DD gerbe with connective structure $(\mathcal{G}, \mathcal{A})$ over M, then we recall the following construction from [Hi]. Let $\{\{U_i\}, \{Q_i\}, \{s_{ij}\}, \{\mu_i\}\}$ be a collection of local trivializations for $(\mathcal{G}, \mathcal{A})$, and let $\{\{g_{ijk}\}, \{A_{ij}\}\}$ be the corresponding Cech data. From the relation

$$A_{jk} - A_{ik} + A_{ij} = d\log(g_{ijk}), \tag{4.6.9}$$

it follows that the linear transformation

$$\begin{pmatrix} 1 & 0 \\ -dA_{ij} & 1 \end{pmatrix} : TU_{ij} \oplus T^*U_{ij} \to TU_{ij} \oplus T^*U_{ij} \tag{4.6.10}$$

defines an extension E of TM by T^*M, where dA_{ij} acts on TU_{ij} by contraction $\xi \mapsto \iota_\xi dA_{ij}$. In other words, a (global) section of E is described by a pair $(\xi, \{a_i\})$ with $a_i \in i\Omega^1(U_i)$ satisfying

$$a_j - a_i = -\iota_\xi dA_{ij} \tag{4.6.11}$$

on the overlaps U_{ij}. Because the 2-forms dA_{ij} are closed, we obtain a well-defined Courant bracket $[\cdot,\cdot]_E$ and bilinear pairing $\langle\cdot,\cdot\rangle_E$ on sections of E.

Suppose that in addition we are given a curving K for $\mathcal{A}$ with corresponding 2-forms

$$B_i = K(\mu_i) \tag{4.6.12}$$

satisfying

$$B_j - B_i = dA_{ij} \tag{4.6.13}$$

as discussed in §3.6. We can then define a splitting $s : TM \oplus T^*M \to E$ by

$$s(\xi + a) = (\xi, \{a - \iota_\xi B_i\}). \tag{4.6.14}$$

We then have

$$\langle s(\xi + a), s(\eta + b)\rangle_E = \langle \xi + a, \eta + b\rangle \tag{4.6.15}$$

and

$$[s(\xi + a), s(\eta + b)]_E = s([\xi + a, \eta + b]_c) + \iota_\xi \iota_\eta C, \tag{4.6.16}$$

where $C = dB_i$ is the curvature form.

4.6.2 The Courant algebroid in terms of infinitesimal connective lifts

We next explain a construction of E in terms of our category of infinitesimal connective lifts. Let

$$\tilde{\pi} : \mathcal{L}_{(\mathcal{G},\mathcal{A})} \to \mathcal{L}_{\mathcal{G}} \tag{4.6.17}$$

be the obvious forgetful functor. Recall from Proposition 4.2.26 that for each object $\hat{\xi} \in \mathcal{L}_{\mathcal{G}}$, the set $\tilde{\pi}^{-1}(\hat{\xi})$ of connective lifts extending $\hat{\xi}$ is a torsor for the group of 1-forms $i\Omega^1(M)$. Furthermore, since we have specified a connective structure $\mathcal{A}$ on $\mathcal{G}$, for each vector field ξ we have the horizontal lift $\hat{\xi}^h \in \mathcal{L}_{\mathcal{G}}(\xi)$ defined in (3.6.18). We may then define

Definition 4.6.18.

$$E = \coprod_{\xi \in C^\infty(TM)} \tilde{\pi}^{-1}(\hat{\xi}^h). \tag{4.6.19}$$

Let $\pi : E \to C^\infty(TM)$ denote the projection map, and for each $\xi \in C^\infty(TM)$ let $E_\xi = \pi^{-1}(\xi)$ denote the set of connective lifts extending ξ^h. Let us describe the sets E_ξ more concretely. Given $\check{\xi} = (\xi^h, \Theta_{\check{\xi}}) \in E_\xi$, for each object $Q \in \mathcal{G}(U)$ and each local section μ of $\mathcal{A}(Q)$, define a 1-form $a_{\check{\xi}}(\mu)$ by

$$a_{\check{\xi}}(\mu) = A_{\Theta_{\check{\xi}}(\mu)}(\iota_\xi[\mu]). \tag{4.6.20}$$

The relations

$$\Theta_{\check{\xi}}(\mu + \alpha) = \Theta_{\check{\xi}}(\mu) - \mathcal{L}_\xi \alpha \tag{4.6.21}$$

together with the Cartan formula for the Lie derivative imply that for each 1-form α we have

$$a_{\check{\xi}}(\mu + \alpha) = a_{\check{\xi}}(\mu) - \iota_\xi d\alpha. \tag{4.6.22}$$

Equivalently, we have an isomorphism of $\underline{i\Omega}^1_U$-torsors

$$-\iota_\xi d[\mathcal{A}(Q)] \xrightarrow{\cong} \underline{i\Omega}^1_U. \tag{4.6.23}$$

In addition, one can check directly that given $\psi : Q \to R$ we have

$$a_{\check{\xi}}(\mu) = a_{\check{\xi}}(\psi_* \mu), \tag{4.6.24}$$

and that given an inclusion $i : V \to U$, we have

$$a_{\check{\xi}}(i^* \mu) = a_{\check{\xi}}(\mu)|_V. \tag{4.6.25}$$

Unwinding definition (.2.16) we see that the conditions (4.6.24) and (4.6.25) imply that the assignment (4.6.20) is equivalent to a 2-morphism from $\iota_\xi d[\mathcal{A}]$ to the constant 1-morphism $\mathcal{G} \to \mathrm{B}(\underline{i\Omega}^1_M)$ taking every $Q \in \mathcal{G}(U)$ to the trivial $\underline{i\Omega}^1_U$-torsor.[3].

Proposition 4.6.26. *The correspondence $\check{\xi} \mapsto a_{\check{\xi}}$ defines a bijection between E_ξ and the set of 2-morphisms*

$$-\iota_\xi d[\mathcal{A}] \overset{\cong}{\Longrightarrow} \underline{i\Omega}^1, \tag{4.6.27}$$

where $\underline{i\Omega}^1$ denotes the constant 1-morphism taking every $Q \in \mathcal{G}(U)$ to $\underline{i\Omega}^1_U$.

Proof. By definition, an element of E_ξ is an equivalence

$$\Theta_{\check{\xi}} : \pounds_{-\xi}[\mathcal{A}] \overset{\cong}{\Longrightarrow} \mathcal{A}_0(F_{\hat{\xi}^h}) = -d[F_{\hat{\xi}^h}], \tag{4.6.28}$$

where by definition

$$F_{\check{\xi}^h} = \iota_\xi[\mathcal{A}]. \tag{4.6.29}$$

[3]For comparison, see remark (3.6.26)

Suppose we are given an equivalence

$$a_{\check{\xi}} : -\iota_\xi d[\mathcal{A}] \overset{\cong}{\Longrightarrow} \underline{i\Omega}^1_M \tag{4.6.30}$$

as in the proposition. Then applying Proposition 2.3.17 repeatedly we can construct an equivalence

$$\begin{aligned} \pounds_{-\xi}[\mathcal{A}] &\overset{\cong}{\Longrightarrow} -\iota_\xi d[\mathcal{A}] \boxplus (-d\iota_\xi[\mathcal{A}]) \overset{\cong}{\Longrightarrow} \underline{i\Omega}^1_M \boxplus (-d\iota_\xi[\mathcal{A}]) \\ &\overset{\cong}{\Longrightarrow} -d\iota_\xi[\mathcal{A}] \overset{\cong}{\Longrightarrow} \mathcal{A}_0(F_{\check{\xi}^h}). \end{aligned} \tag{4.6.31}$$

By Corollary 4.2.26 the set E_ξ is a torsor for the group of global 1-forms on M; on the other hand, it is not hard to see that the set of 2-morphisms described in the proposition is also a torsor for 1-forms and that the map taking $a_{\check{\xi}}$ to $\Theta_{\check{\xi}}$ is by construction a morphism of torsors. This map is therefore both one-to-one and onto. □

Using the characterization of sections of E given in Proposition 4.6.26, we can now endow E with several interesting algebraic structures. Given an element $\check{\xi}$ of E_ξ, let $a_{\check{\xi}}$ denote the corresponding function satisfying (4.6.22), (4.6.24) and (4.6.25). Given $\check{\xi}, \check{\eta} \in E$, we define $\check{\xi} + \check{\eta}$ by

$$a_{\check{\xi}+\check{\eta}} = a_{\check{\xi}} + a_{\check{\eta}}, \tag{4.6.32}$$

and given $f \in C^\infty(M)$, we define $f\check{\xi}$ by

$$a_{f\check{\xi}} = f a_{\check{\xi}}. \tag{4.6.33}$$

It is then readily verified that $a_{\check{\xi}+\check{\eta}}$ and $a_{f\check{\xi}}$ satisfy (4.6.22), (4.6.24) and (4.6.25). Furthermore, we define $0 \in E_0$ to be the function taking every local section μ of $\mathcal{A}(Q)$ to 0. Altogether E obtains the structure of a $C^\infty(M)$-module. Furthermore, it is clear that the projection map $\pi : E \to C^\infty(TM)$ is a map of modules. Therefore we obtain an extension of modules

$$0 \longrightarrow i\Omega^1(M) \longrightarrow E \longrightarrow C^\infty(TM) \longrightarrow 0. \tag{4.6.34}$$

Suppose we are given a curving K. If we then define

$$a_{s(\xi)}(\mu) = -\iota_\xi K(\mu), \tag{4.6.35}$$

then it follows from definition (3.6.22) that $a_{s(\xi)}$ satisfies condition (4.6.22). Therefore a curving determines a splitting of the sequence (4.6.34).

We next construct a non-degenerate pairing $\langle \cdot, \cdot \rangle_E$ on E. Given $a_{\check{\xi}}, a_{\check{\eta}} \in E$ locally over an open set $U \subset M$ by

$$\langle a_{\check{\xi}}, a_{\check{\xi}} \rangle_E|_U = \frac{1}{2}(\iota_\xi a_{\check{\eta}}(\mu) + \iota_\eta a_{\check{\xi}}(\mu)), \tag{4.6.36}$$

for μ is a global section of $\mathcal{A}(Q)$ for some $Q \in \mathcal{G}(U)$. By (4.6.22) and (4.6.24), we see that (4.6.36) is independent of the choice of Q and μ. Furthermore, if we cover M by open sets $\{U_i\}$ such that $\mathcal{G}(U_i)$ is non-empty for each i, then condition (4.6.25) implies that the local the formula (4.6.36) is consistent with restrictions and we therefore obtain a global function on M. Clearly $\langle \cdot, \cdot \rangle_E$ is bilinear over functions and symmetric. To see that it is also non-degenerate, suppose the for some $\check{\xi} \in E$ we have

$$\langle \check{\xi}, \check{\eta} \rangle_E = 0 \tag{4.6.37}$$

for each $\check{\eta} \in E$. In particular, for each 1-form α on M, if we let $\eta = 0$ and $\check{\eta}$ equal the image of α in E_0, we have

$$\frac{1}{2}\iota_\xi \alpha = 0. \tag{4.6.38}$$

Since this holds for arbitrary α we must have $\xi = 0$. Therefore $\check{\xi} \in E_0$, say $\check{\xi} = \beta$ for β a 1-form. But equation (4.6.36) implies that β must pair with each vector field trivially, and must therefore vanish.

There is also a natural bracket on E. Given $\check{\xi}, \check{\eta} \in E$, define $[\check{\xi}, \check{\eta}]_E$ by

$$a_{[\check{\xi},\check{\eta}]} = \pounds_\xi a_{\check{\eta}} - \pounds_\eta a_{\check{\xi}} - \frac{1}{2} d\iota_\xi a_{\check{\eta}} + \frac{1}{2} d\iota_\eta a_{\check{\xi}}. \tag{4.6.39}$$

To verify that this does in fact define an element of $E_{[\xi,\eta]}$, note that

$$\begin{aligned}
&a_{[\check{\xi},\check{\eta}]_E}(\mu + \alpha) - a_{[\check{\xi},\check{\eta}]_E}(\mu) \\
&= -\left(\pounds_\xi \iota_\eta d\alpha - \pounds_\eta \iota_\xi d\alpha + \frac{1}{2} d\iota_\xi \iota_\eta d\alpha - \frac{1}{2} d\iota_\eta \iota_\xi d\alpha\right) \\
&= -\left(\pounds_\xi \iota_\eta d\alpha - \iota_\eta d\iota_\xi d\alpha + d\iota_\eta \iota_\xi d\alpha - d\iota_\eta \iota_\xi d\alpha\right) \\
&= -\left(\pounds_\xi \iota_\eta d\alpha - \iota_\eta \pounds_\xi d\alpha\right) \\
&= -[\pounds_\xi, \iota_\eta] d\alpha \\
&= -\iota_{[\xi,\eta]} d\alpha.
\end{aligned} \tag{4.6.40}$$

Although it is possible to show directly that these structures satisfy the compatibility conditions possessed by the sections of a Courant algebroid, we will instead proceed by comparing our construction to that in [Hi]. We introduce local trivializations $\{\{U_i\}, \{Q_i\}, \{s_{ij}\}, \{\mu_i\}$ for $(\mathcal{G}, \mathcal{A})$ with corresponding

Cech data $(\{g_{ijk}\}, \{A_{ij}\})$. Recall from §3.6 that we obtain Cech data for the horizontal lift of a vector field ξ

$$f_{ij}^{\hat{\xi}^h} = \iota_\xi A_{ij}. \tag{4.6.41}$$

A simple calculation then shows that the collection[4]

$$(\xi, \iota_\xi A_{ij}, a_i^{\check{\xi}}) \tag{4.6.42}$$

is Cech data for a connective lift if and only if

$$a_j^{\check{\xi}} - a_i^{\check{\xi}} = -\iota_\xi dA_{ij}. \tag{4.6.43}$$

In terms of the functions $a_{\check{\xi}}$ given above, the 1-forms a_i are given by

$$a_i^{\check{\xi}} = a_{\check{\xi}}(\mu_i). \tag{4.6.44}$$

Therefore in the Cech picture the pairing is given by

$$\langle(\xi, \iota_\xi A_{ij}, a_i^{\check{\xi}}), (\eta, \iota_\eta A_{ij}, a^{\check{\eta}})\rangle_E = \frac{1}{2}(\iota_\xi a^{\check{\eta}} + \iota_\eta a^{\check{\xi}}), \tag{4.6.45}$$

and the bracket by $[(\xi, \iota_\xi A_{ij}, a_i^{\check{\xi}}), (\eta, \iota_\eta A_{ij}, a^{\check{\eta}}] =$

$$([\xi,\eta], \iota_{[\xi,\eta]}A_{ij}, \pounds_\xi a_i^{\check{\eta}} - \pounds_\eta a_i^{\check{\xi}} - \frac{1}{2}d\iota_\xi a_i^{\check{\eta}} + \frac{1}{2}\iota_\eta a_i^{\check{\eta}}). \tag{4.6.46}$$

We see that these are exactly the formulas used to define the Courant algebroid structure in §4.6.1.

In [RW] it was shown that any Courant algebroid E gives rise to an L_∞-algebra. Moreover, as pointed out in [Ro1], this L_∞-algebra can be restricted to one with 2-terms, which we call L_E.

---

[4] To avoid notational clutter, we will surpress the brackets around $f_{ij}^{\check{\xi}}$ and $a_{ij}^{\check{\xi}}$.

Definition 4.6.47. *In terms of local trivializations* $\{\{U_i\}, \{Q_i\}, \{s_{ij}\}, \{\mu_i\}\}$, *the 2-term* L_∞*-algebra* $L_E = \{W_0 \oplus W_1, d, [\cdot,\cdot], Jac\}$ *is given by*

1. $W_0 = C^\infty(E)$, $W_1 = C^\infty(\underline{i\mathbb{R}}_M)$.

2. $d : W_1 \to W_0$ *is given by*

$$f \mapsto (0, a_i = df|_{U_i}). \tag{4.6.48}$$

3. $[\cdot,\cdot] : W_0 \times W_0 \to W_0$ *is given by the Courant bracket. Given* $(\xi, a_i) \in W_0$ *and* $f \in W_1$, *we define*

$$[(\xi, a_i), f] = -[f, (\xi, a_i)] = \frac{1}{2}\xi \cdot f. \tag{4.6.49}$$

4. $Jac(A, B, C) = -\frac{1}{3}Nij(A, B, C)$.

Suppose that we are given two sections $\hat{\xi}, \hat{\eta}$ of the Courant algebroid E, as described by Cech data. In particular, $\hat{\xi}$ and $\hat{\eta}$ are elements of the category $\mathcal{L}_{(g_{ijk}, A_{ij})}$, and we can take their bracket $[\hat{\xi}, \hat{\eta}]$ as described in equation (4.2.27). This yields another connective lift which is not in general a section of E. On the other hand, as discussed above the Courant bracket of $\hat{\xi}$ and $\hat{\eta}$ does define another section of E. Although these two brackets are not equal, they are related by a natural isomorphism. The following theorem explains this relationship in the language of L_∞-algebras.

Theorem 4.6.50. *Let* E *be the Courant algebroid constructed above. Then there is an isomorphism of 2-term* L_∞ *algebras*

$$\Phi : L_E \xrightarrow{\cong} L_{(g_{ijk}, A_{ij})}. \tag{4.6.51}$$

Proof. First, we define a degree 0 chain map $\phi_0 : W_0 \to V_0$. In degree 0 we define

$$\phi_0 : (\xi, \{a_j\}) \mapsto (\xi, \{\iota_\xi A_{ij}\}, \{a_i\}); \tag{4.6.52}$$

by our earlier discussion right-hand side defines an object of $\mathcal{L}_{(g_{ijk}, A_{ij})}$. In degree 1 we define ϕ_0 to be the identity map on $C^\infty(\underline{i\mathbb{R}}_M)$. The verification that ϕ_0 is a chain map is trivial.

Given $x = (\xi, \{a_i\})$ and $y = (\eta, \{b_i\})$, note that

$$\begin{aligned} \phi([x, y]) &= \phi([\xi, \eta], \{\pounds_\eta b_i - \pounds_\xi a_i - \frac{1}{2} d\iota_\eta b_i + \frac{1}{2} d\iota_\eta a_i\}) \\ &= ([\xi, \eta], \{\iota_{[\xi,\eta]} A_{ij}\}, \{\pounds_\eta b_i - \pounds_\xi a_i - \frac{1}{2} d\iota_\eta b_i + \frac{1}{2} d\iota_\eta a_i\}). \end{aligned} \tag{4.6.53}$$

On the other hand we have

$$[\phi(x), \phi(y)] = ([\xi, \eta], \{\xi \cdot (\iota_\eta A_{ij}) - \eta \cdot (\iota_\xi A_{ij})\}, \{\pounds_\xi b_i - \pounds_\eta a_i\}). \tag{4.6.54}$$

Using the formula

$$\iota_\eta \iota_\xi A_{ij} = \xi \cdot (\iota_\eta A_{ij}) - \eta \cdot (\iota_\xi A_{ij}) - \iota_{[\xi,\eta]} A_{ij}, \tag{4.6.55}$$

we therefore have

$$\phi([x, y]) - [\phi(x), \phi(y)] = (0, \{-\iota_\eta \iota_\xi A_{ij}\}, d(-\frac{1}{2} \iota_\xi b_i + \frac{1}{2} \iota_\eta a_i\}). \tag{4.6.56}$$

If we then define

$$\phi_2 : C^\infty(E) \times C^\infty(E) \to C^\infty(\underline{i\mathbb{R}}_M) \tag{4.6.57}$$

$$(\xi, \{a_i\}), (\eta, \{b_i\}) \mapsto \{-\frac{1}{2} \iota_\xi b_i + \frac{1}{2} \iota_\eta a_i\}, \tag{4.6.58}$$

a simple computation shows that

$$d\phi_2(x, y) = \phi([x, y]) - [\phi(x), \phi(y)] \tag{4.6.59}$$

for all $x, y \in C^\infty(E)$. To finish the proof that $\Phi = (\phi, \phi_2)$ defines a homomorphism of L_∞-algebras, we need to verify that equation (2) in Definition 3.6.32 holds. We omit this computation, which is somewhat tedious but straightforward.

□

Appendices

.1

In this appendix we discuss smooth families of symmetries of gerbes and their relationship to infinitesimal symmetries. This gives a direct connection between equivariant gerbes and the infinitesimal symmetries explored in this paper. We consider both gerbes with and without connective structures.

.1.1 Gerbes without connective structures

Let Φ be a 1-parameter family of diffeomorphisms of M generated by a vector field ξ. We now explain how a family of symmetries of $\mathcal{G}$ lifting Φ gives rise to an infinitesimal symmetry of $\mathcal{G}$ lifting ξ via a process analogous to differentiation. The local version of this functor was discussed in §3.4

Let us briefly return to the case where P is a principal circle bundle over M. Given any diffeomorphism φ of M, we can form the pull-back φ^*E. By definition, for each $m \in M$ the fiber $(\varphi^*E)_m$ can be identified with the fiber $E_{\varphi(m)}$, and a symmetry of E lifting φ is equivalent to an isomorphism $E \xrightarrow{\cong} \varphi^*E$. More generally, given a smooth family of diffeomorphisms

$$\Phi : M \times \mathbb{R} \to M, \tag{.1.1}$$

a smooth family of symmetries of E lifting Φ is equivalent to an isomorphism

$$\hat{\Phi} : p_1^*E \to p_0^*E \tag{.1.2}$$

of bundles over $M^1 = M \times \mathbb{R}$, where as discussed in §2.1, by definition $p_0 = \Phi$ and p_1 is projection onto M. Equivalently, the isomorphism $\hat{\Phi}$ may be encoded as a global section of the bundle

$$\delta E = p_0^*E \otimes p_1^*E^\vee. \tag{.1.3}$$

To generalize to gerbes, recall from §2.3, that in addition to forming the inverse image of a gerbe, we may define the tensor product of two gerbes and the dual $\mathcal{G}^\vee$ of a gerbe $\mathcal{G}$ using the associated gerbe construction. Thus we may define a gerbe

$$\delta\mathcal{G} = p_0^*\mathcal{G} \otimes p_1^*\mathcal{G}^\vee \tag{.1.4}$$

over M^1. Recall also from §2.1 that for each open set $U \in M$ we define an open set

$$\nu(U) = p_0^{-1}(U) \cap p_1^{-1}(U) \subset M^1. \tag{.1.5}$$

Given any sheaf $\mathcal{F}$ over M^1, we can define a sheaf $\nu_*\mathcal{F}$ over M by

$$\nu_*\mathcal{F}(U) = F(\nu(U)) \tag{.1.6}$$

for any open set $U \subset M$; this definition makes sense both for sheaves taking values in a category (e.g. the category of groups) and for gerbes. We can also define a homomorphism of sheaves of groups over M

$$\delta_{\mathbb{T}} : \underline{\mathbb{T}}_M \to \nu_*(\underline{\mathbb{T}}_{M^1}) \tag{.1.7}$$

by

$$\delta_{\mathbb{T}} g = (p_0^* g)(p_1^* g^{-1}). \tag{.1.8}$$

Remark .1.9. By construction we have a canonical morphism of gerbes

$$\delta_{\mathcal{G}} : \mathcal{G} \to \nu_* \delta\mathcal{G} \tag{.1.10}$$

intertwining $\delta_{\mathbb{T}}$.

Definition .1.11. *The category of smooth familes of symmetries of $\mathcal{G}$ lifting Φ is the category $\mathcal{L}_{\mathcal{G}}(\Phi)$ of global sections of $\delta\mathcal{G}$ over M.*

Remark .1.12. Definition .1.11 is similar to that of an $\mathbb{R}$-equivariant gerbe over M; see [Br2], [G2], [Me] for general discussions of equivariant gerbes. In particular, an extension of $\mathcal{G}$ to an $\mathbb{R}$-equivariant gerbe gives an example of a symmetry of $\mathcal{G}$ lifting Φ in the sense of Definition .1.11. Definition .1.11 is more general in that it does not require the specification of any data over $M \times \mathbb{R}^2$.

.1.2 The differentiation functor

We now explain the construction of a functor $\mathcal{L}_{\mathcal{G}}(\Phi) \to \mathcal{L}_{\mathcal{G}}(\xi)$, where the latter is the category of global infinitesimal symmetries of $\mathcal{G}$ lifting the vector field ξ. Our approach parallels the discussion in §2.1; in particular the reader may find it useful to compare the constructions in this section to the derivation of equation (2.1.6).

Given $\hat{\Phi} \in \mathcal{L}_{\mathcal{G}}(\Phi)$ will construct $D(\hat{\Phi}) = F_{\hat{\xi}}$ as a composition of 1-morphisms which are described in the following lemma. The existence of these 1-morphisms is a straightforward consequence of the definitions in Appendix B, and the proof will be omitted.

Lemma .1.13. *(1) Let S be a global section of a DD gerbe $\mathcal{G}$ over a manifold M. Then there is a 1-morphism of gerbes (intertwining the identity on $\underline{\mathbb{T}}_M$)*

$$\underline{Hom}(\cdot, S) : \mathcal{G}^{op} \to B(\underline{\mathbb{T}}_M) \tag{.1.14}$$

sending

$$Q \mapsto \underline{Hom}(Q, S|_U) \tag{.1.15}$$

for each object $Q \in \mathcal{G}(U)$. Moreover, the assignment

$$S \mapsto \underline{Hom}(\cdot, S) \qquad (.1.16)$$

defines an equivalence of categories

$$\Gamma(\mathcal{G}) \xrightarrow{\cong} Hom(\mathcal{G}^{op}, B(\underline{\mathbb{T}}_M)), \qquad (.1.17)$$

where $\Gamma(\mathcal{G})$ denotes the category of global sections of $\mathcal{G}$.

(2) Let $i : N \to M$ be the inclusion of a submanifold. Then there is a restriction *1-morphism*

$$i^*[\cdot] : B(\underline{i\mathbb{R}}_M) \to i_* B(\underline{i\mathbb{R}}_N) \qquad (.1.18)$$

intertwining the restriction homomorphism

$$\underline{i\mathbb{R}}_M \to i_* \underline{i\mathbb{R}}_N \qquad (.1.19)$$

sending $f \in \underline{i\mathbb{R}}_M(U)$ to the restriction of f to $U \cap N$.

Notation .1.20. For convenience, let $\frac{d}{dt} \log : \underline{\mathbb{T}}_{M^1} \to \underline{\mathbb{T}}_{M^1}$ denote the homomorphism given by

$$g \mapsto \iota_{\frac{d}{dt}} d \log g. \qquad (.1.21)$$

Remark .1.22. Recall from the discussion before Definition 3.2.4 that there is a canonical equivalence

$$\mathrm{Hom}_{\iota_\xi d \log}(\mathcal{G}^{op}, \mathrm{B}(\underline{i\mathbb{R}}_M)) \xrightarrow{\cong} \mathrm{Hom}_{-\iota_\xi d \log}(\mathcal{G}, \mathrm{B}(\underline{i\mathbb{R}}_M)) \qquad (.1.23)$$

which is actually a bijection on the level of both objects and morphisms. For convenience we will work here with $\mathrm{Hom}_{\iota_\xi d \log}(\mathcal{G}^{op}, \mathrm{B}(\underline{i\mathbb{R}}_M))$, which we will call $\tilde{\mathcal{L}}_{\mathcal{G}}(\xi)$.

Definition .1.24. *$D : \mathcal{L}_{\mathcal{G}}(\Phi)^{op} \to \tilde{\mathcal{L}}_{\mathcal{G}}(\xi)$ is the functor*

$$S \mapsto i^*[\frac{d}{dt}\log[\underline{Hom}(\delta(\cdot), S)]] \tag{.1.25}$$

for $S \in \Gamma(\delta\mathcal{G}) = \mathcal{L}_{\mathcal{G}}(\Phi)$.

Remark .1.26. In more detail, for each $S \in \Gamma(\delta\mathcal{G})$, $D(S)$ is the composition of 1-morphisms

$$\mathcal{G}^{op} \xrightarrow{\Phi_1} \nu_*(\delta\mathcal{G}^{op}) \xrightarrow{\Phi_2} \nu_*\mathrm{B}(\underline{\mathbb{T}}_{M^1}) \xrightarrow{\Phi_3} \nu_*\mathrm{B}(\underline{i\mathbb{R}}_{M^1}) \xrightarrow{\Phi_4} \mathrm{B}(\underline{i\mathbb{R}}_M). \tag{.1.27}$$

Here Φ_1 is the 1-morphism described in remark (.1.9), Φ_2 is ν_* applied to the 1-morphism $\underline{\mathrm{Hom}}(\cdot, S)$ described in part (1) Lemma .1.13, Φ_3 is $\nu_*(\frac{d}{dt}\log)[\cdot]$, Φ_4 is given by applying ν_* to the restriction 1-morphism described in part (2) of (.1.13); note that Φ_4 is a 1-morphism from

$$\nu_*\mathrm{B}(\underline{\mathbb{T}}_{M^1}) \to \nu_* i_* \mathrm{B}(\underline{i\mathbb{R}}_M) = \mathrm{B}(\underline{i\mathbb{R}}_M). \tag{.1.28}$$

We must check that $D(S)$ does in fact intertwine the homomorphism $\iota_\xi d\log$, i.e. defines an element of $\mathrm{Hom}_{\iota_\xi d\log}(\mathcal{G}^{op}, \mathrm{B}(\underline{i\mathbb{R}}_M))$. The composition $\Phi_2 \circ \Phi_1$ intertwines

$$\delta_{\mathbb{T}} : \underline{\mathbb{T}}_M \to \nu_*\underline{\mathbb{T}}_{M^1}. \tag{.1.29}$$

Φ_3 intertwines $\nu_*\frac{d}{dt}\log$, and Φ_4 intertwines the restriction homomorphism $\nu_*\underline{\mathbb{T}}_{M^1} \to \underline{\mathbb{T}}_M$. Thus $D(S)$ intertwines the composition

$$\zeta = i^* \circ \nu_*(\frac{d}{dt}\log) \circ \delta_{\mathbb{T}} : \underline{\mathbb{T}}_M \to \underline{i\mathbb{R}}_M. \tag{.1.30}$$

Given $g : U \to \mathbb{T}$ we have

$$\zeta(g) = \frac{d}{dt} \log(g(p_0)g^{-1}(p_1))|_U. \tag{.1.31}$$

More concretely, the value of $\zeta(g)$ on $x \in U$ is given by

$$\begin{aligned} &\frac{d}{dt}|_{t=0} \log(g(\varphi_t(x))g^{-1}(x)) \\ =&\frac{d}{dt}|_{t=0} \log(g(\varphi_t(x)) \\ =&\iota_\xi d\log(g)(x). \end{aligned} \tag{.1.32}$$

.1.3 Gerbes with connective structures

In §4.5 we discussed families of connective symmetries of a gerbe $\mathcal{G}$ with connective structure $\mathcal{A}$ using relative 1-forms. To explain why we use relative forms, let us briefly turn to the case of a $\mathbb{T}$-bundle $E \to M$ with connection Θ. Given a symmetry $\Phi : E \to E$ covering $\varphi : M \to M$, Φ is a symmetry of the pair (E, Θ) if

$$\Phi^*\Theta = \Theta, \tag{.1.33}$$

where Θ is viewed as a 1-form on the total space of E. Given a smooth family of symmetries $\{\Phi_t\}$ covering $\{\varphi_t\}$ (or more generally any Lie group of symmetries G), there are two possible ways in which we might wish $\{\Phi_t\}$ to be compatible with Θ. On the one hand, we might ask that for each t, Φ_t is a symmetry of (E, Θ). In terms of the vector field $\hat{\xi}$ corresponding to $\{\Phi_t\}$, this is equivalent to requiring

$$\mathcal{L}_{\hat{\xi}}\Theta = 0. \tag{.1.34}$$

On the other hand, we might want a definition such that, in the good case that the quotient of M by the symmetry is a manifold, a lift of $\{\varphi_t\}$ to (E, Θ) is equivalent to a bundle with connection over the quotient $M/\mathbb{R}$. In this case, in addition to requiring (.1.34), we must also have

$$\iota_{\hat{\xi}} \Theta = 0. \tag{.1.35}$$

This distinction can be understood in terms of simplicial manifolds. Recall that a 1-parameter family of symmetries $\{\Phi_t\}$ of E covering $\{\varphi_t\}$ is equivalent to a global section S of $\delta E = (p_1^{-1} E)^* \otimes p_0^{-1} E$ over M^1. The connection Θ on E determines a connection $\delta\Theta$ over δE, and conditions (.1.34) and (.1.35) are satisfied if and only if

$$\alpha = A_{\delta\Theta} S = 0, \tag{.1.36}$$

i.e. if and only if the section S is flat. On the other hand, the condition (.1.34) by itself holds if and only if the restriction of α to $M_t = M \times \{t\}$ vanishes for each t, i.e. if and only if the relative 1-form determined by α is zero.

In this paper we will be interested only in the condition (.1.34) and the analogous notion for gerbes[5]. We can define a *relative connective structure* on a gerbe over $M \times \mathbb{R}$ by replacing 1-forms with relative 1-forms in Definition 3.6.8.

Let us generalize Definition .1.11 to the situation where $\mathcal{G}$ has a connective structure $\mathcal{A}$. We first observe that the gerbe $\delta\mathcal{G}$ over M^1 naturally

[5]For a related discussion of the relationship between equivariant gerbes and gerbes over quotients, see [G2].

inherits a relative connective structure, which we denote $\delta\mathcal{A}$. We will sketch the construction, which is similar to that on page 211 of [Br1]. For the related notion of an equivariant gerbe with connective structure, we refer the reader to [Br1] and [G2]. By construction, given local sections $Q_0 \in \mathcal{G}(U_0)$ and $Q_1 \in \mathcal{G}(U_1)$, we have an object

$$p_0^* Q_0 \otimes p_1^* Q_1^\vee \tag{.1.37}$$

of $\delta\mathcal{G}(p_0^{-1}(U_0) \cap p_1^{-1}(U_1))$. We then define $\delta\mathcal{A}(p_0^* Q_0 \otimes p_1^* Q_1^\vee)$ to be

$$q[p_0^* \mathcal{A}(Q_0) \boxminus p_1^*(Q_1)], \tag{.1.38}$$

where $q : \underline{i\Omega}^1_{M^1} \to \underline{i\Omega}^1_{M^1,rel}$ is the quotient map. We then complete the construction of $\delta\mathcal{A}$ using the fact that every object of $\delta\mathcal{G}$ is locally isomorphic to one of the form (.1.37). Note that for every object $Q \in \mathcal{G}(U)$ and every connection $\mu \in \mathcal{A}(Q)$, we obtain a relative connection on δQ, which we denote by $\delta\mu$. Furthermore, given any 1-form $\alpha \in i\Omega^1(U)$ we have

$$\delta^i(\mu + \alpha) = \delta^i(\mu) + p_0^*\alpha - p_1^*\alpha, \tag{.1.39}$$

where the relative 1-forms on the right hand side are implicitly restricted to the appropriate open set. More formally, we have the following.

Lemma .1.40. *There is a 2-morphism*

$$\delta_{\underline{i\Omega}^1_{rel}}[\mathcal{A}] \Rightarrow \nu_*(\delta\mathcal{A}). \tag{.1.41}$$

Definition .1.42. *A* connective lift *of Φ to $\mathcal{G}$ is a pair (S, μ_S), where S is a global section of $\delta\mathcal{G}$, and μ_S is a global section of $\delta\mathcal{A}(S)$. Given another connective lift $(S', \mu_{S'})$, an* equivalence of connective lifts *from $(S, \mu) \to (S', \mu_{S'})$ is an isomorphism $\tau : S \to S'$ such that $\tau_*(\mu_S) = \mu_{S'}$. We will denote the corresponding category by $\mathcal{L}_{(\mathcal{G},\mathcal{A})}(\Phi)$.*

In the previous section we constructed a functor from the category of 1-parameter lifts of Φ to $\mathcal{G}$ to the category of global lifts of the corresponding vector field ξ to $\mathcal{G}$, or rather to the equivalent category

$$\tilde{\mathcal{L}}_{\mathcal{G}} = \mathrm{Hom}_{\iota_\xi d\log}(\mathcal{G}^{op}, \mathrm{B}(\underline{i\mathbb{R}_M})). \tag{.1.43}$$

Recall that a connective structure $\mathcal{A}$ is a 1-morphism from $\mathcal{G}$ to $\mathrm{B}(\underline{i\Omega^1_M})$ intertwining $-dlog$. Equivalently, $\mathcal{A}$ determines a 1-morphism

$$\tilde{A} \in \mathrm{Hom}_{d\log}(\mathcal{G}^{op}, \mathrm{B}(\underline{i\Omega^1_M})). \tag{.1.44}$$

We may then define the category $\tilde{\mathcal{L}}_{(\mathcal{G},\mathcal{A})}$ which is equivalent to the category of connective lifts $\mathcal{L}_{(\mathcal{G},\mathcal{A})}$. An object of $\tilde{\mathcal{L}}_{(\mathcal{G},\mathcal{A})}$ is an element $F_{\hat{\xi}} \in \tilde{\mathcal{L}}_{\mathcal{G}}$ together with a 2-morphism $\Theta_{\hat{\xi}} : -\mathcal{L}_\xi[\tilde{\mathcal{A}}] \overset{\cong}{\Longrightarrow} \tilde{A}^0_{\underline{i\mathbb{R}}_M} \circ F_{\hat{\xi}}$.

Recall from part (1) of Lemma .1.13 that for each global section S of $\delta\mathcal{G}$ over $M \times \mathbb{R}$, we have a 1-morphism of gerbes

$$\underline{\mathrm{Hom}}(\cdot, S) : \delta\mathcal{G}^{op} \to \mathrm{B}(\underline{\mathbb{T}}_{M^1}). \tag{.1.45}$$

Furthermore, if we fix $\mu_S \in \delta\mathcal{A}(S)$, then for every object $P \in \delta\mathcal{G}(U)$ and every relative connection $\mu \in \delta\mathcal{A}(P)$, we obtain a relative connection on $\underline{\mathrm{Hom}}(P, S)$.

This connection, which we denote by $\mu_S - \mu$, is described explicitly as follows: given a local section $\psi \in \underline{\mathrm{Hom}}(P, S)$, define

$$A_{\mu_S - \mu}(\psi) = \mu_S - \psi_* \mu, \tag{.1.46}$$

where by definition the right hand side is the unique 1-form α such that $\mu_S = \psi_* \mu + \alpha$.

Next, given a relative connection Θ on a $\mathbb{T}$-bundle $E \to U \subset M^1$, recall from Remark 4.5.8 that we can construct a (relative) connection on $\iota_{\frac{d}{dt}} d \log[E]$, which we call $\pounds_{\frac{d}{dt}}(\Theta)$. This is characterized by the formula

$$A_{\pounds_{\frac{d}{dt}}(\Theta)}(\iota_{\frac{d}{dt}} d \log[\sigma] + f) = \pounds_{\frac{d}{dt}} A_\Theta(\sigma) + df. \tag{.1.47}$$

Finally, note that given a relative connection Θ on a principal $\mathbb{T}$-bundle P over some open set $U \subset M \times \mathbb{R}$, we can restrict Θ to a connection on the restriction of P to $U \cap (M \times \{0\})$. The following lemma summarizes the above discussion in more formal language.

Lemma .1.48. *(1) Let S be a global section of $\delta\mathcal{G}$ over M^1, and let $\underline{Hom}(\cdot, S) : \delta\mathcal{G}^{op} \to B(\underline{\mathbb{T}}_{M^1})$ be the 1-morphism described in (.1.13). For fixed $\mu_S \in \delta\mathcal{A}(S)$, there is a 2-morphism*

$$-1[\tilde{\mathcal{A}}] \overset{\cong}{\Rrightarrow} \tilde{\mathcal{A}}^0_{\underline{\mathbb{T}}_{M^1}, rel} \circ \underline{Hom}(\cdot, S) \tag{.1.49}$$

given by formula (.1.46).

(2) There is a 2-morphism

$$\pounds_{\frac{d}{dt}}(\cdot) : \pounds_{\frac{d}{dt}}[\tilde{\mathcal{A}}^0_{\underline{\mathbb{T}}_{M^1}, rel}] \overset{\cong}{\Rrightarrow} \tilde{\mathcal{A}}^0_{\underline{i\mathbb{R}}_{M^1}, rel} \circ \frac{d}{dt} \log[\cdot]. \tag{.1.50}$$

(3) There is a 2-morphism

$$i^*[\tilde{\mathcal{A}}^0_{\underline{\mathbb{T}}_{M^1},rel}] \overset{\cong}{\Longrightarrow} i_*(\tilde{\mathcal{A}}^0_{\underline{\mathbb{T}}_M} \circ i^*[\cdot]). \tag{.1.51}$$

Recall the construction of the functor $D : \mathcal{L}_{\mathcal{G}}(\Phi) \to \mathcal{L}_{\mathcal{G}}(\xi)$ in Definition .1.24: for each global section $S \in \mathcal{L}_{\mathcal{G}}(\Phi)$ of $\delta\mathcal{G}$, we constructed the corresponding infinitesimal lift $F_{\hat{\xi}} = D(S)$ as a composition of 1-morphisms

$$\mathcal{G}^{op} \xrightarrow{\Phi_1} \nu_*(\delta\mathcal{G}) \xrightarrow{\Phi_2} \nu_*\mathrm{B}(\underline{\mathbb{T}}_{M^1}) \xrightarrow{\Phi_3} \nu_*\mathrm{B}(\underline{i\mathbb{R}}_{M^1}) \xrightarrow{\Phi_4} \mathrm{B}(\underline{i\mathbb{R}}_M). \tag{.1.52}$$

Now, given $(S, \mu_S) \in \mathcal{L}_{(\mathcal{G},\mathcal{A})}(\Phi)$, we wish to construct a 2-morphism

$$\Theta_{\hat{\xi}} : -\pounds_{\xi}[\tilde{\mathcal{A}}] \to \tilde{\mathcal{A}}^0_{\underline{i\mathbb{R}}_M} \circ F_{\hat{\xi}} \tag{.1.53}$$

To do so, consider the diagram

$$\begin{array}{ccc}
\mathcal{G}^{op} & \xrightarrow{\tilde{\mathcal{A}}} & \mathrm{B}(i\Omega^1_M) \\
{\scriptstyle\Phi_1}\downarrow & \Searrow & \downarrow{\scriptstyle\delta_{i\Omega^1}} \\
\nu_*(\delta\mathcal{G}^{op}) & \xrightarrow{\nu_*\delta\tilde{\mathcal{A}}} & \nu_*\mathrm{B}(i\Omega^1_{M^1,rel}) \\
{\scriptstyle\Phi_2}\downarrow & \Searrow & \downarrow{\scriptstyle id} \\
\nu_*\mathrm{B}(\underline{\mathbb{T}}_{M^1}) & \xrightarrow{\nu_*\tilde{\mathcal{A}}^0_{\underline{\mathbb{T}}_{M^1},rel}} & \nu_*\mathrm{B}(i\Omega^1_{M^1,rel}) \\
{\scriptstyle\Phi_3}\downarrow & \Searrow & \downarrow{\scriptstyle\pounds_{\frac{d}{dt}}[\cdot]} \\
\nu_*\mathrm{B}(\underline{i\mathbb{R}}_{M^1}) & \xrightarrow{\nu_*\tilde{\mathcal{A}}^0_{\underline{i\mathbb{R}}_{M^1},rel}} & \nu_*\mathrm{B}(i\Omega^1_{M^1,rel}) \\
{\scriptstyle\Phi_4}\downarrow & \Searrow & \downarrow{\scriptstyle i^*[\cdot]} \\
\mathrm{B}(\underline{i\mathbb{R}}_M) & \xrightarrow{\tilde{\mathcal{A}}^0_{\underline{i\mathbb{R}}_M}} & B(i\Omega^1_M).
\end{array} \tag{.1.54}$$

In this diagram double arrows denote the 2-morphisms from Lemma .1.40 and Lemma .1.48. The counter-clockwise composition of the outer arrows from $\mathcal{G}^{op}$ to $B(\underline{i\Omega}^1_M)$ is by definition

$$\tilde{\mathcal{A}}^0_{\underline{i\mathbb{R}}_M} \circ F_{\hat{\xi}}. \tag{.1.55}$$

On the other-hand, consider the vertical composition of 1-morphisms from $B(\underline{i\Omega}^1_M)$ in the upper right-hand corner to $B(\underline{i\Omega}^1_M)$ in the lower right-hand corner, which we call Ψ. By composing 2-morphisms we obtain a 2-morphism

$$\Psi \circ \tilde{\mathcal{A}} \overset{\cong}{\Longrightarrow} \tilde{\mathcal{A}}^0_{\underline{i\mathbb{R}}_M} \circ F_{\hat{\xi}}. \tag{.1.56}$$

Note that Ψ intertwines

$$-i^* \circ \pounds_{\frac{d}{dt}} \circ \delta_{i\Omega^1} : i\Omega^1_M \to i\Omega^1_M. \tag{.1.57}$$

By definition, this takes a 1-form α on an open set $U \subset M$ to

$$-(\pounds_{\frac{d}{dt}}(p_0^*\alpha - p_1^*\alpha))|_U = -\pounds_\xi \alpha. \tag{.1.58}$$

Therefore, by Proposition 2.3.17 we have a canonical 2-morphism from

$$-\pounds_\xi[\cdot] \overset{\cong}{\Longrightarrow} \Psi. \tag{.1.59}$$

Combining this 2-morphism with the 2-morphism (.1.56), we obtain the desired 2-morphism

$$-\pounds_\xi[\mathcal{A}] \overset{\cong}{\Longrightarrow} \mathcal{A}^0_{\underline{i\mathbb{R}}_M} \circ F_{\hat{\xi}}. \tag{.1.60}$$

If we denote this 2-morphism by $\Theta_{\check{\xi}}$, it is straightforward to check that the assignment $D : (S, \mu_S) \mapsto (F_{\hat{\xi}}, \Theta_{\check{\xi}})$ is functorial.

.2

In order to make this paper self-contained, in the first of this appendix we collect necessary definitions involving stacks and gerbes. We then use these definitions to prove Theorems 3.3.18 and 4.2.25.

.2.1 Gerbes as sheaves of groupoids

Definition .2.1. *Let M be a smooth manifold. A* presheaf of groupoids $\mathcal{G}$ *over M consists of the following.*

1. *For every open set $U \subset M$, a groupoid $\mathcal{G}(U)$.*
2. *For every inclusion of open sets $i : V \to U$, a functor $i^* : \mathcal{G}(U) \to \mathcal{G}(V)$.*
3. *For every sequence of inclusions of open sets*

$$W \xrightarrow{j} V \xrightarrow{i} U \tag{.2.2}$$

a natural transformation $\alpha_{i,j} : j^* i^* \Rightarrow (ij)^*$, such that for every sequence of inclusions

$$T \xrightarrow{k} W \xrightarrow{j} V \xrightarrow{i} U \tag{.2.3}$$

the diagram

$$\begin{array}{ccc} k^* j^* i^* & \xRightarrow{1_{k^*} * \alpha_{i,j}} & k^*(ij)^* \\ {\scriptstyle \alpha_{j,k} * 1_{i^*}} \Downarrow & & \Downarrow {\scriptstyle 1_{k^*} \alpha_{ij,k}} \\ (jk)^* i^* & \xRightarrow[\alpha_{i,jk}]{} & (ijk)^* \end{array} \tag{.2.4}$$

commutes.

Notation .2.5. The notation used in diagram (.2.4) is the following: given categories A, B, C, functors $F, G : A \to B$, $H, I : B \to C$ and natural transformations $\alpha : F \Rightarrow G$, $\beta : H \Rightarrow I$, we denote by $\beta * \alpha : HF \Rightarrow IG$ the horizontal composition of β with α.

Notation .2.6. When we wish to be more explicit, we will sometimes label the functors i^* and the natural transformations $\alpha_{i,j}$ by $\mathcal{G}$; thus a presheaf $\mathcal{G}$ consists of a triple $\{\{\mathcal{G}(U)\}, \{i^*_{\mathcal{G}}\}, \{\alpha_{\mathcal{G},i,j}\}\}$ subject to the conditions given in (.2.1).

Given an open subset $U \subset M$ and objects $P, Q \in \mathcal{G}(U)$, it is an easy consequence of definition (.2.1) that there is a presheaf (of sets) $\underline{\mathrm{Hom}}(P, Q)$ over U which assigns to each inclusion $i : V \hookrightarrow U$ the set $\underline{\mathrm{Hom}}(i^*P, i^*Q)$. The presheaf $\mathcal{G}$ of groupoids is said to be a *prestack* if $\underline{\mathrm{Hom}}(P, Q)$ is a sheaf for all open sets U and all objects P and Q. We are interested in prestacks that satisfy an additional gluing property. Let $\{U_i\}$ be an open cover of an open subset $V \subset M$, and denote the n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$ by $U_{i_1 \cdots i_n}$. Let $\{Q_i\} \in \mathcal{G}(U_i)$ and $s_{ij} : Q_i|_{U_{ij}} \to Q|_{U_{ij}}$. Note that on triple intersection U_{ijk}, we have two morphisms from $Q_i|_{U_{ijk}} \to Q_j|_{U_{ijk}}$. On the one hand, we can compose

$$Q_i|_{U_{ijk}} \xrightarrow{\cong} (Q_i|_{U_{ik}})|_{U_{ijk}} \xrightarrow{s_{ik}|_{U_{ijk}}} (Q_k|_{U_{ik}})|_{U_{ijk}} \xrightarrow{\cong} Q_k|_{U_{ijk}}, \qquad (.2.7)$$

where the first an last isomorphisms are constructed using the natural transformations α described in definition (.2.1). Similarly, we can first use s_{ij} to obtain a morphism from $Q_i|_{U_{ijk}}$ to $Q_j|_{U_{ijk}}$, and then compose with the morphism from $Q_j|_{U_{ijk}}$ to $Q_k|_{U_{ijk}}$ determined by s_{jk}. We say that the collection $\{Q_i\}$ $\{s_{ij}\}$ satisfies the *descent condition* if these two morphisms are equal for each i, j, k. From now on, for notational simplicity we suppress the restriction maps and natural transformations and simply write the descent condition as

$$s_{jk} \circ s_{ij} = s_{ik}. \tag{.2.8}$$

Similarly, given another collection $\{\{Q'_i\}\{s'_{ij}\}\}$ satisfying the descent condition, we say that a collection of morphisms $\{\psi_i : Q_i \to Q'_i\}$ satisfy the descent condition if we have

$$\psi_j s_{ij} = s'_{ij}\psi_i \tag{.2.9}$$

on overlaps. Overall we obtain a *descent category* $\mathrm{Desc}(\mathcal{G}, \{U_i\})$. By construction, there is a restriction functor from $\mathcal{G}(U) \to \mathrm{Desc}(\mathcal{G}, \{U_i\})$, defined using the structure described in Definition .2.1.

Definition .2.10. *A prestack $\mathcal{G}$ is a* stack *if for each open set $V \subset M$ and each open cover $\{U_i\}$ of V, the restriction functor $\mathcal{G}(V) \to Desc(\mathcal{G}, \{U_i\})$ is an equivalence of categories.*

Definition .2.11. *Let $\mathcal{G}$ and $\mathcal{G}'$ be presheaves of groupoids over M. A* 1-morphism $\Phi : \mathcal{G} \to \mathcal{G}'$ *consists of the following.*

1. *For every open set $U \subset M$, a functor $\Phi_U : \mathcal{G}(U) \to \mathcal{G}'(U)$.*

2. *For every inclusion of open sets $i : V \to U$, a natural transformation*

$$\Phi_i : i^*_{\mathcal{G}'}\Phi_U \Rightarrow \Phi_V i^*_{\mathcal{G}}, \tag{.2.12}$$

such that for every sequence of inclusions

$$W \xrightarrow{j} V \xrightarrow{i} U \tag{.2.13}$$

the diagram

$$\begin{array}{ccccc} j^*_{\mathcal{G}'} i^*_{\mathcal{G}'}\Phi_U & \xRightarrow{1_{j^*_{\mathcal{G}'}}*\Phi_i} & j^*_{\mathcal{G}'}\Phi_V i^*_{\mathcal{G}} & \xRightarrow{\Phi_j * 1_{i^*_{\mathcal{G}}}} & \Phi_W j^*_{\mathcal{G}} i^*_{\mathcal{G}} \\ {\scriptstyle \alpha_{\mathcal{G}',i,j}*1_{\Phi_U}}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle 1_{\Phi_W}*\alpha_{\mathcal{G},i,j}} \\ (ij)^*_{\mathcal{G}'}\Phi_U & & \xRightarrow[\Phi_{ij}]{\qquad\qquad} & & \Phi_W(ij)^*_{\mathcal{G}} \end{array} \tag{.2.14}$$

commutes.

Remark .2.15. In the language of 2-categories, Definition .2.11 says that a 1-morphism from $\mathcal{G}$ to $\mathcal{G}'$ is a pseudo-natural transformation between the pseudo-functors corresponding to $\mathcal{G}$ and $\mathcal{G}'$(again, see [Bo]).

Definition .2.16. *Let $\mathcal{G}$, $\mathcal{G}'$ be presheaves of groupoids over M, and let $\Phi, \Psi : \mathcal{G} \to \mathcal{G}'$ be 1-morphisms. A* 2-morphism $\tau : \Phi \Rightarrow \Psi$ *is a collection of natural transformations $\tau(U) : \Phi_U \Rightarrow \Psi_U$ for every open set $U \subset M$, such that for each inclusion of open sets $i : V \hookrightarrow U$ the diagram*

$$\begin{array}{ccc} i^*_{\mathcal{G}'}\Phi_U & \xRightarrow{\Phi_i} & \Phi_V i^*_{\mathcal{G}} \\ {\scriptstyle 1_{i^*_{\mathcal{G}'}}*\tau(U)}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \tau(V)*1_{i^*_{\mathcal{G}}}} \\ i^*_{\mathcal{G}'}\Psi_U & \xRightarrow[\Psi_i]{} & \Psi_V i^*_{\mathcal{G}} \end{array} \tag{.2.17}$$

commutes.

Remark .2.18. Again in the language of 2-categories, a 2-morphism from Φ to Ψ is a modification of pseudo-natural transformations.[Bo]

.2.2 Proofs of Theorems 3.3.18 and 4.2.25.

Proof of Theorem 3.3.18: It will be more convenient to construct equivalences

$$\Gamma : \mathcal{L}_{g_{ijk}}(\xi) \to \tilde{\mathcal{L}}_{\mathcal{G}}(\xi) \tag{.2.19}$$

for each vector field ξ, where we recall from Appendix A that

$$\tilde{\mathcal{L}}_{\mathcal{G}}(\xi) = \underline{\mathrm{Hom}}_{\iota_\xi d\log}(\mathcal{G}^{op}, \mathrm{B}(\underline{i\mathbb{R}}_M)) \tag{.2.20}$$

is the category of contravariant functors from $\mathcal{G}$ to $\mathrm{B}(\underline{i\mathbb{R}}_M)$ intertwining $\iota_\xi d\log$. We may then use the canonical equivalence between $\tilde{\mathcal{L}}_{\mathcal{G}}(\xi)$ and $\mathcal{L}_{\mathcal{G}}(\xi)$.

Given an object $\{f_{ij}^{\hat{\xi}}\} \in \mathcal{L}_{g_{ijk}}(\xi)$, we begin by first constructing an object $F_{\hat{\xi}} = \Gamma(\{f_{ij}^{\check{\xi}}\}) \in \tilde{\mathcal{L}}_{\mathcal{G}}(\xi)$. Given an open set $V \subset M$ and $P \in \mathcal{G}(V)$, let $V_i = V \cap U_i$, $P_i = P|_{V_i}$, $E_i(P) = \underline{\mathrm{Hom}}(P_i, Q_i|_{V_i})$, and

$$F_i(P) = \iota_\xi d\log[E_i(P)]. \tag{.2.21}$$

Using the natural isomorphism $P_i|_{V_i\cap V_j} \xrightarrow{\cong} P_j|_{V_i\cap V_j}$ and the morphisms $s_{ij}: Q_i \to Q_j$ we obtain isomorphisms[6]

$$(s_{ij})_* : E_i(P) \to E_j(P). \tag{.2.22}$$

If we then define

$$\zeta_{ij} = \iota_\xi d\log[(s_{ij})_*] - f_{ij}^{\hat{\xi}} : F_i(P) \to F_j(P), \tag{.2.23}$$

it follows from (3.3.4) and (3.3.9) that the the morphisms $\{\zeta_{ij}\}$ satisfy the descent condition, and we therefore define $F_{\hat{\xi}}(P) \in \mathbf{Tor}_{\underline{i\mathbb{R}}_V}$ to be the torsor obtained by gluing. Similarly, given a morphism $\psi : P \to P'$ in $\mathcal{G}(U)$, the morphisms

$$\iota_\xi d\log[\psi^*] : F_i(P') \to F_i(P) \tag{.2.24}$$

[6]for notational simplicity, from now on we will supress writing restriction maps explicitly.

glue to give a morphism $F_{\hat{\xi}}(\psi) : F_{\hat{\xi}}(P') \to F_{\hat{\xi}}(P)$, and for each $\mathbb{T}$-valued function g we have $F_{\hat{\xi}}(\psi \cdot g) = F_{\hat{\xi}}(\psi) + \iota_\xi d\log(g)$. It is then straightforward to construct the restriction natural isomorphisms $F_{\hat{\xi},i}$ described as part of Definition .2.11. Thus, given $\{f_{ij}^{\hat{\xi}}\} \in \mathcal{L}_{g_{ijk}}(\xi)$, we have produced a lift $F_{\hat{\xi}} = \Gamma(\{f_{ij}^{\hat{\xi}}\}) \in \tilde{\mathcal{L}}_{\mathcal{G}}(\xi)$.

Next, given an isomorphism $\{u_i\}$ from $\{f_{ij}^{\hat{\xi}}\}$ to $\{f_{ij}^{\hat{\xi}'}\}$ in $\mathcal{L}_{g_{ijk}}$, we wish to produce an isomorphism $\Gamma(\{u_i\}) : F_{\hat{\xi}} \xrightarrow{\cong} F_{\hat{\xi}'}$. For each i, let $(\tau_P)_i$ be the automorphism of the $\underline{i\mathbb{R}}_{V_i}$-torsor $\iota_\xi d\log[E_i(P)]$ corresponding to the function $u_i|_{V_i}$. It then follows from equation (3.3.11) that the diagram

$$\begin{array}{ccc} \iota_\xi d\log[E_i(P)] & \xrightarrow{\zeta_{ij}} & \iota_\xi d\log[E_j(P)] \\ \downarrow{\scriptstyle (\tau_P)_i} & & \downarrow{\scriptstyle (\tau_P)_j} \\ \iota_\xi d\log[E_i(P)] & \xrightarrow[\zeta'_{ij}]{} & \iota_\xi d\log[E_j(P)] \end{array} \tag{.2.25}$$

commutes, so that the morphisms $(\tau_P)_i$ satisfy the descent condition (.2.8) and thus glue to define an isomorphism $(\tau)_P : F_{\hat{\xi}}(P) \xrightarrow{\cong} F_{\hat{\xi}'}(P)$. It is easily checked that this isomorphism is natural in P, so that the condition (.2.17) is satisfied, and that Γ is functorial.

To verify that Γ is an equivalence of categories, we must check that it is both essentially surjective and fully faithful. Given $F_{\hat{\xi}} \in \mathcal{L}_{\mathcal{G}}(\xi)$, choose local trivializations $\{r_i\}$ with corresponding Cech data idefined by

$$F_{\hat{\xi}}(s_{ij})(r_j) = r_i + f_{ij}^{\hat{\xi}}. \tag{.2.26}$$

Let $F'_{\hat{\xi}} = \Gamma(\{f_{ij}^{\hat{\xi}}\})$. For each object $P \in \mathcal{G}(U)$, we wish to define isomorphisms

$$\tau_{P,i} : F_{\hat{\xi}}(P) \to F_i(P) \tag{.2.27}$$

such that on overlaps U_{ij} we have

$$\begin{array}{ccc} F_{\hat{\xi}}(P) & \xrightarrow{\tau_{P,i}} & F_i(P) \\ & \searrow^{\tau_{P,j}} & \downarrow \zeta_{ij} \\ & & F_j(P). \end{array} \tag{.2.28}$$

To do so, for each i choose $\psi_i : P \to Q_i$, and let $\tau_{P,i}$ be the unique morphism

$$F_{\hat{\xi}}(P) \to F_i(P) = \iota_\xi d \log[\underline{\mathrm{Hom}}(P, Q_i)] \tag{.2.29}$$

taking

$$F_{\hat{\xi}}(\psi)(r_i) \mapsto \iota_\xi d \log[\psi]. \tag{.2.30}$$

To see that this is independent of the choice of ψ, a simple computation shows that for each $\mathbb{T}$-valued function h we have

$$\tau_{P,i}(F_{\hat{\xi}}(\psi \cdot h)(r_i)) = \iota_\xi d \log[\psi \cdot h]. \tag{.2.31}$$

Similarly, it is easily checked that diagram (.2.28) commutes. This shows essentially surjectivity. Since any two objects in $\mathcal{L}_{g_{ijk}}$ are isomorphic, to show that Γ is fully faithful it is sufficient to check that for every object $\{f_{ij}^{\hat{\xi}}\} \in \mathcal{L}_{g_{ijk}}$ that Γ induces a bijection from $\mathrm{Aut}(\{f_{ij}^{\hat{\xi}}\})$ to $\mathrm{Aut}(\Gamma(\{f_{ij}^{\hat{\xi}}\})$. Let τ be an automorphism of $F_{\hat{\xi}} = \Gamma(\{f_{ij}^{\hat{\xi}}\})$. Then for each object $P \in \mathcal{G}(U)$, τ_P corresponds to a function $f_P : U \to i\mathbb{R}$ via the isomorphism α_P in definition (2.3.2). Furthermore, given another object $Q \in \mathcal{G}(V)$, Definition .2.11 implies that f_P and f_Q agree on their common domain, and thus we obtain a global function $f_\tau : M \to i\mathbb{R}$; conversely, given such a function, we can construct an automorphism of $F_{\hat{\xi}}$. Finally, by construction it is clear that $\Gamma : \mathrm{Aut}(\{f_{ij}^{\hat{\xi}}\}) \to$

$\mathrm{Aut}(F_{\hat{\xi}})$ is consistent with the identification of both groups with $C_M^\infty(i\mathbb{R})$, and is therefore a bijection.

Proof of Theorem 4.2.25: We will use the same notation as that in the proof of (3.3.18). Given $(\xi\{f_{ij}^{\check{\xi}}\}, \{a_i\}) \in \mathcal{L}_{(g_{ijk}, A_{ij})}$, let $F_{\check{\xi}} = \Gamma((\xi, \{f_{ij}^{\check{\xi}}\})$ be the (non-connective lift) constructed in the proof of (3.3.18)) from $(\xi, \{f_{ij}^{\check{\xi}}\})$ In order to extend $F_{\check{\xi}}$ to a connective lift, for each $P \in \mathcal{G}(V)$ and each $\mu \in \mathcal{A}(P)$, we must construct a connection $\Theta_{\hat{\xi}}(\mu)$ on $F_{\check{\xi}}(P)$. Recall that $F_{\check{\xi}}(P)$ is constructed by gluing the $\underline{i\mathbb{R}}_{V_i}$-torsors

$$F_i(P) = \iota_\xi d\log[\underline{\mathrm{Hom}}(P, Q_i)] \tag{.2.32}$$

via maps $\zeta_{ij} : F_i(P) \to F_j(P)$. Thus to specify a connection on $F_{\hat{\xi}}(P)$, we must specify connections ν_i on $F_i(P)$ such that on overlaps we have $\nu_j = (\zeta_{ij})_* \nu_i$. Using lemma (.1.48), we obtain a connection $\mu_i - \mu_P$ on $\underline{\mathrm{Hom}}(Q_i, P)$, and by Lemma 4.1.10 we may define a connection ν_i on $F_i(P)$ as

$$\nu_i = \pounds_\xi(\mu_i - \mu_P) + a_i. \tag{.2.33}$$

We then have

$$\begin{aligned}
(\zeta_{ij})_*(\nu_i) = &(\iota_\xi d\log[(s_{ij})_* - f_{ij}^{\check{\xi}})_*(\pounds_\xi[\mu_i - \mu_P] + a_i) \qquad (.2.34)\\
=&\mathcal{A}^0_{\underline{i\mathbb{R}}}(\iota_\xi d\log[(s_{ij})_*])\pounds_\xi(\mu_i - \mu_P) + df_{ij}^{\check{\xi}} + a_i\\
=&\pounds_\xi((s_{ij})_*(\mu_i) - \mu_P) + df_{ij}^{\check{\xi}} + a_i\\
=&\pounds_\xi[\mu_j - A_{ij} - \mu_P] + df_{ij}^{\check{\xi}} + a_i\\
=&\pounds_\xi(\mu_j - \mu_P) + a_j - (a_j - a_i) + df_{ij}^{\check{\xi}} - \pounds_\xi A_{ij}\\
=&\nu_j.
\end{aligned}$$

We then let $\Theta_{\check{\xi}}(\mu)$ be the connection obtained by gluing; by construction we have $\Theta_{\check{\xi}}(\mu + \alpha) = \Theta_{\check{\xi}}(\mu) - \pounds_{\xi}\alpha$. It is straightforward to check that $\Theta_{\check{\xi}}$ is compatible with restrictions and is suitably natural in P.

Next, suppose are given an isomorphism $\{u_i\}$ from $(\{f_{ij}^{\check{\xi}}\}, \{a_i^{\check{\xi}}\})$ to $(\{f_{ij}^{\check{\xi}'}\}, \{a_i^{\check{\xi}'}\})$. Recall from the proof of (3.3.18) that we obtain a morphism $\Gamma(\{u_i\}) : F_{\hat{\xi}}(P) \to F_{\hat{\xi}'}(P)$ by letting each u_i act as an automorphism of $F_i(P)$. The condition $a_i^{\check{\xi}'} = a_i^{\check{\xi}} + du_i$ implies that $(u_i)_*\nu_i = \nu_i'$ for each i. Therefore $\Gamma(\{u_i\})_*\Theta_{\check{\xi}}(\mu) = \Theta_{\check{\xi}'}(\mu)$, so that $\Gamma(\{u_i\})$ is compatible with the connections and thus defines a connective equivalence. Mimicking the arguments used in the proof of Theorem 3.3.18 above, it is then easily checked that Γ is both essentially surjective and fully faithful.

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