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Theory and Application of Chiral Anomalies in Quantum Field Theory

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Theory and Application of Chiral Anomalies in Quantum Field Theory

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"What one fool can do, another can."

R.P. Feynman

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Abstract

Theory and Application of Chiral Anomalies in Quantum Field Theory

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In this thesis we investigate both the theoretical mechanism by which chiral anomalies arise in quantum field theories, as well as their implications for physical observables. In Chapter 1, we set the stage by providing a brief historical account of the so-called PCAC (partially conserved axial current) puzzle and the theoretical work which spawned from the observed anomalous decay of the π^0 to two photons. Chapter 2 is dedicated to developing the necessary formalism and theoretical tools– namely the effects of symmetry transformations and their associated Noether currents. The path integral formalism is introduced and the $U(1)_A$ anomaly is derived in detail by studying the noninvariance of the measure. In Chapter 3, we introduce the effective chiral Lagrangian of QCD to investigate the dynamics of the meson nonet. We apply our results from Chapter 2 in calculating the width $\Gamma_{\eta' \to \gamma\gamma}$, and consequently the lifetime $\tau_{\eta' \to \gamma\gamma}$ to leading order. We report that our derived expressions yield numerical values of $\Gamma_{\eta' \to \gamma\gamma} = (0.0037 \pm 0.0003)\,\text{MeV}$ and $\tau_{\eta' \to \gamma\gamma} = (1.78 \pm 0.03) \times 10^{-19}\,\text{s}$. Our results lie within 2σ of the experimental values reported by the Particle Data Group[11].

Contents

1 Introduction

To many of its students, the beauty of particle physics lies in its simplicity. In a rudimentary way, its mission is indeed simple: Identify the indivisible constituents of the various types of matter of the universe, and construct the mathematics necessary to describe their interactions. The fulfillment of even a small portion of this mission in practice, however, can be enormously complex. Consider for example, a proton-proton beam collision such as those procured in the Large Hadron Collider (LHC) at CERN. At the high energies involved in an initial pp collision, the running QCD coupling constant α_S resides in the regime of asymptotic freedom in which inter-particle interactions may be described by the tried-and-true methods of analytic perturbation theory. At energies below the confinement scale of Λ_{QCD}, however, the hadronization of ancillary particle showers requires an entirely different phenomenological and computational toolkit for analysis. The rich proliferation of vastly dissimilar dynamics at differing energy scales is a direct consequence of the seemingly elementary SU(3) structure of quantum chromodynamics.

In high energy collider experiments, fundamental particle properties such as the familiar mass, charge, and spin are often most sought after in encountering a new resonance. The overwhelming majority of particles, both composite and fundamental, however, are unstable, thus eventually decaying to some combination of lighter, stable particles. The lifetime τ of a particle then becomes of central importance in understanding not just the average time it takes for a single particle to decay, but also in informing theorists of the underlying *mechanism* by which the decay may take place.

Determining the lifetime of a particle from a theoretical standpoint usually amounts to identifying which fundamental interactions are allowed, constrained by the symmetries of the theory. For example, particles such as the ρ meson may decay to two pions via the strong interaction rather quickly – on the order of 10^{-24} s – due to the large strength of the QCD coupling constant α_S. In contrast, the charged kaons ($K^{\pm}$) decay predominantly to pairs of muons and neutrinos via the electroweak interaction, with lifetimes on the order of 10^{-8} s. The large masses of the weak gauge bosons are to blame here. Internuclear processes such as beta decay are governed entirely by the weak nuclear force, with characteristic timescales of around 10^{-3} s for short-lived isotopes and upwards of billions of years for the most stable.

As the roster of known particles increased steadily through the mid-20th century, a selection of puzzling composite particles presented new challenges in understanding underlying decay mechanisms. Perhaps most infamous was the puzzle of the neutral pion decay, denoted π^0. While the neutral pion's charged cousins, the π^+ and π^-, are able to decay to a muon and neutrino via the electroweak force, the π^0 was found to decay predominately to two photons. Primitive iterations of quantum field theory were not able to explain why this process should be allowed at all, as it explicitly violates parity. The so-called "PCAC puzzle" (partially conserved axial current) was not fully resolved and understood in terms of the axial anomaly until 1969 through the seminal work of Adler [1], Bell, and Jackiw [3].

Discrepancies between theory and experiment such as the neutral pion decay rate hinted not just toward a lack of computational tools, but a need for a new theoretical framework to explain why some symmetries may be conserved classically, but simultaneously violated by quantum corrections. Specifically, the path integral formulation of quantum field theory provides us with a systematic method for calculating anomalies through their effect on the path integral measure, despite leaving the classical action invariant. This non-perturbative approach due to Fujikawa [6] is invaluable as it allows us to calculate the effects of the anomaly at all energy scales.

As quantum field theory has matured, it has become apparent that symmetries are paramount in understanding fundamental physics. While conservation laws are inviolable in classical mechanics, it turns out that some symmetries of the classical action are in fact *not* realized in the quantum theory. For this reason, symmetries that are not respected by the quantum theory are considered to be *anomalous*.

Writing down a bare Lagrangian is not enough to fully specify a quantum field theory—one needs to also specify over what range of energies the theory holds. In practice, this involves a choice of regularization; in effect, a prescription for how to identify and extract the infinities that materialize in the UV. A good theory, however, should not make different predictions depending on which physicist decides to use which prescription. This is the driving idea behind effective field theory: There exist fundamental conclusions about physical observables that do not require a full UV completion of the theory at all. Anomalous symmetries are an example of these non-perturbative phenomena. Not only is the choice of regularization immaterial, there in fact exists no choice of regulator in such a quantum field theory at all that can realize an anomalously broken classical symmetry. Consequently, if an anomaly can be identified, an effective field theory can be used perfectly well to make physical predictions. A concrete example of this is the process

$$\eta' \longrightarrow \gamma\gamma, \tag{1.1}$$

which proceeds via an axial anomaly, but has a decay width that is calculable to high precision from an effective chiral Lagrangian. The core of this thesis is devoted to delving into the underlying mechanisms that make this possible.

In the sections that follow, we will investigate how chiral anomalies emerge in quantum field theory and examine their implications for physical observables, namely the lifetime of the η' meson. In Chapter 2, we will develop the path integral formalism and derive an explicit result relating the divergence of the axial current to the $U(1)_A$ anomaly. In Chapter 3, we will apply our results to the specific case of the effective chiral Lagrangian, which describes the dynamics of mesons at low energies. In introducing the η' as the would-be Goldstone boson associated with the anomalously broken $U(1)_A$ symmetry, we will then calculate the decay rate for the radiative decay $\eta' \to \gamma\gamma$, and in turn the lifetime of the η'.

2 Theory

In this section, we will develop the foundational tools necessary to understand the physical implications of anomalies as they arise in quantum field theories. An anomalous symmetry is one that is classically conserved, but is not respected by the quantum field theory. To understand how this works in practice, we first introduce the concept of classical action in Section 2.1 and show how continuous symmetries give rise to conserved Noether currents in Section 2.2. We then begin our investigation of symmetries in a quantum framework via the path integral formalism in Section 2.3, with an introduction to anomalous symmetries in Section 2.4. Finally, in Section 2.5, we derive the U(1) axial anomaly explicitly by enacting an axial transformation of the fields and following effects of the transformation on the action and the path integral measure. The noninvariance of the measure is shown to be directly responsible for the appearance of the anomaly. Our results from Section 2.5 will give us an operator relation that we will require in Chapter 3 in deriving the width for decay of the η' to two photons.

2.1 The Classical Action

The principle of least action has been invoked in interpreting physical occurrences as far back as the 1600's. Seemingly disparate phenomena such as the path of a ray of light or the motion of celestial bodies were eventually understood by Fermat and Maupertuis, respectively, to be direct consequences of Nature's desire to minimize a particular quantity, namely the action.

In classical mechanics' modern incarnation, due largely to the work of Lagrange and Hamilton in the 18th and 19th centuries, the action S is defined to be the time integral of the difference between the kinetic and potential energies. This difference is of course the Lagrange function, L. If the dynamical variables of the system are the spatial coordinates $\mathbf{x}$, then in symbols we have

$$S[\mathbf{x}(t)] = \int \mathrm{d}t\, L[\mathbf{x}(t), \dot{\mathbf{x}}(t); t]. \tag{2.1}$$

In moving from particle mechanics to field theory, the concept of action remains central. The dynamical variables, however, are different. The spacetime coordinates x now play the role of identifying labels rather than variables, with the fields themselves treated as the dynamical variables of the system. The fields —call them $\phi(x)$— extend infinitely in space and thus inspire us to instead work with the Lagrangian *density* $\mathcal{L}$:

$$S[\phi] = \int \mathrm{d}^4 x \mathcal{L}[\phi, \partial_\mu \phi]. \tag{2.2}$$

Because the determinant of any Lorentz transformation has modulus 1[1], the measure is unaffected by such transformations and thus the action inherits the Lorentz invariance of the Lagrangian. The classical equations of motion can be extracted by the usual method of taking the variational derivative such that $\delta S = 0$ to leading order. We will find the Lagrangian formulation especially useful in what follows in investigating symmetries in general.

2.2 Symmetries and Currents

Constructing the bare Lagrangian for a theory essentially amounts to writing down all possible interactions between the fields involved. In practice, the job is made much easier when considering the symmetries of the theory at hand, as well as by taking account of the mass dimension of each term. Symmetries constrain the allowed interaction terms between fields in the Lagrangian. Because the dynamics are determined by the Lagrangian, knowledge of which symmetries are respected by a theory and which are not can be exploited when calculating physical observables such as the expected rates of various processes.

Noether's theorem[8] tells us that each continuous symmetry of the Lagrangian is associated with a conservation law. More specifically, if there exists a transformation depending on a continuous parameter that leaves the action invariant to leading order, then there also exists an associated conserved current. Take, for example, a single field configuration $\phi(x)$ that is subject to a transformation such that

$$\phi \longrightarrow \phi + \delta\phi\,. \tag{2.3}$$

The variation $\delta\phi$ is assumed to be small. In practice, it may depend on a continuous parameter α in the case of a global symmetry, or it may be a function of spacetime $\alpha(x)$ in the case of a local or gauge transformation. Here, we follow the derivation outlined by

[1] *I.e.*, $\det\{\Lambda\} = \pm 1$.

Schwartz[10]. Varying $\mathcal{L}[\phi]$ directly with respect to α gives

$$\frac{\delta\mathcal{L}}{\delta\alpha} = \frac{\partial\mathcal{L}}{\partial\phi}\frac{\delta\phi}{\delta\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta(\partial_\mu\phi)}{\delta\alpha} \tag{2.4}$$

$$= \frac{\partial\mathcal{L}}{\partial\phi}\frac{\delta\phi}{\delta\alpha} + \partial_\mu\left[\frac{\delta\phi}{\delta\alpha}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\delta\phi}{\delta\alpha}\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] \tag{2.5}$$

$$= \frac{\delta\phi}{\delta\alpha}\left\{\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\right\} + \partial_\mu\left[\frac{\delta\phi}{\delta\alpha}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]. \tag{2.6}$$

So long as ϕ is on shell, the difference in the left brackets vanishes via the Euler-Lagrange equations. The term that remains is a total divergence of what is called a Noether current, denoted $\mathcal{J}^\mu$. For a theory with multiple fields, the definition generalizes as

$$\mathcal{J}^\mu = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta\phi_n}{\delta\alpha}. \tag{2.7}$$

As a simple application, consider the Lagrangian for a complex scalar field ϕ with mass m:

$$\mathcal{L} = |\partial_\mu\phi|^2 - m^2|\phi|^2. \tag{2.8}$$

Treating the field ϕ and its conjugate ϕ^* as the independent degrees of freedom, we see that the Lagrangian is invariant under the transformation $\phi \to \exp\{-i\alpha\}\phi$, and similarly for ϕ^*. By direct computation, we find via Equation 2.7 that the associated Noether current is

$$\mathcal{J}^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\frac{\delta\phi^*}{\delta\alpha} \tag{2.9}$$

$$= -i\left(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right). \tag{2.10}$$

By utilizing the fact that free scalar fields obey the Klein-Gordon equations of motion $(\Box + m^2)\phi = 0$, we see that taking the divergence of equation 2.10 yields $\partial_\mu\mathcal{J}^\mu = 0$, as expected.

2.3 The Path Integral

Although first formally introduced by Feynman [5] in his PhD thesis in 1942, the path integral formulation of quantum mechanics was not fully appreciated until other early techniques in quantum field theory had either proved intractable or had failed outright in describing anomalous processes and symmetry breaking phenomena in the 1950s and onward. While canonical second quantization makes the unitarity of the S matrix readily apparent in a given process, the path integral, as we will see, is manifestly Lorentz covariant. This, as well as the explicit inclusion of the integral measure, makes the path integral formalism an invaluable framework for studying anomalies in quantum field theories.

From a practical standpoint, the path integral approach is straightforward and builds directly on the foundational concepts from classical field theory we have discussed so far. The two main ingredients are the classical action and the path integral measure.

In a (0+1) dimensional QFT —*i.e.* quantum mechanics— we can interpret the probability amplitude for a particle to propagate from one point in spacetime to another as a sum over all possible paths, with each path assigned a phase proportional to it's classical action. In field theory with (3+1) dimensions, we generalize by instead summing over all possible field configurations, with the product of all differential field configurations denoted $\mathcal{D}\phi$. Putting it all together, we have

$$Z[\phi] = \int \mathcal{D}\phi \, e^{iS[\phi]/\hbar}. \tag{2.11}$$

Despite the appearance of the classical action in the exponential, the inclusion of $\hbar$ and the dependence of the measure on the field configurations make $Z[\phi]$ a decidedly quantum object.

2.3.1 N-Point Functions from the Path Integral

In practice, we often wish to calculate explicit matrix elements or correlation functions of a fields in a theory. In this context, the utility of the path integral can be extended by introducing an auxiliary current $J(x)$ that couples to a field $\phi(x)$. We then consider the generating functional $Z[J]$:

$$Z[J] = \int \mathcal{D}\phi \, \exp\left\{ \frac{iS[\phi]}{\hbar} + i \int \mathrm{d}^4x \, J(x)\phi(x) \right\}. \tag{2.12}$$

Note that $Z[0]$ evaluates to 2.11.

There is a common algebraic trick that is employed by field theorists to manipulate the generating functional into a more amenable form. A partial variational derivative with respect to $J(x)$ can isolate a factor of the field $\phi(x)$ in the following way:

$$\frac{\delta}{\delta J(x')} \int \mathrm{d}^4x \, J(x)\phi(x) = \phi(x'). \tag{2.13}$$

Applying this logic to $Z[J]$ and arranging appropriately, we can construct the vacuum expectation value of the field as

$$\frac{1}{iZ[0]}\left.\frac{\delta Z}{\delta J(x_1)}\right|_{J=0} = \frac{\int \mathcal{D}\phi\, e^{iS[\phi]/\hbar}\phi(x_1)}{\int \mathcal{D}\phi\, e^{iS[\phi]/\hbar}} = \langle \phi(x_1) \rangle\,. \tag{2.14}$$

In fact, we can construct any time-ordered N-point correlation function in this fashion by applying derivatives in succession:

$$\left(\frac{1}{i}\right)^n \frac{1}{Z[0]}\left.\frac{\delta^n Z}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} = \langle T\left\{\phi(x_1)\cdots\phi(x_n)\right\}\rangle\,. \tag{2.15}$$

2.3.2 Currents and the Path Integral

These methods can be extended to the evaluation of currents $\mathcal{J}^\mu(x)$ associated with a symmetry transformation applied to a field $\phi(x)$ as well. In this case, we introduce yet another external classical source $K_\mu(x)$ which couples to the classically conserved current $\mathcal{J}^\mu(x)$. The relevant generating functional is then

$$Z[J,K] = \int \mathcal{D}\phi\, \exp\left\{\frac{iS[\phi]}{\hbar} + i\int \mathrm{d}^4x\, \left[J(x)\phi(x) + K_\mu(x)\mathcal{J}^\mu(x)\right]\right\}. \tag{2.16}$$

The vacuum expectation value of the current $\mathcal{J}^\mu$ can then be extracted:

$$\frac{1}{iZ[0,0]}\left.\frac{\delta Z}{\delta K(x)}\right|_{K_\mu=0} = \langle \mathcal{J}^\mu(x)\rangle\,. \tag{2.17}$$

By taking appropriate derivatives of the generating functional, more complicated matrix elements consisting of products of currents and fields can be calculated in an analogous fashion to 2.15.

2.4 Anomalous Symmetries

In Section 2.2, we found that each continuous symmetry of the action has an associated conserved Noether current $\mathcal{J}^\mu$, such that $\partial_\mu\mathcal{J}^\mu = 0$. Classically, this is always the case. As it turns out, under certain conditions, quantum corrections can violate symmetries that would otherwise be conserved classically. Such symmetries are called *anomalous*.

Although anomalies popped up in experimental data as early as the 1950s —most infamously in the case of the $\pi \to \gamma\gamma$ decay rate— they were not understood on a deep or formal level until much later. It was not until 1969 that Adler [1], Bell, and Jackiw [3] independently proposed a theoretical basis for the observed chiral anomaly involving diagrammatic methods. Ten years later, Fujikawa presented an alternative derivation [6] using the path integral as a starting point. Fujikawa's insight was this: A symmetry transformation respected by the classical action but *not* by the path integral measure would give rise to an anomalous current:

$$\partial_\mu \mathcal{J}^\mu \neq 0. \tag{2.18}$$

Fujikawa's method provides a systematic formalism for studying anomalies that essentially amounts to calculating the Jacobian determinant of the transformation due to a redefinition of the fields. In the following section, we will follow Fujikawa in deriving the U(1) axial anomaly directly from the path integral.

2.5 Example: The U(1) Axial Anomaly

In this section[2], we will derive an expression for the U(1) axial anomaly as it materializes the simplest possible case. Consider a theory consisting only of a single massless fermion field $\psi(x)$ that couples to a gauge field $A_\mu(x)$ with strength g. The relevant Lagrangian is

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \bar{\psi} i \left(\not{\partial} + ig \not{A} \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{2.19}$$

where we have employed Feynman's slash notation when contractions of gamma matrices appear; *e.g.*, $\not{\partial} = \gamma^\mu \partial_\mu$, *etc.* With only one species of fermion, this is essentially a single-generation QED model. In Equation 2.19, $\not{D} = (\not{\partial} + ig\not{A})$ is the contracted covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor which acts as the kinetic term for the gauge field $A_\mu(x)$. The full generating functional for our theory is then

$$Z[\bar{\psi}, \psi, A] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A \, \exp\left\{ i \int \mathrm{d}^4 x \left[\bar{\psi} i \left(\not{\partial} + ig\not{A} \right) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \right\}. \tag{2.20}$$

2.5.1 Axial Transformation of the Lagrangian

To extract the U(1) axial anomaly, we will need to consider how the path integral 2.20 is altered by a chiral rotation of the fermion fields. Let the local gauge parameter of the transformation be $\alpha(x)$. The fields $\psi(x)$ and $A_\mu(x)$ then transform as

$$\begin{cases} \psi(x) \longrightarrow \psi'(x) = e^{i\alpha(x)\gamma_5} \psi(x) \\ A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) \end{cases} \tag{2.21}$$

[2] In this and the remaining sections that follow, we will use natural units $\hbar = c = 1$ unless otherwise stated.

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Note that the gauge field is unaffected. The adjoint field $\bar{\psi}$, however, transforms accordingly as

$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = \left[e^{i\alpha(x)\gamma_5}\psi(x)\right]^\dagger \gamma^0 \tag{2.22}$$
$$= \psi^\dagger(x)e^{-i\alpha(x)\gamma_5}\gamma^0 \tag{2.23}$$
$$= \psi^\dagger(x)\gamma^0 e^{i\alpha(x)\gamma_5} \tag{2.24}$$
$$= \bar{\psi}(x)e^{i\alpha(x)\gamma_5}, \tag{2.25}$$

where we have used the fact that $\{\gamma_5, \gamma^\mu\} = 0$. Meanwhile, under these field redefinitions, the Lagrangian transforms as

$$\mathcal{L}' \equiv \mathcal{L}[\bar{\psi}', \psi'] = \bar{\psi}' i \not{D} \psi' - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.26}$$
$$= \left(\bar{\psi}e^{i\alpha\gamma_5}\right) i \left(\not{\partial} + ig\not{A}\right) \left(e^{i\alpha\gamma_5}\psi\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.27}$$
$$= \bar{\psi}e^{i\alpha\gamma_5} i \left[i\gamma^\mu(\partial_\mu\alpha)\gamma_5 + \not{\partial} + ig\not{A}\right] e^{i\alpha\gamma_5}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.28}$$
$$= \bar{\psi}e^{i\alpha\gamma_5} i e^{-i\alpha\gamma_5} \left[i\gamma^\mu(\partial_\mu\alpha)\gamma_5 + \not{\partial} + ig\not{A}\right] \psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2.29}$$
$$= \mathcal{L} - (\partial_\mu\alpha)\bar{\psi}\gamma^\mu\gamma_5\psi. \tag{2.30}$$

We recover the original Lagrangian, along with an extra term proportional to the axial current $j_5^\mu \equiv \bar{\psi}\gamma^\mu\gamma_5\psi$. Inserting our transformed Lagrangian into the expression for the action, we can integrate by parts to re-express the extraneous term as a total divergence of the axial current proportional to $\alpha(x)$:

$$S[\bar{\psi}', \psi'] = \int \mathrm{d}^4x \mathcal{L}' = \int \mathrm{d}^4x \left\{ \mathcal{L} - (\partial_\mu\alpha) j_5^\mu \right\} \tag{2.31}$$
$$= \int \mathrm{d}^4x \left\{ \mathcal{L} - \alpha(\partial_\mu j_5^\mu) \right\}. \tag{2.32}$$

This is only half of the picture, however. To fully understand the implications of the chiral transformation on a quantum level, we will need to investigate how the path integral measure changes as well.

2.5.2 Transformation of the Measure

We now turn our attention to the consequences that the chiral transformation

$$\begin{cases} \psi(x) \longrightarrow \psi'(x) = e^{i\alpha(x)\gamma_5}\psi(x) \\ \bar{\psi}(x) \longrightarrow \bar{\psi}'(x) = \bar{\psi}(x)e^{i\alpha(x)\gamma_5} \end{cases} \tag{2.33}$$

inflict on the fermionic portion of the measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$. [3] In other words, we seek the function $\mathcal{F}(x)$ such that

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} \longrightarrow \mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \mathcal{F}(x)\mathcal{D}\psi\mathcal{D}\bar{\psi}. \tag{2.34}$$

Conventional knowledge borrowed from standard techniques in multivariable calculus would lead us to expect the measure to transform with a factor of the determinant of the Jacobian matrix for each dynamical variable. Considering the transformations 2.33, we might then write[4]

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' \stackrel{?}{=} \det\left[e^{2i\alpha(x)\gamma_5}\right]\mathcal{D}\psi\mathcal{D}\bar{\psi}. \tag{2.35}$$

This is nearly correct, but we have neglected to account for the fact that our path integral is fermionic— the fields ψ and $\bar{\psi}$ anticommute with one another. This is a defining property of Grassmann numbers, the elements of which may be used in the construction of an exterior algebra. Consequently, our fermionic path integral measure must transform instead with an *inverse* factor of the Jacobian. The function we must evaluate is then

$$\mathcal{F}(x) = \det\left[e^{-2i\alpha(x)\gamma_5}\right]. \tag{2.36}$$

2.5.3 Calculation of the Determinant

The object defined by Equation 2.36, having resulted from the transformation of an ill-defined quantity (namely, the measure), is itself not well-defined. Looking back at our bare-bones QED theory described by the Lagrangian 2.19, we see that the determinant must be formally performed over the Dirac indices, as well as the spacetime variables themselves. Being physicists, we will approach the problem by considering a discretization of spacetime, performing the calculation, and hoping that the result holds also when

[3] Because A_μ is left unaffected by the transformation, we will omit its contribution to the measure in this section.

[4] The conspiring of the chiral transformation matrices to add exponentially is in direct contrast to the vector transformation $\psi \to e^{i\beta(x)}\psi$, $\bar{\psi} \to e^{-i\beta(x)}\bar{\psi}$, in which case the exponentials cancel and the measure is unaffected entirely. It is the anticommutation of γ_5 with the adjoint field itself– which contains a factor of γ^0– that gives rise to the axial anomaly.

returning to the continuum limit. This will allow us to employ the usual techniques of finite dimensional linear algebra in intermediate steps. Here, we follow the derivation of Donahue[4].

By applying the following identity from linear algebra to a general matrix M

$$\det\{M\} = e^{\mathrm{tr}\ln M}, \tag{2.37}$$

we may trade our determinant for a trace:

$$\mathcal{F}(x) = \det\Big[e^{-2i\alpha(x)\gamma_5}\Big] = e^{-2i\,\mathrm{tr}\{\alpha(x)\gamma_5\}}. \tag{2.38}$$

Keeping in mind that the trace (tr) is over Dirac and spacetime indices, consider now breaking it into a trace (Tr) over the Dirac indices while treating the trace over spacetime indices as an integral over single-particle position eigenstates $|x\rangle$. We then have

$$\mathcal{F}(x) = \exp\left\{-2i\,\mathrm{Tr}\int \mathrm{d}^4x\,\langle x|\,\alpha(x)\gamma_5\,|x\rangle\right\} \equiv \exp\left\{i\int \mathrm{d}^4x\mathcal{A}(x)\right\}. \tag{2.39}$$

With some amount of foresight, we have gone ahead and implicitly defined $\mathcal{A}(x)$ as the anomaly. To prevent the integral from diverging, we introduce a gauge-invariant regulator due to Fujikawa[6]:

$$\mathcal{A}(x) = -2\lim_{\Lambda\to\infty}\mathrm{Tr}\,\langle x|\,\alpha(x)\gamma_5 e^{-(\not{D}/\Lambda)^2}\,|x\rangle\,, \tag{2.40}$$

where $\not{D}$ is the covariant derivative from before and Λ is some high UV cutoff. Using the identity $\not{D} = D_\mu D^\mu + \frac{g}{2}F_{\mu\nu}\sigma^{\mu\nu}$, we can expand the exponential as

$$\mathcal{A}(x) = -2\lim_{\Lambda\to\infty}\mathrm{Tr}\,\langle x|\,\alpha(x)\gamma_5\exp\left\{-\frac{D_\mu D^\mu + \frac{g}{2}F_{\mu\nu}\sigma^{\mu\nu}}{\Lambda^2}\right\}|x\rangle \tag{2.41}$$

$$= -2\lim_{\Lambda\to\infty}\alpha(x)\,\langle x|\,\mathrm{Tr}\,\gamma_5\left\{1 - \frac{g}{2\Lambda^2}F_{\mu\nu}\sigma^{\mu\nu} + \frac{1}{2!}\left(\frac{g}{2\Lambda^2}F_{\mu\nu}\sigma^{\mu\nu}\right)^2 - \cdots\right\}e^{-\frac{D_\mu D^\mu}{\Lambda^2}}\,|x\rangle\,. \tag{2.42}$$

Note that the trace of the product of γ_5 with any number of gamma matrices less than four is zero. Because each factor of $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$ contributes two gamma matrices, the first two terms of the expansion will then vanish. The first nonzero term then must be of order g^2:

$$\mathcal{A}(x) = -2\frac{1}{2!}\frac{g^2}{4}\lim_{\Lambda\to\infty}\left[\frac{\alpha(x)}{\Lambda^4}\,\langle x|\,\mathrm{Tr}\,\gamma_5\left(F_{\mu\nu}\sigma^{\mu\nu}\right)^2 e^{-D_\mu D^\mu/\Lambda^2}\,|x\rangle + \mathcal{O}\left(\frac{1}{\Lambda^6}\right)\right]. \tag{2.43}$$

With the identity $(F_{\mu\nu}\sigma^{\mu\nu})^2 = 2(F_{\mu\nu})^2 + i\gamma_5\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$, we can finally evaluate the Dirac portion of the trace:

$$\mathcal{A}(x) = -ig^2\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\lim_{\Lambda\to\infty}\left[\frac{\alpha(x)}{\Lambda^4}\langle x|\, e^{-D_\mu D^\mu/\Lambda^2}\,|x\rangle + \mathcal{O}\left(\frac{1}{\Lambda^6}\right)\right]. \tag{2.44}$$

The factor of 4 comes from the idempotence of γ_5.[5]

All that remains for us is to calculate the matrix element of the exponential. This can be evaluated with a familiar trick from single-particle quantum mechanics. Recall that the identity may resolved in terms of momentum eigenstates $|k\rangle$ as

$$\mathbb{1} = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\,|k\rangle\,\langle k|\,. \tag{2.45}$$

Focusing just on the exponential for the moment, we insert 2.45 to obtain

$$\langle x|\, e^{-D_\mu D^\mu/\Lambda^2}\,|x\rangle = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\langle x|\, e^{-D_\mu D^\mu/\Lambda^2}\,|k\rangle\,\langle k|x\rangle \tag{2.46}$$

$$= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\langle x|\, e^{-\partial_\mu\partial^\mu/\Lambda^2}\,|k\rangle\, e^{ik\cdot x} \tag{2.47}$$

In swapping D_μ for ∂_μ, we are exploiting the fact that the chiral anomaly is a gauge-invariant phenomenon and therefore is not dependent on the dynamics or form of A_μ itself. Thus we are free to set $A_\mu = 0$ in what follows. We can complete the integral by using the fact that $\langle x|\,\hat{p}_\mu\,|k\rangle = -i\partial_\mu\,\langle x|k\rangle$ and $\hat{p}\,|k\rangle = k\,|k\rangle$:

$$\langle x|\, e^{-\partial_\mu\partial^\mu/\Lambda^2}\,|x\rangle = \int\frac{\mathrm{d}^4k}{(2\pi)^4}\langle x|\, e^{+k^2/\Lambda^2}\,|k\rangle\, e^{ik\cdot x} \tag{2.48}$$

$$= \lim_{y\to x}\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{+k^2/\Lambda^2}\,\langle y|k\rangle\, e^{ik\cdot x} \tag{2.49}$$

$$= \lim_{y\to x}\int\frac{i\mathrm{d}^4k_E}{(2\pi)^4}e^{-k_E^2/\Lambda^2}e^{-k_E\cdot(x-y)}. \tag{2.50}$$

In the last step we have Wick rotated in momentum space such that $k^0 = ik_E^0$. The remaining integral is then over a four dimensional Euclidean space, which can be solved by the

[5] $\mathrm{Tr}\{(\gamma_5)^2\} = \mathrm{Tr}\{\mathbb{1}\} = 4$.

usual method of completing the square. The result is

$$\langle x| \, e^{-\partial_\mu \partial^\mu / \Lambda^2} \, |x\rangle = \frac{i}{(2\pi)^4} \lim_{y \to x} \left(\sqrt{\pi} \Lambda e^{\Lambda^2 |x-y|/4} \right)^4 \tag{2.51}$$

$$= \frac{i\Lambda^4}{16\pi^2}. \tag{2.52}$$

Combining this result with Equation 2.44 and taking the limit $\Lambda \to \infty$, we find that

$$\mathcal{A}(x) = \alpha(x) \frac{g^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \tag{2.53}$$

or, in terms of the dual field strength $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}/2$,

$$\mathcal{A}(x) = \alpha(x) \frac{g^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}. \tag{2.54}$$

2.5.4 Extraction of the Anomaly

Armed with our results in calculating the effects of the chiral transformation on both the action 2.32 and of the measure 2.54, our generating functional 2.20 becomes

$$Z[\alpha] = \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \mathcal{D}A \, \exp\left\{ i \int \mathrm{d}^4 x \left[\mathcal{L} - \alpha(x) \left(\partial_\mu j_5^\mu + \frac{g^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right) \right] \right\}. \tag{2.55}$$

By straightforward application of the methods of Section 2.3.2, we can vary $Z[\alpha]$ with respect to α, then taking α to zero:

$$\frac{1}{iZ[0]} \frac{\delta Z}{\delta \alpha(x)} \bigg|_{\alpha=0} = 0 = \left\langle \partial_\mu j_5^\mu + \frac{g^2}{8\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \right\rangle \tag{2.56}$$

which implies that

$$\partial_\mu \left\langle j_5^\mu \right\rangle = -\frac{g^2}{8\pi^2} \left\langle F_{\mu\nu} \tilde{F}^{\mu\nu} \right\rangle. \tag{2.57}$$

As a corollary, for any number of operators $\mathcal{O}_n$, we have

$$\left\langle \left(\partial_\mu j_5^\mu \right) \mathcal{O}_1 \cdots \mathcal{O}_n \right\rangle = -\frac{g^2}{8\pi^2} \left\langle F_{\mu\nu} \tilde{F}^{\mu\nu} \mathcal{O}_1 \cdots \mathcal{O}_n \right\rangle. \tag{2.58}$$

What we have shown is that the vacuum expectation value of the divergence of the axial current j_5^μ under a U(1) chiral transformation is nonzero. Instead, we have a term proportional to the square of the gauge field strength squared. This is a direct consequence of the noninvariance of the path integral measure under redefinition of the fermionic fields.

3 Application to the Lifetime of η'

In this section, we will apply our results from Chapter 2 in deriving the contribution of the U(1) axial anomaly to the lifetime of the η' meson. In the massless chiral limit, the pseudoscalar mesons that constitute the octet consisting of the π^0, $\pi^{\pm}$, $K^{\pm}$, K^0, $\bar{K}^0$, and η are the Goldstone bosons resulting from the spontaneous symmetry breaking of the full global symmetry group $\mathcal{G}$ of the QCD Lagrangian:

$$\mathcal{G} \equiv \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_V \times \mathrm{U}(1)_A \Longrightarrow \mathrm{SU}(3)_V \times \mathrm{U}(1)_V \equiv \mathcal{H}. \tag{3.1}$$

Following symmetry breaking, the QCD vacuum admits quark-antiquark condensates of the form $\langle q\bar{q}\rangle \neq 0$, and thus the ground state does not respect the full symmetry of the theory. While the pseudoscalar octet of mesons may be identified as the Goldstone bosons arising directly from $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \Longrightarrow \mathrm{SU}(3)_V$, the η' singlet is the "would-be" Goldstone boson associated with the breaking of $\mathrm{U}(1)_A$. However, as we know, $\mathrm{U}(1)_A$ is anomalous, which explicitly breaks the symmetry and in turn gives the η' a significantly higher mass than the members of the pseudoscalar octet. We will see in Section 3.2 that this greatly affects the lifetime of the η', as the nonzero divergence of the axial current directly provides the mechanism by which the $\eta' \to \gamma\gamma$ process occurs.

3.1 The Effective Chiral Lagrangian

Because we are interested in phenomena related to the dynamics of low-energy excitations of the vacuum in the form of mesons, it would be sensible for us to forgo the full QCD Lagrangian in favor of an effective field theory that still captures the physics we want to investigate. Before writing down our effective Lagrangian, however, we need to compile and parameterize the meson fields appropriately.

Each field excitation π_a corresponds to one of the eight broken generators in the symmetry breaking $\mathcal{G} \Rightarrow \mathcal{H}$. It is natural then to choose the normalized generators $T_a = \lambda_a/2$

of SU(3) as a basis in defining the expansion of the meson matrix $\boldsymbol{\pi}(x)$:

$$\boldsymbol{\pi}(x) = \sum_{a=1}^{8} \pi_a(x) T_a \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix}. \tag{3.2}$$

The generators T_a are normalized such that $\mathrm{Tr}\{T_a T_b\} = \delta_{ab}/2$. We further parameterize our fields with an exponential map $\Sigma(x)$:

$$\Sigma = \exp\left\{\frac{2i\boldsymbol{\pi}}{f}\right\} = \exp\left\{\frac{2i\pi_a T_a}{f}\right\} = \exp\left\{\frac{i\pi_a \lambda_a}{f}\right\}, \tag{3.3}$$

where f is a real constant of mass dimension one. The effective Lagrangian for our theory can then be written down[7] as

$$\mathcal{L}_{eff} = \frac{f^2}{4} \mathrm{Tr}\left\{\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right\}. \tag{3.4}$$

In practice, the Lagrangian 3.4 is used perturbatively to capture the dynamics of mesons to a desired order. For example, expanding Σ to first order, we get

$$\mathcal{L}_0 = \frac{f^2}{4} \mathrm{Tr}\left\{\partial_\mu \left(1 - \frac{2i\boldsymbol{\pi}}{f} + \cdots\right) \partial^\mu \left(1 + \frac{2i\boldsymbol{\pi}}{f} + \cdots\right)\right\} \tag{3.5}$$

$$= \frac{f^2}{4} \left(\frac{-2i}{f}\right) \left(\frac{2i}{f}\right) \mathrm{Tr}\left\{T_a T_b^\dagger\right\} \partial_\mu \pi_a \partial^\mu \pi_b^* \tag{3.6}$$

$$= \frac{1}{2} \partial_\mu \pi_a \partial^\mu \pi_a^*, \tag{3.7}$$

which is just a sum of the kinetic terms for the individual fields. By including more terms in the expansion of Σ, we can in fact describe all the purely mesonic interactions, with all other operators being higher dimensional.

3.1.1 The Mesonic Axial Current

To identify the axial current in the context of the chiral Lagrangian, we need to first identify the relevant $\mathrm{U}(1)_A$ generator. As written, our chiral Lagrangian is invariant under simultaneous left- and right-handed transformations. For unitary transformations L and R in $\mathrm{SU}(3)_L$ and $\mathrm{SU}(3)_R$ respectively, we expect Σ to transform as

$$\Sigma \longrightarrow \Sigma' = L \Sigma R^\dagger \tag{3.8}$$

while still leaving the Lagrangian 3.4 invariant. Under SU(3), the only constraint on R and L is that they be unitary 3×3 matrices, thereby having eight independent degrees of freedom each. By narrowing our focus on a subgroup U(1) contained within SU(3), we limit ourselves to a single parameter for each of our left- and right-handed matrices:

$$L = e^{i\theta_L \mathbb{1}} \qquad\qquad R = e^{i\theta_R \mathbb{1}}. \tag{3.9}$$

With L and R now both proportional to the identity, the phases θ_L and θ_R conspire under simultaneous transformation of Σ. Specifically, under $\mathrm{U}(1)_V$, the vector transformation can be effected by setting $\theta_L = \theta_R \equiv \theta_V$, in which case the phases cancel. Under the chiral $\mathrm{U}(1)_A$ transformation, however, the left- and right-handed fields are oppositely transformed such that $\theta_L = -\theta_R \equiv \theta_A$. It then follows that $L = R^\dagger$, and so

$$\Sigma' = e^{i\theta_A \mathbb{1}} \Sigma \left(e^{-i\theta_A \mathbb{1}} \right)^\dagger = e^{2i\theta_A} \Sigma. \tag{3.10}$$

We see now that the generator of $\mathrm{U}(1)_A$ is just the identity matrix. By finding the variation in the Lagrangian due to an axial transformation, we can identify the associated current as the term proportional to θ_A. The result is

$$j_A^\mu = \frac{if^2}{2} \mathrm{Tr} \Big\{ \Sigma \left(\partial^\mu \Sigma \right)^\dagger - \Sigma^\dagger \left(\partial^\mu \Sigma \right) \Big\}. \tag{3.11}$$

3.1.2 Inclusion of the η'

The meson field matrix $\boldsymbol{\pi}$ as written in Equation 3.2 notably does not include the η' field. As we remarked before, the mesons belonging to the octet arise in the massless limit as Goldstone bosons. Each meson field π_a is identifiable with one of the eight broken generators in the chiral symmetry breaking $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \Longrightarrow \mathrm{SU}(3)_V$.

In the context of $\mathrm{SU}(3)_{flavor}$, however, the decomposition $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ suggests that we construct a nonet of flavor eigenstates consisting of our original pseudoscalar octet plus an additional singlet state. It is this flavor singlet state that we identify as the η', *i.e.*, the would-be Goldstone boson of the anomalously broken $\mathrm{U}(1)_A$ symmetry.

In Section 3.1.1, we identified the $\mathrm{U}(1)_A$ generator as being proportional to the identity matrix. This suggests that we include the η' in our meson field matrix by introducing a normalized singlet generator $T_0 = \mathbb{1}/\sqrt{3}$:

$$\tilde{\boldsymbol{\pi}}(x) = \sum_{a=0}^{8} \tilde{\pi}_a(x) T_a \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} + \frac{\eta'}{\sqrt{3}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} + \frac{\eta'}{\sqrt{3}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} + \frac{\eta'}{\sqrt{3}} \end{pmatrix}. \tag{3.12}$$

Note that in defining $\tilde{\pi}$, our sum over a is now from 0 to 8. With our field matrix extended to include the full nonet of mesons, we are now in a position to calculate the relationship between the axial current and the η'.

Expanding Σ in the axial current given by Equation 3.11 and keeping only leading terms, we find

$$j_A^\mu \approx \frac{if^2}{2}\,\mathrm{Tr}\left\{\left(1+\frac{2i\tilde{\pi}}{f}\right)\left(\frac{-2i\partial^\mu\tilde{\pi}}{f}\right)-\left(1-\frac{2i\tilde{\pi}}{f}\right)\left(\frac{2i\partial^\mu\tilde{\pi}}{f}\right)\right\} \tag{3.13}$$

$$= \frac{if^2}{2}\,\mathrm{Tr}\left\{\frac{-4i\partial^\mu\tilde{\pi}}{f}\right\} \tag{3.14}$$

$$= \sqrt{6}f\,\partial^\mu\eta'. \tag{3.15}$$

Only the η' survives the trace. Taking the divergence of both sides gives us a direct relation between the axial divergence and the field itself:

$$\partial_\mu j_A^\mu = \sqrt{6}f\,\Box\eta' = -\sqrt{6}f\,m_{\eta'}^2\eta'. \tag{3.16}$$

Here, we have used the Klein-Gordon equation in relating action of the d'Alembertian operator on the field to the mass $m_{\eta'}$.

3.2 Calculation of the Lifetime of the η'

Recall our result derived in Chapter 2:

$$\left\langle\left(\partial_\mu j_A^\mu\right)\mathcal{O}_1\cdots\mathcal{O}_n\right\rangle = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\left\langle F_{\mu\nu}F_{\rho\sigma}\mathcal{O}_1\cdots\mathcal{O}_n\right\rangle, \tag{3.17}$$

where e is the charge of the electron. To calculate the average lifetime τ for the decay process

$$\eta' \longrightarrow \gamma\gamma, \tag{3.18}$$

we will need to calculate the the relevant matrix element $\mathcal{M}_{\eta'\to\gamma\gamma}$. This in turn will allow us to calculate the partial width $\Gamma_{\eta'\to\gamma\gamma}$, the inverse of which will give the average lifetime τ for the decay of the η' to two photons.

Naively, we would not expect such a process to be allowed— there is no term[1] in our Lagrangian 3.4 that directly couples the electrically neutral η' to the electromagnetic field strength $F_{\mu\nu}$. Equations 3.16 and 3.17, however, hint at a possible loophole: Both the η'

[1] Even after introducing the gauge field A_μ into the covariant derivative, only the charged mesons will couple directly to the photon.

field and the anomalous electromagnetic field term are directly related to the divergence of the axial current. We can exploit this with a bit of algebraic manipulation.

To relate the η' field directly to the anomaly term, we first multiply both sides of Equation 3.16 by $A_\alpha A^\alpha$. Rearranging, we find

$$\eta' A_\alpha A^\alpha = \frac{-1}{\sqrt{6} f\, m_{\eta'}^2} \partial_\mu j_A^\mu A_\alpha A^\alpha. \tag{3.19}$$

Then, taking the vacuum expectation value and applying the operator identity in Equation 3.17,

$$\begin{aligned} \left\langle \eta' A_\alpha A^\alpha \right\rangle &= \frac{-1}{\sqrt{6} f\, m_{\eta'}^2} \left\langle \partial_\mu j_A^\mu A_\alpha A^\alpha \right\rangle & (3.20) \\ &= \frac{e^2}{16\sqrt{6}\pi^2 f m_{\eta'}^2} \epsilon^{\mu\nu\rho\sigma} \left\langle F_{\mu\nu} F_{\rho\sigma} A_\alpha A^\alpha \right\rangle . & (3.21) \end{aligned}$$

All that remains then is to evaluate the factor $\epsilon^{\mu\nu\rho\sigma} \left\langle F_{\mu\nu} F_{\rho\sigma} A_\alpha A^\alpha \right\rangle$. Expanding the field strengths in terms of the gauge fields, we get

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} \left\langle F_{\mu\nu} F_{\rho\sigma} A_\alpha A^\alpha \right\rangle &= \epsilon^{\mu\nu\rho\sigma} \left\langle \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \left(\partial_\rho A_\sigma - \partial_\sigma A_\rho\right) A_\alpha A^\alpha \right\rangle \\ &= \epsilon^{\mu\nu\rho\sigma} \left\{ \left\langle \partial_\mu A_\nu \partial_\rho A_\sigma A_\alpha A^\alpha \right\rangle - \left\langle \partial_\nu A_\mu \partial_\rho A_\sigma A_\alpha A^\alpha \right\rangle \right. \\ &\quad \left. + \left\langle \partial_\nu A_\mu \partial_\sigma A_\rho A_\alpha A^\alpha \right\rangle - \left\langle \partial_\mu A_\nu \partial_\sigma A_\rho A_\alpha A^\alpha \right\rangle \right\} . \end{aligned} \tag{3.22}$$

Each term can be evaluated in turn by Wick contracting. For example, the first term becomes

$$\begin{aligned} \epsilon^{\mu\nu\rho\sigma} \left\langle \partial_\mu A_\nu \partial_\rho A_\sigma A_\alpha A^\alpha \right\rangle = \epsilon^{\mu\nu\rho\sigma} \big[&\left\langle \partial_\mu A_\nu \partial_\rho A_\sigma \right\rangle \left\langle A_\alpha A^\alpha \right\rangle + \left\langle \partial_\mu A_\nu A_\alpha \right\rangle \left\langle \partial_\rho A_\sigma A^\alpha \right\rangle \\ &+ \left\langle \partial_\mu A_\nu A^\alpha \right\rangle \left\langle \partial_\rho A_\sigma A_\alpha \right\rangle \big] . \end{aligned} \tag{3.23}$$

The presence of the totally antisymmetric Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$ will simplify this expression greatly. Recall that the propagator for a photon with four momentum k is given by

$$D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2 + i\epsilon}, \tag{3.24}$$

where $g_{\mu\nu}$ is the Minkowski metric tensor. In light of this, we then see immediately that only the first term in Equation 3.23 involving the two-point function $\left\langle A_\alpha A^\alpha \right\rangle \sim g_\alpha^\alpha$ survives. Any other two-point function involving an A_μ with a free index will contribute a factor of the metric tensor, each of which will be killed off when any remaining free indices are contracted with the Levi-Civita symbol. Meanwhile, the two derivatives in $\langle \partial_\mu A_\nu \partial_\rho A_\sigma \rangle$ each contribute a four momenta and a polarization vector:

$$\epsilon^{\mu\nu\rho\sigma} \langle \partial_\mu A_\nu \partial_\rho A_\sigma \rangle \langle A_\alpha A^\alpha \rangle = \epsilon^{\mu\nu\rho\sigma}(-ip_\mu)\varepsilon^*_\nu(p)(-iq_\rho)\varepsilon^*_\sigma(q) g^\alpha_\alpha \tag{3.25}$$

$$= -4\epsilon^{\mu\nu\rho\sigma} p_\mu q_\rho \varepsilon^*_\nu(p)\varepsilon^*_\sigma(q). \tag{3.26}$$

Here, p and q are the momenta of the outgoing photons. Looking back to the four terms in 3.22, we see that by permuting the indices and picking up relative minus signs from the Levi-Civita symbol in each term of the form 3.26, the four terms add together to give

$$\epsilon^{\mu\nu\rho\sigma} \langle F_{\mu\nu} F_{\rho\sigma} A_\alpha A^\alpha \rangle = 16\epsilon^{\mu\nu\rho\sigma} p_\mu q_\rho \varepsilon^*_\nu(p)\varepsilon^*_\sigma(q). \tag{3.27}$$

Our original matrix element then reduces to

$$\mathcal{M}_{\eta' \to \gamma\gamma} = \frac{e^2}{\sqrt{6}\pi^2 f} \epsilon^{\mu\nu\rho\sigma} p_\mu q_\rho \varepsilon^*_\nu(p)\varepsilon^*_\sigma(q). \tag{3.28}$$

Finally, we can calculate the width[9] in the center of momentum frame as

$$\Gamma_{\eta' \to \gamma\gamma} = \frac{1}{2m_{\eta'}} \frac{1}{2} \sum_{pols.} \int d\Pi \, |\mathcal{M}_{\eta' \to \gamma\gamma}|^2 \tag{3.29}$$

$$= \frac{1}{16\pi m_{\eta'}} \left(\frac{e^2}{\sqrt{6}\pi^2 f} \right)^2 \sum_{pols.} |\epsilon^{\mu\nu\rho\sigma} p_\mu q_\rho \varepsilon^*_\nu(p)\varepsilon^*_\sigma(q)|^2 \tag{3.30}$$

$$= \frac{e^4}{96\pi^5 m_{\eta'} f^2} (p \cdot q)^2 \tag{3.31}$$

$$= \frac{e^4 m^3_{\eta'}}{384\pi^5 f^2} \tag{3.32}$$

where the extra factor of $\frac{1}{2}$ accounts for the fact that the outgoing photons are identical particles, each with energy $p^0 = q^0 = m_{\eta'}/2$. In terms of the fine structure constant $\alpha = e^2/4\pi \approx 1/137$, the width is given by

$$\Gamma_{\eta' \to \gamma\gamma} = \frac{\alpha^2 m^3_{\eta'}}{24\pi^3 f^2}. \tag{3.33}$$

Inserting the physical values $m_{\eta'} = (957.78 \pm 0.06)$ MeV and $f = (130 \pm 5)$ MeV provided by the Particle Data Group[11] (PDG), we get

$$\Gamma_{\eta' \to \gamma\gamma} = (0.0037 \pm 0.0003)\,\text{MeV}. \tag{3.34}$$

Comparing to the experimental value of (0.0043 ± 0.0002) MeV for the partial width[2] provided by the PDG, we see that our analytic result 3.33 agrees with current experimental data within 2σ. The predicted lifetime for the process is then

$$\tau_{\eta' \to \gamma\gamma} = \frac{\hbar}{\Gamma_{\eta' \to \gamma\gamma}} = (1.78 \pm 0.03) \times 10^{-19}\,\mathrm{s}. \tag{3.35}$$

[2] (2.20 ± 0.08) % of the full width $\Gamma = (0.198 \pm 0.009)$ MeV[11].

4 Conclusion

In calculating of the lifetime of the η', we have managed to show that anomalous decays do indeed occur in a regime outside of those mediated by the standard interactions of the known forces. In the introduction, we noted that typical decays that proceed via the strong force, such as the process

$$\rho \longrightarrow \pi\pi \tag{4.1}$$

occur on a timescale of $\tau_\rho \sim 10^{-24}\,\mathrm{s}$, while a typical mesonic decay proceeding via the electroweak force such as

$$K^+ \longrightarrow \mu^+\nu_\nu \tag{4.2}$$

occurs much more slowly, with a lifetime on the order of $\tau_{K^+} \sim 10^{-8}\,\mathrm{s}$. Note the vast gulf between these two decays: A whopping 16 orders of magnitude.

Strikingly at odds with either of these two timescales is the decay

$$\eta' \longrightarrow \gamma\gamma, \tag{4.3}$$

which we showed in Chapter 3 to proceed at $\tau_{\eta'} \sim 10^{-19}\,\mathrm{s}$. This does not fit cleanly into either the strong or weak decay time regimes. Instead, it lies logarithmically between the two, closer instead to the original harbinger of the PCAC problem:

$$\pi^0 \longrightarrow \gamma\gamma, \tag{4.4}$$

with a lifetime $\tau_{\pi^0} \sim 10^{-16}\,\mathrm{s}$ that lies precisely at the geometric mean of the ρ and the K^+. Both the η' and the π^0, despite the three orders of magnitude[1] standing between their lifetimes, decay through the same mechanism: The chiral QCD anomaly.

In this thesis, we narrowed our scope to the effect of chiral U(1) anomalies in the low energy regime of QCD. As it turns out, anomalies in quantum field theory are ubiquitous. In our own Standard Model, anomaly cancellation is necessitated by gauge invariance. As

[1] This is to be expected. Equation 3.33 implies that $\tau \sim m^{-3}$ for all anomalous decays of the mesons. The η' and the π^0 are the heaviest and lightest mesons of the nonet, respectively.

they did in the case of the Bell-Jackiw-Adler anomaly responsible for the η' decay, anomalies can point towards new physics. For example, the strong CP problem postulates that axions may be responsible for the suppression of CP violation in QCD. More recently, precision measurements[2] of the anomalous magnetic moment of the muon point towards discrepancies between theory and experiment once again. If the tension between experimental data and theory persists, this opens the door to explanations involving physics beyond the Standard Model, potentially involving a new dark sector portal. In summary, quantum anomalies have very tangible implications on physical observables stemming from theories which contain anomalously broken symmetries.

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