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Mapping Tori and Stable Pairs

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Mapping Tori and Stable Pairs

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Dedicated to Michelle Lee.

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Mapping Tori and Stable Pairs

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We construct an invariant of mapping tori from a moduli space of stable pairs, where a pair is a rank 2 vector bundle over a closed Riemann surface along with a nonzero section. In the case of genus 1 fibers, we calculate this invariant for a large subset of such mapping tori.

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Chapter 1

Introduction

1.1 Background and Motivation

This thesis focuses on problems in symplectic geometry that arise from gauge theory. A motivating correspondence in this field comes from the Atiyah-Floer conjecture, which aims to relate gauge-theoretic constructions to symplectic geometry. This relationship begins with a *Heegaard surface* Σ_g for a closed 3-manifold Y, meaning a closed genus g surface in Y that bounds genus g handlebodies on both sides. The Atiyah-Floer conjecture aims to relate two different constructions of homology theories due to Andreas Floer. Atiyah in [1] observed that instantons, or representations of the fundamental group of Y, naturally correspond to intersections of Lagrangian "submanifolds" in a singular symplectic quotient $\mathcal{M}$ of flat connections over Σ_g. He also conjectured that this could be promoted to an equivalence of topological quantum field theories; namely, that the gauge-theoretic instanton Floer homology IF_* should be isomorphic to Lagrangian Floer homology in $\mathcal{M}$. This TQFT structure relates instanton Floer to Donaldson invariants, which detect exotic smooth structures on 4-manifolds. Unfortunately this correspondence was impossible in general, as there are 3-manifolds where IF_* is well-defined but $\mathcal{M}$ is too singular to define the symplectic version.

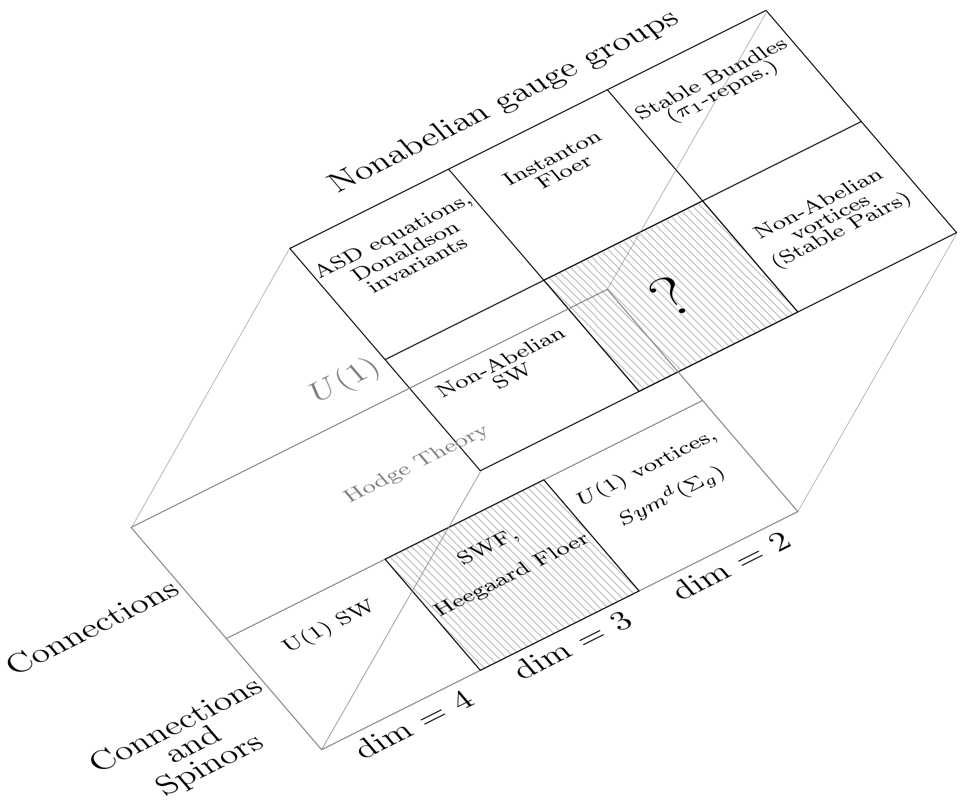

Figure 1.1: Dimensional reductions in the Seiberg-Witten and instanton settings.

However, analogues of this conjecture present interesting geometric problems while also giving tools for studying low-dimensional manifolds. In the case of Seiberg-Witten Floer homology (SWF), a related gauge-theoretic invariant for 3-manifolds, the symplectic side involves the space of U(1)-vortices, which can be interpreted as a symmetric product $Sym^d(\Sigma_g)$ with a natural symplectic form. Using a strategy closely related to the Atiyah-Floer approach, Osváth and Szábo developed Heegaard Floer homology [17], a symplectic model for SWF. Intersections of Lagrangians in $Sym^g(\Sigma_g)$, determined by a Heegaard surface, generate a chain complex whose homology is isomorphic to SWF but is significantly more computable and accessible. Their invariant provides powerful tools for studying knots and 3-manifolds as well as proofs of key conjectures such as the Weinstein Conjecture on contact manifolds in dimension 3. The spectacular success of this theory motivates further investigation of the relationship between gauge-theoretic equations on a 3-manifold, and symplectic geometry in the space of solutions to the reduced equations on a Riemann surface or 2-manifold.

We investigate an Atiyah-Floer-type correspondence in the context of $PU(2)$-monopoles, using a generalization of the spaces used for Heegaard Floer theory. On 4-manifolds, an analogue of the SW equations with gauge group $PU(2)$ [15, 16] relates the instanton and $U(1)$ SW theories, depicted in Figure 1. Over a surface these equations reduce to the one defining stable pairs in [2]. These stable pairs can be thought of as a "non-Abelian" version of symmetric products.

1.2 Summary of Results

A version of Heegaard Floer for stable pairs would require much technical investment to define a general 3-manifold invariant. This thesis draws on the subset of the work on Atiyah-Floer regarding mapping tori of surface diffeomorphisms. Let T_ϕ be the Σ_g-bundle over S^1 defined as the mapping torus of an orientation-preserving diffeomorphism $\phi : \Sigma_g \to \Sigma_g$:

$$T_\phi := \mathbb{R} \times \Sigma_g/(t, x) \sim (t + 1, \phi(x))$$

A special form of Atiyah and Floer's original conjecture was proven in [19] for mapping tori, where one can choose parameters so the space of stable bundles $\mathcal{M}$ is smooth. They establish an isomorphism between instanton Floer homology with the fixed-point Floer homology $HF_*(\Phi)$ of a symplectomorphism Φ of $\mathcal{M}$ induced by ϕ. Here $HF_*(\Phi)$ is the homology of a chain complex generated by fixed points of Φ, which we describe in detail in Chapter 3. In the Seiberg-Witten setting, work in [31] establishes a parallel result, constructing an isomorphism between SWF of T_ϕ with fixed-point Floer homology of a symmetric product of Σ_g. In a similar vein [21] recovers the Turaev torsion of a mapping torus by analyzing the Lefschetz number of a symplectomorphism of the symmetric product of the surface fiber.

In the current work, we use the moduli spaces of stable pairs to construct an invariant of mapping tori $HSP_*(T_\phi)$, in the spirit of the works mentioned above, and calculate it in certain genus 1 examples.

Theorem 1.2.1. *Suppose that T_ϕ is a mapping torus, and let $L \to T_\phi$ be a smooth complex line bundle with $\langle c_1(L), \Sigma_g\rangle = 2g + 2$. Then there is an S^1-family of monotone stable pair spaces $\mathcal{M}_g$ with symplectic connection Ω and fiber M_g such that, if $\Phi : M_g \to M_g$ is the monodromy,*

$$HSP_*(\phi, L) := HF_*(\Phi)$$

is an invariant of L and the isotopy class of ϕ.

The construction used in Theorem 1.2.1 in fact generates a symplectomorphism of any stable pair space. However, we make choices to ensure that Floer theory on the M_g is well-behaved; with these restrictions $HF_*(\Phi)$ is a finitely generated $\mathbb{Z}$-module as a direct consequence of the construction in [19].

We will explain in Chapter 2 that when the genus of the fiber is 1, we have $2g + 2 = 4$, so $L \to T_\phi$ induces an S^1 family of projective embeddings $\Sigma_1 \hookrightarrow \mathbb{P}H^1(L^{-1})$. A reference fiber Σ_1 then carries a symplectic form ω, and ϕ preserves ω. In this case M_1 can be identified with the blowup $Bl_{\Sigma_1}(\mathbb{CP}^3)$ of $\mathbb{CP}^3$ along this genus 1 curve.

Theorem 1.2.2. *Let T_ϕ be as above, and suppose that $Id - [\phi]^*$ is invertible as an endomorphism of $H^1(X;\mathbb{Q})$, so $L \to T_\phi$ is determined. Then, fixed-point Floer homology of ϕ on (Σ_g, ω) is well-defined, and we have an following isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded Abelian groups:*

$$HSP_*(\phi, L) \cong \mathbb{Z}^4 \oplus HF_*(\phi) \tag{1.1}$$

where $HF_*(\phi)$ is the Floer homology of a symplectomorphism isotopic to ϕ.

The proof of these facts draws on a variety of tools from symplectic geometry, including the (monotone) Fukaya category, quantum cohomology and Gromov-Witten invariants, as well as invariance of fixed-point Floer homology under certain symplectic isotopies altering the cohomology class of the symplectic form.

While the monotone form κ_0 on the blowup M of $\mathbb{P}^3$, with monodromy Φ_0, ensures that Floer theory is well-behaved, it is insufficiently explicit for determining generators of $CF_*(\Phi_0)$. On the other hand, one can effect the blowup symplectically to produce a diffeomorphic but only weakly monotone Kähler manifold (M, κ_1) and a symplectomorphism Φ_1 isotopic to Φ_0. In this setting the fixed points can be identified, but robustly defining $HF_*(\Phi_1)$ in the weakly monotone setting requires an argument ruling out bubbling in moduli spaces of trajectories. A continuity principle for the change in Floer homology under certain symplectic isotopies [11, 12] provides a homotopy equivalence between the chain complexes.

We then compute the differential ∂ using both constructions. An explicit formula for the symplectic connection inducing Φ_1 allows us to determine generators of $CF^*(\Phi_1)$ and their gradings. On the monotone side, interpreting $(M, \kappa_0$ as a fiber of a Lefschetz fibration places Φ into a Dehn twist exact sequence. The induced sequences of generalized eigenspaces for the action of $c_1 \in QH^*(M, \kappa_0)$ isolate the summands in (1.1). Determining the generalized eigenspaces for the c_1-action involves a calculation of the quantum cohomology ring of the blowup, which is itself a subject of independent interest.

Theorem 1.2.3. *The eigenvalues of c_1 acting on $QH_*(Bl_{\Sigma_1}(\mathbb{CP}^3))$ are -1 with multiplicity 4 and $\{0, 8, 4(-1+i), 4(-1-i)\}$ each with multiplicity 1. There are isomorphisms of $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}$-algebras*

$$QH^*(Bl_{\Sigma_1}(\mathbb{CP}^3); \mathbb{Z}) \cong \frac{\mathbb{Z}[t]}{1-t^4} \oplus QH^*(Bl_{\Sigma_1}(\mathbb{CP}^3); \mathbb{Z})_{-1}$$

and

$$QH^*(Bl_{\Sigma_1}(\mathbb{CP}^3); \mathbb{Z})_{-1} \cong H^*(X; \mathbb{Z})$$

where $QH^*(Bl_{\Sigma_1}(\mathbb{CP}^3);\mathbb{Z})_{-1}$ *is the subalgebra of* $QH^*(Bl_{\Sigma_1}(\mathbb{CP}^3);\mathbb{Z})$ *where* $c_1 + Id$ *acts nilpotently.*

Combining Theorem 1.2 and the isomorphism of [31] yields the following corollary. $HF_*(\Phi)$ contains a summand isomorphic to fixed-point Floer homology in $Sym^1(\Sigma_1) = \Sigma_1$, which corresponds to $SWF(T_\phi;\mathfrak{s})$ in appropriate Spin^c structures. Thus, in this test case, the invariant constructed above does in fact detect SWF.

1.3 Organization of Material

Chapter 2 introduces the moduli spaces of stable pairs described above. Each such space depends on a real parameter Σ, a holomorphic line bundle L, and a closed Riemann surface X. For appropriate choices of these parameters, these moduli spaces admit Kähler forms which are particularly amenable to defining Floer theory. We describe the possible such forms as well as how the spaces vary with the real parameter.

In Chapter 3, we give a precise definition for Floer theory of symplectic fixed points for monotone symplectic manifolds, which we use in defining the invariant HSP. Both the formulation in terms of path spaces and a formulation in terms of mapping tori are given, as both viewpoints are useful for our later constructions. We also describe a continuity result which relates symplectomorphisms connected by a symplectic isotopy, which will be needed for later calculations. We then illustrate how this machinery works with some specific examples in the case of surfaces.

Chapter 4 constructs the invariant for general surface bundles and establishes its invariance properties. We first describe a construction which applies to surface bundles over the circle of any genus g. Following the examples in the instanton and Seiberg-Witten cases, we produce a symplectomorphism Φ of a Fano stable pair space (M_g, κ) whose fixed-point Floer homology defines an invariant $HSP_*(T_\phi)$ for mapping tori of surfaces(here ϕ is the monodromy of the mapping torus).

The final three chapters are dedicated to calculations in the genus 1 case. Chapter 5 describes the general construction of symplectic blowups, which is employed in Chapter 6 to calculate the generators of the chain complex for HSP_*. Finally, in Chapter 7 we calculate the quantum cohomology ring of the blowup, as Floer homology is naturally a module over this ring. We then argue that differential is trivial by decomposing the chain complex into eigenspaces for the action of $c_1(TM_1)$.

Chapter 2

Moduli of Stable Pairs

In this section we describe the construction of moduli spaces of stable pairs given in [27]. These spaces vary with a positive real parameter and change by certain birational transformations. We also describe their Kähler cones.

2.1 The Definition

Let X be a compact Riemann surface of genus g, and let L be a holomorphic line bundle of degree d.

Definition 2.1.1. *A* σ***-stable pair*** (E, ϕ) *over* X *is a rank 2 holomorphic vector bundle* $E \to X$ *together with a nonzero section* $\phi \in H^0(E)$*, such that for every line sub-bundle* $F \subset E$*, we have*

$$\deg(F) < \frac{1}{2}\deg(\Lambda^2 E) - \sigma \quad \mathit{if} \sigma \in H^0(F) \tag{2.1}$$

$$\deg(F) < \frac{1}{2}\deg(\Lambda^2 E) + \sigma \quad \mathit{if} \sigma \notin H^0(F). \tag{2.2}$$

We say that a pair (E, σ) *is* σ***-semistable*** *if the inequalities above hold non-strictly.*

This notion of stability is familiar from the theory of vector bundles over Riemann surfaces; a 0-stable pair is a (semi)stable bundle with an arbitrary nonzero section ϕ. Any nonzero $\phi \in H^0(E)$ lies in a unique line subbundle $F_\phi \subset E$; even if E is a stable bundle, (E, ϕ) may be σ-unstable if F_ϕ has high degree.

Example 2.1.2. *Taking $\sigma = \frac{degE}{2} - \epsilon, 0 < \epsilon \ll 1$, (2.1) implies that the section ψ sweeps out a degree 0 line bundle L, yielding an exact sequence*

$$0 \to L \to E \to \Lambda \otimes L^{-1} \to 0.$$

But if ψ is a holomorphic section of a degree 0 line bundle, $\psi \in \Gamma(\mathcal{O}_{\Sigma_g})$. Moreover, there is a nonzero map $E \to E/L = \Lambda$.

By the second inequality E can have no line sub-bundles of degree $deg(E)$ or greater. This implies that E is a nonsplit extension of $\mathcal{O}_X$ by L, for if $E = M \oplus M'$, then we have a nonzero map of line bundles $M \to E \to \Lambda$, which must be an isomorphism. But then $deg(M) = deg(\Lambda) = d$, contradicting the inequality (2). So $M_{\sigma,L}$ for such σ is $\mathbb{P}H^1(L^{-1})$, classifying such nonsplit extensions of $\mathcal{O}_X$ by Λ.

On the other hand, for $\sigma = \epsilon$, (2.2) implies that E is a semi-stable, relating stable pairs to stable bundles (equivalently, flat connections).

A GIT construction carried out in [27] builds a compact complex-analytic space $M_{\sigma,d}$ (in fact, projective algebraic variety) which is a coarse moduli space for σ-stable pairs with $\deg \Lambda^2 E = d$ containing a fine moduli space for σ-stable pairs. There is a natural map

$$\det : M_{\sigma,d} \to Pic^d(X), \quad (E, \phi) \mapsto \Lambda^2 E \tag{2.3}$$

and we denote by $M_{\sigma,L}$ its fiber over a line bundle L. For a fixed d, we say that σ is *critical* if there exist pairs which are σ-semistable but not σ-stable. There are finitely many d-critical values σ, and Thaddeus shows that $M_{\sigma,d}$ is singular precisely along the locus of non-σ stable pairs. Thus when σ is non-critical, $M_{\sigma,d}$ is a compact complex manifold; det is a submersion, and thus $M_{\sigma,L}$ is also a compact complex manifold. We also have $\dim(M_{\sigma,L}) = d + g - 2$.

There is an alternate construction of these moduli spaces in a more gauge-theoretic setting, carried out on [2, 22], rather than the algebro-geometric approach of [27]. The necessary tools are still solidly in the realm of geometric invariant theory, but their construction begins with infinite dimensional configuration spaces and uses the language of differential geometry and gauge theory, where Hilbert Lie groups act on configuration spaces of smooth connections and smooth sections. However, the results describe the same underlying complex manifolds, and in fact this gauge-theoretic approach will be useful later in constructing families of stable pairs spaces.

2.2 Variation with σ

Of particular interest is the change in the moduli spaces as σ changes. $M_{\sigma,d}$ is nonempty exactly when when $0 \leq \sigma \leq \frac{d-1}{2}$. It is clear from the definition that $M_{\sigma,L}$ is constant on intervals containing no critical values. Thus we simplify the notation slightly by setting $M_i = M_{\sigma,L}$ for all σ where $i = \lceil d + g - 2 + \sigma \rceil$. So we have nonempty spaces $M_0, M_1, ..., M_w$ where $w = \lfloor \frac{d-1}{2} \rfloor$.

M_0 can be canonically identified with the projective space $\mathbb{P}H^1(L^{-1})$. The curve X naturally embeds in M_0, and M_1 is then the blowup of M_0 along X. Making the transition through the remaining critical values corresponds to a "complex surgery"; that is, for $2 \leq i \leq w$, we have the following diagram.

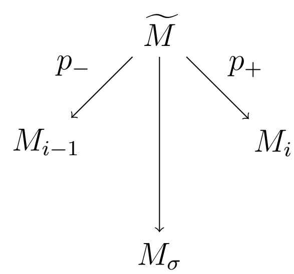

Figure 2.1: Moduli spaces of stable pairs related by birational transformations.

The maps $p_\pm$ are blow-down maps, contracting a common divisor $E \subset \widetilde{M}$ to smooth subvarieties $V_i \subset M_i$, . The loci $V_\pm$ take the form of projective vector bundles over symmetric products of X. The canonical bundle $K_{\widetilde{M}}$ is respectively positive and negative on the projective-space fibers of $p_\pm$; thus, the diagram expresses M_{i+1} as a *flip* of M_i. This space M_w admits a "Abel-Jacobi" map to the moduli space N of semistable bundles of degree d(given by forgetting the section ϕ). The fiber of this Abel-Jacobi map over a stable bundle E is $\mathbb{P}H^0(E)$.

2.3 Their Kähler Cones

The description of M_i as a flip of a blowup of projective space has some straightforward topological consequences. For one, M_i is always simply connected. Also, for $i \geq 1$ we have

$$H^2(M_i;\mathbb{Z}) = H^2(M_1;\mathbb{Z}) \cong \mathbb{Z}^2 \tag{2.4}$$

and for $i = 1$ the generators are the hyperplane class H and the class of the exceptional divisor E. Then since flipping does not change H^2, and Hodge theory implies that $H^1(\mathcal{O}) = H^2(\mathcal{O}) = 0$, we see that $H^2(M_i;\mathbb{Z}) = Pic(M_i)$.

Proposition 2.3.1. *When $d = 2g+2$ and w is maximal, M_w is Fano.*

[27] completely determines the ample cone, as well as the canonical class. Thus, one reads directly from this calculation that when $d = 2g+2$ and w is maximal, $-K_{M_w}$ is ample.

Chapter 3

Fixed-Point Floer Homology

In this section we provide a brief description of the symplectic geometry tools necessary for the proof of Theorem 1.1: the construction of fixed-point Floer theory (suppressing many details including proofs of invariance under Hamiltonian perturbations and changes in complex structure, smoothness and compactness of the moduli spaces, etc.) as well as the role of continuation maps and perturbations. We then explore some consequences in the case of surface symplectomorphisms. The presentation here closely follows the treatments given in [8], [23], [24].

Fixed-point Floer theory (in particular for Hamiltonian diffeomorphisms) is the tool originally developed by Floer to solve the Arnold conjecture. For (M, ω) a symplectic manifold, the observation that the graph of a symplectomorphism ϕ in $M \times M$ and the diagonal are Lagrangian relates fixed points to the better-known Lagrangian intersection Floer theory, and it can be shown that the two are isomorphic. Since fixed-point Floer theory will be the main tool in defining our invariant, we review its definition and the necessary background material here.

3.1 Fixed-Point Floer Homology in the Monotone Case

As with Lagrangian intersection Floer homology, fixed-point Floer homology of a symplectomorphism admits a variational formulation in terms of the "Morse theory" of an action functional on a configuration space of paths. Though this can be defined for several types of symplectic manifolds and symplectomorphisms, we focus on those which satisfy a convenient technical assumption (also satisfied by the stable pair spaces and maps we work with).

Definition 3.1.1. *Let (M, ω) be a compact symplectic manifold of dimension $2n$. We say that (M, ω) is* ***monotone*** *if $c_1(TM) = \lambda[\omega]$, with $\lambda > 0$.*

Similarly, for a general symplectic manifold (M, ω), consider a symplectomorphism $\phi : M \to M$. This map determines a symplectic connection 2-form ω_ϕ on its mapping torus T_ϕ via the projection of ω from $M \times \mathbb{R}$.

Definition 3.1.2. *We say that ϕ is* ***monotone*** *if $c_1(T_\phi^{vert}) = \lambda[\omega] \in H^2(T_\phi; \mathbb{R})$ for $\lambda > 0$.*

Now for monotone (M, ω) or monotone $\phi : M \to M$, there exist homology groups with the following properties, as constructed by Floer in the path space setting and interpreted as a TQFT in [23] to emphasize its functorial properties.

Proposition 3.1.3. *For every monotone symplectomorphism $\phi \in Aut(M, \omega)$ as above, one can construct a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded Abelian group $HF_*(\phi)$, called* fixed-point Floer homology*, with the following properties:*

1. ***Fixed Points:*** *When* ϕ *has nondegenerate fixed points,* $HF_*(\phi) := H_*(CF(\phi), \partial)$ *where* $CF(\phi) = \mathbb{Z}^{Fix(\phi)}$*, with the mod 2 grading of a fixed point* x *given by the Lefschetz number* $\text{sign}(\det(D_x\phi - Id)$*, and* ∂ *is an odd-degree differential. This differential* $\partial = \partial_J$ *depends on an auxiliary choice of almost complex structure* J *in* $T^{vert}(T_\phi)$ *but the homology groups are independent of* J *up to a canonical isomorphism.*

2. ***Naturality:*** *There are canonical isomorphisms* $HF(\theta \circ \phi \circ \theta^{-1}) \cong HF_*(\phi)$ *for any diffeomorphism* θ*.*

3. ***Invariance:*** *Let* $\{\psi_t\}$ *be a symplectic isotopy of* M *generated by 1-forms* b_t *such that* $b_{t+1} = \psi^*(b_t)$ *whose flux* $\int_0^1 b_t dt$ *lies in* $Im(\phi^* - Id) \subset H^1(M; \mathbb{R})$*. The isotopy then defines a canonical isomorphism* $HF(\psi_1 \circ \phi) \cong HF_*(\phi)$*.*

We now briefly review the construction of $HF_*(\phi)$ from two perspectives, starting with Floer's original formulation.

3.1.1 Fixed Points via Action Functionals and the Twisted Path Space.

Consider the space $\mathcal{P}(\phi)$ of paths

$$\mathcal{P}(\phi) = \{\gamma \in C^\infty(\mathbb{R}, M) |\, \gamma(t) = \phi(\gamma(t+1))\}$$

so that fixed points of ϕ correspond to paths in this configuration space where $\gamma(0) = \gamma(1)$.

A choice of compatible complex structure J in M furnishes an L^2-metric on the tangent spaces to $\mathcal{P}_\phi$, which we use to define the 1-form $\mathfrak{Y}_\gamma : T_\gamma\Omega_\phi \to \mathbb{R}$ given by

$$\mathcal{Y}_\gamma(X) = \int_0^1 \omega(J(\gamma'(t)), \gamma'(t))dt$$

where $X \in \Gamma(\gamma^*(TM)$

Then fixed points of ϕ correspond to zeroes of this one-form. In general, however, $\mathcal{Y}$ is not exact, so in addition we choose a cover $p : \mathcal{C} \to \Omega_\phi$ with deck transformation group $\mathfrak{h}$ such that the pullback of $\mathcal{Y}$ to $\mathcal{C}$ is exact.

This covering space $\mathcal{C}$ has an explicit construction, the details of which can be found in [14]. For a choice of basepoint $\gamma_0 \in \Omega_\phi$, consider the pairs $(x, [w])$ where $x \in \mathcal{C}$ and $[w]$ is a homotopy class of path from γ_0 to x. We identify $\mathcal{C}$ with the set of equivalence classes of such pairs, where $(x, [w])$ and $(x', [w'])$ are equivalent if and only if $x = x'$ and $[w] - [w'] = 0 \in \pi_1(\mathcal{P}_\phi)$. In this cover, then, we can integrate $\mathcal{Y}$ to a function $\tilde{\mathcal{A}}$. Its values on fibers of p differ by integer multiples of λ, by monotonicity.

$\tilde{\mathcal{A}}$ then descends to a circle-valued function $\mathcal{A} : \mathcal{P}_\phi \to \mathbb{R}/\mathbb{Z}$ whose critical points are the fixed points of ϕ. The gradient flow of this function with respect to the L^2-metric chosen above is denoted $\mathcal{X}$, a vector field dual to $\mathcal{Y}$. We then define a chain complex $CF(\phi, \partial)$ as follows. This is generated by critical points of $\mathcal{A}$. Non-degeneracy of the fixed points above corresponds to non-degeneracy of these critical points.

To define the differential, we require an analogue of Morse flow lines, consisting of paths in Ω_ϕ connecting x^+, x^- in the fixed-point set $\mathcal{P}_\phi$ described above. These will be denoted by

$$\mathcal{P}(x^+, x^-) = \{u : \mathbb{R} \times [0,1] \to M | \lim_{s \to \pm\infty} u(s,t) = x^{\pm}(t), u(1,s) = \phi(0,s)\} \tag{3.1}$$

The subset of $\mathcal{P}(x^+, x^-)$ corresponding to gradient flow trajectories of $\mathcal{A}$ are the solutions to Floer's equation

$$\frac{\partial u}{\partial s} + J(u(s,t))\frac{\partial u}{\partial t} - \nabla H(t,u) = 0$$

In general we will refer to solutions of this PDE as trajectories. Note that since $\mathcal{A}$ is circle-valued, x^+ could equal x^-, yielding J-holomorphic tori in M.

Existence of a well-defined differential for $CF_*(\phi)$ follows from Gromov compactness applied to holomorphic maps in $\mathcal{P}(x^+, x^-)$, where we fix the index(and thus the area, by monotonicity) of the trajectories being considered. The Lefschetz $\mathbb{Z}/2\mathbb{Z}$-grading mentioned in Proposition 3.1.3 is necessarily related to the dimensions of the spaces of such maps. In general there is a relative $\mathbb{Z}$-grading on fixed points determining the dimensions of the moduli space of trajectories called the Conley-Zehnder index, but the grading it induces on CF is only defined modulo the minimal Chern number. We will see that this is 2 for all stable pair spaces in question, so the CZ grading is equivalent to the Lefschetz grading in our case.

With these definitions in place we are ready to describe the compactified moduli spaces of trajectories. For trajectories $\{u\}$ with fixed index $CZ(u) = d$, monotonicity allows us to conclude that the areas of these trajectories are bounded, hence we may apply Gromov compactness. A priori, in this compactification we may see degenerations, i.e. broken trajectories or sphere bubbles. However, suppose that the complex structure J is **regular**, so the moduli spaces $\mathcal{M}(x^+, x^-)$ are transversely cut out. Then for a limit $\{u_n\}$ of trajectories with index d in the Gromov compactification we have the index formula

$$d = \sum CZ(u_i) + \sum \langle c_1(TM), \beta_j \rangle$$

where the first sum is taken over the trajectories into which the u_n degenerate, and the second over sphere bubbles β_j.

Since the relevant moduli spaces are cut out transversely, all the terms in this sum are positive. So if we consider the differential ∂_J defined above, we have $d = 1$, so there can be at most 1 trajectory and no sphere bubbles by monotonicity. This ensures that the differential is well-defined. A similar argument shows that $\partial_J^2 = 0$; in the compactification of index two trajectories, we see only broken trajectories each with index one.

3.1.2 Fixed Points via Horizontal Sections of the Mapping Torus.

There is another formulation of the configuration space in fixed-point Floer homology, described in [23, 24], which we will also review for later use. Instead of considering the twisted free loop space Ω_ϕ, we work on the mapping torus T_ϕ and consider sections $\gamma : S^1 \to T_\phi$. To connect this to the previous construction, points in the configuration space Ω_ϕ correspond to sections γ which are horizontal with respect to the symplectic connection form

ω introduced in Definition 3.1; that is,

$$\omega\left(\frac{d\gamma}{dt}, v\right) = 0 \quad \forall v \in T^{vert}(T_\phi) \tag{3.2}$$

Taking these horizontal sections of the symplectic fibration (T_ϕ, ω) as generators of a chain complex, we will define the differential in this setting by counting holomorphic sections rather than trajectories as above. On the cylinder

$$Z_\phi := T_\phi \times \mathbb{R}$$

where $Z := \mathbb{R} \times S^1$ with coordinates (s,t), and j is a complex structure for which $J(\frac{d}{dt}) = \frac{d}{ds}$. We consider the space $\underline{\mathfrak{J}}$ of smooth families $(J_t)_{t\in\mathbb{R}}$ of complex structures on the fiber M such that $J_{t+1} = \phi^*(J_t)$, so each such family is a complex structure in $T^{vert}(T_\phi)$. Note that this differs from the action functional construction in that we allow the complex structure on the fiber M to vary in an S^1-family, whereas the previous construction was defined in terms of a fixed J.

To define the differential in this framework we then consider two-parameter families of complex structures $J_{s,t}$ on Z_ϕ which are constant in s. Given two ω-horizontal sections of (T_ϕ, ω) denoted $x^\pm$, the matrix coefficient of the differential $\langle \partial x^+, x^- \rangle$ counts J_t-holomorphic sections of Z_ϕ. The space of such maps with index k is denoted

$$M(x^+; x^-)_k = \{u : Z \to Z_\phi | \lim_{t\to\infty} u(t,\cdot) = x^\pm(\cdot), du \circ j = J \circ Du\} \tag{3.3}$$

In this setting we can define a Conley-Zehnder index for horizontal sections $x^\pm$ similar to the previous construction. The difference $CZ(x^+) - CZ(x^-)$ is again the dimension of moduli spaces of trajectories for generically chosen $J_{s,t}$. For a generic J these moduli spaces are smooth and compact.

Let $\tilde{\mathcal{M}}_k(x^+, x^-, J)$ be the space of solutions after taking the quotient by the obvious $\mathbb{R}$-action on such solutions. As before, these moduli spaces are not connected, nor are they of uniform dimension. In fact $\pi_2(M)$ acts on trajectories by connected sum.

Now we may define a chain complex generated by the above fixed points, with differential given by

$$\partial(x) = \sum_{y \in \mathcal{P}_\phi} \#\tilde{\mathcal{M}}_1(x, y, J)$$

where the # denotes the number of points in this compact 0-dimensional moduli space(using the dimension formula above). The fact that $\partial_J^2 = 0$ follows from the Gromov compactness argument mentioned above, following the argument involving (3.4) to show that the boundaries of the index two moduli spaces correspond to broken trajectories only.

The result is a $\mathbb{Z}/2N\mathbb{Z}$-graded complex over $\mathbb{Z}$, and its homology of this complex is $HF_*(\phi)$.

3.1.3 Continuation Maps and Perturbations

Having established a well-defined chain complex, it remains to show that the assumptions regarding generic complex structures and non-degenerate fixed points can be done away with, as they do not appear in Theorem 3.2.

In both the mapping torus and loop space settings, one can define the maps relating these choices (called continuation maps) in similar ways. We will use the mapping torus setting as it is the viewpoint we will take in later computations of HF.

For example, to construct the continuation maps for Hamiltonian perturbations to the symplectomorphism ϕ, one considers forms on the mapping cylinder Z_ϕ, a bundle over the cylinder $Z = \mathbb{R} \times S^1$ with coordinates (s, t) as before. Following [24] we define a class of perturbations to the symplectic connection.

Definition 3.1.4. *Let $\mathcal{B}$ be the space of smooth families $b = (b_t)_{t\in\mathbb{R}}$ of closed one-forms on M satisfying $b_{t+1} = \phi^*(b_t)$ and*

$$\int_0^1 [b_t]dt \in im(Id - \phi^*) \subset H^1(M;\mathbb{R}). \tag{3.4}$$

Clearly all such b are defined on $M \times \mathbb{R}$ and descend to the mapping torus by construction. This condition is equivalent to requiring that $dt \wedge b_t$ be exact when considered as a form on $M \times \mathbb{R}$.

Now, in the case where the symplectic connection ω on T_ϕ has degenerate fixed points, there is an open dense subset $\mathcal{B}_{reg}$ of the above such that monodromy in $(T_\phi, \omega + b)$ has nondegenerate fixed ponts. To relate these back to (T_ϕ, ω), note that this perturbation induces an isotopy of the monodromy symplectomorphism $\psi_t \circ \phi$, where ψ_t is the monodromy of the symplectic connection b_t. This determines a map $\Psi : \mathcal{P}_\phi \to \mathcal{P}_{\psi_1\circ\phi}$ given by

$$\Psi(\gamma(t)) = \psi_1 \circ \gamma(t) \tag{3.5}$$

which induces a canonical isomorphism of chain complexes $CF_*(\phi) \to CF(\psi_1 \circ \phi)$ as needed.

To relate chain complexes associated to two regular complex structures J_+, J_-, recall that each one is defined on Z_ϕ but is constant in the $\mathbb{R}$-direction. We now consider $\underline{J}_{s,t}$ a complex structure which coincides with J_+ (respectively, J_-) for sufficiently large positive (negative) t, and interpolates smoothly between them in the middle. The continuation maps then count sections $u : Z \to Z_\phi$ which are $\underline{J}_{s,t}$-holomorphic, index zero, and asymptotic to a generator $x^+ \in CF_*(\phi, \partial_{J_+})$ as $t \to \infty$ and similarly $x^- \in CF(\phi, \partial_{J_-})$ as $t \to -\infty$. These index 0 moduli spaces(for appropriately chosen $\underline{J}_{s,t}$ so this is cut out transversely) determine the given matrix coefficient.

The naturality property of Floer homology can be established in a similar fashion. Noting that conjugate symplectomorphisms determine isomorphic mapping tori, we can consider the bundle $Z_\phi \to S^1 \times [0,1]$ where bundle over 0 corresponds to T_ϕ and over 1 corresponds to $T_{\theta\circ\phi\circ\theta^{-1}}$. As above, counting index zero J-holomorphic sections asymptotic to horizontal orbits provides the required map between chain complexes.

Note that in the case where the fixed points are not nondegenerate, a perturbation by a Hamiltonian vector field as above is sufficient to guarantee nondegeneracy; we may replace ϕ by $\psi \circ \phi$ (where ψ is a Hamiltonian symplectomorphism) so as to make the fixed points non-degenerate. By applying this perturbation in the form of some $b \in \mathcal{B}$, Floer homology for this perturbation is then defined to be $HF_*(\phi)$.

3.1.4 Module Structure

With the continuation maps described as above, it is straightforward to describe an action of the quantum homology $QH_*(M)$ on $HF_*(\phi)$, as shown in [20] and described in more detail in [24]. Informally, one can consider holomorphic maps with a marked point:

$$\sigma : (\mathbb{R} \times S^1, z) \to Z_\phi.$$

For a given reference fiber (M, ω) of T_ϕ, there is a reference fiber of Z_ϕ providing a symplectic isomorphism $F_\phi : (M, \omega) \to Z_\phi|_{(0,0)}$. Thus it makes sense to consider the image of a representative of some $x \in QH_*(M, \omega)$ in Z_ϕ. The module structure is then given by counting sections σ as above, with the additional condition that $\sigma(z)$ lies on $F_\phi(x)$.

Though we will later work with quantum cohomology rather than homology, one may extend the Poincaré duality of classical cohomology to the quantum version. Some references may refer to the quantum module structure of Floer cohomology rather than homology, but for our purposes there is no substantial difference between Floer cohomology and what we describe other than that the differential increases gradings rather than decreasing them.

3.1.5 Coherent Orientations

In order for the Floer homology groups to take coefficients in $\mathbb{Z}$ rather than $\mathbb{Z}/2\mathbb{Z}$, the moduli spaces in question must be oriented in a way which is compatible with the gluing maps. For Floer homology for intersecting Lagrangians this requires an additional choice of Spin structures, but in the case of fixed points there is a canonical such choice, described by Floer in [6].

3.1.6 Monotone Surfaces

In this section we describe some applications of the above theory to surface symplectomorphisms. Note that a closed surface Σ_g carries a monotone symplectic form if and only if $g = 1$, for if $g > 1$ then $\langle c_1(T\Sigma_g), [\Sigma_g]\rangle = 2-2g < 0$. However, in the mapping torus formulation we say that ϕ is *monotone* if there is a symplectic connection Ω inducing a symplectomorphism of the reference fiber $\phi : \Sigma_g \to \Sigma_g$ such that $c_1(T_\phi^{vt}) = \lambda[\Omega]$, for $\lambda = \frac{\int_{\Sigma_g}\Omega}{\chi(\Sigma_g)}$. This definition allows symplectomorphisms to be monotone, even when $g \neq 0$.

Example 3.1.5. *Consider the sphere S^2, identified with $\mathbb{CP}^1$, with some area form ω. By Moser's argument, Symp(S^2, ω) deformation retracts to $SO(3)$, so we see that any symplectomorphism must be isotopic to the identity in $SO(3)$. Since $\mathbb{CP}^1$ is simply connected, every such isotopy must be Hamiltonian, so every such symplectomorphism is Hamiltonian. All of these symplectomorphisms are monotone.*

Thus it suffices to consider symplectomorphisms which are rotations about a fixed axis, generated by some function H. Each fixed point has Lef-

schetz number (and hence CZ index) equal to the Morse index of the corresponding critical point of H. Since their index difference is 2, the differential is trivial and HF is isomorphic to $H^*(S^2)$.

Example 3.1.6. *Now for closed surfaces Σ_g for $g > 1$, note that by Riemann-Roch, for any Riemann surface structure on Σ_g there exists a k so that $(T\Sigma_g)^{\otimes k}$ has cohomology concentrated in degree 1, providing an embedding in projective space $\iota : \mathbb{P}(H^0((T\Sigma_g)^{\otimes k}$ so that $(T\Sigma_g)^{\otimes k} = \iota(\mathcal{O}(1))$. The fixed-point Floer homology of such symplectomorphisms is calculated in [3].*

3.2 The Non-Monotone Case

We will later consider symplectomorphisms which are not monotone. In such cases, it is no longer possible to control areas of bubbles and trajectories purely in terms of their index. In fact there may be trajectories of arbitrarily large area for a given Conley-Zehnder index, preventing a direct application of Gromov compactness. We will be concerned with the following weaker notion of montonicity.

Definition 3.2.1. *A symplectic manifold (M, ω) of dimension $2n$ is **weakly monotone** if one of the following conditions is satisfied.*

1. *$\omega(A) = \lambda c_1(A)$ for every $A \in \pi_2(M)$ where $\lambda \geq 0$.*
2. *$c_1(A) = 0$ for every $A \in \pi_2(M)$.*
3. *For homology classes A in the image of the Hurewicz map $\pi_2(M) \to H_2(M;\mathbb{Z})$ such that $\omega(A) > 0$, $c_1(A) \geq 3 - n$ implies that $c_1(A) \geq 0$.*

These conditions are sufficient to rule out bubbling off of spheres with positive area whose index is sufficiently large, but they do not entirely exclude the existence of holomorphic spheres or periodic trajectories with index not proportional to their area. As such, for a given pair of periodic orbits with index difference one, for example, an trajectory connecting these could a priori have area unrelated to the index difference, preventing a direct application of Gromov compactness. A given convergent sequence of trajectories could then have index zero components in the limit with arbitrarily large positive area. To keep track of such degenerations we will need the following definition, following [11, 12, 8].

Definition 3.2.2. *Let G be an Abelian group, R a ring, and $N : G \to \mathbb{R}$ a homomorphism. The **Novikov ring** Nov(G,N;R) is the set of formal sums*

$$\sum_{g \in G} a_g \cdot g, a_g \in R$$

such that for every $C \in \mathbb{R}$, the set $\{g \in G | N(g) < C, a_g \neq 0\}$ is finite.

This is a ring with the convolution product

$$(\sum_{g \in G} a_g \cdot g)(\sum_{g \in G} b_g \cdot g) := \sum_{g \in G} \sum_{h \in G} a_h b_{hg^{-1}} \cdot h$$

In the path space setting above, then, we will take

$$\Lambda_F = \mathrm{Nov}(\mathrm{Ker}\psi, -\langle \mathfrak{Y} \rangle; \mathbb{Z}).$$

The appropriate Novikov ring for the situation at hand is most easily described in the path space setting for Floer homology. Suppose that (M, ω) is a weakly monotone symplectic manifold, and ϕ a symplectomorphism, with Ω_ϕ the twisted path space. We interpret the flux α as a perturbuation of the action 1-form $\mathfrak{Y}$ on the configuration space $\mathcal{C}$ - for the closed 1-form $\alpha \in H^1(M;\mathbb{R})$, there exists a cover $\tilde{M} \to M$ on which α is an exact form dH. The action functional we seek will then be defined on a cover of the path space for $\tilde{M}$, where we perturb the action functional by H.

$$\Lambda_\alpha := \mathrm{Nov}(\mathrm{Ker}\psi,\ -[\mathfrak{Y} + [\tilde{H}];\mathbb{Z}). \tag{3.6}$$

One can think of $\tilde{H}$ as determining a local system on $\mathcal{C}$, which in fact descends to a local system on the fixed-point set of ϕ by restricting to the constant paths.

The elements of the kernel of the spectral flow correspond to index zero periodic trajectories in the base configuration space, and the Novikov ring allows us to keep track of the (infinite) possible number of such trajectories by filtering them with respect to their area. The resulting chain complex is a module over this ring. With these coefficients in place, for ϕ homologically degenerate the differential in $HF_*(\phi)$ is well-defined as before. The resulting group $HF(\phi, \alpha)$ necessarily depends on some $\alpha \in H^1(T^2;\mathbb{R}^2/\mathbb{Z}^2)$ encoding the flux of the perturbation used to define the Novikov ring.

As this weaker formulation of monotonicity does not rule out the possibility of sphere bubbles, an argument from [8] rules out any hypothetical sphere bubbles arising in convergent sequences of trajectories with fixed area and index. Briefly, there exist generic choices of Hamiltonian and complex structure such that the the orbits of fixed points are a codimension $2n$ set, images of $c_1 = 1$ spheres are codimension 2, and images of $c_1 = 0$ spheres are codimension 4. Thus it is shown that the set of points where candidate sphere bubbles intersect trajectories of index $\mu(u) \leq 2$ is of dimension $2c_1 + \mu(u) - 3$, negative for spheres of index 0, and thus generically there are no such points.

So, in considering a limit of index 2 trajectories, these may degenerate into broken trajectories and sphere bubbles, with index sum 2. Generically there are no trajectories with negative index and the above rules out index $2c_1 = 0$ bubbles. So if positive index bubbles appear then the limit trajectories must have nonpositive index sum, but genericity implies that the only possibility is then a periodic index zero trajectory. But the limiting sphere must have index $2c_1 = 2$, and by the above argument index sphere do no intersect $\mu(u) = 0$ trajectories. This is the required contradiction.

Example 3.2.3. *When $g = 1$, we take the linear symplectic form on T^2 from $\mathbb{R}^2$ so that the induced symplectic form in T^2 is weakly monotone. Note that Hamiltonian isotopy classes of area-preserving maps of the torus with this form are in bijection with lattice-preserving linear maps, i.e. $SL_2(\mathbb{Z})$. Then we may calculate the fixed points directly for some $A \in SL_2(\mathbb{Z})$ by considering the equation $Ax = x$ over $\mathbb{Q}^2$. By weak monotonicity the coefficients of Floer homology should lie in a Novikov ring. However, when $Id - A$ is injective, an argument with the long exact sequence on homotopy implies that $\pi_1(\Omega_A)$ is trivial, and so the period homomorphism is automatically zero.*

3.3 Families of Symplectic Forms and Variation of HF with Parameters

In the previous section we saw that if sphere bubbles do not appear, the sole difference between the monotone and non-monotone cases is the appearance of periods, a phenomenon which can be described precisely by considering the algebraic topology of Ω_ϕ. The index homomorphism is defined on elements of $\pi_1(\Omega_\phi)$ for a choice of $\gamma \in \pi_0(\Omega_\phi)$, and we are interested in its kernel. By the above argument and the fact that $c_1(T^{vt}(\mathcal{X}) = 0$ since the fibers are genus 1 Riemann surfaces, hence the Chern number is 0 and we see that the kernel of the index is the entirety of $\pi_1(\Omega_\phi)$.

On the other hand, we may calculate the area of these periods by recalling that perturbations b_t of the symplectic connection correspond to elements of the set of co-invariants for ϕ, which lie in $H^2(T_\phi)$. To show that b_t pairs positively with some element k of Ker(ind), we restrict to neighborhoods in which T_ϕ is trivializable. The construction of the b_t ensures that we may write them as $d(\chi_i \cdot a_t \wedge dt)$, for a_t a closed 1-form on T^2. If this 1-form is Poincaré dual to a curve $\gamma \in T^2$, note that γ fits into the Wang sequence for homology and hence can be thought of as an element of $\pi_1(\Omega_\phi)$. Put differently, γ descends from $\mathbb{R} \times T^2$, since γ is ϕ-invariant. So $\frac{dk}{dt}$ is the sum of $\frac{d\gamma}{dt}$ and $\frac{d}{dt}$ in the trivial neighborhood $U_i \times T^2$, so $d(\chi_i a_t \wedge dt)$ must evaluate positively. It follows that the integral of b_t over k is positive.

To see that this cannot be canceled by the integral along k of any symplectic connection Ω with monodromy representing ϕ, note simply that since

k is transverse to the fibers while in trivial neighborhoods the symplectic connection is simply the pullback to $U_i \times T^2$ of the symplectic form on T^2, which vanishes on Tk since $\frac{d}{dt}$ is tangent to Tk.

Example 3.3.1. *To futher develop the picture of how Floer homology depends on these parameters, we describe the possible Floer homology groups for a given power of a Dehn twist τ and 1-form α on T^2 an incorporate the previous example where $\alpha = 0$ and $HF(\tau) \cong H_*(S^1)$.*

When $\alpha \neq 0$, the critical points of the perturbed action functional still give a chain complex corresponding to $C_*(S^1)$, but the homology will be twisted by a local system determined by α. This is proven in [18] Corollary 3.5.7 in the following form. For a tubular neighborhood of a Lagrangian L, thought of as a neighborhood of the zero section in T^*L, let α be a closed 1-form with graph another Lagrangian L' in T^*L. Then the Floer homology $HF(L, L')$ is isomorphic to the Novikov homology $HN(L, [\alpha], \mathbb{Z}_2)$ with coefficients twisted by $\alpha \in H^1(M)$ pulled back to L.

In this way, when $\alpha \neq 0$ one can calculate homology with twisted coefficients of S^1, which vanishes. On the other hand, when $\alpha = 0$ we recover the homology of S^1 with untwisted coefficients, which does not vanish.

This failure of continuity owes to the fact that the kernel of the action functional jumps at $t = 0$. However, when this kernel does not drop in rank, Floer homology for different parameter values can be related in a precise way.

3.3.1 One-parameter Families of Floer Systems

In order to compute the invariants to be constructed later, we will need to relate the fixed-point Floer homologies induced by symplectomorphisms preserving distinct symplectic forms. A description of such a change in Floer homology is given in [11, 12], and [28] describes an application of the result for a specific type of variation in the symplectic form.

Theorem 3.3.2. *Let $\{\kappa_t\}$ be a smooth family of symplectic forms on a manifold Y such that the symplectic manifolds (Y, κ_t) are weakly monotone, let ϕ_t be a smooth family of diffeomorphisms such that $\phi_t^*(\kappa_t) = \kappa_t$, and let $\mathcal{P} \in \pi_0(\Gamma(\mathcal{M}_1))$ be such that the action functionals $\mathcal{Y}^t$ for the Floer complexes $CF(\phi_t, \mathcal{P})$ are H^1-**codirectional**, in the sense that for $K := Ker(\psi)$ where ψ is the spectral flow homomorphism, we have*

$$[\mathcal{Y}^t]|_K = f(t)[\mathcal{Y}]|_K$$

where f is a nonnegative continuous function; thus where Λ_t is the Novikov ring over which $CF(\phi_t, \mathcal{P})$ is naturally defined, Λ_0 is a module over Λ_t with $\Lambda_0 = \Lambda_t$ when $f(t) \neq 0$. Then $CF(\phi_0, \mathcal{P})$ is chain homotopy equivalent to $CF(\phi_1, \mathcal{P}) \otimes_{\Lambda_1} \Lambda_0$.

Note that a choice of path of complex structures is implicit in the definition of the action functionals $\mathcal{Y}^t$. In fact, for symplectic manifolds as in the above theorem, there is a Baire set of complex structures which determine an appropriate path of action 1-forms $\mathcal{Y}^t$. WIth this path the above theorem induces a identification of Floer cochain complexes for two symplectomorphisms of monotone(or weakly monotone) symplectic manifolds and a symplectic isotopy between them.

Chapter 4

Defining the Invariant HSP_*

In this section we define the invariant HSP_* using the construction of the stable pair spaces and the machinery of fixed-point Floer homology. We also establish the invariance of HSP_* with respect to the many choices made in the construction.

4.1 Families of Stable Pair Spaces

While we will want to work with S^1-families of complex-analytic stable pair spaces, the constructions of [27] (1.11), (1.16) produce an algebraic moduli space of stable pairs. To bridge the gap between the algebraic and analytic settings, we adapt the (analytic) GIT construction in [22] to the relative case. The following definition describes the class of objects we will construct.

Definition 4.1.1. *A* semi-holomorphic structure *on a fiber bundle $F \hookrightarrow E \to B$ consists of a cover of E by neighborhoods $U_i \times V_i$ where $U_i \subset B, V_{i_j} \subset F$, and the restrictions of transition functions to the V_i are holomorphic.*

Proposition 4.1.2. *Let $\mathcal{X} \xrightarrow{\pi} B$ be a smooth fiber bundle over a compact base B whose fibers are closed surfaces Σ_g, equipped with a vertical complex structure J inducing a semi-holomorphic structure. Also let $\Lambda \to \mathcal{X}$ be a* *fiberwise holomorphic line bundle over $\mathcal{X}$. Then there is a smooth manifold $\mathcal{M}_{\Lambda,\mathcal{X}} \to B$ with semi-holomorphic structure whose fiber over $b \in B$ is the space of σ-stable pairs on X_b with determinant Λ_b.*

Proof. Let E be a smooth complex rank 2 vector bundle over $\mathcal{X}$ with Hermitian metric such that $c_1(det(E)) = c_1(\Lambda)$. We also suppose also that the degree of E is sufficiently large so that the space of sections of E over $\mathcal{X}_b$ (note here that $\mathcal{X}_b$ has a holomorphic structure) is globally generated (c.f. [22], Assumption A2], [27] (1.9)]). We will construct the family $\mathcal{M}_{\Lambda,\mathcal{X}} \to S^1$ as a quotient of the configuration space

$$\mathcal{C} := \mathcal{A}(E) \times C^\infty(\Sigma; E) \tag{4.1}$$

where $\mathcal{A}$ is the space of connections in E, and $C^\infty(\Sigma; E)$ is the space of smooth sections of E. As in the absolute case, choices of complex structure and connection in E determine a fiberwise holomorphic structure on fibers of $\Lambda \to \mathcal{X}$.

Then, families $\mathcal{M}_i$ with semi-holomorphic structures will arise by first producing a bundle H over B whose fibers consist of L^2_k-sections of $\Omega(E)$ or $\Omega(EndE)$. Although this construction of families of Hilbert spaces is not new, we outline it here for completeness. Given $\{U_i\}$ a cover of B by trivial neighborhoods for $\mathcal{X} \to B$, we have diffeomorphisms $\phi_i : \pi^{-1}(U_i) \to U_i \times \Sigma_g$. Note also that by restricting E to $\pi^{-1}(U_i)$, we obtain families of vector bundles $\phi_i^*(E_{\pi^{-1}(U_i)}) \to \Sigma_g$.

In each local trivialization the vertical complex structure J pulls back to a family of complex structures on Σ_g parametrized by U_i. Then for every

$b \in U_i$, we have a complex structure j_b on Σ_g given by

$$j_b := \rho_{ij}(b)^*(J).$$

The complex structure determines a conformal class of metric on Σ_g, and a choice of representative in this class yields a family of reference volume forms $\{dvol_{g_t}\}$ on Σ_g in each trivial neighborhood $U_i \times \Sigma_g$. Without loss of generality we choose the g_i so that each fiber has the same volume, and also choose a reference metric g on Σ_g giving each $\Sigma_g \times \{x\}$ the same volume as $\pi^{-1}(x)$.

Now for each $b \in U_i$, $dvol_g - \phi^*(dvol_{g_b})$ must be an exact form, since they determine the same volume on Σ_g, so a Moser argument yields a diffeomorphism ψ_b of Σ_g such that $(\psi_b \circ \phi_i)^*(g_b) = g$ which depends smoothly on $b \in B$. These metrics and diffeomorphisms will induce L^2_k metrics on the space of smooth forms $\Omega^0(\mathcal{X}_b; E_b)$ and $\Omega^{0,1}(X; End(E))$ as follows. Over $\mathcal{X}|_{U_i}$ the maps ρ_{ij} determine Hilbert spaces over any $b \in B$ by completing the space of smooth forms $\Omega^0(X; E)$, with respect to the metrics

$$\int_{\Sigma_g} \langle \phi, \phi \rangle (\rho_{ij}(b)^{-1})^*(dvol_g)$$

and similarly for $\Omega^{0,1}(X; End(E))$. More precisely, in each U_i there is a trivialization

$$\begin{aligned} & \coprod_{b \in U_i} (t, L^2_k(\Sigma_g^{j_b}; End(E), vol_{g_b}) \oplus L^2_k(\Sigma_g^{j_b}; E), vol_{g_b}) \\ \to \; & \coprod_{b \in U_i} (t, L^2_k(\Sigma_g^{j}; End(E), vol_g) \oplus L^2_k(\Sigma_g^{j}; E), vol_g) \end{aligned} \tag{4.2}$$

of the Hilbert bundle which is an isometry on fibers. In each trivial neighborhood this allows us to consider the variation with respect to b in a fixed ambient Hilbert space.

It is then straightforward to verify that these transition functions determine sequences of topological bundles of Hilbert spaces over B

$$0 \longrightarrow L^2_{k+1}(\Omega^0(\Sigma_g; End\, E|_{\Sigma_g})) \xrightarrow{d_1} L^2_k(\Omega^{0,1}(\Sigma_g; End\, E|_{\Sigma_g})) \oplus L^2_{k+1}(\Omega^0(\Sigma_g; End\, E|_{\Sigma_g})) \xrightarrow{d_2} L^2_k(\Omega^{0,1}(\Sigma_g; End\, E|_{\Sigma_g})) \longrightarrow 0$$

These bundles fail to have a global smooth structure, since the action of the ρ_{ij} fails to be smooth on L^2_k; pre-composing an L^2_k-function with a smooth diffeomorphism may not even yield a continuous function for general k. There is, however, a sub-bundle of H which does admit a natural smooth structure. We denote by $\mathcal{H} \subset \mathcal{C}$ the subset of *holomorphic pairs*:

$$\mathcal{H} = \{(\partial_J, \phi) \,|\, \overline{\partial}_J(\phi) = 0\}$$

This sub-bundle is the solutions to the equation $\overline{\partial}_J = 0$, whose restriction to each fiber is an elliptic equation. Elliptic regularity then ensures that each fiber $\mathcal{H}_b$ consists of smooth connections in E paired with smooth sections of E, since this is precisely the absolute case. It follows that the transition functions defined for the bundles above act on smooth objects.

Then, trivializations ψ_i on trivial neighborhoods $U_i \times \Sigma_g$ of $\mathcal{X}$ pull back the semi-holomorphic structure of $\mathcal{X}$ to families of complex structures j_b on Σ_g, denoted by $\Sigma_g^{j_b}$. Similarly the bundle E pulls back to a smooth complex vector bundle on each $\Sigma_g^{j_b}$ via ψ_i. Since the holomorphic pairs depend on the complex structures j_b in trivial neighborhoods, we write $\mathcal{H}_b^i$ for the holomorphic pairs over a point $b \in U_i$.

Note that $\mathcal{H}$ is necessarily still infinite-dimensional, as we have yet to take any quotient by the gauge group of holomorphic fiber-preserving general linear automorphisms of E, denoted $\mathfrak{G}_{\mathbb{C}}$. Following the GIT template, we quotient by a smooth action on $\mathcal{H}$ by some Hilbert Lie group. This can be constructed with standard methods, defining holomorphic gauge transformations as a subset of the linear space $L_k^2(End(E))$.

Now, by adapting the slice theorem in [22] we construct a neighborhood of a holomorphic stable pair whose fibers over an open set in S^1 are are slices for the action in each fiber. First, we restrict to a subset of each fiber on which the $\mathfrak{G}_{\mathbb{C}}$-action is well-behaved, namely the holomorphic irreducible pairs:

$$\hat{\mathcal{H}} = \{(\partial_J, \phi) \,|\, \mathfrak{G}_{\mathbb{C}} \text{ acts freely}\}.$$

The group $\mathcal{G}_{\mathbb{C}}$ acts smoothly on $\hat{\mathcal{H}}$ and holomorphically on the fibers, preserving them. Consider the bundle $\mathcal{S}_{(\overline{\partial}_E, \phi)}$ over $\hat{\mathcal{H}}$ with the following fibers.

$$\mathcal{S}_{(\overline{\partial}_E, \phi)} = \{(\alpha, \eta) \in \hat{\mathcal{H}} \,|\, d_1^*(\alpha, \eta) = 0, ||(\alpha, \eta)|| < \epsilon\}$$

As $\hat{\mathcal{H}}$ is open and dense, take a neighborhood V in $\hat{\mathcal{H}}$ containing a holomorphic irreducible pair $(\overline{\partial}_E, \phi)$, lying in some trivializing neighborhood $U_i \times \hat{\mathcal{H}}_t$. That is, we intersect V with the orthogonal complement to the tangent space to the orbit of $(\overline{\partial}_E, \phi)$ in $\hat{\mathcal{H}}$. Having already restricted to a trivial neighborhood, it is no loss to restrict V to a subset which is of the form $(-\epsilon, \epsilon)^n \times B_\epsilon(\sigma)$ for

$dim(B) = n$ and σ a local section of $\hat{\mathcal{H}}$ with $(\overline{\partial}_E, \phi)$ in its image.

Then the arguments of [22] apply in each fiber of $(-\epsilon, \epsilon)^n \times B_\epsilon(\sigma)$, producing for each $\mathcal{X}_b$ a slice for the fiberwise action of $\mathcal{G}_{\mathbb{C}}$ and a family of Kuranishi maps. That is, each $(-\epsilon, \epsilon) \times B_\epsilon(\sigma)$ splits as $(-\epsilon, \epsilon) \times \mathcal{G}^b_{\mathbb{C}} \times \mathcal{S}$, where $\mathcal{S}$ is a slice for the action on each fiber. By construction these slices correspond to neighborhoods in each fiber of $\mathcal{M}_i$ of the orbits containing the holomorphic pairs σ.

Lastly, to see that these charts vary smoothly in the base directions, recall that we are simply using the isometric trivializations of (2.4). Having constructed fiberwise coordinate charts in the local trivializations which depend on the smooth metrics and complex structures chosen in $E \to \mathcal{X}$, composing them with the smooth maps back into $\hat{\mathcal{H}}$ determines slices that vary smoothly with b. As this slice neighborhood is mapped diffeomorphically to an open neighborhood in $\mathcal{M}_i$, we have constructed a coordinate chart on $\mathcal{M}_i$ which is semi-holomorphic by the above argument, as required. □

4.1.1 The Action of the Mapping Class Group on Stable Pairs

We now use Proposition 4.1.2 to describe a homomorphism from the mapping class group $\pi_0(Diff^+(\Sigma_g))$ to the group of symplectomorphisms of a stable pair space.

Fix a closed oriented surface X of genus g and a complex line bundle L. Also let $\Gamma = \pi_0(\mathrm{Diff}^+(X))$. Given X, $\phi : X \to X$, we define the mapping torus over S^1

$$T_\phi := (X \times \mathbb{R})/(x,t) \sim (\phi(x), t+1) \tag{4.3}$$

which is a fiber bundle over S^1 with fiber X and the isomorphism class of T_ϕ only depends on $[\phi]$ in Γ.

We also require an extension of the complex line bundle $L \to X$ to $L_\phi \to T_\phi$. To specify such an L_ϕ it suffices to give a bundle isomorphism $\widetilde{\phi} : L \to \phi^*(L)$. The isotopy classes of pairs $(\phi, \widetilde{\phi})$ form the **extended mapping class group** $\widetilde{\Gamma}$ which fits into the following exact sequence:

$$0 \longrightarrow H^1(X;\mathbb{Z}) \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 0 \tag{4.4}$$

Here we make the standard identification of $H^1(X;\mathbb{Z})$ with homotopy classes of maps from X to S^1.

However, this method for specifying L_ϕ has some redundancy; by the Wang sequence, the kernel of the restriction map is the group of co-invariants

$$H^1(X)_\phi := H^1(X)/(Id - \phi^*)$$

The compatibility of the short exact sequence (2.4) with the Wang sequence is expressed by commutativity of the following diagram:

$$\begin{array}{ccccccc}
 & & H^1(X;\mathbb{Z}) & \longrightarrow & \widetilde{\Gamma} & \longrightarrow & \Gamma \\
 & & \updownarrow & & \downarrow c_1 & & \downarrow c_1 \\
H^1(X;\mathbb{Z}) & \xrightarrow{Id - \phi^*} & H^1(X;\mathbb{Z}) & \xrightarrow{\delta} & H^2(T_\phi;\mathbb{Z}) & \xrightarrow{restr.} & H^2(X;\mathbb{Z})
\end{array}$$

where the vertical maps marked c_1 take the Chern class of the associated line bundle on T_ϕ.

We now fix a complex structure on X and a holomorphic structure on L. The data (X, L, σ) determines a space of stable pairs using the notation of Chapter 2. If σ is a non-critical value, $M_{\sigma,L}$ is a smooth complex manifold whose name we abbreviate to M. Also choose a Kähler form κ on M.

Construction 4.1.3. *We will describe a homomorphism from $\tilde{\Gamma}$ to the symplectic mapping class group of the stable pair space M,*

$$\theta : \widetilde{\Gamma} \to \pi_0(Symp(M, \kappa)) \tag{4.5}$$

where $\theta([\phi, \tilde{\phi}])$ depends only on $[\phi] \in \Gamma$ together with $c_1(L_{\widetilde{\phi}}) \in H^2(T_\phi; \mathbb{Z})$ where $L_{\widetilde{\phi}}$ is the line bundle determined by $\widetilde{\phi}$.

The construction essentially goes as follows: diffeomorphism types of mapping tori equipped with line bundles (and additional contractible choices) determine a semi-holomorphic structure on L_ϕ. Then Proposition 2.3 ensures that there is a semi-holomorphic family of stable pair spaces $\mathcal{M}_i$. Choosing a Kähler form κ on a reference fiber M_i and possibly an element $\alpha \in H^1(X)^\phi$ the set of co-invariants, Lemma 2.6 shows that (κ, α) determines a cohomology class of symplectic connection in $\mathcal{M}_i$. After producing a connection representing this class, the resulting parallel transport map Φ will be the required symplectomorphism. So given a mapping torus as in (2.3) and a smooth complex line bundle L_ϕ over it, we choose a complex structure J on the (real) rank 2 vector bundle $T^{vt}(T_\phi)$, producing a circle's worth of smooth closed genus g

Riemann surfaces $\{X_t\}$. The bundle L_ϕ then determines a smooth complex line bundle $\mathcal{L}$ over each Riemann surface fiber. The space of pairs of Hermitian metric in L_ϕ and unitary connection in L_ϕ is (weakly) contractible, and any such pair produces an S^1-family of holomorphic line bundles $\mathcal{L}_t \to X_t$. Any choice of reference fiber (X, Λ) determines a basepoint for this family. Applying Proposition 2.3, we have a family $\mathcal{M}_i$ with reference stable pair space M_i.

Lemma 4.1.4. *Let $\mathcal{M}_i$ be a family of stable pair spaces with reference fiber M_i.*

- *The restriction map $H^2(\mathcal{M}) \to H^2(M_i)$ is surjective.*
- *For families $\mathcal{M}$ of stable pairs over S^1, the action of Φ on $H^2(M)$ is trivial, and $H^2(M) \cong H^2(\mathcal{M})$.*

Proof. For the first statement, by (5.1) in [27], the ample cone of each space M_i for $i \geq 1$ is generated by the hyperplane class in $\mathbb{P}H^1(\Lambda^{-1})$ and the exceptional divisor in M_1. The flips relating these spaces to the remaining M_i induce isomorphisms on H^2, as birational maps are isomorphisms away from sets of codimension at least 2. Thus it suffices to show that these classes are preserved by the monodromy, as their images under the flips generate $H^2(M_i)$.

Note first that $\mathcal{M}_0$ is the projectivization of a (trivial) family of complex vector spaces $H^1(\mathcal{L})$ over S^1. Then, on a reference fiber of the vector bundle, a linear subspace descending to a hyperplane in a reference fiber M_0 extends naturally to a sub-bundle of $H^1(\mathcal{L})$. By construction, the projectivization of this sub-bundle is Poincaré dual to a cohomology class on $\mathcal{M}_0$ which restricts to the hyperplane class on M_0 the reference fiber.

In the case of the divisor, we take a similar approach and construct a Poincaré dual submanifold in $\mathcal{M}_1$ representing the divisor class. In the setting of Construction 2.5 we have an embedding in each fiber of $\mathcal{M}_0$ of a curve X_t. Also, by [27] (3.19) we have an identification of $\mathcal{M}_1$ with the blowup of $\mathcal{M}_0$ along $\mathcal{X}$. Then the bundle of exceptional divisors represents a submanifold of codimension 2 in $\mathcal{M}_1$, and by construction it restricts to the exceptional divisor in M.

For the second statement, recall that exactness of the Wang sequence for the fibration $\mathcal{M} \to S^1$ implies that those classes in the image of the restriction map $H^2(\mathcal{M}) \to H^2(M)$ coincides with the kernel of the map $Id_{H^2(M)} - \Phi^*$. By the above paragraph, the classes generating $H^2(M)$ are induced from classes on $H^2(\mathcal{M})$, so they lie in this image and hence Φ^* acts as the identity. Then the fact that the second cohomology groups coincide follows since $Id_{H^2(M)} - \Phi^*$ is zero and $H^1(M) = 0$.

$\square$

Now, when one has a smooth fiber bundle $Y \to B$ and smoothly varying symplectic forms ω_b in the fibers, one can find a closed 2-form $\eta \in \Omega^2(Y)$ restricting to ω_b on each fiber Y_b if there is a class $w \in H^2(Y)$ with $w|_{Y_b} = [\omega_b]$ for each $b \in B$ (see [MS, Chapter 6]).

Given a Kähler form κ on the reference fiber M, this determines a cohomology class $[\Omega]$ in $\mathcal{M}$ by the isomorphism of Lemma 2.4, with a family of restrictions $[\Omega_b]$ to all other fibers. The space of Kähler forms in a given cohomology class is convex, so the space of forms representing $[\Omega_b]$, then, is a bundle with contractible fiber over K. Any section of this bundle gives a family of forms representing the $[\Omega_b]$, and the previous paragraph establishes that there is no obstruction to extending κ to a closed and fiberwise Kähler 2-form Ω on $\mathcal{M}$. The independence of the construction with respect to these choices will be proven later.

We then define $\theta(\widetilde{\gamma})$ to be the symplectic monodromy Φ of Ω. By symplectic monodromy we mean that Ω induces a vector field on $\mathcal{M} \to S^1$ which is a section of the Ω-complement to $T^{vt}(\mathcal{M})$. Parallel transport with respect to this complement induces a diffeomorphism of the fiber M which preserves $\Omega|_M = \kappa$.

To show that this construction produces a homomorphism $\tilde{\Gamma} \to Symp/Ham(M_i)$, consider two symplectic fiber bundles $(\mathcal{M}, \Omega), (\mathcal{M}', \Omega')$ over S^1 with monodromies Φ, Φ'. We will produce a new fiber bundle $\mathcal{M}''$ with monodromy Hamiltonian isotopic to $\Phi \circ \Phi'$.

Fix any two semi-holomorphic structures on $\mathcal{M}, \mathcal{M}'$, with reference fibers both identified with (M, κ). Without loss of generality the charts $\{V_k \times \Sigma_g\}$, $\{V_k' \times \Sigma_g\}$ have the property that there are unique charts $V \times \Sigma_g$, $V' \times \Sigma_g$ which contain the reference fiber, and the intersection of any two $V_i \cap V_j, V_i' \cap V_j'$ is either empty or in a connected open subset of S^1. Then, to construct the required bundle, we can form a cover of S^1 by rescaled neighborhoods

$\{\frac{1}{2}V_k\}, \frac{1}{2}\{V_k' + \frac{\pi}{2}\}$, with the same transition functions as in $\mathcal{M}, \mathcal{M}'$. Then, this new symplectic fiber bundle $\mathcal{M}''$ will have monodromy isotopic to $\Phi \circ \Phi'$, note that without loss of generality the forms Ω, Ω' are identical in a neighborhood of the reference fiber, and so we may patch them together to obtain a form Ω'' with the desired properties.

Lastly we consider the choices made in the above construction. Given any two such choices for all possible parameters (complex structures, Hermitian metrics etc.), they induce symplectic connections Ω_0, Ω_1. The path interpolating between them yields an exact deformation of Ω in the following standard way. Denote these by $\mathcal{J}_t$ and g_t for $t \in [0,1]$. These determine almost complex structures and metrics on the cylinders

$$Z_\phi := T_\phi \times [0,1]$$

so Proposition 2.3 produces families of stable pair spaces over this cylinder Z_ϕ from these choices. Then Construction 2.5 gives a path of symplectic connections Ω_t determining a closed 2-form on Z_ϕ which restricts on the boundary to Ω_i. The forms Ω_i must then be cohomologous, hence their induced symplectic parallel transports are Hamiltonian isotopic symplectomorphisms of M. So the image of the map $\tilde{\theta}$ is a well-defined element of the symplectic mapping class group.

End of construction.

It is interesting to consider the symplectomorphism induced on M when $[\phi] = Id$ but the corresponding element of $H^1(X;\mathbb{Z})$ is nontrivial. In this case we have an injection $H^1(X;\mathbb{R}) \to H^2(S^1 \times X;\mathbb{R})$ by (2.7), so the Hamiltonian isotopy class of the induced map Φ is completely determined. In general, however, the homomorphism θ constructed above does not factor through the mapping class group.

Proposition 4.1.5. *Up to N-torsion in $H^2(\mathcal{X})$, any nontrivial element of the group $H^1(X)$ (the kernel of $\tilde{\Gamma} \to \Gamma$), acts nontrivially on M_g.*

Proof. Beginning with $\mathcal{M}_0$ and $\mathcal{M}_1$, our strategy will be to localize along the exceptional divisor, whose bundle structure is amenable to computations with Chern polynomials. By calculating $c_{n-1}(T^{vt}\mathcal{M}_i)$, we construct a map $\tau : H^*(\mathcal{M}_{i,L}) \to H^2(\mathcal{X})$ with image $Nc_1(L_\phi)$, for N a constant independent of $\mathcal{X}$, $\mathcal{M}_i$ and element of $H^1(X)$. Then since the monodromy Φ determines the isomorphism class (and hence the cohomology) of $\mathcal{M}_1$, isomorphic $\mathcal{M}_1$ must come from identical line bundles up to torsion elements we specify later.

For the remaining spaces, the birational maps to other $\mathcal{M}_i$ preserve the next-to-top Chern class $c_{n-1}(TM_1)$ where $n = dim(M_0)$, so it suffices to show that $c_{n-1}(T^{vt}\mathcal{M}_1)$ distinguishes the choice of line bundle embedding $\mathcal{X} \hookrightarrow \mathcal{M}_0$, as all other $\mathcal{M}_i$ carry the same class.

Recall from Lemma 2.3 that the family $\mathcal{M}_0$ arises as the projectivization of a (trivial) vector bundle $V \to S^1$ of rank $n+1 = \deg \Lambda +$ genus $(\mathcal{X}) - 1$. The Chern polynomial of the normal bundle ν to $\mathcal{X}$ in $\mathcal{M}_0$ is determined by the relation

$$c_t(T^{vt}(\mathcal{M}_0)|_{\mathcal{X}} = c_t(T^{vt}(\mathcal{X}))c_t(\nu) \tag{4.6}$$

Now, if $u = c_1(\mathcal{O}_{\mathbb{P}V(-1))} \in H^2(\mathcal{M}_0)$, this class pulls back to the hyperplane class on each projective space fiber, the left hand side can be expressed straightforwardly as $(1-ut)^{n+1}|_{\mathcal{X}}$. Note that $-u$ restricts to $\mathcal{X}$ as $-c_1(L_\phi)$. Moreover, restricting to $\mathcal{X}$, all but the degree 0 and 2 classes vanish, so

$$c_t(T^{vt}\mathcal{M}_0)|_{\mathcal{X}} = 1 + (n+1)c_1(\mathcal{L}_\phi)t.$$

Now since $c_t(T^{vt}(\mathcal{X}) = 1 + c_1(T^{vt}(\mathcal{X})t$ is a unit in $H^*(\mathcal{X})[t]$ with inverse $1 - c_1(T^{vt}(\mathcal{X})t$, we arrive at

$$\begin{aligned} c_t(\nu) &= (1 + (n+1)c_1(\mathcal{L}_\phi)t)(1 - c_1(T^{vt}(\mathcal{X})t)) \\ &= 1 + ((n+1)c_1(\mathcal{L}_\phi) - c_1(T^{vt}(\mathcal{X})t \end{aligned} \tag{4.7}$$

A similar observation in M_1 yields

$$0 \to T^{vt}(\mathbb{P}\nu) \to T(\mathcal{M}_1)^{vt}|_E \to \mathcal{O}_{\mathbb{P}\nu(\mathcal{X})}(-1) \to 0 \tag{4.8}$$

From this we can deduce the Chern polynomial of the vertical tangent bundle to the projectivization $\mathbb{P}\nu$. Recall that the tangent bundle to $\mathbb{P}\nu$ fits in the Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}\nu} \to p^*\nu \otimes \mathcal{O}_{\mathbb{P}\nu}(-1) \to T^{vt}(\mathbb{P}\nu) \to 0$$

with $p^*(\nu) \otimes \mathcal{O}_{\mathbb{P}\nu}(-1))$, where $p : \mathbb{P}\nu \to \mathcal{X}$. It follows that

$$c_t(T^{vt}(\mathbb{P}\nu)) = c_t(p^*(\nu) \otimes \mathcal{O}_{\mathbb{P}\nu}(-1)).$$

For $\mathcal{M}_1$ the blowup of $\mathcal{M}_0$ relative to $\mathcal{X}$, let $e \in H^2(\mathcal{M}_1)$ be the cohomology class Poincaré dual to the exceptional divisor E, and let $v = e|_E \in H^2(\mathcal{M}_1)$.

Expressing the Chern polynomial of $T^{vt}(p^*\nu \otimes \mathcal{O}_{\mathbb{P}\nu}(-1))$ as a product of monomials involving the Chern roots ν_i of ν, we get

$$\begin{aligned} & c_t(p^*(\nu) \otimes \mathcal{O}_{\mathbb{P}\nu}(-1)) \\ = & \prod_{i=1}^{n-1}(1 + \nu_i - v) \end{aligned}$$

But then, by (2.9), all Chern classes c_k for $k > 1$ vanish, so any product containing terms $\nu_i \cup \nu_j$ vanishes. We may then express the above product as

$$c_t(p^*\nu \otimes \mathcal{O}_{\mathbb{P}\nu}(-1)) = (1 - vt)^{n-1} + c_1(\nu)t(1 - vt)^{n-2} \tag{4.9}$$

so the exact sequences (2.8), (2.10) give the full Chern polynomial $c_t(T\mathcal{M}_1)$ as

$$\begin{aligned} & \left[(1 - vt)^{n-1} + c_1(\nu)t(1 - vt)^{n-2}\right](1 - c_1(T^{vt}(\mathcal{X}))^2(1 - v) \\ = & (1 - vt)^n + tc_1(\nu)(1 - vt)^{n-1} \end{aligned} \tag{4.10}$$

Determining $c_{n-1}(T^{vt}\mathcal{M}_1)$ amounts to isolating the degree $n - 1$ terms.

To complete the calcluation of $c_t(T\mathcal{M}_1)$, we adapt the formula for the cohomology of a blowup to the relative case, obtaining an isomorphism

$$F_i : H^{2i}(\mathcal{M}_1) \to H^{2i}(\mathcal{M}_0) \oplus \left(\oplus_{j+k=i} v^j \cup H^{2k}(\mathcal{X})\right) \tag{4.11}$$

We may use F to define a map

$$\tau : H^{2i}(\mathcal{M}) \to H^2(\mathcal{X})$$

by identifying the image of a given class in the summand $v^{i-2}H^2(\mathcal{X})$ above. That is, for $x \in H^{2i}(\mathcal{M}_i)$, $v^{2i-2}p^*\tau(x) = x|_{\mathbb{P}\nu}$. Without loss of generality the monodromy Φ in $\mathcal{M}_i$ preserves the exceptional divisor, and hence the Wang exact sequence applied to the mapping torus $\mathcal{M}_i$ is compatible with this splitting (i.e. it preserves the summands). Now, as the reference fiber M_i is independent of the choice of L_ϕ, then F acting on $H^*(M_i)$ does not depend on the choice of L_ϕ. It then follows that the decomposition $H^*(\mathcal{M}_i)$ is also independent of L_ϕ.

Now the calculation of (2.12) shows that in the case $\mathcal{X} = \Sigma_g \times S^1$, if $c_1(L_\phi)$ is nonzero, it must have image $Nc_1(L) \in H^2(\mathcal{X})$, for N a universal constant coming from the binomial coefficients. Thus τ distinguishes the choices of line bundle (up to N-torsion in $H^2(\mathcal{X})$, and the $\mathcal{M}_i$ for different such choices cannot be isomorphic. □

This is one of several reasons which inform our choice to restrict to the subset of homologically non-degenerate elements of the mapping class group as in Theorem 1.2.

4.2 Relating θ and $MCG(X)$ in the Genus 1 Case

With the goal of relating the symplectomorphism Φ to a symplectomorphism of the Riemann surface X, we describe how the above method also produces a symplectic form on X and a symplectic connection on T_ϕ. Uniqueness of the resulting symplectic connection will again depend on the criterion that $Id - \phi^*$ be injective.

Note that by the construction of the stable pair spaces, the reference

Riemann surface X embeds into a projective space, inducing a symplectic form ω. In this way, we obtain a family of symplectic forms on the fibers of T_ϕ. We would like to produce a closed 2-form on the total space of T_ϕ restricting on the reference fiber to ω, so that symplectic parallel transport produces a map in class $[\phi] \in MCG^+(X)$. The remainder of this section will treat the case of genus 1 fibers. Homological non-degeneracy has a particularly clear interpretation here; since $MCG^+(X) \cong SL(2;\mathbb{Z})$, the condition from Theorem 1.2 that $Id - \phi^*$ be injective corresponds to $|Tr(\phi)| > 2$.

4.2.1 The Homologically Non-Degenerate Case

As for the stable pair spaces, the Wang exact sequence allows us to characterize the possible choices of symplectic connection. When ϕ is homologically non-degenerate, the form ω on X determines a class in $H^2(X;\mathbb{R})$, so the symplectic connection in $H^2(T_\phi)$ is determined by the kernel of $Id - \phi^* : H^1(X;\mathbb{R}) \to H^1(X;\mathbb{R})$, i.e. those classes on which ϕ^* acts trivially. The end result is then a symplectic fibration over S^1 whose monodromy produces a symplectomorphism ϕ preserving ω.

Now to construct the symplectic Floer homology groups for a symplectomorphism of (X, ω) representing $[\phi]$, note that the symplectic manifold in question is aspherical, and the fundamental group of the twisted loop space $\pi_1(\Lambda_\phi)$ is trivial for any connected component of Λ_ϕ as a consequence of injectivity of $Id - \phi^*$. It follows that the groups in question are necessarily defined over $\mathbb{Z}$, since periods and sphere bubbles cannot occur as a consequence of the underlying algebraic topology. The term $HF_*(\phi)$ in the statement of Theorem 1 is the Floer homology of this symplectomorphism ϕ.

Independence of $HF(\phi)$ from the choices made in its construction follows from the same continuation map arguments as in Chapter 2. We have already shown that the line bundle L_ϕ and the symplectic connection on T_ϕ) are completely determined by the element of the extended mapping class group $\tilde{\Gamma}$. The only choices left are again the metric used to determine the holomorphic structure on L and the vertical complex structure on T_ϕ, and these clearly induce a Hamiltonian isotopy of ϕ. We state this as a proposition below.

Proposition 4.2.1. *The map θ induces a homomorphism*

$$\rho : \widetilde{\Gamma} \to (Symp/Ham)(X, \omega) \tag{4.12}$$

where ω is the symplectic form on X induced by its projective embedding.

4.2.2 The Homologically Degenerate Case on the Torus

In the setting where $Id - \phi^*$ is not invertible, the above arguments involving the Wang exact sequence no longer hold, so we are in the situation of Section 3.2 involving flux and Novikov rings. These symplectomorphisms fail to be monotone, but we may still define Floer homology with an additional choice.

Proposition 4.2.2. *Consider a symplectic fibration $T_\phi \to S^1$ with T^2-fibers and symplectic connection Ω. There is a perturbation to Ω generated by a* *choice of nontrivial element α in $H^1(X;\mathbb{Z})$, and its monodromy is a symplectomorphism of X isotopic to the identity but not Hamiltonian isotopic, with flux α.*

Proof. We first show that it suffices to consider the case of $T_\phi \cong S^1 \times T^2$. For a perturbation Ω_α on a non-trivial family T_ϕ whose monodromy has flux α, without loss of generality its induced horizontal vector field is constant in a neighborhood of $1 \in S^1$. There is an isotopy of this vector field to one which is non-constant only within a trivial neighborhood for T_ϕ. Using the procedure of Construction 3.3 in reverse, we think of the connection in this neighborhood as the fiber sum of $(S^1 \times \Sigma_g, \Omega_\alpha)$ with (T_ϕ, Ω). The fact that these forms are isotopic to one another ensures that $(T_\phi, \Omega + \Omega_\alpha)$ and this fiber sum are cobordant as symplectic fiber bundles, and hence a continuation map argument shows that their parallel transport maps are at least Hamiltonian isotopic. Hence if we show that $(S^1 \times \Sigma_g, \Omega_\alpha)$ has monodromy with flux α, the claim follows.

In this case, the space of co-invariants $H^1(X)_\phi = H^1(X)$. To produce a perturbation to the symplectic connection on the total space representing α, the Thurston patching argument gives a closed 2-form on the total space. Pulled back to $\mathbb{R} \times T^2$ from T_ϕ, the connection is necessarily of the form $dt \wedge b_t$, where b_t is a family of 1-forms on T^2 satisfying $b_{t+1} = \phi^*(b_t)$. Using a straight-line path from this Ω to $0 \in H^2(T_\phi)$ it is clear that the parallel transport induced by the original is isotopic to the identity.

Now we verify that this connection gives a symplectomorphism with flux $\alpha \neq 0$, so it cannot be Hamiltonian isotopic to the identity. In the trivial bundle there is a canonical splitting $TS^1 \times T\Sigma_g$ so we can project out the base directions from the horizontal vector field, and what remains is a family of vector fields on the fiber whose time-1 flow induces the parallel transport. But then there is a standard method for obtaining the flux of a symplectic isotopy by integrating the dual 1-forms to these vector fields. By construction this is the form α, since the connection was of the form $\alpha \wedge dt$. $\square$

A similar argument establishes that the space of symplectic connections in T_ϕ which restrict to a given form on the fiber is parametrized by $H^1(T_\phi; \mathbb{Z})_\phi$. Variations within this class produce isotopic symplectomorphisms, but since the perturbations are not exact, the Floer homology groups are not necessarily naturally isomorphic.

Indeed, Example 3.3.2 shows precisely why we should not expect these groups to be related for all perturbations $\alpha \in H^1(T^2)$. Nonzero choices of α correspond to composing Dehn twists by a translation in T^2, which has no fixed points.

Chapter 5

The Symplectic Blowup

Having defined the invariant HSP_* for surface bundles of any genus, we now focus on calculations in the genus 1 case. Recall that in this situation the desired Fano stable pair space is simply the blowup of $\mathbb{P}^3$ along an embedded genus 1 curve. The divisor is a symplectic submanifold which one can arrange to be preserved by parallel transport, and by analogy with the blowup formula in ordinary cohomology, one would hope that the fixed-point Floer homology of M_1 splits into two summands: fixed points on the exceptional divisor, and Hamiltonian fixed points in the complement of the curve in $\mathbb{P}^3$. In order to realize this decomposition, we will require a more concrete construction of the symplectic form on the blowup.

5.1 The Kähler Cone of M_1

The first stable pair space M_0 is always a projective space, and by the argument of Lemma 4.14 the family $\mathcal{M}_0$ is the projectivization of a rank 4 vector bundle over S^1. So if Λ is the reference holomorphic line bundle over the reference Riemann surface X, we take $\mathcal{M}_0$ to be the trivial $\mathbb{P}H^1(\Lambda^{-1})$-bundle over the circle with projection maps

$$\mathcal{M}_0 = \mathbb{P}H^1(\Lambda^{-1}) \times S^1$$
$$pr_1 : \mathcal{M}_0 \to \mathbb{P}H^1(\Lambda^{-1})$$
$$pr_2 : \mathcal{M}_0 \to S^1$$

As before let $\mathcal{X}$ be the family of genus 1 Riemann surfaces produced from T_ϕ in Chapter 2. Construction 2.3 determines a fiberwise holomorphic embedding of $\mathcal{X}$ into $\mathcal{M}_0$, so that the family $\mathcal{M}_1$ of Fano stable pair spaces is the result of a fiberwise complex-analytic blowup of $\mathcal{M}_0$ along $\mathcal{X}$. Each fiber is a complex manifold which admits a Kähler form. Indeed, [Th] characterizes the Kähler cone of M_1 as the open subset of $H^2(M_0)$ bounded by the rays containing $(0, 4)$ and $(1, 2)$, depicted as the shaded region in Figure 2, In this picture the Fano form then corresponds to the point $(1, 4)$.

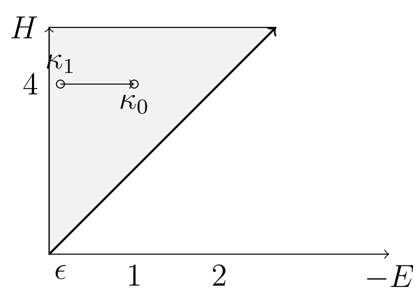

Figure 5.1: The ample cone of M_1, the monotone blowup form κ_0, and the low-weight blowup κ_1. The shaded region between the indicated rays (unbounded above and without its boundary) is the Kähler cone of M_1.

While the above discussion considers those forms which are Kähler with respect to a given integrable complex structure, there is a standard method [4] for effecting the blowup symplectically without reference to a complex structure. For the projective spaces described above, this amounts to removing a tubular neighborhood of $\mathfrak{X}$ with boundary an S^3-bundle and collapsing the circle fibers of the Hopf map in each S^3 (i.e. perform a symplectic cut). Suppose that each S^3 bounds some ball in the normal bundle with volume t. As a consequence of this construction (which we describe in more detail below), each fiber of the exceptional divisor has area equal to the excised volume t, determining a 1-parameter family of cohomology classes corresponding to possible blowups. There is an obvious upper bound for t given by the total volume, as we cannot remove more volume than is contained in $\mathbb{P}^3$. To achieve a better bound, by integrating a symplectic form in the cohomology class $4H - tE$ one can verify that the Kähler forms resulting from a symplectic blowup correspond in Figure 5.2 to points $(t, 4)$ for $0 < t < 2$.

Our discussion in this Chapter will in some sense interpolate between these viewpoints. First, in Section 5.2.1 we describe a construction of a Kähler form δ on a neighborhood of the divisor in the complex-analytic blowup which allows us to realize the cohomology classes on the line segment depicted in Figure 5.1. The resulting Kähler manifold is symplectomorphic to the smooth manifold given by the construction of [4] in the previous paragraph. However, this form will still be insufficiently explicit for our purposes, and so Section 5.2.2 we construct a symplectic local model for a neighborhood of the exceptional divisor that does not involve the holomorphic structure. In this local model, we will see that δ has an explicit form which will be employed in the calculation of fixed points in Chapter 6.

For this chapter and the next, we think of the blowup as a smooth manifold and vary the possible complex structures and Kähler forms. For this reason we change our notation from the previous sections to focus attention on the parameter t, as opposed to the index of the stable pair spaces $\mathcal{M}_i$. We will also refer to the base projective space as $\mathcal{M}$, and the symplectic blowup $Bl_{\mathcal{X}}(\mathcal{M}_0)$ with weight t as $\tilde{\mathcal{M}}_t$. The family of curves will still be denoted by $\mathcal{X}$.

5.2 The Low-Weight Blowup

5.2.1 A Representative for the Low-Weight Blowup Form

Let D denote the submanifold given by the exceptional divisor, isomorphic to $\mathbb{P}N$. Recall that $pr_1^*(\omega_{FS})$ is a closed 2-form on $\mathcal{M}$ which we pull back to $\tilde{\mathcal{M}}$ via the blowdown map $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$. The main issue is to construct a form δ on $\tilde{\mathcal{M}}$ over S^1 which is defined on a neighborhood of D, for which $\pi^* \circ pr_1^*(\omega_{FS}) - t\delta$ remains fiberwise symplectic.

In fact δ will arise as the curvature of a connection γ on a line bundle which we push forward onto the total space of the blowup. [29, 3.3.3] produces this form by constructing a line bundle on $\mathcal{M}$ which is trivial in the complement of the divisor, and coincides with the tautological bundle $\mathcal{O}_{\mathbb{P}N}(-1)$ on D. Then for a choice of metric in this line bundle, we may apply a cutoff function to any connection in it, and the resulting Chern form is symplectic near D but zero outside a neighborhood of D.

This construction of δ induces a Kähler form κ_t on the blowup which gives the exceptional sphere area t. We call this form the *symplectic blowup with weight* t:

$$\Omega_t := pr_1^*(\omega_{FS}) - t\delta \tag{5.1}$$

where t is the positive real number corresponding to the area of a fiber of the exceptional divisor. For a small t, Ω is guaranteed to remain non-degenerate, and hence fiberwise Kähler on $\tilde{\mathcal{M}}_t$.

5.2.2 An Alternate Construction of a Blowup Form

It is important to note that the construction from [29] of a Kähler form only holds for a small t-values. When adding in the form δ, in order to maintain non-degeneracy of $\omega + t\delta$, we must choose δ small compared to ω, so it not clear that we can realize the monotone form using this construction. On the other hand, there is an alternate construction of a form on the blowup which is monotone. The space $\tilde{M} = Bl_X(\mathbb{P}^3)$ can be described as a divisor in $\mathbb{CP}^3 \times \mathbb{CP}^1$, representing the class $2H_{\mathbb{P}^3} - H_{\mathbb{P}^1}$ as the blowup occurs along an intersection of degree 2 divisors. This is described in [25] Lemma 5.5, but we briefly review the construction here.

Starting with $L \to \mathcal{X}$ thought of as a semi-holomorphic family of line bundle-curve pairs over S^1, we take some extension of this to a holomorphic family over $\mathbb{C}^*$, still denoted by $\mathcal{X}$, with generic smooth fiber X_w. As L is very ample, it induces an embedding of $\mathcal{X}$ in $\mathbb{C}^* \times \mathbb{P}^3$, with X_w a degree 4 curve embedded as an intersection of quadrics

$$\mathcal{X} \to \mathbb{C}^* \quad X_w \to w \in \mathbb{C} \tag{5.2}$$

Suppose first that the monodromy in the family $\mathcal{X}$ consists of a single Dehn twist about a curve γ. It will be helpful to think of $\mathcal{X}$ as the restriction of a family of curves over $\mathbb{C}$ to a loop about the origin, where the fiber over 0 is a nodal curve whose singular point corresponds to the vanishing cycle γ.

In fact, we can be even more concrete in constructing this family by interpreting the degree 2 homogenous polynomials as symmetric matrices. Since each element of $\mathcal{X}$ is given by an intersection of quadrics, there is a linear change of coordinates so that one quadric is given by Q^0 the 4×4 identity matrix. Then $X_w = Q^0 \cap Q^1_w$ where Q^1_w is a family of 4×4 matrices. In addition, for each w, Q^0 and Q^1_w generate a pencil of quadrics whose common base locus for $w \neq 0$ is the smooth genus 1 curve X_w. When $w = 0$, we get a pencil of quadrics whose common intersection is a nodal curve X_0. The existence of such families is guaranteed by an analysis of the locus of singular quadrics in $\mathbb{P}^6$, as in [9] pg. 302.

By blowing up $\mathbb{P}^3 \times \mathbb{C}$ along the embedded family $\mathcal{X}$, we obtain a family M_w over $\mathbb{C}$ consisting of blowups along smooth curves for all $w \neq 0$, and the fiber over 0 is the blowup along a nodal curve.

For any w this construction produces a divisor in $\mathbb{P}^3 \times \mathbb{P}^1$ representing $2H_{\mathbb{P}^3} - H_{\mathbb{P}^1}$. Moreover, using the explicit construction of this family using the matrices Q^0 and Q^1_w it is straightforward to verify that the projection to the $\mathbb{P}^1$ factor gives a Lefschetz pencil of such divisors. Away from the base locus this projection is simply the map given by the pencil of quadrics, and along the exceptional divisor each $\mathbb{P}^1$-fiber is mapped isomorphically to the base $\mathbb{P}^1$

of this Lefschetz fibration.

If we then equip $\mathbb{P}^1$ with a 2-form with total area t, inducing a symplectic form on $(\mathbb{P}^3 \times \mathbb{P}^1, \omega_{FS} \oplus \omega_t)$, then for $0 < t < 2$ the blowup inherits a Kähler form which gives each fiber of the exceptional divisor area t. Taking $t = 1$ then realizes the monotone form on the blowup.

It is important to note that the form constructed via this pullback differs from ω_{FS} even away from the exceptional divisor. The inverse image of $\lambda \in \mathbb{P}^1$ is a quadric, and adding the pullback of ω_t perturbs ω_{FS} in the normal directions to the quadric. Though it is plausible that this perturbation does not change Hamiltonian Floer homology of $\mathbb{P}^3$ and the count of fixed points away from the divisor, our analysis in Chapter 6 employs the low-weight blowup.

5.3 Defining HF and Interpolating between the Low-Weight and Fano Forms

In the case of the low-weight blowup, since $t \neq 1$ it is clear that such forms κ_t fail to be monotone. However, since the dimension of $\tilde{\mathcal{M}}$ is 6, it satisfies condition (3) of Definition 3.3.4, and thus is weakly monotone. As such, the groups $HF(\Phi, J)$ are well-defined over $\mathbb{Z}((t))$ for a generic choice of complex structure J, but a priori they may depend on J. In the setting of Hamiltonian Floer homology the argument from [8] excludes the possibility of any hypothetical sphere bubbles arising in convergent sequences of trajectories with fixed area and index.

We summarize their argument for completeness. Suppose that Φ is a Hamiltonian diffeomorphism of a symplectic manifold M of dimension $2n$. [8] shows that the space of holomorphic maps $s : S^2 \to M$ in homology class A is a smooth manifold of dimension

$$2n + \langle 2c_1(A), s\rangle - 4. \tag{5.3}$$

The evaluation map to M is well-defined on elements of this space, and [8] show that its image is a countable union of images of smooth maps defined on manifolds with dimension given by (5.3). So the image of the space of $c_1 = 1$ spheres has codimension at most 2, and images of $c_1 = 0$ spheres has codimension at most 4. In addition, it is shown that these images of evaulation maps are compact for fixed index and homology class.

On the other hand, moduli spaces of trajectories of index $\mu(u) \leq 2$ have dimension $\mu(u)$, and evaluating these sweeps out $(2 + \mu(u) - 1)$-dimensional submanifolds of M. Generically, then, spheres of index 0 will not intersect trajectories of index at most 2, and similarly for index 2.

When establishing independence of the choice of complex structure for Hamiltonian Floer homology, it is crucial that one can take a single complex structure on M, rather than a 1-parameter family as in our case. For arbitrary weakly monotone (M, ω) there may be exceptional complex structures J which admit non-constant spheres s with $c_1(s) = -1$. To define continuation maps relating Floer homology groups for distinct complex structures, we consider 1-parameter families of holomorphic cylinders of index 0. By the arguments in the previous paragraph these sweep out subsets of $M \times \mathbb{R}$ of dimension 3, which avoid the codimension 4 sets of $c_1 = 0$ spheres. Generic 2-parameter families, however, sweep out a subset of M which has dimension at least 4, and thus may intersect the exceptional spheres of index -1, obstructing compactness.

In this section, we check that the hypotheses for the invariance result of Chapter 3 hold, and give an argument which produces a generic 2-parameter family of complex structures for which this compactness issue does not arise. As we will see, this family not only induces the appropriate continutation maps for establishing independence of the choice of complex structure, but also allows us to apply the invariance result of Section 3.3.1 and relate Floer homology for the monotone form to that of the low-weight blowup.

5.3.1 H^1-Codirectionality

To show that the analysis in the local model with the low-weight blowup can be related to a monotone symplectic form on the spaces M_w, we appeal to the invariance result in Chapter 3. Let $\mathcal{Y}^0$ be the action 1-form on the configuration space $\mathcal{C}$ for a Fano symplectic form, and $\mathcal{Y}^1$ be a 1-form for the blowup symplectic form. Then there is a family interpolating between them given by

$$\mathcal{Y}^t|_{Ker\, c_1} = t\mathcal{Y}^0|_{Ker\, c_1}\,,\; t \in [0, \epsilon]$$

where ϵ is the area of an exceptional sphere. Clearly $\mathcal{Y}^0$ is not identically zero, but its restriction to the kernel K of the spectral flow is zero by monotonicity. Moreover, this interpolation clearly satisfies the non-negativity condition in Theorem 3.3.3, and thus the family of forms described above is H^1-codirectional.

Let $CF^i := CF(\mathcal{C}, \mathfrak{h}, \psi, \mathcal{Y}_i, \mathcal{X}_i)$ denote the chain complexes associated to the action 1-forms $\mathcal{Y}^0, \mathcal{Y}^1$, and let HF^i denote the homology groups. Note that although these are naturally defined over appropriate Novikov rings, we have $\Lambda_0 \simeq \mathbb{Z}$ but $\Lambda_1 = \mathrm{Nov}(\mathfrak{h}, \psi; \mathbb{Z}) = \mathbb{Z}((t))$.

For the symplectic manifolds in question, however, the minimal Chern number is exactly 1.

Lemma 5.3.1. *For the stable pair spaces M_g, the minimal Chern number N is always 2.*

Proof. Recall that [Th] shows that M_i admits a birational map f to M_1, so when evaluating $c_1(TM_i)$ on some $\alpha \in \pi_2(M_i)$, $\langle c_1(TM_i), \alpha \rangle = \langle c_1(TM_1), f_*(\alpha) \rangle$ where $f_*(\alpha) \in \pi_2(M_1)$. But since the minimal Chern number in M_1 is 2, we see that the same is true for all other M_i. □

So, $\mathrm{Ker}(\psi)$ is in fact isomorphic to $\mathbb{Z}$ as the fundamental group of the configuration space is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. As this kernel is rank 1 it follows that we may take our Novikov coefficients to be the ring of Laurent polynomials over $\mathbb{Z}$, denoted by $\mathbb{Z}((t))$.

5.3.2 Admissible Families of Complex Structures

In order to apply Lee's invariance result, we must verify not only that the relevant Floer complexes are defined for all $t \in [0, 1]$ outside of the bifurcation and handleslide points, but that the path of Hamiltonian perturbations and complex structures itself can be made "admissible" in the sense that the moduli spaces are cut out transversely. Now the codimension argument from [8] described in Section 3.2 fails, as the complex structures on T_Φ are not constant in the circle parameter. The complex structure on each fiber of T_Φ determining $\mathcal{X}$ varies with the circle parameter, and also varies as the symplectic form changes.

The set of points in M_1 in the images of such spheres may intersect with trajectories, and multiple covers of these spheres are problematic in attempting to define Floer homology. To rule these out, we appeal to the monotonicity of the symplectic form κ_0. Define the space

$$\underline{\mathcal{J}} := \{\underline{J} \in \mathcal{J}(T^{vert}(T_\phi))\} \tag{5.4}$$

so an element $\underline{J}$ corresponds to a 1-parameter family of complex structures on M, and a choice of admissible homotopy $\underline{J}_s$ from $\underline{J}$ to some $\underline{J}'$ determines a path in $\underline{J}$. A standard identification of $\mathbb{R} \cup \{\pm\infty\}$ with $[0, 1]$ puts this in the framework of the continuation maps described in Section 3, so that $\underline{J}_s$ corresponds to a t-dependent complex structure on Z_ϕ which is $\mathbb{R}$-invariant on the ends.

Proposition 5.3.2. *There exists a path of complex structures $\underline{J}_s$ on the vertical tangent bundle to T_ϕ, parametrized by $[0,1]$, for which*

1. *For $i = 0, 1$, J_i is regular for the symplectomorphism Φ_i, and*
2. *for all $s \in [0,1]$, J_s is regular for the symplectomorphism of $(\tilde{M}, \kappa_s)$, so there are no nonnpositive index $J_{s,t}$ holomorphic spheres with positive κ_t area, and*
3. *the path connecting J_0 to J_1 is such that the handleslide and bifurcation moduli spaces are cut out transversely.*

Proof. We are given a fixed integrable complex structure J_{int} on Y for which all κ_t are Kähler. Interpret this as a constant path in the space of vertical complex structures $\mathcal{J}^{vert}(T_\Phi)$ on the mapping torus. Note that each complex structure in this path is compatible with all κ_t.

It is a standard result that there exists J_0 which is C^∞-close to J_{int} in $\mathcal{J}^{vert}(T_\Phi$, regular for Φ_0, and tame for κ_0. Similarly, to ensure that the moduli spaces for Φ_1 are transversely cut out, we may perturb J_{int} to some J_1 which is regular for Φ_1. Now by Proposition 6.12 in [11] there exists a set of admissible paths from J_0 to J_1 which is Baire in $\underline{\mathcal{J}}$. Thinking of J_0, J_1 as endpoints of a path in $\mathcal{J}^{vert}(T_\Phi$, a path connecting them is a perturbation of the constant path J_{int} to a path $\underline{J}_{s,t}$ with endpoints J_i such that for every $s \in [0,1]$, $\underline{J}_{s,t}$ is regular for κ_t. This guarantees that the moduli spaces defining ∂_{J_s} are cut out transversely, and establishes (1) and (3).

To exclude the possibility of nonconstant spheres with nonpositive Chern number, we first restrict the entire path $\underline{J}_s$ to be tame/compatible with κ_0. Since J_{int} was tame for all κ_t(in particular κ_0) and tameness is an open condition, all small perturbations of J will remain tame for the monotone symplectic form κ_0. So if $\underline{\mathcal{J}}$ is the space of paths of complex structures from J_0 to J_1, the space of paths through κ_0-tame complex structures constitutes an open subset.

Now, note that a candidate $J_{s,t}$-holomorphic sphere with nonpositive index but positive κ_t-area remains $J_{s,t}$-holomorphic in (M_1, κ_0). This sphere necessarily has zero area as it has nonpositive index in a monotone symplectic manifold, so it is constant. □

5.3.3 Applying the Invariance Result

This path of complex structures avoids sphere bubbles, and thus allows us to apply the invariance result as above, establishing the following theorem.

Proposition 5.3.3. *The groups $HF^0 := HSP_*(T_\gamma)$ and HF^1 have the same rank over the Novikov ring Λ_1. In fact there is an isomorphism $HSP_*(T_\gamma) \simeq H_*(CF^1, \partial_{J_1})$ as $\mathbb{Z}((t))$-modules.*

Proof. The results of [11, 12] give a chain homotopy

$$(CF^0, \partial_{J_0}) \otimes \Lambda_0 \simeq (CF^1, \partial_{J_1}) \tag{5.5}$$

so in particular their ranks are the same over the Novikov ring.

□

Chapter 6

Generating the Chain Complex $CSP_*(\phi, L)$

The main advantage of the symplectic blowup construction is that it allows us to localize the calculation of fixed points to a neighborhood of the divisor, since the symplectic form away from the divisor is simply the Fubini-Study form on projective space. In this section we use a local model for a neighborhood of the divisor to describe the horizontal sections, then calculate them explicitly.

6.0.1 A Normal Form for δ Near the Divisor

The construction of δ in Section 5.2 is insufficiently for determining fixed points. However, we will see by the following theorem that in a smaller neighborhood of the divisor δ must take a particular form.

Theorem 6.0.1. *[26] Let X be a symplectic manifold with symplectic form ω. Let G be a Lie group and $\rho : P \to X$ be a principal G-bundle. Let (F, η) be a Hamiltonian G-space with moment map $\mu : F \to \mathfrak{g}^*$. Then, there is a unique form β on $P \times_G F$ so that for any connection α on P,*

$$\rho^*(\beta) = \eta + d\langle \alpha, \mu \rangle \tag{6.1}$$

Moreover, if

$$\omega + \beta \tag{6.2}$$

is non-degenerate, then this determines a symplectic form on the associated bundle $P \times_G F$.

Let N be the normal bundle to $\mathcal{X}$ in $\tilde{\mathcal{M}}$. For any blowup form κ_t, N carries a symplectic form as $\mathcal{X}$ is a symplectic submanifold and N is its orthogonal complement. In our case, the frame bundle in the normal bundle N is the principal $U(2)$-bundle $P \to \mathcal{X}$. Here we may take $U(2)$ as the structure group for N, since the Hermitian metric on $\mathcal{M}$ induces a metric on the normal directions to $\mathcal{X}$.

Now there is a Kähler form η_t on $\tilde{\mathbb{C}}^2$ which gives the divisor area t, given by the form on $(\mathbb{CP}^1 \times \mathbb{C}^2, t\omega_{FS} \oplus \omega_{std})$. The action of $U(2)$ on the blowup $\widetilde{\mathbb{C}^2}$ of $\mathbb{C}^2$ at the origin is Hamiltonian with respect to this form; the group $U(2)$ certainly acts symplectically on each factor of $\mathbb{CP}^1 \times \mathbb{C}^2$, and as this space is simply connected we see that the action must also be Hamiltonian.

With this in hand, we see that there exists an associated bundle over $\mathcal{X}$ whose fibers are precisely $(\widetilde{\mathbb{C}^2}, \eta_t)$. For a choice of connection in P, then, the theorem above produces a closed 2-form on this bundle, and different choices of a connection result in exact perturbations of δ (and hence Ω). We organize this information into the diagram below, setting notation for later.

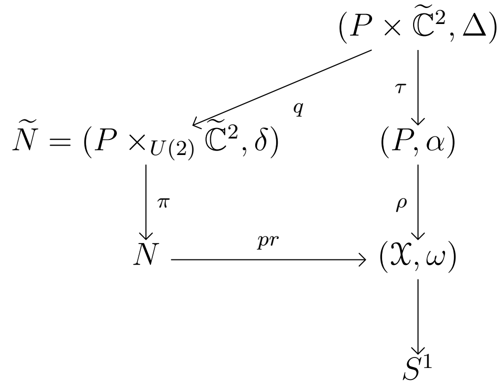

Figure 6.1: A diagram of the various principal and associated bundles used in the construction of δ.

There is a moment map $\mu : \widetilde{\mathbb{C}^2} \to \mathfrak{u}(2)^*$ for the action of $U(2)$ on $(\tilde{\mathbb{C}}^2, \eta_t)$ as follows. The moment map for the $U(2)$ action on $\mathbb{CP}^1$ evaluates on $\xi \in \mathfrak{u}(2)$ as

$$\mu_{\mathbb{CP}^1}([z_0; z_1])[\xi] = \frac{1}{||(z_0, z_1)||}(z_0, z_1)^T \xi(z_0, z_1)$$

and similarly the moment map for the $U(2)$-action on $\mathbb{C}^2$ is

$$\mu_{\mathbb{C}^2}(z_0; z_1)[\xi] = (z_0, z_1)^T \xi(z_0, z_1).$$

Then we define $\mu = \mu_{\mathbb{C}^2} + t\mu_{\mathbb{CP}^1}$.

We now apply Theorem 6.0.1 to this relative case, where we P is a principal bundle over a family of symplectic manifolds. Fix a connection $\alpha \in \Omega^1(P; \mathfrak{u}(2))$. Then Δ on $P \times \widetilde{\mathbb{C}}^2$ is given by

$$\begin{aligned} \Delta &= \rho^*(\omega) + \eta + d(\langle \mu, \alpha(\cdot) \rangle) \\ &= \rho^*(\omega) + \eta + \langle F_\alpha, \mu \rangle - \langle \frac{1}{2}[\alpha \wedge \alpha], \mu \rangle + \langle \alpha, d\mu \rangle \end{aligned}$$

where we suppress the pullbacks, and use the formula for the curvature $F_\alpha - \frac{1}{2}[\alpha \wedge \alpha]$. Here the brackets $\langle \cdot, \cdot \rangle$ denote the pairing of element of $\mathfrak{u}(2)$ with elements of the dual. This form is equivariant with respect to the diagonal $U(2)$ action, descending to the necessary closed 2-form δ on the associated bundle.

Choosing η_t to give the exceptional sphere the same area as κ_t in $\tilde{\mathcal{M}}$, we invoke Lemma 3.26 of [29], which establishes that the normal bundle to D in $\tilde{\mathcal{M}}$ is in fact isomorphic to the total space of $\mathcal{O}_{\mathbb{P}N}(-1)$. For appropriate choice of metric in this line bundle, the total space is symplectomorphic to $\tilde{N}$. D is a symplectic submanifold of $\tilde{M}_t$ and the divisor in $\tilde{N}$ is a symplectic submanifold, so the symplectic neighborhood theorem of [30] provides a neighborhood of D which is symplectomorphic to a neighborhood of the exceptional divisor in $\tilde{N}$. We organize this information the following diagram, where $\widetilde{N}$ is the blowup of the total space of the normal bundle N to $\mathcal{X}$ along the zero-section.

$$
\begin{array}{ccc}
\widetilde{N} & \xrightarrow{\;\widetilde{\iota}\;} & (\tilde{\mathcal{M}}_t, \pi^*(\omega_{FS}) - t\delta) \\
\Big\downarrow b & & \beta \Big\downarrow \\
N & \xrightarrow[\;\iota\;]{} & (\mathcal{M}, \omega_{FS})
\end{array}
$$

Figure 6.2: A tubular neighborhood for the blowup locus, and the exceptional divisor.

Now, pulling back δ to $\tilde{N}$, by construction it restricts to a symplectic form on a neighborhood of the divisor for which the exceptional sphere has area t, and we have a local model for δ near the exceptional divisor; it is a quotient of Δ on $P \times \widetilde{\mathbb{C}^2}$ for some choice of connection α. It is this formula for the symplectic connection in $\tilde{N}$ which will allow us to calculate the fixed points of Φ.

6.1 Identifying Horizontal Sections

Restricting our attention now to a tubular neighborhood of D, here we show that horizontal sections of $\tilde{N}$ are parametrized by points of $\widetilde{\mathbb{C}^2}$, the blowup of $\mathbb{C}^2$ at the origin.

Let σ^f be a section of $\mathcal{X} \times \widetilde{\mathbb{C}^2}$ of the form

$$\sigma^{f(t)} := (\sigma(t), f(t))$$

which is horizontal with respect to

$$\omega_f = \omega + \langle F_\alpha, \mu(f) \rangle \tag{6.3}$$

For such sections, let $\widetilde{\sigma}^f$ be its lift using the connection α to $P \times \widetilde{\mathbb{C}^2}$.

Proposition 6.1.1. *Let $f \in \widetilde{\mathbb{C}}^2$ be a point in a sufficiently small neighborhood of the exceptional divisor. All sections*

$$\widetilde{\sigma}^f \in P \times \widetilde{\mathbb{C}^2}, \tag{6.4}$$

where $f(t) = f$ is a constant map, descend to a δ-horizontal section of the associated bundle $\widetilde{N} = P \times_{U(2)} \widetilde{\mathbb{C}^2}$.

Conversely, any δ-horizontal section of $\widetilde{N}$ over the circle has a lift to $P \times \widetilde{\mathbb{C}^2}$ of the form $\widetilde{\sigma}^f$ above.

Proof. First use the given connection α to fix a gauge in $P \times \widetilde{\mathbb{C}^2}$, identifying a distribution in $T(P \times \widetilde{\mathbb{C}^2})$ which maps isomorphically to the vertical tangent space in $\widetilde{N}$. Decomposing the tangent space to $P \times \widetilde{\mathbb{C}^2}$ as $TP \oplus T\widetilde{\mathbb{C}}^2$, the first summand splits (using both the connection α and the symplectic connection ω_f on $\mathfrak{X}$) as follows, using the notation in Figure 6.1.

$$TP \oplus T\widetilde{\mathbb{C}}^2 = T^\alpha P \oplus Ker(d\tau) \oplus T\widetilde{\mathbb{C}}^2 = T^v\mathfrak{X} \oplus T^h_\omega\mathfrak{X} \oplus Ker(d\tau) \oplus T\widetilde{\mathbb{C}}^2 \quad (6.5)$$

Let $\dot{\sigma}$ denote the tangent vector $\frac{d}{dt}(\widetilde{\sigma}(t), f(t))$. Horizontality amounts to the condition $\Delta(v, \dot{\sigma}) = 0$ for all vertical v, so by making specific choices for v we show that $\dot{\sigma} \in T^h_\omega\mathfrak{X}$.

Consider tangent vectors v_τ in the kernel of $d\tau$.

$$\begin{aligned} & \Delta(\dot{\sigma}, v_\tau) \\ = \; & \omega(\dot{\sigma}, v_\tau) + \eta(\dot{\sigma}, v_\tau) + \langle d\alpha(\dot{\sigma}, v_\tau), \mu\rangle + \langle \alpha \wedge d\mu\rangle(\sigma, v_\tau) \\ = \; & \langle F_\alpha(\dot{\sigma}, v_\tau), \mu\rangle - \langle \frac{1}{2}[\alpha \wedge \alpha](\dot{\sigma}, v_\tau), \mu\rangle + \langle \alpha \wedge d\mu\rangle(\dot{\sigma}, v_\tau) \\ = \; & 0 \end{aligned}$$

In the first line, the first two terms vanish irrespective of $\dot{\sigma}$ because v_τ is vertical in the principal bundle, so it vanishes when pulled back to F or projected to the base $\mathfrak{X}$. Similarly, the curvature term in the second line vanishes as it is a form on $\mathfrak{X}$. Now the remaining terms vanish for any v_τ as $\dot{\sigma}$ is horizontal with respect to the connection α.

Next we consider tangent vectors v_F to the fiber $\widetilde{\mathbb{C}^2}$.

$$
\begin{aligned}
& \Delta(\dot\sigma, v_F) \\
= \ & \omega(\dot\sigma, v_F) + \eta(\dot\sigma, v_F) + d\langle \alpha, \mu\rangle(\dot\sigma, v_F) \\
= \ & \eta(\dot\sigma, v_F) + \langle F_\alpha(\dot\sigma, v_F), \mu\rangle - \langle \frac{1}{2}[\alpha \wedge \alpha](\dot\sigma, v_F), \mu\rangle + \langle \alpha(\dot\sigma), d\mu(v_F)\rangle \\
& + \langle \alpha(v_F), d\mu(\dot\sigma)\rangle \\
= \ & 0
\end{aligned}
$$

Again the last four terms vanish independent of $\dot\sigma$. For the first, since $f(t)$ is constant, its tangent vector lying in $\dot\sigma$ has zero TF-component, so this vanishes as well.

Last we consider v_α, meaning vectors lifted using α that are in the kernel of $d\rho$, that is, tangent vectors to the fibers of $\mathcal{X}$.

$$
\begin{aligned}
& \Delta(\dot\sigma, v_\alpha) \\
= \ & \omega(\dot\sigma, v_\alpha) + \eta(\sigma, v_\alpha) + d\langle \alpha, \mu\rangle(\dot\sigma, v_\alpha) \\
= \ & \langle F_\alpha(\dot\sigma, v_\alpha), \mu\rangle - \langle \frac{1}{2}[\alpha \wedge \alpha](\dot\sigma, v_\alpha], \mu\rangle + \langle \alpha(\dot\sigma), d\mu(v_\alpha)\rangle \\
& + \langle \alpha(v_\alpha), d\mu(\dot\sigma)\rangle \\
= \ & \langle F_\alpha(\dot\sigma, v_\alpha), \mu(f)\rangle
\end{aligned} \tag{6.6}
$$

Every term with an α vanishes as both vectors are horizontal with respect to the connection, but the curvature term does not necessarily vanish. Indeed, there exist choices of connection such that F_α induces a symplectic form on each fiber of $\mathcal{X}$, and hence the curvature term in this case will certainly fail to vanish for all v_α.

So if the given path is to descend to a δ-horizontal section, we must show that under the given hypotheses the pairing with μ vanishes identically. First consider the exceptional divisor in the associated bundle $\widetilde{N}$.

Lemma 6.1.2. *Let P be the principal frame bundle for the normal bundle N constructed above. Any connection $\alpha \in \Omega^1(P; \mathfrak{u}(2))$ takes values in the subalgebra of scalar matrices, and so the expression (6.6) vanishes for f on the exceptional divisor of $\widetilde{\mathbb{C}}^2$.*

Proof. For each $t \in S^1$, the fiber X_t of $\mathcal{X}$ is an intersection of quadrics. As such, its normal bundle can be identified with $(\mathcal{O}(2) \oplus \mathcal{O}(2))|_{X_t}$, meaning that N_Z in fact splits as a direct sum of two isomorphic line bundles (of degree 8). It follows then that the structure group can be reduced to S^1, and the curvature of any connection in the principal frame bundle must take values in the subalgebra of scalar matrices.

Turning now to the moment map, the action of $U(2)$ on the exceptional divisor factors through $PU(2)$, inducing a map $\mathfrak{pu}(2)^* \to \mathfrak{u}(2)^*$ whose kernel is the set of scalar matrices. At points on the exceptional divisor, then, the pairing vanishes. $\square$

So, for every ω-horizontal section σ of $\mathcal{X}$ and fixed $f \in \mathbb{P}^1 \subset \widetilde{\mathbb{C}}^2$, we may lift σ to a section $\widetilde{\sigma}$ of P. Then $\widetilde{\sigma}^f$ is an Δ-horizontal section of $P \times \widetilde{\mathbb{C}}^2$ for any connection α, since the curvature term vanishes, and $\widetilde{\sigma}^f$ descends to a horizontal section of $\widetilde{N}$.

To extend this to a neighborhood of the divisor as claimed, we linearize the equation $\Delta = 0$ and show that this linearized equation continues to hold for tangent vectors to $\widetilde{\mathbb{C}^2}$. Consider a vector field $\xi \in \Gamma(T\widetilde{\mathbb{C}^2})$ lying in the direction λ normal to the divisor and let $\psi_t : P \times \widetilde{\mathbb{C}^2} \to P \times \widetilde{\mathbb{C}^2}$ be the flow in $P \times \widetilde{\mathbb{C}^2}$ at time t generated by $(0, \xi) \in \Gamma(TP \oplus T\widetilde{\mathbb{C}^2})$. Also take $\widetilde{\sigma}_t^f = \psi_t^* \widetilde{\sigma}^f$ to be the pullback under this flow. Then the condition that $\widetilde{\sigma}_t^f$ remain horizontal amounts to the following.

$$\begin{aligned} 0 &= \left.\frac{d}{dt}\right|_{t=0} \widetilde{\sigma}_t^{f*} \Delta(v, \cdot) \\ &= \left.\frac{d}{dt}\right|_{t=0} \widetilde{\sigma}_0^{f*} (\psi_t^* \Delta(v, \cdot)) \\ &= \widetilde{\sigma}_0^{f*} \left(\left.\frac{d}{dt}\right|_{t=0} (\psi_t^* \Delta(v, \cdot)) \right) \\ &= \widetilde{\sigma}_0^{f*} (\mathcal{L}_\xi \Delta(v, \cdot)) \qquad (6.7) \end{aligned}$$

Applying Cartan's formula to each term of Δ implies that $\widetilde{\sigma}_t^f$ remains horizontal if we have

$$0 = \sigma_0^{f*} \left[d\eta(v, \xi) + d\langle \alpha(v), d\mu(\xi) \rangle + \iota_\xi (d\eta(v, \cdot) + \langle \alpha(v), d\mu(\xi) \rangle) \right]$$

We claim that any vector field $\xi \in \Gamma(T\widetilde{\mathbb{C}^2}) \subset \Gamma(H \oplus T\widetilde{\mathbb{C}^2})$ satisfies the above equation for every $v \in T^{vt}(\widetilde{N})$, so long as the initial choice of f in $\widetilde{\sigma}_0^f$ is constant and on the divisor. Note that for such ξ and f, this equation is a pullback of 1-forms, and since σ_0^f is constant on $\widetilde{\mathbb{C}^2}$ these terms vanish. Similarly, the contraction of $d\eta$ vanishes as well. The last two terms vanish by a similar argument combined with Lemma 6.1.2.

To explain the need for the definition of ω_f, the above argument only considers the Δ-horizontality of $\widetilde{\sigma}_0^f$, and not the effect of the section of $\mathcal{X}$. In fact, for points off the divisor in $\widetilde{\mathbb{C}}^2$, the restriction of Ω to $\mathcal{X}$ is the symplectic connection ω_f as in (6.3). As the form ω_f changes depending on the point $f \in \widetilde{\mathbb{C}^2}$, after perturbing by ξ, $\widetilde{\sigma}_t^f$ may not lie over an ω-horizontal section.

Note that this perturbation of ω on $\mathcal{X}$ remains symplectic so long as we we ensure that ξ is sufficiently small; the form $\langle F_\alpha, \mu(f)\rangle$ is simply a scalar multiple of the curvature of α, which determines the symplectic form on $\mathcal{X}$.

Now suppose that we have a section of the form specified in the statement of Proposition 6.1.1, a δ-horizontal section of $\widetilde{N}$, and any lift to $P \times \widetilde{\mathbb{C}}^2$ given by $\widetilde{\sigma}^f(t)$ By assumption if f lies sufficiently close to the divisor, it is in the image of a point on the exceptional divisor under the flow ψ_1. For an ω-horizontal section σ, we can take its flow along ξ as above, taking place in $\mathcal{X} \times \widetilde{\mathbb{C}}^2$, to obtain for each t perturbed sections $(\sigma_t, \psi_t(f))$. By construction $\frac{d}{ds}(\sigma_t)$ has no $T\widetilde{\mathbb{C}^2}$ component, so lifts of these perturbations to $P \times \widetilde{\mathbb{C}^2}$ remain Δ-horizontal. Under the assumption that the section $\widetilde{\sigma}^f$ we started with lies sufficiently close to the divisor, it can be constructed as such a perturbation, so the calculations that the conditions of Proposition 5.1 imply δ-horizontality suffice to show that $\tilde{\sigma}^f(t)$ must be a lift of an ω_f-horizontal section using α.

Similarly, the above calculations and Lemma 5.2 establish that $f(t)$ must be a constant map to a point in $\widetilde{\mathbb{C}^2}$ by non-degeneracy of η. Moreover, if f is the image of a point on the exceptional divisor under the flow of a vector field ξ as above then $\Delta = 0$ continues to hold and thus the section is

Δ-horizontal.

$\square$

6.2 Perturbations and Non-Degeneracy

We have shown that the δ-horizontal sections in the blowup are precisely those lying over ω_f-horizontal sections of $\mathcal{X}$, so we have a manifold of horizontal sections which can be identified with a neighborhood of the divisor in $\widetilde{\mathbb{C}}^2$. To force non-degeneracy and isolate a finite set of generators for $CF_*(\Phi)$, we add in a Hamiltonian perturbation in $\tilde{N}$ pulled back from N (induced by a Morse function on $\tilde{\mathcal{M}}$ pulled back from $\mathcal{M}$). We will see that this restricts the possible horizontal sections considerably.

Proposition 6.2.1. *Fix $W : \mathcal{X} \to \mathbb{R}$ an S^1 family of Morse functions as constructed in Theorem A, and α a connection as in the previous section. A section of the associated bundle $\widetilde{N}$ is horizontal with respect to $\Omega_W = \Omega + dW \wedge dt$ (where W is pulled back to $P \times F$) if and only if it has a representative in $P \times F$ of in the set*

$$\{(\widetilde{\sigma}, 0, \lambda) \,|\, \dot{\widetilde{\sigma}} \in H,\ (0, \lambda) \in \widetilde{\mathbb{C}^2}\} \tag{6.8}$$

and λ has the property that for every $t \in S^1$, the quotient map to the associated bundle sends λ to a unique element of $\mathbb{P}N$.

Proof. First suppose that $\widetilde{\sigma}_f$ descends to a $\delta + dW \wedge dt$)-horizontal section. We see how adding $dW \wedge dt$ to the form δ above affects the previous computations.

Let h be an $\delta + dW \wedge dt$-horizontal vector, and $v \in T^{vert}(\mathcal{M}_1)$.

$$\begin{aligned} & \pi^*(dM \wedge dt)(h, v_\alpha) \\ = \; & \pi^*(dW(h)) \cdot dt(v_\alpha) - \pi^*(dW)(v_\alpha) \cdot dt(h) \\ = \; & -\pi^*(dW)(v_\alpha) \cdot 1 \end{aligned} \tag{6.9}$$

For this expression to vanish for all v, we must have $d(W \circ \pi)(v) = 0$. Stated differently, a horizontal section for $\Omega + dW \wedge dt$ must pass through critical points of the Hamiltonian pulled back to the blowup, i.e. the critical set of the Morse-Bott function $W \circ \pi$, restricted to X_t. Theorem A, proven in the appendix, describes this critical set precisely.

Theorem A. *Let $\pi : Y \to Z$ be the blowup of a complex manifold Z along a complex submanifold X. Suppose also that W is a Morse function on Z with no critical points on X. Then $W \circ \pi$ is Morse-Bott, and its critical manifolds are given by critical points of W in Z/W and projective hyperplanes $Ker(dW) \cap J(Ker(dW) \subset \pi^{-1}(c)$, where $c \in Crit(W|_Z)$, J is the integrable complex structure on X, and $\pi^{-1}(c)$ is a fiber of the exceptional divisor.*

In the case at hand, a projective hyperplane is a connected complex submanifold of $\mathbb{CP}^1$, i.e. a point, so $\widetilde{f}$ is actually Morse. It follows that $\widetilde{\sigma}_f$ must lie on the exceptional divisor, and that it lies in the preimage of a unique point on $\widetilde{N}$.

On the other hand, suppose we have a section of $P \times \widetilde{\mathbb{C}}^2$ which is $\Omega + dW \wedge dt$-horizontal. Applying the same computations as in Lemma 5.3, we see that it must be of the form $(\widetilde{\sigma}, \lambda)$ where the $\widetilde{\sigma}$ is a lift via α of an ω-horizontal section of $\mathfrak{X} \to S^1$ and λ lies on the divisor. Note that the unperturbed form ω on $\mathfrak{X}$ is sufficient here, as the sections in question lie on the exceptional divisor.

It remains to establish non-degeneracy of these sections of the associated bundle. Examining the monodromy of Φ on $H \oplus T\widetilde{\mathbb{C}^2}$, note that adding the Hamiltonian perturbation term to the equation (5.4) results in the linearized horizontality equation below, where again ξ is a vector field along $\widetilde{\mathbb{C}^2}$.

$$0 = \tilde{\sigma}_0^{f*}(\mathcal{L}_\xi(\Delta + dW \wedge dt))$$

Now most of the terms vanish as in (5.4), but linearizing the Hamiltonian term results in $Jd^2W(\sigma)(\dot{\sigma})$, the Hessian of W at the critical point in question. As this derivative calculates the linearized return map $d\Phi$ at the fixed point, we see that by construction of W ρ is nondegenerate in all directions except for X, the blowup locus.

Along the blowup locus $\mathfrak{X}$, the decomposition of the tangent spaces to $\widetilde{N}$ imposed by (6.5) and the construction in Figure 6.1 implies that on the reference fiber, the projection of $d\Phi$ to X is simply $d\phi$, the derivative of the symplectomorphism of X.

Since the horizontal sections are constant on the divisor, $d\Phi$ preserves TF and H and has block form

$$d\Phi : H \oplus TF \to H \oplus TF$$

$$\begin{pmatrix} d\phi & 0 \\ 0 & Hess(W) \end{pmatrix}$$

By Theorem A, W is Morse on the blowup, with critical set on each fiber consisting of isolated points. Thus non-degeneracy of the critical points of W and the assumption that $(\mathcal{X}, \omega)$ had non-degenerate fixed points suffices to conclude the same fact for $(\mathcal{M}_1, \Omega)$ as needed. □

6.2.1 Determining $CSP_*(\Phi)$

We are now ready to show that the rank of CF^1 over the Novikov ring determines the rank over $\mathbb{Z}$ of the $\mathbb{Z}$-module CF^0, recall that for CF^0 with the monotone form, we think of this as a $\mathbb{Z}[t, t^{-1}]$ module by extension of scalars. Using Theorem 6.2.1, this explicit description of the generators of the chain complex as $\mathbb{Z}[t, t^{-1}]^{Fix(\gamma)}$ shows that when $Tr(\gamma) = Tr\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) > 2$,

$$\det(\gamma - Id) = \det \begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} = 2 - Tr(\gamma) \tag{6.10}$$

and thus when $Tr(\gamma) > 2$, the fixed points on Z are of even degree. Moreover, both CF^0 and CF^1 are torsion-free modules by definition.

Then since CF^0, CF^1 are free modules of the same rank and the differential acts trivially, the chain homotopy provides an isomorphism

$$\mathbb{Z}[t, t^{-1}]^{Fix(\phi_1)} \cong \mathbb{Z}^{Fix(\phi_0)} \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}] \tag{6.11}$$

It is then a standard result in commutative algebra that the rank of the free modules on both sides must be the same - that is, if $R \subset S$ are unital commutative rings, then $S \otimes_R R^{\oplus n} \cong S^{\oplus n}$ as free S-modules.

Computations in the local model which determine CF^1_* over the Novikov ring Λ_1 then suffice to determine $HSP_*(\gamma)$ over $\mathbb{Z}$, as both are free modules.

Now, when $Tr(\gamma) < 2$, all fixed points of γ are of even degree (by calculating their Lefschetz number), so the differential in CF^0 must be trivial. However, if $Tr(\gamma) > 2$, this degree argument is insufficient to determine the differential as the fixed points on the divisor have odd degree. This point will be addressed in Chapter 7.

Chapter 7

Quantum Cohomology and Calculating the Differential

To establish the rest of Theorem 1.2, we must show that in the remaining case where $Tr(\phi) < 2$, the Floer differential cannot map odd degree fixed points, i.e. horizontal sections on the divisor, to even generators, i.e. horizontal sections in the complement.

Our strategy here is to determine eigenspaces for the quantum cup action on $HF(\Phi)$. There is a particular summand of $HF_*(\Phi)$ which corresponds to horizontal sections coming from a Hamiltonian symplectomorphism on $\mathbb{CP}^3$, while a different summand corresponds to the horizontal sections coming from $HF(\phi)$ on Σ_1. As quantum multiplication is a chain map, it preserves these eigenspaces, and the odd and even fixed points correspond to different eigenspaces. Then an argument involving the exact sequence for Dehn twists in monotone symplectic manifolds shows that the differential cannot map these eigenspaces to one another.

7.1 The Quantum Cohomology Ring of (M, κ)

To this end, we will first require a description of the action of $c_1(M)$ on $QH^*(M)$. We define the Gromov-Witten invariants as follows, following [5].

Theorem 7.1.1. *The **Gromov-Witten invariant** for a homology class $A \in H_2(Y;\mathbb{Z})$ and $k \geq 3$ is a homomorphism*

$$GW^Y_{A,k} : H^*(Y;\mathbb{Z})^{\otimes k} \otimes H_*(\overline{\mathcal{M}}_{0,k}) \to \mathbb{Q}$$

defined by

$$GW^Y_{A,k}(a_1, \cdots, a_k; \beta) := \int_{\overline{\mathcal{M}, (A,J)}} ev_1^*(a_1) \cup \cdots \cup ev_k^*(a_k) \cup \pi^*(PD(\beta))$$

for J an ω-tame complex structure, $ev_k : \mathcal{M}_{0,k} \to Y$ the evaluation at the kth marked point, and $\pi : \mathcal{M}_{0,k}(A, J) to \mathcal{M}_{0,k}$ the forgetful map which sends a stable map to the underlying curve. This homomorphism has the following properties.

1. *(Effective) If $\omega(A) < 0$ then $GW^Y_{A,k} = 0$.*

2. *(Symmetry) For each permutation $\sigma \in S_k$,*

   $$GW^M(A, k)(a_{\sigma_1}, \cdots, a_{\sigma_k}; \sigma_*(\beta)) = \epsilon(\sigma; \{a_i\}) GW^Y_{A,k}(a_1, \cdots, a_k; \beta)$$

   where ϵ depends on the sign of the permutation σ and the degrees of the cohomology classes $\{a_i\}$, and $\sigma_*(\beta)$ is the image of the homology class β under the map which permutes marked points in $\mathcal{M}_{0,k}$.

3. *(Grading) If $GW^Y_{A,k}(a_1, \cdots, a_k; \beta) \neq 0$ then*

   $$\sum_{i=1}^{k} deg(a_i) - deg(\beta) = 2n + 2c_1(A)$$

4. *(Homology) For every $A \in H_2(Y;\mathbb{Z})$ and every integer $k \geq 3$ there exists a homology class*

$$\sigma_{A,k} \in H_{2n+2c_1(A)+2k-6}(Y^k \times \mathcal{M}_{0,k})$$

such that

$$GW^Y_{A,k}(a_1, \cdots, a_k; \beta) = \langle \pi_1^*(a_1) \cup \cdots \cup \pi_k^*(a_k) \cup \pi_0^*(PD(\beta)), \sigma_{A,k} \rangle$$

where $\pi_i : Y^k \times \overline{\mathcal{M}_{0,k}} \to Y$ denotes projection onto the ith factor and π_0 is projection onto the last factor.

5. *(Fundamental Class) If $\pi_{0,k} : \mathcal{M}_{0,k} \to \mathcal{M}_{0,k-1}$ is the map forgetting the last marked point, and $(A,k) \neq (0,3)$ then*

$$GW^Y_{A,k}(a_1, \cdots, a_k; \beta) = GW^Y_{A,k-1}(a_1, \cdots, a_{k-1}; (\pi_{0,k})_*(\beta))$$

6. *(Zero) If $A = 0$ then $GW^Y_{A,k}(a_1, \cdots, a_k; \beta) = 0$ whenever $deg(\beta) > 0$ and*

$$GW^Y_{A,k}(a_1, \cdots, a_k; [pt]) = \int_Y a_1 \cup \cdots \cup a_k$$

7. *(Splitting) Fix a basis $e_0, ...e_N$ for $H^*(Y;\mathbb{Z})$ and consider the $N \times N$ matrix whose g_{ij}th entry is the pairing $\int_Y e_i \cup e_j$. This has inverse which we denote g^{ij}. For any partition $S_0 \cup S_1 = \{1, ..., k\}$ of the index set, there is a canonical map*

$$\phi_S : \overline{\mathcal{M}}_{0,k_0+1} \times \overline{\mathcal{M}}_{0,k_1+1} \to \overline{\mathcal{M}}_{0,k}$$

obtained by gluing the last marked point k_0+1 to the first marked point of k_1+1, preserving the relative ordering otherwise. Fixing two homology classes $\beta_i \in H_*(\overline{\mathcal{M}}_{0,k_i+1};\mathbb{Z})$, we have

$$\begin{aligned} & GW^Y_{A,k}(a_1,...,a_k;\phi_{S*}(\beta_0\otimes\beta_1)) \\ = \ & \epsilon(S,\{a_i\}) \sum_{A_0+A_1=A} \\ & GW^Y_{A_0,k_0+1}(\{a_i\}_{i\in S_0},e_\nu;\beta_0)g^{\nu\mu}GW^Y_{A_0,k_1+1}(\{a_i\}_{i\in S_1},e_\mu;\beta_1) \end{aligned}$$

These axioms will allow us to calculate the structure coefficients for the quantum cohomology of M_1.

Though the definition of Gromov-Witten invariants above requires a generic ω-tame complex structure, we are in a setting where the "automatic regularity" criterion in Lemma 3.3.1 of [MS2] applies. We recall its statement for completeness.

Theorem 7.1.2. *Let $E \to S^2$ be a complex vector bundle of rank n and*

$$D : \Omega^0(S^2;E) \to \Omega^{0,1}(S^2;E)$$

be a real linear Cauchy-Riemann operator. Suppose that there exists a splitting $E = \sum_i L_i$ into complex line bundles such that each sub-bundle $L_1 \oplus \cdots \oplus L_k$, $k = 1,...,n$ is invariant under D. Then D is surjective if and only if $c_1(L_k) \geq -1$ for every k.

7.2 Geometry of Quadrics in Projective Space

In order to calculate the Gromov-Witten invariants for the blowup in question, it will be crucial to understand how the given genus 1 curve lies in $\mathbb{P}^3$, as well as the constraints on lines which intersect it. Recall that we have a genus 1 curve X embedded in $\mathbb{P}^3$ as an intersection of quadric (degree 2) hypersurfaces. We briefly review some facts about the geometry of quadric surfaces which will inform our computation of Gromov-Witten invariants later on.

Definition 7.2.1. *A* quadric *hypersurface in $\mathbb{P}^n$ is the set of points with homogenous coordinates $[z_0; \cdots ; z_n]$ which satisfy a homogenous polynomial of the form*

$$0 = \sum_{0 \leq i,j \leq n} M_{ij} v_i v_j$$

where M_{ij} are entries of a symmetric $n \times n$ matrix. A quadric Q is singular *if there is at least one point $p \in Q$ such that every line in Q intersects p.*

Note that a matrix M representing a non-singular quadric necessarily has full rank, and all non-singular quadrics are related by projective transformations (changes of basis). The quadric represented by a singular matrix of rank $n-1$ is a cone on a quadric of lower dimension. The following fact in dimension 3 will prove especially useful.

Proposition 7.2.2. *A quadric hypersurface in $\mathbb{P}^3$ is isomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.*

Proof. As all non-singular quadrics are related by projective transformations, it suffices to exhibit two distinct rulings of a particular quadric. Thus we consider the Segre map

$$\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$$

$$([x_0; x_1], [y_0, y_1]) \mapsto [x_0 y_0; x_0 y_1; x_1 y_0; x_1 y_1]$$

and note that the image lies in the quadric $Q = \{z_{00} z_{11} = z_{01} z_{10}\}$, where the homogenous coordinates on $\mathbb{CP}^3$ are $[z_{00}; z_{01}; z_{10}; z_{11}]$. To construct Φ^{-1}, assume first that $z_{00} \neq 0$. Then since neither x_0, y_0 are zero, we can define Φ^{-1} here to be

$$[z_{00}; z_{01}; z_{10}; z_{11}] \mapsto ([1; \frac{z_{10}}{z_{00}}], [1, \frac{z_{01}}{z_{00}}])$$

and clearly $\Phi^{-1} \circ \Phi = Id$. A similar construction for $z_{11} \neq 0$ defines Φ^{-1} for the remaining subset of Q, and thus Q is doubly ruled. □

7.3 Computation of the Quantum Cohomology Ring

Let X be the blowup locus, and $\pi : M \to \mathbb{P}^3$ be the blowdown map. With respect to the ordered basis for even cohomology given by

$$H^{even}(M_1; \mathbb{C}) = \langle 1, H, E, H^2, \varphi, h^3 \rangle$$

we have the following relations for the cohomology of the blowup [7]

1. $E^2 = -4h^2 + \langle c_1(N_{\mathbb{P}^3}(X), [X] \rangle \cdot 4f$
2. $E \cdot \varphi = -1$

3. $\forall D \in H^2(\mathbb{P}^3), E \cdot \pi^*(D) = (X \cdot D)\varphi$ and $\pi^*(D) \cdot \varphi = 0$

4. $\forall C \in H^4(\mathbb{P}^3), \pi^*(C) \cdot E = 0 = \pi^*(C) \cdot \varphi$

This determines the matrices for the classical action of H, E on the cohomology. To determine the quantum action, we first enumerate the homology classes represented by holomorphic spheres. The dimension of the GW moduli space intersecting three arbitrary cocycles in a symplectic manifold of real dimension $2n$ is

$$dim_{\mathbb{R}}\mathcal{M}_\beta(a, b, c) = 2c_1(\beta) + 2n - |a| - |b| - |c| \tag{7.1}$$

As the invariants are only nonzero for 0-dimensional moduli spaces, the largest possible Chern number needed to define the quantum product occurs when the degrees of a, b, c are maximal, which in this case is degree 6. However, since we only care about the action of the degree 2 class $c_1(M)$, this restricts the degrees even further:

$$0 = 2c_1(\beta) + 6 - 2 - 6 - 6$$

and hence $c_1(\beta) = 4$ is the largest relevant Chern number. By monotonicity, there are no nonconstant spheres with nonpositive Chern number, so in fact we need only describe those spheres with $c_1 = 1, 2, 3, 4$.

To accomplish this, represent the c_1 homomorphism geometrically as the intersection product with the following anticanonical divisor. In the blown-up manifold, there are total transforms $\tilde{Q}_\lambda$ of the quadrics whose intersection determines the curve X. We can express these in our basis as $[\tilde{Q}_\lambda] = 2H - E$.

Thus the anticanonical divisor is given by

$$-K = 2\tilde{Q}_\lambda + E.$$

Now for a holomorphic curve u in homology class $\beta = ml - kf$, by examining how a representative of a curve with given index intersects this anticanonical divisor, we can completely characterize spheres with the given Chern numbers.

First suppose u has Chern number 1. Clearly any fiber of the exceptional divisor satisfies this condition, so we show that these are the only possible curves with index 1. If u is not contained entirely in the exceptional divisor or the relative quadric $\tilde{Q}_0$, it will intersect each term in $-K$ positively, so

$$2\tilde{Q}_0 \cdot u + E \cdot u = 1 \tag{7.2}$$

and since each term is nonnegative, $E \cdot u = 1$. Then since $4h \cdot u - E \cdot u = 4m - 1 = 1$, we have $m = 0$. It follows that the only such spheres with Chern number 1 are those spheres in the class f.

Now if the curve u is completely contained in $\tilde{Q}_0$, recall that there are only two classes of lines on a quadric surface. These are the ruling spheres, and each point in the quadric lies on exactly 2 such lines. Moreover, each such line intersects the curve twice, so its proper transform is in homology class $l - 2f$. As this doesn't have Chern number 1, we see that the only index 1 spheres are exceptional spheres.

Turning now to Chern number 2, the $l - 2f$ spheres mentioned above are candidates, as well as any $ml-(4m-2)f$ sphere. But a similar argument to the above eliminates most of them. If an index 2 sphere u is transverse to both E and $\tilde{Q}_0$, by positivity of intersections $u\cdot\tilde{Q}_0$ is 0 or 1, in which case the intersection with E is 2 or 0, respectively. On the other hand, we see that

$$\begin{aligned}
-K.u &= \langle 2\tilde{Q}_0+E, ml-kf\rangle \\
&= 4m-2k+k \\
&= 4m-k \\
&= 2
\end{aligned}$$

Since $u\cdot E=k$ can be 0 or 2, we see that $m=0,1$. These correspond to the classes $l-2f, 2f$. Note that in class $l-2f$, all such lines lie in a generic quadric of the given pencil - as $u\cdot E=2$, the line u must intersect X in two points, represented by vectors z_i in $\mathbb{C}^4$ which satisfy the equation

$$z_i^T(Q_t)z_i=0 \tag{7.3}$$

defining points on any quadric. A straightforward calculation then shows that there is a unique t such that all linear combinations of the z_i also satisfy (7.5).

A similar argument works for Chern numbers 3 and 4, where the possibilities are $l-f$ and $3f$, and Chern number 4 corresponds to classes $l, 2l-4f, 4f$.

Note also that these calculations allow us to deduce the Chern numbers to the normal bundles of these spheres. For each possible homology class above, the exact sequence defining the normal bundle allows us to express the

Chern class of the normal bundle in terms of the Chern numbers of the sphere and the Chern numbers of $u^*(TM)$. As the only spheres we consider have positive Chern number, and $c_1(u^*(TM))$ is a direct sum of positive multiples of $c_1(TM)$. Thus, holomorphic representatives have normal bundles whose Chern numbers are at least -1, and by Theorem 7.1.2 the curves in question are automatically regular.

7.3.1 Calculating GW Invariants

Having characterized the relevant homology classes of spheres, so now we may calculate the quantum contributions of the actions of H and E by way of the following lemmas.

Lemma 7.3.1. *Using the notation and basis for cohomology given above, we have*

1. $I_{l-2f}(H^2 - 2\varphi, H^2 - 2\varphi) = 0$,
2. $I_{l-2f}(\varphi, \varphi) = 1$,
3. $I_{l-2f}(H^2 - 2\varphi, \varphi) = 1$
4. $I_{l-2f}(H^2, H^2) = 8$, $I_{l-2f}(H^2, \varphi) = 3$.

Proof. (1) Choosing two generic representatives Λ_i of $H^2 - 2\varphi$, the previous lemma showed that they are proper transforms of lines lying in distinct quadrics Q_i of the pencil. On the other hand, the argument characterizing

Chern number 2 spheres above established that they must lie in some quadric of the pencil and have intersection number 0 with the exceptional divisor. Taking the Λ_i to be in distinct quadrics, no line in another fiber of the pencil can simultaneously intersect these two lines, as it would have different intersection with the exceptional divisor.

(2) As two fibers of the exceptional divisor determine two points on X, there is a unique line passing through them.

(3) We ask for lines Λ connecting a generic point on X and a line l in a generic quadric $Q \in \mathbb{CP}^3$ of the pencil, whose proper transform intersects the divisor in points x_1, x_2. These three points determine two lines in $\mathbb{CP}^3$ whose proper transforms are the only candidates for Λ. However, only one such line can lie in Q; they must be in the same ruling distinct from the one generated by l. It follows that $I(H^2 - 2\varphi, \varphi) = 1$.

(4) This follows directly from the above statements and multilinearity of the Gromov-Witten invariant. □

Lemma 7.3.2. $I_{l-2f}(H^3) = 2$.

Proof. Without loss of generality the generic point p dual to H^3 lies away from X and on a smooth quadric Q_0 of the pencil. Any line l contributing to $I_{l-2f}(H^3)$ necessarily intersects X twice, so it follows that l must also lie in Q_0(any line intersecting a quadric in three distinct points must lie on the quadric). Now there are two lines through p contained in Q_0, one for each ruling, and these exhaust the possible lines in Q_0. □

Lemma 7.3.3. $I_{l-f}(h^2, h^3) = 4$

Proof. Given a generic point p and a generic line l in $\mathbb{P}^3$, we count how many lines intersecting p and l also intersect X once. The set of such lines in projective space sweeps out a plane, and thus has intersection number 4 with the degree 4 curve X. Each such point has a unique line through it intersecting p, so we find in total 4 lines satisfying the given conditions. □

Lemma 7.3.4. $I_{l-f}(\varphi, h^3) = 1$.

Proof. Given a fixed fiber and generic point p, the image of f in X determines a point on the curve, through which there is a unique line through p. □

Lemma 7.3.5. $I_f(\varphi) = -1$, $I_{2f}(\varphi, \varphi) = 0$

Proof. Consider curves u_i in $\mathbb{P}^3$ whose strict transforms intersect the exceptional divisor transversely in i points. Then the cohomology class of $PD(u_i)$ satisfies $I_f(PD(u_i)) = i$, one for each intersection point. But since $PD(u_0) = PD(u_1) + \varphi$, the claim follows from multilinearity. For the second claim, since all fibers in the exceptional divisor are disjoint from one another, no fiber can be incident to two distinct fibers. □

A direct computation then produces the following matrices for the action of E, H.

$$H\cdot = \begin{pmatrix} 0 & 2q^2 & 4q^2 & 4q^3 & q^3 & q^4 \\ 1 & 0 & 0 & 8q^2 & 3q^2 & 4q^3 \\ 0 & 0 & 0 & -3q^2 & -q^2 & -q^3 \\ 0 & 1 & 0 & 0 & 0 & 2q^2 \\ 0 & 0 & 4 & 0 & 0 & -4q^2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{7.4}$$

Similarly the matrix for the action of E is given by

$$E\cdot = \begin{pmatrix} 0 & 4q^2 & 8q^2 & 4q^3 & q^3 & 0 \\ 0 & 0 & 0 & 16q^2 & 6q^2 & 4q^3 \\ 1 & 0 & 0 & -6q^2 & -2q^2 & -q^3 \\ 0 & 0 & -4 & 0 & 0 & 4q^2 \\ 0 & 4 & 16 & 0 & q & -8q^2 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{7.5}$$

Since $c_1(M_1) = 4H - E$, a computer algebra system allows us to deduce generalized eigenspaces and eigenvalues.

Proposition 7.3.6. *There is a generalized eigenspace decomposition for $c_1(M)$ of $QH^*(M)$ into summands which are subalgebras:*

$$\begin{aligned} QH^*(M)\otimes\mathbb{C} &= QH_{\pm 1} && (rk_{\mathbb{C}} = 4) \\ &\oplus QH_0 \oplus QH_{-(4i+4)} \oplus QH_{4i-4} \oplus QH_8 && (\text{Each } rk_{\mathbb{C}} = 1) \end{aligned}$$

Having determined the eigenvalue decomposition of $QH^*(M)$, it follows from [10] that the $QH^*(M)$-module $HF(\Phi)$ must split into summands corresponding to the above eigenvalues. We take advantage of this result as well as the following theorem. Though it is proven in a significantly more general situation than this, we will only require the special case of a Dehn twist and its effect on fixed-point Floer homology, so the statement is reinterpreted for our setting.

Theorem 7.3.7. *([10], Theorem 5.14) Let V be a Lagrangian sphere in Y, $\tau_V : M \to M$ a Dehn twist about V, and $\Phi : M \to M$ a symplectomorphism. Then there is an exact sequence*

$$\begin{array}{ccc} HF_*(V,V) & \longrightarrow & HF(\Delta, \Gamma_\Phi) \\ & \searrow \quad \swarrow & \\ & HF(\Delta, (Id \times \tau_V)(\Gamma_\Phi)) & \end{array}$$

of $QH^*(M)$-modules.

Since passing to generalized eigenspaces is an exact functor, we see that for any eigenvalue λ for the action of $c_1(M)$ on $HF_*(\Phi)$, there is an exact sequence as above consisting of the generalized λ-eigenspaces.

Proposition 7.3.8. *For $tr(\gamma) < 2$, the differential in $CF^0(\Phi)$ is trivial.*

Proof. We proceed by induction on the number of Dehn twists needed to generate ϕ. First note that the reference blowup $M = Bl_X(\mathbb{P}^3)$ can be described as a divisor in $\mathbb{CP}^3 \times \mathbb{CP}^1$, representing the class $2H_{\mathbb{P}^3} - H_{\mathbb{P}^1}$ as the blowup occurs along an intersection of quadrics. This is described in [Smi] Lemma 5.5, but we review the construction here briefly. Starting with $L \to \mathfrak{X}$ thought of as a family of holomorphic line bundle-curve pairs over S^1, we extend this to a family over $\mathbb{C}^*$, still denoted by $\mathcal{X}$, with generic smooth fiber X_w. Suppose the monodromy in the family $\mathcal{X}$ consists of a single Dehn twist about a curve γ. As L is very ample, it induces an embedding of $\mathcal{X}$ in $\mathbb{C}^* \times \mathbb{P}^3$, with X_w embedded as an intersection of quadrics Q_0, Q_1^w where Q_i are symmetric and full rank. Asking that the monodromy about 0 induce a Dehn twist in the fiber determines a two-parameter family of quadrics

$$\{wQ_0 + (1-w)Q_1^t\}$$

whose fiber over $w = 0$ is a pencil of quadrics whose common intersection is a nodal curve X_0. The existence of such families is guaranteed by an analysis of the locus of singular quadrics in $\mathbb{P}^6$, as in [9, p. 302].

Then, it is straightforward to construct a family of blowups M_w along X_w over $\mathbb{C}$ which are blowups along smooth curves for all $w \neq 0$, and the fiber over center is the blowup along a nodal curve. These quadrics can be written explicitly as symmetric 4×4 matrices varying in two parameters λ and w. A calculation with the explicit model above shows that the degeneration of the blowup loci Z_w in fact corresponds to a degeneration of the blowups Y_t which is a Lefschetz fibration over $\mathbb{C}$. In particular this establishes that the vanishing cycles are Lagrangian S^3's which lie over the vanishing cycle γ in X_w. Further, the loop about $0 \in \mathbb{C}$ producing the Dehn twist about γ induces a Dehn twist about a Lagrangian S^3 in the Fano blowup M_w. Following the arguments in [25] Section 5.1, these vanishing cycles satisfy the intersection condition

$$\langle V_\gamma, V_{\gamma'} \rangle_{M_w} = \langle \gamma, \gamma' \rangle_{X_w}$$

We now require a fact regarding the action of c_1 on the cohomology classes represented by these vanishing cycles.

Lemma 7.3.9. *([25], Lemma 5.4) For $V_\gamma \in H^3(M_1)$ a class of a vanishing cycle as above, $c_1 \cdot V_\gamma = V_\gamma$, so these Lagrangian spheres lie in the ± 1-summand for the action of c_1.*

Thus, when considering the c_1-eigenspaces not corresponding to $\lambda = \pm 1$, every third term in the above exact sequence is 0. Since $HF(Id) \cong QH(Y)$ this means that $HF(\phi)$ contains a distinguished rank 4 summand corresponding to those λ-eigenspaces for the c_1-action for which $\lambda \neq \pm 1$.

For general ϕ, consider the composition of the above with one or more Dehn twists about another curve. By the above construction, if $\gamma, \gamma' \subset X_t$ are vanishing cycles that intersect transversely in a point, then by Lemma 6.10, $HF(V_\gamma, V_{\gamma'})$ lies in the ± 1 summand of the Fukaya category, so the same exact sequence argument as above, but with different Lagrangians, gives the same rank 4 summand. Thus, no even degree generators of $CF(\Phi)$ can lie in the image of the differential, so the rank of the even part of $HF(\Phi)$ is at least 4.

However, Proposition 5.6 shows that $HF(\Phi)$ is isomorphic to the homology of a complex with even dimension at most 4 when $Tr(\phi) < 2$, so it follows that the even summand of $HF(\Phi)$ has dimension precisely 4, and the result follows. □

Appendix

Morse-Bott Functions on Blow-ups

Let $X \subset Z$ be a closed submanifold of a Kähler manifold Z, and let Y be the blowup of Z along X. Then, for a Morse function $f : X \to \mathbb{R}$, we have $\tilde{f} : Y \to \mathbb{R}$ its pullback via the blowdown map $\pi : Y \to Z$, which is in general a Morse-Bott function. As before we denote the exceptional divisor by E. Then we have the following description of its critical manifolds.

Theorem A. *Let X, Y, Z, f be as above, and suppose that f has no critical points on X. Then $\widetilde{f}$ is Morse-Bott, and its critical manifolds are given by critical points of f and $Ker(df) \cap J(Ker(df) \subset \pi^{-1}(c)$, where $c \in Crit(f|_X)$, J is the integrable complex structure on Y, and $\pi^{-1}(c)$ is a fiber of the exceptional divisor.*

Proof. Away from the divisor, $d\pi$ is an isomorphism, and hence $d\widetilde{f}$ vanishes only at critical points of f and $\widetilde{f}$ remains Morse away from E, so we focus our attention on E. Note that the tangent space to Y at a point (x, λ) on E fits into compatible exact sequences as below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_{(x,\lambda)}E & \xrightarrow{\text{incl.}} & T_{(x,\lambda)}Y & \longrightarrow & \lambda & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \text{incl.}} & & \updownarrow{\scriptstyle Id} & & \downarrow{\scriptstyle \text{incl.}} & & \\ 0 & \longrightarrow & T_{(x,\lambda)}\mathbb{P}NX_x & \xrightarrow{\text{incl.}} & T_{(x,\lambda)}Y & \xrightarrow{d\pi} & T_xZ & \longrightarrow & 0 \end{array} \tag{6}$$

Here a fiber of the exceptional divisor is given by $\pi^{-1}(x) = \mathbb{P}NX_x$. These diagrams relate the behavior of df and $d\widetilde{f}$, allowing us to characterize the set of points on the divisor at which $d\widetilde{f} = df \circ d\pi = 0$.

At points on the divisor, implies that $df|_X = 0$ if $(x, \lambda) \in Crit(\tilde{f})$, so the critical submanifold of Y must lie in $\pi^{-1}(Crit(f|_X)$. Note that *df* is well-defined on NX at critical points of $f|_X$.

However, $\pi(x, \lambda) \in Crit(f|_X)$ does not imply that $(x, \lambda) \in Crit(\tilde{f})$. By (7.1), the map $d\pi$ sends normal vectors v to the exceptional divisor into TX by inclusion. So if *df* does not vanish on $d\pi(v) \in T_{\pi(x,\lambda)}(E)$, then $\widetilde{f}$ fails to be critical at (x, λ). It follows that $x \in Crit(\widetilde{f})$ if and only if $\pi(x) \in Crit(f|_Z)$ and *df* vanishes on $\lambda \in N_{X,Z}$. More precisely, $(x, \lambda) \in E$ is a critical point of $d\widetilde{f}$ when λ is a complex line in Ker(*df*).

$Crit(\tilde{F})$ then consists of all lines in the maximal complex linear subspace of the kernel of *df*, which we denote $L_x := Ker(df) \cap J(Ker(df))$. This is a projective hyperplane in the projectivized normal bundle over $c \in Crit(f|_X)$. □

To prove Theorem 6.2.1, then, it only remains to construct a Morse function $f : Z \to \mathbb{R}$ which has no critical points on X but for which $df|_X = 0$. A generic Morse function has no critical points on X by a codimension argument, so we arrange for a perturbation so the former condition holds. Recall that $N_{X,Z}$ has $c_1(N_{X,Z})$ even, so as a real oriented vector bundle it must be trivial as all Stiefel-Whitney classes vanish. This argument applies equally well to the family $\mathfrak{X}$, and so as a real vector bundle $N_{X,Z} \cong \mathfrak{X} \times \mathbb{R}^4$.

With this trivialization it suffices to consider functions $\mathfrak{X} \times \mathbb{R}^4 \to \mathbb{R}$ which are constant along X but has no critical points. One can verify that projection to the first coordinate clearly satisfies these conditions; the gradient restricted to Z is zero, and it has no critical points in a neighborhood of $X \times \{0\}$. Multiplying by an appropriate cutoff function, this satisfies the given property in a neighborhood of $\mathcal{X} \subset \mathcal{M}_0$.

Adding this to any function which vanishes on a neighborhood of X results in a Morse function with the desired properties near X. Note that for the non-degeneracy result in Chapter 6, the perturbed symplectic connection is given by

$$\omega_f = \omega + \langle F_\alpha, \mu \rangle + dW|_X \wedge dt \tag{7}$$

and we will require that this remain a symplectic connection. However, given some section σ of $\mathcal{X} \times F$ which is horizontal with respect to the first two terms, this may no longer be horizontal after perturbation; if dW_X fails to vanish identically, then contracting with an ω_f-horizontal lift of $\frac{d}{dt}$ means that $dW|_X(v)$ must vanish identically for all vertical v, i.e. dW must be constant on X. But the function constructed above does in fact satisfy this property, as $\mathcal{X}$ lies in its normal bundle as the zero section.

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