Nontraditional Approximation in Geophysical Fluid Dynamics

by

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Nontraditional Approximation in Geophysical Fluid Dynamics

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In the conventional approach to geophysical fluid dynamics, only the horizontal components of the Coriolis force due to horizontal motions of the fluid are taken into account. All the other components of the Coriolis force, which are called the non-traditional (NT) terms, are considered to be small second order quantities and are usually dropped. This effectively simplifies the system and the nice and clean quasi-geostrophic (QG) equation can be obtained, which is widely used in analytical studies of climate systems. Interest has been drawn to the dropped terms in recent studies. It is shown that in some special cases these second order terms actually have a noticeable influence on the dynamics of the system. However, a full picture of these terms in the dynamics of the real ocean is still lacking. Here, we will start from the fundamental equations of fluid dynamics, and through careful scaling analysis conduct a detailed study of the governing equations of geophysical fluid dynamics while keeping the NT terms. We will specifically investigate the

influence of these NT terms on equatorial waves, since near the equator the NT components of the Coriolis force are the most significant.

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Chapter 1

Introduction

1.1 Motivation

The climate system of the earth is one of the most complicated systems of today's scientific research. It includes rich and vast phenomena across many time and spatial scales, involving many aspects of scientific research. However, the attempt to understand climate started more than two thousand years ago, in ancient Babylon, Greek, and China, for example, where weather prediction was attempted from experiential knowledge of cloud patterns. The scientific approach to describe the climate system appeared along with the development of Newtonian physics, but it was mostly limited to theoretical understanding without practical application at that time. Not until the 1830's-1840's, when the invention of electric telegraph first enabled the near instantaneous transfer of weather information among different localities could truly useful weather forecasting be done. In the twentieth century, the invention of computers made it possible to run computer models to simulate the climate system and thus usher research of the climate system into a new age.

In the past few decades, with the rapid growth of computer technology and other modern techniques to obtain climate data, researchers were building more and more complicated computer climate models that captive more and more aspects of the climate system. However, despite the significant growth of complexity of the computer models, the accuracy of predictions made by the computer models has not improved that significantly. This is evidenced by the inaccurate predictions made by the weather forcasts on TV, newspapers, and on websites.

There are two kinds of errors of computer simulations. One is random error, which is due to the limited accuracy of the computer, while the other is systematic error, which is introduced by the computer model itself. As long as the growth of computation power continues, the random errors will continue to be effectively subdued. However, no matter how fast computers become, how fine the grid is, or how many processors are running, the systematic errors will still exist unless the dynamics of the computer model itself is improved. Moreover, unlike random errors, the systematic error usually drives the system with a consistent bias. Even though the deviation is initially tiny, it may grow large exponentially and significantly reduce the predictability of the models.

Since the climate system is such a complex one, in order to get any useful result, assumptions and approximations must be made. On the one hand, these methods help to understand the problem or to give some useful results, but on the other hand, any assumptions and approximations may introduce systematic errors to such simplified physical models. Recently, more attention has been drawn from building giant climate models to reexamining the accuracy and the validity of the widely adopted assumptions and approximations used in the climate research. One of these approximations is so called *traditional approximation.*

1.2 Traditional Approximation

In any rotating reference frame with angular velocity Ω, a moving object with velocity v has a non-inertial acceleration which is twice the cross produce of the velocity of the object and the angular velocity of the rotating frame. This force is called the Coriolis force, $F = 2mv \times \Omega$, where m is the mass of the object. In the context of geophysical fluid dynamics, the rotating reference frame is that of the earth. The Coriolis force can be decomposed into four terms: two are proportional to the sine of the latitude and the other two are proportional to the cosine. For the two terms proportional to the sine, only horizontal motion is involved, while the two cosine terms are related to vertical motion. The two cosine terms are traditionally neglectable, a pratice dating to Laplace, who developed a series of reasonings to argue that these two terms are insignificant. For this reason the approximation of neglecting the two cosine terms of the Coriolis force is called the *traditional approximation* (TA).

Indeed it is reasonable to adopt the TA in most circumstances. One reason is that both the atmosphere and the ocean are thin layers, i.e. their vertical scales are much smaller than their horizontal scales, which limits the vertical motion. The Coriolis force is important for motions with time scales larger than the rotation period of the earth. A large time scale implies a large spatial scale as well. Therefore, large scale motion has to be horizontal.

To leading order, the horizontal Coriolis force is the one that balances the horizontal pressure gradient. While in the vertical direction, the cosine terms contribute little to the hydrostatic balance. Moreover, both the atmosphere and the ocean are stratified, which further suppresses the vertical motion. Apart from these reasons, the true reason might be simplicity at the expense of accuracy, since the TA greatly simplifies the problem.

However, even though the TA maybe a very accurate approximation, due to the reasons discussed above, it may lead to large systematic error when the climate system is integrated over a long time span. In some special cases, or near the equator of the earth, the cosine terms of the Coriolis force can have more significance. Recently, researchers [6] have investigated the effect of these dropped terms in the traditional approximation. Models with these non-traditional (NT) terms included have been developed. These terms have been integrated into numerical simulations in non-hydrostatic or a quasi-hydrostatic setting. The effects of these terms have also been analytically studied. Some research was done to investigate the influence of these terms on Ekman dynamics, internal waves, and some other special cases. Similarly, I am going to further investigate the influence of the nontraditonal terms, especially on equatorial waves in the ocean, starting with the basic theory of fluid dynamics.

Chapter 2

Fundamental Fluid Dynamics

This Chapter will start from basic fluid dynamics and derive fundamental governing equations of geophysical fluid dynamics, from the point of view of real fluids in the atmosphere and oceans of the earth. I will firstly give a very brief review of general fluid dynamics. Then, rotation is introduced to derive the basic equations of geophysical fluid dynamics.

2.1 Two Time Derivatives

Like classical mechanics for rigid bodies, the motion of classical fluids is governed by the set of fundamental laws of physics, i.e. Newton's Laws and the laws of thermodynamics. However, the equations of motion of rigid bodies and fluids are quite different.

When we study the motion of rigid bodies, the very first thing we need to do is usually to find a function to describe the time derivative of a certain quantity ϕ associated with a certain object, i.e. $d\phi/dt$. No matter what forms the function can take, the time derivative itself must always be tagged to the same object for all time. In principle it is certainly not wrong to stay with the same approach to describe the motion of a fluid, by tracking the motion of each infinitesimal fluid parcel through the dynamical process. But as a continuum, fluids deform while flowing. This Lagrangian variable approach can be extremely complicated to be implemented. Moreover, when studying the motions of a fluid, usually people are not interested in how a particular fluid element moves through time, but rather information of the motion of the local fluid at a certain point in space and time, the Eulerian variable approach. For example, a weather forecast does not tell what will happen to a certain portion of the air tomorrow, instead, it generally reveals what will happen at a given locality. That is, we are more likely to use the time derivatives in the form of, $\partial\phi(x, y, z, t)/\partial t$, where ϕ is a dynamical field variable (e.g. velocity, temperature, density) which is defined in an arbitrary volume in space and an arbitrary period in time. The time derivative $d\phi(x, y, z, t)/dt$ follows a fluid element and is called the material or total derivative (with respect to time), while $\partial\phi(x, y, z, t)/\partial t$ is called the Eulerian or partial derivative. Nevertheless, the material derivatives sometimes are useful to describe some laws, especially certain conservation laws and are also helpful when deriving the equations. So we are going to find the relationship between these two time derivatives in the following section.

2.1.1 Material Derivative

Assume the overall motion of the fluid can be described by a velocity field $v(x, t)$, the material derivative or total derivative of a scalar property associated with any fluid element can be derived by the chain rule:

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{\partial\phi}{dt} + \frac{\partial\phi}{dx}\frac{dx}{dt} + \frac{\partial\phi}{dy}\frac{dy}{dt} + \frac{\partial\phi}{dz}\frac{dz}{dt} \\ &= \frac{\partial\phi}{dt} + \frac{dx}{dt}\cdot\nabla\phi \\ &= \frac{\partial\phi}{dt} + v\cdot\nabla\phi. \end{aligned} \tag{2.1}$$

Because of its heavy usage, the material derivative is usually denoted by D/Dt. Then we can rewrite the material derivative of scalar field ϕ as:

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{dt} + v\cdot\nabla\phi. \tag{2.2}$$

When the material derivative acts on a vector field, for each component b_i of a vector b, we still have:

$$\frac{Db_i}{Dt} = \frac{\partial b_i}{dt} + v\cdot\nabla b_i. \tag{2.3}$$

Therefore, the material derivative of a vector field b can be written as:

$$\frac{Db}{Dt} = \frac{\partial b}{dt} + (v\cdot\nabla)b. \tag{2.4}$$

2.2 Conservation Laws

The equations of motion of a general fluid can be either derived from basic Newtonian mechanics and some intuitive observations on the general properties of a fluid, or from the most fundamental conservation laws. We are going to following the latter approach in this thesis.

We need to use some fundamental result from calculus. The first is the Leibniz's rule for differentiation under the integral sign, which tells us that:

$$\frac{d}{dt}\int_V \phi\, dV = \int_V \frac{\partial}{\partial t}\phi\, dV, \tag{2.5}$$

where V is an arbitrary volume fixed in space and ϕ can be any kind of field defined locally within the fixed volume. In the content of basic fluid dynamics, usually a scalar or a vector is used here.

The second is the divergence theorem, which tells us that:

$$\int_V \nabla \cdot F dV = \int_S F \cdot n\, dA, \tag{2.6}$$

where S is the surface of the volume and n is the unit normal vector of the differential surface element dA.

Now, we are going to use the Reynolds transport theorem, which states that the rate of change for the integral of some intensive property defined over a control volume is equal to what is lost or gained through the flux across the boundaries of that volume plus the creation or consumption by the sources and sinks within the volume. This theorem can be expressed by the following integral equation:

$$\frac{d}{dt}\int_V \phi\, dV = -\int_S \phi\, v \cdot\, ndA - \int_V Q\, dV, \tag{2.7}$$

where Q is the source or sink within the liquid.

By applying the divergence theorem to change the surface integral to a volume integral, and using the Leibniz's rule, we have:

$$\int_V \frac{\partial}{\partial t}\phi\, dV = -\int_V \nabla \cdot (\phi v)\, dV - \int_V Q\, dV, \tag{2.8}$$

or equivalently

$$\int_V (\frac{\partial}{\partial t}\phi + \nabla \cdot (\phi v) + Q)\, dV = 0. \tag{2.9}$$

The volume V is arbitrary in space, which implies that the integrand must be identically zero,

$$\frac{\partial}{\partial t}\phi + \nabla \cdot (\phi v)\, dV + Q = 0. \tag{2.10}$$

2.3 Equations of Motion

From the general conservation law above, we can obtain the equations of motion of fluid mechanics. By taking ϕ as the mass density, the momentum density, and the energy density, the above equation will become the conservation of mass, momentum and energy, respectively. The conservation of mass and momentum can be further developed into the mass continuity equation and the Navier-Stokes equation of motion. In order to close the system, conservation of energy is required as well, which will require knowledge of some laws of thermodynamics. Since our focus is on the mechanical aspects of fluid dynamics, we only briefly review the thermodynamics of fluids.

2.3.1 The Continuity Equation

The density of a fluid may vary and for any given fixed volume, fluid may flow in and flow out. However, the total mass should always be conserved. Therefore, the total mass flow out of any given fixed volume V, which is bounded by a close surface S, must be equal to the total mass loss within that volume, assuming there is no source or sink inside the volume. (This is always true for fluid dynamics in the classical realm.) Mathematically, by plugging the density ρ into Eq. (2.10) as ϕ, we can write down the mass continuity equation as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{2.11}$$

which can also be written as:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot v = 0. \tag{2.12}$$

2.3.2 The Momentum Equation

According to Newton's Second Law, the rate of change of an object's momentum is equal to the force on it. Let $\rho(x, y, z, t)$ be the density of the fluid, so ρv is the momentum density. Let B be the density of the force within the fluid, which serves as source or sink for the momentum (per volume) within the liquid. By replacing ϕ with ρv and Q with B in Eq. (2.10), we have:

$$\frac{\partial (\rho v)}{\partial t} + \nabla \cdot (\rho v v) = B. \tag{2.13}$$

Note that vv is not a dot product, instead, it is a vector outer product producing a second order tensor. Expanding the derivatives completely, yields

$$v\frac{\partial \rho}{\partial t} + \rho \frac{\partial v}{\partial t} + vv \cdot \nabla \rho + \rho v \cdot \nabla v + \rho v \nabla \cdot v = B, \tag{2.14}$$

and regrouping implies

$$v\left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v)\right) + \rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) = B. \tag{2.15}$$

By Eq. (2.10), the leftmost expression enclosed in parentheses is zero. So, we have the momentum equation, knows as the Navier-Stokes Equation:

$$\rho\left(\frac{\partial v}{\partial t}+v\cdot\nabla v\right)=B, \tag{2.16}$$

which can also be written in form of the material derivative as:

$$\rho\frac{Dv}{Dt}=B. \tag{2.17}$$

This is simply an expression of Newton's second law in terms of body forces.

The body force B can be complicated for some fluids, however, for a Newtonian fluid, like air and water near the surface of the earth, the body force B can be described by the sum of three terms: the pressure force $-\nabla p$, the viscosity F, and gravity force $-\rho g$. Therefore, the Navier-Stokes equation in an inertial reference frame is:

$$\rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)=-\nabla p-\rho g+F. \tag{2.18}$$

It is shown in many fluid text books that the viscous force per unit volume F for a Newtonian fluid can be expressed as $\nu\nabla^2 v$, where ν is the kinematic viscosity. As far as large scale motion is considered, the viscosity can usually be dropped. We will get back to this idea later.

2.4 Thermal Dynamics

In order to complete the dynamics of fluid, the laws of thermodynamics must be introduced here. Since in our focus of study, all the thermal dynamical effects will be dropped, only the key results will be listed below.

The first Law of thermodynamics will give us the conservation of energy, which is:

$$\rho\frac{De}{Dt} = -p\rho\frac{d}{dt}\rho^{-1} + \kappa\nabla^2 T + \chi + \rho Q, \tag{2.19}$$

where e is the internal energy per unit mass, T is the temperature, κ is the thermal conductivity, χ is the addition of heat due to viscous dissipation, and Q is the rate of heat addition per unit mass by internal heat sources. For all circumstances, the addition of heat due to viscous dissipation is negligible. By introducing the specific entropy s, we have:

$$T\Delta s = \Delta e + p\Delta\left(\frac{1}{\rho}\right). \tag{2.20}$$

Then, Eq. (2.19) can be written as:

$$T\frac{Ds}{Dt} = \frac{\kappa}{\rho}\nabla^2 T + Q. \tag{2.21}$$

To complete the system, thermodynamical equations of state are required. For example:

$$\rho = \rho(p, T) \quad \text{and} \quad s = s(p, T).$$

A simple equation of state could be in form of:

$$\rho = \rho_0(1 - \alpha(T - T_0)), \tag{2.22}$$

and the corresponding thermodynamical equation will be

$$\frac{d\rho}{dt} = \kappa\nabla^2\rho - \frac{\alpha\rho_0}{C_p}Q, \tag{2.23}$$

where α is the coefficient of thermal expansion and C_p is the specific heat at constant pressure.

2.5 Rotating Frames

One of the major differences between geophysical fluid dynamics and general fluid dynamics is that the most natural reference frame for studying the motions of atmospheres and oceans on the surface of the earth is the rotating earth itself. Therefore, we adopt the equation of motion of fluid dynamics in a rotating frame.

In an inertial frame, if a vector A, fixed in magnitude, is rotating at a constant angular velocity Ω, we have:

$$\left(\frac{dA}{dt}\right)_I = \Omega \times A. \tag{2.24}$$

The subscript I denotes that the derivative is taken in an inertial frame. Therefore, in a rotating frame with angular velocity Ω, a fixed vector A is rotating at angular velocity $-\Omega$, and we have:

$$\left(\frac{dA}{dt}\right)_R = -\Omega \times A, \tag{2.25}$$

and for any vector A, we have:

$$\left(\frac{dA}{dT}\right)_I = \left(\frac{dA}{dt}\right)_R + \Omega \times A. \tag{2.26}$$

Let r be the position vector of an arbitrary fluid element. According to Eq. (2.26), we have

$$\left(\frac{dr}{dT}\right)_I = \left(\frac{dr}{dt}\right)_R + \Omega \times r \tag{2.27}$$

or

$$u_I = u_R + \Omega \times r, \tag{2.28}$$

where u_I and u_R are the velocities of the fluid element observed in the inertial frame and the rotating frame, respectively. Applying Eq. (2.26) to u_I gives

$$\begin{aligned} \left(\frac{du_I}{dT}\right)_I &= \left(\frac{du_I}{dt}\right)_R + \Omega \times u_I \\ &= \left(\frac{d(u_R + \Omega \times r)}{dt}\right)_R + \Omega \times (u_R + \Omega \times r) \\ &= \left(\frac{du_R}{dt}\right)_R + 2\Omega \times u_R + \Omega \times (\Omega \times r) + \frac{d\Omega}{dt} \times r. \end{aligned} \tag{2.29}$$

Since Ω is regarded as a constant, the last term in Eq. (2.29) vanishes. Moreover,

$$\Omega \times (\Omega \times r) = -\nabla \phi_c,$$

where

$$\phi_c = \frac{|\Omega \times r|^2}{2}.$$

Since the acceleration due to gravity, g, can be written as the gradient of a gravitational potential ϕ_g, we can define a $\Phi = \phi_g + \phi_c$ so that the Navier-Stokes equation in a rotating reference frame with angular velocity Ω can be written as:

$$\rho\left(\frac{Du}{Dt} + 2\Omega \times u\right) = -\nabla p - \rho \nabla \Phi + F. \tag{2.30}$$

Since rotation has no influence on scalars in general, the continuity equation (2.11), the energy conservation equation (2.19), and the equations of state remain the same in a rotating reference frame.

Chapter 3

Geophysical Fluid Dynamics

3.1 Introduction

Geophysical fluid dynamics deals with the fundamental and essential dynamical concepts for understanding the atmosphere and the oceans, based on the principles of general fluid dynamics, with some special conditions demanded by the specific geophysical setting. Some characteristics of the atmosphere and the ocean water determine the dynamics. For example, the first concern is the spatial dimension. For large scale or meso scale motion of the atmosphere or ocean water, an especially slow process, the horizontal scale of the motion is usually much larger than the vertical scale, which limits the vertical motion of the liquid. This property leads to the thin layer approximation. The second is that both the atmosphere and the ocean water are stratified, which effectively subdues the vertical motion of the liquid, and further enables the existence of geophysical waves in the atmosphere and the oceans. The third is rotation. When the time scale of motion is much shorter than the period of the earth's rotation, namely a day, the rotational effect is insignificant. However, when the time scale is comparable to or larger than a day, then, the influence of rotation must be taken into account.

The complete set of governing equations, in the rotating frame of the earth, as we discussed in the previous chapter, are:

- The Continuity Equation

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot u = 0. \tag{3.1}$$

- The Momentum Equation (Navier-Stokes Equation)

$$\rho\left(\frac{Du}{Dt} + \underline{2\Omega \times u}\right) = -\nabla p - \rho\nabla\Phi + F. \tag{3.2}$$

- The Thermodynamic Equation (The simplest version)

$$\frac{D\rho}{Dt} = \kappa\nabla^2\rho - \frac{\alpha\rho_0}{C_p}Q. \tag{3.3}$$

where D/Dt is the material derivative, ρ stands for the density, u is the velocity of the fluid, $\Omega = 7.292 \times 10^{-5}$ rad/s is the angular velocity of the earth's rotation, p stands for the pressure, Φ is the sum of the gravitational and the centripetal potentials, and F is the shear stress due to viscosity. In Eq. (3.3), κ is the thermal conductivity, Q is the rate of heat addition per unit mass by internal heat sources, α is the coefficient of thermal expansion, and C_p is the specific heat at constant pressure.

In this chapter, I am going to start with these basic equations and develop the basic theory of geophysical fluid dynamics with careful scaling analysis, tailored to the configuration of oceans, while retaining the terms dropped in the traditional approximation.

3.2 Equations of Motion

Since we are interested in the effects of the dropped Coriolis terms in the traditional approximation, rather than attempting to include everything in our model, we will firstly follow some general steps to simplify the system and get rid of irrelevant small quantities in the equation of motion.

During long geological times, the Earth's surface has stabilized in the form of an oblate sphere by self-adjustment according to its state of rotation. Thus, it is natural and accurate to adopt an oblate spherical coordinate system [5] to describe the motion near the surface of the Earth. However, in order to simplify the equations, a spherical coordinate system is commonly used instead, with an error that is small and far beyond any concerns of the present research [17].

The viscosity of the sea water in the real ocean is very small and we also neglect it in our study. Then, in the spherical coordinate system, we have the inviscid Navier-Stokes momentum Eq. (3.2) as

$$\begin{aligned}
\frac{D_s u_\lambda}{D_s t} + \frac{u_\lambda w_r - u_\lambda v_\phi \tan\phi}{r} + \underline{2\Omega w_r \cos\phi} - 2\Omega v_\phi \sin\phi &= -\frac{1}{\rho r \cos\phi}\frac{\partial p}{\partial \lambda} \\
\frac{D_s v_\phi}{D_s t} + \frac{v_\phi w_r + u_\lambda^2 \tan\phi}{r} + 2\Omega u_\lambda \sin\phi &= -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} - \frac{1}{r}\frac{\partial \Phi}{\partial \phi} \\
\frac{D_s w_r}{D_s t} - \frac{u_\lambda^2 + v_\phi^2}{r} - \underline{2\Omega u_\lambda \cos\phi} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{\partial \Phi}{\partial r}.
\end{aligned}$$

where λ is the longitude, ϕ the latitude, r the radius, and u_λ, v_ϕ and w_r are the three velocity components. The material derivative in the spherical coordinate system is defined as:

$$\frac{D_s}{D_s t} = \frac{\partial}{\partial t} + \frac{u_\lambda}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v_\phi}{r}\frac{\partial}{\partial\phi} + w_r\frac{\partial}{\partial r}.$$

The equations above in the earth's rotating spherical coordinate system can be further simplified by choosing a local Cartesian coordinate system. $x = R\cos\phi_0\lambda$, $y = R(\phi-\phi_0)$, and $z = r-R$, where ϕ_0 is the reference latitude and R is the average radius of the earth, and u, v, w are the associated velocity components. Since the radius of the earth is quite large, $R = 6365\,\text{km}$, we can replace r with R and further drop those nonlinear terms with R as denominators. In order to keep the variation of the Coriolis force with latitude, which is due to the curvature of the earth's surface, we use a linear approximation to represent the Coriolis parameters $f = f_0 + \beta y$, with $f_0 = 2\Omega\sin\phi_0$, and $\beta = 2\Omega\cos\phi_0/R$. We also let $\tilde{f} = 2\Omega\cos\phi_0$. Note, $\tilde{f}$ only appears in the underlined terms which are dropped in the traditional approximation. Here we only track the leading order effects of keeping these terms by making them constant. In fact, it is shown in [7] that to keep $\tilde{f}$ constant is necessary in order to keep the conservation of angular momentum and vorticity. We further assume the local variation of gravity can be ignored. So we let the constant effective gravitational acceleration $g = \partial\Phi/\partial r$, and $\partial\Phi/\partial\phi = 0$. Then, we can rewrite the Navier-Stokes equation in the β–plane as:

$$\begin{aligned}
\frac{Du}{Dt} + \tilde{f}w - fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} & (3.4)\\
\frac{Dv}{Dt} \qquad + fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} & (3.5)\\
\frac{Dw}{Dt} - \tilde{f}u \qquad &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. & (3.6)
\end{aligned}$$

Where the material derivative is now defined as:

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

When studying the daily or seasonal cycles in the ocean, the thermodynamic equation (3.3) is important, especially in the upper layer of the ocean, which is greatly influenced by the sun's radiation and the air temperature. But, in the rest of the ocean, like viscosity it can be neglected, because the thermal conductivity is too small to make any significant difference in the system within the time span of interest. The density of the ocean has slight variation and the compressibility of water is very small ($4.6 \times 10^{-10}\,\mathrm{m}^2/\mathrm{N}$), which means that even at the ocean bottom at 4000 m below the surface, the pressure can only introduce a 1.8% change of a volume of water. As far as the motion of the ocean water is concerned, the water can be treated as incompressible, i.e. $\nabla \cdot v = 0$. We use this condition instead of the thermodynamic equation to close the system. The mass continuity equation is thus simplified by the incompressibility of water. Therefore, the last two equations needed to close our system are:

$$\frac{D\rho}{Dt} = 0 \tag{3.7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3.8}$$

3.3 Scale Analysis

3.3.1 Leading order balance in the Equation of Motion

Now we take a look at the vertical equation (3.6). In the ocean, except in areas with a lot of sinking and upwelling, the vertical motion of the sea water is very small. In fact the vertical velocity has a magnitude of about 1 cm/s or less. The typical time scale of interest is days ($10^5 - 10^6$ s) and the typical horizontal speed of motion is about .1 m/s – 1 m/s, so all the w terms in Eq. (3.6) are of order 10^{-7} m/s^2 – 10^{-6} m/s^2. The $\tilde{f}u$ term is about 10^{-4} m/s^2, and $(\rho)^{-1}\partial p/\partial z$ and g are of order 10 m/s^2. Therefore, the leading order of the vertical equation (3.6) is:

$$\frac{\partial p}{\partial z} = -\rho g, \tag{3.9}$$

which is just the hydrostatic balance approximation. It is indeed a very accurate approximation, since all the other terms are at most 10^{-5} smaller. This is one of the reasons why the TA is widely adopted. Equation (3.9) is a static balance, however, for the motion of water, but what matters is the deviation away from this balance. The term $\tilde{f}u$ actually is only two orders of magnitude smaller than this leading terms when dynamical aspects are considered. We will return to this later in this chapter.

When only meso or larger scales of motion in the ocean are considered, the ocean actually is in a quasi-static state. The acceleration and nonlinear terms are relatively small. Therefore, the leading orders[1] of Eqs. (3.4) and (3.5) are:

$$-fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.10}$$

$$fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{3.11}$$

By taking the x and y derivatives of the hydrostatic balance equation (3.9) and taking the z derivatives of the Eq. (3.10) and Eq. (3.11), and dropping the variations of density, we can obtain the famous thermal wind relationships:

$$f\frac{\partial v}{\partial z} = \frac{g}{\rho}\frac{\partial \rho}{\partial x} \tag{3.12}$$

$$-f\frac{\partial u}{\partial z} = \frac{g}{\rho}\frac{\partial \rho}{\partial y}. \tag{3.13}$$

3.3.2 Scales in the ocean

Now we identify the typical scales of variability in the β–plane version of the Navier-Stokes equation. The scaling analysis is based on the mesoscale waves/eddies in the real ocean. The typical horizontal length scale of mesoscale motion, i.e. the size of mesoscale eddies is about $L = 10^5\,\mathrm{m}$, the typical depth of the ocean is about $D = 4000\,\mathrm{m}$, the typical horizontal velocity is $U = 1\,\mathrm{m/s}$, and the typical vertical velocity is scaled as $W = UD/L$. For

[1]This leading order balance is not valid near the equator where f vanishes. This is the reason why the nontraditional terms play an important role in equatorial wave theory.

mesoscale eddies, the revolution period is about several days, which is about $2\pi L/U$, For simplicity, we define the typical time scale to be $T = L/U$.

The scaling of density and pressure are subtle. In the case of slow relative motion, both the pressure and the density are only slightly disturbed from their values when the ocean is at rest. Let $\rho_s(z)$ be the density profile of the stratified ocean at rest and $p_s(z)$ be the pressure generated by this density profile. From the hydrostatic balance equation (3.9) we have:

$$\frac{dp_s(z)}{dz} = -\rho_s(z)g. \tag{3.14}$$

The typical density deviation from rest, $\Delta\rho$, can be determined through the thermal wind relations of Eq. (3.12) or Eq. (3.13). By plugging in the scales of each variable, we have:

$$f_0 \frac{U}{D} = \frac{g}{\rho_s(z)} \frac{\Delta\rho}{L},$$

and therefore,

$$\Delta\rho = \frac{f_0 U L \rho_s(z)}{gD}. \tag{3.15}$$

The typical pressure deviation Δp can be determined from Eq. (3.10) or Eq. (3.11):

$$f_0 U = \frac{1}{\rho_s(z)} \frac{\Delta p}{L},$$

and therefore,

$$\Delta p = f_0 U L \rho_s(z). \tag{3.16}$$

Now reconsider the vertical momentum Eq. (3.6). The leading order balance is just the hydrostatic approximation of Eq. (3.9). The w terms are more than two orders of magnitude smaller than the $\tilde{f}u$ term and all of them are dropped. It is true that $|\tilde{f}u| \sim 2\Omega U \sim 10^{-4}\,\mathrm{m/s^2}$ is also about 5 orders of magnitude smaller than $|g| \sim 10\,\mathrm{m/s^2}$. However, the vertical pressure gradient is not related to the mesoscale fluid motion directly. Instead, only the horizontal gradient of the pressure field appears in Eqs. (3.4) and (3.5). So, we compute the $\partial^2 p/\partial y \partial z$ by taking the y derivative of Eq. (3.6) without the w terms, yielding

$$
\begin{aligned}
\frac{\partial^2 p}{\partial y \partial z} &= \frac{\partial}{\partial y}[\rho(\tilde{f}u - g)] \\
&= \frac{\partial \rho}{\partial y}\tilde{f}u + \frac{\partial u}{\partial y}\tilde{f}\rho - \frac{\partial \rho}{\partial y}g
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \rho}{\partial y}\tilde{f}u \sim \frac{\Delta\rho f_0 U}{L} = \frac{f_0^2 U^2 \rho_m}{gD} &\sim 10^{-9}\,\mathrm{Pa/m} \\
\frac{\partial u}{\partial y}\tilde{f}\rho \sim \frac{U f_0 \rho_m}{L} &\sim 10^{-6}\,\mathrm{Pa/m} \\
\frac{\partial \rho}{\partial y}g \sim \frac{\Delta\rho g}{L} = \frac{f_0 U \rho_m}{D} &\sim 10^{-4}\,\mathrm{Pa/m}.
\end{aligned}
$$

We can see from the estimations above that despite the smallness of the $\tilde{f}$ term, as one part of 10^5 in the leading order hydrostatic balancing terms, this term can influence the horizontal pressure gradient by about 1% of the leading order terms. A quasi-hydrostatic balance abandons the TA by keeping the $\tilde{f}u$ terms [10] [18] [19], which is

$$
\frac{1}{\rho}\frac{\partial p}{\partial z} = -g + \tilde{f}u. \qquad (3.17)
$$

3.3.3 Dimensionless Equations

By using the quasi-hydrostatic approximation, we have another complete set of equations: Eqs. (3.4), (3.5), (3.17), (3.7), (3.8). We can define a new set of starred dimensionless variables with magnitude of about O(1), shown in Table 3.1.

Table 3.1: The Scaling of Dimensionless Variables

Variables	Typical Values	Dimensionless Variables
x, y	$L \sim 10^5 - 10^6\,\mathrm{m}$	$x_* = x/L$, $y_* = y/L$
z	$D \sim 4000\,\mathrm{m}$	$z_* = z/D$
u, v	$U \sim 1\,\mathrm{m/s}$	$u_* = u/U$, $v_* = v/U$
w	$W = UD/L$	$w_* = w/W = (wL)/(UD)$
t	$T = L/U$	$t_* = t/T$
ρ	$\Delta\rho = fUL\rho_s(z)/(gD)$	$\rho_* = (\rho - \rho_s(z))/\Delta\rho$
p	$\Delta p = fUL\rho_s(z)$	$p_* = (p - p_s(z))/\Delta p$

Fome Table 3.1, we have

$$\frac{D}{Dt} = \frac{U^2}{L}\left(\frac{\partial}{\partial t_*} + u_*\frac{\partial}{\partial x_*} + v_*\frac{\partial}{\partial y_*} + w_*\frac{\partial}{\partial z_*}\right),$$

and we can further define:

$$\frac{D_*}{Dt} = \frac{\partial}{\partial t_*} + u_*\frac{\partial}{\partial x_*} + v_*\frac{\partial}{\partial y_*} + w_*\frac{\partial}{\partial z_*},$$

Then, we have:

$$\frac{D}{Dt} = \frac{U}{L}\frac{D_*}{Dt_*}. \tag{3.18}$$

We also have:

$$\frac{\partial p}{\partial x} = \frac{\Delta p}{L}\frac{\partial p_*}{\partial x_*} \quad \text{And} \quad \frac{\partial p}{\partial y} = \frac{\Delta p}{L}\frac{\partial p_*}{\partial y_*}$$
$$\frac{\partial p}{\partial z} = \frac{\partial (p_s(z_*) + \Delta p p_*)}{D\partial z_*} = -g\rho_s(z_*) + \frac{\Delta p}{D}\frac{\partial p_*}{\partial z_*}. \tag{3.19}$$

Since $\Delta\rho/\rho_s << 1$,

$$\frac{1}{\rho} = \frac{1}{\rho_s + \Delta\rho\ \rho_*} = \frac{1}{\rho_s}(1 - \frac{\Delta\rho}{\rho_s}\rho_*) = \frac{1}{\rho_s}(1 - \frac{fUL}{gD}\rho_*). \tag{3.20}$$

Thus Eq. (3.7) becomes

$$\frac{D\rho}{Dt} = \frac{U}{L}\frac{D_*}{Dt_*}(\rho_s + \Delta\rho\,\rho_*). \tag{3.21}$$

But, since ρ_s depends only on z_*, $D_*\rho_s/Dt_* = w_* d\rho_s/dz_*$, and

$$\frac{D\rho}{Dt} = \frac{U}{L}\left(w_*\frac{d\rho_s(z_*)}{dz_*} + \Delta\rho\frac{D_*\rho_*}{Dt_*}\right) = 0. \tag{3.22}$$

Defining

$$s(z_*) = -\frac{1}{\Delta\rho}\frac{d\rho_s(z_*)}{dz_*} = -\frac{gD^2}{fUL\rho_s}\frac{d\rho_s(z)}{dz}, \tag{3.23}$$

we will have:

$$\frac{D_*\rho_*}{Dt_*} - w_* s(z_*) = 0. \tag{3.24}$$

With the Rossby number $\varepsilon = U/f_0L$ and the Rossby deformation radius L_D defined by $L_D^2 = gD/f_0^2$, we have the following two parameters:

$$\gamma = \frac{1}{\varepsilon}\frac{\beta L}{f_0} = \frac{\beta L^2}{U} \quad \text{and} \quad \delta = \frac{1}{\varepsilon}\frac{D\tilde{f}}{Lf_0} = \frac{D\tilde{f}}{U}.$$

By dropping all the second or higher order terms, we can further simplify the equations (dropping the stars on the dimensionless variables):

$$v - \frac{\partial p}{\partial x} = \varepsilon\left(\frac{Du}{Dt} - \gamma y v + \delta w - \left(\frac{L}{L_D}\right)^2 \rho \frac{\partial p}{\partial x}\right) \quad (3.25)$$

$$-u - \frac{\partial p}{\partial y} = \varepsilon\left(\frac{Dv}{Dt} + \gamma y u - \left(\frac{L}{L_D}\right)^2 \rho \frac{\partial p}{\partial y}\right) \quad (3.26)$$

$$\frac{\partial p}{\partial z} = -\rho - \delta\varepsilon u \quad (3.27)$$

$$w = \frac{1}{s(z)}\frac{D\rho}{Dt} \quad (3.28)$$

$$\frac{\partial w}{\partial z} = -\Big(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\Big), \quad (3.29)$$

where the terms with γ are related to the β effect, i.e. the variation of local Coriolis force with respect to the latitude, and the terms with δ are related to the nontraditional terms that are dropped in the TA.

At this point it is still too hard to work directly on these equations to see how important the NT terms are to the overall dynamics of the system. Therefore, we will move to a more specific setting where the NT terms are the largest - the equator.

Chapter 4

Equatorial Trapped Waves

4.1 Introduction

In the northern hemisphere, for horizontal motion, the Coriolis force acts to the right of the direction of motion, while it acts to the left of the direction of motion in the southern hemisphere. At the equator the Coriolis force vanishes. If an object is moving to the east at the equator, a deviation to the north or to the south will be brought back toward the equator by the Coriolis force. So the equatorial zone essentially serves as a wave guide, causing a disturbance to be trapped near the equator. Such waves that are trapped and propagate within the equatorial zone are called *equatorial trapped waves*.

For example, the equatorial Kelvin wave, which has no meridional velocity and no dispersion, propagates eastward in the equatorial zone. It has been well observed in both the atmosphere and the oceans. When it travels over the Pacific ocean, it carries its variation from the West Pacific to the East Pacific, which plays an important role in the dynamics of the El Nino-Southern Oscillation [8].

The dynamics of equatorial trapped waves have been well studied. A classical theory of equatorial waves was developed in 1970's, from which dispersion relationships were derived, was in recent times verified by the satellite observations of the sea surface height and sea surface temperature. In this classical approach to explain the mechanism of equatorial trapped waves, the TA was adopted. However, at the equator, the NT terms achieve their maximum. One may ask how the NT terms affect the dynamics of equatorial trapped waves.

In this chapter we first briefy present the classical theory of equatorial waves. After that, the influence of the NT terms on the dynamics will be analyzed and discussed.

4.2 Equation of Motion

We start with the equations (3.4 - 3.8). Recall that $f = f_0 + \beta y$, where $f_0 = 0$ at the equator, $\beta = 2\Omega/R$, and $\tilde{f} = 2\Omega$. Our focus is the dynamics of wave-like motion of oscillation close to the equilibrium state, so all the non-linear terms are dropped. Then the linearized equations we have are

$$\frac{\partial u}{\partial t} + \underline{2\Omega w} - 2\Omega y v/R = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{4.1}$$

$$\frac{\partial v}{\partial t} + 2\Omega y u/R = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{4.2}$$

$$-\underline{2\Omega u} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \tag{4.3}$$

$$\frac{D\rho}{Dt} = 0 \tag{4.4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.5}$$

where the nontraditional terms are underlined.

Now consider the scaling procedures of the previous chapter, in order to obtain dimensionless equations. Since $f_0 = 0$, we use βL instead of f_0 as the scaling factor. The typical horizontal length scale $L \sim 10^5 - 10^6\,\mathrm{m}$, $\beta = 2.2 \times 10^{-11}\,\mathrm{s}^{-1}\mathrm{m}^{-1}$. The typical horizontal velocity $U \sim 1\,\mathrm{m/s}$, which is close to βL^2, so we just choose $U = \beta L^2$. The relationships between the dimensional variables and dimensionless variables are shown in Table 4.1.

Table 4.1: The Scaling of Dimensionless Variables

Variables	Scaling factors	Dimensionless Variables
x_*, y_*	L	$x_* = Lx$, $y_* = Ly$
z_*	D	$z_* = Dz$
u_*, v_*	$U = \beta L^2$	$u_* = Uu$, $v_* = Uv$
w_*	$W = UD/L$	$w_* = Ww = \frac{D}{L}Uw$
t_*	$T = L/U = 1/\beta L$	$t_* = Tt = \frac{t}{\beta L}$
ρ_*	$\Delta\rho = \frac{\beta U L^2 \rho_s(z)}{gD}$	$\rho_* = \rho_s(z) + \Delta\rho\rho = \rho_s(z)\big(1 + \frac{\beta U L^2}{gD}\rho\big)$
p_*	$\Delta p = \beta U^2 L \rho_s(z)$	$p_* = p_s(z) + \Delta p p = p_s(z) + \beta U L^2 \rho_s(z) p$

The function $s(z)$ is now

$$s(z) = -\frac{gD}{\beta^2 L^4 \rho_s}\frac{\partial \rho_s}{\partial z},$$

and we define:

$$r = DR/L^2.$$

Then, we have the complete set of linearized dimensionless equations for the equatorial waves

$$\frac{\partial u}{\partial t} - yv + \underline{rw} = -\frac{\partial p}{\partial x} \tag{4.6}$$

$$\frac{\partial v}{\partial t} + yu = -\frac{\partial p}{\partial y} \tag{4.7}$$

$$\rho = -\frac{\partial(\rho_s\, p)}{\rho_s\, \partial z} + \underline{ru} \tag{4.8}$$

$$\frac{\partial \rho}{\partial t} - ws(z) = 0 \tag{4.9}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{4.10}$$

If only the ocean is considered, Eq. (4.11) can be simplified as:

$$\rho = -\frac{\partial p}{\partial z} + \underline{ru}. \tag{4.11}$$

4.3 Classical Equatorial Wave Theory

In the case of $r << 1$, which means that $L^2 >> R\,D$, i.e. $L > 10^6$ m, the NT terms are small, and the traditional approximation is valid. In this case the equations can be solved by separation of varables [11] by seeking solutions of the form :

$$u = U(x, y, t)\, G(z) \tag{4.12}$$

$$v = V(x, y, t)\, G(z) \tag{4.13}$$

$$p = P(x, y, t)\, G(z) \tag{4.14}$$

$$w = W(x, y, t)\, \frac{dG}{s\, dz} \tag{4.15}$$

$$\rho = P(x, y, t)\, \frac{dG}{dz}. \tag{4.16}$$

Using Eqs. (4.12)- (4.16), Eqs. (4.6) and (4.7) become

$$\frac{\partial U}{\partial t} - yV = -\frac{\partial P}{\partial x} \tag{4.17}$$

$$\frac{\partial V}{\partial t} + yU = -\frac{\partial P}{\partial y}. \tag{4.18}$$

Equation (4.9) becomes

$$\frac{\partial P}{\partial t} + W = 0, \tag{4.19}$$

while Eq. (4.10) becomes

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{W}{G}\frac{d}{dz}\left[\frac{dG}{s\,dz}\right] = 0. \tag{4.20}$$

In order to successfully separate variables, Eq. (4.20) implies that G must satisfies the vertical structure equation:

$$\frac{d}{dz}\left[\frac{dG}{s\,dz}\right] + m^2 G = 0, \tag{4.21}$$

where m^2 is the separation constant. By Eq. (4.21) and Eq. (4.19), we can rewrite Eq. (4.20) as:

$$m^2\frac{\partial P}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{4.22}$$

This equation together with Eq. (4.17) and Eq. (4.18) form a close system for U, V, P. We can further eliminate P and get:

$$L_1(U) = \frac{\partial^2 U}{\partial t^2} - \frac{1}{m^2}\frac{\partial^2 U}{\partial x^2} \quad = \quad y\frac{\partial V}{\partial t} + \frac{1}{m^2}\frac{\partial V^2}{\partial x \partial y} \tag{4.23}$$

$$L_2(V) = \frac{\partial^2 V}{\partial t^2} - \frac{1}{m^2}\frac{\partial^2 V}{\partial x^2} \quad = \quad -y\frac{\partial U}{\partial t} + \frac{1}{m^2}\frac{\partial U^2}{\partial x \partial y}, \tag{4.24}$$

where L_1 and L_2 are linear operators.

4.3.1 Kelvin Wave

There is a trivial solution of Eq. (4.31) of $V \equiv 0$, which is the case of the Kelvin wave. For this case, Eq. (4.23) and Eq. (4.24) become

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} - \frac{1}{m^2}\frac{\partial^2 U}{\partial x^2} &= 0 & (4.25) \\ -y\frac{\partial U}{\partial t} + \frac{1}{m^2}\frac{\partial U^2}{\partial y \partial x} &= 0. & (4.26) \end{aligned}$$

This first equation is nothing but an ordinary wave equation, which has a general solution composed of the linear combination of two independent solutions of the form

$$U_{\pm} = U_{\pm}(x \pm \frac{t}{m}, y). \tag{4.27}$$

Inserting $U_{\pm}$ into Eq. (4.26), yields

$$\frac{\partial}{\partial y}\left(\frac{\partial U_{\pm}}{\partial x}\right) \mp my \left(\frac{\partial U_{\pm}}{\partial x}\right) = 0. \tag{4.28}$$

Solving for the y-dependence gives

$$U_{\pm} = A_{\pm}\left(x \pm \frac{t}{m}\right) e^{\pm my^2/2}, \tag{4.29}$$

where $A\pm$ denotes free function.

Without loss of generality, we can take $m > 0$, since changing the sign of m is the same as exchanging the two solutions. The solution U_+ explodes when y gets large, so it is abandoned. The only reasonable solution is therefore

$$U = A\left(x - \frac{t}{m}\right) e^{-my^2/2}, \tag{4.30}$$

where m is determined by solving the eigenvalue problem of Eq. (4.21). It is not hard to determine the rest of the variables from here. The final solution is a Kelvin wave traveling eastward with no dispersion as in [11].

4.3.2 General Case

Now consider the case when $V \neq 0$. To this cud we apply the operator L_1 of Eq. (4.23) to both sides of Eq. (4.24) to eliminate U and get:

$$\frac{\partial}{\partial t}\left[\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V - m^2y^2V - m^2\frac{\partial^2 V}{\partial t^2}\right] + \frac{\partial V}{\partial x} = 0. \tag{4.31}$$

Then assume that

$$V = Re\left[e^{i(kx-\sigma t)}\psi(y)\right] \tag{4.32}$$

and, without the loss of generality that $\sigma > 0$. Thus the equation for the meridional structure function $\psi(y)$ becomes

$$\frac{d^2\psi}{dy^2} + \psi\left[m^2(\sigma^2 - y^2) - \frac{k}{\sigma} - k^2\right] = 0. \tag{4.33}$$

With the help of the theory of the harmonic oscillator in quantum physics, we know the solution to this equation is:

$$\psi_j(y) = \frac{e^{-my^2/2}H_j(m^{1/2}y)}{(2^j j!\pi^{1/2})^{1/2}}, \tag{4.34}$$

where H_j is the Hermite polynomial defined by:

$$H_j(\eta) = (-1)^j e^{\eta^2}\frac{d^j}{d\eta^j}e^{\eta^2}, \tag{4.35}$$

and σ, k, and m must satisfy the dispersion relation

$$m^2\sigma^2 - \frac{k}{\sigma} - k^2 = (2j+1)m, \qquad j = 0, 1, 2... \quad . \tag{4.36}$$

Note that when $j = -1$, the dispersion relation becomes

$$\frac{k}{\sigma} = \frac{1}{m}, \tag{4.37}$$

which is simply the same as the dispersion relation for the Kelvin wave.

4.4 Keeping the NT terms

When the NT terms are retained, it is no longer possible to separate variables and find the vertical structure function $G(z)$. An analytical solution is not generally available. So we do not deal with the NT terms in the most general case. Instead, our goal here is to identify the contribution of the NT terms to the dynamics of equatorial waves. Even though we only study some special cases, we obtain an idea of how much influence the NT terms have. In the case of constant stratification, i.e. $s(z) = constant$, the method of separation of variables still applies. So, we to limit our problem to a constant stratified ocean. We start form Eqs. (4.6)- (4.10), using Eq. (4.11) instead of Eq. (4.8) for the case of the ocean. The NT terms are those with a coefficient of r. In the classical theory with the TA, $r = 0$. So, although r could be large, in our derivation, we only keep first order in r. This is good enough to give a perturbative view of the NT terms upon the classical theory.

Plugging Eq. (4.11) into Eq. (4.9),

$$ws = -\frac{\partial^2 p}{\partial z \partial t} + r\frac{\partial u}{\partial t}, \tag{4.38}$$

substituting w with Eq. (4.38) into Eq. (4.6), and dropping the second order term in r gives

$$\frac{\partial u}{\partial t} - yv - \frac{r}{s}\frac{\partial^2 p}{\partial z \partial t} = -\frac{\partial p}{\partial x}. \tag{4.39}$$

Substituting w with Eq. (4.38) into Eq. (4.10) yields

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial}{\partial z}\left(\frac{1}{s}\left(-\frac{\partial^2 p}{\partial z \partial t} + r\frac{\partial u}{\partial t}\right)\right) = 0. \tag{4.40}$$

Equations (4.39), (4.7), and (4.40) are a closed set for u, v, p. Now, taking $s(z) = s$ which is a constant, we assume that:

$$\begin{aligned} u &= U(y)e^{i(kx-\sigma t)}e^{i\sqrt{s}mz} \\ v &= V(y)e^{i(kx-\sigma t)}e^{i\sqrt{s}mz} \\ p &= P(y)e^{i(kx-\sigma t)}e^{i\sqrt{s}mz}. \end{aligned} \tag{4.41}$$

Therefore, Eqs. (4.39), (4.7), and (4.40) become:

$$-i\sigma U - yV - \frac{rm\sigma}{\sqrt{s}}P = -ikP \tag{4.42}$$

$$-i\sigma V + yU = -P' \tag{4.43}$$

$$ikU + V' + \frac{rm\sigma}{\sqrt{s}}U - i\sigma m^2 P = 0. \tag{4.44}$$

With Eq. (4.44) and Eq. (4.42), we eliminate P and further express U by V according to

$$U = \frac{m^2 sy\sigma V + (irm\sigma\sqrt{s}1ks)V'}{im^s\sigma^2 s + krm\sigma\sqrt{s} - iks^2}. \tag{4.45}$$

Again with Eq. (4.43) and Eq. (4.44) to eliminate P, we obtain

$$yU - i\sigma V = \frac{i((iks - rm\sigma\sqrt{s})U' + sV''}{m^2\sigma s}. \tag{4.46}$$

Substituting U with Eq. (4.45) into Eq. (4.46) and simplifying the result, gives

$$V'' + V\left[\frac{kms - im^2r\sqrt{s} + k^2ms\sigma - m^3s\sigma(\sigma^2 - y^2)}{-m\sigma s - 2ikr\sqrt{s}}\right] = 0. \tag{4.47}$$

Without the NT terms, i.e. when $r = 0$, this equation is exactly the same as Eq. (4.4) that we obtained in the last section.

Scaling the dependent variable, $\eta = \sqrt{m}y$, Eq. (4.47) becomes

$$\psi''(\eta) + \psi(\eta)\left[\frac{\frac{1}{m}(m^2\sigma^2 - \frac{k}{\sigma} - k^2) - \eta^2 - \frac{ir}{\sqrt{s}}}{1 + \frac{2ikr}{m\sigma\sqrt{s}}}\right] = 0. \tag{4.48}$$

When $r = 0$, we can solve the eigenvalue problem and we have the same dispersion relation as that of Eq. (4.36)

$$\frac{1}{m}(m^2\sigma^2 - \frac{k}{\sigma} - k^2) = 2j + 1, \tag{4.49}$$

where j is any non negative integer.

In the case of $r \neq 0$, we solve the differential equation numerically, by taking $k = 1$, $\sigma = 1$, $m = 1$, and $s = 1$, For r values of 0, 0.003, 0.02, and 0.1, our results are shown in Fig 4.1. Whenever $r > 0$, no matter how small, there is no longer a finite solution to Eq. (4.48). But, when r is small (in this case, 0.003), $\psi(\eta)$ almost vanishes in the region around $\eta = 4$, for $j = 0,\ 1$. This suggests that even though there is no longer a stable finite solution for $\psi(\eta)$, if initially there is a wave trapped in the equatorial zone, it will be hard for the wave to escape. However, when r is large (0.1), the solution $\psi(\eta)$ intersects the η axis at a sharp angle. This indicates that there will be no trapped wave at all in this setting. The energy in the equatorial zone will be transmitted either northward or southward.

Recall that $r = DR/L^2$, so that a wave can only be effectively trapped in the equatorial zone when $r < 0.01$[1]. In particular, this means $L > \sqrt{RD} = 1.6 \times 10^6$ m. This result appears to explain the reason for the absence of mesoscale length ($\sim 10^5$ m) equatorial trapped waves in all observations [3] [16].

Our results also show that a wave will be better trapped for smaller j, when r is the same. This means that a higher baroclinic mode will be scattered out more easily than a lower baroclinic mode near the equator. Equation (4.48) also indicates that greater stratification can help trap waves and therefore prevent them from escaping the equatorial zone.

[1] This is not a hard limit, but rather an order of magnitude estimation.

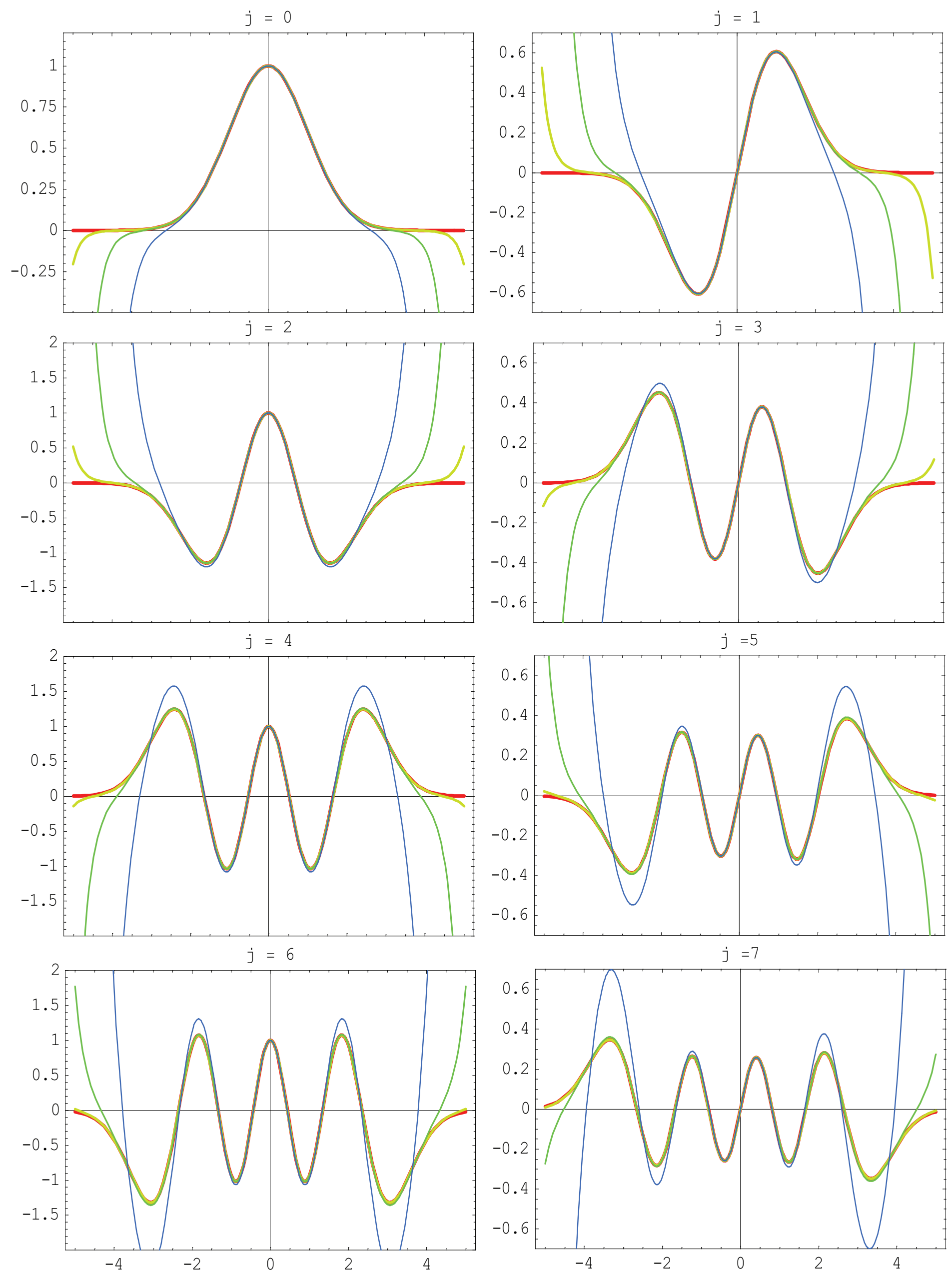

Figure 4.1: Plots of the numerical solutions to Eq. 4.48 for j = 0, 1, ... 7. For r values of 0, 0.003, 0.02, and 0.1, the curves are shown in red, yellow, green, and blue, respectively.

Chapter 5

Short Conclusion

From the basic principles of fluid dynamics, through a careful scaling analysis, we have derived the equations of motion while keeping the NT terms. The influence of these NT terms on equatorial trapped waves has been studied. Our results show that the NT terms make it impossible for equatorial trapped waves with meso-scale wavelengths to exist. This we offer as an explanation for the absence of waves of wavelength less than 10^6 meters propagating in the equatorial zone. When L is large, comparable with the earth radius R, the effect of the NT terms can be neglected again, even near the equator.

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