The Loop Theorem using Hierarchies

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This report will build up the machinery of special hierarchies by discussing normal surfaces and boundary patterns. Then the report will use this construction to prove the Loop Theorem, following closely the proof presented by Marc Lackenby.

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Chapter 1

Introduction

The Loop Theorem is a conerstone of 3-manifold theory that originally developed as a generalization of Dehn's Lemma presented in 1910 [2]. Dehn's original proof was shown to be flawed and the validity of the lemma was in doubt until 1956, when Papakyriakopolous proved the lemma using a tower construction [6]. This was then generalized to the Loop Theorem by John Stallings [7] who used double covers.

Throughout the 1960's Haken was applying his theory of normal nurfaces to various 3-Manifold problems [3] and in 1968 Waldhausen extended his ideas to prove, in particular, the solvability of the word problem for the fundamental groups of 3-manifolds containing an incompressible surface [8]. With some modification this proof could be used as a proof for the Loop Theorem, but was not explicitly done so by Waldhausen. The specific proof of the Loop Theorem using hierarchies was obtained for Haken manifolds by Johannson in 1994 [4]and later by Lackenby [5] and by Aitchison and Rubinstein working jointly [1]. The Loop Theorem is a widely used fundamental tool of 3-manifold topology. One particularly elegant application is to show that a knot with abelian fundamental group is necessarily the unknot.

Chapter 2

Incompressible and Normal Surfaces

A convenient starting place for our purpose of presenting the Loop Theorem is to study submanifold surfaces of a 3-manifold M. This is a fairly natural course of action since surfaces are completely classified and we will see it is a particularly useful method to gain information about M. In particular we will be studying embedded *incompressible* surfaces.

Note: All 3-manifolds discussed are assumed to be orientable.

2.1 Incompressible Surfaces

Definition: Let S be a properly embedded surface in a 3-manifold M. *A compression disc* D for S is a disc D embedded in M such that $D \cap S = \partial D$, but with ∂D not bounding a disc in S.

Definition: A properly embedded surface S in 3-manifold M is said to be *incompressible* if no such compression discs exist.

Suppose D is a compression disc for S. We can *compress S along D* in the following way. We can assume D lies in the interior of M and therefore we may find an embedding of $D \times [-1, 1]$ in $\text{int}(M)$ with $(D \times [-1, 1]) \cap S =$

$\partial D \times [-1,1]$. Then

$$S \cup (D \times \{-1,1\}) - (\partial D \times (-1,1)) \tag{2.1}$$

is a new surface properly embedded in M.

Remark: Define the *complexity* of a surface S to be the sum of $-\chi(S)$, the number of components of S and the number of 2-sphere components of S. Then the complexity of a 2-manifold is positive and compressing S reduces its complexity.

Lemma 2.1: *Let M be a compact connected 3-manifold. Then any homomorphism $\pi_1(M,p) \to \mathbb{Z}$ is induced by a map $(M,p) \to (S^1,x)$.*

Proof. Choose a triangulation for M such that p is at a vertex. Let T be a maximal tree of the 1-skeleton. Map T to x. For each 1-simplex $\triangle^1$ not in T orienting $\triangle^1$ together with a path through T connecting its endpoints and $\triangle^1$ represents an element of $\pi_1(M,p)$ and the given homomorphism sends this to an integer n. So we wrap $\triangle^1$ n times around S^1 with endpoints at x. For every 2-simplex $\triangle^2$ in M, we know its boundary is trivial in $\pi_1(M,p)$ so its boundary maps trivially in $\pi_1(S^1)$ and we can extend the map on $\partial\triangle^2$ to a map over all of $\triangle^2$. For any 3-simplex $\triangle^3$ we use the triviality of $\pi_2(S^1)$ and repeat the above argument to extend the map over all of $\triangle^3$. $\square$

Lemma 2.2: *Let M be a compact irreducible 3-manifold. Then any map $f : M \to S^1$ is homotopic to a map g such that for any point $y \in S^1$ each* *component of $g^{-1}(y)$ is a properly embedded 2-sided incompressible surface in M, not a 2-sphere.*

Proof. Choose a triangulation of S^1 so that $y \times [-1, 1]$ is a 1-simplex in the triangulation and $y = y \times 0$. We may subdivide the triangulation for M and then perform a homotopy to f so that it is simplicial. Now y is a regular value and $f^{-1}(y)$ is a properly embedded 2-sided surface in M. Also, the map $f|_{\mathcal{N}(f^{-1}(y))}$ maps fibers homeomorphically to fibers of $\mathcal{N}(y)$. Suppose $f^{-1}(y)$ is compressible with compression disc D. Take a regular neighborhood $\mathcal{N}(D)$ in M such that $\mathcal{N}(D) \cap f^{-1}(y)$ is an annulus A properly embedded in $\mathcal{N}(D)$. Now we can find two disjoint discs D_1 and D_2 that are properly embedded in $\mathcal{N}(D)$ such that $\partial D_1 \cup \partial D_2 = \partial A$. We can define a map $f_1 : M \to S^1$, homotopic to f, in the following way. Set $f_1|_{M-\text{int}(\mathcal{N}(D))} = f|_{M-\text{int}(\mathcal{N}(D))}$. We can use the trivializing homotopy of $f|_{\partial D_i}$ to extend to a map $f|_{D_i}$ and homotoping the image so that it is y giving us an extension of $f_1|_{\partial D_i}$ to a map $f_1|_{D_i}$ whose image is y. We can further extend f_1 to a small neighborhood $\mathcal{N}(D_i)$ by using the fact that f_1 respects the product structure of $\mathcal{N}(y)$.

Now we see that f_1 is defined on all of M except for $\mathcal{N}(D) - \mathcal{N}(D_1 \cup D_2)$, which is homeomorphic to three 3-balls. Since f_1 is defined on their boundaries and $\pi_2(S^1) = 0$, we can use the trivializing homotopies to extend f_1 over these 3-balls. Thus we have defined f_1 for all of M and $f_1^{-1}(y)$ is obtained from $f^{-1}(y)$ by a compression and thus has smaller complexity. This means that we will at some point be able to stop and be left with a map f_n such that $f_n^{-1}(y)$ is incompressible. Also note that f_1 differs from f only within a 3-ball, implying that they are homotopic, since $\pi_3(S^1) = 0$.

In a case in which $f^{-1}(y)$ has a 2-sphere component, which would bound a 3-ball B in M, we define $f_1 : M \to S^1$ in the following way. Let $f_1|_{M-\mathrm{int}(B)} = f|_{M-\mathrm{int}(B)}$. Using the fact that $\pi_2(S^1) = 0$ we can extend $f|\partial B$ to a map $f_1|_B : B \to y$. Now we simply push the image off of y so $f_1(B) \cap y = \emptyset$, removing this 2-sphere component of $f^{-1}(y)$. Continuing in this fashion we can remove all 2-sphere components and get a map g as required. □

Theorem 2.3: *Let M be a compact irreducible 3-manifold with $H_1(M)$ infinite. Then M contains a 2-sided non-separating properly embedded incompressible surface S.*

Proof. Since $H_1(M)$ is infinite and finitely generated, it has a $\mathbb{Z}$ summand. Hence there is a surjection $\pi_1(M) \to \mathbb{Z}$, and by Lemma 2.1, this is induced by a map $M \to S^1$. Then applying Lemma 2.2 we get a map $g : M \to S^1$ such that $g^{-1}(y)$ is a 2-sided non-separating incompressible surface in M. □

2.2 Haken Manifolds

Definition: A compact orientable 3-manifold is *Haken* if it is prime and contains a connected orientable incompressible properly embedded surface other than a 2-sphere.

2.3 Hierarchies

Definition: Let M be a 3-manifold containing an incompressible surface S. Then a new 3-manifold M_S can be obtained from M by *cutting* M along S.

Definition: A *partial hierarchy* for a Haken 3-manifold M_1 is a sequence of 3-manifolds $M_1, ..., M_n$ where M_{i+1} is obtained from M_i by cutting along an orientable incompressible surface in M_i, no component of which is a 2-sphere. If M_n is a collection of of 3-balls then we have a *hierarchy.*

$$M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} M_n.$$

2.4 Normal Surfaces

Definition: A *triangle* in a 3-simplex Δ^3, is a disc D properly embedded in Δ^3 such that $\partial\, D$ intersects exactly three 1-simplices transversely, each at a single point, and is disjoint from the remaining 1-simplices and all of the vertices. A *square* is defined analogously, intersecting exactly four 1-simplices transversely, each at a single point, and disjoint from the other 1-simplices and the vertices.

Definition: Let M be a 3-manifold with a triangulation T. Then a surface in M is said to be in *normal form* with respect to T if it is properly embedded and intersects each 3-simplex in a finite collection of disjoint triangles and squares. A surface in normal form is called a *normal surface.*

The following theorem will be stated without proof, but is due to Haken and can be found in various sources[3].

Theorem 2.4: *Let M be a compact irreducible 3-manifold and S a properly embedded closed incompressible surface in M such that no component of S is a 2-sphere. Then, for any triangulation of M, S may be ambient isotoped into normal form.*

Theorem 2.5: *Let M be a compact irreducible 3-manifold. Then there exists an integer n such that if S is a closed properly embedded incompressible surface in M with more than n components, none of which is a 2-sphere, then there exists at least one pair of components S_i, S_j of S that are parallel with no component of S in the product region between them.*

Proof. Let $n = 2\beta_1(M;\mathbb{Z}_2) + 6t$ where t is the number of 3-simplices in the triangulation of M. Let S be a closed properly embedded incompressible surface in M with components $S_1, ..., S_k$ with $k > n$. Theorem 2.1 lets us ambient isotope S into normal form. Since at most $\beta_1(M;\mathbb{Z}_2)$ components of S are nonseparating, there are strictly more than $\beta_1(M;\mathbb{Z}_2) + 6t$ components of M_S. Note that for each 3-simplex, $\triangle^3 - S$ has at most 6 regions that are not product regions. We can get up to four tetrahedral regions lying near each vertex and an additional two regions from the *square* intersections of S with $\triangle^3$. This leaves more than $\beta_1(M;\mathbb{Z}_2)$ components of M_S that are entirely composed of product regions. Each such component is either a product I-bundle or and I-bundle over a non-orientable surface. If X is an I-bundle over a non-orientable surface, then $H_1(X, \partial X; \mathbb{Z}_2) \neq 0$, therefore X contributes to $\beta_1(M; \mathbb{Z}_2)$. So at most $\beta_1(M; \mathbb{Z}_2)$ of these components are of this form, which means there is at least one product I-bundle component of M_S and its two boundary components are parallel in M. $\square$

Chapter 3

Boundary Patterns

3.1 Boundary Patterns

Definition: For a 3-manifold M, a *boundary pattern* Γ is a collection of disjoint simple closed curves and trivalent graphs in ∂M, such that no simple closed curve in ∂M intersects Γ in a single point. We will use (M, Γ) to refer to a 3-manifold with boundary pattern.

Let M_S be obtained from a 3-manifold with boundary pattern, (M, Γ), by cutting along a properly embedded 2-sided surface S that intersects Γ transversely. Then we define a boundary pattern Γ' for M_S inherited from (M, Γ) in the following way. Note that ∂M_S contains two copies of S, S' and S'', and a portion of ∂M, namely $\partial M_S \cap \partial M$. Then $\partial S' \cup \partial S''$ is a disjoint collection of closed simple curves in ∂M_S and together with $\Gamma \cap \partial M_S$ forms the boundary pattern Γ' for M_S inherited from M.

It is clear that a boundary pattern may be inherited by this method throughout a hierarchy:

$$(M_1, \Gamma_1) \xrightarrow{S_1} (M_2, \Gamma_2) \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} (M_n, \Gamma_n).$$

Note that Γ_n is the union of the boundaries of $S_1, \cdots, S_{n-1}$ and the part of Γ_1 lying on M_n.

3.2 Essential and Homotopically Essential Boundary Patterns

Of special importance is the notion of an *essential boundary pattern.*

Definition: A boundary pattern Γ for M is *essential*, if for each disc D properly embedded in M such that ∂D meets Γ in at most three points, there exists a disc $D' \subset \partial M$ such that the boundaries of D, D' agree, D' contains no more than one vertex of Γ and no simple closed curves of Γ.

Definition: A boundary pattern Γ is *homotopically essential* if for any mapping f of the pair $(D, \partial D) \to (M, \partial M)$ with $\partial D \cap \Gamma$ consisting of at most three points none of which are self intersection points of $f(\partial D)$, there exists homotopy F of f to an embedding of D in ∂M so that $f(D)$ contains at most one vertex of Γ and no simple closed curves of Γ, where F keeps $\partial D \cap \Gamma$ fixed while introducing no new points of intersection of ∂D with Γ and keeps ∂D in ∂M.

It is not hard to see that a homotopically essential boundary pattern is necessarily an essential boundary pattern. Though harder to show, the converse is also true and proving so will occupy a significant portion of this report.

3.3 Pattern Incompressibilty

Definition: Let S be a surface properly embedded in a 3-manifold M with boundary pattern P. Then a *pattern-compression disc* for S is a disc D embedded in M such that

- $D \cap S$ is an arc α in ∂D
- $\partial D - \text{int}(\alpha) = D \cap \partial M$ intersects Γ at most once.
- α does not separate off a disc from S intersecting P at most once.

If no such pattern-compression disc exists, then S is *pattern-incompressible.*

Definition: A *special hierarchy* for a compact irreducible 3-manifold (M, Γ) is a hierarchy for M such that each of the surfaces S_i is a properly embedded connected pattern-incompressible incompressible surface in (M_i, Γ_i), no component of which is a 2-sphere or a boundary-parallel disc.

$$(M_1, \Gamma_1) \xrightarrow{S_1} (M_2, \Gamma_2) \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} (M_n, \Gamma_n).$$

Chapter 4

The Loop Theorem

4.1 Statement of Theorems

The Loop Theorem: *Let M be a compact irreducible 3-manifold. Then ∂M is a incompressible surface if and only if for each component F of ∂M the map $\pi_1(F) \to \pi_1(M)$ is an injection.*

One direction can and will be shown immediately. *Proof:* Since ∂M is a 2-manifold we know that a simple closed curve is homotopically trivial if and only if it bounds a disc in ∂M. So, if a compression disc exists for a component F of ∂M then $\pi_1(F) \to \pi_1(M)$ cannot be injective.

The other direction will follow directly from the following theorem, which we will prove shortly.

4.1.1 Theorem 4.1

Theorem 4.1: *Let (M, Γ) be a compact irreducible 3-manifold with essential boundary pattern Γ then Γ is also homotopically essential.*

Proof of Loop Theorem from Theorem 4.1: Suppose that ∂M is incompressible and let Γ be the empty boundary pattern in ∂M. Γ is then essential and by Theorem 4.1 it is also homotopically essential. Therefore any closed curve c in ∂M homotopically trivial in M is also homotopically trivial in ∂M, hence the map $\pi_1(F) \to \pi_1(M)$ is injective. □

4.2 Proof of Theorem 4.1

4.2.1 Sketch of Proof

At this point the proof of the Loop Theorem is complete once we prove Theorem 4.1. In order to do so, we prove the following four Theorems, which, when taken together, prove Theorem 4.1.

Theorem 4.2: *Let (M, Γ) be a compact irreducible 3-manifold with essential boundary pattern Γ. Let S be a connected pattern-incompressible incompressible surface properly embedded in M. Then the inherited boundary pattern Γ' is an essential boundary pattern of M_S, the 3-manifold obtained from M by cutting along S.*

Theorem 4.3: *Let (M, Γ) be a 3-ball with essential boundary pattern Γ. Then Γ is homotopically essential.*

Theorem 4.4: *Let (M, Γ) be a compact irreducible 3-manifold with boundary pattern Γ. Let S be a connected pattern-incompressible incompressible surface properly embedded in M. If the inherited boundary pattern Γ' for M_S is homotopically essential then Γ is homotopically essential.*

Theorem 4.5: *Let M be an irreducible 3-manifold with boundary and an essential boundary pattern Γ. Then there exists a special hierarchy:*

$$(M_1, \Gamma_1) \xrightarrow{S_1} (M_2, \Gamma_2) \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} (M_n, \Gamma_n).$$

4.2.2 Descension of Essential Boundary Patterns

Here we will show that given a special hierarchy for a 3-manifold (M, Γ) with essential boundary pattern Γ, the inherited boundary patterns remain essential as we move down the hierarchy.

Theorem 4.2: *Let (M, Γ) be a compact irreducible 3-manifold with essential boundary pattern Γ. Let S be a connected pattern-incompressible incompressible surface properly embedded in M with no boundary-parallel disc components. Then the inherited boundary pattern Γ' is an essential boundary pattern for M_S, the 3-manifold obtained from M by cutting along S.*

Proof. Let D be a properly embedded disc in M_S with $D \cap \Gamma'$ having at most three points none of which is a vertex of Γ'. There are two possibilities here, either ∂D is disjoint from S or it is not.

In the case where ∂D is disjoint from S, $\partial D \subset \partial M$ and therefore bounds a disc D' in ∂M by the essentiality of Γ. If this intersects S it must do so in a collection of boundary components of S. An innermost such curve bounds a disc that cannot be a compression disc for S, so this component of S must be a boundary parallel disc contradicting the assumption. So D' is disjoint from S and does not violate the essentiality of Γ'.

For the case where ∂D intersects S we notice that at most one arc of $\partial D \setminus \Gamma'$ can lie in S since at most one side of any point $\partial D \cap \Gamma'$ lies in S.

So $D \cap S$ is an arc α in ∂D, and $\partial D - (\alpha) = D \cap \partial M$ intersects Γ at most once. But D cannot be a pattern compression disc for S and therefore it must separate a disc D_1 from S with at most one intersection point with Γ. So $D \cup D_1$ is a properly embedded disc in M with at most two intersection points with Γ implying there is a disc D_2 embedded in ∂M with boundary equal to $\partial(D \cup D_1)$, containing no vertices or simple closed curves of Γ. Therefore $D_1 \cup D_2$ is a disc embedded in ∂M_S containing at most one vertex and no simple closed curves of Γ' and $\partial(D_1 \cup D_2) = \partial D$. This shows that Γ' is essential. □

4.2.3 Theorem 4.1 for B^3

We can prove that Theorem 4.1 holds for B^3.

Theorem 4.3: *Let (M, Γ) be a 3-ball with essential boundary pattern Γ. Then Γ is homotopically essential.*

Proof. Consider a component C of $\partial M - \Gamma$; its boundary must be a collection of disjoint simple closed curves made up of line segments in Γ. Pick any one of these curves α and perform a small ambient isotopy to push it into C. We can embed a disc in M with its boundary sent to α and since $\alpha \cap \Gamma = \emptyset$, α must bound a disc in ∂M satisfying essentiality. So either this disc in ∂M lies towards the interior of C or towards the boundary of C, but it cannot lie toward the boundary since we know it would contain at least a simple closed curve of Γ or at least two vertices. So this disc must lie towards the interior of this component and cannot contain any component of Γ. Showing that this component is homeomorphic to a disc and therefore every component of $M - \Gamma$ is homeomorphic to a disc.

Consider a map $(D, \partial D) \longrightarrow (M, \partial M)$ with $\partial D \cap \Gamma$ at most three points. Then $(\partial D - \Gamma)$ is a collection of arcs each in a distinct component of $\partial M - \Gamma$ or a single closed curve lying entirely in a component of $\partial M - \Gamma$. Since these curves lie in discs we may homotope them so they are embeddings. This makes ∂D embedded in ∂M intersecting Γ at most three times and therefore there exist a disc D' embedded in ∂M containing no simple closed curves of and at most one vertex of Γ. Now $D \cup D'$ is the image of a 2-sphere into a 3-ball and since $\pi_2(M)$ is trivial there exist a homotopy taking D to D'. □

4.2.4 Ascension of Homotopically Essential Boundary Patterns

Theorem 4.4: *Let M be a compact irreducible 3-manifold with boundary pattern Γ. Let S be an orientable incompressible pattern-incompressible surface properly embedded in M. Let Γ' be the boundary pattern inherited by M_S. Then, if Γ' is homotopically essential so is Γ.*

Proof. Consider a map $h : (D, \partial D) \to (M, \partial M)$ with ∂D intersecting Γ in at most three points. After a small homotopy $h^{-1}(S)$ is a collection of properly embedded arcs and simple closed curves in D. An innermost such curve lies in the interior of S, hence disjoint from Γ', and bounds a disc D' in the image of D. Since Γ' is homotopically essential, we may homotope D' to embed in ∂M_S, specifically in S. Now a further small homotopy reduces $|h^{-1}(S)|$ and so we may assume that $h^{-1}(S)$ contains no simple closed curves.

Now we only have to worry about potential arcs in $h^{-1}(S)$, one or more of these will cut D into discs. Focus on two extreme most in D. Since $D \cap \Gamma$ is at most three points, one of these discs, D_1, will have at most one intersection point with Γ. D_1 has an additional two intersection points with Γ' at the endpoints of its intersection with S, hence D_1 has at most three intersection points with Γ'. Since D_1 is properly embedded in M_S we can use the homotopy essentiality of Γ' to homotope D_1 to a disc D' embedded in ∂M_S containing at most one vertex and no simple closed curves of Γ'. Replace D_1 with D' and then use a small homotopy to pull it off of S entirely.

Repeat this process until $h^{-1}(S) = \emptyset$ and we have now homotoped D to be disjoint from S and therefore only has at most three intersection points with Γ'. We use the homotopy essentiality of Γ' to give us a homotopy of D to an embedded disc D'' in ∂M_S containing at most one vertex and no simple closed curves of Γ', but this is in fact the desired homotopy of D to an embedded disc in ∂M since D'' is disjoint from S, otherwise it would contain a boundary component of S violating essentiality. $\square$

4.2.5 Existence of Special Hierarchies

Here we profit from being able to work entirely with normal surfaces by making use of Theorem 2.5 to prove:

Lemma 4.5: *Let M be a compact 3-manifold with partial hierarchy*

$$M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \cdots \xrightarrow{S_{n-1}} M_n$$

.

Let $X = \mathcal{N}(\partial M \cup S_1 \cup S_2 \cup \cdots \cup S_{n-1})$ and let $F = \partial X - \partial M$. Then F is an incompressible surface in X.

Proof. Assume false, then there exists a compression disc D for F in X. Let S_i be the first surface in the sequence $\{S_1, S_2, \cdots, S_{n-1}\}$ with non-empty intersection with D. Notice that $D \cap S_i$ must be a collection of disjoint simple closed curves in D. Choose any such curve c such that the disc D_1 it bounds does not additionally intersect S_i. Since S_i is incompressible, D_1 is not a compression disc for S_i, therefore c bounds a disc D_2 embedded in S_i. D_2 may additionally intersect D, but will have an innermost curve of intersection bounding a disc D' in D and another disc D'_2 in S_i. Replacing D' with D'_2 in D and isotoping it off S_i allows us to reduce $|D \cap S_i|$ without introducing new intersections with $S_1 \cup S_2 \cup \cdots \cup S_{i-1}$. Proceeding in this fashion we may assume D is disjoint from S_i and therefore disjoint from all the surfaces in the partial hierarchy. Then D is a compression disc for F embedded in X less the interior of a small neighborhood of $S_1 \cup S_2 \cup \cdots \cup S_{n-1}$. In fact, this is a product region $F' \times I$ with $F' \times \{0\}$ identified with F. Clearly F is π_1 injective, so by the easy part of the loop theorem F is incompressible, contradicting our assumption. □

Lemma 4.7 *Let F be a compact orientable surface. Then $F \times I$ contains no non-separating surface S with $\partial S \cap F \times \{1\} = \emptyset$.*

Proof. Let $S \cap F \times \{1\} = \emptyset$; then S is a surface properly embedded in $F \times [0, 1)$. We may perform an isotopy to S so that $\partial S \subset F \times \{0\}$. Since there exists a homotopy equivalence $F \times \{0\} \to F \times I$ we get $H_*(F \times \{0\}) \to H_*(F \times I)$ is an isomorphism in the long exact sequence of pairs, so $H_*(F \times I, F \times \{0\}) = 0$. In particular $H_2(F \times I, F \times \{0\}) = 0$, therefore S is homologous rel ∂ to a 2-chain $C \subset F \times \{0\}$. If S were non-separating there would be a loop $\gamma \subset F \times (0,1)$ whose mod 2 intersection with S would be 1, but we know that $\gamma \cap C = \emptyset$. Therefore we can conclude that S is non-separating. □

Theorem 4.5: *Let M be an irreducible 3-manifold with boundary and an essential boundary pattern Γ. Then there exists a special hierarchy:*

$$(M_1, \Gamma_1) \xrightarrow{B_1} (M_2, \Gamma_2) \xrightarrow{S_1} (M_3, \Gamma_3) \overset{S_2}{\cdots} \xrightarrow{B_n} (M_{2n}, \Gamma_{2n}).$$

where each B_i is the union of embedded pattern-incompressible discs and each S_i is a non-separating pattern-incompressible incompressible surface.

Proof. Supppose ∂M has a compression disc D. If there is a pattern compression disc for D, then this disc separates D into two discs, each of which has at least two intersection points with Γ. Then compressing along this disc yields two discs each having fewer intersections with Γ than D and one of which must still be a compression disc for ∂M. Repeat the process for this disc until we have a pattern incompressible compression disc for ∂M and then decompose M along it. This reduces the complexity of ∂M and results in the following partial hierarchy

$$M = M^1 \xrightarrow{D_1} M^2 \xrightarrow{D_2} \cdots \xrightarrow{D_{i-1}} M_1$$

with ∂M_1 incompressible in M_1. Pushing ∂M_1 into M_1 gives a properly embedded surface F_1 incompressible in M_1. F_1 is also incompressible in M since it separates M into two components, M_1 and $X = \mathcal{N}(\partial M \cup D_1 \cup \cdots \cup D_{i-1})$ and it is incompressible in both these components, one by assumption and the other by Lemma 4.6 respectively.

Suppose that at least one component of ∂M_1 is not a 2-sphere; this implies that $H_1(M_1)$ is infinite. Applying Theorem 2.3 gives us an incompressible non-separating surface S_1. We want S_1 to be pattern incompressible so we can continue the special hierarchy, so let's assume it isn't. Then it has a pattern compression disc and compressing S_1 along this disc leaves us with at least one component S_1' remaining non-separating, which we will focus on. The discarded component is either a disc, in which case it intersects the boundary pattern at least twice, or it is not a disc, in which case we cannot say anything about its intersection with the boundary pattern. If it is a disc, then the pattern compression disc we replaced it with has at least one less intersection with the boundary pattern and if it is not a disc then, $\chi(S_1') > \chi(S_1)$, so either way we have a bound on how many times we can compress along a pattern compression. Therefore we can assume that S_1 is pattern incompressible.

Cut along S_1 giving M_{i+1}, and now repeat the process from the beginning of the proof, compressing ∂M_{i+1} until it is incompressible, giving us F_2, which is disjoint from F_1. Continue in this fashion until we get all 3-ball components or until we have constructed $F_{n(M)+1}$. Applying Theorem 2.5 implies that at least one pair of surfaces, say F_i and F_j, are parallel for $i < j$ with no F_k lying in the product region between them. Note this product region is necessarily homeomorphic to $F_i \times I$. F_i is the pushed in boundary of some manifold in the sequence M', in which we needed to have found a properly embedded incompressible non-separating surface S in order to continue and S must be disjoint from all surfaces in the sequence save F_i. This implies that S can be properly embedded in $F_i \times I$ disjointly from $F_i \times \{1\}$, violating Lemma 4.7. This proves that the process must terminate in 3-balls prior to this and thereby establishes the existence of a special hierarchy. $\square$

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Vita

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