The Dissertation Committee for Ignacio Tomasetti
certifies that this is the approved version of the following dissertation:

Fully Nonlinear Equations with Applications to Grad Equations in Plasma Physics. Interaction Between a One Phase Free Boundary Problem and an Obstacle Problem. Optimal Trace Sobolev Inequalities

Committee:

Luis A. Caffarelli, Supervisor

Francesco Maggi, Co-Supervisor

Stefania Patrizi

Pablo R. Stinga

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Dedicated to my grandparents,

Julio, Vicen and Bruno.

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Fully Nonlinear Equations with Applications to Grad Equations in Plasma Physics. Interaction Between a One Phase Free Boundary Problem and an Obstacle Problem. Optimal Trace Sobolev Inequalities

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Supervisors: Luis A. Caffarelli
Francesco Maggi

In this thesis we address three different problems.

First, we prove existence and regularity for a fully nonlinear and non-local equation which arises in plasma physics. This is a generalization of Grad Equations which model the behavior of plasma confined in a toroidal vessel called TOKAMAK. We prove existence of a $W^{2,p}$-viscosity solution and regularity up to $C^{1,\alpha}(\overline{\Omega})$ for any $\alpha < 1$. Then we elaborate in how to improve this regularity near the boundary. The main ingredient to study is the nonlocality due to the presence of the measure of the superlevel sets in the equations.

Second, we address a problem which models a reaction-diffusion process. Existence is proved, and the solution solves a one phase free boundary

problem in the lower half space $\mathbb{R}^3_-$. Its trace in $\mathbb{R}^2$ solves an obstacle problem for a given obstacle. We study the exchange between the diffusion in the horizontal plane and the lower half space.

Third, we work with a family of variational problems with critical volume and trace constraints. This arises from the study of "best p-Sobolev inequalities" for $n \geq 2$ and $p \in (1, n)$. We extend the analysis from [MV05] and [MN17] for an open set $\Omega \subset \mathbb{R}^n$. We prove existence of minimizers for Ω bounded and $n > 2\,p$, and existence of generalized minimizers for $n > p$. We also establish rigidity results for the comparison theorem "balls have the worst best Sobolev inequalities" from [MV05].

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Chapter 1

Introduction

As we mentioned in the abstract, three unrelated problems will be studied in this thesis work. Therefore, we will separate accordingly into three chapters, each of them consisting in the development of a particular problem. In Chapter 2: Fully Nonlinear Equations with Applications to Grad Equations in Plasma Physics. In Chapter 3: Interaction Between a One Phase Free Boundary Problem and an Obstacle Problem. In Chapter 4: Optimal Trace Sobolev Inequalities. For clarity of the presentation, instead of a general introduction, we will address an overview at the beginning of each of them.

Chapter 2

Fully Nonlinear Equations with Applications to Grad Equations in Plasma Physics

2.1 Overview

Grad equations arise in plasma physics to model plasma, which is confined under magnetic forces in a toroidal container, a TOKAMAK. These equations were named after Harold Grad, who introduced them in the seventies ([GHS75], [Gra77], [Gra80]). They also appear often in the literature as Queer Differential Equations(QDE), or directly as Plasma Equations. Grad noticed that a simplified version of plasma equations was possible using the increasing rearrangement of the solution, or equivalently, its inverse, the measure of the sublevel sets of the solution. Many authors attacked the problem trying to approximate these equations. The first one was introduced by Roger Temam in [Tem79], and then improved by Mossino and Temam in [MT81]. They studied properties of directional derivatives of the rearrangement function, and proved existence results. Years later, Laurence and Stredulinsky in [LS85], studied a model equation, closer to Grad's formulation. Even this simplified case presents many difficulties. The authors introduced a very interesting approach to the problem: they described an approximation with solutions to a N-free boundary problem. In order to apply this process they assumed extra regularity for the level sets of a solution, and obtained existence and smoothness for it.

The research presented is based on the study of a fully nonlinear version of these approximations to Grad equations, and explain in detail the results obtained in [CT21]. The study of plasma equations is of great interest; development of the models may follow by direct applications for the plasma industry. On the other hand, the study of these fully nonlinear PDEs with nonlocal right hand side have importance by itself; the nonlocal dependence in the sublevel sets naturally appears in many problems. Possible relations with financial math will be analyzed in the future. In addition to the problem itself, the method applied has its own relevance. A fixed point theorem was used in the construction of a solution; a method that could be replicated for other problems. This is the case in the work of Araujo and Teymurazyan in [AT20], in which they use this to find solutions to fully nonlinear dead-core systems.

As we mentioned, the undertaking of this research consists in a generalization to the fully nonlinear case, and this procedure has two main reasons. First, all the papers in the literature addressed the problem with a variational method for the linear case, i.e, with the Laplacian as the operator. Second, several authors mention that a nonlinearity would improve the accuracy of the model ([Tem76]) but, as expected, the complexity increases. The work with professor Caffarelli in [CT21], started the study of the fully nonlinear case, but with a viscosity approach. For the simplest case, similar to the model from Mossino and Temam, we proved existence and regularity, which improves the regularity obtained before even for the Laplacian case.

This topic is analyzed in Chapter 2 and is organized as follows: In the first section we present a chronology of the main results in the literature focusing on the equations that motivate our work. In the second section we cite some preliminary definitions. Mainly, we state the basics of $W^{2,p}$-viscosity solutions. The classic theory of viscosity solutions does not apply to this particular problem because of our right-hand side in our main equation. Disregarding the regularity of this right hand side and the one of the solution itself, we notice that having flat region for our solution makes the right-hand side of our equation discontinuous. Therefore, we adopt this $W^{2,p}$-viscosity notion defined in [CCKS96] by Caffarelli, Crandall, Kocan, and Swiech, which allows merely measurable "ingredients". In their paper they proved existence and interior $W^{2,p}$-estimates for solutions to an equation with a fixed right-hand side. Strongly based on their results, Winter in [Win09] extended this regularity up to the boundary proving global $W^{2,p}$-estimates for viscosity solutions and an existence result for $W^{2,p}$-strong solutions. In Section 3 we state and prove the main theorem of existence and global regularity. The idea of the proof is to first freeze our solution and solve the equation for a fixed right-hand side. Then we obtain a bounded sequence of right-hand sides and solutions. And finally we use a fixed point argument and a convergence theorem to find a solution. In Section 4 we prove more regularity under additional hypotheses. As long as we have a uniform bound by below for the gradient, or equivalently, if we have a uniform interior ball condition for the level sets of the solution. Then

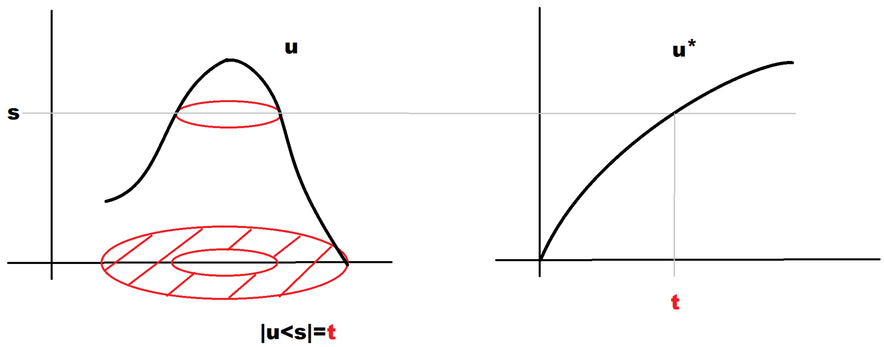

Figure 2.1: Increasing rearrangement of a function u.

we will gain $C^{0,\alpha}$ regularity for the right-hand side. This estimate turns into $C^{2,\alpha}$ regularity for the solution using classic Schauder estimates. In principle, we cannot ensure regularity for the level sets, but if we start with a regular enough domain, say with a uniform interior ball condition, then we gain $C^{2,\alpha}$ regularity in a neighborhood of the boundary.

2.2 Grad Equations in Plasma Physics

Queer differential equations arise from the model of confined plasma in equilibrium due to Harold Grad (see [GHS75], [Gra77], [Gra80]). A thorough survey for the subject can be found in the literature from P. Laurence and E. Stredulinsky, in particular in [LS85] and [LS88]. The main equation governing this model is the Grad-Shafranov equation (widely known as the plasma equation)

$$\Delta u = -p'(u) - \{\frac{1}{2}f^2(u)\}' \tag{2.1}$$

in $\Omega \subset \mathbb{R}^2$. Here Ω is the two dimensional cross section of the toroidal axisymmetric container. The pressure profile p, and the total poloidal current f, depend on the flux function u, the solution. Traditionally p and f were assumed prescribed by the dynamics of the plasma. Instead, Grad proposed a dependence on the measure of the sublevel sets of u

$$V_u(s) = |u < s|,$$

where $|\cdot|$ is the n-dimensional Lebesgue measure and $|u < s|$ is the short notation for $|\{x \in \Omega : u(x) < s\}|$. Indeed, Grad rewrites the equation in terms of the increasing rearrangement of u

$$u^*(t) := \inf\{s : V_u(s) \geq t\}.$$

Notice that heuristically, u^* is the inverse of the measure of the sublevel sets of u (see Figure 2.1); this is whenever u does not have "flat regions" (we will go back to this observation and elaborate in detail). In [Gra80] he demonstrates that the prescribed quantities should be

$$\mu(u) = \frac{p}{(u^{*\prime})^\gamma}$$

and

$$\nu(u) = \frac{f}{u^{*\prime}},$$

for some power γ ($\gamma = 2$ in the two dimensional case). Consequently, if we rewrite (2.1) in terms of μ and ν we obtain

$$\Delta u = -\mu'(u)(u^{*\prime})^{\gamma} - \gamma\mu(u)(u^{*\prime})^{\gamma-2}u^{*\prime\prime} - \frac{1}{2}(\nu^2(u))'(u^{*\prime})^2 - \nu^2(u)u^{*\prime\prime}. \quad (2.2)$$

For clarity we avoided the arguments: u and its derivatives are evaluated at some point x while the rearrangements and its derivatives are evaluated at $t := |u < u(x)|$.

Due to the complexity of (2.2), many authors started by simplifying this problem. In 1978, Roger Temam [Tem79] continued the work of Grad and Claude Mercier [Mer74]. He proposed a reasonable assumption considering the particular case when $\gamma = 2$, $\mu(u) \equiv \frac{1}{2}$, and $\nu(u) \equiv 0$ leading to

$$\Delta u(x) = -u^{*\prime\prime}(|u < u(x)|). \quad (2.3)$$

This case maintains a high level of difficulty because it keeps the second derivatives of the rearrangement, u^*, which forces to study the regularity of both u and u^* in parallel. It also keeps the dependence on the measure of the sub-level sets which makes the problem nonlinear and nonlocal. In [Tem79], he worked on the properties of the rearrangement function and proved existence. Later in [MT81], J. Mossino and R. Temam improved the understanding on the directional derivatives of the rearrangement and generalized (2.3) into the n-dimensional Dirichlet problem

$$\begin{cases} \Delta u(x) = g(|u < u(x)|) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (2.4)$$

for some prescribed functions f and g, and Ω a open bounded regular set in $\mathbb{R}^n$. Their approach was variational, and they proved existence by minimizing the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx + \int_{[0,|\Omega|]} g(s) v^*(s) ds$$

in $H_0^1(\Omega)$. In this paper they even proved existence when g also depends on $u(x)$, under certain grow restrictions on g.

A few years later, P. Laurence and E. Stredulinsky enriched the literature in the topic with a series of papers [LS85, LS89, LS87, LS90, LS88], where they studied a model equation, closer to original Grad's formulation. Again with a variational approach, and using some of the results from Mossino and Temam's work, the authors introduced several interesting strategies to prove regularity. First, in addition to the zero Dirichlet boundary data, they prescribed the difference in the flux, u, between a maximum $u^*(|\Omega|)$ and a minimum $u^*(0)$. In this case existence and gradient bounds were proved. Second, for a convex domain Ω, they discretized the problem in a sequence of rings, and built an approximation with solutions to an N-free boundary problem. And third, they proved a bootstrap argument to gain smoothness, assuming some initial regularity ($W^{2,2}$) and a uniform lower bound on the gradient, $|\nabla u| \geq \delta > 0$. The argument behind this result is that a lower bound on the gradient prevents from forming flat regions and this forces regularity on V_u. This last argument will be adapted and used later in the regularity proofs. Therefore we cite it here in its original version in [LS89]. Let $W^{2,2}_{t,t'}(\Omega)$ be the space of functions $u \in W_0^{2,2}(\Omega)$, with $0 < t < t' < \operatorname{ess\,sup} u$, for which there exists a sequence $u_m \in C_0^\infty(\Omega)$, and a constant $\delta > 0$ such that

$$(a) \quad u_m \to u \quad \text{ in } W^{2,2}(\Omega),$$

and

$$(b) \quad |\nabla u_m| \geq \delta > 0 \quad \text{ in } \{t < u_m < t'\} \quad \text{for all } m.$$

Notice that this implies $|\nabla u| \geq \delta > 0$ almost everywhere in $\{t < u < t'\}$. In this context, we have Theorem 3.1 in [LS89]

Theorem 2.2.1. *Let $u \in W_{t,t'}^{2,2}(\Omega)$ then V_u and V_u' are absolutely continuous in $[t,t']$ with $0 < \min V_u'$ in $[t,t']$ and $V_u'' \in L^2(t,t')$. This also implies regularity on the rearrangement function*

$$u^{*\prime\prime} \in L^2\left(V_u(t), V_u(t')\right)$$

and

$$u^{*\prime\prime}(V_u) = -\frac{V_u''}{V_u'^3} L^2\left(V_u(t), V_u(t')\right) \quad \textit{almost everywhere in } (t,t').$$

2.3 Viscosity Solutions for Fully Nonlinear Equations with Measurable Ingredients

As we mentioned before, one of the main ideas of the paper is to generalize (2.4) for a fully nonlinear operator, instead of the laplacian on the left hand side. Indeed, we will work on the problem

$$\begin{cases} F(D^2u(x)) = g(|u \geq u(x)|) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega, \end{cases}$$

for a convex fully nonlinear elliptic operator F. Unlike the classic literature based on variational techniques, this nonvariational approach will rely on the vast machinery developed in the last decades for viscosity solutions. This will allow us to consider solutions that are, at first, merely continuous. But we will not only gain generality on the equations, but under certain reasonable conditions, this technique will permit us to bootstrap higher regularity, even for the case of the laplacian.

As a first observation, the argument on the right hand side $g(|u \geq u(x)|)$, could be in principle not even continuous. And this is disregarding the regularity of g and u, because it could be the case where $|u = c| > 0$ for some constant c. Unfortunately, the classic theory of viscosity solutions from [CC95] and [CIL92] does not apply directly. Instead, we will adopt the notion of $W^{2,p}$-viscosity that can be found fully described in [CCKS96] by L.A. Caffarelli, M.G. Crandall, M. Kocan, and A. Swiech, which allows merely measurable "ingredients".

We point out that throughout this section we will focus on stating definitions, existence, uniqueness, regularity and basic results for the problem

$$\begin{cases} F(D^2u(x)) = f(x) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega, \end{cases}$$

where the right-hand side is a fixed measurable function f.

2.3.1 Definitions and Basic Notation

We start with some classic notation that can be found in [CC95]. Let $\mathcal{S} \subset \mathbb{R}^{n\times n}$ be the space of real $n \times n$ symmetric matrices. For some matrix $M \in \mathcal{S}$ we note by $e_i = e_i(M)$, for $i = 1, ..., n$, the eigenvalues of M. Given two positive constants $0 < \lambda \leq \Lambda$, that we will call ellipticity constants, we will define the Pucci extremal operators, $\mathcal{M}^-$ and $\mathcal{M}^+ : \mathcal{S} \longrightarrow \mathbb{R}$, as

$$\mathcal{M}^-(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

and

$$\mathcal{M}^+(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i.$$

We say that $F : \mathcal{S} \longrightarrow \mathbb{R}$ is uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$, if it satisfies the structure condition

$$\mathcal{M}^-(M-N) \leq F(M) - F(N) \leq \mathcal{M}^+(M-N) \tag{2.5}$$

for all $M, N \in \mathcal{S}$.

Now we will state the general equation in which we will work from now on

$$\begin{cases} F(D^2u(x)) = f(x) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

with the following setting: $\Omega \subset \mathbb{R}^n$ is an open, bounded and connected set. The fully nonlinear operator $F : \mathcal{S} \longrightarrow \mathbb{R}$ is uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$. For simplicity we will assume $F(0) = 0$. In the right-hand side, $g : [0; |\Omega|] \longrightarrow \mathbb{R}$ is a continuous function. Finally, we consider a boundary value $\psi \in W^{2,p}(\Omega)$ for $p > n$.

Under all these assumptions we can state the definition of $W^{2,p}$-viscosity solution. This follows the theory in [CCKS96] (see also the work of N. Winter in [Win09] for an updated version).

Definition 2.3.1. Let be F a uniformly elliptic operator, $f \in L^p(\Omega)$ for $p > \frac{n}{2}$. Let $u : \Omega \longrightarrow \mathbb{R}$ be a continuous function; we say it is a $W^{2,p}$*-viscosity subsolution* of (2.6) in Ω if $u \leq \psi$ on $\partial\Omega$ and the following holds: for all $\varphi \in W^{2,p}(\Omega)$ such that $u - \varphi$ has a local maximum at $x_0 \in \Omega$, then

$$\operatorname*{ess\,lim\,sup}_{x \to x_0} F(D^2\varphi(x)) - f(x) \geq 0.$$

We define supersolutions in the same way; u is a $W^{2,p}$*-viscosity supersolution* of (2.6) in Ω if $u \geq \psi$ on $\partial\Omega$ and the following holds: for all $\varphi \in W^{2,p}(\Omega)$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$, then

$$\operatorname*{ess\,lim\,inf}_{x \to x_0} F(D^2\varphi(x)) - f(x) \leq 0.$$

Remark 2.3.2. We can also use this alternative definition for $W^{2,p}$-viscosity subsolutions. For all $\varphi \in W^{2,p}_{\text{loc}}(\Omega)$, for all $\varepsilon > 0$, and $O \subset \Omega$ open such that

$$F(D^2\varphi(x)) - f(x) \leq -\varepsilon,$$

a.e. in O, then $u - \varphi$ cannot have a local maximum in O.

We also add the definition of $W^{2,p}$-strong subsolutions.

Definition 2.3.3. In the same setting as before, u is a $W^{2,p}$*-strong subsolution* of (2.6) in Ω, if $u \in W^{2,p}_{\text{loc}}(\Omega)$, $u \leq \psi$ on $\partial\Omega$ and

$$F(D^2u(x)) \geq f(x)$$

a.e. in Ω.

2.3.2 Existence, Uniqueness and Regularity Results

First we state an existence, uniqueness and regularity result for (2.6) proved in [CCKS96], when we have a convex operator F. For this, we refer to Winter's version of the theorem (see Theorem 4.6 in [Win09]) because it includes additional $W^{2,p}$ bounds for the unique solution.

Theorem 2.3.4. *Let F be a uniformly elliptic convex operator with $F(0) = 0$, $f \in L^p(\Omega)$ for $p > n$, $\psi \in W^{2,p}(\Omega)$, and Ω with $C^{1,1}$ boundary. Then, there exists a unique $W^{2,p}$-viscosity solution to (2.6). Moreover, u is a $W^{2,p}$-strong solution, $u \in W^{2,p}(\Omega)$ and*

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left[\|u\|_{L^\infty(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} + \|f\|_{L^p(\Omega)}\right]$$

for $C = C(n, \lambda, \Lambda, p, \Omega)$.

If we observe the bound on the $W^{2,p}$ norm, we have a term depending on $\|u\|_{L^\infty(\Omega)}$. Therefore it will be useful to state the Alexandroff-Bakelman-Pucci (ABP) maximum principle for sub and supersoultions. The proof for this ABP version for measurable ingredients can be found in Proposition 3.3 in [CCKS96] (a previous version appears in Theorem 6.3 in [Wan92])

Proposition 2.3.5. *Let u be a $W^{2,p}$-viscosity subsolution to (2.6) and $f \in L^n(\Omega)$, then there exists a positive constant $C = C(n, \lambda, \Lambda, \operatorname{diam}(\Omega))$ such that*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C\|f\|_{L^n\left(\Gamma_u^+ \cap \{u>0\}\right)}.$$

And we have the equivalent for supersolutions

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^{-} + C\|f\|_{L^n\left(\Gamma_u^- \cap \{u<0\}\right)},$$

where Γ_u^+ *and* Γ_u^- *are the upper and lower contact sets of* u *respectively, defined by*

$$\Gamma_u^+ := \{x \in \Omega : \exists p \in \mathbb{R}^n : u(y) \leq u(x) + p(y-x) \text{ for all } y \in \Omega\},$$

and

$$\Gamma_u^- := \{x \in \Omega : \exists p \in \mathbb{R}^n : u(y) \geq u(x) + p(y-x) \text{ for all } y \in \Omega\}.$$

Note that, even though this is the sharp version, we can obtain a simpler one, assuming that $f \in L^p(\Omega)$ for $p > n$ and therefore

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^{+} + C\|f\|_{L^p(\Omega)}.$$

We will also need the next powerful convergence result (see Theorem 3.8 in [CCKS96]). More precisely, it will be used for proving continuity in a fixed point argument and later for proving convergence of the solutions to auxiliary problems.

Theorem 2.3.6. *Let* $\Omega_k \subset \Omega_{k+1}$ *be a sequence of subdomains of* Ω *converging to* Ω. *Let* F *and* F_k *be uniformly elliptic operators with the same ellipticity constants and satisfying the structure condition* (2.5). *Let* $f \in L^p(\Omega)$ *and* $f_k \in L^p(\Omega_k)$ *for* $p > n$. *Let* $u_k \in C^0(\Omega_k)$ *be* $W^{2,p}$*-viscosity subsolutions (supersolutions) of*

$$F_k(D^2u(x)) = f_k(x)$$

in Ω_{k+1}*, with* u_k *converging locally uniformly to* u *in* Ω*. Finally, assume that for every* $B_r(x_0) \subset \Omega$ *and for every* $\varphi \in W^{2,p}(B_r(x_0))$*, we have*

$$\left\| [F_k(D^2\varphi(x)) - f_k(x) - F(D^2\varphi(x)) + f(x)]^+ \right\|_{L^p(B_r(x_0))} \longrightarrow 0,$$
$$\left(\left\| [F_k(D^2\varphi(x)) - f_k(x) - F(D^2\varphi(x)) + f(x)]^- \right\|_{L^p(B_r(x_0))} \longrightarrow 0 \right),$$

Then u *is a* $W^{2,p}$*-viscosity subsolution (supersolution) of*

$$F(D^2u(x)) = f(x)$$

in Ω*.*

Finally, we note the following

Remark 2.3.7. Due to a result by L. Escauriaza in [Esc93], we can extend p to the case where $p > n - \varepsilon_0$ for some universal $\varepsilon_0 = \varepsilon_0(\Lambda/\lambda, n, \Omega)$ in Theorems 2.3.4 and 2.3.6. This extension is possible due to fundamental estimates for Green's functions proved by E. Fabes and D. Stroock in [FS84].

2.4 Fully Nonlinear Grad Equations

2.4.1 Existence and Uniqueness

In this section we state and prove our main existence and a first global regularity result.

Theorem 2.4.1. *We have existence of a* $W^{2,p}$*-viscosity solution* u *for the problem*

$$\begin{cases} F(D^2u(x)) = g(|u \geq u(x)|) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Furthermore, $u \in W^{2,p}(\Omega)$, and we have the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} \leq C\left[\|u\|_{L^\infty(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} + \|g(|u \geq u(x)|)\|_{L^p(\Omega)}\right].$$

We work under the same assumptions as stated in the previous chapter; $\Omega \subset \mathbb{R}^n$ is an open, bounded and connected set. The fully nonlinear operator $F : \mathcal{S} \longrightarrow \mathbb{R}$ is uniformly elliptic with ellipticity constants $0 < \lambda \leq \Lambda$, and $F(0) = 0$. The function $g : [0; |\Omega|] \longrightarrow \mathbb{R}$ is continuous and the boundary value ψ is in $W^{2,p}(\Omega)$ for $p > n$.

Corollary 2.4.2. *Using the Sobolev embedding theorem, we get that a solution is in $C^{1,\alpha}(\overline{\Omega})$ for any $\alpha < 1$ provided that $\psi \in W^{2,p}$ for every $p > n$.*

The structure of the proof for Theorem 2.4.1 is somehow simple; we set an approximating problem (2.10), we prove the existence of a solution for it, and then we take the limit to obtain the solution to (2.7). Before presenting this approximating problem in Lemma 2.4.3, we give a quick explanation on the reasoning behind it. Note that the three fundamental results that will be used next are existence and uniqueness, a fixed point theorem, and the convergence theorem. If in (2.7) we freeze a function $v \in \mathrm{Lip}(\Omega)$ for the right-hand side, i.e., $f_v(x) := g(|\{y \in \Omega : v(y) \geq v(x)\}|)$, we get

$$\begin{cases} F(D^2u(x)) = f_v(x) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Then the hypotheses of Theorem 2.3.4 are satisfied, and there exists a unique $W^{2,p}$-viscosity solution u to (2.8). The next step would be to apply Schaefer's Fixed Point Theorem (Theorem A.0.1 in the Appendix A) for the application

$T(v) = u$. The problem is that we cannot ensure continuity for T because of the right-hand side of (2.8), not even if we require more regularity for v (not even C^∞ works). We will overcome this inconvenience by solving an auxiliary problem with a smoothed right-hand side, which allows us to perform the fixed point argument. Given $v \in \mathrm{Lip}(\Omega)$, $\varepsilon > 0$, consider

$$\begin{cases} F(D^2u(x)) = f_v^\varepsilon(x) := g\big(\frac{1}{\varepsilon}\int_0^\varepsilon |v \geq v(x) - h| dh\big) & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Because $f_v^\varepsilon \in L^p(\Omega)$, using Theorem 2.3.4 we have existence and uniqueness of a $W^{2,p}$-viscosity solution $u \in W^{2,p}(\Omega)$ to (2.9) with the estimate

$$\|u\|_{W^{2,p}(\Omega)} \leq C\big[\|u\|_{L^\infty(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} + \|f_v^\varepsilon\|_{L^p(\Omega)}\big].$$

Now we can state our approximation lemma.

Lemma 2.4.3. *Given $\varepsilon > 0$, there exists a $W^{2,p}$-viscosity solution u_ε to*

$$\begin{cases} F(D^2u(x)) = f_u^\varepsilon(x) & \textit{in } \Omega, \\ u = \psi & \textit{on } \partial\Omega. \end{cases} \tag{2.10}$$

Proof. The existence is proved, as we remarked, using Schaefer's Fixed Point Theorem A.0.1. We define $T : \mathrm{Lip}(\Omega) \longrightarrow \mathrm{Lip}(\Omega)$ as the application implicitly defined by (2.9) and the existence and uniqueness theorem, i.e., given $v \in \mathrm{Lip}(\Omega)$, $T(v) = u$ for u the unique solution to (2.9). In order to prove the hypothesis required for T, we will make use of the convergence Theorem 2.3.6.

Continuity of T: If we consider $v_k \xrightarrow{\mathrm{Lip}} v$, then we will prove $u_k := T(v_k) \xrightarrow{\mathrm{Lip}} T(v)$. We know that $u_k \in W^{2,p}(\Omega)$ and

$$\|u_k\|_{W^{2,p}(\Omega)} \leq C\big[\|u_k\|_{L^\infty(\Omega)} + \|\psi\|_{W^{2,p}(\Omega)} + \|f_v^\varepsilon\|_{L^p(\omega)}\big] \leq \widetilde{C}$$

with $\widetilde{C}$ independent on k. This is achieved using ABP estimate (Proposition 2.3.5), then

$$\sup_{\Omega} u_k \leq \sup_{\partial\Omega} u_k^+ + C\|f_v^\varepsilon\|_{L^p(\Omega)}$$

and the equivalent for the $\sup_\Omega u_k^-$. We also have the uniform estimate

$$\|f_v^\varepsilon\|_{L^p(\Omega)} \leq |\Omega|^{1/p} g(|\Omega|),$$

which makes $\widetilde{C}$ also independent on ε. Now consider u_{k_j} any subsequence of u_k. Using the Rellich-Kondrachov theorem, we can find a subsequence $u_{k_{j_i}}$ of u_{k_j}(for simplicity we will use the notation $u_i := u_{k_{j_i}}$) converging to some u_ε in the Lipschitz norm, i.e.,

$$u_i \xrightarrow{\text{Lip}} u_\varepsilon.$$

If we can prove that u_ε is the unique $W^{2,p}$-viscosity solution to (2.9) ($T(v) = u_\varepsilon$), then we have the convergence $u_k = T(v_k) \xrightarrow{Lip} u_\varepsilon = T(v)$. Therefore, we obtain the continuity for T.

Every $u_i \in C^0(\Omega)$ is the unique $W^{2,p}$-viscosity solution to (2.9) with v_i in the right-hand side. We have $\Omega_i = \Omega$, and $F_i = F$ fixed for every i. The convergence

$$u_i \xrightarrow{\text{Lip}} u_\varepsilon$$

implies the locally uniformly convergence. So we only need to check the convergence

$$\|f_v^\varepsilon(x) - f_{v_i}^\varepsilon(x)]\|_{L^p(B_r(x_0))} \longrightarrow 0$$

in order to satisfy all the hypotheses in the Convergence Theorem 2.3.6. We know that $v_i \xrightarrow{\text{Lip}} v$, then $\delta_i := \|v_i - v\|_{L^\infty} \longrightarrow 0$. Thus, letting x and h be fixed,

$$|v_i \geq v_i(x) - h| \leq |v + \delta_i \geq v(x) - \delta_i - h| \searrow |v \geq v(x) - h|$$

as $i \longrightarrow \infty$, and also

$$|v_i \geq v_i(x) - h| \geq |v - \delta_i \geq v(x) + \delta_i - h| \nearrow |v > v(x) - h|.$$

We can show that $|v > v(x) - h| = |v \geq v(x) - h|$ for a.e. h in $[0, \varepsilon]$. This happens if and only if $|v = v(x) - h| = |v^{-1}(v(x) - h)| = 0$ for a.e. $h \in [0, \varepsilon]$. A corollary of Rademacher theorem says that if v is a Lipschitz function, then for a.e. $y \in v^{-1}(\alpha)$, $\nabla v(y) = 0$. Therefore

$$|v^{-1}(v(x) - h)| = |v^{-1}(v(x) - h) \cap \{\nabla v(y) = 0\}|.$$

Using a corollary of the coarea formula, we also get that

$$\mathcal{H}^{n-1}(v^{-1}(v(x) - h) \cap \{\nabla v(y) = 0\}) = 0.$$

Here $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure. Then for every $x \in \Omega$ we get the convergence

$$|v_i \geq v_i(x) - h| \longrightarrow |v \geq v(x) - h|$$

for a.e. $h \in [0, \varepsilon]$. Applying the dominated convergence theorem first and the continuity of g, we get

$$g\left(\frac{1}{\varepsilon}\int_0^\varepsilon |v_i \geq v_i(x) - h| dh\right) \longrightarrow g\left(\frac{1}{\varepsilon}\int_0^\varepsilon |v \geq v(x) - h| dh\right) \tag{2.11}$$

as $i \longrightarrow \infty$. This last result is the pointwise convergence of $f^{\varepsilon}_{v_i}$ to f^{ε}_{v}. Again, applying the dominated convergence theorem, we get the L^p convergence needed. So all the hypotheses are satisfied to apply the theorem and therefore T is continuous.

Compactness of T: Let v_k be a bounded sequence in $\mathrm{Lip}(\Omega)$. Then $u_k := T(v_k) \in W^{2,p}(\Omega)$ is bounded as we proved. After Rellich-Kondrachov there exists a convergent subsequence.

Boundedness of the eigenvectors: We have to prove that the set

$$\Gamma := \{v \in \mathrm{Lip}(\Omega) : \exists \gamma \in [0;1] \text{ such that } v = \gamma T(v)\}$$

is bounded. Suppose by contradiction that it is not. First we note that $0 \in \Gamma$ with $\gamma = 0$, and for every $0 \neq v \in \Gamma$ the γ associated with v is not zero. Suppose then that there exists a sequence of nonzero elements $v_k \in \Gamma$, and a respective sequence γ_k such that $v_k = \gamma_k T(v_k)$ and $\|v_k\|_{\mathrm{Lip}(\Omega)} \longrightarrow \infty$. Because $v_k \in \mathrm{Lip}(\Omega)$, then $v_k/\gamma_k \in W^{2,p}(\Omega)$ and

$$\|v_k\|_{\mathrm{Lip}(\Omega)} \leq \left\| \frac{v_k}{\gamma_k} \right\|_{\mathrm{Lip}(\Omega)} \leq C \left\| \frac{v_k}{\gamma_k} \right\|_{W^{2,p}(\Omega)} \leq \widetilde{C},$$

which is a contradiction. Therefore Γ is bounded.

The hypotheses of Schaefer's theorem are satisfied, so there exists a Lipschitz fixed point u_ε for T, i.e., $u_\varepsilon = T(u_\varepsilon)$. Moreover, by Theorem 2.3.4, u_ε is a $W^{2,p}$-viscosity solution to (2.10), which is in $W^{2,p}(\Omega)$. □

The purpose of finding such a u_ε was to approximate a solution for

(2.7). Then the following question is whether we can take the limit $\varepsilon \longrightarrow \infty$. We will answer this question proving the existence theorem.

Proof of Theorem 2.4.1. For every $\varepsilon > 0$ we have a solution $u_\varepsilon \in W^{2,p}(\Omega)$ with uniformly bounded $W^{2,p}$ norm (with respect to ε). Then there exists a subsequence (that we will also call u_ε) and a Lipschitz function u such that

$$u_\varepsilon \xrightarrow{\text{Lip}} u.$$

So $u_\varepsilon \longrightarrow u$ locally uniformly and we will be able to apply again the Convergence Theorem 2.3.6. In this case we have on the right-hand sides, the L^p functions

$$f^\varepsilon_{u_\varepsilon}(x) := g\left(\frac{1}{\varepsilon}\int_0^\varepsilon |u_\varepsilon \geq u_\varepsilon(x) - h| dh\right),$$

and

$$f_u(x) := g\big(|u \geq u(x)|\big)$$

respectively. We are left to prove the convergence

$$\|f_u - f^\varepsilon_{u_\varepsilon}(x)\|_{L^p(B_r(x_0))} \longrightarrow 0.$$

By the triangle inequality we have

$$\big\|f_u - f^\varepsilon_{u_\varepsilon}(x)\big\|_{L^p(\Omega)} \leq \big\|f_u - f^\varepsilon_u(x)\big\|_{L^p(\Omega)} + \big\|f^\varepsilon_u - f^\varepsilon_{u_\varepsilon}(x)\big\|_{L^p(\Omega)}.$$

The second term tends to zero as we already observed in previous calculations like in (2.11). We will use a similar argument for bounding the first term, as follows

$$
\begin{aligned}
|u \geq u(x)| \leq |u \geq u(x) - h| & \\
&= |u \geq u(x)| + |u(x) > u \geq u(x) - h| \\
&\leq |u \geq u(x)| + |u(x) > u \geq u(x) - \varepsilon|.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_0^\varepsilon |u \geq u(x) - h| dh \geq \frac{1}{\varepsilon}\int_0^\varepsilon |u \geq u(x)| dh \\
= |u \geq u(x)|
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_0^\varepsilon |u \geq u(x) - h| dh \leq \frac{1}{\varepsilon}\int_0^\varepsilon |u \geq u(x)| + |u(x) > u \geq u(x) - \varepsilon| dh \\
= |u \geq u(x)| + |u(x) > u \geq u(x) - \varepsilon|.
\end{aligned}
$$

Therefore, taking the limit in the last two inequalities leads us to

$$
\frac{1}{\varepsilon}\int_0^\varepsilon |u \geq u(x) - h| dh \searrow |u \geq u(x)|
$$

as $\varepsilon \longrightarrow \infty$. Accordingly, we obtained pointwise convergence for f_u^ε to f_u, which after the dominated convergence theorem implies the convergence on the L^p-norm.

The hypotheses of Theorem 2.3.6 are satisfied, and we finally obtain our main result: u is a $W^{2,p}$-viscosity solution to (2.7), in $W^{2,p}(\Omega)$ and with the corresponding $W^{2,p}$-estimates. □

The last remark of this section is that we obtain an explicit formula for the Dirichlet problem in a ball with 0 boundary data.

Example 2.4.4. When $\Omega = B_r(x_0)$, $F = \Delta$, $g(t) = -t$, and $\psi = 0$, we have the explicit solution

$$\tilde{u}(x) = \frac{\omega_n}{2n(n+2)}[r^{n+2} - |x - x_0|^{n+2}],$$

where ω_n is the measure of the n-dimensional unit ball. In a similar way we can prove that $\frac{1}{\Lambda}\tilde{u}$ is a solution when $F = \mathcal{M}^-$ (respectively, $\frac{1}{\lambda}\tilde{u}$ for $F = \mathcal{M}^+$). We will use this example in the next section to build subsolutions that can be used as barriers to prove gradient bounds.

2.4.2 Regularity

In order to gain more regularity for our unique solution u to (2.7), we probably need to get some regularity for the right-hand side f_u. So far, in the case when u has flat regions, f_u is not even continuous. In principle, this discontinuity does not depend on the regularity of u but on its flat regions. We can prove that under the negativity of g, u is not allowed to have these flat regions with positive measure.

Lemma 2.4.5. *Let u be a solution for (2.7) with right-hand side $g < 0$ in $(0, |\Omega|]$; then*

$$|u = a| = 0$$

for every constant $a \in \mathbb{R}$.

Proof. Suppose that there exists an $a \in \mathbb{R}$ such that $A := \{u = a\}$ satisfies that $|A| > 0$. Then, by a classic result from Stampacchia (see [Sta66]), we obtain that the Hessian vanishes almost everywhere in A. Indeed, we have

$$\nabla u(x) = 0 \quad \text{for a.e. } x \in A.$$

Now we can define $A' := A \cap \{\nabla u = 0\}$ and apply the same argument again, then

$$D^2 u(x) = 0 \quad \text{for a.e. } x \in A'.$$

We are left with the set $A'' := A' \cap \{D^2 u = 0\}$ with the same measure $|A''| = |A'| = |A| > 0$. By the definition of u being a $W^{2,p}$-strong supersolution of (2.7), we have that for a.e. x in Ω

$$F(D^2 u(x)) - g(|u \geq u(x)|) \leq 0.$$

Moreover, for $x_0 \in A''$, the argument inside g is strictly positive:

$$|u \geq u(x_0)| \geq |A''| > 0.$$

So in the particular case when g is strictly positive in $(0, |\Omega|]$, we get the contradiction

$$F(D^2 u(x_0)) - g(|u \geq u(x_0)|) > 0.$$

□

Then, for this specific case we obtain continuity for f_u. But we need at least $C^{0,\alpha}$ regularity on f_u in order to apply Schauder-type estimates to obtain

$u \in C^{2,\alpha}$. We will be able to obtain this regularity in two particular cases listed in the next two theorems. The first one is an adaptation (simplification) of Laurence and Stredulinsky's theorem (see Theorem 3.1 in [LS89]) and requires an additional lower bound for the gradient.

Theorem 2.4.6. *Let $u \in W_0^{2,p}(\Omega)$ have a uniform lower bound on the gradient, $|\nabla u| > c_0 > 0$ in the set $\Omega_{t_0} := \{y \in \Omega : u(y) < t_0\}$. Where $t_0 < \|u\|_{L^\infty}$ and $c_0 = c_0(t_0)$. Then $f_u \in C^1(\Omega_{t_0})$.*

In other words, the theorem asserts that if we have an uniform lower bound for the gradient (away from the maximum of u), then we get: regularity for the level sets of u and we discard a possible "flatness" that ruins the smoothness of f_u. The proof presented in [LS89] includes an approximation argument by C_0^∞ functions and coarea formula.

This theorem translates into regularity for our problem. We state this in the following corollary.

Corollary 2.4.7. *If we have a $W^{2,p}$-viscosity solution u to our problem (2.7), with boundary condition $\psi \equiv 0$ and a gradient lower bound as in Theorem 2.4.6, then $f_u \in C^1(\Omega_{t_0})$, and therefore $u \in C^{2,\alpha}(\Omega_{t_0})$.*

Proof. In order to get $C^{2,\alpha}$ estimates we just need to apply the classical theory of viscosity solutions for fully nonlinear equations as in Chapter 8 from [CC95]. Recall that at this point we have a right-hand side in $C^1(\Omega_{t_0})$ that allows us to use classical viscosity solutions instead of $W^{2,p}$-viscosity solutions. □

The second theorem states that, under certain conditions, a barrier argument implies lower bounds as in Theorem 2.4.6. In this case we need some reasonable extra regularity assumption on Ω, as having a uniform inner ball condition.

Definition 2.4.8. We say that Ω has a uniform inner ball condition if there exists an $\varepsilon_0 > 0$, such that for any point y in $\partial\Omega$, there exists a ball $B_\varepsilon \subset \Omega$ with $\varepsilon > \varepsilon_0$ and $y \in \partial B_\varepsilon$.

Theorem 2.4.9. *Let be u a $W^{2,p}$-viscosity solution to (2.7), with $g(t) = -t$ and boundary condition $\psi \equiv 0$. If Ω has a uniform inner ball condition then $|\nabla u| > c_0 > 0$ in a neighborhood of $\partial\Omega$, where $c_0 = c_0(\|u\|_{C^{1,\alpha}(\overline{\Omega})})$.*

Proof. If we pick any point $y \in \partial\Omega$, we can touch it with a ball $B_{\varepsilon_0} \subset \Omega$, which actually means $y \in \partial B_{\varepsilon_0}$. As in Example 2.4.4, we can build a an explicit solution $\tilde{u}$ in B_{ε_0} for $F = \mathcal{M}^-$, $g(t) = -t$ and $\tilde{u} = 0$ on $\partial B_{\varepsilon_0}$,

$$\mathcal{M}^-(D^2\tilde{u}(x)) = -|\{\tilde{u} \geq \tilde{u}(x)\} \cap B_{\varepsilon_0}|$$

We write the intersection with the ball to highlight that $\tilde{u}$ is only defined in B_{ε_0} and therefore the measure is consider over $x \in B_{\varepsilon_0}$. Now we apply a comparison between u and $\tilde{u}$ in order to get gradient estimates. Without loss of generality we can consider ε_0 small enough such that

$$|u \geq u(x)| \geq \frac{1}{2}|\Omega| \geq |B_{\varepsilon_0}|$$

for every $x \in B_{\varepsilon_0}$. This is possible because u vanishes at the boundary, and due to the continuity of u and of the right-hand side (by Lemma 2.4.5), i.e.,

$$|u \geq t| \xrightarrow[t \longrightarrow 0]{} |\Omega|.$$

With this assumption we have

$$\begin{aligned} \mathcal{M}^-(D^2\tilde{u}(x) &\geq -|B_{\varepsilon_0}| \\ &\geq -|u \geq u(x)| \\ &= F(D^2u(x)) \\ &\geq \mathcal{M}^-(D^2u(x)) \end{aligned}$$

in B_{ε_0}. In addition, we have that $0 = \tilde{u} \leq u$ at $\partial B_{\varepsilon_0}$. So comparison applies and forces $\tilde{u} \leq u$ in B_{ε_0}. Here with comparison we refer to the maximum principle for classical viscosity solutions, which can be derived from ABP estimates (see Corollary 3.7 in [CC95]). Therefore we also have a lower bound for the gradient at the boundary, with the estimate

$$|\nabla u| \geq \tilde{u}_{-\nu} = \frac{\omega_n}{2n\Lambda}{\varepsilon_0}^{n+1} = c_0 > 0$$

where $-\nu$ is the inner normal to $\partial\Omega$. Finally, we can extend a lower bound for the gradient (say $c_0/2$) to a neighborhood of the boundary of Ω that will depend of course on the $C^{1,\alpha}(\overline{\Omega})$ norm of u. □

Remark 2.4.10. We can repeat this argument as long we have uniform inner ball conditions for the level sets $\{u = t\}$, and so $C^{2,\alpha}$ regularity for the solution in that annulus.

Remark 2.4.11. We expect this condition to be satisfied for convex domains, where we deduce that the solutions will have convex level sets. On the other hand, for nonconvex domains, in particular for dumbbell-shaped domains, we expect to have a singular critical point where the superlevel sets separate into two components.

This fact will be studied in detail in an ongoing project with Daniel Restrepo and Luis Caffarelli. In this paper (to appear) we study the same problem but under extra conditions on the domain (convexity, radial symmetry) or under extra conditions on the right hand side. We will prove optimal regularity and radial symmetry for the radial case. Also with professor Caffarelli we have been analyzing variations of this plasma problem, and looking for relationships with other problems and applications. For instance, it could be possible to do a mimic of the method used in this first attempt, but for the Laurence and Stredulinsky's approximation, and for the more general Grad equation. Also a possible adaptation of the N-free boundary problem (see [LS85]) could be studied with this viscosity approach. Additionally, the parabolic case and the relationship with American Options, or Obstacle problems are of interest.

Chapter 3

Interaction Between a One Phase Free Boundary Problem and an Obstacle Problem

3.1 Overview

In this chapter we will introduce a different topic, unrelated to Plasma. This is the result of the joint work with L.A. Caffarelli and J.-M. Roquejoffre in [CRT22]. The main idea consist on finding a solution u to a one phase free boundary problem in three dimensions, actually, in the lower half space, and with trace v solving an obstacle problem for a given smooth obstacle φ. The complexity of these equations relies not only on finding such a coupled spatial-solution/trace, but also on the interaction between them. We will start with some notation, and the statement of the equations, followed by some of the motivation. In the next two sections we will study the equations from two different approaches, variational and nonvariational.

Throughout this chapter we will be working in the half space

$$\mathbb{R}^3_- := \mathbb{R}^2 \times (-\infty, 0).$$

We will use the notation (x_1, x_2, y) for an element of $\mathbb{R}^3_-$ and we will refer to $\mathbb{R}^2$ as the boundary of $\mathbb{R}^3_-$, i.e., the elements of the form $(x_1, x_2, 0)$. We also adopt the notation for disks in $\mathbb{R}^2$, $D_r(x_1^0, x_2^0) = \{(x_1, x_2) \in \mathbb{R}^2 : |(x_1, x_2) -$

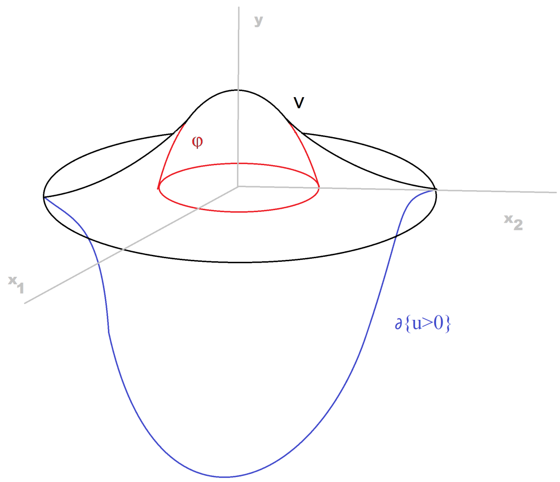

Figure 3.1: Free boundary of u in $\mathbb{R}^3_-$, and the graph of its respective trace $v : \mathbb{R}^2 \to \mathbb{R}_+$ solving an obstacle problem.

$(x_1^0, x_2^0)| < r\}$, and balls in $\mathbb{R}^3$, $B_r(x_1^0, x_2^0, y^0) = \{(x_1, x_2, y) \in \mathbb{R}^2 : |(x_1, x_2, y) - (x_1^0, x_2^0, y^0)| < r\}$, for some positive radius $r > 0$. We look for a nonnegative function $u : \mathbb{R}^3_- \longrightarrow \mathbb{R}_+$, with trace $v(x_1, x_2) = u(x_1, x_2, 0)$ in $\mathbb{R}^2$. The trace v solves an obstacle problem for the given obstacle $\varphi : \mathbb{R}^2 \to [0, \infty)$. We can think it as φ^+ where φ is a smooth enough obstacle with $\Delta\varphi < 0$ in its compact support. The equations (u, v) solves are

$$\begin{cases} \Delta u = 0, & \text{in } \{u > 0\} \cap \{y < 0\} \\ u_\nu = 1, & \text{on } \partial\{u > 0\} \cap \{y < 0\} \\ & \\ v \geq \varphi, & \text{on } \mathbb{R}^2 \\ \Delta v = u_y, & \text{on } \mathbb{R}^2 \end{cases} \tag{3.1}$$

where ν is the inward normal direction on $\partial\{u > 0\}$.

Some variants of (3.1) could be considered leading to similar results.

For example, the obstacle condition

$$v(x_1, x_2) = u(x_1, x_2, 0) \geq \varphi(x_1, x_2)$$

could be replaced by a Dirichlet condition in a disc $D_R(0)$ for some fixed $R > 0$

$$v(x_1, x_2) \equiv 1 \quad \text{for all} \quad (x_1, x_2) \in D_R(0).$$

Another generalization that could be take into account is changing the laplacian in the diffusion equation for the plane, $\Delta u = u_y$. For instance, with

$$-\sum_{1\leq i,j\leq N} \partial_{x_i}\big(a_{ij}(\nabla u)\big) + u_y = 0,$$

where the matrix $\big(a_{ij}(p)\big)$ satisfies ellipticity conditions, as in [BK74]. The results for (3.1) could probably be extended for this case without many modifications.

Back to our original statement (3.1), we study diffusion on a horizontal plane which also has an exchange with the lower half space. The free boundary problem there, model a reaction-diffusion process. The motivation behind this problem is to extend the work of L. Caffarelli and J.-M. Roquejoffre in [CR20] and [CR21]. In these previous articles they consider a two dimensional system that models the influence of a line of fast diffusion on the propagation of reaction-diffusion waves. The setting is

$$\begin{cases} -\Delta u + c\partial_x u = & 0 \quad (x,y) \in \{u > 0\} \\ |\nabla u| = & 1 \quad ((x,y) \in \Gamma := \partial\{u > 0\} \\ & \\ -u_{xx} + c\partial_x u + u_y = & 0 \quad \text{for } x \in \mathbb{R},\ y = 0 \\ u(-\infty, y) = & 1, \quad \text{uniformly in } y \in \mathbb{R}_- \\ u(+\infty, y) = & 0 \quad \text{pointwise in } y \in \mathbb{R}_-. \end{cases}$$

At the same time, the study of this problem is motivated by a two-dimensional reaction diffusion model initially proposed by J.-M. Roquejoffre, H. Berestycki and L. Rossi. In [BRR13], the authors studied a model on the influence of transportation networks on biological invasions (see also [BCRR14]). After this model, A.-C. Coulon established some counter-intuitive numerical simulations in [Cou14]. The work in [CR20], [CR21] transforms this model into a free boundary interacting with a line of fast diffusion, which explains the simulations.

3.2 Existence for the Variational case

The problem

$$\begin{cases} \Delta u = 0, & \text{in } \{u > 0\} \cap \{y < 0\} \\ u_\nu = 1, & \text{on } \partial\{u > 0\} \cap \{y < 0\} \\ & \\ v \geq \varphi, & \text{on } \mathbb{R}^2 \\ \Delta v = u_y, & \text{on } \mathbb{R}^2 \end{cases}$$

is variational and a solution can be found as a minimizer for the functional

$$\mathcal{J}(u) := \int_{\mathbb{R}^3_-} |\nabla u|^2 + |u > 0| + \int_{\mathbb{R}^2} |\nabla v|^2.$$

Here, as in the previous chapters, $|u > 0|$ stands for the Lebesgue measure of $\{(x_1, x_2, y) \in \mathbb{R}^3_- : u(x_1, x_2, y) > 0\}$. The minimization is considered over all functions in

$$\mathcal{K} := \left\{ u \in H^1_0(\mathbb{R}^3_-) \text{ such that } v \in H^1_0(\mathbb{R}^2) \text{ and } v \geq \varphi \right\} \tag{3.2}$$

for the given obstacle $\varphi \geq 0$.

We remark that in [CRT22] we will take a detour before arriving to this minimization because we want also to obtain a solution with nice geometrical properties that allow us to study and obtain regularity of the free boundary. Therefore, we will get a minimizer to (3.2) as the limit of a sequence of solutions to a semilinear approximation.

3.3 Existence for the Viscosity Case

In this section we will define and prove the existence of a viscosity solution for (3.1). This nonvariational approach is the first step into extending (3.1) into more general (possibly nonlinear) operators instead of the laplacian. However, as this is a first approach, we will be working with the laplacian throughout the chapter. For this case we are going to construct a solution using Perron like method, i.e., building a viscosity solution as the infimum of viscosity supersolutions. Then, we start stating the definitions.

Definition 3.3.1. We say that a continuous function $u : \mathbb{R}^3_- \to \mathbb{R}$, with trace $v(x_1, x_2) := u(x_1, x_2, 0)$, is a *viscosity supersolution* for (3.1) if

$$\begin{cases} \Delta u \leq 0, & \text{in } \{u > 0\} \cap \{y < 0\} \\ u_\nu \leq 1, & \text{on } \partial\{u > 0\} \cap \{y < 0\} \\ \\ v \geq \varphi, & \text{on } \mathbb{R}^2 \\ \Delta v \leq u_y, & \text{on } \mathbb{R}^2 \end{cases}$$

in the classic viscosity sense. In an analogous way we define a *viscosity subso-*

lution solving

$$\begin{cases} \Delta u \geq 0, & \text{in } \{u > 0\} \cap \{y < 0\} \\ u_\nu \geq 1, & \text{on } \partial\{u > 0\} \cap \{y < 0\} \\ \\ v \geq \varphi, & \text{on } \mathbb{R}^2 \\ \Delta v \geq u_y, & \text{on } \mathbb{R}^2 \end{cases}$$

A *viscosity solution* for (3.1) will be, a continuous function that is both a viscosity subsolution and supersolution.

Because we are in the context of classic viscosity solutions (as in [CC95], [CIL92]), and because it will not cause confusion, we will avoid using the term viscosity and refer to them every time as supersolutions, subsolutions and solutions respectively. As a first observation we will prove the existence of a subsolution and a supersolution for (3.1).

Lemma 3.3.2. *There exist a subsolution $\underline{u}$ and a supersolution $\overline{u}$ for* (3.1) *with traces $\underline{v}$ and $\overline{v}$ respectively.*

Proof. Let $0 < M$ be a big enough constant to be chosen later, and a big enough radius $R > 0$, such that the disk $D_R \subset \mathbb{R}^2$ contains the $\text{supp}(\varphi)$, and let $\underline{v}$ be a solution for the obstacle problem

$$\begin{cases} \underline{v} \geq \varphi, & \text{in } \mathbb{R}^2 \\ \Delta \underline{v} = M, & \text{in } \{\underline{v} > \varphi\} \\ \text{supp}(\underline{v}) \subset D_R. \end{cases}$$

Then for φ regular enough we can get regularity for $\underline{v}$ up to $C^{1,1}$. Now $\underline{v}$ will be a low boundary value with big laplacian and we want to extend it to $\underline{u}$ in the interior, with $\underline{u}$ being a subsolution. But first, we will perform a rescaling that will play in our favor. Lets consider

$$\underline{\tilde{v}}(x_1, x_2) := M\underline{v}\left(\frac{(x_1, x_2)}{M}\right),$$

and

$$\tilde{\varphi}(x_1, x_2) := M\varphi\left(\frac{(x_1, x_2)}{M}\right).$$

Note that with this change of variables we obtain

$$\Delta\underline{\tilde{v}}(x_1, x_2) = \frac{1}{M}\Delta\underline{v}\left(\frac{(x_1, x_2)}{M}\right) = 1$$

over the set $\{\underline{\tilde{v}} > \tilde{\varphi}\}$. The next step is to solve $\underline{\tilde{u}}$ in the interior with fixed boundary data $\underline{\tilde{v}}$

$$\begin{cases} \Delta\underline{\tilde{u}} = 0, & \text{in } \{\underline{\tilde{u}} > 0\} \cap \{y < 0\} \\ \underline{\tilde{u}}_\nu = 1, & \text{on } \partial\{\underline{\tilde{u}} > 0\} \cap \{y < 0\} \\ \underline{\tilde{u}} = \underline{\tilde{v}}, & \text{in } \mathbb{R}^2. \end{cases}$$

After the rescaling we still have $\underline{\tilde{v}} \in C^{1,1}$, and because $\underline{\tilde{u}}$ is its harmonic extension inside, we get at least $\underline{\tilde{u}} \in C^{1,\alpha}$ with normal derivatives of order one: $\underline{\tilde{u}}_y \approx 1$. Once we have extended $\underline{\tilde{v}}$ into $\underline{\tilde{u}}$ we only need to rescale back. We define

$$\underline{u}(x_1, x_2) := \frac{\underline{\tilde{u}}\left(M(x_1, x_2)\right)}{M}$$

This rescaling back maintain the order of the first derivatives,

$$\nabla\underline{u}(x_1, x_2) = \nabla\underline{\tilde{u}}\left(M(x_1, x_2)\right).$$

So $\underline{u}$ will satisfy $\Delta\underline{u} = 0$ in the interior, $\underline{u}_\nu = 1$ at the free boundary and still $\underline{u}_y \approx 1$ which makes $\underline{u}$ satisfy

$$1 \approx \underline{u}_y \leq M = \Delta\underline{v}$$

for big enough M. Therefore we have constructed a subsolution $\underline{u}$

We can do the analogous argument with a small laplacian M^{-1} in order to construct a supersolution $\overline{u}$. □

Then we need to prove that the comparison principle holds since this will be one of the foundations of our construction.

Lemma 3.3.3. ***Comparison Principle*** *Let $\underline{u}$ be a subsolution and $\overline{u}$ be a supersolution for (3.1) with $\{\underline{u} > 0\} \subset \{\overline{u} > 0\}$ and $\underline{v} \leq \overline{v}$, then we have $\underline{u} \leq \overline{u}$.*

Proof. We work under the assumption $\{\underline{u} > 0\} \subset \{\overline{u} > 0\}$ because it appears naturally in our construction of a solution to the main problem. Indeed we expect the condition $\underline{u}_\nu \geq 1 \geq \overline{u}_\nu$ to "push the free boundary further" for the supersolution. If we also assume that we have the bound for the fixed boundary, $\underline{v} \leq \overline{v}$, then we can extend it to the interior, using the comparison for subharmonic and superharmonic functions. We have in that $\underline{u} - \overline{u}$ is subharmonic in $\{\underline{u} > 0\}$ with nonpositive boundary condition, therefore $\underline{u} \leq \overline{u}$.

Therefore we must have $\underline{u} \leq \overline{u}$. □

We also have following lemma

Lemma 3.3.4. *The minimum of two supersolutions is a supersolution. Analogously, the maximum of two subsolutions is a subsolution.*

Proof. Let u_1 and u_2 two supersolutions with traces v_1 and v_2 respectively. And let's define $u := \min(u_1; u_2)$ with trace $v := \min(v_1; v_2)$. Then we have immediately that u is continuous and $v \geq \varphi$. The subharmonicity of u is granted in $\{u > 0\} \cap \{y < 0\} = \{u_1 > 0\} \cap \{u_2 > 0\} \cap \{y < 0\}$ because u is the minimum of two subharmonic functions. To prove the normal derivatives' conditions at a point (x_1^0, x_2^0, y^0) at the boundary, we can assume by continuity that without loss of generality u is u_1 in a neighborhood of (x_1^0, x_2^0, y^0). In the case when (x_1^0, x_2^0, y^0) lays in the intersection of both boundaries, we can either use the conditions for u_1 or u_2. □

Now we can state our main existence result

Theorem 3.3.5. *Let $w := \inf\{u(x_1, x_2, y) : u \text{ is a supersolution}\}$, then w is a solution for (3.1).*

This theorem will be proved later, after we describe the construction of this infimum based on Perron's method.

Idea of the proof: We will start working with our supersolution $\overline{u}$ from Lemma 3.3.2. We will decrease the value of $\overline{u}$ and $\overline{v}$ maintaining its property of being a supersolution. This will be done by following three iterative steps. In the first one we will replace $\overline{u}$ by a harmonic function in the set $\{\overline{u} > 0\}$ with same boundary conditions. On the second step replace $\overline{v}$ in a disk(when possible) by the solution of a bigger laplacian and same boundary conditions. And the third one consists in moving the free boundary $\{\overline{u} = 0\} \cap \{y < 0\}$ by replacing the condition $\overline{u}_\nu \leq 1$ by $\overline{u}_\nu = 1$. These three steps will be performed in adequate ball coverings and they will be explained in detail in the following lemmas.

Lemma 3.3.6. ***Step 1:*** *Let u be a supersolution with boundary value v. We can replace u by harmonic inside $\{u > 0\}$ with the same boundary values, and we obtain a smaller supersolution.*

Proof. We have a fixed domain $\{u > 0\}$, fixed boundary values(0 and v respectively), and we can solve harmonic inside:

$$\begin{cases} \tilde{u} = v, & \text{in } \mathbb{R}^2 \\ \tilde{u} = 0, & \text{on } \partial\{u > 0\} \cap \{y < 0\} \\ \Delta\tilde{u} = 0, & \text{in } \{u > 0\}. \end{cases}$$

Finally, we need to check that $\tilde{u}$ is indeed a supersolution. By comparison, we have $\tilde{u} \leq u$. This implies $\tilde{u}_\nu \leq u_\nu \leq 1$ at the free boundary, and $\tilde{u}_y \geq u_y \geq \Delta v$ at the fixed boundary $\{y = 0\}$. □

Lemma 3.3.7. ***Step 2:*** *Let u be a supersolution with boundary value v, and consider a half ball $B_r(x) \cap \{y \leq 0\}$ centered at $x = (x_1, x_2, 0)$. Let also assume that $\Delta u = 0$ in $\{u > 0\}$. We can build a supersolution modifying u, v in $B_r(x) \cap \{y \leq 0\}$, increasing Δv, as long as we can maintain $\Delta v \leq u_y$.*

Proof. Suppose there exist some $\delta_1 > \delta > 0$ such that $\Delta v \leq \delta$ in $D_r(x_1, x_2) = B_r(x) \cap \{y = 0\}$. We solve the problem

$$\begin{cases} \tilde{v} = v, & \text{in } \mathbb{R}^2 \setminus D_r(x_1, x_2) \\ \tilde{v} \geq \varphi, & \text{in } D_r(x_1, x_2) \\ \Delta\tilde{v} = \delta_1, & \text{in } \{\tilde{v} > \varphi\}. \end{cases}$$

Then, by comparison principle we have that $\tilde{v} \leq v$. Now we replace u by harmonic in $B_r(x) \cap \{u > 0\}$ but with boundary $\tilde{v}$, i.e.,

$$\begin{cases} \tilde{u} = \tilde{v}, & \text{on } D_r(x_1, x_2) \\ \tilde{u} = u, & \text{in } \mathbb{R}^3_- \setminus \Big[B_r(x) \cap \{u > 0\}\Big] \\ \Delta \tilde{u} = 0, & \text{in } B_r(x) \cap \{u > 0\}. \end{cases}$$

In the interior, $\{\tilde{u} > 0\}$, $\tilde{u}$ is the minimum of two superharmonic functions, so therefore it is superharmonic. At the free boundary, by comparison, we have $\tilde{u} \leq u$, which makes $\tilde{u}_\nu \leq u_\nu \leq 1$. At the fixed boundary, $\{y = 0\}$, $\tilde{v} \leq v$ forces the "wrong" inequality, $\tilde{u}_y \leq u_y$. So, in the case we can maintain the condition $\Delta \tilde{v} \leq \tilde{u}_y$ on $B_r(x) \cap \{y = 0\}$, we end up with a smaller supersolution $\tilde{u}, \tilde{v}$. □

Lemma 3.3.8. ***Step 3:*** *Let u be a supersolution such that $u_\nu(z) < 1$ for some $z \in \partial\{u > 0\} \cap \{y < 0\}$. Then, for some ball $B_r \subset \{y < 0\}$ containing z we can solve*

$$\begin{cases} \tilde{u} = u, & \text{in } \mathbb{R}^3_- \setminus B_r \\ \Delta \tilde{u} = 0, & \text{in } \{\tilde{u} > 0\} \cap B_r \\ \tilde{u}_\nu = 1, & \text{on } \partial\{\tilde{u} > 0\} \cap B_r \end{cases}$$

where $\tilde{u}$ is still a supersolution with smaller support.

Proof. This last condition $u_\nu \leq \tilde{u}_\nu = 1$ forces $u \geq \tilde{u}$. So, we have $\text{supp}(\tilde{u}) \subset$ *supp*(u). As before, in the interior $\tilde{u}$ is the minimum of two harmonic functions, therefore, superharmonic. Because the fixed boundary is not modified, $\tilde{u}$ is still a supersolution. □

Proof. of Theorem 3.3.5 As we anticipated, our starting point is the initial supersolution $\overline{u}$ from Lemma 3.3.2. We are going to perform the modifications described in the previous lemmas, over a covering of $\overline{\{u > 0\}}$. We first assume, using *Step 1*, that $\overline{u}$ is harmonic in $\{\overline{u} > 0\}$.

Then, we proceed to cover the fix boundary $\{\overline{v} > 0\}$. We consider a covering with countably many half balls $B_r(x) \cap \{y \leq 0\}$ with rational radius r and centers $x = (x_1, x_2, 0) \in \{\overline{v} > 0\}$ with rational coordinates. Now, we are ready to increase $\Delta\overline{v}$ in an iterative way. Recall that the laplacian of $\overline{v}$ is small

$$\Delta\overline{v} = \delta = M^{-1}$$

by the construction in Lemma 3.3.2. The first step in the iteration is the following: we increase (if possible) $\Delta\overline{v}_2 = 2\delta$ in $\{\overline{v}_2 > \varphi\} \cap B_r(x)$. We solve for each half ball, as in (3.3) in *Step 2*, and replace harmonic in $B_r(x) \cap \{\overline{u} > 0\}$ as in (3.3). Then we check if $\Delta\overline{v}_2 \leq \overline{u}_{2y}$ is maintained. If not, we keep $\overline{u}$,$\overline{v}$ as they were. Instead, if $\Delta\overline{v}_2 \leq \overline{u}_{2y}$ holds, we follow to shrink the domain $\{\overline{u}_2 > 0\}$ by performing *Step 3*. So we get a lower supersolution, harmonic in the inside, with bigger laplacian at the free boundary and with $\overline{u}_{2\nu} = 1$ on the free boundary. And this conditions hold, for every half ball in the covering. We will call again $\overline{u}_2$ to the resulting supersolution. The next step is to repeat this process, building $\overline{u}_3$, satisfying $\Delta\overline{v}_3 = 3\delta$ in each of the half balls(whenever is possible).

If we repeat this iterative process, we will end up with a supersolution u that is harmonic in the inside, $u_\nu = 1$ on the free boundary, and v was lowed down as much as it could (Δv was enlarged as much as it could). But also, in each step $\Delta v \leq u_y$ was ensured. Therefore, because we enlarged Δv with u_y as its upper bound, we will end up with u being a solution. Suppose this is not true, then it will exist some point $x = (x_1, x_2, 0) \in \mathbb{R}^3_-$ for which $\Delta v(x) < u_y(x)$. And this cannot be the case, because whenever this happens, there is room for increasing Δv in a sufficiently small ball containing x. So, with this iteration, we force our "supersolution" to satisfy the equality at the fix boundary condition, making it a solution. $\square$

Chapter 4

Optimal Trace Sobolev Inequalities

4.1 Introduction

In this chapter we work on a minimization problem that is related to the study of "best p-Sobolev inequalities" for $n \geq 2$ and $p \in (1, n)$. The results presented here derive from a project which was a joint work with Francesco Maggi and Robin Neumayer. This problem was studied in [MNT22]. The main idea is to continue with the work from [MV05] and [MN17], where the minimization of the family $\{\Phi_\Omega(T)\}_{T\geq 0}$ was thoroughly described when Ω is a ball or a half-space. In these previous papers, existence of minimizers was proved for those cases, but some open questions were left for a general bounded open set $\Omega \subset \mathbb{R}^n$. The difficulty here is that the minimization is performed under critical volume and trace constraints, so compactness fails. Therefore, we are going to prove existence of generalized minimizers with possible concentration at the boundary. If the boundary of Ω is C^1 then, after a boundary flattening, this concentration will behave as a re scaled minimizer for the half-space. Moreover, if the boundary of Ω is C^2 and $n > 2\,p$, we will have existence of classical minimizers. We also address a comparison theorem from [MV05] which states that "balls have the worst best Sobolev inequalities". In this context, we will be able to prove rigidity results conjectured in [MV05].

4.1.1 Overview

The goal of this chapter is to answer some basic open questions concerning a "doubly critical" family $\{\Phi_\Omega(T)\}_{T\ge 0}$ of minimization problems on Sobolev functions, which, in a precise sense to be clarified below, can be interpreted as collectively defining the best Sobolev inequality on an open set $\Omega\subset\mathbb{R}^n$ with C^1-boundary. Given an integer $n\ge 2$ and $p\in(1,n)$, these problems are defined as

$$\Phi_\Omega(T)=\inf\Big\{\Big(\int_\Omega|\nabla u|^p\Big)^{1/p}:\int_\Omega|u|^{p^\star}=1\,,\int_{\partial\Omega}|u|^{p^\#}=T^{p^\#}\Big\}\,,$$

and their minimizers, whenever they exist, satisfy the Euler–Lagrange equation

$$\begin{cases}-\Delta_p u=\lambda\,u^{p^\star-1}\,, & \text{on }\Omega\,,\\ |\nabla u|^{p-2}\,\dfrac{\partial u}{\partial\nu_\Omega}=\sigma\,u^{p^\#-1}\,, & \text{on }\partial\Omega\,,\end{cases}\tag{4.1}$$

for suitable Lagrange multipliers $\lambda,\sigma\in\mathbb{R}$. We call $\int_\Omega|u|^{p^\star}=1$ and $\int_{\partial\Omega}|u|^{p^\#}=T^{p^\#}$ the "volume" and "trace" constraints of $\Phi_\Omega(T)$. The critical Sobolev exponents associated to n and p, $p^\star$ and $p^\#$, are defined by

$$p^\star=\frac{np}{n-p}\,,\qquad p^\#=\frac{(n-1)p}{n-p}=\frac{n-1}{n}\,p^\star\,,$$

and their precise values guarantee the scale invariance of Φ_Ω, i.e.

$$\Phi_{x+r\,\Omega}(T)=\Phi_\Omega(T)\qquad\forall x\in\mathbb{R}^n\,,r>0\,,T\ge 0\,.$$

When Ω is bounded, the C^1-regularity of $\partial\Omega$ guarantees that every $u\in L^1_{\rm loc}(\Omega)$ with $\nabla u\in L^p(\Omega;\mathbb{R}^n)$ lies in the competition class of $\Phi_\Omega(T)$ for some $T\ge 0$. In particular,

$$\mathrm{Epi}(\Phi_\Omega)=\big\{(T,G)\in\mathbb{R}^2:T\ge 0\,,G\ge\Phi_\Omega(T)\big\}\,,$$

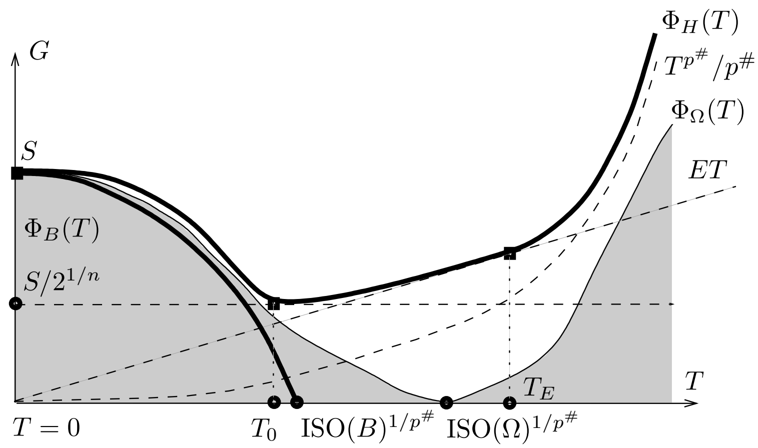

Figure 4.1: The state of the art about Φ_Ω. The "exclusion zone" for the values of $\|\nabla u\|_{L^p(\Omega)}$ (under the constraint $\|u\|_{L^{p^\star}(\Omega)} = 1$) is depicted in gray. It always contains the subgraph of $\Phi_B(T)$ on $T \in [0, \mathrm{ISO}(B)^{1/p^\#}]$, see (4.9), but is always smaller than the one of a half-space H, see (4.14). The Sobolev inequality on $\mathbb{R}^n$ is equivalent to $\Phi_\Omega(0) = S(n,p)$, the Euclidean isoperimetric inequality to the fact that the only zero of Φ_Ω (i.e. $T = \mathrm{ISO}(\Omega)^{1/p^\#}$) is achieved to the right of the only zero of Φ_B, and the Escobar inequality (4.5) is equivalent to the linear bound $\Phi_H(T) \geq E\,T$. Both $\{(T, \Phi_B(T)) : T \in [0, \mathrm{ISO}(B)^{1/p^\#}]\}$ and $\{(T, \Phi_H(T)) : T \geq 0\}$ can be implicitly parametrized by looking at the explicit families of minimizers given in (4.8), (4.10), (4.11) and (4.12).

(the epigraph of Φ_Ω) collects the best possible information on the range of values achievable by $\|\nabla u\|_{L^p(\Omega)}$ when $\|u\|_{L^{p^\star}(\Omega)}$ is fixed: from this peculiar viewpoint, which is reminiscent of the one adopted in the study of Blaschke–Santaló diagrams, $\mathrm{Epi}(\Phi_\Omega)$ *is* "the best Sobolev inequality on Ω". The following list of results, summarized in Figure 4.1, aims to provide a hopefully complete state of the art on Φ_Ω, and illustrates the wealth of information stored in this family of variational problems. As a disclaimer: here we are definitely *not* attempting to exhaustively frame the study of Φ_Ω into the incredibly vast and layered context of the theory of Sobolev-type inequalities (see e.g. [Maz85]), as that would be a long and delicate exercise, lying well beyond the scope of this introduction.

(1) Sobolev inequality on $\mathbb{R}^n$: A scaling and localization argument shows that, for every open set Ω, one has

$$\Phi_\Omega(0) = S(n,p) := \inf\Big\{\Big(\int_{\mathbb{R}^n}|\nabla u|^p\Big)^{1/p} : \int_{\mathbb{R}^n}|u|^{p^\star}=1\Big\}, \tag{4.2}$$

that is, $\Phi_\Omega(0)$ is the best constant in the L^p-Sobolev inequality on $\mathbb{R}^n$. Minimizers of (4.2) are exactly given by the family $\{\tau_{x_0}[U_S^{(\alpha)}]\}_{x_0\in\mathbb{R}^n,\alpha>0}$ generated by

$$U_S(x) = \big(1+|x|^{p/(p-1)}\big)^{1-(n/p)}, \qquad x\in\mathbb{R}^n, \tag{4.3}$$

see [Aub76, Tal76, CENV04], we see that $\Phi_\Omega(0)$ is attained if and only if $\Omega = \mathbb{R}^n$. Here and in the following, we set

$$\tau_{x_0}[v](x) = v(x-x_0), \qquad v^{(\alpha)}(x) = \alpha^{-n/p^\star}\,v(x/\alpha) \tag{4.4}$$

whenever $x_0\in\mathbb{R}^n$ and $\alpha>0$.

(2) Escobar inequality: The Escobar inequality ([Esc88, for $p=2$], [Naz06, for $p\in(1,n)$]) states that if H is an (open) half-space in $\mathbb{R}^n$ with outer unit normal ν_H, then

$$\Big(\int_H|\nabla u|^p\Big)^{1/p} \ge E(n,p)\Big(\int_{\partial H}|u|^{p^\#}\Big)^{1/p^\#} \tag{4.5}$$

with equality if and only if $u=\tau_{x_0}[U_E^{(\alpha)}]$ for some $x_0\in\mathbb{R}^n\setminus\overline{H}$, and where

$$U_E(x) = |x|^{-(n-p)/(p-1)}, \qquad x\in\mathbb{R}^n\setminus\{0\},$$

is a multiple of the fundamental solution of the p-Laplacian. The "only if" statement in the equality for $p \neq 2$ was left open in [Naz06], but was proven in [MN17, Theorem 2.3]. The quantity

$$T_E = \|\tau_{x_0}[U_E^{(\alpha)}]\|_{L^{p^\#}(\partial H)} \Big/ \|\tau_{x_0}[U_E^{(\alpha)}]\|_{L^{p^\star}(H)} \tag{4.6}$$

is independent of $x_0 \in \mathbb{R}^n \setminus \overline{H}$, and is such that

$$\Phi_H(T_E) = E(n,p)T_E\,.$$

The Escobar inequality (4.5) can be equivalently reformulated in the "Φ-setting" as a linear lower bound for Φ_H, i.e.

$$\Phi_H(T) \geq E(n,p)\,T\,, \qquad \forall T \geq 0\,.$$

This bound is sharp only if $T = T_E$, and is nearly optimal only if T is close to T_E; but it largely suboptimal away from T_E, see Figure 4.1.

(3) Euclidean isoperimetry: It is easily seen that $\Phi_\Omega(T) = 0$ for some $T > 0$ if and only if $|\Omega| < \infty$ and $T = \mathrm{ISO}(\Omega)^{1/p^\#}$, where $\mathrm{ISO}(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)/|\Omega|^{(n-1)/n}$ stands for the isoperimetric ratio of Ω. Since the Euclidean isoperimetric inequality states that

$$\mathrm{ISO}(\Omega) \geq \mathrm{ISO}(B)\,, \tag{4.7}$$

(with equality if and only if Ω is a ball), in the Φ-setting, (4.7) is equivalent to saying that Φ_B has the left-most zero among all Φ_Ω.

(4) Balls have the worst best Sobolev inequalities: In [CL90, CL94, $p = 2$] (by symmetrization methods and conformal invariance) and in [MV05,

$p \in (1,n)$] (via the mass transportation method pioneered in [Kno57, MS86, CENV04]) it is shown that if B is a ball, then for every $T \in (0, \mathrm{ISO}(B)^{1/p^{\#}})$ there is a unique $\alpha > 0$ such that

$$\Phi_B(T) = \|\nabla U_S^{(\alpha)}\|_{L^p(B)} \Big/ \|U_S^{(\alpha)}\|_{L^{p^\star}(B)}\,, \tag{4.8}$$

with U_S defined in (4.3). Further elaborating on the proof of this partial characterization of Φ_B, again in [MV05] the comparison theorem that *balls have the worst best Sobolev inequalities*

$$\Phi_\Omega(T) \geq \Phi_B(T)\,, \qquad \forall T \in [0, \mathrm{ISO}(B)^{1/p^{\#}}] \tag{4.9}$$

is proved. This sharp lower bound, combined with (4.8), allows one to infer some sharp and more traditional-looking Sobolev-type inequalities, like the following sharp interpolation between (4.2) and (4.7)

$$\frac{\|\nabla u\|_{L^p(\Omega)}}{S(n,p)} + \frac{\|u\|_{L^{p^{\#}}(\partial\Omega)}}{\mathrm{ISO}(B)^{1/p^{\#}}} \geq \|u\|_{L^{p^\star}(\Omega)}\,,$$

and the following sharp Sobolev inequality, additive in the domain of the p-Dirichlet energy,

$$\frac{\|\nabla u\|^p_{L^p(\Omega)}}{S(n,p)^p} + \frac{\|u\|^p_{L^{p^{\#}}(\partial\Omega)}}{C(n,p)} \geq \|u\|^p_{L^{p^\star}(\Omega)}\,,$$

which was first conjectured by Brezis and Lieb in [BL85].

(5) Full characterization of Φ_H: In [CL90, CL94, for $p = 2$] and, again by optimal mass transport arguments, in [MN17, for $p \in (1,n)$], it is proven that if H is half-space in $\mathbb{R}^n$, then: for each $T \in (0, T_E)$ (Sobolev range) there is a unique $t_T \in \mathbb{R}$ such that

$$\Phi_H(T) = \|\nabla(\tau_{t_T\,\nu_H} U_S)\|_{L^p(H)} \Big/ \|\tau_{t_T\,\nu_H} U_S\|_{L^{p^\star}(H)}\,; \tag{4.10}$$

if $T = T_E$ (Escobar point), then

$$\Phi_H(T_E) = E(n,p) T_E = \|\nabla(\tau_{\nu_H} U_E)\|_{L^p(H)} \Big/ \|\tau_{\nu_H} U_E\|_{L^{p^\star}(H)}\,; \tag{4.11}$$

for each $T > T_E$ (beyond Escobar range), there is a unique $s_T > 1$ s.t.

$$\Phi_H(T) = \|\nabla(\tau_{s_T\,\nu_H} U_{BE})\|_{L^p(H)} \Big/ \|\tau_{s_T\,\nu_H} U_{BE}\|_{L^{p^\star}(H)}\,,$$
$$\text{where } U_{BE}(x) = (|x|^{p/(p-1)} - 1)^{1-(n/p)}\,, \qquad |x| > 1\,.$$

Up to the natural dilation and translation invariances, these functions are the unique minimizers of $\Phi_H(T)$. Moreover, again by [MN17]: **(a):** $\inf_{T\ge 0}\Phi_H(T)$ is achieved at $T = T_0 \in (0, T_E)$, where $t_{T_0} = 0$ and $\Phi_H(T_0) = S(n,p)/2^{1/n}$; **(b):** by the divergence theorem, $\Phi_H(T) > T^{p^\#}/p^\#$ for every $T > 0$, and this lower bound is sharp as $T \to \infty$; **(c):** finally, Φ_H has the *best* best Sobolev inequality, i.e.

$$\Phi_\Omega(T) \le \Phi_H(T)\,, \qquad \forall T \ge 0\,. \tag{4.14}$$

4.1.2 Statements of the main results

With this summary on the state of the art for Φ_Ω in mind, there are two fundamental open questions that form the subject of our paper:

Question 1: When does $\Phi_\Omega(T)$ $(T > 0)$ admit minimizers?

Question 2: Does rigidity hold in the comparison theorem (4.9)?

The main idea of this paper is attacking these two closely related questions by systematically exploiting the complete characterization of Φ_H obtained in [MN17].

Concerning Question 1, a classical concentration-compactness argument characterizes the limit behavior of minimizing sequences of $\Phi_\Omega(T)$ as the superposition of a standard weak limit plus at most countably many concentration points, located either in the interior of Ω, or on its boundary. By exploiting properties of Φ_H we are able to (i): exclude all interior concentrations and all but *at most one* boundary concentration, thus proving existence of minimizers for a suitable "relaxed problem" $\Phi^*_\Omega(T)$; and (ii): completely exclude concentrations, and thus establish the existence of minimizers of $\Phi_\Omega(T)$, as soon as $\partial\Omega$ is of class C^2 and $n > 2p$. To give precise statements, it is convenient to let $\mathcal{X}_\Omega(T)$ denote the competition class of $\Phi_\Omega(T)$, and let $\mathcal{Y}_\Omega(T)$ denote the set of triples $(u, \mathsf{v}, \mathsf{t})$ with either $u \in \mathcal{X}_\Omega(T)$ and $\mathsf{v} = \mathsf{t} = 0$, or $u \in W^{1,p}(\Omega)$, $\mathsf{v} \in (0,1]$, $\mathsf{t} \in (0,T]$, and

$$\mathsf{v}^{p^\star} + \int_\Omega u^{p^\star} = 1\,, \qquad \mathsf{t}^{p^\#} + \int_{\partial\Omega} u^{p^\#} = T^{p^\#}\,.$$

The relaxed problem associated to $\Phi_\Omega(T)$ is then given by

$$\Phi^*_\Omega(T) = \inf_{\mathcal{Y}_\Omega(T)} \mathcal{E}\,, \quad \text{where} \quad \mathcal{E}(u, \mathsf{v}, \mathsf{t})^p = \int_\Omega |\nabla u|^p + \mathsf{v}^p\, \Phi_H\Big(\frac{\mathsf{t}}{\mathsf{v}}\Big)^p\,, \qquad (4.15)$$

with the convention that $\mathsf{v}^p \Phi_H(\mathsf{t}/\mathsf{v})^p = 0$ if $(\mathsf{v}, \mathsf{t}) = (0,0)$.

Theorem 4.1.1 (Existence of minimizers of Φ_Ω)**.** *If $n \ge 2$, $p \in (1,n)$, and Ω is a bounded open set with C^1-boundary in $\mathbb{R}^n$, then:*

(i): *for every $T > 0$, there is a minimizer $(u, \mathsf{v}, \mathsf{t})$ of $\Phi^*_\Omega(T)$, and*

$$\Phi_\Omega(T) = \Phi^*_\Omega(T)\,;$$

moreover, if $\int_\Omega u^{p^\star} > 0$, then we have that $u/\|u\|_{L^{p^\star}(\Omega)}$ is a minimizer of $\Phi_\Omega(\|u\|_{L^{p^\#}(\partial\Omega)}/\|u\|_{L^{p^\star}(\Omega)})$;

(ii): *if Ω has boundary of class C^2, $n > 2p$, $T > 0$, and $(u, \mathsf{v}, \mathsf{t})$ is a minimizer of $\Phi^*_\Omega(T)$, then $\mathsf{v} = \mathsf{t} = 0$, and thus u is a minimizer of $\Phi_\Omega(T)$.*

Remark 4.1.2. Minimizers of $\Phi_\Omega(T)$ solve the Euler–Lagrange equation (4.1). For the Euler–Lagrange equation satisfied by minimizers $(u, \mathsf{v}, \mathsf{t})$ of the relaxed problem $\Phi^*_\Omega(T)$, see Theorem 4.4.1 below.

Question 2 is motivated by the various rigidity statements associated to comparison theorems in Riemannian geometry (see, e.g. [CE08]). In that setting, a certain model space provides a universal bound on a certain global geometric quantity (comparison theorem), which is then shown to be saturated by the model space alone (rigidity statement). With this analogy in mind, we can reformulate more precisely Question 2 as follows:

Question 2, weak form: Does $\Phi_\Omega = \Phi_B$ on $(0, \mathrm{ISO}(B)^{1/p^\#})$ imply that Ω is a ball?

Question 2, strong form: Does $\Phi_\Omega(T) = \Phi_B(T)$ at just one value of $T \in (0, \mathrm{ISO}(B)^{1/p^\#})$ imply that Ω is a ball?

Concerning the *weak form* of Question 2, through a careful use of the properties of Φ_H we answer affirmatively whenever Ω is bounded and connected. These conditions are optimal, as shown by unbounded or disconnected non-rigidity examples presented in [MV05]. In fact, the argument we propose gives rigidity under the mere assumption that $\Phi_\Omega = \Phi_B$ holds on an open neighborhood of $T = 0$. Concerning the *strong form* of Question 2, which was originally formulated in [MV05, Section 1.9], we can answer in the affirmative as a direct by-product of our existence result for minimizers of $\Phi_\Omega(T)$ (thus, when Ω has C^2-boundary and $n > 2p$) thanks to the following "conditional rigidity" statement, which is proved in [MV05] as a direct by-product of the proof of (4.9):

if Ω is connected (possibly unbounded),

if $\Phi_\Omega(T) = \Phi_B(T)$ for a value of $T \in (0, \mathrm{ISO}(B)^{1/p^\#})$,

and if $\Phi_\Omega(T)$ is known to admit minimizers (possibly just for that T),

then Ω is a ball.

(We notice for future use an important consequence of (4.16), namely, we have

$$\Phi_B(T) < \Phi_H(T)\,, \qquad \forall T \in \left(0, \mathrm{ISO}(B)^{1/p^\#}\right); \tag{4.17}$$

indeed, by [MN17], $\Phi_H(T)$ admits minimizers for every $T > 0$.) With these premises, we now state our main results concerning Question 2.

Theorem 4.1.3 (Rigidity of "Balls have the worst best Sobolev inequalities")**.** *Let $n \geq 2$, $p \in (1, n)$, Ω an open, bounded, connected set with C^1-boundary in $\mathbb{R}^n$, and assume that* **one** *of the following two conditions holds:*

(i): *there is $T_* > 0$ such that $\Phi_\Omega(T) = \Phi_B(T)$ for every $T \in (0, T_*)$; or*

(ii): *$n > 2p$, the boundary of Ω is of class C^2, and there is $T \in (0, \mathrm{ISO}(B)^{1/p^\#})$ such that $\Phi_\Omega(T) = \Phi_B(T)$.*

Then, Ω is a ball.

4.1.3 Strategy of proof

Concentration-compactness arguments and the use of sharp Sobolev-type inequalities (like the Sobolev and Escobar inequalities (4.2) and (4.5)) are the standard tools of the trade in the analysis of variational problems with critical growth. As seen, if interpreted as assertions about Φ_H, (4.2) and (4.5) contain only very partial information (respectively, "$\Phi_H(0) = S(n, p)$" and "$\Phi_H(T) \geq E(n, p)\, T$ for every $T > 0$"). From this viewpoint, our arguments provide an interesting example of the potentialities of using, in the familiar context of concentration-compactness, the full characterization of Φ_H obtained in [CL94, MN17]. We now explain how this characterization is used in this paper.

We have already mentioned how the mere knowledge of the existence of minimizers in $\Phi_H(T)$ for every $T > 0$ allows one to reduce the analysis of concentrations to the simplest possible case of a *single* boundary concentration (thus leading to Theorem 4.1.1-(i)). Finer properties of Φ_H are exploited in the proof of Theorem 4.1.1-(ii), which goes as follows. We consider the existence of a minimizer $(u, \mathsf{v}, \mathsf{t})$ of $\Phi^*_\Omega(T)$ with $\mathsf{v} > 0$, and, keeping in mind that $\Phi_\Omega(T) = \Phi^*_\Omega(T)$, we aim to obtain a contradiction to $\mathsf{v} > 0$ by constructing a competitor v of $\Phi_\Omega(T)$ with

$$\int_\Omega |\nabla v|^p < \int_\Omega |\nabla u|^p + \mathsf{v}^p\, \Phi_H\Big(\frac{\mathsf{t}}{\mathsf{v}}\Big)^p .$$

We seek v in the form $v = u_\varepsilon$, for the *Ansatz* given by

$$u_\varepsilon(x) = (1 - \varphi_\varepsilon(x))\, u(x) + \varphi_\varepsilon(x)\, \big(U^{(\varepsilon)} \circ g\big)(x)\,, \qquad x \in \Omega\,. \tag{4.18}$$

Here $x_0 \in \partial\Omega$ is a boundary point of Ω with positive mean curvature, i.e. $\mathrm{H}_{\partial\Omega}(x_0) > 0$; φ_ε is a cut-off function between $B_{\varepsilon^\beta}(x_0)$ and $B_{2\,\varepsilon^\beta}(x_0)$ for $\beta = \beta(n,p) \in (0,1)$ to be suitably chosen (the condition $n > 2\,p$ enters in this choice); g is a boundary flattening diffeomorphism near x_0; and, finally, $U = U_\tau + b\,\varepsilon\, V_\tau$ for $\tau = \mathsf{t}/\mathsf{v}$, V_τ a standard perturbation of U_τ, and b a constant suitably chosen depending on n, p, $H_{\partial\Omega}(x_0)$ and τ. Our convention here is that the scalar mean curvature of $\partial\Omega$ is computed with respect to the outer unit normal to Ω, so that every bounded open set with C^2-boundary has at least one boundary point of positive mean curvature. The energy, volume and trace expansions for u_ε as $\varepsilon \to 0^+$ are computed to be

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^p \le \int_\Omega |\nabla u|^p + \mathsf{v}^p\, \Phi_H\Big(\frac{\mathsf{t}}{\mathsf{v}}\Big)^p \\ -\Big\{\mathcal{L}(U_\tau) - \frac{(n-p)}{n}\, \lambda_H(\tau)\, \mathcal{M}(U_\tau)\Big\}\, \mathrm{H}_{\partial\Omega}(x_0)\, \mathsf{v}^p \varepsilon + \mathrm{o}(\varepsilon)\,,\end{aligned}$$

$$\int_\Omega u_\varepsilon^{p^\star} = 1 + \mathrm{o}(\varepsilon)\,, \qquad \int_{\partial\Omega} u_\varepsilon^{p^\#} = T^{p^\#} + \mathrm{o}(\varepsilon)\,,$$

where $\lambda_H(T)$ is the volume Lagrange multiplier of U_T (see (4.49) below), and where $\mathcal{L}$ and $\mathcal{M}$ are functionals defined on $U : H \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}(U) &= \int_H x_n\,|\nabla U|^p - p\,x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,,\\ \mathcal{M}(U) &= \int_H x_n\,U^{p^\star}\,.\end{aligned}$$

Modulo o(ε)-perturbations of u_ε aimed at correcting the volume and trace constraints to the exact values needed for inclusion in $\mathcal{X}_\Omega(T)$, we have constructed the required competitors, and proved Theorem 4.1.1-(ii), if we can show the existence of $c(n,p,T) > 0$ such that

$$\mathcal{L}(U_T) - \frac{(n-p)}{n}\,\lambda_H(T)\,\mathcal{M}(U_T) \geq c(n,p,T)\,, \qquad \forall T > 0\,. \tag{4.23}$$

Of course, the full characterization of Φ_H plays a crucial role in our proof of (4.23), see Lemma 4.3.1 below.

While Theorem 4.1.3-(ii) is immediate from Theorem 4.1.1-(ii) thanks to the rigidity criterion (4.16), the proof of Theorem 4.1.3-(i) requires an additional argument, which once more exploits several fine properties of Φ_B and Φ_H: these include (4.16), (4.17), and information on the signs of the Lagrange multipliers $\lambda_H(T)$ and $\sigma_H(T)$ for minimizers U_T of $\Phi_H(T)$ (see (4.54) and (4.53) below).

4.1.4 Organization of the paper

After collecting a few preliminary results in section 4.2, in section 4.3 we study in detail various properties of Φ_H and of its minimizers: in particular, we prove the key inequality (4.23) (see Lemma 4.3.1), and discuss in detail the *Ansatz* (4.18) (see Lemma 4.3.4). Sections 4.4 and 4.5 contain, respectively, the proofs of Theorem 4.1.1 and Theorem 4.1.3. Finally, we collect some auxiliary, routine proofs in an appendix.

4.2 Notation and preparations

Some basic notation is presented in section 4.2.1. Afterwards, we discuss, in separate subsections, the following four useful technical lemmas: a concentration-compactness lemma with boundary terms (Lemma 4.2.1); a second order expansion for the boundary flattening diffeomorphisms used in the *Ansatz* (4.18) (Lemma 4.2.3); some basic regularity information on minimizers of $\Phi_\Omega(T)$ (Lemma 4.2.5); and the basic technique of "volume/trace correcting variations" (Lemma 4.2.6). Some proofs are postponed to the appendix.

4.2.1 Notation

Throughout the paper we always assume that $n \geq 2$ and $p \in (1, n)$. We denote by $\mathcal{L}^n$ and $\mathcal{H}^k$ the Lebesgue measure and the k-dimensional Hausdorff measure of $\mathbb{R}^n$, although we simply set $|E|$ in place of $\mathcal{L}^n(E)$. We denote by $B_r(x)$ the open ball of center $x \in \mathbb{R}^n$ and radius $r > 0$, and set $B_r = B_r(0)$, while B denotes a ball of unspecified center and radius.

Following a standard shorthand notation, by "$f(x) = g(x) + \mathrm{O}_{a,b}(|x|)$ for $|x| > R$" we mean that $|f(x) - g(x)| \leq C(a,b)\,|x|$ if $|x| > R$; by "$f(x) =$

$g(x) + o_{a,b}(|x|)$ as $|x| \to 0$" we mean that $\lim_{|x|\to 0} |f(x) - g(x)|/|x| = 0$ at a rate that is uniform with respect to the parameters a and b.

In general, we will use capital letters (e.g. U, V, Ψ) to denote functions defined on the half space H and lowercase letters (e.g. u, v, φ) to denote functions defined on an open bounded domain Ω.

4.2.2 Concentration-compactness

The following lemma is a version of Lions' celebrated concentration-compactness lemma and provides a natural starting point to study minimizing sequences of $\Phi_\Omega(T)$.

Lemma 4.2.1 (Concentration-compactness)**.** *Let $n \geq 2$, $p \in (1, n)$, and let $\Omega \subset \mathbb{R}^n$ be open and bounded with C^1-boundary. If $\{u_j\}_j$ is a sequence in $L^1_{\text{loc}}(\Omega)$, $\{\nabla u_j\}_j$ is bounded in $L^p(\Omega; \mathbb{R}^n)$ and $u_j \rightharpoonup u$ as distributions in Ω, then the Radon measures on $\mathbb{R}^n$ defined by*

$$\mu_j = |\nabla u_j|^p \, \mathcal{L}^n \llcorner \Omega \,, \qquad \nu_j = |u_j|^{p^\star} \, \mathcal{L}^n \llcorner \Omega \,, \qquad \tau_j = |u_j|^{p^\#} \, \mathcal{H}^{n-1} \llcorner \partial\Omega \,,$$

have subsequential weak-star limits μ, ν and τ which satisfy

$$\begin{aligned} \nu &= |u|^{p^\star} \mathcal{L}^n \llcorner \Omega + \sum_{i \in I} \mathsf{v}_i^{p^\star} \, \delta_{x_i} \,, \\ \tau &= |u|^{p^\#} \mathcal{H}^{n-1} \llcorner \partial\Omega + \sum_{i \in I} \mathsf{t}_i^{p^\#} \, \delta_{x_i} \,, \\ \mu &\geq |\nabla u|^p \mathcal{L}^n \llcorner \Omega + \sum_{i \in I} \mathsf{g}_i^p \, \delta_{x_i}, \end{aligned}$$

where $\{x_i\}_{i\in I} \subset \overline{\Omega}$ *is at most countable set,* $\mathsf{v}_i > 0$ *and* $\mathsf{t}_i \geq 0$ *for every* $i \in I$*,* $\mathsf{t}_i > 0$ *only if* $x_i \in \partial\Omega$*, and*

$$\mathsf{g}_i \geq \mathsf{v}_i \, \Phi_H\left(\frac{\mathsf{t}_i}{\mathsf{v}_i}\right), \qquad \forall i \in I \,. \tag{4.28}$$

In particular, $\mathsf{g}_i \geq S\,\mathsf{v}_i$ *whenever* $x_i \in \Omega$*.*

Proof. See appendix B.1. □

4.2.3 Near-boundary coordinates

In this section, we introduce two types of coordinates for a neighborhood of a boundary point of a domain Ω: one that requires minimal regularity of the boundary of Ω and will suffice in the proofs of Theorem 4.1.1**(i)** and Theorem 4.1.3**(i)**, and a second that requires C^2 regularity of the boundary of Ω and will be used in the proof of Theorem 4.1.1**(ii)** and Theorem 4.1.3**(ii)**.

Given an open set Ω with C^1-boundary, we denote by ν_Ω its outer unit normal and by $T_x(\partial\Omega)$ the tangent space to $x \in \partial\Omega$. When Ω has C^2-boundary, we denote by A_Ω and H_Ω the second fundamental form and the scalar mean curvature of $\partial\Omega$ defined by ν_Ω. To define coordinates near boundary points of Ω, for $x \in \mathbb{R}^n$ we set $\mathbf{p}(x) = x - x_n\, e_n$, $\mathbf{D}_r = \{x : x_n = 0\,, |\mathbf{p}x| < r\}$, and $\mathbf{C}_r = \big\{x : |x_n| < r\,, |\mathbf{p}(x)| < r\big\}$. In particular, if Ω is an open set with C^1-boundary such that

$$0 \in \partial\Omega\,, \qquad T_0\,(\partial\Omega) = \{x_n = 0\}\,, \qquad \nu_\Omega(0) = -e_n\,, \tag{4.29}$$

then we can find $r_0 > 0$ and $\ell : \mathbf{D}_{r_0} \to (-r_0, r_0)$ such that $\ell(0) = 0$, $\nabla\ell(0) = 0$, and

$$\begin{aligned} \Omega \cap \mathbf{C}_{r_0} &= \left\{x + t\, e_n : x \in \mathbf{D}_{r_0}\,, r_0 > t > \ell(x)\right\}, \\ (\partial\Omega) \cap \mathbf{C}_{r_0} &= \left\{x + \ell(x)\, e_n : x \in \mathbf{D}_{r_0}\right\}. \end{aligned}$$

We then define the maps $F : \mathbf{D}_{r_0} \to \partial\Omega$, $f : \mathbf{C}_{r_0} \to \mathbb{R}^n$ and $\hat{f} : \mathbf{C}_{r_0} \to \mathbb{R}^n$ by setting

$$\begin{aligned} F(x) &= x + \ell(x)\, e_n\,, && x \in \mathbf{D}_{r_0}\,, \\ \hat{f}(x) &= F(\mathbf{p}x) + x_n e_n\,, && x \in \mathbf{C}_{r_0}\,. \\ f(x) &= F(\mathbf{p}x) - x_n\, \nu_\Omega(F(\mathbf{p}x))\,, && x \in \mathbf{C}_{r_0}\,. \end{aligned}$$

In this way, for every $y \in (\partial\Omega) \cap \mathbf{C}_{r_0}$, if we set $y = F(\mathbf{p}x)$, then

$$\nu_\Omega(y) = \frac{\nabla\ell(x) - e_n}{\sqrt{1 + |\nabla\ell(x)|^2}}\,, \qquad \mathrm{H}_\Omega(y) = \mathrm{div}\left(\frac{\nabla\ell}{\sqrt{1 + |\nabla\ell|^2}}\right)(x)\,.$$

Notice that the map f need not be of class C^1 if the boundary of Ω is only of class C^1, while the map $\hat{f}$ will be as regular as the boundary of Ω. The following lemma summarizes basic properties about the map $\hat{f}$.

Lemma 4.2.2 (Near-boundary coordinates, one)**.** *If $H = \{x_n > 0\}$, Ω is an open set with C^1-boundary and (4.29) holds, then there exist r_0 and C_0 positive such that the map $\hat{f}$ in (4.31) defines a C^1-diffeomorphism from $\mathbf{C}_{r_0}$ to its image, taking $\mathbf{D}_{r_0}$ into $\partial\Omega$ and with*

$$\hat{f}(\mathbf{C}_{r/C_0} \cap H) \subset \Omega \cap B_r \subset \hat{f}(\mathbf{C}_{C_0\, r} \cap H) \qquad \forall r < r_0/C_0\,.$$

Moreover, letting $\hat{g} = \hat{f}^{-1}$ *denote the inverse of* $\hat{f}$*, we have*

$$\nabla \hat{f} = \mathrm{Id}_{\mathbb{R}^n} + \mathrm{o}(1)\,, \qquad (\nabla \hat{g})^* \circ \hat{f} = \mathrm{Id}_{\mathbb{R}^n} + \mathrm{o}(1)\,,$$
$$J\hat{f} = 1 + \mathrm{o}(1)\,, \qquad 1 \le J^{\partial H}\hat{f} \le 1 + \mathrm{o}(1)\,, \qquad \text{for } x \in \mathbf{C}_{r_0}.$$

The orders in (4.34) *and* (4.35) *depend on* $\partial\Omega$ *and on* $0 \in \partial\Omega$*.*

Proof. See appendix B.2. □

The map f defined in (4.32) has the advantage that, when the boundary of Ω is at least of class C^2, curvature quantities appear in expansions of the metric coefficients and the volume form in these coordinates. These properties are the content of the following lemma.

Lemma 4.2.3 (Near-boundary coordinates, two)**.** *If* $H = \{x_n > 0\}$*,* Ω *is an open set with* C^2*-boundary and* (4.29) *holds, then there exist* r_0 *and* C_0 *positive such that the map* f *in* (4.32) *defines a* C^1*-diffeomorphism from* $\mathbf{C}_{r_0}$ *to its image, taking* $\mathbf{D}_{r_0}$ *into* $\partial\Omega$ *and with*

$$f(\mathbf{C}_{r/C_0} \cap H) \subset \Omega \cap B_r \subset f(\mathbf{C}_{C_0\,r} \cap H) \qquad \forall r < r_0/C_0\,.$$

Moreover, for $x \in \mathbf{C}_{r_0}$ *and* $x \in \mathbf{D}_{r_0}$ *respectively, we have*

$$Jf(x) = 1 - x_n\,\mathrm{H}_\Omega(0) + \mathrm{O}(|x|^2)\,, \qquad 1 \le J^{\partial H} f(x) \le 1 + \mathrm{O}(|x|^2)\,,$$

and if $\{e_i\}_{i=1}^{n-1}$ *is an orthonormal basis of* $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ *of eigenvectors of* $\nabla^2\ell(0)$ *and* $\{\kappa_i\}_{i=1}^{n-1}$ *denote the corresponding eigenvalues (so that, by* (4.29)*, they are the principal curvatures of* $\partial\Omega$ *with respect to* ν_Ω *computed at* $0 \in \partial\Omega$*, and in* *particular* $\mathrm{H}_\Omega(0) = \sum_{i=1}^{n-1} \kappa_i$*), then, letting* $g = f^{-1}$ *denote the inverse of* f*, we have*

$$(\nabla g)^* \circ f = \mathrm{Id}_{\mathbb{R}^n} + \big(\nabla \ell \otimes e_n - e_n \otimes \nabla \ell\big) + x_n \sum_{i=1}^{n-1} \kappa_i e_i \otimes e_i + \mathrm{O}(|x|^2)\,.$$

The orders in (4.37) *and* (4.38) *depend on* $\partial\Omega$ *and on* $0 \in \partial\Omega$*.*

Proof. See appendix B.2. □

Remark 4.2.4. Given $x_0 \in \partial\Omega$, we denote by π_{x_0} the rigid motion of $\mathbb{R}^n$ that maps x_0 to 0 such that (4.29) holds with $\pi_{x_0}(\Omega)$ in place of Ω. Then we set, for $\hat{f}$ and f defined as in (4.31) and (4.32) respectively but with $\pi_{x_0}(\Omega)$ in place of Ω,

$$\hat{f}_{x_0} = \pi_{x_0}^{-1} \circ \hat{f}\,, \qquad f_{x_0} = \pi_{x_0}^{-1} \circ f\,.$$

Clearly these maps are diffeomorphisms on $\mathbf{C}_{r_0}$, mapping $H \cap \mathbf{C}_{r_0}$ into a neighborhood of x_0 in Ω and $\mathbf{D}_{r_0} = (\partial H) \cap \mathbf{C}_{r_0}$ into a neighborhood of x_0 in $\partial\Omega$, and satisfies proper reformulations of the estimates in Lemmas 4.2.2 and 4.2.3. Here r_0 and C_0 depend also on the choice of x_0, and can of course be assumed uniform across $x_0 \in \partial\Omega$ if $\partial\Omega$ is bounded.

4.2.4 Properties of minimizers

The following lemma gathers some fundamental properties of minimizers of Φ_Ω that will be needed in the sequel.

Lemma 4.2.5. *If* $n \geq 2$*,* $p \in (1, n)$*,* $T > 0$*,* Ω *is a bounded open set with* C^2*-boundary, and* u *is a minimizer of* $\Phi_\Omega(T)$*, then* u *is bounded and Lipschitz* *continuous in Ω, there are λ and σ such that the Euler–Lagrange equation (4.1) holds in the weak sense, and the balance condition*

$$\lambda \int_{\Omega} u^{p^\star-1} + \sigma \int_{\partial\Omega} u^{p^\#-1} = 0\,, \tag{4.39}$$

holds.

Proof. By a standard argument, based on similar considerations to the one presented in Lemma 4.2.6 below, one sees that a minimizer u of $\Phi_\Omega(T)$ is a $W^{1,p}(\Omega)$-distributional solution of the Euler-Lagrange equation (4.1) for some $\lambda, \sigma \in \mathbb{R}$. As soon as Ω is bounded and has Lipschitz boundary, one can exploit (4.1) in conjunction with a Moser iteration argument to prove that $u \in L^\infty(\Omega)$ (see, e.g. [MW19, Theorem 3.1]; their result applies to (4.1) by taking, in the notation of their paper, $\mathcal{A}(x,u,\nabla u) = |\nabla u|^{p-2}\nabla u$, $\mathcal{B}(x,u,\nabla u) = \lambda u^{p^\star-1}$, and $\mathcal{C}(x,u) = \sigma u^{p^\#-1}$). On further assuming that $\partial\Omega$ is of class C^2, then the classical result [Lie92, Theorem 1.7] can be applied to deduce that $u \in C^{1,\beta}(\overline{\Omega})$ for a suitable $\beta = \beta(n,p) \in (0,1)$ (for more details, see [MW19, Theorem 3.9]). In particular, u is bounded and Lipschitz continuous on Ω, as claimed. $\square$

4.2.5 Volume/trace correcting variations

At various stages in our arguments we will need to slightly modify certain competitors so to restore the volume and trace constraints defining $\mathcal{X}_\Omega(T)$. The following lemma describes the basic mechanism used to this end.

Lemma 4.2.6 (Volume/trace correcting variations)**.** *If $n \geq 2$, $p \in (1,n)$, $M > 0$, Ω is an open set with C^1-boundary, $v \in L^1_{\mathrm{loc}}(\Omega)$ with $\nabla v \in L^p(\Omega;\mathbb{R}^n)$,* *and if* $x_0 \in \mathbb{R}^n$ *and* $r > 0$ *are such that*

$$\int_{\Omega \setminus B_r(x_0)} v^{p^\star} \text{ and } \int_{(\partial\Omega)\setminus B_r(x_0)} v^{p^\#} \text{ are positive and finite}, \tag{4.40}$$

then there exist positive constants η *and* C*, and functions* $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \overline{B}_r(x_0))$ *and* $\psi \in C_c^\infty(\Omega \setminus \overline{B}_r(x_0))$*, all depending on* n*,* p*,* v *and* M *only, and with the following property.*

If $\{v_\varepsilon\}_{\varepsilon<\varepsilon_0} \subset L^1_{\rm loc}(\Omega)$ *is such that, for every* $\varepsilon < \varepsilon_0$*,*

$$v_\varepsilon = v \text{ on } \Omega \setminus B_r(x_0)\,, \qquad \int_\Omega |\nabla v_\varepsilon|^p \le M\,, \tag{4.41}$$

then for every (a, b) *with* $|a|, |b| < \eta/C$*, we can find* (s, t) *with* $|s|, |t| < \eta$ *such that*

$$w_\varepsilon = v_\varepsilon + s\,\varphi + t\,\psi$$

satisfies

$$\int_{\partial\Omega} |w_\varepsilon|^{p^\#} = a + \int_{\partial\Omega} |v_\varepsilon|^{p^\#}\,, \qquad \int_\Omega |w_\varepsilon|^{p^\star} = b + \int_\Omega |v_\varepsilon|^{p^\star}\,, \tag{4.42}$$

$$\left| \int_\Omega |\nabla w_\varepsilon|^p - \int_\Omega |\nabla v_\varepsilon|^p \right| \le C\left(|a| + |b|\right).$$

Proof. By (4.40) there are $\xi \in C_c^\infty(\mathbb{R}^n \setminus \overline{B}_r(x_0))$ and $\psi \in C_c^\infty(\Omega \setminus \overline{B}_r(x_0))$ such that

$$\int_{\partial\Omega} v^{p^\#-1}\,\xi = 1\,, \qquad \int_\Omega v^{p^\star-1}\,\psi = 1\,. \tag{4.43}$$

Setting $\varphi = \xi - (\int_\Omega v^{p^\star-1}\xi)\,\psi$, we have $\varphi \in C_c^\infty(\mathbb{R}^n \setminus \overline{B}_r(x_0))$ with

$$\int_\Omega v^{p^\star-1}\,\varphi = 0\,, \qquad \int_{\partial\Omega} v^{p^\#-1}\,\varphi = 1\,. \tag{4.44}$$

We now define $h_\varepsilon:\mathbb{R}^2\to\mathbb{R}^2$ by

$$h_\varepsilon(s,t)=\Big(\int_{\partial\Omega}|v_\varepsilon+s\,\varphi+t\,\psi|^{p^\#}-\int_{\partial\Omega}|v_\varepsilon|^{p^\#},\ \int_\Omega|v_\varepsilon+s\,\varphi+t\,\psi|^{p^\star}-\int_\Omega|v_\varepsilon|^{p^\star}\Big)\,.$$

By (4.41) we have $h_\varepsilon\in C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)$ for some $\alpha=\alpha(n,p)\in(0,1)$, with

$$\sup_{\varepsilon<\varepsilon_0}\|h_\varepsilon\|_{C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)}<\infty\,;$$

moreover, $h_\varepsilon(0,0)=0$ and, by (4.43) and (4.44),

$$\nabla h_\varepsilon(0,0)=\left(\begin{array}{cc} p^\#\int_{\partial\Omega}|v_\varepsilon|^{p^\#-1}\varphi & p^\#\int_{\partial\Omega}|v_\varepsilon|^{p^\#-1}\psi \\ p^\star\int_\Omega|v_\varepsilon|^{p^\star-1}\varphi & p^\star\int_\Omega|v_\varepsilon|^{p^\star-1}\psi\end{array}\right)=\left(\begin{array}{cc} p^\# & 0\\ 0 & p^\star\end{array}\right).$$

We can thus apply the inverse function theorem *uniformly* in ε, to find positive constants η and C_1 depending on n, p, and v so that each h_ε is invertible on $E=\{(s,t):|s|,|t|<\eta\}$, with $\{(a,b):|a|,|b|<\eta/C_1\}\subset h_\varepsilon(E)$, and $\nabla h_\varepsilon^{-1}(a,b)\ge \mathrm{Id}_{2\times2}/C_1$ (in the sense of positive definite matrices) for every $(a,b)\in h_\varepsilon(E)$. In particular, if we let $(s,t)=h_\varepsilon^{-1}(a,b)$ for a pair (a,b) with $|a|,|b|<\eta/C_1$, then the function $w_\varepsilon=v_\varepsilon+s\,\varphi+t\,\psi$ satisfies (4.42) and $|(a,b)|=|h_\varepsilon^{-1}(s,t)|\ge|(s,t)|/C_1$. Moreover, by the elementary inequality

$$\big||X+Y|^p-|X|^p\big|\le p\,\max\big\{|X|,|Y|\big\}^{p-1}|Y|\qquad\forall X,Y\in\mathbb{R}^n\,,$$

we see that, setting $\gamma=\max\big\{|\nabla v_\varepsilon|,|\nabla\varphi|,|\nabla\psi|\big\}^p$,

$$\begin{aligned}\Big|\int_\Omega|\nabla w_\varepsilon|^p-\int_\Omega|\nabla v_\varepsilon|^p\Big|&\le p\int_\Omega\gamma^{(p-1)/p}\big(|s|\,|\nabla\varphi|+|t|\,|\nabla\psi|\big)\\&\le C\Big(\int_\Omega\gamma\Big)^{(p-1)/p}\Big(\int_\Omega\big(|s|^p\,|\nabla\varphi|^p+|t|^p\,|\nabla\psi|^p\big)\Big)^{1/p}\\&\le C\,|(s,t)|\le C_2\,|(a,b)|\,,\end{aligned}$$

for a constant C_2 depending on n, p, v, and M. Letting $C=\max\{C_1,C_2\}$ concludes the proof of the lemma. $\square$

4.3 Boundary concentrations

4.3.1 Properties of Φ_H-minimizers

We recall some facts proved in [MN17] about Φ_H and its minimizers. Recall that we denote by T_0 the minimum point of Φ_H, so that

$$T_0 \in (0,T_E)\,, \qquad \Phi_H(T_0) = 2^{-1/n}\,S(n,p)\,,$$

where T_E is the "Escobar trace" defined in (4.6). If we set $H = \{x_n > 0\}$, the minimizers of U_T of $\Phi_H(T)$ for $T > 0$ are characterized (modulo the obvious scaling and translation invariance of Φ_H) as

$$U_T(x) = c_T \begin{cases} \tau_{t_T\,e_n} U_S(x) = (1 + |x - t_T\,e_n|^{p'})^{1-(n/p)}\,, & T \in (0,T_E)\,, \\ \tau_{e_n} U_E(x) = |x + e_n|^{(p-n)/(p-1)}\,, & T = T_E\,, \\ \tau_{s_T\,e_n} U_{BE}(x) = (|x - s_T\,e_n|^{p'} - 1)^{1-(n/p)}\,, & T > T_E\,, \end{cases}$$

where the constants c_T, t_T and s_T are chosen in such a way that

$$\int_H U_T^{p^\star} = 1\,, \qquad \int_{\partial H} U_T^{p^\sharp} = T^{p^\sharp}\,, \qquad \int_H |\nabla U_T|^p = \Phi_H(T)^p\,.$$

It is convenient to keep in mind that the various formulas for U_T listed in (4.45) all share the same decay behavior at infinity, that is (see (4.60) below), we have

$$U_T(x) \sim |x|^{(p-n)/(p-1)}, \qquad |\nabla U_T| \sim |x|^{(1-n)/(p-1)} \qquad \text{as } |x| \to \infty. \tag{4.46}$$

(where the rate depends on the specific value of T under consideration). The constants t_T and s_T have the following properties: $T \in (0,T_E) \mapsto t_T$ is continuous and strictly decreasing, with $t_T > 0$ if and only if $T \in (0,T_0)$, and

$$\lim_{T \to 0^+} t_T = +\infty\,, \qquad \lim_{t \to (T_E)^-} t_T = -\infty\,, \qquad t_{T_0} = 0\,, \tag{4.47}$$

while $T \in (T_E,\infty) \mapsto s_T$ is continuous, negative, strictly increasing, with

$$\lim_{T\to(T_E)^+} s_T = -\infty\,, \qquad \lim_{T\to+\infty} s_T = -1\,. \tag{4.48}$$

Denoting by $\Delta_p v = \mathrm{div}\,(|\nabla v|^{p-2}\nabla v)$ the p-Laplace operator, we have

$$\begin{cases} -\Delta_p U_T = \lambda_H(T)\, U_T^{p^\star-1} & \text{on } H\,,\\ |\nabla U_T|^{p-2}\,\dfrac{\partial U_T}{\partial \nu_H} = \sigma_H(T)\, U_T^{p^\sharp-1} & \text{on } \partial H\,, \end{cases} \qquad \forall T>0\,, \tag{4.49}$$

where $\lambda_H, \sigma_H : (0,\infty) \to \mathbb{R}$ are continuous and satisfy the relations

$$\Phi_H(T)^p = \lambda_H(T) + \sigma_H(T)\, T^{p^\#}\,, \qquad \sigma_H(T) = \frac{\Phi_H(T)^{p-1}\,\Phi_H'(T)}{T^{p^\#-1}}\,, \tag{4.50}$$

(see [MN17, Lemma 3.3][1]) as well as

$$\lim_{T\to0^+} \sigma_H(T) = -\infty\,, \qquad \lim_{T\to+\infty} \sigma_H(T) = +\infty$$
$$\lim_{T\to0^+} \lambda_H(T) > 0\,, \qquad \lim_{T\to+\infty} \lambda_H(T) = -\infty\,.$$

The signs of σ_H and λ_H can be easily deduced from (4.45), and satisfy

$$(0,T_0) = \{\sigma_H < 0\}\,, \qquad (T_0,\infty) = \{\sigma_H > 0\}\,, \qquad \sigma_H(T_0) = 0\,,$$
$$(0,T_E) = \{\lambda_H > 0\}\,, \qquad (T_E,\infty) = \{\lambda_H < 0\}\,, \qquad \lambda_H(T_E) = 0\,.$$

[1] Notice that in [MN17, (3.16)] it is incorrectly stated that $\sigma_H(T) = \Phi_H(T)^{p-1}\,\Phi_H'(T)/(p^\#\,T^{p^\#-1})$, where the extra $1/p^\#$-factor is wrongly introduced in the penultimate displayed equation in the proof of Lemma 3.3, where $\tau'(0) = T^{1-p^\#}/p^\#$ should be replaced by $\tau'(0) = T^{1-p^\#}$. This error is inconsequential for the arguments in [MN17], since this expression for $\sigma_H(T)$ is only used in equation (3.22) and subsequent displayed equations, and since, in all these subsequent identities, a generic multiplicative factor $c(n,p)$ is used (in particular, two functions of (n,p) differing by a $1/p^\#$-factor are both $c(n,p)$). It will instead be important in the proof of (4.90) to work with correct expression for $\sigma_H(T)$.

4.3.2 A key inequality and further properties of U_T

In this section, we prove the key inequality (4.55) for the functions $\mathcal{L}$ and $\mathcal{M}$ introduced in (4.21) and (4.22), namely

$$\mathcal{L}(U) = \int_H x_n \, |\nabla U|^p - p \, x_n \, (\partial_1 U)^2 \, |\nabla U|^{p-2} \, ,$$
$$\mathcal{M}(U) = \int_H x_n \, U^{p^\star} \, .$$

Whenever U satisfies the decay properties (4.46) (e.g., when U is a compactly supported perturbation of some U_T), we have that $\mathcal{M}(U) < \infty$; however, $\mathcal{L}(U) < \infty$ under (4.46) if and only if $n > 2\,p - 1$; see (4.64) and (4.65) below.

Lemma 4.3.1 (Key inequality)**.** *If $n \geq 2$, $p \in (1, n)$, $n > 2p - 1$, and $T > 0$, then there is a positive constant $c(n, p, T)$ such that*

$$\mathcal{L}(U_T) - \frac{n-p}{n} \, \lambda_H(T) \, \mathcal{M}(U_T) \geq c(n, p, T) \, .$$

The following lemma will be useful in proving Lemma 4.3.1.

Lemma 4.3.2. *If $H = \{x_n > 0\}$, $U : H \to \mathbb{R}$ is radially symmetric with respect to $t\, e_n$ for some $t \in \mathbb{R}$, and $\int_H x_n |\nabla U|^p$ is finite, then*

$$\int_H x_n \, |\nabla U|^{p-2} \big\{ (\partial_n U)^2 - (\partial_1 U)^2 \big\} > 0 \, .$$

Proof of Lemma 4.3.2. We have $U(x) = \eta(|x - t\, e_n|)$, $y = x - t\, e_n$, $r = |y|$, and $\hat{y} = y/|y|$, so that

$$x_n \, |\nabla U|^{p-2} \big\{ (\partial_n U)^2 - (\partial_1 U)^2 \big\} = (y_n + t) \, |\eta'(r)|^p \big\{ (\hat{y}_n)^2 - (\hat{y}_1)^2 \big\} \, ,$$

and, setting $y = r\,z$,

$$\int_H x_n\,|\nabla U|^{p-2}\big\{(\partial_n U)^2-(\partial_1 U)^2\big\} = \int_{\{y_n>-t\}}(y_n+t)\,|\eta'(r)|^p\big\{(\hat y_n)^2-(\hat y_1)^2\big\}$$

$$= \int_0^\infty |\eta'(r)|^p\,dr \int_{\{y_n>-t\}\cap\partial B_r}(y_n+t)\,\big\{(\hat y_n)^2-(\hat y_1)^2\big\}\,d\mathcal{H}^{n-1}_y$$

$$= \int_0^\infty r^n\,|\eta'(r)|^p\,dr \int_{\{z_n>-t/r\}\cap\partial B_1}\Big(z_n+\frac{t}{r}\Big)\,\big\{z_n^2-z_1^2\big\}\,d\mathcal{H}^{n-1}_z\,.$$

We conclude the proof by showing that

$$\int_{\{z_n>-s\}\cap\partial B_1}\big(z_n+s\big)\,\big\{z_n^2-z_1^2\big\}\,d\mathcal{H}^{n-1}_z>0\qquad\text{for all } s\in(-1,1), \tag{4.56}$$

noting that for each $s\in[1,\infty)$, this integral vanishes by symmetry while for $s\in(-\infty,-1]$ the domain of integration is empty. To see (4.56), let $\mathbf{p}:\mathbb{R}^n\to\mathbb{R}^{n-2}$ denote the projection map $\mathbf{p}(x)=(x_2,...,x_{n-1})$ (if $n=2$ there is no need to introduce $\mathbf{p}$). The tangential coarea factor of $\mathbf{p}$ along ∂B_1 defines a positive function $K:\partial B_1\to(0,\infty]$ which is $\mathcal{H}^{n-1}$-a.e. finite on ∂B_1, and which is *independent* of the variables (x_1,x_n), i.e. $K(x_1,w,x_n)=K(w)$ for every $(x_1,w,x_n)\in\partial B_1$. Therefore, setting for brevity $M_s=\{z_n>-s\}\cap\partial B_1$,

$$\int_{M_s}\big(z_n{+}s\big)\,\big\{z_n^2{-}z_1^2\big\}\,d\mathcal{H}^{n-1}_z=\int_{\mathbf{p}(M_s)}\frac{d\mathcal{L}^{n-2}_w}{K(w)}\int_{M_s\cap\mathbf{p}^{-1}(w)}\big(z_n{+}s\big)\,\big\{z_n^2{-}z_1^2\big\}\,d\mathcal{H}^1_{(z_1,z_n)}\,,$$

where

$$M_s\cap\mathbf{p}^{-1}(w)=\Big\{(z_1,z_n):z_1^2+z_n^2=1-|w|^2\,,z_n>-s\Big\}\,,\qquad s\in(-1,1)\,.$$

As before, the inner integral above vanishes when $1-|w|^2\le s$ by symmetry and the domain of integration is empty when $-s\ge 1-|w|^2$. We are thus left to prove that

$$\int_{-\alpha}^{\pi+\alpha}\big(\sin\theta+\sin\alpha\big)\,\big\{\sin^2\theta-\cos^2\theta\big\}\,d\theta>0$$

for $\alpha \in (0, \pi/2)$ (corresponding to the case when $s \geq 0$) and

$$\int_{\alpha}^{\pi-\alpha} \big(\sin\theta - \sin\alpha\big)\,\big\{\sin^2\theta - \cos^2\theta\big\}\,d\theta > 0$$

for $\alpha \in (0, \pi/2)$ (corresponding to the case when $s < 0$). Direct computation shows that both of these integrals are equal to the positive quantity $(2/3)\,(\cos\alpha)^3$. $\square$

We now prove Lemma 4.3.1.

Proof of Lemma 4.3.1. Testing (4.49) with $x_n\,U_T$ we find

$$\lambda_H(T)\int_H x_n\,U_T^{p^\star} = -\int_H x_n\,U_T\,\Delta_p U_T = \int_H |\nabla U_T|^{p-2}\,\nabla U_T \cdot \nabla(x_n\,U_T)\,.$$

Here the integration by parts is justified since $x_n = 0$ on ∂H and since by (4.46),

$$\Big|\int_{H\cap\partial B_R} x_n\,U_T\,|\nabla U_T|^{p-2}(\nu_{B_R}\cdot\nabla U_T)\Big| \leq C\,\frac{R^{n-1}\,R}{R^{(n-p)/(p-1)}}\Big(\frac{1}{R^{(n-1)/(p-1)}}\Big)^{p-1} \to 0$$

like $R^{-(n+1)/(p-1)}$ as $R \to \infty$. We thus find that

$$\begin{aligned}\mathcal{L}(U_T) - \frac{n-p}{n}\,\lambda_H(T)\,\mathcal{M}(U_T) = &\int_H x_n\,|\nabla U_T|^p - p\,x_n\,(\partial_1 U_T)^2\,|\nabla U_T|^{p-2}\\ &-\frac{n-p}{n}\Big\{\int_H x_n\,|\nabla U_T|^p + \int_H U_T\,|\nabla U_T|^{p-2}\,\partial_n U_T\Big\}\,.\end{aligned}$$

Now, since $|\nabla U_T|$ is symmetric by reflection with respect to the hyperplanes $\{x_i = 0\}$, $i = 1, \ldots, n-1$, we see that

$$\begin{aligned}\int_H x_n\,|\nabla U_T|^p &= \sum_{i=1}^{n-1}\int_H x_n\,(\partial_i U_T)^2|\nabla U_T|^{p-2} + \int_H x_n\,(\partial_n U_T)^2|\nabla U_T|^{p-2}\\ &= (n-1)\int_H x_n\,(\partial_1 U_T)^2|\nabla U_T|^{p-2} + \int_H x_n\,(\partial_n U_T)^2|\nabla U_T|^{p-2}\end{aligned}$$

so that continuing from above we have

$$
\begin{aligned}
&\mathcal{L}(U_T) - \frac{n-p}{n}\,\lambda_H(T)\,\mathcal{M}(U_T)\\
&= \frac{p}{n}\int_H x_n\,|\nabla U_T|^p - p\int_H x_n\,(\partial_1 U_T)^2\,|\nabla U_T|^{p-2} - \frac{n-p}{n}\int_H U_T\,|\nabla U_T|^{p-2}\,\partial_n U_T\\
&= \frac{p}{n}\int_H x_n\,|\nabla U_T|^{p-2}\Big\{(\partial_n U_T)^2 - (\partial_1 U_T)^2\Big\} + \frac{n-p}{n}\int_H U_T\,|\nabla U_T|^{p-2}\,\big(-\partial_n U_T\big)\,.
\end{aligned}
$$

In particular, the lemma is proved by showing that

$$
\begin{aligned}
&\int_H x_n\,|\nabla U_T|^{p-2}\big\{(\partial_n U_T)^2 - (\partial_1 U_T)^2\big\} > 0\,,\\
&\int_H U_T\,|\nabla U_T|^{p-2}\,\big(-\partial_n U_T\big) > 0\,,
\end{aligned}
$$

where the first inequality, (4.57), is immediate from Lemma 4.3.2 (recall that $n > 2p-1$ and U_T is radially symmetric with respect to te_n for some $t \in \mathbb{R}$).

We are thus left to prove (4.58). This is immediate in the case when $T \ge T_0$, because in that case, by (4.47) and (4.48), U_T has center of symmetry at $t\,e_n$ for some $t \le 0$, and thus $\partial_n U_T < 0$ on H. By (4.47), if $T \in (0, T_0)$, then U_T has center of symmetry at $t\,e_n$ for some $t > 0$. Correspondingly, $U_T\,\partial_n U_T$ is odd with respect to $\{x_n = t\}$, with $U_T\,\partial_n U_T < 0$ on $\{x_n > t\}$ and $U_T\,\partial_n U_T > 0$ on $\{0 < x_n < t\}$: in particular, if p_t denotes the reflection with respect to $\{x_n = t\}$, then

$$
\begin{aligned}
&\int_{2t > x_n > t} x_n\,U_T\,(-\partial_n U_T) = \int_{t > x_n > 0} (p_t(x)\cdot e_n)\ [U_T\,(-\partial_n U_T)](p_t(x))\,dx\\
&= \int_{t > x_n > 0} (p_t(x)\cdot e_n)\ [U_T\,\partial_n U_T)](x)\,dx \ge \int_{t > x_n > 0} x_n\ U_T\,\partial_n U_T\,,
\end{aligned}
$$

so that

$$
\int_H U_T\,|\nabla U_T|^{p-2}\,\big(-\partial_n U_T\big) \ge \int_{\{x_n > 2t\}} U_T\,|\nabla U_T|^{p-2}\,\big(-\partial_n U_T\big)\,,
$$

and the latter integral is positive because $\partial_n U_T < 0$ on $\{x_n > t\}$. $\square$

4.3.3 Standard variations of Φ_H-minimizers

We now introduce a "class of standard variations" of minimizers of Φ_H. With $H = \{x_n > 0\}$, we define $\zeta = \zeta(r,T) : [0,\infty) \times (0,\infty) \to [0,\infty)$, so that, setting $V_T(x) = \zeta(|x - e_n|, T)$ for $x \in \mathbb{R}^n$, we have

$$V_T \in C_c^\infty(H;[0,\infty)) , \qquad \int_H U_T^{p^\star - 1} V_T = 1 , \qquad \int_{\partial H} U_T^{p^\# - 1} V_T = 0 .$$

Given $T > 0$ we denote by

$$\mathcal{U}_T$$

the family of functions $U : H \to \mathbb{R}$ of the form

$$U = U_T + t\, V_T , \qquad |t| \le 1 .$$

The following lemma contains some basic properties of functions in $\mathcal{U}_T$. We notice that

$$\text{every } U \in \mathcal{U}_T \text{ is symmetric by reflection} \\ \text{with respect to the coordinates } x_1, \ldots, x_{n-1} . \tag{4.59}$$

Lemma 4.3.3 (Standard variations of U_T). *If $n \ge 2$, $p \in (1,n)$, and $T > 0$, then there are positive constants R_0 and C_0 depending on n, p, T, and V_T such that the following properties hold:*

(i): *if $U \in \mathcal{U}_T$, then for every $|x| > R_0$ we have*

$$\frac{1}{C_0 |x|^{(n-p)/(p-1)}} \le U(x) \le \frac{C_0}{|x|^{(n-p)/(p-1)}} ,$$
$$\frac{1}{C_0 |x|^{(n-1)/(p-1)}} \le |\nabla U(x)| \le \frac{C_0}{|x|^{(n-1)/(p-1)}} ,$$

and for every $R > R_0$,

$$\int_{H\setminus B_R} U^{p^\star} \le \frac{C_0}{R^{n/(p-1)}}\,, \qquad \int_{H\setminus B_R} U^{p^\star-1} \le \frac{C_0}{R^{p/(p-1)}}\,,$$

$$\int_{H\cap(B_{2R}\setminus B_R)} U^p \le \frac{C_0}{R^{(n-p^2)/(p-1)}}\,, \quad \int_{H\setminus B_R} |\nabla U|^p \le \frac{C_0}{R^{(n-p)/(p-1)}}\,,$$

$$\int_{(\partial H)\setminus B_R} U^{p^\#} \le \frac{C_0}{R^{(n-1)/(p-1)}}\,, \qquad \int_{(\partial H)\setminus B_R} U^{p^\#-1} \le \frac{C_0}{R}\,,$$

$$\int_{H\setminus B_R} |x|\, U^{p^\star} \le \frac{C_0}{R^{[1+n-p]/(p-1)}}\,,$$

$$\int_{H\setminus B_R} |x|\, |\nabla U|^p \le \frac{C_0}{R^{(n+1-2p)/(p-1)}}\,, \qquad \text{if } n > 2\,p-1\,.$$

(ii): *for every $U \in \mathfrak{U}_T$ we have*

$$\int_H |\nabla U|^p = \Phi_H(T)^p + p\,\lambda_H(T)\,t + \mathrm{o}(t)\,,$$

$$\int_H U^{p^\star} = 1 + p^\star\,t + \mathrm{o}(t)\,,$$

$$\int_{\partial H} U^{p^\#} = T^{p^\#}\,.$$

Proof of Lemma 4.3.3. Since V_T is assumed to be compactly supported, statement (i) follows immediately from the corresponding properties for U_T. More specifically, we deduce (4.60) from (4.45), and (4.61)–(4.65) from (4.60). Statement (ii) follows from (4.49). □

4.3.4 The *Ansatz* for boundary concentrations

We next use the standard variations of minimizers of Φ_H described in Lemma 4.3.3 to define certain competitors for Φ_Ω that provide us with a notion of "standard boundary concentration." Recall the notation $U^{(\varepsilon)}$ for dilations introduced in (4.4).

Lemma 4.3.4. *Fix $n \geq 2$, $p \in (1,n)$, $T > 0$, $U \in \mathcal{U}_T$. Let Ω be an open set with C^1-boundary, $x_0 \in \partial\Omega$, and let $\hat{f} = \hat{f}_{x_0}$, $\hat{g} = \hat{f}^{-1}$, $f = f_{x_0}$ and $g = f^{-1}$ be determined as in Remark 4.2.4 starting from Ω and x_0. Then the following statements hold:*

(i): *If $v \in W^{1,p}(\Omega)$, $\beta \in (0,1)$, $r_1 = \varepsilon^\beta$, $r_2 = 2\,\varepsilon^\beta$, and φ_ε is a cut-off function between $B_{r_1}(x_0)$ and $B_{r_2}(x_0)$ with $|\nabla\varphi_\varepsilon| \leq C\varepsilon^{-\beta}$, then*

$$v_\varepsilon(x) = (1 - \varphi_\varepsilon(x))\, v(x) + \varphi_\varepsilon(x)\, \big(U^{(\varepsilon)} \circ \hat{g}\big)(x)\,, \qquad x \in \Omega\,,$$

satisfies

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_\Omega |\nabla v_\varepsilon|^p &= \int_H |\nabla U|^p + \int_\Omega |\nabla v|^p\,,\\
\lim_{\varepsilon\to 0^+} \int_\Omega v_\varepsilon^{p^\star} &= \int_H U^{p^\star} + \int_\Omega v^{p^\star}\,,\\
\lim_{\varepsilon\to 0^+} \int_{\partial\Omega} v_\varepsilon^{p^\#} &= \int_{\partial H} U^{p^\#} + \int_{\partial\Omega} v^{p^\#}\,.
\end{aligned}$$

(ii): *If $n > 2p$, $v \in \mathrm{Lip}(\Omega)$ and Ω has C^2-boundary, then there exists a choice of $\beta = \beta(n,p) \in (0,1)$ (used in the definition of $r_1 = \varepsilon^\beta$ and $r_2 = 2\,\varepsilon^\beta$), such that the function*

$$v_\varepsilon(x) = (1 - \varphi_\varepsilon(x))\, v(x) + \varphi_\varepsilon(x)\, \big(U^{(\varepsilon)} \circ g\big)(x)\,, \qquad x \in \Omega\,,$$

satisfies (4.70), (4.71), and (4.72) in the more precise form

$$\int_\Omega |\nabla v_\varepsilon|^p = \int_H |\nabla U|^p + \int_\Omega |\nabla v|^p - \mathrm{H}_{\partial\Omega}(x_0)\, \mathcal{L}(U)\, \varepsilon + \mathrm{o}(\varepsilon)\,, \tag{4.74}$$

$$\int_\Omega v_\varepsilon^{p^\star} = \int_H U^{p^\star} + \int_\Omega v^{p^\star} - \mathrm{H}_{\partial\Omega}(x_0)\, \mathcal{M}(U)\, \varepsilon + \mathrm{o}(\varepsilon)\,, \tag{4.75}$$

$$\int_{\partial\Omega} v_\varepsilon^{p^\#} = \int_{\partial H} U^{p^\#} + \int_{\partial\Omega} v^{p^\#} + \mathrm{o}(\varepsilon)\,, \tag{4.76}$$

as $\varepsilon \to 0^+$. Here $\mathcal{L}(U)$ and $\mathcal{M}(U)$ are defined in (4.21) *and* (4.22) *and the orders in* (4.74)*,* (4.75)*, and* (4.76) *depend on n, p, T, and v.*

Proof. Without loss of generality we assume that $x_0 = 0 \in \partial\Omega$, $T_0\,(\partial\Omega) = \{x_n = 0\}$ and $\nu_\Omega(0) = -e_n$. We carry out the proof in several steps.

Step one: We start by noticing the following estimates for the energy, volume and trace of v_ε in transition region for the cut-off function φ_ε. The estimates in this step hold in identical form with the same proofs for v_ε defined from $\hat{f}$ as in (4.69) and for v_ε defined from f as in (4.73); we write the proof for (4.73). First, with $v \in W^{1,p}(\Omega)$,

$$\lim_{\varepsilon\to 0} \max\left\{\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})} |\nabla v_\varepsilon|^p, \int_{\Omega\cap(B_{r_2}\setminus B_{r_1})} v_\varepsilon^{p^\star}, \int_{(\partial\Omega)\cap(B_{r_2}\setminus B_{r_1})} v_\varepsilon^{p^\#}\right\} = 0\,, \tag{4.77}$$

and, second, under the additional assumption that $v \in \mathrm{Lip}(\Omega)$,

$$\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})} |\nabla v_\varepsilon|^p \le C\,\max\left\{\varepsilon^{(1-\beta)\,(n-p)/(p-1)}, \varepsilon^{\beta(n-p)}\right\},$$
$$\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})} v_\varepsilon^{p^\star} \le C\,\max\left\{\varepsilon^{(1-\beta)\,n/(p-1)}, \varepsilon^{\beta\,n}\right\}$$
$$\int_{(\partial\Omega)\cap(B_{r_2}\setminus B_{r_1})} v_\varepsilon^{p^\#} \le C\,\max\left\{\varepsilon^{(1-\beta)\,(n-1)/(p-1)}, \varepsilon^{\beta(n-1)}\right\}.$$

Indeed, we have $\nabla v_\varepsilon = a_\varepsilon + b_\varepsilon$ for

$$a_\varepsilon = \varphi_\varepsilon\,(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g] + (U^{(\varepsilon)}\circ g)\,\nabla\varphi_\varepsilon\,,\qquad b_\varepsilon = (1-\varphi_\varepsilon)\,\nabla v - v\,\nabla\varphi_\varepsilon\,.$$

By (4.36), and thanks to $|\nabla g|, Jf \le 2$ on $\mathbf{C}_{r_0}$, we find

$$
\begin{aligned}
&\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|a_\varepsilon|^p \le C\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p+\frac{U^{(\varepsilon)}(g)^p}{\varepsilon^{\beta\,p}}\\
\le\ & C\int_{H\cap(B_{Cr_2}\setminus B_{r_1/C})}|\nabla U^{(\varepsilon)}|^p+\frac{(U^{(\varepsilon)})^p}{\varepsilon^{\beta\,p}}\\
=\ & C\int_{H\cap(B_{Cr_2/\varepsilon}\setminus B_{r_1/C\varepsilon})}|\nabla U|^p+\varepsilon^{np/p^\star}\,\frac{U^p}{\varepsilon^{\beta\,p}}\,\varepsilon^n\\
\le\ & C\left\{\varepsilon^{(1-\beta)(n-p)/(p-1)}+\frac{\varepsilon^p\,\varepsilon^{(\beta-1)(p^2-n)/(p-1)}}{\varepsilon^{\beta\,p}}\right\}=C\,\varepsilon^{(1-\beta)(n-p)/(p-1)}\,,
\end{aligned}
$$

where in the last inequality we have used (4.62). Concerning b_ε, we notice that if we only know that $v\in W^{1,p}(\Omega)$ then by $v\in L^{p^\star}(\Omega)$ and $\nabla v\in L^p(\Omega)$ we find that

$$
\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|b_\varepsilon|^p\le\int_{\Omega\cap B_{r_2}}|\nabla v|^p+\int_{\Omega\cap B_{r_2}}\frac{|v|^p}{\varepsilon^{\beta\,p}}\le\int_{\Omega\cap B_{r_2}}|\nabla v|^p+\Big(\int_{\Omega\cap B_{r_2}}|v|^{p^\star}\Big)^{p/p^\star}
$$

where the latter quantity converges to 0 at an non-quantified rate as $\varepsilon\to0^+$ (as stated in (4.77)); while, if $v\in\mathrm{Lip}(\Omega)$, then

$$
\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|b_\varepsilon|^p\le C\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|v|^p\,|\nabla\varphi_\varepsilon|^p+|\nabla v|^p\le C\,r_2^n\,\mathrm{Lip}(\varphi_\varepsilon)^p\le C\varepsilon^{\beta\,(n-p)}\,,
$$

and (4.78) is proved. The other two limits in (4.77) follow similarly (with non-quantified rates), while if $v\in\mathrm{Lip}(\Omega)$, then (4.79) and (4.80) follow from (4.61), (4.63), and

$$
\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}u_\varepsilon^{p^\star}\le C\varepsilon^{\beta n}+C\int_{H\cap(B_{Cr_2/\varepsilon}\setminus B_{r_1/C\varepsilon})}U^{p^\star}\le C\varepsilon^{\beta n}+C\,\varepsilon^{(1-\beta)n/(p-1)}
$$

$$
\int_{(\partial\Omega)\cap(B_{r_2}\setminus B_{r_1})}u_\varepsilon^{p^\#}\le C\varepsilon^{\beta(n-1)}+C\int_{(\partial H)\cap(B_{Cr_2/\varepsilon}\setminus B_{r_1/C\varepsilon})}U^{p^\#}\le C\varepsilon^{\beta(n-1)}+C\,\varepsilon^{(1-\beta)\frac{(n-1)}{(p-1)}}\,.
$$

Step two: We prove statement (i). By (4.77),

$$\int_\Omega |\nabla v_\varepsilon|^p = \int_{\Omega\cap B_{r_1}} |(\nabla\hat{g})^*[(\nabla U^{(\varepsilon)})\circ\hat{g}]|^p + \int_{\Omega\setminus B_{r_2}} |\nabla v|^p + \mathrm{o}(1)\,,$$

and, similarly,

$$\int_\Omega |\nabla v|^p \geq \int_{\Omega\setminus B_{r_2}} |\nabla v|^p \geq \int_\Omega |\nabla v|^p + \mathrm{o}(1)\,. \tag{4.81}$$

Moreover, if we set $E_\varepsilon = g(B_{r_1}) \subset H$ and $\tilde{E}_\varepsilon = E_\varepsilon/\varepsilon \subset H$, then keeping in mind (4.36), (4.34), (4.35), and (4.62), we have

$$\begin{aligned}\int_{\Omega\cap B_{r_1}} |(\nabla\hat{g})^*[(\nabla U^{(\varepsilon)})\circ\hat{g}]|^p &= \int_{E_\varepsilon} |((\nabla\hat{g})\circ\hat{f})^*[\nabla U^{(\varepsilon)}]|^p\, J\hat{f} = (1+\mathrm{o}(1))\int_{E_\varepsilon} |\nabla U^{(\varepsilon)}|^p \\ &= (1+\mathrm{o}(1))\Big\{\int_H |\nabla U|^p - \int_{H\setminus\tilde{E}_\varepsilon} |\nabla U|^p\Big\} = (1+\mathrm{o}(1))\int_H |\nabla U|^p\,.\end{aligned}$$

This proves (4.70). Entirely analogous arguments prove (4.71) and (4.72).

Step three: We now start the proof of statement (ii); in particular, from now on, Ω has C^2-boundary, $n > 2p$, and v_ε is defined as in (4.73); moreover, for the sake of brevity, we set $h = \mathrm{H}_{\partial\Omega}(0)$. In this step, we discuss the choice of $\beta = \beta(n,p) \in (0,1)$, which is determined by the rates in (4.78), (4.79) and (4.80), and by the fact that in (4.74), (4.75) and (4.76) we want errors of size $\mathrm{o}(\varepsilon)$: therefore, by

$$\begin{aligned}\beta\ \min\Big\{n, n-1, n-p\Big\} > 1 \qquad &\text{iff} \qquad \beta > \frac{1}{n-p}\,,\\ (1-\beta)\ \min\Big\{\frac{n-p}{p-1}, \frac{n-1}{p-1}, \frac{n}{p-1}\Big\} > 1 \qquad &\text{iff} \qquad \beta < \frac{n+1-2p}{n-p}\,,\end{aligned}$$

we are led to choose

$$\beta \in \Big(\frac{1}{n-p}, \min\Big\{1, \frac{n+1-2p}{n-p}\Big\}\Big)\,, \tag{4.82}$$

(where the interval appearing in (4.82) is non-empty thanks to $n > 2p$). With this choice of β, we have $\min\{\beta\,(n-p),(1-\beta)\,\frac{n-p}{p-1}\} > 1$, and thus deduce from (4.78), (4.79) and (4.80) that

$$\max\Big\{\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}|\nabla v_\varepsilon|^p,\int_{\Omega\cap(B_{r_2}\setminus B_{r_1})}v_\varepsilon^{p^\star},\int_{(\partial\Omega)\cap(B_{r_2}\setminus B_{r_1})}v_\varepsilon^{p^\#}\Big\}=\mathrm{o}(\varepsilon)\,. \tag{4.83}$$

Step four: We prove (4.74). We first notice that by (4.83) and (4.81) we have

$$\int_\Omega|\nabla v_\varepsilon|^p=\int_{\Omega\cap B_{r_1}}|(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p+\int_\Omega|\nabla v|^p+\mathrm{o}(\varepsilon)\,. \tag{4.84}$$

Now, by (4.38) we have

$$\begin{aligned}
&\nabla(U^{(\varepsilon)}\circ g)\circ f=[(\nabla g)\circ f]^*\,\nabla U^{(\varepsilon)}\\
=\;&\nabla U^{(\varepsilon)}+\partial_nU^{(\varepsilon)}\,\nabla\ell-(\nabla\ell\cdot\nabla U^{(\varepsilon)})\,e_n+x_n\sum_{i=1}^{n-1}\kappa_i\,\partial_iU^{(\varepsilon)}\,e_i+\mathrm{O}(|x|^2)\,|\nabla U^{(\varepsilon)}|\,,
\end{aligned}$$

so that, recalling that $|\nabla\ell|=\mathrm{O}(|x|)$,

$$\begin{aligned}
\Big|\nabla(U^{(\varepsilon)}\circ g)\circ f\Big|^2 &= |\nabla U^{(\varepsilon)}|^2+2\left((\nabla\ell\cdot\nabla U^{(\varepsilon)})\,e_n-\partial_nU^{(\varepsilon)}\,\nabla\ell\right)\cdot\nabla U^{(\varepsilon)}\\
&\quad+2\,x_n\sum_{i=1}^{n-1}\kappa_i\,(\partial_iU^{(\varepsilon)})^2+\mathrm{O}(|x|^2)\,|\nabla U^{(\varepsilon)}|^2\\
&= |\nabla U^{(\varepsilon)}|^2+2\,x_n\sum_{i=1}^{n-1}\kappa_i\,(\partial_iU^{(\varepsilon)})^2+\mathrm{O}(|x|^2)\,|\nabla U^{(\varepsilon)}|^2\,.
\end{aligned}$$

Now set $a=|\nabla U^{(\varepsilon)}|$ and $b=[2\sum_{i=1}^{n-1}\kappa_i\,(\partial_iU^{(\varepsilon)})^2]^{1/2}$, so that $0\le b\le C\,a$ for a constant depending on $|A_{\partial\Omega}(x_0)|$. Since $z\mapsto(1+z)^{p/2}$ is smooth in a neighborhood of $z=0$, we see that if $|x|<1/C$ for a constant C depending on $|A_{\partial\Omega}(x_0)|$, then

$$\begin{aligned}&\left(a^2+x_n\,b^2+\mathrm{O}(|x|^2)\,a^2\right)^{p/2}=a^p\left(1+(b/a)^2\,x_n+\mathrm{O}(|x|^2)\right)^{p/2}\\&=a^p\left(1+\frac{p}{2}(b/a)^2\,x_n+\mathrm{O}(|x|^2)\right)=a^p+\frac{p}{2}\,a^{p-2}\,b^2\,x_n+\mathrm{O}(|x|^2)\end{aligned}$$

and thus

$$\begin{aligned}\left|\nabla(U^{(\varepsilon)}\circ g)\circ f\right|^p Jf&=|\nabla U^{(\varepsilon)}|^p\Big(1+p\,x_n\sum_{i=1}^{n-1}\kappa_i\frac{(\partial_i U^{(\varepsilon)})^2}{|\nabla U^{(\varepsilon)}|^2}+\mathrm{O}(|x|^2)\Big)\Big(1-x_n\,h+\mathrm{O}(|x|^2)\Big)\\&=|\nabla U^{(\varepsilon)}|^p-x_n\left(h-p\sum_{i=1}^{n-1}\kappa_i\frac{(\partial_i U^{(\varepsilon)})^2}{|\nabla U^{(\varepsilon)}|^2}\right)|\nabla U^{(\varepsilon)}|^p+\mathrm{O}(|x|^2)\,|\nabla U^{(\varepsilon)}|^p\,.\end{aligned}$$

Then, by (4.36),

$$\begin{aligned}&\int_{\Omega\cap B_{r_1}}|(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p\le\int_{H\cap B_{C\,r_1}}\left|[(\nabla g)\circ f]^*\,(\nabla U^{(\varepsilon)})\right|^p Jf\\&\le\int_{H\cap B_{C\,r_1}}|\nabla U^{(\varepsilon)}|^p-h\int_{H\cap B_{C\,r_1}}x_n\,|\nabla U^{(\varepsilon)}|^p\\&+p\sum_{i=1}^{n-1}\kappa_i\int_{H\cap B_{C\,r_1}}x_n\,(\partial_i U^{(\varepsilon)})^2\,|\nabla U^{(\varepsilon)}|^{p-2}+C\int_{H\cap B_{C\,r_1}}|x|^2\,|\nabla U^{(\varepsilon)}|^p\\&=\int_{H\cap B_{Cr_1/\varepsilon}}|\nabla U|^p-h\,\varepsilon\int_{H\cap B_{Cr_1/\varepsilon}}x_n\,|\nabla U|^p\\&+p\,\varepsilon\sum_{i=1}^{n-1}\kappa_i\int_{H\cap B_{Cr_1/\varepsilon}}x_n\,(\partial_i U)^2\,|\nabla U|^{p-2}+C\,\varepsilon^2\int_{H\cap B_{Cr_1/\varepsilon}}|x|^2\,|\nabla U|^p\,.\end{aligned}$$

Now, by the reflection symmetries of U with respect to $\{x_i=0\}$, $i=1,...,n-1$ (recall (4.59)), we have

$$\int_{H\cap B_R}x_n\,(\partial_i U)^2\,|\nabla U|^{p-2}=\int_{H\cap B_R}x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,,\qquad\forall i=1,...,n-1\,,\forall R>0\,,$$

and therefore

$$\sum_{i=1}^{n-1}\kappa_i\int_{H\cap B_{Cr_1/\varepsilon}}x_n\,(\partial_i U)^2\,|\nabla U|^{p-2}=h\int_{H\cap B_{Cr_1/\varepsilon}}x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,.$$

Setting

$$\mathcal{L}(U, B_R) = \int_{H\cap B_R} x_n\,|\nabla U|^p - p\,x_n\,(\partial_1 U)^2\,|\nabla U|^{p-2}\,,$$

we can thus rewrite (4.85) as

$$\int_{\Omega\cap B_{r_1}} |(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p \le \int_H |\nabla U|^p - h\,\varepsilon\,\mathcal{L}(U, B_{C\,r_1/\varepsilon}) + C\,\varepsilon^2 \int_{H\cap B_{C\,r_1/\varepsilon}} |x|^2\,|\nabla U|^p\,.$$

At the same time, $\varepsilon^2\,|x|^2 \le C\,\varepsilon\,r_1\,|x|$ for any $x \in B_{C\,r_1/\varepsilon}$, so

$$\begin{aligned}
&\varepsilon^2 \int_{H\cap B_{C r_1/\varepsilon}} |x|^2\,|\nabla U|^p \le C\,\varepsilon\,r_1 \int_{H\cap B_{C r_1/\varepsilon}} |x|\,|\nabla U|^p \\
&\le C\,\varepsilon^{1+\beta} \int_H |x|\,|\nabla U_T|^p + C\,\varepsilon^{1+\beta} \int_H |x|\,|\nabla V_T|^p \le C(n,p,T)\,\varepsilon^{1+\beta},
\end{aligned}$$

where we have used the facts that V_T is compactly supported and that $n > 2p-1$ to guarantee the convergence of the integrals in the final line. Hence,

$$\begin{aligned}
\int_{\Omega\cap B_{r_1}} |(\nabla g)^*[(\nabla U^{(\varepsilon)})\circ g]|^p &= \int_H |\nabla U|^p - h\,\mathcal{L}(U, B_{C\,r_1/\varepsilon})\,\varepsilon + \mathrm{o}(\varepsilon) \\
&= \int_H |\nabla U|^p - h\,\mathcal{L}(U)\,\varepsilon + \mathrm{o}(\varepsilon)\,,
\end{aligned}$$

where the $\mathrm{o}(\varepsilon)$ term depends on n, p, and T, and in the second line we have applied (4.65). By (4.84) we deduce (4.74).

Step five: We prove (4.75). We first notice that by (4.83), $v \in \mathrm{Lip}(\Omega)$, $r_2^n = \varepsilon^{\beta\,n} = \mathrm{o}(\varepsilon)$ and our choice of β we have

$$\int_\Omega v_\varepsilon^{p^\star} = \int_{\Omega\cap B_{r_1}} (U^{(\varepsilon)}\circ g)^{p^\star} + \int_\Omega v^{p^\star} + \mathrm{o}(\varepsilon)\,. \tag{4.87}$$

Let $E_\varepsilon = g(B_{r_1}\cap\Omega)\subset H$ and $\tilde E_\varepsilon = E_\varepsilon/\varepsilon\subset E$ as in step two. Then keeping in mind (4.36) and (4.37),

$$\begin{aligned}\int_{\Omega\cap B_{r_1}}(U^{(\varepsilon)}\circ g)^{p^\star} &= \int_{E_\varepsilon}(U^{(\varepsilon)})^{p^\star} - h\int_{E_\varepsilon} x_n\,(U^{(\varepsilon)})^{p^\star} + \int_{E_\varepsilon}\mathrm{O}(|x|^2)\,(U^{(\varepsilon)})^{p^\star}\\ &= \int_{\tilde E_\varepsilon} U^{p^\star} - h\,\varepsilon\int_{\tilde E_\varepsilon} x_n\,U^{p^\star} + \int_{E_\varepsilon}\mathrm{O}(|x|^2)\,(U^{(\varepsilon)})^{p^\star}\\ &= \int_H U^{p^\star} - h\,\varepsilon\mathcal{M}(U) + \Big\{-\int_{H\setminus\tilde E_\varepsilon}U^{p^\star} + h\varepsilon\int_{H\setminus\tilde E_\varepsilon}x_nU^{p^\star} + \int_{E_\varepsilon}\mathrm{O}(|x|^2)\,(U^{(\varepsilon)})^{p^\star}\Big\}\,.\end{aligned}$$

By (4.61) and (4.64), along with our choice of β, we see that

$$-\int_{H\setminus\tilde E_\varepsilon}U^{p^\star} = o(\varepsilon),\qquad h\,\varepsilon\int_{H\setminus\tilde E_\varepsilon}x_nU^{p^\star} = o(\varepsilon)\,.$$

Moreover, since $U = U_T + tV_T$ with V_T compactly supported in H and $|t|\le 1$, we have

$$\begin{aligned}\int_{E_\varepsilon}|x|^2\,(U^{(\varepsilon)})^{p^\star} &\le C\,r_1\,\varepsilon\int_{\tilde E_\varepsilon}|x|\,U^{p^\star} \le \varepsilon^{1+\beta}\int_H|x|\,U^{p^\star}\\ &\le C\varepsilon^{1+\beta}\int_H|x|\,U_T^{p^\star} + C\varepsilon^{1+\beta}\int_H|x|V^{p^\star} = \mathrm{o}(\varepsilon),\end{aligned}$$

with $\mathrm{o}(\varepsilon)$ depending on n, p, and T. So, the entire term in brackets above can be written as $\mathrm{o}(\varepsilon)$. Combining this estimate with (4.87), we deduce (4.75).

Step six: We finally prove (4.76). Notice that, by (4.83), $v\in\mathrm{Lip}(\Omega)$, $r_2^{n-1} = \varepsilon^{\beta\,(n-1)} = \mathrm{o}(\varepsilon)$ (by the choice of β), we have

$$\int_{\partial\Omega}v_\varepsilon^{p^\#} = \int_{(\partial\Omega)\cap B_{r_1}}(U^{(\varepsilon)}\circ g)^{p^\#} + \int_{\partial\Omega}v^{p^\#} + \mathrm{o}(\varepsilon)\,.$$

Now, by $J^{\partial H}f\ge 1$, (4.63) and our choice of β we have

$$\int_{(\partial\Omega)\cap B_{r_1}}(U^{(\varepsilon)}\circ g)^{p^\#} \ge \int_{(\partial H)\cap B_{r_1/C}}(U^{(\varepsilon)})^{p^\#} = \int_{\partial H}U^{p^\#} - \int_{(\partial H)\setminus B_{r_1/C\varepsilon}}U^{p^\#} \ge \int_{\partial H}U^{p^\#} + \mathrm{o}(\varepsilon)\,.$$

At the same time, by $J^{\partial H} f \le 1 + C\,|x|^2$, we have

$$\int_{(\partial\Omega)\cap B_{r_1}} (U^{(\varepsilon)} \circ g)^{p^\#} \le \int_{(\partial H)\cap B_{C\,r_1}} (1 + C\,|x|^2)\,(U^{(\varepsilon)})^{p^\#}$$
$$\le \int_{\partial H} U^{p^\#} + C\,\varepsilon^2 \int_{(\partial H)\cap B_{C\,r_1/\varepsilon}} |x|^2\,U^{p^\#}$$

where

$$\varepsilon^2 \int_{(\partial H)\cap B_{C\,r_1/\varepsilon}} |x|^2\,U^{p^\#} \le C\,\varepsilon^2 \int_0^{C\,r_1/\varepsilon} \frac{r^2\,r^{n-2}\,dr}{(r^{(n-p)/(p-1)})^{p^\#}}$$
$$\le C\,\varepsilon^2\,\big(r_1/\varepsilon\big)^{1+n-(n-1)\,[p/(p-1)]}$$
$$\le C\,\varepsilon^2\,\big(r_1/\varepsilon\big)^{(2p-n-1)/(p-1)} \le C\,\varepsilon^2\,\varepsilon^{(1-\beta)\,(n+1-2p)/(p-1)} \le C\,\varepsilon^2 = \mathrm{o}(\varepsilon)\,,$$

thanks to $n > 2p-1$. This completes the proof. □

4.4 Existence of minimizers

We first establish the existence of generalized minimizers. Recall that $\Phi^*_\Omega(T)$ was defined in (4.15).

Theorem 4.4.1. *Let $n \ge 2$, $p \in (1,n)$, and let Ω be a bounded open set with C^1-boundary in $\mathbb{R}^n$. Then:*

(i): *for every $T > 0$, $\Phi_\Omega(T) = \Phi^*_\Omega(T)$;*

(ii): *there is a minimizer $(u, \mathsf{v}, \mathsf{t})$ of $\Phi^*_\Omega(T)$;*

(iii): *if $(u, \mathsf{v}, \mathsf{t})$ is a minimizer of $\Phi^*_\Omega(T)$ with $\int_\Omega u^{p^\star} > 0$, then $\int_{\partial\Omega} u^{p^\#} > 0$,*

$$u\Big/\|u\|_{L^{p^\star}(\Omega)} \text{ is a minimizer of } \Phi_\Omega\Big(\|u\|_{L^{p^\#}(\partial\Omega)}\Big/\|u\|_{L^{p^\star}(\Omega)}\Big)\,, \tag{4.88}$$

and there exists $\lambda, \sigma \in \mathbb{R}$ *such that*

$$\begin{cases} -\Delta_p u = \lambda\, u^{p^\star - 1}\,, & \textit{on } \Omega\,,\\ |\nabla u|^{p-2}\,\dfrac{\partial u}{\partial \nu_\Omega} = \sigma\, u^{p^\# - 1}\,, & \textit{on } \partial\Omega\,, \end{cases} \tag{4.89}$$

In particular, $u \in \mathrm{Lip}(\Omega)$*. If, in addition,* $\mathsf{v} > 0$*, then* λ *and* σ *are given by*

$$\lambda = \mathsf{v}^{p-p^\star}\,\lambda_H(\mathsf{t}/\mathsf{v})\,, \qquad \sigma = \mathsf{v}^{p-p^\#}\,\sigma_H(\mathsf{t}/\mathsf{v})\,, \tag{4.90}$$

and, in particular,

$$\frac{\mathsf{t}}{\mathsf{v}} \in (0, T_0) \cup (T_E, \infty)\,. \tag{4.91}$$

Proof. Step one: Since $(u, 0, 0) \in \mathcal{Y}_\Omega(T)$ if $u \in \mathcal{X}_\Omega(T)$, we have $\Phi^*_\Omega(T) \le \Phi_\Omega(T)$. To prove the converse inequality it is enough to show that for every $(u, \mathsf{v}, \mathsf{t}) \in \mathcal{Y}_\Omega(T)$,

$$\exists\, u_j \in \mathcal{X}_\Omega(T) \text{ s.t. } \lim_{j\to\infty} \int_\Omega |\nabla u_j|^p = \int_\Omega |\nabla u|^p + \mathsf{v}^p\,\Phi_H\Big(\frac{\mathsf{t}}{\mathsf{v}}\Big)^p\,. \tag{4.92}$$

Looking back at the definition of $\mathcal{Y}_\Omega(T)$ in the paragraph preceding the statement of Theorem 4.1.1, we can assume without loss of generality that $\mathsf{v} > 0$ and $\mathsf{t} > 0$. Moreover, given $(u, \mathsf{v}, \mathsf{t}) \in \mathcal{Y}_\Omega(T)$ with v and t positive we can easily find $(u_j, \mathsf{v}_j, \mathsf{t}_j) \in \mathcal{Y}_\Omega(T)$ with v_j, t_j, $\int_\Omega u_j^{p^\star}$, and $\int_{\partial\Omega} u_j^{p^\#}$ positive and such that $\mathcal{E}(u_j, \mathsf{v}_j, \mathsf{t}_j) \to \mathcal{E}(u, \mathsf{v}, \mathsf{t})$. By a diagonal argument, it is thus sufficient proving (4.92) under the assumption that $\int_\Omega u^{p^\star}$ and $\int_{\partial\Omega} u^{p^\#}$ are positive. This said, we apply Lemma 4.3.4**(i)** with

$$v = \frac{u}{\mathsf{v}}\,, \qquad U = U_{\mathsf{t}/\mathsf{v}}\,,$$

to find functions v_j with $v_j = v$ on $\Omega \setminus B_{2\varepsilon_j}(x_0)$ for some $x_0 \in \partial\Omega$ and $\varepsilon_j \to 0^+$, and with

$$\begin{aligned}
\int_\Omega |\nabla v_j|^p &= \frac{1}{\mathsf{v}^p}\int_\Omega |\nabla u|^p + \Phi_H(\mathsf{t}/\mathsf{v})^p + G_j\,,\\
\int_\Omega v_j^{p^\star} &= \frac{1}{\mathsf{v}^{p^\star}}\int_\Omega u^{p^\star} + 1 + V_j\\
\int_{\partial\Omega} v_j^{p^\#} &= \frac{1}{\mathsf{v}^{p^\#}}\int_{\partial\Omega} u^{p^\#} + (\mathsf{t}/\mathsf{v})^{p^\#} + T_j\,,
\end{aligned}$$

where G_j, V_j, $T_j \to 0$ as $j \to \infty$ at a rate depending on n, p, Ω, t/v and u only. By Lemma 4.2.6, there exist η and C depending on n, p, Ω, t/v and u, but independent from j, such that for any (a_j, b_j) with $|a_j| + |b_j| < \eta$, we have functions w_j such that

$$\int_{\partial\Omega} |w_j|^{p^\#} = a_j + \int_{\partial\Omega} |v_j|^{p^\#}\,, \qquad \int_\Omega |w_j|^{p^\star} = b_j + \int_\Omega |v_j|^{p^\star}\,,$$

$$\Big|\int_\Omega |\nabla w_j|^p - \int_\Omega |\nabla v_j|^p\Big| \le C\left(|a_j| + |b_j|\right).$$

For j large enough we can apply this statement with $a_j = -T_j$ and $b_j = -V_j$ to find a sequence $\{w_j\}_j$ with

$$\int_{\partial\Omega} |w_j|^{p^\#} = \frac{1}{\mathsf{v}^{p^\#}}\int_{\partial\Omega} u^{p^\#} + (\mathsf{t}/\mathsf{v})^{p^\#} = \frac{T^{p^\#}}{\mathsf{v}^{p^\#}}\,, \qquad \int_\Omega |w_j|^{p^\star} = \frac{1}{\mathsf{v}^{p^\star}}\int_\Omega u^{p^\star} + 1 = \frac{1}{\mathsf{v}^{p^\star}}\,,$$

$$\Big|\int_\Omega |\nabla w_j|^p - \frac{1}{\mathsf{v}^p}\int_\Omega |\nabla u|^p - \Phi_H(\mathsf{t}/\mathsf{v})^p - G_j\Big| \le C\left(|T_j| + |V_j|\right).$$

Setting $u_j = \mathsf{v}\, w_j$, we obtain a sequence in $\mathcal{X}_\Omega(T)$ that satisfies (4.92).

Step two: We prove that there is a minimizer for the generalized problem $\Phi^*_\Omega(T)$. By the argument in step one we can find a sequence $\{u_j\}_j$ in $\mathcal{X}_\Omega(T)$

such that $\int_\Omega |\nabla u_j|^p \to \Phi^*_\Omega(T)^p$. By Lemma 4.2.1, the measures μ_j, ν_j and τ_j defined in (4.24) have subsequential weak-star limits μ, ν and τ satisfying (4.25), (4.26) and (4.27) and (4.28). In particular, there is an at most countable set $\{x_i\}_{i\in I} \subset \overline{\Omega}$ and corresponding $\mathsf{v}_i > 0$ and $\mathsf{t}_i \geq 0$ for every $i \in I$, such that

$$\Phi^*_\Omega(T)^p = \lim_{j\to\infty} \int_\Omega |\nabla u_j|^p \geq \int_\Omega |\nabla u|^p + S^p \sum_{i\in I\setminus I_{\rm bd}} \mathsf{v}_i^p + \sum_{i\in I_{\rm bd}} \mathsf{v}_i^p \, \Phi_H(\mathsf{t}_i/\mathsf{v}_i)^p \,, \quad (4.93)$$

where u is the subsequential weak limit of u_j, and

$$1 = \int_\Omega u^{p^\star} + \sum_{i\in I} \mathsf{v}_i^{p^\star} \,, \qquad T^{p^\#} = \int_{\partial\Omega} u^{p^\#} + \sum_{i\in I_{\rm bd}} \mathsf{t}_i^{p^\#} \,. \quad (4.94)$$

Now set

$$\mathsf{v}_c^{p^\star} = \sum_{i\in I} \mathsf{v}_i^{p^\star} \,, \qquad \mathsf{t}_c^{p^\#} = \sum_{i\in I_{\rm bd}} \mathsf{t}_i^{p^\#} \,.$$

By an immediate adaptation of the proof of Lemma 4.3.4 we can easily construct a sequence $\{W_j\}_j$ in $\mathfrak{X}_H(\mathsf{t}_c/\mathsf{v}_c)$ with the property that

$$\int_H |\nabla W_j|^p \to \sum_{i\in I} \left(\frac{\mathsf{v}_i}{\mathsf{v}_c}\right)^p \Phi_H(\mathsf{t}_i/\mathsf{v}_i)^p \,.$$

Since $W_j \in \mathfrak{X}_H(\mathsf{t}_c/\mathsf{v}_c)$ implies $\int_H |\nabla W_j|^p \geq \Phi_H(\mathsf{t}_c/\mathsf{v}_c)^p$, we deduce from (4.93) that

$$\Phi^*_\Omega(T)^p \geq \int_\Omega |\nabla u|^p + \mathsf{v}_c^p \, \Phi_H(\mathsf{t}_c/\mathsf{v}_c)^p \,,$$

while (4.94) gives $(u, \mathsf{v}_c, \mathsf{t}_c) \in \mathcal{Y}_\Omega(T)$. This proves that $(u, \mathsf{v}_c, \mathsf{t}_c)$ is a minimizer of $\Phi^*_\Omega(T)$.

Step three: We finally prove statement (iii). If $(u, \mathsf{v}, \mathsf{t})$ is a minimizer of $\Phi^*_\Omega(T)$ with $\int_\Omega u^{p^\star} > 0$, it is immediate to deduce (4.88), and since $\Phi_\Omega(0)$ does not admit minimizers, it must also be $\int_{\partial\Omega} u^{p^\#} > 0$. By Lemma 4.2.5, the Euler-Lagrange equation (4.89) for u holds for some $\lambda, \sigma \in \mathbb{R}$ and $u \in \mathrm{Lip}(\Omega)$. Assuming now that $\mathsf{v} > 0$, we can prove (4.90) by noticing that, given $\varphi \in C_c^\infty(\mathbb{R}^n)$, if we define

$$\alpha(\delta) = \Big(1 - \int_\Omega (u + \delta\,\varphi)^{p^\star}\Big)^{1/p^\star} - \mathsf{v}\,, \qquad \beta(\delta) = \Big(T^{p^\#} - \int_{\partial\Omega} (u + \delta\,\varphi)^{p^\#}\Big)^{1/p^\#} - \mathsf{t}\,,$$

then there is $\delta_0 > 0$ such that $(u + \delta\,\varphi, \mathsf{v} + \alpha(\delta), \mathsf{t} + \beta(\delta)) \in \mathcal{Y}_\Omega(T)$ for every $|\delta| < \delta_0$. In particular,

$$0 = \frac{d}{d\delta}\Big|_{\delta=0} \int_\Omega |\nabla(u + \delta\varphi)|^p + (\mathsf{v} + \alpha(\delta))^p\, \Phi_H\Big(\frac{\mathsf{t} + \beta(\delta)}{\mathsf{v} + \alpha(\delta)}\Big)^p,$$

and exploiting (4.89) as well as

$$\alpha'(0) = -\mathsf{v}^{1-p^\star} \int_\Omega u^{p^\star - 1}\varphi\,, \qquad \beta'(0) = -\mathsf{t}^{1-p^\#} \int_{\partial\Omega} u^{p^\# - 1}\varphi\,,$$

and (4.50) (i.e. $\Phi_H(T) = \lambda_H(T) + \sigma_H(T)\, T^{p^\#}$ and $\sigma_H(T)\, T^{p^\#-1} = \Phi_H(T)^{p-1}\Phi_H'(T)$ for every $T > 0$), we see that

$$\begin{aligned}
0 &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi + \mathsf{v}^{p-1}\,\Phi_H(\mathsf{t}/\mathsf{v})^{p-1}\,\Phi_H'(\mathsf{t}/\mathsf{v})\,\beta'(0) \\
&\qquad\qquad + \Big\{\mathsf{v}^{p-1}\,\Phi_H(\mathsf{t}/\mathsf{v})^p - \mathsf{t}\,\mathsf{v}^{p-2}\,\Phi_H(\mathsf{t}/\mathsf{v})^{p-1}\,\Phi_H'(\mathsf{t}/\mathsf{v})\Big\}\alpha'(0) \\
&= \lambda\int_\Omega u^{p^\star-1}\,\varphi + \sigma\int_{\partial\Omega} u^{p^\#-1}\varphi - \mathsf{v}^{p-1}\,\mathsf{t}^{1-p^\#}\,\Phi_H(\mathsf{t}/\mathsf{v})^{p-1}\,\Phi_H'(\mathsf{t}/\mathsf{v})\int_{\partial\Omega} u^{p^\#-1}\varphi \\
&\quad - \mathsf{v}^{p-p^\star}\Big\{\Phi_H(\mathsf{t}/\mathsf{v})^p - (\mathsf{t}/\mathsf{v})\,\Phi_H(\mathsf{t}/\mathsf{v})^{p-1}\,\Phi_H'(\mathsf{t}/\mathsf{v})\Big\}\int_\Omega u^{p^\star-1}\varphi \\
&= \big(\sigma - \mathsf{v}^{p-p^\#}\sigma_H(\mathsf{t}/\mathsf{v})\big)\int_{\partial\Omega} u^{p^\#-1}\varphi + \big(\lambda - \mathsf{v}^{p-p^\star}\lambda_H(\mathsf{t}/\mathsf{v})\big)\int_\Omega u^{p^\star-1}\varphi\,.
\end{aligned}$$

Testing with $\varphi \in C_c^\infty(\Omega)$ we find $\lambda = \mathsf{v}^{p-p^\star}\lambda_H(\mathsf{t}/\mathsf{v})$, and testing with $\varphi = 1$ on $\overline{\Omega}$ then gives $\sigma = \mathsf{v}^{p-p^\#}\sigma_H(\mathsf{t}/\mathsf{v})$. We finally prove (4.91), i.e. $\mathsf{t}/\mathsf{v} \notin [T_0, T_E]$.

Indeed, combining the balance condition (4.39) with (4.90) we find

$$\mathsf{v}^{p-p^\star}\,\lambda_H(\mathsf{t}/\mathsf{v})\int_\Omega u^{p^\star-1}+\mathsf{v}^{p-p^\#}\,\sigma_H(\mathsf{t}/\mathsf{v})\int_{\partial\Omega}u^{p^\#-1}=0\,, \tag{4.95}$$

where by (4.53) and (4.54) we have $\lambda_H>0$ in $[T_0,T_E)$ and $\sigma_H>0$ on $(T_0,T_E]$ (with $\lambda_H(T_E)=\sigma_H(T_0)=0$ by continuity). If $\mathsf{t}/\mathsf{v}\in[T_0,T_E)$, then (4.95) implies $\int_\Omega u^{p^\star}=0$, a contradiction; if $\mathsf{t}/\mathsf{v}=T_E$, then (4.95) gives $u=0$ on $\partial\Omega$, so that, by (4.88), u is a minimizer of $\Phi_\Omega(0)$, and thus u is optimal in the Sobolev inequality on $\mathbb{R}^n$, so that $\Omega=\mathbb{R}^n$, contradicting the fact that Ω is bounded. □

We are now ready to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. Statement (i) is an immediate consequence of Theorem 4.4.1. We thus focus on statement (ii), and assume that Ω is of class C^2 and that $n>2p$. We want to prove that if $(u,\mathsf{v},\mathsf{t})$ is a minimizer of $\Phi^*_\Omega(T)$, then $\mathsf{v}=\mathsf{t}=0$. We assume by way of contradiction that either $\mathsf{v}>0$ *or* $\mathsf{t}>0$; recalling the definition of $\mathcal{Y}_\Omega(T)$, this implies that $\mathsf{v}>0$ *and* $\mathsf{t}>0$. We apply Lemma 4.3.4 **(ii)** with the choice $(v,T)=(u/\mathsf{v},\tau)$ at a point $x_0\in\partial\Omega$ of positive mean curvature, noting that if $\mathsf{v}=1$ then $v\equiv 0$ and that v is Lipschitz continuous if $\mathsf{v}\in(0,1)$ thanks to (4.88) and Lemma 4.2.5. Then, for every $U\in\mathcal{U}_\tau$, we have

$$\begin{aligned}
\int_\Omega|\nabla v_\varepsilon|^p &\le \int_H|\nabla U|^p+\int_\Omega|\nabla v|^p-\mathrm{H}_{\partial\Omega}(0)\,\mathcal{L}(U)\,\varepsilon+\mathrm{o}(\varepsilon)\,,\\
\int_\Omega v_\varepsilon^{p^\star} &= \int_H U^{p^\star}+\int_\Omega v^{p^\star}-\mathrm{H}_{\partial\Omega}(0)\,\mathcal{M}(U)\,\varepsilon+\mathrm{o}(\varepsilon)\,,\\
\int_{\partial\Omega} v_\varepsilon^{p^\#} &= \int_{\partial H} U^{p^\#}+\int_{\partial\Omega} v^{p^\#}+\mathrm{o}(\varepsilon)\,,
\end{aligned}$$

where $\mathcal{L}(U)$ and $\mathcal{M}(U)$ are defined in (4.86) and (4.22). We apply this with $U \in \mathcal{U}_\tau$ given by

$$U = U_\tau + b\,\varepsilon\, V_\tau\,, \qquad |\varepsilon| < \frac{1}{|b|}\,, \qquad b = \frac{\mathrm{H}_{\partial\Omega}(0)\,\mathcal{M}(U_\tau)}{p^\star}\,,$$

The reason for the choice of b will become apparent in a moment. Indeed, thanks to (4.66), (4.67) and (4.68), we have

$$\int_H |\nabla U|^p = \Phi_H(\tau)^p + p\,b\,\lambda_H(\tau)\varepsilon + \mathrm{o}\big(\varepsilon\big)\,,$$
$$\int_H U^{p^\star} = 1 + p^\star\,b\,\varepsilon + \mathrm{o}\big(\varepsilon\big)\,, \qquad \int_{\partial H} U^{p^\#} = \tau^{p^\#}\,,$$

which, combined with (4.96), (4.97), (4.98) and

$$\frac{\Phi_\Omega(T)^p}{\mathsf{v}^p} = \Phi_H(\tau)^p + \int_\Omega |\nabla v|^p, \qquad \frac{1}{\mathsf{v}^{p^\star}} = 1 + \int_\Omega v^{p^\star}\,, \qquad (T/\mathsf{v})^{p^\#} = \tau^{p^\#} + \int_{\partial\Omega} v^{p^\#}\,,$$

implies that $w_\varepsilon = \mathsf{v}\,v_\varepsilon$ satisfies

$$\int_\Omega |\nabla w_\varepsilon|^p \le \Phi_\Omega(T)^p + \Big\{p\,b\,\lambda_H(\tau) - \mathrm{H}_{\partial\Omega}(0)\,\mathcal{L}(U)\Big\}\,\mathsf{v}^p\varepsilon + \mathrm{o}(\varepsilon)\,,$$
$$\int_\Omega w_\varepsilon^{p^\star} = 1 + \big\{p^\star\,b - \mathrm{H}_{\partial\Omega}(0)\,\mathcal{M}(U)\big\}\,\mathsf{v}^{p^\star}\varepsilon + \mathrm{o}(\varepsilon) = 1 + \mathrm{o}(\varepsilon)$$
$$\int_{\partial\Omega} w_\varepsilon^{p^\#} = T^{p^\#} + \mathrm{o}(\varepsilon)\,,$$

where in (4.99) we have used the choice of b to deduce

$$\mathcal{M}(U) = \mathcal{M}(U_\tau) + p^\star \int_H x_n\,U_\tau^{p^\star-1}\,V_\tau\,b\,\varepsilon + \mathrm{o}(\varepsilon)\,, \quad \big\{p^\star\,b - \mathrm{H}_{\partial\Omega}(0)\,\mathcal{M}(U)\big\}\,\varepsilon = \mathrm{o}(\varepsilon)\,.$$

In the same spirit, by

$$\lim_{\varepsilon\to 0^+} |\mathcal{L}(U_\tau + b\,\varepsilon\,V_\tau) - \mathcal{L}(U_\tau)| = 0\,,$$

we deduce from (4.99) that

$$\begin{aligned}\int_\Omega |\nabla w_\varepsilon|^p &\le \Phi_\Omega(T)^p - \Big\{\mathcal{L}(U_\tau) - \frac{(n-p)}{n}\,\mathcal{M}(U_\tau)\,\lambda_H(\tau)\Big\}\,\mathrm{H}_{\partial\Omega}(0)\,\mathsf{v}^p\varepsilon + \mathrm{o}(\varepsilon)\\ &\le \Phi_\Omega(T)^p - C(n,p,\tau)\,\mathrm{H}_{\partial\Omega}(0)\,\mathsf{v}^p\varepsilon + \mathrm{o}(\varepsilon)\,,\end{aligned}$$

where in the second line we apply Lemma 4.3.1. We thus conclude that

$$\int_\Omega |\nabla w_\varepsilon|^p \le \int_\Omega |\nabla u|^p + \mathsf{v}^p\,\Phi_H(\mathsf{t}/\mathsf{v})^p - M\,\mathsf{v}^p\,\varepsilon + \mathrm{o}(\varepsilon)\,.$$

It remains to modify the functions w_ε to obtain $w_\varepsilon^* \in \mathcal{X}_\Omega(T)$ also satisfying (4.102), allowing us to conclude the proof of the theorem by choosing ε sufficiently small. We will distinguish between two cases, applying Lemma 4.2.6 in different ways in the two cases.

Case one: Suppose first that $\mathsf{v} < 1$ and thus $\int_\Omega u^{p^\star} > 0$. This also implies that $\int_{\partial\Omega} u^{p^\#} > 0$ by Theorem 4.4.1-(iii). Taking into account (4.100) and (4.101), we can thus apply Lemma 4.2.6 in an analogous way to step one of the proof of Theorem 4.4.1 in order to slightly modify w_ε into $w_\varepsilon^* \in \mathcal{X}_\Omega(T)$ with

$$\begin{aligned}\Phi_\Omega(T)^p &\le \int_\Omega |\nabla w_\varepsilon^*|^p = \int_\Omega |\nabla w_\varepsilon|^p + \,(\mathsf{v}^{p^\#} + \mathsf{v}^{p^\star})\,\mathrm{o}(\varepsilon)\\ &\le \int_\Omega |\nabla u|^p + \mathsf{v}^p\,\Phi_H(\mathsf{t}/\mathsf{v})^p - \frac{M}{2}\,\mathsf{v}^p\,\varepsilon = \Phi_\Omega(T)^p - \frac{M}{2}\,\mathsf{v}^p\,\varepsilon < \Phi_\Omega(T)^p\,,\end{aligned}$$

thus reaching a contradiction.

Case two: Next, suppose that $\mathsf{v} = 1$. So, $\Phi_\Omega(T) = \Phi_H(T)$, $u = v \equiv 0$ and $v_\varepsilon = \varphi_\varepsilon\,(U^{(\varepsilon)} \circ g)$. In this case, we will pull the relevant quantities back to the half space H and apply Lemma 4.2.6 there to correct the volume and trace constraints. More specifically, for $\varepsilon < \varepsilon_0$, the support of φ_ε is entirely contained in the domain of the diffeomorphism f_{x_0}, and so we can define $\Psi_\varepsilon : H \to \mathbb{R}$ by $\Psi_\varepsilon(y) = \varphi_\varepsilon \circ f(\varepsilon\, y)$. In this way, we can rewrite

$$w_\varepsilon = v_\varepsilon = (\Psi_\varepsilon\, U)^{(\varepsilon)} \circ g.$$

Thanks to (4.36), Ψ_ε is identically equal to one in $B_{\varepsilon^{\beta-1}/C} \cap H$ and vanishes outside of $B_{C\,\varepsilon^{\beta-1}} \cap H$. Using the area formula, we rewrite (4.100), (4.101), and (4.102) as

$$\int_H (\Psi_\varepsilon\, U)^{p^\star}\, m_\varepsilon = \int_\Omega w_\varepsilon^{p^\star}\, dx = 1 + o(\varepsilon)\,,$$
$$\int_{\partial H} (\Psi_\varepsilon\, U)^{p^\sharp}\, \hat{m}_\varepsilon = \int_{\partial\Omega} w_\varepsilon^{p^\sharp} = T^{p^\sharp} + o(\varepsilon).$$

$$\int_H |A_\varepsilon\, [\nabla(\Psi_\varepsilon\, U)]|^p\, m_\varepsilon = \int_\Omega |\nabla w_\varepsilon|^p \le \Phi_H(T) - M\varepsilon + o(\varepsilon),$$

where we have set

$$m_\varepsilon(x) = Jf(\varepsilon\, x)\,, \qquad \hat{m}_\varepsilon(x) = J^{\partial H} f(\varepsilon\, x)\,, \qquad A_\varepsilon(x) = (\nabla g \circ f(\varepsilon\, x))^*\,.$$

We now repeat the argument used in the proof of Lemma 4.2.6: exploiting the fact that

$$\Psi_\varepsilon\, U = U_T \qquad \text{on } (H \cap B_R) \setminus (\mathrm{spt} V_T)\,, \qquad R = \frac{\varepsilon_0^{\beta-1}}{C}\,, \tag{4.103}$$

as well as that both $\int_H (\Psi_\varepsilon\, U)^{p^\star}\, m_\varepsilon$ and $\int_{\partial H} (\Psi_\varepsilon\, U)^{p^\sharp}\, \hat{m}_\varepsilon$ are positive and finite, we can easily find $\psi \in C_c^\infty((H \cap B_R) \setminus (\mathrm{spt} V_T))$ and $\varphi \in C_c^\infty((\mathbb{R}^n \cap B_R) \setminus (\mathrm{spt} V_T))$ such that

$$\int_H (\Psi_\varepsilon\, U_T)^{p^\star - 1}\, m_\varepsilon\, \psi = \int_{\partial H} (\Psi_\varepsilon\, U_T)^{p^\# - 1}\, \hat{m}_\varepsilon \varphi = 1\,,$$
$$\int_H (\Psi_\varepsilon\, U_T)^{p^\star - 1}\, m_\varepsilon\, \varphi = \int_{\partial H} (\Psi_\varepsilon\, U_T)^{p^\# - 1}\, \hat{m}_\varepsilon\, \psi = 0\,.$$

Correspondingly, we consider the maps $h_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$h_\varepsilon(s,t) = \Big(\int_{\partial H} \big(|v_\varepsilon + s\,\varphi + t\,\psi|^{p^\#} - |v_\varepsilon|^{p^\#}\big)\,\hat{m}_\varepsilon,\ \int_H \big(|v_\varepsilon + s\,\varphi + t\,\psi|^{p^\star} - |v_\varepsilon|^{p^\star}\big)\,m_\varepsilon \Big).$$

By (4.103), $h_\varepsilon \in C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)$ for some $\alpha = \alpha(n,p) \in (0,1)$, with

$$\sup_{\varepsilon<\varepsilon_0} \|h_\varepsilon\|_{C^{1,\alpha}(\mathbb{R}^2;\mathbb{R}^2)} < \infty\,;$$

moreover, $h_\varepsilon(0,0) = 0$ and, by (4.104),

$$\nabla h_\varepsilon(0,0) = \begin{pmatrix} p^\# & 0 \\ 0 & p^\star \end{pmatrix}.$$

We can thus apply the inverse function theorem uniformly in ε, to obtain functions $W_\varepsilon^* : H \to \mathbb{R}$ with support in $B_{C\varepsilon^{\beta-1}}$ such that

$$\begin{aligned} &\int_H (W_\varepsilon^*)^{p^\star} m_\varepsilon = 1\,, \qquad \int_{\partial H} (W_\varepsilon^*)^{p^\sharp} \hat{m}_\varepsilon = T^{p^\sharp}\,, \\ &\left| \int_H |A_\varepsilon[\nabla(\Psi_\varepsilon U)]|^p m_\varepsilon \ - \ \int_H |A_\varepsilon[\nabla W_\varepsilon^*]|^p m_\varepsilon \right| = o(\varepsilon). \end{aligned} \tag{4.105}$$

Finally, for $\varepsilon < \varepsilon_0$, define $w_\varepsilon^* : \Omega \to \mathbb{R}$ by $w_\varepsilon^* = (W_\varepsilon^*)^{(\varepsilon)} \circ g$. Changing variables once again, (4.105) tells us that $w_\varepsilon \in \mathfrak{X}_\Omega(T)$ and that

$$\begin{aligned} \Phi_\Omega(T)^p &\le \int_\Omega |\nabla w_\varepsilon^*|^p = \int_\Omega |\nabla w_\varepsilon|^p + o(\varepsilon) \\ &\le \Phi_H(T)^p - M\varepsilon + \mathrm{o}(\varepsilon) \le \Phi_H(T)^p - \frac{M\varepsilon}{2} < \Phi_H(T) = \Phi_\Omega(T), \end{aligned}$$

giving us a contradiction in this case as well. This completes the proof of the theorem. □

4.5 Rigidity theorems for best Sobolev inequalities

In this section, we prove Theorem 4.1.3.

Proof of Theorem 4.1.3. Rigidity under assumption (ii) is immediate by combining Theorem 4.1.1-(ii) with (4.16) and (4.17). Let us now consider assumption (i), namely, there is $T_* > 0$ such that

$$\Phi_\Omega(T) = \Phi_B(T) \qquad \forall T \in (0, T_*)\,. \tag{4.106}$$

Without loss of generality we can assume that $T_* < \mathrm{ISO}(B)^{1/p^\#}$. We argue by contradiction and assume that Ω is not a ball.

By Theorem 4.4.1, for every $T > 0$ there is $(u_T, \mathsf{v}_T, \mathsf{t}_T)$ a minimizer of $\Phi^*_\Omega(T)$. The basic idea of the proof will be to show that the trace-to-volume ratio of u_T must be, on one hand, uniformly positive and, on the other hand, tending to zero as $T \to 0$, giving us a contradiction. Since Ω is connected and is not a ball by assumption, the rigidity criterion (4.16) together with (4.17) and (4.106) tell us that a classical minimizer for $\Phi_\Omega(T)$ cannot exist for $T \in (0, T_*)$, and so we immediately deduce that $\mathsf{v}_T < 1$ for all such T. In other words, if we set

$$\nu_T = \big(1 - \mathsf{v}_T^{p^\star}\big)^{1/p^\star} = \|u_T\|_{L^{p^\star}(\Omega)}\,, \qquad \tau_T = (T^{p^\#} - \mathsf{t}^{p^\#})^{1/p^\#} = \|u\|_{L^{p^\#}(\partial\Omega)}\,,$$

then $\nu_T > 0$ for all $T \in (0, T_*)$. So, Theorem 4.4.1-(iii) implies that

$$u_T/\nu_T \text{ is a minimizer of } \Phi_\Omega(\tau_T/\nu_T)\,. \tag{4.107}$$

In particular, this means that

$$\frac{\tau_T}{\nu_T} \geq T_* \qquad \text{for all } T \in (0, T_*)\,, \tag{4.108}$$

since as we noted above, no minimizer of $\Phi_\Omega(\tilde{T})$ can exist for $\tilde{T} = \tau_T/\nu_T < T_*$. Since $T \geq \tau_T$, (4.108) tells us that $T/\nu_T \geq T_*$; rearranging this inequality gives us the following lower bound on v_T:

$$\mathsf{v}_T^{p^\star} \geq 1 - (T/T_*)^{p^\star} \qquad \forall T \in (0, T_*)\,.$$

From this, an upper bound on the ratio $\mathsf{t}_T/\mathsf{v}_T$ follows immediately:

$$\frac{\mathsf{t}_T}{\mathsf{v}_T} \leq \big(1 - (T/T_*)^{p^\star}\big)^{-1/p^\star}\, T \qquad \forall T \in (0, T_*)\,.$$

In particular $\mathsf{t}_T/\mathsf{v}_T \to 0$ as $T \to 0^+$.

On the other hand, we will now use the Euler-Lagrange equation for u_T to show that

$$\lim_{T \to 0^+} \frac{\tau_T}{\nu_T} = 0\,, \tag{4.111}$$

which is a clear contradiction to (4.108). Indeed, by Theorem 4.4.1-(iii) and (4.107), u_T satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta_p u_T = \mathsf{v}_T^{p-p^\star}\, \lambda_H(\mathsf{t}_T/\mathsf{v}_T)\, u_T^{p^\star-1} & \text{on } \Omega\,, \\ |\nabla u_T|^{p-2}\, \dfrac{\partial u}{\partial \nu_\Omega} = \mathsf{v}_T^{p-p^\#}\, \sigma_H(\mathsf{t}_T/\mathsf{v}_T)\, u_T^{p^\#-1} & \text{on } \partial\Omega\,; \end{cases} \tag{4.112}$$

see (4.89), (4.90) and (4.49) (for the definition of$\lambda_H(T)$ and $\sigma_H(T)$). Testing (4.112) with u_T, we find that

$$\begin{aligned} \int_\Omega |\nabla u_T|^p &= \mathsf{v}_T^{p-p^\star}\, \lambda_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big) \int_\Omega u_T^{p^\star} + \mathsf{v}_T^{p-p^\#}\, \sigma_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big) \int_{\partial\Omega} u_T^{p^\#} \\ &= \mathsf{v}_T^{p-p^\star}\, \lambda_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\, \nu_T^{p^\star} + \mathsf{v}_T^{p-p^\#}\, \sigma_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\, \tau_T^{p^\#}\,. \end{aligned}$$

After rearranging and multiplying through by $\mathsf{v}_T^{p^\sharp - p}\nu_T^{-p^\sharp} > 0$, we arrive at the inequality

$$-\sigma_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\left(\frac{\tau_T}{\nu_T}\right)^{p^\#} \leq \lambda_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\left(\frac{\nu_T}{\mathsf{v}_T}\right)^{p^\star - p^\#} . \tag{4.113}$$

By (4.54) and the continuity of $T \mapsto \lambda_H(T)$, there are $C > 0$ and $T_{**} > 0$ such that $\lambda_H(T) \in (1/C, C)$ for every $T \in (0, T_{**})$. Moreover, thanks to (4.109), we can ask that $\mathsf{v}_T \geq 1/C$ for $T \in (0, T_{**})$. In particular, by (4.110) and up to further increasing C, if $T < 1/C$, then $\mathsf{t}_T/\mathsf{v}_T < T_{**}$ and thus (4.113), $\nu_T \leq 1$ and $\mathsf{v}_T \geq 1/C$ give

$$-\sigma_H\Big(\frac{\mathsf{t}_T}{\mathsf{v}_T}\Big)\left(\frac{\tau_T}{\nu_T}\right)^{p^\#} \leq C\,. \tag{4.114}$$

By (4.53) and the fact that $\mathsf{t}_T/\mathsf{v}_T \to 0$ as $T \to 0$, we see that $\sigma_H(T) \to -\infty$ as $T \to 0^+$, so that (4.114) implies (4.111). We reach a contradiction to (4.108), completing the proof. □

Appendices

Appendix A

to Chapter 2

In this appendix we state a classic fixed point theorem that will be crucial to extract a solution to our problem, out of a family of approximations. This theorem is known as the Leray-Schauder or Schaefer's Fixed Point Theorem in [Sch55]. We state it, for example, as in Theorem 11.3 in [GT01].

Theorem A.0.1. *Let $T : V \longrightarrow V$ be a continuous and compact mapping, with V a Banach space such that the set*

$$\{v \in V : \exists \gamma \in [0;1] \text{ such that } v = \gamma T(v)\}$$

is bounded. Then T has a fixed point.

Appendix B

to Chapter 4

B.1 Proof of Lemma 4.2.1

We will use the *Brezis–Lieb lemma* (if (X, μ) is a measure space, $q \geq 1$, and $\{f_j\}_j$ is bounded in $L^q(X)$, then

$$\int_X |f|^q \, d\mu = \lim_{j \to \infty} \int_X |f_j|^q \, d\mu - \int_X |f_j - f|^q \, d\mu \,, \tag{B.1}$$

provided f is a μ-a.e. limit of $\{f_j\}_j$ on X), and the two Sobolev-type inequalities

$$\begin{aligned} \|u\|_{L^{p^\star}(\Omega)} &\leq C_1 \left(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right) , \\ \|u\|_{L^{p^\#}(\partial\Omega)} &\leq C_2 \left(\|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right) , \end{aligned}$$

which are valid, with constants C_1 and C_2 depending on n, p and Ω only, as soon as Ω is bounded and has Lipschitz boundary.

Proof of Lemma 4.2.1. Step one: Since Ω is a bounded open set with Lipschitz boundary, $u_j \rightharpoonup u$ as distributions in Ω, and $\{\nabla u_j\}_j$ is bounded in $L^p(\Omega)$, standard considerations show that $u \in W^{1,p}(\Omega)$, $\{u_j\}_j$ is bounded in $L^{p^\star}(\Omega)$ and in $L^{p^\#}(\partial\Omega)$, and, up to extracting subsequences, $u_j \to u$ in $L^q(\Omega)$ for every $q \in [1, p^\star)$, $u_j \to u$ in $L^r(\partial\Omega)$ for every $r < p^\#$, $u_j \to u$ pointwise $\mathcal{L}^n$-a.e.

on Ω and $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, and that the sequences of Radon measures $\{\nu_j\}_j$, $\{\tau_j\}_j$ and $\{\mu_j\}_j$ defined in (4.24) admits weak-$*$ limits ν, τ and μ, with $\operatorname{spt}\nu$ and $\operatorname{spt}\mu$ contained in $\overline{\Omega}$, and $\operatorname{spt}\tau$ contained in $\partial\Omega$.

Step two: We let $\tilde\mu$ denote the weak-$*$ limit of $|\nabla(u_j-u)|^p\,\mathcal{L}^n\llcorner\Omega$ (which exists up to possibly extracting a further subsequence), and claim that $\tilde\mu(\{x\}) = \mu(\{x\})$ for every $x\in\mathbb{R}^n$. Indeed, by the elementary inequality

$$\big||v+w|^p-|w|^p\big|\le\varepsilon\,|v|^p+C(p,\varepsilon)\,|w|^p\,,\qquad v,w\in\mathbb{R}^n\,,\varepsilon>0\,,$$

given $x\in\mathbb{R}^n$ and $r>0$ we see that

$$\Big|\int_{B_r(x)}|\nabla u_j|^p-\int_{B_r(x)}|\nabla(u_j-u)|^p\Big|\le\varepsilon\int_{B_r(x)}|\nabla(u_j-u)|^p+C(p,\varepsilon)\int_{B_r(x)}|\nabla u|^p\,,$$

so that letting first $j\to\infty$ (for $r>0$ such that $\mu(\partial B_r(x))=\tilde\mu(\partial B_r(x))=0$) and then $r\to 0^+$ (along a generic sequence of radii), we find indeed $|\mu(\{x\})-\tilde\mu(\{x\})|\le\varepsilon\,\tilde\mu(\{x\})$ for every $\varepsilon>0$.

Step three: If now pick $\eta\in C^\infty_c(\mathbb{R}^n)$ and set $\tilde\nu=\nu-|u|^{p^\star}\,\mathcal{L}^n\llcorner\Omega$ (notice that, by lower semicontinuity, $\tilde\nu$ is a Radon measure), then, by exploiting, in order, $\nu_j\overset{*}{\rightharpoonup}\nu$, (B.1), (B.2), and $u_j\to u$ in $L^p(\Omega)$, we find

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\eta|^{p^\star}\,d\tilde\nu\le\liminf_{j\to\infty}\int_\Omega|\eta|^{p^\star}\,(|u_j|^{p^\star}-|u|^{p^\star})\;=\lim_{j\to\infty}\int_\Omega|\eta\,(u_j-u)|^{p^\star}\\
&\le C_1^{p^\star}\lim_{j\to\infty}\Big(\|\nabla(\eta(u_j-u))\|_{L^p(\Omega)}+\|\eta(u_j-u)\|_{L^p(\Omega)}\Big)^{p^\star}\\
&=C_1^{p^\star}\lim_{j\to\infty}\|\nabla(\eta(u_j-u))\|_{L^p(\Omega)}^{p^\star}=C_1^{p^\star}\lim_{j\to\infty}\|\eta\,\nabla(u_j-u))\|_{L^p(\Omega)}^{p^\star}\\
&\le C_1^{p^\star}\Big(\int_{\mathbb{R}^n}|\eta|^p\,d\tilde\mu\Big)^{p^\star/p}.
\end{aligned}$$

In particular, for every $x \in \mathbb{R}^n$ one has

$$\tilde{\nu}(B_r(x)) \le C_1^{p^\star}\, \tilde{\mu}(B_r(x))^{p^\star/p} \qquad \text{for a.e. } r > 0\,. \tag{B.4}$$

By (B.4), $\tilde{\nu}$ is absolutely continuous with respect to $\tilde{\mu}$, so that $\tilde{\nu} = f\,\tilde{\mu}$ where f is such that, for $\tilde{\mu}$-a.e. $x \in \mathbb{R}^n$,

$$f(x) = \lim_{r\to 0^+} \frac{\tilde{\nu}(B_r(x))}{\tilde{\mu}(B_r(x))}\,;$$

in particular, again by (B.4), for $\tilde{\mu}$-a.e. $x \in \mathbb{R}^n$ we have

$$f(x) \le C_1^{p^\star} \lim_{r\to 0^+} \tilde{\mu}(B_r(x))^{(p^\star/p)-1} = C_1^{p^\star}\, \mu(\{x\})^{(p^\star/p)-1}\,.$$

In particular, as $p^\star > p$, if $X = \{x \in \mathbb{R}^n : \mu(\{x\}) > 0\} = \{x_i\}_{i\in I} \subset \overline{\Omega}$ (I at most countable) denotes the set of atoms of μ, then $f(x) = 0$ μ-a.e. on $\mathbb{R}^n \setminus X$, and we have proved that

$$\mathrm{spt}\tilde{\nu} \subset \{x_i\}_{i\in I}\,.$$

An entirely analogous argument, this time based on (B.3) rather than on (B.2), shows that, if $\tilde{\tau} = \tau - |u|^{p^\#}\, \mathcal{H}^{n-1} \llcorner \partial\Omega$, then

$$\mathrm{spt}\tilde{\tau} \subset \{x_i\}_{i\in I} \cap \partial\Omega\,.$$

We have thus proved the validity of (4.25), (4.26) and (4.27) for suitable $\mathsf{v}_i, \mathsf{t}_i \ge 0$ and $\mathsf{g}_i > 0$: and of course we can discard possible points x_i with $\mathsf{v}_i = 0$ from these decompositions, and directly assume that $\mathsf{v}_i > 0$ for every i. The fact that $\mathsf{g}_i \ge S\,\mathsf{v}_i$ if $x_i \in \Omega$ is immediate by repeating the above argument with arbitrary $\eta \in C_c^\infty(\Omega)$ (in which case C_1 can be replaced by $1/S$). An analogous argument, this time using Lemma 4.2.2, shows that $\mathsf{g}_i \ge \mathsf{v}_i\, \Phi_H(\mathsf{t}_i/\mathsf{v}_i)$ if $x_i \in \partial\Omega$. □

B.2 Proof of Lemmas 4.2.2 and 4.2.3

This section is dedicated to the proof of Lemmas 4.2.2 and 4.2.3. We recall the standard Taylor expansions for the inverse and determinant of a matrix that is a perturbation of the identity:

$$(\mathrm{Id}_{\mathbb{R}^n} + tA)^{-1} = \mathrm{Id}_{\mathbb{R}^n} - tA + O(t^2)\,, \tag{B.5}$$

$$\det(\mathrm{Id}_{\mathbb{R}^n} + tA) = 1 + t\,\mathrm{trace}A + O(t^2); \tag{B.6}$$

see for instance [**?**, Lemma 17.4].

Proof of Lemma 4.2.2. Note that $\hat{f}$ can equivalently be written as $\hat{f}(x) = x + \ell(\mathbf{p}(x))e_n$. We directly see that $\hat{f}$ maps C^1-diffeomorphically onto its image with inverse $\hat{g}(y) = y - \ell(\mathbf{p}(y))x_n$ and (4.33) holds because $\ell(0) = |\nabla\ell(0)| = 0$ and ℓ is C^1. We compute, in the standard basis for $\mathbb{R}^n$, that

$$\nabla\hat{f}(\mathbf{p}(x), x_n) = \begin{pmatrix} Id & 0 \\ \nabla\ell(\mathbf{p}(x)) & 1 \end{pmatrix} = Id + \begin{pmatrix} 0 & 0 \\ \nabla\ell(\mathbf{p}(x)) & 0 \end{pmatrix}.$$

Since ℓ is C^1 with $\nabla\ell(0) = 0$, this gives the first estimate in (4.34). The second estimate in (4.34) follows in the same way using the explicit form of g above. The first estimate in (4.35) follows from (B.6) and the expression for $\nabla\hat{f}$ above. The second estimate in (4.35) follows because $\hat{f} = F$ on $\mathbf{D}_{r_0}$ (compare with (4.30)), and so

$$J^{\partial H}\hat{f} = J^{\partial H}F = \sqrt{1 + |\nabla\ell|^2}\,.$$

This completes the proof of the lemma. □

Proof of Lemma 4.2.3. Step one: We compute some geometric quantities for $\partial\Omega$ using the graphical coordinates defined by the map F given in (4.30). We start with first order quantities. For $x \in \mathbf{D}_{r_0}$ and $i = 1, \ldots, n-1$, we set $\tau_i := dF_x(e_i)$, i.e.,

$$\tau_i = \partial_i F(x) = e_i + \partial_i \ell(x) e_n, \tag{B.7}$$

so that $\{\tau_1, \ldots, \tau_{n-1}\}$ forms a basis for $T_{F(x)}\partial\Omega$. Since ℓ is C^2 and $\nabla\ell(x) = 0$, we have

$$g_{ij} = \langle \tau_i, \tau_j \rangle_{\mathbb{R}^n} = \delta_{ij} + \partial_i \ell \, \partial_j \ell = \delta_{ij} + \mathrm{O}(|x|^2)\,.$$

In other words, the metric coefficients in graphical coordinates are Euclidean up to second order in $|x|$. The volume measure of $\partial\Omega$ is given by

$$J^{\partial H} F = \sqrt{\det g_{ij}} = \sqrt{1 + |\nabla \ell|^2}\,;$$

this immediately gives the second estimate in (4.37) since $f = F$ for $x \in \mathbf{D}_{r_0}$. The inverse metric coefficients are

$$g^{ij} = \delta^{ij} + \frac{\delta^{ia}\,\delta^{ib}\,\partial_a \ell\,\partial_b \ell}{1 + |\nabla \ell|^2} \delta^{ij} + O(|x|^2)\,. \tag{B.8}$$

Recall that, without loss of generality, we have chosen an orthonormal basis for $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ that diagonalizes the Hessian of ℓ at $x = 0$. Let $\{\kappa_1, \ldots, \kappa_{n-1}\}$ denote the eigenvalues. The second fundamental form of $\partial\Omega$ at x is defined by $A_x(v, w) = -\langle d\nu_\Omega(v), w \rangle$ for tangent vectors $v, w \in T_x(\partial\Omega)$. Using the shorthand $\nu := \nu_\Omega \circ F : \mathbf{D}_{r_0} \to S^{n-1}$, the coefficients of the second fundamental form in the coordinates defined by F are given by $A_{ij} = \langle \partial_{ij} F, \nu \rangle_{\mathbb{R}^n}$. Differentiating

(B.7) above, we have $\partial_{ij} F = \partial_j \tau_i = \partial_{ij}\ell \, e_n$, and thus for $i, j \in \{1, \ldots, n-1\}$ we have

$$A_{ij} = \langle \partial_{ij} F, \nu \rangle = \partial_{ij}\ell \, \langle e_n, \nu \rangle = \frac{-\partial_{ij}\ell}{\sqrt{1 + |\nabla \ell|^2}} = -\partial_{ij}\ell + \mathrm{O}(|x|^2) \, . \tag{B.9}$$

We have

$$\partial_i \nu = -A_i^j \tau_j \, , \tag{B.10}$$

and from (B.8) and (B.9) we directly compute that $A_i^j = g^{ik} A_{kj}$ is given by

$$\begin{aligned} A_i^j = \frac{-\partial_{ij}\ell}{\sqrt{1 + |\nabla \ell|^2}} + O(|x|^2) &= -\partial_{ij}\ell + O(|x|^2) \\ &= -\partial_{ij}\ell(0) + O(|x|^2) = -\kappa_j \delta_i^j + O(|x|^2). \end{aligned}$$

Step 2: Next, we use the previous step to compute geometric quantities associated to the coordinates defined by f. For $x \in \mathbf{C}_{r_0}$ and $i = 1, \ldots, n-1$, we note that $\partial_i \mathbf{p}(x) = e_i$ and thus from (B.10), In what follows we will suppress the composition with $\mathbf{p}$ in our notation, writing for instance τ_i in place of $\tau_i \circ \mathbf{p}(x)$. For $i = 1, \ldots, n-1$, we have

$$\begin{aligned} \partial_i f(x) &= \tau_i - x_n \partial_i \nu(\mathbf{p}(x)) \qquad \text{(B.11)} \\ &= e_i + \partial_i \ell \, e_n + x_n A_i^j \tau_j = e_i + \partial_i \ell \, e_n - x_n \kappa_i e_i + O(|x|^2), \end{aligned}$$

and $$\partial_n f(x) = -\nu = \frac{e_n - \nabla \ell}{\sqrt{1 + |\nabla \ell|^2}} = e_n - \nabla \ell + O(|x|^2). \tag{B.12}$$

Together (B.11) and (B.12) can be expressed in consolidated form as

$$\nabla f = \sum_{i=1}^{n} \partial_i f \otimes e_i = \sum_{i=1}^{n} e_i \otimes e_i + e_n \otimes \nabla \ell - \nabla \ell \otimes e_n - x_n \sum_{i,j=1}^{n-1} \kappa_i(0) e_i \otimes e_i + O(|x|^2)$$

$$= \mathrm{Id}_{\mathbb{R}^n} + e_n \otimes \nabla \ell - \nabla \ell \otimes e_n - x_n \sum_{i,j=1}^{n-1} \kappa_i(0) e_i \otimes e_i + O(|x|^2).$$

In particular, from (B.6) we see that the volume form is given by

$$Jf = \sqrt{\det g_{ij}} = 1 - x_n H_\Omega(0) + O(|x|^2)\,,$$

giving us the first estimate in (4.37). We see directly from the definition that f is a C^1 map, and since we see from the expression for ∇f above that $\nabla f(0) = \mathrm{Id}_{\mathbb{R}^n}$, we may apply the inverse function theorem to see that, up to decreasing r_0, f defines a C^1 diffeomorphism onto its image. Letting $g = f^{-1}$ and using the expansion of the inverse (B.5), we find

$$(\nabla g) \circ f = (\nabla f)^{-1} = \mathrm{Id}_{\mathbb{R}^n} - e_n \otimes \nabla \ell + \nabla \ell \otimes e_n + x_n \sum_{i,j=1}^{n-1} \kappa_i(0) e_i \otimes e_i + O(|x|^2).$$

Finally, (4.36) follows from these expressions for ∇f and ∇g, along with the assumptions that $\nabla \ell(0) = 0$. This completes the proof of the lemma. □

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