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Well-posedness for the space-time Monopole Equation and Ward Wave Map.

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Well-posedness for the space-time Monopole Equation and Ward Wave Map.

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Well-posedness for the space-time Monopole Equation and Ward Wave Map.

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We study local well-posedness of the Cauchy problem for two geometric wave equations that can be derived from Anti-Self-Dual Yang Mills equations on $\mathbb{R}^{2+2}$. These are the space-time Monopole Equation and the Ward Wave Map. The equations can be formulated in different ways. For the formulations we use, we establish local well-posedness results, which are sharp using the iteration methods.

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Chapter 1

Introduction

In this dissertation we study two geometric wave equations that can be obtained from Anti-Self-Dual Yang Mills (ASDYM) equations on $\mathbb{R}^{2+2}$. We begin by introducing the equations and discussing the main results. Subsequently we provide summary of all the chapters.

1.1 Space-time Monopole and Ward Wave Map equations

The space-time Monopole Equation is given by

$$\text{(ME)} \qquad F_A = *D_A\phi,$$

where F_A is the curvature of a one-form connection A on $\mathbb{R}^{2+1}$, $D_A\phi$ is a covariant derivative of the Higgs field ϕ, and $*$ is the Hodge star operator with respect to the Minkowski $\mathbb{R}^{2+1}$ metric.

The Ward Wave Map equation is

$$\text{(WWM)} \qquad (J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0$$

where J is a map from $\mathbb{R}^{2+1}$ into a Lie group, typically taken to be $SU(n)$ or $U(n)$, and $[\cdot,\cdot]$ is a Lie bracket. By selecting a proper gauge we can reduce

(ME) to (WWM). We show this in Chapter 4 whereas in Chapter 3 we show how to obtain (ME) from (ASDYM).

Both equations were introduce by Richard Ward: (ME) in [30] and (WWM) in [29]. We now provide some historical background.

Space-time Monopole Equation

Electric charge is quantized, which means it appears in integer multiples of an electron. This is called the principle of quantization and has been observed in nature. The only theoretical proof so far was presented by Paul Dirac in 1931 [7]. In the proof Dirac introduced the concept of a magnetic monopole, of an isolated point-source of a magnetic charge. Despite extensive research magnetic monopoles have not been found in nature. The magnetic monopole equations are also called Bogomolny Equations. The space-time monopole equation can be viewed as the space-time analog of Bogomolny Equations. In fact, both equations have exactly the same coordinate free form. The difference is that in Bogomolny Equations the base manifold is $\mathbb{R}^3$ instead of R^{2+1}. In addition, both equations are examples of integrable systems and have equivalent formulations as Lax pairs [5].

It is needless to say a lot of work has been done for the magnetic monopoles. For example, see books by Jaffe and Taubes [11] and Atiyah and Hitchin [1]. The literature for (ME) is less abundant. Ward studies it in [30] from the point of view of twistor theory, and investigates its soliton solutions in [31]. Recently, Dai, Terng and Uhlenbeck gave a broad survey on (ME) in [5]. In particular, using scattering transform they show global existence and uniqueness up to a gauge transformation for small initial data in $W^{2,1}$. All in all, the equation has received some interest in the recent years, but its well-posedness theory remains widely open. The objective of this thesis is to try to fill this gap by specifically treating the Cauchy problem for rough initial data in H^s.

Ward Wave Map

(WWM) was introduced by Ward in [29] to provide an example of an integrable model in $2+1$ dimensions that would exhibit traveling solitons. In [28] Villarroel used inverse scattering methods to construct soliton solutions. This was followed by Fokas and Ioannidou [8]. Dai and Terng constructed all solitons in [4] (also see [5]). Ioannidou and Ward presented an infinite sequence of conserved quantities in [10]. Nevertheless, and similarly to (ME), questions of well-posedness have not been considered before.

The main result concerning (ME) is contained in the following theorem.

Main Theorem 1. *The space-time monopole equation (ME) in a Coulomb gauge is locally well-posed for initial data sufficiently small in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{4}$.*

The corresponding result for (WWM) is

Main Theorem 2. *Ward Wave Map is locally well-posed for initial data in $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ for $s > \frac{5}{4}$.*

We make the notion of local well-posedness precise in Chapter 2. For now, we motivate the above results.

Written in coordinates, (ME) is a system of first order hyperbolic partial differential equations. The unknowns are a pair (A, ϕ). If (A, ϕ) solve the equation, then so do

$$\lambda A(\lambda t, \lambda x) \quad \text{and} \quad \lambda \phi(\lambda t, \lambda x),$$

for any $\lambda > 0$. This results in the critical exponent $s_c = 0$ such that the homogeneous Sobolev space $\dot{H}^{s_c}(\mathbb{R}^n)$ is invariant under the above scaling. (WWM) scales the same as a wave map i.e., if J solves (WWM), then so does

$$J(\lambda t, \lambda x),$$

for any $\lambda > 0$. Hence (WWM) is critical in $\dot{H}^1(\mathbb{R}^2)$. Since in general one expects local well-posedness for $s > s_c$ the goal is to show (ME) is well-posed for $s > 0$ and (WWM) for $s > 1$. However, the two spatial dimensions create an obstacle, which so far only allows $s > \frac{1}{4}$ and $s > \frac{5}{4}$ respectively. We explain this now.

In Section 3.3 (ME) is reformulated as a system of semilinear wave equations coupled with an elliptic equation. Schematically it looks as follows

$$\begin{aligned} \triangle A_0 &= \mathcal{E}(\partial u, \partial v, A_0), \\ \Box u &= \mathcal{B}_+(\partial u, \partial v, A_0), \qquad (1.1) \\ \Box v &= \mathcal{B}_-(\partial u, \partial v, A_0), \end{aligned}$$

where $\mathcal{E}, \mathcal{B}_\pm$ are bilinear forms[1], and A_0 is the nondynamical part of the connection A. $\partial u, \partial v$ denote space-time derivatives of u and v respectively, and are given in terms of ϕ and spatial part of A. As a result, showing well-posedness of (ME) for $s > 0$ can follow from showing (1.1) is well-posed for $s > 1$. Also, the most difficult nonlinearity that we have to handle is contained in $\mathcal{B}_\pm(\partial u, \partial v, A_0)$. Luckily, it exhibits a structure of a null form. There are two standard null forms

$$Q_0(u, v) = -\partial_t u \partial_t v + \nabla u \cdot \nabla v, \tag{1.2}$$

$$Q_{\alpha,\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v. \tag{1.3}$$

For these kind of nonlinearities one can assume much less regularity of the initial data than for general products. (See counterexamples for general products found in Lindblad [19] [20].) We uncover the null forms $Q_{\alpha\beta}$ in our system of wave equations as well as a new type of a null form which is related to $Q_{\alpha\beta}$. Unfortunately, the results in two spatial dimensions for $Q_{\alpha\beta}$ are not as optimal as they are in higher dimensions or as they are for Q_0. In fact, the best result in literature so far for $Q_{\alpha\beta}$ in two dimensions is due to Zhou [32]. He establishes local well-posedness for initial data in $H^s \times H^{s-1}$ for $s > \frac{5}{4}$. In addition, by examining the first iterate Zhou shows that this is as close as one can get to the critical level using iteration methods. On the other hand, for dimensions $n \geq 3$ Klainerman and Machedon [16] showed almost optimal local well-posedness in $H^s \times H^{s-1}$ for $s > \frac{n}{2}$. Work of Klainerman and Machedon

[1]See Section 3.3 for the precise formula for $\mathcal{E}$ and $\mathcal{B}_\pm$.

[15] and Klainerman and Selberg [17] gives as satisfying results for Q_0, and in all dimensions $n \geq 2$.

Now, one of the nonlinearities in the system (1.1) is $Q_{\alpha\beta}$, so showing (1.1) is locally well-posed for $s > \frac{5}{4}$ would be sharp by iteration methods. This is what we do in this thesis, and as a result we obtain local well-posedness of (ME) in the Coulomb gauge for $s > \frac{1}{4}$ (See the full statement of the theorem in Chapter 3). However, (1.1) is not exactly (ME), so we hope to treat (ME) directly in the near future and improve the results. What should be mentioned here is that we have considered other traditional gauges such as Lorentz and Temporal, but they have not been as nearly useful as the Coulomb gauge. Perhaps other, less traditional gauges could be used. Moreover, we note that even the estimates involving the nondynamical variable A_0 seem to require $s > \frac{1}{4}$.

To finish the discussion on (ME) we add that our system (1.1) resembles a system considered by Selberg in [22] for the Maxwell-Klein-Gordon (MKG) equations, where he successfully obtains almost optimal local well-posedness in dimensions $1 + 4$. Besides the dimension considered, there are two fundamental technical differences from the point of view of our problem. First comes from the fact that the monopole equation is an example of a system in the non-abelian gauge theory whereas MKG is an example of a system in the abelian gauge theory. The existence of a global Coulomb gauge requires smallness of initial data in the former, but is not needed in the latter. Another technical difference arises from Selberg being able to solve the elliptic equation for his nondynamical variable using Riesz Representation theorem, where he does not require smallness of the initial data. Our elliptic equation is more difficult, and so far we need the restriction on the size of the initial data. Finally, we should point out that the proof of our estimates involving the nondynamical variable A_0 is modeled after Selberg's proof in [22]. However, the two dimensions again complicate matters, and we have to work harder to obtain needed estimates (see Sections 3.4.5 and 3.5).

The above exposition is closely related to (WWM). This is because we can we rewrite (WWM) so it has a form of a wave equation together with Q_0 and Q_{tj} nonlinearities. General framework developed by Selberg [22] makes the proof of Main Theorem 2 very easy and with no need for small initial data. However, with the presence of Q_{tj}, $s > \frac{5}{4}$ is the best we can do with the iteration methods.

1.2 Chapter Summaries

Chapter 2: We review the classical results for semilinear wave equations and the improvements one obtains when the nonlinearity has a structure of a null form. We also introduce the function spaces and the main estimates used as well as rewrite the needed null form estimates in the context of the spaces we use here. Finally we provide some new estimates related to null forms which are required in the later chapters.

Chapter 3: This chapter is devoted to the space-time Monopole Equation.

We take a closer look at the equations and consider its gauge invariance. We write (ME) as a system of wave equations coupled with an elliptic equation and establish the local well-posedness result. In the process we uncover a new null form for which we prove estimates needed to close the Picard iteration. We also present a variety of elliptic estimates for the nondynamical variable A_0.

Chapter 4: In this chapter we discuss the Ward Wave Map. We show the derivation of the equation from (ASDYM), establish conservation of energy and prove the local well-posedness theorem.

Appendices: In appendix A we verify conditions of two theorems that are extensively used throughout several proofs. In Appendix B we show some simple bilinear estimates, which we quote during the proof of Theorem 3.4.2.

Chapter 2

Preliminaries

In this chapter we would like to introduce function spaces and estimates used, as well as give an overview of null forms. Therefore, the first half of the chapter is mostly a review of a well known material. In the second half we establish known null form estimates in the context of the spaces used in this dissertation. We also add some new estimates related to null forms.

2.1 Notation

$a \lesssim b$ means $a \leq Cb$ for some positive constant C. $\hat{u}$ denotes the Fourier transform of u, and $u \precsim v$ means $|\hat{u}| \leq C\hat{v}$ for some $C > 0$. A point in the $2+1$ dimensional Minkowski space is written as $(t, x) = (x^\alpha)_{0 \leq \alpha \leq 2}$. Greek indices range from 0 to 2, and Roman indices range from 1 to 2. We raise and lower indices with the Minkowski metric diag$(-1, 1, 1)$. We write $\partial_\alpha = \partial_{x^\alpha}$ and $\partial_t = \partial_0$, and we also use the Einstein notation. Therefore, $\partial^i \partial_i = \triangle$, and $\partial^\alpha \partial_\alpha = -\partial_t^2 + \triangle = \Box$. When we refer to spatial and time derivatives of a function f, we write ∂f, and when we consider only spatial derivatives of f, we write ∇f. Also, $D^\alpha = (-\triangle)^{\frac{\alpha}{2}}$. Finally, d denotes the exterior differentiation operator and d^* its dual given by $d^* = (-1)^k * * * d *$, where $*$ is the Hodge

$*$ operator and k comes from d^* acting on some given k-form. It will be clear from the context, when $*$ and d^* operators act with respect to the Minkowski metric and when with respect to the Euclidean metric.

2.2 Function Spaces & Inversion of the Wave Operator

We use Picard iteration to find solutions for our equations. Here we introduce the spaces in which we perform the iteration.

First we define following Fourier multiplier operators

$$\begin{aligned}
\widehat{\Lambda^\alpha f}(\xi) &= (1+|\xi|^2)^{\frac{\alpha}{2}}\hat{f}(\xi),\\
\widehat{\Lambda^\alpha_+ u}(\tau,\xi) &= (1+\tau^2+|\xi|^2)^{\frac{\alpha}{2}}\hat{u}(\tau,\xi),\\
\widehat{\Lambda^\alpha_- u}(\tau,\xi) &= \left(1+\frac{(\tau^2-|\xi|^2)^2}{1+\tau^2+|\xi|^2}\right)^{\frac{\alpha}{2}}\hat{u}(\tau,\xi),
\end{aligned} \tag{2.1}$$

where the symbol of Λ^α_- is comparable to $(1+\big||\tau|-|\xi|\big|)^\alpha$. The corresponding homogeneous operators are D, D_+, and D_- respectively. We set the following notation for the symbols

$$\begin{aligned}
w_+(\tau,\xi) = 1+|\tau|+|\xi|, &\quad \dot{w}_+(\tau,\xi) = |\tau|+|\xi|,\\
w_-(\tau,\xi) = 1+\big||\tau|-|\xi|\big|, &\quad \dot{w}_-(\tau,\xi) = \big||\tau|-|\xi|\big|.
\end{aligned} \tag{2.2}$$

The spaces of interest are $H^{s,\theta}$ and $\mathcal{H}^{s,\theta}$ with norms given by

$$\|u\|_{H^{s,\theta}} = \|\Lambda^s\Lambda^\theta_- u\|_{L^2(\mathbb{R}^{2+1})}, \tag{2.3}$$

$$\|u\|_{\mathcal{H}^{s,\theta}} = \|u\|_{H^{s,\theta}} + \|\partial_t u\|_{H^{s-1,\theta}}. \tag{2.4}$$

An equivalent norm for $\mathcal{H}^{s,\theta}$ is $\|u\|_{\mathcal{H}^{s,\theta}} = \|\Lambda^{s-1}\Lambda_+\Lambda^\theta_- u\|_{L^2(\mathbb{R}^{2+1})}$.

These spaces, together with results in [23], allowed Klainerman and

Selberg to present a unified approach to local well-posedness for Wave Maps, Yang-Mills and Maxwell-Klein-Gordon types of equations in [17]. See [25] for a general exposition.

By results in Selberg's thesis [21] if $\theta > \frac{1}{2}$ we have

$$H^{s,\theta} \hookrightarrow C_b(\mathbb{R}, H^s), \tag{2.5}$$

$$\mathcal{H}^{s,\theta} \hookrightarrow C_b(\mathbb{R}, H^s) \cap C_b^1(\mathbb{R}, H^{s-1}). \tag{2.6}$$

This is a crucial fact needed to localize our solutions in time[1]. We denote the corresponding restrictions to the time interval $[0, T]$ for some T by

$$H_T^{s,\theta} \quad \text{and} \quad \mathcal{H}_T^{s,\theta}.$$

$\mathcal{H}^{s,\theta}$ spaces are a very appropriate setting for a local well-posedness of wave equations. The main idea is that when solving a wave equation locally in time, we can replace $\Box^{-1}$ by $\Lambda_+^{-1}\Lambda_-^{-1}$. This goes back to the papers of Bourgain for the Schrödinger and KdV equations [2], and subsequently to the work of Kenig-Ponce-Vega for the KdV [12]. Klainerman-Machedon proved the first estimates for the wave equation in [15]. However, in their paper they require small initial data. This assumption was removed by Selberg in [23], where he showed that by introducing ϵ small enough in the invertible version of the wave operator i.e., $\Lambda_+^{-1}\Lambda_-^{-1+\epsilon}$ we can use initial data as large as we wish[2]. In [23] Selberg also gives a very useful, general framework for local well-posedness of wave equations. Indeed

[1]See [23] for more details.

[2]See also [17] Section 5 for an excellent discussion and motivation of the issues involved in the Picard iteration.

Theorem 2.2.1. ([23] Theorem 2),([17] Theorem 5.3 and 5.4) Given

$$(*) \quad \Box u = \mathcal{N}(u),$$

where u takes values in $\mathbb{R}^N$ and $\mathcal{N}$ is a map

$$\mathcal{N} : \mathcal{H}^{s+1,\theta} \to \mathcal{D}',$$

which is

- time-translation invariant: $\mathcal{N}(u(\cdot + t, \cdot)) = \mathcal{N}(u)(\cdot + t, \cdot)$,
- local in time: if $u|_I = v|_I, I$ an open interval, then $\mathcal{N}(u)|_I = \mathcal{N}(v)|_I$,
- $\mathcal{N}(0) = 0$.

If for some $\epsilon > 0$ we have

$$\|\Lambda_+^{-1}\Lambda_-^{-1+\epsilon}\mathcal{N}(u)\|_{\mathcal{H}^{s+1,\theta}} \lesssim A(\|u\|_{\mathcal{H}^{s+1,\theta}}),$$
$$\|\Lambda_+^{-1}\Lambda_-^{-1+\epsilon}(\mathcal{N}(u) - \mathcal{N}(v))\|_{\mathcal{H}^{s+1,\theta}} \lesssim A'\big(\max(\|u\|_{\mathcal{H}^{s+1,\theta}}, \|v\|_{\mathcal{H}^{s+1,\theta}})\big)\|u - v\|_{\mathcal{H}^{s+1,\theta}},$$

where A and A' are continuous and $A(0) = 0$, then $(*)$ is locally well-posed[3] for initial data in $H^{s+1} \times H^s$.

2.3 Estimates Used

There are many estimates that are fundamental for our results. We state them without a proof and refer the reader to the original sources for the

[3]Local well-posedness is meant here in the sense defined in Section 2.4 with $\mathcal{H}_T^{s+1,\theta} = Y(T)$.

details.

The first estimate is a consequence of a theorem by Klainerman and Tataru [18]. We state it for two dimensions only (the original result holds for $n \geq 2$), and as it was given in [17].

Klainerman-Tataru Theorem. Let $1 \leq p \leq \infty$, $1 \leq q < \infty$. Assume that

$$\frac{1}{p} \leq \frac{1}{2}\left(1 - \frac{1}{q}\right), \tag{2.7}$$

$$0 < \sigma < 2\left(1 - \frac{1}{q} - \frac{1}{p}\right), \tag{2.8}$$

$$s_1, s_2 < 1 - \frac{1}{q} - \frac{1}{2p}, \tag{2.9}$$

$$s_1 + s_2 + \sigma = 2(1 - \frac{1}{q} - \frac{1}{2p}). \tag{2.10}$$

Then

$$\|D^{-\sigma}(uv)\|_{L_t^p L_x^q(\mathbb{R}^2)} \lesssim \|u\|_{H^{s_1,\theta}} \|v\|_{H^{s_2,\theta}}, \tag{2.11}$$

provided $\theta > \frac{1}{2}$.

The theorem was first established for the time-spatial operator D_+. The proof for the spatial operator D was shown by Selberg in [21].

Another important estimate is a version of the Sobolev embedding in the context of the $H^{s,\theta}$ spaces.

Klainerman-Selberg Theorem [17] The embedding

$$H^{\frac{n}{2}-\frac{n}{q}-\frac{1}{p},\theta}(\mathbb{R}^{n+1}) \hookrightarrow L_t^p L_x^q$$

holds whenever $2 \leq p \leq \infty, 2 \leq q < \infty, \frac{2}{p} \leq (n-1)(\frac{1}{2} - \frac{1}{q})$ and $\theta > \frac{1}{2}$.

This is a simple, but a very useful result proved using the triangle inequality.

Lemma 2.3.1. (Product Rule) [17] If $\alpha > 0$, then

$$\Lambda^\alpha(uv) \precsim (\Lambda^\alpha u)v + u\Lambda^\alpha v,$$

for all u, v with $\hat{u}, \hat{v} \geq 0$. Moreover, the same estimate holds with Λ^α replaced by either of the operators D, D_+ or Λ_+.

The previous lemma suffices when we work in L^2. However for a general product rule in L^p, we need a more sophisticated lemma

Lemma 2.3.2. (Liebniz Rule in L^p) [26] Let $s > 0, 1 < p < \infty$,

$$\|fg\|_{W^{s,p}} \leq C\|f\|_{L^{q_1}}\|g\|_{W^{s,q_2}} + C\|g\|_{L^{r_1}}\|f\|_{W^{s,r_2}}$$

provided

$$\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}, \quad q_2, r_2 \in (1, \infty), q_1, r_1 \in (1, \infty].$$

The following follows from Lemma 2(ii) on p. 133 in Stein [24] and is stated in [22] as Lemma 3, which we now reproduce.

Lemma 2.3.3. For $\alpha > 0$ and $1 \leq p \leq \infty$,

$$\|\Lambda^\alpha u\|_{L^p} \lesssim \|u\|_{L^p} + \|D^\alpha u\|_{L^p},$$

where the suppressed constant only depends on α.

Lemma 2.3.4. Let $n = 2$ and $\theta > \frac{1}{2}$, then

$$\|u\|_{L_t^p L_x^2} \lesssim \|u\|_{H^{0,\theta}}, \quad 2 \leq p \leq \infty,$$

Proof. Interpolate between

$$H^{0,\theta} \hookrightarrow L_t^2 L_x^2$$

and (2.5) with $s = 0$. □

Theorem 2.3.5. ([17], Theorem 7.2) Let $s > \frac{n}{2}$ and $\frac{1}{2} < \theta \leq s - \frac{n-1}{2}$. Then

$$H^{a,\alpha} \cdot H^{s,\theta} \hookrightarrow H^{a,\alpha}$$

for all a, α satisfying

$$0 \leq \alpha \leq \theta,$$

$$-s + \alpha < a \leq s.$$

Hence, by duality, for all $-\theta \leq \alpha \leq 0$ and $-s \leq a < s + \alpha$.

One special case is the following

Theorem 2.3.6. ([17], Theorem 7.3) $H^{s,\theta}$ is an algebra if $s > \frac{n}{2}$ and $\frac{1}{2} < \theta \leq s - \frac{n-1}{2}$.

We have a definition before we state the final theorem.

Definition 2.3.1. For $\alpha > 0$ define operator R^α by

$$\widehat{R^\alpha(u)}(\tau,\xi) = \iint r^\alpha(\tau-\lambda,\lambda,\xi-\eta,\eta)\hat{u}(\tau-\lambda,\xi-\eta)\hat{v}(\lambda,\eta)d\lambda$$

where

$$r(\tau,\lambda,\xi,\eta) = \begin{cases} |\xi| + |\eta| - |\xi+\eta| & \text{if} \quad \tau\lambda \geq 0, \\ |\xi+\eta| - \big||\xi| - |\eta|\big| & \text{if} \quad \tau\lambda < 0. \end{cases}$$

We give the result for $n = 2$. The original holds for $n \geq 2$.

Theorem 2.3.7. [17] Let $n = 2$, The estimate

$$\|D^{\gamma} R^{\gamma_-}(u,v)\|_{L^2(\mathbb{R}^{1+2})} \lesssim \|D^{s_1} u\|_{H^{0,\theta}} \|D^{s_2} v\|_{H^{0,\theta}},$$

holds whenever $\theta > \frac{1}{2}$ and $s_1, s_2, \gamma, \gamma_-$ satisfy the following conditions:

$$\begin{aligned}
&\gamma + \gamma_- = s_1 + s_2 - \frac{1}{2}, \\
&\gamma_- \geq \frac{1}{4}, \\
&\gamma > -\frac{1}{2}, \\
&s_i \leq \gamma_- + \frac{1}{2}, \quad i = 1, 2, \\
&s_1 + s_2 \geq \frac{1}{2}, \\
&\quad (s_i, \gamma_-) \neq \left(\frac{3}{4}, -\frac{1}{4}\right), \\
&(s_1 + s_2, \gamma_-) \neq \left(\frac{1}{2}, \frac{1}{4}\right).
\end{aligned} \tag{2.12}$$

2.4 Classical Results

We begin by making precise what we mean by local well-posedness.

Definition. Local Well-Posedness (LWP) Given initial data (f, g) in $H^s \times H^{s-1}$ the Cauchy problem

$$(*) \qquad \begin{cases} \Box u = F(u, \partial u), \\ (u, u_t)|_{t=0} = (f, g), \end{cases}$$

is locally well posed in $H^s \times H^{s-1}$ if:

- **(Local Existence)** There exist time $T = T(\|f\|_{H^s} + \|g\|_{H^{s-1}}) > 0$, a space $Y(T) \hookrightarrow C([0,T], H^s) \cap C^1([0,T], H^{s-1})$, and a function $u \in Y(T)$ which solves $(*)$ on $S_T = [0,T] \times \mathbb{R}^n$ in the sense of distributions and such that the initial conditions are satisfied.

- **(Uniqueness)** u is the unique solution of $(*)$ in $Y(T)$.

- **(Continuous Dependence on Initial Data)** For any (f', g') sufficiently close to (f, g) there exists $u' \in Y(T)$ which solves $(*)$ on S_T and

$$\|u - u'\|_{Y(T)} \leq C(\|f - f'\|_{H^s} + \|g - g'\|_{H^{s-1}}).$$

The classical result relying only on energy estimates can be stated as follows

Classical Local Well-Posedness Theorem. Consider the system

$$(*) \qquad \Box u = F(u, \partial u),$$

where $u : \mathbb{R}^{n+1} \to \mathbb{R}^N$ and F is a smooth $\mathbb{R}^N$-valued function satisfying $F(0) = 0$. Then $(*)$ is locally well-posed for initial data in $H^s \times H^{s-1}(\mathbb{R}^n)$ for all $s > \frac{n}{2} + 1$.

Proof. See [17]. □

In 2D this translates to $s > 2$. Further improvement can come from Strichartz estimates which allow us to only assume $s > \frac{7}{4}$. However as it was shown by Lindblad [19] [20] this is sharp for general products. One example is

$$\Box u = (\partial u)^2$$

which is critical in $\dot{H}^1$, but we need $s > \frac{7}{4}$ to obtain LWP.

2.5 Null Forms

The null condition was introduced by Klainerman in [13], and it was first applied to produce better local well-posedness results for wave equations with a null form by Klainerman and Machedon in [14]. As mentioned in the introduction, wave equations with a null form

$$Q_0(u,v) = -\partial_t u \partial_t v + \nabla u \cdot \nabla v$$

as the nonlinearity allow for optimal results [15] [17] in all dimensions $n \geq 2$, whereas the presence of

$$Q_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v$$

in 2D stops us $\frac{1}{4}$ away from the critical level if we wish to use Picard iteration. This was first showed by Zhou in [32]. Here is the result

Proposition. [32] Consider in $\mathbb{R}^{2+1}$ the Cauchy problem

$$\Box u = Q_{12}(\phi, \varphi), \qquad (u, u_t)|_{t=0} = (0,0),$$

where ϕ, φ solve

$$\Box \phi = 0, \qquad (\phi, \phi_t)|_{t=0} = (f, 0),$$
$$\Box \varphi = 0, \qquad (\varphi, \varphi_t)|_{t=0} = (g, 0), \tag{2.13}$$

and

$$f, g \in H^{s+1}.$$

If $0 < s \leq \frac{1}{4}$, then the first iterate u fails to be in H^{s+1} and more precisely the following estimate fails

$$\|\partial_t u\|_{H^s} + \|\nabla u\|_{H^s} \leq C(t)\|f\|_{H^{s+1}}\|g\|_{H^{s+1}}.$$

The necessity of $s > \frac{1}{4}$ can be also seen from the results of Foschi and Klainerman (see Sect. 13 [9]), and from the appendix of Klainerman and Selberg [17].

2.5.1 Symbols

Here we only review symbols and their estimates related to $Q_{\alpha\beta}$. Consider

$$\begin{aligned} \widehat{Q_{tj}(u,v)}(\tau,\xi) &= \widehat{\partial_t u} * \widehat{\partial_j v}(\tau,\xi) - \widehat{\partial_j u} * \widehat{\partial_t v}(\tau,\xi) \\ &= -\iint \big((\tau-\lambda)\eta_j - (\xi_j-\eta_j)\lambda\big)\hat{u}(\tau-\lambda,\xi-\eta)\hat{v}(\lambda,\eta)d\lambda d\eta \end{aligned}$$

It follows the symbol of Q_{tj}, denoted by q_{tj}, is

$$q_{tj}(\tau,\xi,\lambda,\eta) = \tau\eta_j - \lambda\xi_j.$$

Similarly, the symbol of Q_{ij}, denoted by q_{ij}, is

$$q_{ij}(\xi,\eta) = \xi_i\eta_j - \eta_i\xi_j.$$

We have the following estimates

Lemma 2.5.1. If $q_{ij}(\xi,\eta) = \xi_i\eta_j - \eta_i\xi_j$, then

$$q_{ij}^2 \leq 2|\xi||\eta|(|\xi||\eta| \mp \xi\cdot\eta) = \begin{cases} |\xi||\eta|\big((|\xi|+|\eta|)^2 - |\xi+\eta|^2\big) \\ |\xi||\eta|\big((|\xi+\eta|^2 - \big||\xi|-|\eta|\big|^2\big) \end{cases} \tag{2.14}$$

and

$$q_{ij}^2 \leq C|\xi||\eta|\Big(||\xi| \pm |\eta|| + |\xi+\eta|\Big)$$
$$\times(w_-(\tau+\lambda,\xi+\eta) + w_-(\tau,\xi) + w_-(\lambda,\eta)), \tag{2.15}$$

where $w_-(\cdot,\cdot)$ is as in (2.2).

Proof. The first bound in (2.14) is obvious once one observes that

$$q_{ij}^2 \leq |\xi\times\eta|^2 = |\xi|^2|\eta|^2 - (\xi\cdot\eta)^2 = (|\xi||\eta| - \xi\cdot\eta)(|\xi||\eta| + \xi\cdot\eta),$$

and the second part of the inequality can be checked by direct computation [32][13][16].

To show (2.15) one uses (2.14). The proof can be found in [32] (compare with [13] [16].) □

Lemma 2.5.2. [16]

$$Q_{ij}(u,v) \precsim D^{\frac{1}{2}}D_-^{\frac{1}{2}}(D^{\frac{1}{2}}uD^{\frac{1}{2}}v) + D^{\frac{1}{2}}(D_-^{\frac{1}{2}}D^{\frac{1}{2}}uD^{\frac{1}{2}}v) + D^{\frac{1}{2}}(D^{\frac{1}{2}}uD_-^{\frac{1}{2}}D^{\frac{1}{2}}v). \tag{2.16}$$

Now we would like to examine the symbol q_{tj} more carefully. First of all, if we were to consider Q_{tj} in the first iterate or in general, for any functions u, v, whose Fourier transform is supported on a light cone $\tau = \pm|\xi|$, q_{tj} would reduce to

$$\pm|\xi|\eta_j \mp |\eta|\xi_j \quad \text{or} \quad \pm|\xi|\eta_j \pm |\eta|\xi_j,$$

depending if $\hat{u}, \hat{v}$ are both supported on a forward light cone ($++$ interactions), or on a backward light cone ($--$ interactions), or $\hat{u}$ is supported on a forward light cone and $\hat{v}$ on a backward light cone ($+-$ interactions) or finally, $\hat{u}$ is supported on a backward light cone and $\hat{v}$ on a forward light cone ($-+$ interactions). It is enough to just consider $++$ and $+-$ interactions. The corresponding symbols are

$$\sigma_{\pm}(\xi, \eta) = |\xi|\eta_j \mp |\eta|\xi_j.$$

$\sigma_{\pm}$ is usually referred to as the reduced symbol. There is a relationship between q_{tj} and $\sigma_{\pm}$ that can be seen in the following lemma.

Lemma 2.5.3.

$$q_{tj}(\tau, \xi, \lambda, \eta) \leq \begin{cases} \big||\tau| - |\xi|\big||\eta| + \big||\lambda| - |\eta|\big||\xi| + \sigma_+(\xi, \eta) & \text{if} \quad \tau\lambda \geq 0, \\ \big||\tau| - |\xi|\big||\eta| + \big||\lambda| - |\eta|\big||\xi| + \sigma_-(\xi, \eta) & \text{if} \quad \tau\lambda < 0. \end{cases}$$

Proof. Suppose $\tau, \lambda \geq 0$, then

$$\tau\eta_j - \xi_j\lambda = (\tau - |\xi|)\eta_j + (|\eta| - \lambda)\xi_j + \sigma_+.$$

The rest of the signs follows similarly. □

We can establish similar estimates for $\sigma_{\pm}$ as the ones we have for q_{ij}.

Lemma 2.5.4. Let $\sigma_{\pm} = |\xi|\eta_j \mp |\eta|\xi_j$, then

$$\sigma_{\pm}^2 \leq 4|\xi||\eta|(|\xi||\eta| \mp \xi \cdot \eta), \tag{2.17}$$

and

$$\begin{aligned}
&\text{if} \quad \tau\lambda \geq 0, \quad \sigma_+^2 \leq C|\xi||\eta|\Big(|\xi| + |\eta| + |\xi+\eta|\Big) \\
&\qquad\qquad\qquad \times (w_-(\tau+\lambda, \xi+\eta) + w_-(\tau,\xi) + w_-(\lambda,\eta)). \\
&\text{if} \quad \tau\lambda < 0, \quad \sigma_-^2 \leq C|\xi||\eta|\Big(||\xi| - |\eta|| + |\xi+\eta|\Big) \\
&\qquad\qquad\qquad \times (w_-(\tau+\lambda, \xi+\eta) + w_-(\tau,\xi) + w_-(\lambda,\eta)).
\end{aligned} \tag{2.18}$$

Proof. To show (2.17) we first observe that $\sigma_\pm$ can be viewed as one of the components of a cross product of $X = (\xi, |\xi|)$ with $Y = (\eta, \pm|\eta|)$. Therefore

$$\begin{aligned}
|\sigma_\pm(\xi,\eta)|^2 &\leq |X \times Y|^2 \\
&= |X|^2|Y|^2 - (X\cdot Y)^2 \\
&= (|X||Y| - X\cdot Y)(|X||Y| + X \cdot Y).
\end{aligned}$$

Then for σ_+ we have,

$$\begin{aligned}
|\sigma_+(\xi,\eta)|^2 &= (|X||Y| - X\cdot Y)(|X||Y| + X\cdot Y) \\
&= (2|\xi||\eta| - \xi\cdot\eta - |\xi||\eta|)(|X||Y| + X\cdot Y) \\
&\leq 4|\xi||\eta|(|\xi||\eta| - \xi\cdot\eta).
\end{aligned}$$

And for σ_-,

$$\begin{aligned}
|\sigma_-(\xi,\eta)|^2 &= (|X||Y| - X\cdot Y)(|X||Y| + X\cdot Y) \\
&= (|X||Y| - X\cdot Y)(2|\xi||\eta| + \xi\cdot\eta - |\xi||\eta|) \\
&\leq 4|\xi||\eta|(|\xi||\eta| + \xi\cdot\eta),
\end{aligned}$$

as needed.

Now (2.15) follows from Proposition 5.1 in [13] that says

$$\begin{aligned}
\sigma_+(\xi,\eta) &\leq c|\xi|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}(|\xi| + |\eta| - |\xi+\eta|)^{\frac{1}{2}}(|\xi| + |\eta|)^{\frac{1}{2}} \\
\sigma_-(\xi,\eta) &\leq c|\xi|^{\frac{1}{2}}|\eta|^{\frac{1}{2}}(|\xi+\eta| - ||\xi| - |\eta||)^{\frac{1}{2}}(|\xi+\eta|)^{\frac{1}{2}},
\end{aligned}$$

and from Corollary 1 also in [13], which gives that if $\tau\lambda \geq 0$, then

$$\frac{1}{3}||\xi| + |\eta| - |\xi + \eta|| \leq \dot{w}(\tau, \xi) + \dot{w}(\lambda, \eta) + \dot{w}(\tau + \lambda, \xi + \eta),$$

and if $\tau\lambda \leq 0$, then

$$\frac{1}{3}|\xi + \eta| - ||\xi| - |\eta|| \leq \dot{w}(\tau, \xi) + \dot{w}(\lambda, \eta) + \dot{w}(\tau + \lambda, \xi + \eta).$$

□

2.5.2 Null Form Estimates in 2D

In several places in this thesis we use that we have appropriate estimates for null forms Q_0 and $Q_{\alpha\beta}$. In this section we review the main estimates needed in the context of well-posedness as well as establish some related ones that we have not seen in the literature before, but will be needed in our proofs later (Part of theorem 2.5.6 involving the first iterate and theorem 2.5.8).

Start at the beginning. By the discussion in Section 2.2, if the Picard iteration is done in $\mathcal{H}^{s+1,\theta}$, where

$$u \in \mathcal{H}^{s+1,\theta} \Leftrightarrow \Lambda^s \Lambda_+ \Lambda_-^\theta u \in L^2(\mathbb{R}^{2+1}),$$

and if Q denotes any of the null forms in question, we would like to show

$$\Lambda_+^{-1} \Lambda_-^{-1+\epsilon} Q(\mathcal{H}^{s+1,\theta}, \mathcal{H}^{s+1,\theta}) \hookrightarrow \mathcal{H}^{s+1,\theta}. \tag{2.19}$$

Since $\Lambda_+ \Lambda_-^{1-\epsilon} \mathcal{H}^{s+1,\theta} = H^{s,\theta-1+\epsilon}$, (2.19) is equivalent to showing

$$Q(\mathcal{H}^{s+1,\theta}, \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s,\theta-1+\epsilon}.$$

As mentioned before, in 2D Q_0 is best behaved out of the null forms we consider. We have the following optimal result, which can be found in [17] as a part of the proof of the LWP for Wave Maps.

Theorem 2.5.5. [17] Let $n \geq 2, s > \frac{n}{2}$. Suppose

$$\frac{1}{2} < \theta \leq \min(1, s - \frac{n-1}{2}),$$
$$0 \leq \epsilon \leq \min\left(1 - \theta, s - \frac{n-1}{2} - \theta\right),$$

then

$$Q_0(\mathcal{H}^{s+1,\theta}, \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s-1,\theta+\epsilon-1}.$$

For $Q_{\alpha\beta}$ norms we rely on work of Zhou [32]. Zhou's proof is done using spaces $N^{s+1,\theta}$, where the norm is given by[4]

$$N^{s+1,\theta}(u) = \|\Lambda_+^{s+1}\Lambda_-^{\theta} u\|_{L^2}. \tag{2.20}$$

We state the result. Note $\theta = s + \frac{1}{2}$.

Theorem. [32] Consider in $\mathbb{R}^{2+1}$ the space time norms (2.20) and functions φ, ψ defined on $\mathbb{R}^{2+1}$. The estimates

$$N_{s,s-\frac{1}{2}}(Q_{\alpha,\beta}(\varphi,\psi)) \lesssim N_{s+1,s+\frac{1}{2}}(\varphi) N_{s+1,s+\frac{1}{2}}(\psi)$$

hold for any $\frac{1}{4} < s < \frac{1}{2}$.

[4]see [21] Section 3.5 for a comparison with $\mathcal{H}^{s,\theta}$ spaces.

Our iteration is done using spaces $\mathcal{H}^{s+1,\theta}$. Inspection of Zhou's proof shows that it be could be easily modified to be placed in the context of $\mathcal{H}^{s+1,\theta}$ spaces. Zhou's proof works for $\frac{1}{4} < s < \frac{1}{2}$, but studying of his proof motivated an alternate proof that works for all values of $s > \frac{1}{4}$. The proof is closely related to the original proof in [32], but on the surface it appears to be less technical than the original. The reason for this is that we use Theorem 2.3.7 proved in [17], which involves all the technicalities.

Finally, in the proof below we include the enterprise of ϵ as well, and we make minimal assumptions on the regularity of our functions involved.

Theorem 2.5.6. Let $s > \frac{1}{4}$ and

$$\begin{aligned}&\frac{3}{4} - \frac{\epsilon}{2} < \theta \leq s + \frac{1}{2} - \epsilon \quad \text{and} \quad \theta < 1 - \epsilon \\ &0 \leq \epsilon < \min(2s - \frac{1}{2}, \frac{1}{2}).\end{aligned}$$

Also let

$$q = q_{ij} \quad \text{or} \quad q = \begin{cases} \sigma_+(\xi, \eta) & \text{if} \quad \tau\lambda \geq 0, \\ \sigma_-(\xi, \eta) & \text{if} \quad \tau\lambda < 0, \end{cases}$$

and

$$\widehat{Q(u,v)}(\tau, \xi) = \iint q(\xi - \eta, \eta)\hat{u}(\tau - \lambda, \xi - \eta)\hat{v}(\lambda, \eta) d\lambda d\eta,$$

then

$$\|Q(u,v)\|_{H^{s,\theta-1+\epsilon}} \lesssim \|Du\|_{H^{s,\theta}} \|Dv\|_{H^{s,\theta}}$$

Proof. We would like to establish

$$Q(\mathcal{H}^{s+1,\theta}, \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s,\theta-1+\epsilon},$$

where we use the notation

$$u \in H^{s+1,\theta} \quad \text{iff} \quad Du \in H^{s,\theta}.$$

Claim, if $\tau\lambda \geq 0$

$$Q(u,v) \precsim D^{\frac{1}{2}} R^{\theta+\epsilon-\frac{1}{2}} \{ D_-^{1-\theta-\epsilon}(D^{\frac{1}{2}}u D^{\frac{1}{2}}v) + D_-^{1-\theta-\epsilon} D^{\frac{1}{2}}u D^{\frac{1}{2}}v + D^{\frac{1}{2}}u D_-^{1-\theta-\epsilon} D^{\frac{1}{2}}v \},$$

and if $\tau\lambda < 0$,

$$Q(u,v) \precsim \begin{cases} R^{\theta+\epsilon-\frac{1}{2}} \{ D_-^{1-\theta-\epsilon}(Du D^{\frac{1}{2}}v) + D_-^{1-\theta-\epsilon} Du D^{\frac{1}{2}}v + Du D_-^{1-\theta-\epsilon} D^{\frac{1}{2}}v \} & \text{if} \quad |\eta| \leq |\xi|, \\ R^{\theta+\epsilon-\frac{1}{2}} \{ D_-^{1-\theta-\epsilon}(D^{\frac{1}{2}}u Dv) + D_-^{1-\theta-\epsilon} D^{\frac{1}{2}}u Dv + D^{\frac{1}{2}}u D_-^{1-\theta-\epsilon} Dv \} & \text{if} \quad |\xi| < |\eta|, \end{cases}$$

where $R^{\theta+\epsilon-\frac{1}{2}}$ is the operator defined right before Theorem 2.3.7. To see this write[5]

$$q = q^{2-2\theta-2\epsilon} q^{2\theta+2\epsilon-1}$$

and for q_{ij} use estimate (2.15) for the term with the power $2-2\theta-2\epsilon$ and (2.14) for the power $2\theta+2\epsilon-1$. For $\sigma_\pm$ use estimate (2.18) for the term with the power $2-2\theta-2\epsilon$ and (2.17) for the power $2\theta+2\epsilon-1$. Then the claim follows. Now we use that u and v have the same regularity, so by symmetry of the estimates it is enough to show

$$R^{\theta+\epsilon-\frac{1}{2}}(H^{s,\theta} \cdot H^{s+\frac{1}{2},\theta}) \hookrightarrow H^{s,0}$$

$$R^{\theta+\epsilon-\frac{1}{2}}(H^{s,2\theta+\epsilon-1} \cdot H^{s+\frac{1}{2},\theta}) \hookrightarrow H^{s,0}$$

$$R^{\theta+\epsilon-\frac{1}{2}}(H^{s,\theta} \cdot H^{s+\frac{1}{2},2\theta+\epsilon-1}) \hookrightarrow H^{s,0}$$

[5]This is the idea borrowed directly from Zhou except that we keep θ general.

Since $2\theta+\epsilon-\frac{1}{2}\leq\theta$ it is enough to show the first two estimates. By applying product rule, Lemma 2.3.1 we reduce them to

$$R^{\theta+\epsilon-\frac{1}{2}}(H^{0,\theta}\cdot H^{s+\frac{1}{2},\theta})\hookrightarrow L^2 \tag{2.21}$$

$$R^{\theta+\epsilon-\frac{1}{2}}(H^{s,\theta}\cdot H^{\frac{1}{2},\theta})\hookrightarrow L^2 \tag{2.22}$$

$$R^{\theta+\epsilon-\frac{1}{2}}(H^{0,2\theta+\epsilon-1}\cdot H^{s+\frac{1}{2},\theta})\hookrightarrow L^2 \tag{2.23}$$

$$R^{\theta+\epsilon-\frac{1}{2}}(H^{s,2\theta+\epsilon-1}\cdot H^{\frac{1}{2},\theta})\hookrightarrow L^2 \tag{2.24}$$

Then again it suffices to only show estimates (2.23)-(2.24), which follow from Theorem 2.3.7. □

Corollary 2.5.7. Let s,θ and ϵ be as in the above theorem, then

$$\|Q(u,v)\|_{H^{s,\theta-1+\epsilon}}\lesssim\|u\|_{\mathcal{H}^{s+1,\theta}}\|v\|_{\mathcal{H}^{s+1,\theta}}.$$

Next, Zhou shows details for Q_{ij} and remarks that Q_{tj} can be handled similarly. We found a way to estimate Q_{tj}, and while we do not know if that is what author had in mind, the result below is very useful for estimates in Chapter 3. The estimate for Q_{tj} follows as a corollary. Note $s>0$ suffices below.

Theorem 2.5.8. Let $s>0$ and

$$\max(\frac{1}{2},1-s)<\theta<1,$$
$$0\leq\epsilon\leq1-\theta,$$

then

$$\|D_+uD_-v\|_{H^{s,\theta-1+\epsilon}}\lesssim\|u\|_{\mathcal{H}^{s+1,\theta}}\|v\|_{\mathcal{H}^{s+1,\theta}}$$

Proof. We would like to show

$$\|\Lambda^s \Lambda_-^{\theta-1+\epsilon}(D_+ u D_- v)\|_{L^2(\mathbb{R}^{2+1})} \lesssim \|u\|_{\mathcal{H}^{s+1,\theta}} \|v\|_{\mathcal{H}^{s+1,\theta}}.$$

By the product rule, Lemma 2.3.1, and using $\theta - 1 + \epsilon \le 0$ this follows from (stronger) estimates

$$\|uv\|_{L^2(\mathbb{R}^{2+1})} \lesssim \|u\|_{H^{0,\theta}} \|v\|_{\mathcal{H}^{s+1,\theta-1}}, \tag{2.25}$$

$$\|uv\|_{L^2(\mathbb{R}^{2+1})} \lesssim \|u\|_{H^{s,\theta}} \|v\|_{\mathcal{H}^{1,\theta-1}}. \tag{2.26}$$

Let $\|\cdot\|$ denote the $L^2(\mathbb{R}^{2+1})$ norm, and let

$$F(\tau,\xi) = w_-^\theta(\tau,\xi)\hat{u}(\tau,\xi),$$

$$G(\tau,\xi) = (1+|\xi|)^s w_+(\tau,\xi) w_-^{\theta-1}(\tau,\xi)\hat{v}(\tau,\xi).$$

Using duality the corresponding integral for (2.25) is

$$\begin{aligned}
&\iint \frac{F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\xi d\lambda d\eta}{w_-^\theta(\tau,\xi)(1+|\eta|)^s w_+(\lambda,\eta) w_-^{\theta-1}(\lambda,\eta)} \\
&\le \iint \frac{w_+(\lambda,\eta)^{1-\theta} F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\xi d\lambda d\eta}{w_-^\theta(\tau,\xi)(1+|\eta|)^s w_+(\lambda,\eta)} \quad \text{since} \quad \theta < 1 \\
&\le \iint \frac{F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\xi d\lambda d\eta}{w_-^\theta(\tau,\xi)(1+|\eta|)^s w_+^\theta(\lambda,\eta)}, \\
&\le \|F\|\|G\| \left\{ \iint \frac{H^2(\tau+\lambda,\xi+\eta)}{w_-^{2\theta}(\tau,\xi)(1+|\eta|)^{2(s+\theta)}} d\tau d\xi d\lambda d\eta \right\}^{\frac{1}{2}} \\
&\le \|F\|\|G\|\|H\| \left\{ \iint \frac{du d\eta}{(1+|u|)^{2\theta}(1+|\eta|)^{2(s+\theta)}} \right\}^{\frac{1}{2}}, \quad u = |\tau| - |\xi|,
\end{aligned}$$

which is bounded by our conditions on s and θ. The integral for (2.26) is bounded by

$$
\begin{aligned}
&\iint \frac{w_+(\lambda,\eta)^{1-\theta}F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\xi d\lambda d\eta}{w_-^{\theta}(\tau,\xi)(1+|\xi|)^s w_+(\lambda,\eta)} \\
&\le \iint \frac{F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\xi d\lambda d\eta}{w_-^{\theta}(\tau,\xi)(1+|\xi|)^s w_+^{\theta}(\lambda,\eta)}, \\
&\le \|F\|\|G\| \left\{ \iint \frac{H^2(\tau+\lambda,\xi+\eta)}{w_-^{2\theta}(\tau,\xi)(1+|\xi|)^{2s}(1+|\eta|)^{2\theta}} d\tau d\xi d\lambda d\eta \right\}^{\frac{1}{2}} \\
&\le \|F\|\|G\| \left\{ \iint \frac{H^2(\lambda,\eta)}{(1+|u|)^{2\theta}(1+|\xi|)^{2s}(1+|\eta-\xi|)^{2\theta}} du d\xi d\lambda d\eta \right\}^{\frac{1}{2}}, \quad u = |\tau|-|\xi|,
\end{aligned}
$$

which is also bounded by our conditions on s and θ. □

Remark 2.5.1. **Embedding approach for $D_+ uD_- v$.**

There is an alternate proof one can give for Theorem 2.5.8. We would like to show

$$
\|\Lambda^s \Lambda_-^{\theta-1+\epsilon}(D_+ uD_- v)\|_{L^2(\mathbb{R}^{2+1})} \lesssim \|u\|_{\mathcal{H}^{s+1,s+\frac{1}{2}}} \|v\|_{\mathcal{H}^{s+1,s+\frac{1}{2}}}.
$$

This is equivalent to showing

$$
H^{s,\theta} \cdot \mathcal{H}^{s+1,\theta-1} \hookrightarrow H^{s,\theta-1+\epsilon},
$$

which by the product rule for the operator Λ^s in turn follows from

$$
\begin{aligned}
H^{0,\theta} \cdot \mathcal{H}^{s+1,\theta-1} &\hookrightarrow H^{0,\theta-1+\epsilon}, \\
H^{s,\theta} \cdot \mathcal{H}^{1,\theta-1} &\hookrightarrow H^{0,\theta-1+\epsilon}.
\end{aligned}
$$

It is easy to check

$$
\mathcal{H}^{s+1,\theta-1} \hookrightarrow H^{s+1+\theta-1,0} \quad \text{and} \quad \mathcal{H}^{1,\theta-1} \hookrightarrow H^{\theta,0},
$$

so we just need to show

$$H^{0,\theta} \cdot H^{s+\theta,0} \hookrightarrow H^{0,\theta-1+\epsilon},$$
$$H^{s,\theta} \cdot H^{\theta,0} \hookrightarrow H^{0,\theta-1+\epsilon},$$

which are weaker than

$$H^{0,\theta} \cdot H^{s+\theta,0} \hookrightarrow L^2,$$
$$H^{s,\theta} \cdot H^{\theta,0} \hookrightarrow L^2,$$

but those follow from Proposition A.1 in [17] as long as $s+\theta > 1$, which follows from the conditions we impose on s and θ.

Corollary 2.5.9. Let $s > 0$ and

$$\max(\frac{1}{2}, 1-s) < \theta < 1,$$
$$0 \leq \epsilon \leq 1-\theta,$$

then

$$\|DuD_-v\|_{H^{s,\theta-1+\epsilon}} \lesssim \|u\|_{\mathcal{H}^{s+1,\theta}} \|v\|_{\mathcal{H}^{s+1,\theta}}.$$

Corollary 2.5.10. Let $s > \frac{1}{4}$ and

$$\frac{3}{4} - \frac{\epsilon}{2} < \theta \leq s + \frac{1}{2} - \epsilon \quad \text{and} \quad \theta < 1 - \epsilon$$
$$0 \leq \epsilon < \min(2s - \frac{1}{2}, \frac{1}{2})$$

then

$$\|Q_{tj}(u,v)\|_{H^{s,\theta-1+\epsilon}} \lesssim \|u\|_{\mathcal{H}^{s+1,\theta}} \|v\|_{\mathcal{H}^{s+1,\theta}}$$

Proof. Use lemma 2.5.3. Then by theorem 2.5.8 and 2.5.6 the result follows. □

2.5.3 Estimates with more regularity.

As one would expect, the further we are from the critical level, the easier the estimates become. Here we quickly show the simpler proof for Q_{ij} if $s > \frac{1}{2}$. Note the same proof can work for a null form defined using $\sigma_\pm$ when we apply estimate (2.18). We need to show

$$Q_{ij}(\mathcal{H}^{s+1,\theta}, \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s,\theta-1+\epsilon}. \tag{2.27}$$

By lemma (2.5.2) and symmetry, this would follow from

$$D^{\frac{1}{2}} D_-^{\frac{1}{2}} (D^{\frac{1}{2}} \mathcal{H}^{s+1,\theta} D^{\frac{1}{2}} \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s,\theta-1+\epsilon}, \tag{2.28}$$

$$D^{\frac{1}{2}} (D^{\frac{1}{2}} D_-^{\frac{1}{2}} \mathcal{H}^{s+1,\theta} D^{\frac{1}{2}} \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s,\theta-1+\epsilon}, \tag{2.29}$$

which in turn reduce to showing

$$H^{s+\frac{1}{2},\theta} \cdot H^{s+\frac{1}{2},\theta} \hookrightarrow H^{s+\frac{1}{2},\theta-\frac{1}{2}+\epsilon}, \tag{2.30}$$

$$H^{s+\frac{1}{2},\theta-\frac{1}{2}} \cdot H^{s+\frac{1}{2},\theta} \hookrightarrow H^{s+\frac{1}{2},\theta-1+\epsilon}. \tag{2.31}$$

(2.30) is weaker than,

$$H^{s+\frac{1}{2},\theta} \cdot H^{s+\frac{1}{2},\theta} \hookrightarrow H^{s+\frac{1}{2},\theta}. \tag{2.32}$$

Now, for $s > \frac{1}{2}$, by Theorem 2.3.6 $H^{s+\frac{1}{2},\theta}$ is an algebra, so (2.30) follows. (2.31) is a special case of Theorem 2.3.5 with $a = s + \frac{1}{2}$ and $\alpha = \theta - \frac{1}{2} + \epsilon$.

Chapter 3

Space-time Monopole Equation

The objective of this chapter is to establish Main Theorem 1 stated in the introduction. To that end we give a precise statement of the theorem.

Main Theorem 1. *Consider the space-time monopole equation*

$$\text{(ME)} \qquad F_A = *D_A\phi,$$

with initial data

$$(A_1, A_2, \phi)|_{t=0} = (a_1, a_2, \phi_0),$$

then (ME) in a Coulomb gauge is locally well-posed for initial data sufficiently small in $H^s(\mathbb{R}^2)$ for $s > \frac{1}{4}$ in the following sense:

- ***(Local Existence)*** *For all $a_1, a_2, \phi_0 \in H^s(\mathbb{R}^2)$ sufficiently small with $s > \frac{1}{4}$ there exist $T > 0$ depending continuously on the norm of the initial data, and functions*

  $$A_0 \in C_b([0,T], \dot{H}^r), r \in (0, \min(2s, 1+s)],$$
  $$A_1, A_2, \phi \in C_b([0,T], H^s),$$

  which solve (ME) in a Coulomb Gauge on $[0,T] \times \mathbb{R}^2$ in the sense of distributions and such that the initial conditions are satisfied.

- ***(Uniqueness)*** *If* $T > 0$ *and* (A, ϕ) *and* (A', ϕ') *are two solutions in a Coulomb gauge of (ME) on* $(0,T) \times \mathbb{R}^2$ *belonging to*

$$C_b([0,T], \dot{H}^r) \times (H^{s,\theta}_T)^3, r \in (0, \min(2s, 1+s)]$$

with the same initial data, then $(A, \phi) = (A', \phi')$ *on* $(0,T) \times \mathbb{R}^2$*.*

- ***(Continuous Dependence on Initial Data)*** *For any* $a_1, a_2, \phi_0 \in H^s(\mathbb{R}^2)$ *there is a neighborhood* U *of* a_1, a_2, ϕ_0 *in* $(H^s(\mathbb{R}^2))^3$ *for* $s > \frac{1}{4}$ *such that the solution map* $(a, \phi_0) \to (A, \phi)$ *is continuous from* U *into* $C_b([0,T], \dot{H}^r) \times (C_b([0,T], H^s))^3, r \in (0, \min(2s, 1+s)]$.

Remark 3.0.2. We do not prescribe initial data for A_0, because when A is in a Coulomb gauge, $A_0(t)$ can be determined at any time by solving an elliptic equation. See Section 3.3 for more details.

We start by taking a closer look at the equations. Next we discuss gauge transformations. In 3.3 we rewrite (ME) as a system of wave equations coupled with an elliptic equation. Section 3.4 is devoted to the proof of Main Theorem 1.

3.1 Closer look at the Monopole Equation

Given a space-time Monopole Equation

$$\text{(ME)} \qquad F_A = *D_A\phi,$$

the unknowns are a pair (A, ϕ). A is a connection one-form given by

$$A = A_0 dt + A_1 dx + A_2 dy,$$

where

$$A_\alpha : \mathbb{R}^{2,1} \to \mathfrak{g}.$$

$\mathfrak{g}$ is the Lie algebra of a Lie group G, which is typically taken to be a matrix group $SU(n)$ or $U(n)$. We consider $G = SU(n)$, but everything we say here should generalize to any compact Lie group.

To be more general we could say A is a connection on a principal G-bundle. Then observe that the G-bundle we deal here with is a trivial bundle $\mathbb{R}^{2+1} \times G$.

Next, Higgs field ϕ is a section of a vector bundle associated to the G-bundle by a representation. We use the adjoint representation. Since we have a trivial bundle, we just think of ϕ as a map

$$\phi : \mathbb{R}^{2,1} \to \mathfrak{g},$$

F_A is the curvature of A. It is a Lie algebra valued 2-form on $\mathbb{R}^{2,1}$

$$F_A = dA + A \wedge A = \sum_{\alpha<\beta} (\partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]) dx^\alpha \wedge dx^\beta.$$

In the physics language, frequently adopted by the mathematicians, A is called a gauge potential, ϕ a scalar field and F_A is called an electromagnetic field. D_A is the covariant derivative associated to A

$$D_A = d + A,$$

and $D_A\phi$ is given by

$$D_A\phi = d\phi + [A \wedge \phi] = D_\alpha \phi dx^\alpha = (\partial_\alpha \phi + [A_\alpha, \phi]) dx^\alpha$$

The space-time monopole equation (ME) is obtained by a dimensional reduction of the Anti-Self-Dual Yang Mills Equations on $\mathbb{R}^{2+2}$

$$\text{(ASDYM)} \qquad F_A = - * F_A.$$

Remark 3.1.1. If the curvature of a connection A satisfies (ASDYM), then A is called an anti-self dual connection. If $F_A = *F_A$, A is called self-dual. It is worth noting that the equations are called Anti-Self-Dual Yang Mills and Self-Dual Yang Mills respectively, because if

$$F_A = \pm * F_A,$$

then F_A satisfies the Yang Mills equation: $D^*F = 0$ since then

$$D^*F = \pm * D * F = \pm * DF = 0,$$

where the last equality follows from the second Bianchi identity.

We now present the details of the derivation of the Monopole Equations from (ASDYM), which are outlined in [5]. Let

$$dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$$

be a metric on $\mathbb{R}^{2,2}$, then in coordinates (ASDYM) is

$$F_{13} = -F_{24}, \quad F_{12} = -F_{34}, \quad F_{23} = F_{14}. \tag{3.1}$$

We show the computation for how to obtain $F_{13} = -F_{24}$. The rest follows in the same way.

First, recall that if α is a k-form, then $*\alpha$ is the unique $(n-k)$-form such that

$$(*\alpha, \omega)\,\mathrm{vol} = \alpha \wedge \omega,$$

where vol is a volume form, and $(\cdot, \cdot)$ is the inner product on the forms induced by the $\mathbb{R}^{2+2}$ metric. To compute $*F_{13} dx_1 \wedge dx_2$ it suffices to consider

$$(*dx_1 \wedge dx_3, dx_2 \wedge dx_4)\,\mathrm{vol} = dx_1 \wedge dx_3 \wedge dx_2 \wedge dx_4.$$

Observe the right hand side is equal to $-\,\mathrm{vol}$. Since

$$(dx_2 \wedge dx_4, dx_2 \wedge dx_4) = -1,$$

we must have

$$*F_{13} dx_1 \wedge dx_3 = F_{13} dx_2 \wedge dx_4.$$

Because $*F = -F$,

$$F_{13} = -F_{24}$$

as needed.

We proceed to the dimensional reduction, where we assume the connection A is independent of x_3, and set $A_3 = \phi$. Then (3.1) becomes

$$D_0\phi = F_{12}, \quad D_1\phi = F_{02}, \quad D_2\phi = F_{10}, \tag{3.2}$$

where we use index 0 instead of 4. This is exactly (ME) written out in components.

There is another way to write (ME), which turns out to be extremely useful for computations [5]. (ME) is an equation involving two-forms on both sides. By taking the parts corresponding to $dt \wedge dx$ and $dt \wedge dy$, and the parts corresponding to $dx \wedge dy$ we can obtain the following two equations respectively

$$\partial_t A + [A_0, A] - dA_0 = *d\phi + [*A, \phi], \tag{3.3}$$

$$dA + [A, A] = *(\partial_t \phi + [A_0, \phi]). \tag{3.4}$$

Observe that now operators d and $*$ act only with respect to the spatial variables. Similarly, A now denotes only the spatial part of the the connection, i.e. $A = (A_1, A_2)$. Finally note (3.3) is an equation involving one-forms, and (3.4) involves two-forms.

3.2 Gauge Transformations

(ME) is invariant under gauge transformations. Indeed, if we have a smooth map g, with compact support such that $g : \mathbb{R}^{2+1} \to G$, and

$$A \to A_g = gAg^{-1} + gdg^{-1},$$
$$\phi \to \phi_g = g\phi g^{-1},$$

then a computation shows

$$F_A \to gF_A g^{-1},$$
$$D_A\phi \to gD_A\phi g^{-1}.$$

Therefore if a pair (A, ϕ) solves (ME), so does (A_g, ϕ_g).

We would like to discuss regularity of the gauge transformations. If $A \in X, \phi \in Y$ where X, Y are some Banach spaces, the smoothness and compact support assumption on g can be lowered just enough so the gauge transformation defined above is a continuous map from X back into X, and from Y back into Y.

First note that since we are mapping into a compact Lie group, we can assume $g \in L^\infty_{t,x}$, and we have

$$\|g\|_{L^\infty_{t,x}} = \|g^{-1}\|_{L^\infty_{t,x}}.$$

Next, Main Theorem 1 produces a solution so that ϕ and the spatial parts of the connection $A_1, A_2 \in C_b(I, H^s)$, and $A_0 \in C_b(I, \dot{H}^r)$, where $r \in (0, \min(2s, 1+s)]$. We have the following

Lemma 3.2.1. Let $\alpha \geq 0$, then the gauge action is a continuous map from

$$\begin{gathered} C_b(I, H^\alpha) \times C_b(I, \dot{H}^1 \cap \dot{H}^{\alpha+1}) \cap L^\infty \to C_b(I, H^\alpha) \\ (h, g) \mapsto ghg^{-1} + gdg^{-1}, \end{gathered} \tag{3.5}$$

and the following estimate holds:

$$\|h_g\|_{C_b(I,H^\alpha)} \lesssim (\|h\|_{C_b(I,H^\alpha)} + 1)\|g\|_Y^2, \tag{3.6}$$

where $Y = C_b(I, \dot{H}^1 \cap \dot{H}^{\alpha+1}) \cap L^\infty$.

Proof. The continuity of the map is an exercise, which follows from the inequalities we obtain when we show (3.6).

Case 0: $\alpha = 0$. For fixed t we have

$$\begin{aligned} \|h_{g(t)}(t)\|_{L^2} &\lesssim \|g(t)h(t)g^{-1}(t)\|_{L^2} + \|g(t)dg^{-1}(t)\|_{L^2} \\ &\lesssim \|h(t)\|_{L^2}\|g(t)\|^2_{L^\infty} + \|g(t)\|_{L^\infty}\|dg^{-1}(t)\|_{L^2}, \end{aligned}$$

and (3.6) follows as needed.

Case 1: $0 < \alpha < 1$. By the previous case it is enough to consider

$$\|D^\alpha h_{g(t)}(t)\|_{L^2} \lesssim \|D^\alpha(ghg^{-1})(t)\|_{L^2} + \|D^\alpha(gdg^{-1})(t)\|_{L^2}. \tag{3.7}$$

For the first term we have

$$\|D^\alpha(ghg^{-1})\|_{L^2} \lesssim \|(D^\alpha g)h\|_{L^2}\|g\|_{L^\infty} + \|hD^\alpha g^{-1}\|_{L^2}\|g\|_{L^\infty} + \|h\|_{\dot{H}^\alpha}\|g\|^2_{L^\infty},$$

where for the ease of notation we eliminate writing of the variable t. The third term is bounded by the right-hand side of (3.6). g and g^{-1} have the same regularity, so we only look at the first term. By Hölder's inequality and Sobolev embedding

$$\|D^\alpha gh\|_{L^2} \leq \|D^\alpha g\|_{L^{2/\alpha}}\|h\|_{L^{(1/2-\alpha/2)^{-1}}} \lesssim \|D^{1-\alpha}D^\alpha g\|_{L^2}\|h\|_{\dot{H}^\alpha}, \tag{3.8}$$

where we use that $\frac{\alpha}{2} = \frac{1}{2} - \frac{1-\alpha}{2}$. Finally for the last term in (3.7) we have

$$\|gdg^{-1}\|_{\dot{H}^\alpha} \lesssim \|(D^\alpha g)dg^{-1}\|_{L^2} + \|g\|_{\dot{H}^{\alpha+1}}\|g\|_{L^\infty}, \tag{3.9}$$

and we are done if we observe that the first term can be handled exactly as (3.8).

Case 2: $\alpha = 1$. Again we start with

$$\begin{aligned}\|Dh_{g(t)}(t)\|_{L^2} &\lesssim \|D(ghg^{-1})(t)\|_{L^2} + \|D(gdg^{-1})(t)\|_{L^2} \\ &\lesssim \|(Dg)h\|_{L^2}\|g\|_{L^\infty} + \|h\|_{H^1}\|g\|^2_{L^\infty} + \|D(gdg^{-1})\|_{L^2}\end{aligned} \tag{3.10}$$

The second term is bounded by the right-hand side of (3.6). For the first term we have

$$\|(Dg)h\|_{L^2} \lesssim \|Dg\|_{L^4}\|h\|_{L^4}.$$

To finish observe

$$\|dg\|_{H^\beta} \leq \|dg\|_{L^2} + \|dg\|_{\dot{H}^1}$$

for $0 < \beta < 1$ so in particular for $\beta = \frac{1}{2}$, so we can use $H^{\frac{1}{2}} \hookrightarrow L^4$. By the same reasoning, for the third term in (3.10) we have

$$\|D(gdg^{-1})\|_{L^2} \lesssim (\|dg\|_{L^2} + \|dg\|_{\dot{H}^1})^2 + \|g\|_{\dot{H}^2}\|g\|_{L^\infty}.$$

Case 3: $\alpha > 1$. There is nothing to prove since now $h, dg \in L^\infty$. Hence

$$\|D^\alpha(ghg^{-1})(t)\|_{L^2} \lesssim \|D^\alpha g(t)\|_{L^2}\|h\|_{L^\infty}\|g\|_{L^\infty} + \|h\|_{H^\alpha}\|g\|^2_{L^\infty},$$

and

$$\|D^\alpha g(t)\|_{L^2} \leq \|g(t)\|_{\dot{H}^1} + \|g(t)\|_{\dot{H}^{1+\alpha}}.$$

Similarly

$$\begin{aligned}\|D^\alpha(gdg^{-1})(t)\|_{L^2} &\lesssim \|D^\alpha(gdg^{-1})(t)\|_{L^2} \\ &\lesssim \|D^\alpha g(t)\|_{L^2}\|dg^{-1}\|_{L^\infty} + \|g\|_{\dot{H}^{1+\alpha}}\|g\|_{L^\infty} \\ &\lesssim (\|g(t)\|_{\dot{H}^1} + \|g(t)\|_{\dot{H}^{1+\alpha}})\|dg\|_{L^\infty} + \|g\|_{\dot{H}^{1+\alpha}}\|g\|_{L^\infty}.\end{aligned}$$

□

From the lemma, we trivially obtain the following corollary.

Corollary 3.2.2. Let $0 < r, s$, $X = C_b(I, \dot{H}^r) \times C_b(I, H^s) \times C_b(I, H^s)$ and $Y = C_b(I, \dot{H}^1 \cap \dot{H}^{s+1} \cap \dot{H}^{r+1}) \cap L^\infty$. Then the gauge action is a continuous map from

$$\begin{aligned} X \times Y &\to X \\ (A_0, A_1, A_2) &\mapsto A_g, \end{aligned} \tag{3.11}$$

as well as from

$$\begin{aligned} C_b(I, H^s) \times Y &\to C_b(I, H^s) \\ \phi &\mapsto \phi_g = g\phi g^{-1}, \end{aligned} \tag{3.12}$$

and the following estimates hold

$$\|A_g\|_X \lesssim \|g\|_Y^2 (1 + \|A\|_X), \tag{3.13}$$

and

$$\|\phi_g\|_{C_b(I,H^s)} \lesssim \|g\|_Y^2 (1 + \|\phi\|_{C_b(I,H^s)}). \tag{3.14}$$

Since we have gauge freedom, we are allowed to choose any representative of a given equivalence class. The traditional gauge conditions are

- Coulomb: $\partial^i A_i = 0$,
- Lorentz: $\partial^\alpha A_\alpha = 0$,
- Temporal: $A_0 = 0$.

In this thesis we work in the Coulomb gauge. Using Hodge theory the Coulomb gauge could be also written as

$$d^* A = 0.$$

We ask: given any initial data $a_1, a_2, \phi_0 \in H^s(\mathbb{R}^2)$, can we find a gauge transformation so that the initial data is placed in the Coulomb gauge? Dell'Antonio and Zwanziger produce a global $\dot{H}^1$ Coulomb gauge using variational methods [6]. Here, we also require $g \in \dot{H}^{s+1}$, and two dimensions are tricky. Fortunately, if the initial data is small, we can obtain a global gauge with the additional regularity as needed. This is considered by the author and Uhlenbeck for two dimensions and higher in [3]. The result in two dimensions is the following

Theorem 3.2.3. [3] Given $A(0) = a$ sufficiently small in $H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$, there exists a gauge transformation $g \in \dot{H}^{s+1}(\mathbb{R}^2) \cap \dot{H}^1(\mathbb{R}^2) \cap L^\infty$ so that $\partial^i(g a_i g^{-1} + g\partial_i g^{-1}) = 0$.

3.3 The Monopole Equation in a Coulomb Gauge as a system of Wave & Elliptic Equations

We begin with a proposition, where we show how we can rewrite the monopole equation in the Coulomb gauge as a system of wave equations coupled with an elliptic equation, to which from now on we refer to as the auxiliary monopole equation (aME).

Proposition 3.3.1. The Monopole Equation, $F_A = *D_A\phi$ on $\mathbb{R}^{2+1}$ in a Coulomb gauge with initial data

$$A_i|_{t=0} = a_i, \quad i = 1, 2 \quad \text{and} \quad \phi|_{t=0} = \phi_0 \tag{3.15}$$

with $\partial^i a_i = 0$ can be rewritten as the following system

$$\text{(aME)} \quad \begin{cases} \triangle A_0 = d^*[A_0, *df] + d^*[df, \phi], \\ \Box u = \mathcal{B}_+(\phi, df, A_0), \\ \Box v = \mathcal{B}_-(\phi, df, A_0), \end{cases}$$

where $\mathcal{B}_\pm$

$$\mathcal{B}_+ = -\mathcal{B}_1 + \mathcal{B}_2 + \mathcal{B}_3 + \mathcal{B}_4,$$

$$\mathcal{B}_- = -\mathcal{B}_1 - \mathcal{B}_2 + \mathcal{B}_3 - \mathcal{B}_4,$$

where

$$\begin{aligned} \mathcal{B}_1 &= [\partial_1 f, \partial_2 f], \\ \mathcal{B}_2 &= R_1[\partial_2 f, \phi] - R_2[\partial_1 f, \phi], \\ \mathcal{B}_3 &= [A_0, \phi], \\ \mathcal{B}_4 &= R_j[A_0, \partial^j f], \end{aligned} \tag{3.16}$$

with R_j denoting Riesz transform, $(-\triangle)^{-\frac{1}{2}}\partial_j$. The initial data for (aME) is given by

$$\begin{aligned} u(0) = v(0) &= 0, \\ \partial_t u(0) &= \phi_0 + ih, \\ \partial_t v(0) &= \phi_0 - ih, \end{aligned} \tag{3.17}$$

where $h = R_1 a_2 - R_2 a_1$.

Proof. Recall equations (3.3) and (3.4)

$$\partial_t A + [A_0, A] - dA_0 = *d\phi + [*A, \phi] \tag{3.18}$$

$$dA + [A, A] = *(\partial_t \phi + [A_0, \phi]), \tag{3.19}$$

where d and $*$ act only with respect to the spatial variables, and A denotes only the spatial part of the connection. If we impose the Coulomb gauge condition, then

$$d^*A = 0. \tag{3.20}$$

By equivalence of closed and exact forms on $\mathbb{R}^n$, we can further suppose that

$$A = *df, \tag{3.21}$$

for some $f : \mathbb{R}^{2+1} \to \mathfrak{g}$. Observe

$$\begin{aligned} d * df &= \triangle f dx \wedge dy \\ [*df, *df] &= [df, df] = \frac{1}{2}[\partial_i f, \partial_j f] dx^i \wedge dx^j \end{aligned}$$

It follows (3.18) and (3.19) become

$$\partial_t * df + [A_0, *df] - dA_0 = *d\phi - [df, \phi], \tag{3.22}$$

$$\triangle f + [\partial_1 f, \partial_2 f] = *(\partial_t \phi + [A_0, \phi]). \tag{3.23}$$

Take d^* of (3.22) to obtain

$$\triangle A_0 = d^*[A_0, *df] + d^*[df, \phi].$$

This is the first equation in (aME). Now take d of (3.22)

$$\partial_t \triangle f + \partial^j [A_0, \partial_j f] = \triangle \phi + \partial_2[\partial_1 f, \phi] - \partial_1[\partial_2 f, \phi]. \tag{3.24}$$

Consider (3.24) and (3.23) on the spatial Fourier transform side

$$-\partial_t |\xi|^2 \hat{f} + |\xi|^2 \hat{\phi} = i(\xi_2 \widehat{[\partial_1 f, \phi]} - \xi_1 \widehat{[\partial_2 f, \phi]} - \xi_j \widehat{[A_0, \partial^j f]}) \tag{3.25}$$

$$-|\xi|^2 f - \partial_t \hat{\phi} = -\widehat{[\partial_1 f, \partial_2 f]} + \widehat{[A_0, \phi]}. \tag{3.26}$$

This allows us to write (3.25) and (3.26) as a system for ϕ and df

$$(\partial_t - i|\xi|)(\hat{\phi} + i|\xi|\hat{f}) = -\hat{\mathcal{B}}_+(\phi, df, A_0), \tag{3.27}$$

$$(\partial_t + i|\xi|)(\hat{\phi} - i|\xi|\hat{f}) = -\hat{\mathcal{B}}_-(\phi, df, A_0), \tag{3.28}$$

where

$$\hat{\mathcal{B}}_\pm = -\widehat{[\partial_1 f, \partial_2 f]} + \widehat{[A_0, \phi]} \pm (\frac{\xi_1}{|\xi|}\widehat{[\partial_2 f, \phi]} - \frac{\xi_2}{|\xi|}\widehat{[\partial_1 f, \phi]} + \frac{\xi_j}{|\xi|}\widehat{[A_0, \partial^j f]}). \tag{3.29}$$

Indeed, multiply (3.25) by $\frac{i}{|\xi|}$, and first add the resulting equation to (3.26) to obtain (3.27), and then subtract it from (3.26) to obtain (3.28). To uncover the wave equation, we let

$$\hat{\phi} + i|\xi|\hat{f} = (\partial_t + i|\xi|)\hat{u} \quad \text{and} \quad \hat{\phi} - i|\xi|\hat{f} = (\partial_t - i|\xi|)\hat{v}, \tag{3.30}$$

where $u, v : \mathbb{R}^{2,1} \to \mathfrak{g}$. u and v are our new unknowns. Note, once we know what u and v are, we can determine ϕ and df using

$$\begin{aligned} \hat{\phi} &= \frac{(\partial_t + i|\xi|)\hat{u} + (\partial_t - i|\xi|)\hat{v}}{2}, \\ i|\xi|\hat{f} &= \frac{(\partial_t + i|\xi|)\hat{u} - (\partial_t - i|\xi|)\hat{v}}{2}. \end{aligned} \tag{3.31}$$

So given $i|\xi|\hat{f} = \hat{g}$ for some g, we get $A = *df = *R_i g dx^i$. In addition, since ϕ and df can be written in terms of derivatives of u and v we sometimes write $\mathcal{B}_\pm(\phi, df, A_0)$ as $\mathcal{B}_\pm(\partial u, \partial v, A_0)$.

Now we discuss initial data. From (3.30)

$$\partial_t \hat{u}(0) = \hat{\phi}_0 + i|\xi|\hat{f}(0) - i|\xi|\hat{u}(0), \tag{3.32}$$

and

$$\partial_t \hat{v}(0) = \hat{\phi}_0 - i|\xi|\hat{f}(0) + i|\xi|\hat{v}(0). \tag{3.33}$$

Note, we are free to choose the initial data for u and v as long as in the end we can recover initial data for ϕ and A. Hence we can just let $u(0) = v(0) = 0$. We still need to say what $|\xi|\hat{f}(0)$ is. Let $\hat{h} = |\xi|\hat{f}(0)$. Then by (3.15) and (3.21) we need

$$R_1 h = a_2$$

$$R_2 h = -a_1.$$

Differentiate the first equation with respect to x, the second with respect to y, and add them together to obtain

$$\triangle D^{-1} h = \partial_1 a_2 - \partial_2 a_1, \tag{3.34}$$

as promised. □

Next we have an important result that states that LWP for (ME) in a Coulomb gauge can be obtained from LWP of the system (aME). For completeness we state exactly what we mean by LWP of (aME).

Let $r \in (0, \min(2s, 1+s)]$, $s > 0$. Consider the system (aME) with initial data

$$(u, u_t)|_{t=0} = (u_0, u_1) \quad \text{and} \quad (v, v_t)|_{t=0} = (v_0, v_1)$$

in $H^{s+1} \times H^s$, then (aME) is LWP if

(Local Existence) There exist $T > 0$ depending continuously on the norm of the initial data, and functions

$$A_0 \in C_b([0,T], \dot{H}^r),$$

$$u, v \in \mathcal{H}_T^{s+1,\theta} \hookrightarrow C_b([0,T], H^{s+1}) \cap C_b^1([0,T], H^s),$$

which solve (aME) on $[0,T]\times\mathbb{R}^2$ in the sense of distributions and such that the initial conditions are satisfied.

(Uniqueness) If $T>0$ and (A_0,u,v) and (A_0,u,v) are two solutions of (aME) on $(0,T)\times\mathbb{R}^2$ belonging to

$$C_b([0,T],\dot{H}^r)\times\mathcal{H}_T^{s+1,\theta}\times\mathcal{H}_T^{s+1,\theta},$$

with the same initial data, then (A_0,u,v) and (A_0',u',v') on $(0,T)\times\mathbb{R}^2$.

(Continuous Dependence on Initial Data) For any $(u_0,u_1),(v_0,v_1)\in H^{s+1}\times H^s$ there is a neighborhood U of the initial data such that the solution map $(u_0,u_1),(v_0,v_1)\to(A_0,u,v)$ is continuous from U into $C_b([0,T],\dot{H}^r)\times(C_b([0,T],H^{s+1})\cap C_b^1([0,T],H^s))^2$.

In fact by the results in [23] combined with estimates for the elliptic equation, we can show these stronger estimates

$$\begin{aligned}&\|u-u'\|_{\mathcal{H}_T^{s+1,\theta}}+\|v-v'\|_{\mathcal{H}_T^{s+1,\theta}}+\|A_0-A_0'\|_{C_b([0,T],\dot{H}^r)}\\&\qquad\lesssim\|u_0-u_0'\|_{H^{s+1}}+\|u_1-u_1'\|_{H^s}+\|v_0-v_0'\|_{H^{s+1}}+\|v_1-v_1'\|_{H^s},\end{aligned}\tag{3.35}$$

where $(u_0',u_1'),(v_0',v_1')$ are sufficiently close to $(u_0,u_1),(v_0,v_1)$.

Theorem 3.3.2. (**Return to the Monopole Equation**) Consider (ME) in a Coulomb gauge with the following initial data in H^s for $s>0$

$$A_i|_{t=0}=a_i,\quad i=1,2\quad\text{and}\quad\phi|_{t=0}=\phi_0\tag{3.36}$$

with $\partial^i a_i=0$. Then local well-posedness of (aME) with initial data as in (3.17) implies local well-posedness of (ME) in a Coulomb gauge with initial data given by (3.36).

Proof. Begin by observing that given initial data in the Coulomb gauge, the solutions of (aME) imply A remains in a Coulomb gauge. Indeed, solutions of (aME) produce $i|\xi|\hat{f} = \hat{g}$ for some g, so we get $A = *df = *R_i g dx^i$, and $d^*A = d^* * df = 0$ as needed.

(Local Existence) From (3.31) we have

$$\begin{aligned} \phi &= \frac{(\partial_t + iD)u + (\partial_t - iD)v}{2}, \\ \partial_i f &= R_i\Big(\frac{(\partial_t + iD)u - (\partial_t - iD)v}{2}\Big). \end{aligned} \tag{3.37}$$

Hence

$$u, v \in \mathcal{H}_T^{s+1,\theta}$$

implies

$$\phi, A = *df \in H_T^{s,\theta}$$

as needed. Next we verify that solutions of (aME) produce the solutions to the Monopole Equation in the Coulomb gauge. The starting point for the monopole equation in the Coulomb gauge are equations (3.22) and (3.23). Suppose (A, ϕ) solve (3.27) and (3.28). Add (3.27) to (3.28) to recover (3.26), which is equivalent to (3.23).

Next given (aME) we need to show (3.22) holds. Write (3.22) in coordinates,

$$\partial_x A_0 - \partial_y \phi + \partial_t \partial_y f = [\partial_x f, \phi] - [A_0, \partial_y f], \tag{3.38}$$

$$\partial_y A_0 + \partial_x \phi - \partial_t \partial_y f = [\partial_y f, \phi] + [A_0, \partial_x f]. \tag{3.39}$$

From the elliptic equation in (aME) we have

$$A_0 = \triangle^{-1}(-\partial_x[A_0, \partial_y f] + \partial_y[A_0, \partial_x f] + \partial_x[\partial_x f, \phi] + \partial_y[\partial_y f, \phi]). \tag{3.40}$$

Also subtract (3.27) from (3.28) and take D on both sides to obtain (3.24), which implies

$$\phi - \partial_t f = \triangle^{-1}(\partial^i[A_0, \partial_i f] - \partial_y[\partial_x f, \phi] + \partial_x[\partial_y f, \phi]). \tag{3.41}$$

In order to recover (3.38), first use (3.40) to get $\partial_x A_0$

$$\begin{aligned} \partial_x A_0 = \triangle^{-1}(&-\partial_x^2[A_0, \partial_y f] + \partial_x\partial_y[A_0, \partial_x f] \\ &+ \partial_x^2[\partial_x f, \phi] + \partial_x\partial_y[\partial_y f, \phi]). \end{aligned} \tag{3.42}$$

Next use (3.41) to get $\partial_y(\phi - \partial_t f)$:

$$\begin{aligned} \partial_y(\phi - \partial_t f) = \triangle^{-1}(&\partial_y\partial^x[A_0, \partial_x f] + \partial_y^2[A_0, \partial_y f] \\ &- \partial_y^2[\partial_x f, \phi] + \partial_y\partial_x[\partial_y f, \phi]), \end{aligned} \tag{3.43}$$

and subtract it from (3.42) to get (3.38) as needed. We recover (3.39) in the exactly same way.

(Continuous Dependence on Initial Data) We would like to show

$$\begin{aligned} \|A_0 - A_0'\|_{C_b([0,T],\dot{H}^r)} + &\|A_1 - A_1'\|_{H_T^{s,\theta}} + \|A_2 - A_2'\|_{H_T^{s,\theta}} + \|\phi - \phi'\|_{H_T^{s,\theta}} \\ &\lesssim \|a_1 - a_1'\|_{H^s} + \|a_2 - a_2'\|_{H^s} + \|\phi_0 - \phi_0'\|_{H^s} \end{aligned} \tag{3.44}$$

for any a_1', a_2', ϕ_0' sufficiently close to a_1, a_2, ϕ_0. In view of LWP for (aME) with data given by

$$\begin{aligned} u(0) = v(0) = 0, \\ \partial_t u(0) = \phi_0 + ih, \\ \partial_t v(0) = \phi_0 - ih, \end{aligned} \tag{3.45}$$

where $h = R_1 a_2 - R_2 a_1$ and by (3.35) we have

$$\begin{aligned} \|u-u'\|_{\mathcal{H}_T^{s+1,\theta}} + \|v - v'\|_{\mathcal{H}_T^{s+1,\theta}} + \|A_0 - A_0'\|_{C_b([0,T],\dot{H}^r)} \\ \lesssim \|u_0'\|_{H^{s+1}} + \|\phi_0 + ih - u_1'\|_{H^s} + \|v_0'\|_{H^{s+1}} + \|\phi_0 - ih - v_1'\|_{H^s}, \end{aligned} \tag{3.46}$$

for all u_0', v_0', u_1', v_1' satisfying

$$\|u_0'\|_{H^{s+1}} + \|\phi_0 + ih - u_1'\|_{H^s} + \|v_0'\|_{H^{s+1}} + \|\phi_0 - ih - v_1'\|_{H^s} \leq \delta \tag{3.47}$$

for some $\delta > 0$. In particular choose

$$\begin{aligned} &u_0' = v_0' = 0, \\ &u_1' = \phi_0' + ih' \quad \text{and} \quad v_1' = \phi_0' - ih' \end{aligned} \tag{3.48}$$

such that

$$\begin{aligned} \|\phi_0 + ih - \phi_0' - ih'\|_{H^s} &+ \|\phi_0 - ih - \phi_0' + ih'\|_{H^s} \\ &\leq \|\phi_0 - \phi_0'\|_{H^s} + \|iR_1(a_2 - a_2')\|_{H^s} + \|iR_2(a_1 - a_1')\|_{H^s} \\ &\leq \|\phi_0 - \phi_0'\|_{H^s} + \|a_1 - a_1'\|_{H^s} + \|a_2 - a_2'\|_{H^s} \\ &\leq \delta. \end{aligned}$$

Then by (3.46), (3.47), and (3.48), $\|A_0 - A_0'\|_{C_b([0,T],\dot{H}^r)}$ is bounded by the right hand side of (3.44). Next observe

$$\begin{aligned} \|A_1 - A_1'\|_{H_T^{s,\theta}} &\lesssim \|R_2(\partial_t + iD)(u - u')\|_{H_T^{s,\theta}} + \|R_2(\partial_t - iD)(v - v')\|_{H_T^{s,\theta}} \\ &\lesssim \|u - u'\|_{\mathcal{H}_T^{s+1,\theta}} + \|v - v'\|_{\mathcal{H}_T^{s+1,\theta}}. \end{aligned}$$

So again by (3.46), (3.47), and (3.48) $\|A_1 - A_1'\|_{H_T^{s,\theta}}$ is bounded by the right hand side of (3.44). We bound the difference for A_2 and ϕ in a similar fashion. **(Uniqueness)** By LWP of (aME), A_0 is unique in the required class. We need to show A and ϕ are unique in $H_T^{s,\theta}$. However, this is obvious in view of (3.44). □

3.4 Proof of Main Theorem 1

By Theorem 3.3.2 it is enough to show LWP for (aME). We start by explaining how we are going to perform our iteration.

3.4.1 Set up of the Iteration

Equations (aME) are written for functions u and v. Nevertheless, functions u and v are only our auxiliary functions, and we are really interested in solving for df and ϕ. In addition, the nonlinearities $\mathcal{B}_\pm$ are a linear combination of $\mathcal{B}_i$'s, $i = 1, 2, 3, 4$ given by (3.16), and $\mathcal{B}_i$'s are written in terms of ϕ, df and A_0. Also, when we do our estimates, it is easier to keep the $\mathcal{B}_i$'s in terms of ϕ and df with the exception of $\mathcal{B}_2$, which we rewrite in terms of ∂u and ∂v[1]. These comments motivate the following procedure for our iteration. Start with $\phi_{-1} = df_{-1} = 0$. Then $\mathcal{B}_\pm \equiv 0$. Solve the homogeneous wave equations for u_0, v_0 with the initial data given by (3.17). Then to solve for df_0, ϕ_0, use (3.37)

$$\begin{aligned} \phi &= \frac{(\partial_t + iD)u + (\partial_t - iD)v}{2}, \\ \partial_i f &= R_i\big(\frac{(\partial_t + iD)u - (\partial_t - iD)v}{2}\big). \end{aligned} \tag{3.49}$$

Then feed ϕ_0 and df_0 into the elliptic equation,

$$\triangle A_{0,0} = d^*([A_{0,0}, *df_0] + [df_0, \phi_0]), \tag{3.50}$$

and solve for $A_{0,0}$. Next we take $df_0, \phi_0, A_{0,0}$ and plug them into $\mathcal{B}_1, \mathcal{B}_3, \mathcal{B}_4$, but rewrite $\mathcal{B}_2$ in terms of $\partial u_0, \partial v_0$. We continue in this manner, so at the j'th

[1]See Section 3.4.4 for the details.

step of the iteration, $j \geq 1$, we solve

$$\begin{aligned}\Box u_j &= -\mathcal{B}_1(\nabla f_{j-1}) + \mathcal{B}_2(\partial u_{j-1}, \partial v_{j-1}) + \mathcal{B}_3(A_{0,j-1}, \phi_{j-1}) + \mathcal{B}_4(A_{0,j-1}, \nabla f_{j-1}),\\ \Box v_j &= -\mathcal{B}_1(\nabla f_{j-1}) - \mathcal{B}_2(\partial u_{j-1}, \partial v_{j-1}) + \mathcal{B}_3(A_{0,j-1}, \phi_{j-1}) - \mathcal{B}_4(A_{0,j-1}, \nabla f_{j-1}),\\ \triangle A_{0,j} &= d^*([A_{0,j}, *df_j] + [df_j, \phi_j]).\end{aligned}$$

3.4.2 Estimates Needed

The elliptic equation is discussed in section 3.5. By results in [23] which are mentioned in Section 2.2, the proof of Main Theorem 1 reduces to establishing following estimates for the nonlinearities $\mathcal{B}_\pm$ and combining them with appropriate elliptic estimates from section 3.5

$$\|\Lambda_+^{-1}\Lambda_-^{-1+\epsilon}\mathcal{B}_\pm(\partial u, \partial v, A_0)\|_{\mathcal{H}^{s+1,\theta}} \lesssim \|u\|_{\mathcal{H}^{s+1,\theta}} + \|v\|_{\mathcal{H}^{s+1,\theta}}, \tag{3.51}$$

$$\|\Lambda_+^{-1}\Lambda_-^{-1+\epsilon}\big(\mathcal{B}_\pm(\partial u, \partial v, A_0) - \mathcal{B}_\pm(\partial u', \partial v', A_0')\big)\|_{\mathcal{H}^{s+1,\theta}} \lesssim \|u - u'\|_{\mathcal{H}^{s+1,\theta}} + \|v - v'\|_{\mathcal{H}^{s+1,\theta}}, \tag{3.52}$$

where the suppressed constants depend continuously on the $\mathcal{H}^{s+1,\theta}$ norms of u, u', v, v'. Since $\mathcal{B}_\pm$ are bilinear, (3.52) can follow from (3.51). Moreover, since we require small initial data[2], we do not need ϵ in our estimates. Next observe $\Lambda_+\Lambda_-\mathcal{H}^{s+1,\theta} = H^{s,\theta-1}$, as well as that

$$\|df\|_{H^{s,\theta}}, \|\phi\|_{H^{s,\theta}} \lesssim \|u\|_{\mathcal{H}^{s+1,\theta}} + \|v\|_{\mathcal{H}^{s+1,\theta}}.$$

[2]See Theorem 3.2.3 and Section 3.5.

Therefore, by (3.16), it will be enough to prove the following

$$\|\mathcal{B}_1\|_{H^{s,\theta-1}} = \|[\partial_1 f, \partial_2 f]\|_{H^{s,\theta-1}} \lesssim \|\nabla f\|^2_{H^{s,\theta}} \tag{3.53}$$

$$\|\mathcal{B}_2\|_{H^{s,\theta-1}} \lesssim \|[df, \phi]\|_{H^{s,\theta-1}} \lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}} \tag{3.54}$$

$$\|\mathcal{B}_3\|_{H^{s,\theta-1}} \lesssim \|A_0 \phi\|_{H^{s,\theta-1}} \lesssim \|A_0\| \|\phi\|_{H^{s,\theta}} \tag{3.55}$$

$$\|\mathcal{B}_4\|_{H^{s,\theta-1}} \lesssim \|A_0 df\|_{H^{s,\theta-1}} \lesssim \|A_0\| \|df\|_{H^{s,\theta}}, \tag{3.56}$$

where the norm we are using for A_0 is immaterial, mainly because we show in Section 3.5,

$$\|A_0\| \lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}}. \tag{3.57}$$

Few remarks are in order. Estimate (3.53) corresponds to estimates for the null form Q_{ij} (this is shown in the next section). Estimate (3.54) gives rise to a new null form Q that we discuss in the next section. A_0 in estimates (3.55) and (3.56) solves the elliptic equation in (aME), which results in a quite good regularity for A_0. As a result, we do not have to look for any special structures to make estimates (3.55) and (3.56) hold, so we can drop the brackets, and also treat these estimates as equivalent since ϕ and df exhibit the same regularity. Finally, Riesz transforms are clearly bounded on L^2, so we ignore them in the estimates needed in (3.54) and (3.56). The estimates (3.53) and (3.54) for the null forms are the most interesting. Hence we discuss them first, and then we consider the elliptic terms.

3.4.3 Null Forms–Proof of Estimate (3.53)

$[\partial_1 f, \partial_2 f]$ has a structure of a null form Q_{ij} :

$$[\partial_1 f, \partial_2 f] = \partial_1 f \partial_2 f - \partial_2 f \partial_1 f = Q_{12}(f,f).$$

It follows (3.53) is equivalent to

$$\|Q_{12}(f,f)\|_{H^{s,\theta-1}} \lesssim \|\nabla f\|_{H^{s,\theta}} \|\nabla f\|_{H^{s,\theta}}.$$

From (3.49) we have

$$\nabla f \in H^{s,\theta} \Rightarrow \|\Lambda^s \Lambda_-^\theta Df\|_{L^2(\mathbb{R}^{2+1})} < \infty, \tag{3.58}$$

so the estimate follows from Corollary 2.5.7.

3.4.4 Null Forms–Proof of Estimate (3.54)

We need

$$\|[df, \phi]\|_{H^{s,\theta-1}} \lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}}.$$

However analysis of the first iterate shows that for this estimate to hold we need $s > \frac{3}{4}$, so we need to work a little bit harder, and use (3.37)[3]

$$\begin{aligned} [\partial_i f, \phi] &= \frac{1}{4}[R_i(\partial_t u + iDu - \partial_t v + iDv), \partial_t u + iDu + \partial_t v - iDv] \\ &= \frac{1}{4}[(R_i\partial_t + \partial_i)u - (R_i\partial_t - \partial_i)v, (\partial_t + iD)u + (\partial_t - iD)v]. \end{aligned} \tag{3.59}$$

If we use the bilinearity of the bracket, we can group (3.59) by terms involving brackets of u with itself, v with itself, and then also by the terms that are

[3]The obvious way is to just substitute for ϕ and leave df the same, but it is an exercise to see that this does not work (for several reasons!).

mixed i.e., involve both u and v. So we have

$$4[\partial_i f, \phi] = [(R_i\partial_t + \partial_i)u, (\partial_t + iD)u] - [(R_i\partial_t - \partial_i)v, (\partial_t - iD)v]$$
$$+ [(R_i\partial_t + \partial_i)u, (\partial_t - iD)v] - [(R_i\partial_t - \partial_i)v, (\partial_t + iD)u].$$

The needed estimates are contained in the following theorem, which we state involving the ϵ.

Theorem 3.4.1. Let $s > \frac{1}{4}$ and

$$\frac{3}{4} - \frac{\epsilon}{2} < \theta \le s + \frac{1}{2} - \epsilon \quad \text{and} \quad \theta < 1 - \epsilon$$
$$0 \le \epsilon < \min(2s - \frac{1}{2}, \frac{1}{2}).$$

and let $Q(\varphi, \psi)$ be given by

$$Q(\varphi, \psi) = (R_i\partial_t \pm \partial_i)\varphi(\partial_t \pm iD)\psi - (\partial_t \pm iD)\varphi(R_i\partial_t \pm \partial_i)\psi$$
$$= (\partial_t \pm iD)R_i\varphi(\partial_t \pm iD)\psi - (\partial_t \pm iD)\varphi(\partial_t \pm iD)R_i\psi$$

or

$$Q(\varphi, \psi) = (R_i\partial_t \pm \partial_i)\varphi(\partial_t \mp iD)\psi + (\partial_t \mp iD)\varphi(R_i\partial_t \pm \partial_i)\psi$$
$$= (\partial_t \pm iD)R_i\varphi(\partial_t \mp iD)\psi + (\partial_t \mp iD)\varphi(\partial_t \pm iD)R_i\psi$$

Then

$$Q(\mathcal{H}^{s+1,\theta}, \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s,\theta-1+\epsilon} \tag{3.60}$$

or equivalently, the following estimate holds

$$\|Q(\varphi, \psi)\|_{H^{s,\theta-1+\epsilon}} \lesssim \|\varphi\|_{\mathcal{H}^{s+1,\theta}} \|\psi\|_{\mathcal{H}^{s+1,\theta}}.$$

Proof. We show the details only for

$$Q(\varphi, \psi) = (R_i \partial_t + \partial_i)\varphi(\partial_t - iD)\psi + (\partial_t - iD)\varphi(R_i \partial_t + \partial_i)\psi$$

as the rest follows similarly.

Observe the symbol of Q is

$$q(\tau, \xi, \lambda, \eta) = \big(\frac{\xi_i}{|\xi|} + \frac{\eta_i}{|\eta|}\big)(\tau + |\xi|)(\lambda - |\eta|).$$

Suppose $\tau\lambda \geq 0$, then

$$q \leq 2(\tau + |\xi|)(\lambda - |\eta|) \leq \begin{cases} 2\big||\tau| + |\xi|\big|\big||\lambda| - |\eta|\big| & \text{if} \quad \tau, \lambda \geq 0, \\ 2\big||\tau| - |\xi|\big|\big||\eta| + |\lambda|\big| & \text{if} \quad \tau, \lambda < 0. \end{cases}$$

It follows

$$\iint_{\tau\lambda \geq 0} |\Lambda^s \Lambda^{\theta-1+\epsilon} Q(u,v)|^2 d\tau d\xi \lesssim \|D_+ u D_v\|^2_{H^{s,\theta-1+\epsilon}} + \|D_- u D_+ v\|^2_{H^{s,\theta-1+\epsilon}}$$

and the estimate follows by Theorem 2.5.8.

Suppose $\tau\lambda < 0$. If we break down the computations into two regions

$$\{(\tau, \xi), (\lambda, \eta) : |\tau| \geq 2|\xi| \text{or} |\lambda| \geq 2|\eta|\} \quad \text{and} \quad \text{otherwise}, \tag{3.61}$$

then in the first region, we bound q by

$$q \leq 2(|\tau| + |\xi|)(|\lambda| + |\eta|)$$

since there we do not need any special structure[4].

In the second region, we have

$$q \leq 4|\xi||\eta|\big|\frac{\xi_i}{|\xi|} + \frac{\eta_i}{|\eta|}\big| = 4\sigma_-$$

and the estimate follows then by theorem 2.5.6. □

[4]It is a simple exercise in the first region. See Appendix B.

Remark 3.4.1. The symbols for which we did not show details are

$$(\frac{\xi_i}{|\xi|} - \frac{\eta_i}{|\eta|})(\tau + |\xi|)(\lambda + |\eta|),$$
$$(\frac{\xi_i}{|\xi|} - \frac{\eta_i}{|\eta|})(\tau - |\xi|)(\lambda - |\eta|),$$
$$(\frac{\xi_i}{|\xi|} + \frac{\eta_i}{|\eta|})(\tau - |\xi|)(\lambda + |\eta|).$$

From above we could extract another null form and show the following

Theorem 3.4.2. Let $s > \frac{1}{4}$ and

$$\frac{3}{4} - \frac{\epsilon}{2} < \theta \leq s + \frac{1}{2} - \epsilon \quad \text{and} \quad \theta < 1 - \epsilon$$
$$0 \leq \epsilon < \min(2s - \frac{1}{2}, \frac{1}{2}),$$

and let $Q(\varphi, \psi)$ be given by

$$Q(\varphi, \psi) = \partial_t R_i \varphi (\partial_t + iD)\psi - (\partial_t + iD)\varphi \partial_t R_i \psi. \tag{3.62}$$

when $\tau\lambda \geq 0$ and

$$Q(\varphi, \psi) = \partial_t R_i \varphi (\partial_t + iD)\psi + (\partial_t + iD)\varphi \partial_t R_i \psi. \tag{3.63}$$

when $\tau\lambda < 0$.

Then

$$Q(\mathcal{H}^{s+1,\theta}, \mathcal{H}^{s+1,\theta}) \hookrightarrow H^{s,\theta-1+\epsilon}. \tag{3.64}$$

Proof. When $\tau\lambda \geq 0$, the symbol of Q can be written as a sum of

$$\tau\lambda \left(\frac{\xi_i}{|\xi|} - \frac{\eta_i}{|\eta|} \right) \tag{3.65}$$

and

$$\tau|\eta|\frac{\xi_i}{|\xi|} - \lambda|\xi|\frac{\eta_i}{|\eta|}. \tag{3.66}$$

We compare (3.65) with the symbol of Q_{tj} for the first iterate

$$|\xi||\eta|\left(\frac{\xi_i}{|\xi|} - \frac{\eta_i}{|\eta|}\right). \tag{3.67}$$

If we break down the computations into two regions

$$\{(\tau,\xi),(\lambda,\eta) : |\tau| \geq 2|\xi|, |\lambda| \geq 2|\eta|\} \quad \text{and} \quad \text{otherwise}, \tag{3.68}$$

then in the first region, again we do not need any special structure, and in the second, we just bound $\tau\lambda$ by $|\xi||\eta|$ and use Theorem 2.5.6.

Next we discuss (3.66). We add and subtract terms to rewrite it as something we recognize. We consider different signs of τ and λ. For instance, when $\tau, \lambda \geq 0$ we have

$$\frac{\xi_i}{|\xi|}\left(\tau|\eta| - \lambda|\xi|\right) + \lambda|\xi|\left(\frac{\xi_i}{|\xi|} - \frac{\eta_i}{|\eta|}\right) \tag{3.69}$$

The term on the right can be taken care of in the same way as (3.65). For the term on the left we subtract and add $\tau\lambda$. This results in

$$\frac{\xi_i}{|\xi|}(\tau|\eta| - \tau\lambda + \tau\lambda - \lambda|\xi|) \leq |\tau|\big||\lambda| - |\eta|\big| + |\lambda|\big||\tau| - |\eta|\big|. \tag{3.70}$$

The estimates follow from theorem 2.5.8. When $\tau, \lambda \leq 0$ can be handled in a similar way. □

3.4.5 Elliptic Piece: Proof of Estimate (3.55)

Recall we wish to show

$$\|A_0 w\|_{H^{s,\theta-1}} \lesssim \|A_0\| \|w\|_{H^{s,\theta}}. \tag{3.71}$$

We need this estimates during out iteration, so we really mean $A_{0,j}$, but for simplicity we omit writing of the index j. Now we choose a norm for A_0 to be anything that makes (3.71) possible to establish. This results in

$$\|A_0\| = \|A_0\|_{L_t^{\tilde{p}} L_x^\infty} + \|D^s A_0\|_{L_t^p L_x^q},$$

where

$$\begin{gathered} \tilde{p} \in (1-2s, \frac{1}{2}), \\ \frac{2}{p} = 1 - \frac{1}{q}, \quad \max(\frac{1}{3}(1-2s), \frac{s}{2}) < \frac{1}{q} < s. \end{gathered} \tag{3.72}$$

For now we assume we can show $A_0 \in L_t^{\tilde{p}} L_x^\infty \cap L_t^p \dot{W}_x^{s,q}$ and delay the proof to section 3.5, where the reasons for our choices of $\tilde{p}, p, q$ should become clear. We start by using $\theta - 1 < 0$

$$\begin{aligned} \|A_0 w\|_{H^{s,\theta-1}} &\le \|\Lambda^s (A_0 w)\|_{L^2(\mathbb{R}^{2+1})} \lesssim \|A_0 w\|_{L^2(\mathbb{R}^{2+1})} + \|D^s (A_0 w)\|_{L^2(\mathbb{R}^{2+1})} \\ &\le \underbrace{\|(D^s A_0) w\|_{L^2(\mathbb{R}^{2+1})}}_{I} + \underbrace{\|A_0 D^s w\|_{L^2(\mathbb{R}^{2+1})} + \|A_0 w\|_{L^2(\mathbb{R}^{2+1})}}_{II}. \end{aligned} \tag{3.73}$$

We discuss I. By Hölder's inequality applied twice

$$I \le \|D^s A_0\|_{L_t^p L_x^q} \|w\|_{L_t^{p'} L_x^{q'}} \le \|A_0\| \|w\|_{L_t^{p'} L_x^{q'}}, \tag{3.74}$$

with $\frac{1}{p}+\frac{1}{p'}=\frac{1}{2}=\frac{1}{q}+\frac{1}{q'}$ where p,q are as in(3.72). Then Klainerman-Selberg Theorem applies[5] and gives

$$I \leq \|A_0\|\|w\|_{L_t^{p'}L_x^{q'}} \lesssim \|A_0\|\|w\|_{H^{1-\frac{2}{q'}-\frac{1}{p'},\theta}}. \tag{3.75}$$

From (3.72) we also have

$$I \lesssim \|A_0\|\|w\|_{H^{1-\frac{2}{q'}-\frac{1}{p'},\theta}} \lesssim \|A_0\|\|w\|_{H^{s,\theta}}. \tag{3.76}$$

To II we also apply Hölder's inequality

$$II \leq \|A_0\|_{L_t^{\tilde{p}}L_x^{\infty}}(\|w\|_{L_t^{\tilde{p}'}L_x^2}+\|D^s w\|_{L_t^{\tilde{p}'}L_x^2}) \lesssim \|A_0\|\|\Lambda^s w\|_{L_t^{\tilde{p}'}L_x^2}, \tag{3.77}$$

where $\frac{1}{\tilde{p}}+\frac{1}{\tilde{p}'}=\frac{1}{2}$. Then by Lemma 2.3.4 we have

$$II \lesssim \|A_0\|\|\Lambda^s w\|_{L_t^{p'}L_x^2} \lesssim \|A_0\|\|w\|_{H^{s,\theta}} \tag{3.78}$$

(3.71) follows now from (3.76) and (3.78).

3.5 Elliptic Regularity: Estimates for A_0.

Here we present a variety of a priori estimates for the nondynamical variable A_0. At each point we could add the index j to A_0, df and ϕ. Therefore the presentation also applies to the iterates $A_{0,j}$. It is an exercise to show that the estimates we obtain here are enough to solve for $A_{0,j}$ at each step as well as to close the iteration for A_0.

Let A_0 solve

$$\triangle A_0 = d^*[A_0, *df] + d^*[df, \phi].$$

[5]See the discussion following Theorem 3.5.3 in 3.5 for more details.

Hence there is a wide range of estimates A_0 satisfies. Nevertheless, the two spatial dimensions limit our "range of motion". For example, it does not seem possible to place $A_0(t)$ in L^2. We add that the proofs of both of the following theorems were originally inspired by Selberg's proof of his estimate (45) in [22]. We start with the homogeneous estimates.

Theorem 3.5.1. Let $s > 0$, and let $0 \leq a \leq s+1$ be given. Suppose $1 \leq p \leq \infty$ and $1 < q < \infty$ satisfy

$$\max\left(\frac{1}{3}(1+2a-4s), \frac{1}{2}(1+a-4s), \frac{1}{2}\min(a,1)\right) < \frac{1}{q} < \frac{1+a}{2}, \tag{3.79}$$

$$1 - \frac{2}{q} + a - 2s \leq \frac{1}{p} \leq \frac{1}{2}(1-\frac{1}{q}) \quad \text{and} \quad \frac{2}{p} < 1 - \frac{2}{q} + a. \tag{3.80}$$

i) If $0 \leq a \leq 1$ and the $H^{s,\theta}$ norm of df is sufficiently small, then $A_0 \in L_t^p \dot{W}_x^{a,q}$ and we have the following estimate

$$\|A_0\|_{L_t^p \dot{W}_x^{a,q}} \lesssim \|\phi\|_{H^{s,\theta}} \|df\|_{H^{s,\theta}}. \tag{3.81}$$

ii) If $1 < a \leq s+1$ and $A_0 \in L_t^p L_x^{(1/q-1/2)^{-1}}$, then $A_0 \in L_t^p \dot{W}_x^{a,q}$ and we have the following estimate

$$\|A_0\|_{L_t^p \dot{W}_x^{a,q}} \lesssim (\|A_0\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} + \|\phi\|_{H^{s,\theta}}) \|df\|_{H^{s,\theta}} \tag{3.82}$$

Proof. Let $a = 0$. Then we have

$$\begin{aligned} \|A_0\|_{L_t^p L_x^q} &= \|\triangle^{-1}(d^*[A_0, *df] + d^*[df, \phi])\|_{L_t^p L_x^q} \\ &\lesssim \|D^{-1}(A_0 df)\|_{L_t^p L_x^q} + \|D^{-1}(df\phi)\|_{L_t^p L_x^q} \\ &\lesssim \|A_0 df\|_{L_t^p L_x^r} + \|D^{-1}(df\phi)\|_{L_t^p L_x^q}, \end{aligned} \tag{3.83}$$

where we use the Sobolev embedding with $\frac{1}{q} = \frac{1}{r} - \frac{1}{2}$. The second term is handled by the Klainerman-Tataru Theorem [18] (see Appendix A for details), which gives

$$\|D^{-1}(df\phi)\|_{L_t^p L_x^q} \lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}}. \tag{3.84}$$

For the first term we have by Hölder's inequality

$$\|A_0 df\|_{L_t^p L_x^r} \leq \|A_0\|_{L_t^p L_x^q} \|df\|_{L_t^\infty L_x^2}. \tag{3.85}$$

Combine (3.84) and (3.85) to get

$$\|A_0\|_{L_t^p L_x^q} \lesssim \|A_0\|_{L_t^p L_x^q} \|df\|_{H^{s,\theta}} + \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}}. \tag{3.86}$$

So if the $H^{s,\theta}$ norm of df is sufficiently small, we obtain

$$\|A_0\|_{L_t^p L_x^q} \lesssim \|\phi\|_{H^{s,\theta}} \|df\|_{H^{s,\theta}}, \tag{3.87}$$

as needed.

Now let $0 < a < 1$. Then beginning as for $a = 0$

$$\begin{aligned} \|A_0\|_{L_t^p \dot{W}_x^{a,q}} = &\lesssim \|D^{-1}(A_0 df)\|_{L_t^p \dot{W}_x^{a,q}} + \|D^{-1}(df\phi)\|_{L_t^p \dot{W}_x^{a,q}} \\ &\lesssim \|D^{a-1}(A_0 df)\|_{L_t^p L_x^q} + \|D^{a-1}(df\phi)\|_{L_t^p L_x^q} \\ &\lesssim \|A_0 df\|_{L_t^p L_x^r} + \|D^{a-1}(df\phi)\|_{L_t^p L_x^q}, \qquad \frac{1}{q} = \frac{1}{r} - \frac{1-a}{2}. \end{aligned} \tag{3.88}$$

The latter term is again bounded using the Klainerman-Tataru theorem. For the former we use $\frac{1}{r} = \frac{1}{q} + \frac{1-a}{2} = (\frac{1}{q} - \frac{a}{2}) + \frac{1}{2}$.

$$\|A_0 df\|_{L_t^p L_x^r} \leq \|A_0\|_{L_t^p L_x^{(}1/q-a/2)^{-1}} \|df\|_{L_t^\infty L_x^2} \leq \|A_0\|_{L_t^p \dot{W}_x^{a,q}} \|df\|_{H^{s,\theta}}. \tag{3.89}$$

Then again if the $H^{s,\theta}$ norm of df is sufficiently small, we obtain

$$\|A_0\|_{L_t^p \dot{W}_x^{a,q}} \lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}} \tag{3.90}$$

as needed.

Now, let $a = 1$. Then

$$\begin{aligned}\|A_0\|_{L_t^p \dot{W}_x^{1,q}} &\lesssim \|D^{-1}(A_0 df)\|_{L_t^p \dot{W}_x^{1,q}} + \|D^{-1}(df\phi)\|_{L_t^p \dot{W}_x^{1,q}} \\ &= \|A_0 df\|_{L_t^p L_x^q} + \|df\phi\|_{L_t^p L_x^q} \\ &\leq \|A_0\|_{L_t^p L_x^r} \|df\|_{L_t^\infty L_x^2} + \|df\|_{L_t^{2p} L_x^{2q}} \|\phi\|_{L_t^{2p} L_x^{2q}}, \qquad \frac{1}{r} = \frac{1}{q} - \frac{1}{2}. \\ &\lesssim \|A_0\|_{L_t^p \dot{W}_x^{1,q}} \|df\|_{H^{s,\theta}} + \|df\|_{L_t^{2p} L_x^{2q}} \|\phi\|_{L_t^{2p} L_x^{2q}}.\end{aligned} \tag{3.91}$$

The estimate follows from $H^{s,\theta} \hookrightarrow H^{1-\frac{2}{2q}-\frac{1}{2p},\theta} \hookrightarrow L_t^{2p} L_x^{2q}$, where the first embedding holds by the left hand side of (3.80), and the last by Theorem D [17] (See Appendix A).

Now, let $1 < a \leq s+1$. Then

$$\begin{aligned}\|A_0\|_{L_t^p \dot{W}_x^{a,q}} &\lesssim \|D^{-1}(A_0 df)\|_{L_t^p \dot{W}_x^{a,q}} + \|D^{-1}(df\phi)\|_{L_t^p \dot{W}_x^{a,q}} \\ &\lesssim \|D^{a-1}(A_0 df)\|_{L_t^p L_x^q} + \|D^{a-1}(df\phi)\|_{L_t^p L_x^q}\end{aligned} \tag{3.92}$$

For the first term we have

$$\begin{aligned}\|D^{a-1}(A_0 df)\|_{L_t^p L_x^q} &\lesssim \|D^{a-1}A_0\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} \|df\|_{L_t^\infty L_x^2} \\ &\quad + \|A_0\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} \|D^{a-1} df\|_{L_t^\infty L_x^2} \\ &\lesssim \|D^a A_0\|_{L_t^p L_x^q} \|df\|_{H^{s,\theta}} + \|A_0\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} \|df\|_{H^{s,\theta}}.\end{aligned} \tag{3.93}$$

For the second term we have

$$\begin{aligned}\|D^{a-1}(df\phi)\|_{L_t^p L_x^q} &\lesssim \|D^{a-1} df\|_{L_t^{p_1} L_x^{q_1}} \|\phi\|_{L_t^{p_2} L_x^{q_2}} + \|df\|_{L_t^{p_2} L_x^{q_2}} \|D^{a-1}\phi\|_{L_t^{p_1} L_x^{q_1}}, \\ &\qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2},\end{aligned} \tag{3.94}$$

and since df and ϕ have the same regularity, it is enough to just show we can control $\|D^{a-1} df\|_{L_t^{p_1} L_x^{q_1}} \|\phi\|_{L_t^{p_2} L_x^{q_2}}$. Now if p and q satisfy (3.79) and (3.80), it can be checked (see Appendix A) that we can find q_1 and q_2 so that Theorem D in [17] again applies, and therefore gives us

$$\begin{aligned} \|D^{a-1}df\|_{L_t^{p_1}L_x^{q_1}}\|\phi\|_{L_t^{p_2}L_x^{q_2}} &\lesssim \|df\|_{H^{1-2/q_1-1/p_1+a-1,\theta}}\|\phi\|_{H^{1-2/q_2-1/p_2,\theta}} \\ &\lesssim \|df\|_{H^{s,\theta}}\|\phi\|_{H^{s,\theta}} \end{aligned} \tag{3.95}$$

□

Remark 3.5.1. In every place where we use the Klainerman-Tataru theorem we could use the Sobolev embedding, and then Klainerman-Selberg theorem. However, then the range of p and q would be much more restricted.

Corollary 3.5.2. In particular, if $s > 0$, we have $A_0 \in C_b(I : \dot{H}^a_x)$ where

$$0 < a \leq \begin{cases} 2s & \text{if} \quad 0 < s \leq 1 \\ 1+s & \text{if} \quad 1 < s \end{cases}$$

Proof. Suppose $0 < s < \frac{1}{2}$. Then use part i) of the theorem with $q = 2$ and $p = \infty$ to obtain $A_0 \in L_t^\infty \dot{H}^a_x$ for $a \leq 2s$. So we just need to show A_0 is continuous as a function of time, but that easily follows from a contraction argument in $C_b(I : \dot{H}^a_x)$ using $L_t^\infty \dot{H}^a_x$ estimates.

Suppose $\frac{1}{2} \leq s \leq 1$. Start with $s = \frac{1}{2}$. If $a < 1$, the statement follows again from part i) of the theorem, so consider $a = 2s = 1$. Then

$$\begin{aligned} \|A_0\|_{L_t^\infty \dot{H}^1_x} &\lesssim \|A_0 df\|_{L_t^\infty L_x^2} + \|df\phi\|_{L_t^\infty L_x^2} \\ &\lesssim \|A_0\|_{L_t^\infty L_x^4}\|df\|_{L_t^\infty L_x^4} + \|df\|_{L_t^\infty L_x^4}\|\phi\|_{L_t^\infty L_x^4} \\ &\lesssim \|A_0\|_{L_t^\infty \dot{H}^{\frac{1}{2}}}\|df\|_{H^{1-\frac{2}{4},\theta}} + \|df\|_{H^{1-\frac{2}{4},\theta}}\|\phi\|_{H^{1-\frac{2}{4},\theta}}, \end{aligned} \tag{3.96}$$

where we use Sobolev embedding and Theorem D. Since $\frac{1}{2} < 1, A_0 \in L_t^\infty \dot{H}^{\frac{1}{2}}$ Now let $\frac{1}{2} < s < 1$. $A_0 \in L_t^\infty \dot{H}^a$ for $a \leq 1$ follows by the previous arguments.

So let $1 < a \leq \min 2s$.

$$\begin{aligned} \|A_0\|_{L_t^\infty \dot{H}_x^a} &\lesssim \|D^{a-1}(A_0 df)\|_{L_t^\infty L_x^2} + \|D^{a-1}(df\phi)\|_{L_t^\infty L_x^2} \\ &\lesssim \|(D^{a-1}A_0)df\|_{L_t^\infty L_x^2} + \|A_0 D^{a-1}df\|_{L_t^\infty L_x^2} + \|D^{a-1}(df\phi)\|_{L_t^\infty L_x^2}. \end{aligned} \tag{3.97}$$

To the last term we apply Leibniz rule and use Theorem D (see Appendix A). For the first term we have

$$\begin{aligned} \|(D^{a-1}A_0)df\|_{L_t^\infty L_x^2} &\lesssim \|D^{a-1}A_0\|_{L_t^\infty L_x^{(\frac{1}{2}-\frac{1-s}{2})^{-1}}} \|df\|_{L_t^\infty L_x^{(\frac{1}{2}-\frac{s}{2})^{-1}}} \\ &\lesssim \|D^{a-s}A_0\|_{L_t^\infty L_x^2} \|df\|_{H^{s,\theta}}, \end{aligned} \tag{3.98}$$

which is bounded by part i) of the Theorem since $0 < a - s < 1$. For the second term we have

$$\begin{aligned} \|A_0 D^{a-1}df\|_{L_t^\infty L_x^2} &\lesssim \|A_0\|_{L_t^\infty L^{(\frac{1}{2}-\frac{s}{2})^{-1}}} \|D^{a-1}df\|_{L_t^\infty L_x^{(\frac{1}{2}-\frac{1-s}{2})^{-1}}} \\ &\lesssim \|A_0\|_{L_t^\infty L_x^{\dot{H}^s}} \|df\|_{H^{a-s,\theta}}, \end{aligned} \tag{3.99}$$

which is bounded since $0 < s < 1$ and $a - s \leq s < 1$.

Now let $s = 1$. If $a < 2$, the statement follows again from part i) of the theorem and previous arguments, so consider $a = 2$. Then

$$\begin{aligned} \|A_0\|_{L_t^\infty \dot{H}_x^2} &\lesssim \|D(A_0 df)\|_{L_t^\infty L_x^2} + \|D(df\phi)\|_{L_t^\infty L_x^2} \\ &\lesssim \|(DA_0 df\|_{L_t^\infty L_x^2} + \|A_0 Ddf\|_{L_t^\infty L_x^2} + \|D^{a-1}(df\phi)\|_{L_t^\infty L_x^2}. \end{aligned} \tag{3.100}$$

The last term again is bounded by Theorem D. For the first term we pick $0 < \alpha < 1$ to obtain

$$\begin{aligned} \|(DA_0)df\|_{L_t^\infty L_x^2} &\lesssim \|DA_0\|_{L_t^\infty L_x^{(\frac{1}{2}-\frac{1-\alpha}{2})^{-1}}} \|df\|_{L_t^\infty L_x^{(\frac{1}{2}-\frac{\alpha}{2})^{-1}}} \\ &\lesssim \|D^{2-\alpha}A_0\|_{L_t^\infty L_x^2} \|df\|_{H^{s,\theta}}, \end{aligned} \tag{3.101}$$

which is bounded by previous arguments since $0 < 2 - \alpha < 2$. For the second term, by Corollary 3.5.4

$$\begin{aligned} \|A_0 Ddf\|_{L_t^\infty L_x^2} &\lesssim \|A_0\|_{L_{t,x}^\infty} \|Ddf\|_{L_t^\infty L_x^{(\frac{1}{2}-\frac{1-s}{2})^{-1}}} \\ &\lesssim \|A_0\|_{L_{t,x}^\infty} \|df\|_{H^{1,\theta}}. \end{aligned} \tag{3.102}$$

Suppose $s > 1$. If $a \leq 2$, the statement follows again from part i) of the theorem and previous arguments, so consider $2 < a \leq s+1$.

$$\begin{aligned}
\|A_0\|_{L_t^\infty \dot{H}_x^a} &\lesssim \|D^{a-1}(A_0 df)\|_{L_t^\infty L_x^2} + \|D^{a-1}(df\phi)\|_{L_t^\infty L_x^2} \\
&\lesssim \|(D^{a-1}A_0)df\|_{L_t^\infty L_x^2} \|A_0 D^{a-1}df\|_{L_t^\infty L_x^2} + \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}} \\
&\lesssim \|D^{a-1}A_0\|_{L_t^\infty L_x^2} \|df\|_{L_{t,x}^\infty} + \|A_0\|_{L_{t,x}^\infty} \|D^{a-1}df\|_{L_t^\infty L_x^2} + \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}} \\
&\lesssim \|A_0\|_{L_t^\infty \dot{H}_x^{a-1}} \|df\|_{L_{t,x}^\infty} + \|A_0\|_{L_{t,x}^\infty} \|df\|_{H^{s,\theta}} + \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}},
\end{aligned} \tag{3.103}$$

where we use that $df, \phi \in L_{t,x}^\infty$.[6] Now if $a-1 \leq 2$, then we already have established $A_0(t) \in \dot{H}^a$, and we are done. If $a-1 > 2$, then we estimate $\|A_0\|_{L_t^\infty \dot{H}_x^{a-1}}$ and "backtrack" till we have needed regularity for A_0. This completes the proof of the corollary. □

So far we just need $s > 0$ in order to make the estimates work. The requirement for $s > \frac{1}{4}$ does not come in till we start looking at the nonhomogeneous spaces, where also the range of p and q is smaller. However, we can distinguish two cases $aq < 2$ and $aq > 2$.

Theorem 3.5.3. Let $s > 0$, and suppose the $H^{s,\theta}$ norm of df is sufficiently small.

i) If $aq < 2$ for $0 < a < (2s, 1)$ and if p and q satisfy

$$\max\left(\frac{1}{2} + a - 2s, \frac{a}{2}\right) < \frac{1}{q} < \frac{1}{2}, \tag{3.104}$$

$$1 - \frac{2}{q} + a - 2s \leq \frac{1}{p} < \frac{1}{2} - \frac{1}{q}, \tag{3.105}$$

[6]We can do the proof directly or use that $\|D^{a-1}u\|_{L_t^\infty L_x^2} \leq \|u\|_{H^{s,\theta}}$ and use that $H^{s,\theta}$ is now an algebra(see [17] Thm 7.3).

then $A_0 \in L_t^p W_x^{a,q}$ and we have the following estimate

$$\|A_0\|_{L_t^p W_x^{a,q}} \lesssim \|\phi\|_{H^{s,\theta}} \|df\|_{H^{s,\theta}}. \tag{3.106}$$

ii) If $aq > 2$, then we need $s > \frac{1}{4}$ and $0 < a < \min(4s-1, 1+s, 2s)$. Suppose p and q also satisfy

$$\max\left(\frac{a-s}{2}, \frac{1}{2} + a - 2s\right) \leq \frac{1}{q} < \frac{1}{2}\min(a,1), \tag{3.107}$$

$$1 - \frac{2}{q} + a - 2s \leq \frac{1}{p} < \frac{1}{2} - \frac{1}{q}, \tag{3.108}$$

then $A_0 \in L_t^p W_x^{a,q}$ and we have the following estimate

$$\|A_0\|_{L_t^p W_x^{a,q}} \lesssim \|\phi\|_{H^{s,\theta}} \|df\|_{H^{s,\theta}}. \tag{3.109}$$

Proof. Let $aq < 2$ for $0 < a < \min(2s, 1)$. We need $A_0, D^a A_0 \in L_t^p L_x^q$, but this follows from Theorem 3.5.1 when we use part *i*) twice: first for A_0 and then for $D^a A_0$ with the same values of p and q.[7]

Now let $aq > 2$ and $s > \frac{1}{4}$ and $0 < a < 4s - 1$. Here, Theorem 3.5.1 does not apply anymore, so we return to estimating A_0 directly.

$$\begin{aligned} \|A_0\|_{L_t^p W_x^{a,q}} &= \|\triangle^{-1}(d^*[A_0, df] + d^*[df, \phi])\|_{L_t^p W_x^{a,q}} \\ &\lesssim \|D^{-1}(A_0 df)\|_{L_t^p W_x^{a,q}} + \|D^{-1}(df\phi)\|_{L_t^p W_x^{a,q}} \\ &\lesssim \|D^{-1}(A_0 df)\|_{L_t^p L_x^q} + \|D^{-1}(df\phi)\|_{L_t^p L_x^q} \\ &\quad + \|D^{a-1}(A_0 df)\|_{L_t^p L_x^q} + \|D^{a-1}(df\phi)\|_{L_t^p L_x^q} \end{aligned} \tag{3.110}$$

[7] The conditions for p and q in (3.104) and (3.105) are obtained by making sure (3.79) and (3.80) hold for both $a = 0$ and $a > 0$. This results in new conditions for a so (3.104) can hold.

The first term is handled in the same way as in the proof of Theorem 3.5.1 part i) with $a = 0$. Klainerman-Tataru theorem takes care of the second term. The last term is handled by Klainerman-Tataru Theorem for $a < 1$, and for $a \geq 1$ in the same way as it was in the proof of Theorem 3.5.1 part iii). So we just need to consider the third term, where we look at three cases: $0 < a \leq s < 1$, $s < a < \min(4s-1, 1+s, 2s) < 1$, and $1 \leq a < \min(4s-1, 1+s, 2s)$. Let $0 < a \leq s < 1$, then

$$\begin{aligned} \|D^{a-1}(A_0 df)\|_{L_t^p L_x^q} &\lesssim \|A_0 df\|_{L_t^p L_x^r}, & \frac{1}{q} = \frac{1}{r} - \frac{1-a}{2} \\ &\leq \|A_0\|_{L_t^p L_x^q} \|D^a df\|_{L_t^\infty L_x^2}, & \frac{1}{r} = \frac{1}{q} + (\frac{1}{2} - \frac{a}{2}) \\ &\lesssim \|A_0\|_{L_t^p W_x^{a,q}} \|df\|_{H^{s,\theta}}. \end{aligned}$$

For $s < a < \min(4s-1, 1+s, 2s) < 1$ we have

$$\begin{aligned} \|D^{a-1}(A_0 df)\|_{L_t^p L_x^q} &\lesssim \|A_0 df\|_{L_t^p L_x^r}, & \frac{1}{q} = \frac{1}{r} - \frac{1-a}{2}. \\ &\leq \|D^{a-s} A_0\|_{L_t^p L_x^q} \|D^s df\|_{L_t^\infty L_x^2} & \frac{1}{r} = (\frac{1}{q} - \frac{a-s}{2}) + (\frac{1}{2} - \frac{s}{2}) \\ &\lesssim \|A_0\|_{L_t^p W_x^{a,q}} \|df\|_{H^{s,\theta}}. \end{aligned}$$

Now let $1 \leq a < \min(4s-1, 1+s, 2s)$. We look at $a = 1$ and $1 < a < \min(4s-1, 1+s, 2s)$ separately. Let $a = 1$ and suppose $s > 1$, then the proof is trivial since then $H^{s,\theta} \hookrightarrow L_{t,x}^\infty$, so

$$\begin{aligned} \|A_0 df\|_{L_t^p L_x^q} &\lesssim \|A_0\|_{L_t^p L_x^q} \|df\|_{L_{t,x}^\infty} \\ &\lesssim \|A_0\|_{L_t^p W_x^{1,q}} \|df\|_{H^{s,\theta}}. \end{aligned} \tag{3.111}$$

If $s = 1$, pick $1 > \alpha > 0$ so that $\frac{1}{q} - \frac{\alpha}{2} > 0$. Then

$$\begin{aligned} \|A_0 df\|_{L_t^p L_x^q} &\le \|A_0\|_{L_t^p L_x^{(1/q-\alpha/2)^{-1}}} \|df\|_{L_t^\infty L_x^{2/\alpha}} \\ &\lesssim \|A_0\|_{L_t^p W_x^{\alpha,q}} \|D^{1-\alpha} df\|_{L_t^\infty L_x^2}, \quad since \quad \frac{\alpha}{2} = \frac{1}{2} - \frac{1-\alpha}{2} \\ &\lesssim \|A_0\|_{L_t^p W_x^{1,q}} \|df\|_{H^{s,\theta}}, \end{aligned} \tag{3.112}$$

Now suppose $s < 1$. We have

$$\begin{aligned} \|A_0 df\|_{L_t^p L_x^q} &\le \|A_0\|_{L_t^p L_x^{(1/q-(1-s)/2)^{-1}}} \|df\|_{L_t^\infty L_x^{2/1-s}} \\ &\lesssim \|A_0\|_{L_t^p W_x^{1-s,q}} \|D^s df\|_{L_t^\infty L_x^2} \\ &\lesssim \|A_0\|_{L_t^p W_x^{1,q}} \|df\|_{H^{s,\theta}}. \end{aligned} \tag{3.113}$$

This completes the case $a = 1$. For $1 < a < \min(4s-1, 1+s, 2s)$ we look at $s < a$ and $s \ge a$. For $s < a$ we obtain

$$\begin{aligned} \|D^{a-1}(A_0 df)\|_{L_t^p L_x^q} &\lesssim \|D^{a-1} A_0\|_{L_t^p L_x^{(1/q-(a-s)/2)^{-1}}} \|df\|_{L_t^\infty L_x^{(a-s)/2^{-1}}} \\ &\quad + \|A_0\|_{L_t^p L_x^{(1/q-(a-s)/2)^{-1}}} \|D^{a-1} df\|_{L_t^\infty L_x^{(a-s)/2^{-1}}} \\ &\lesssim \|D^{a-s+a-1} A_0\|_{L_t^p L_x^q} \|D^{1-a+s} df\|_{L_t^\infty L_x^2} \quad \frac{a-s}{2} = \frac{1}{2} - \frac{1-a+s}{2} \\ &\quad + \|D^{a-s} A_0\|_{L_t^p L_x^q} \|D^{a-1+1-a+s} df\|_{L_t^\infty L_x^2} \\ &\lesssim \|A_0\|_{L_t^p W_x^{a,q}} \|df\|_{H^{s,\theta}} \end{aligned}$$

If $s \ge a > 1$, again pick $1 > \alpha > 0$ so that $\frac{1}{q} - \frac{\alpha}{2} > 0$. Then

$$\begin{aligned} \|D^{a-1}(A_0 df)\|_{L_t^p L_x^q} &\lesssim \|D^{a-1} A_0\|_{L_t^p L_x^q} \|df\|_{L_{t,x}^\infty} \\ &\quad + \|A_0\|_{L_t^p L_x^{(1/q-\alpha/2)^{-1}}} \|D^{a-1} df\|_{L_t^\infty L_x^{\alpha/2^{-1}}} \\ &\lesssim \|A_0\|_{L_t^p W_x^{a,q}} \|df\|_{H^{s,\theta}} \\ &\quad + \|D^\alpha A_0\|_{L_t^p L_x^q} \|D^{1-\alpha} df\|_{L_t^\infty L_x^2}, \quad since \quad \frac{\alpha}{2} = \frac{1}{2} - \frac{1-\alpha}{2} \\ &\lesssim \|A_0\|_{L_t^p W_x^{a,q}} \|df\|_{H^{s,\theta}} \end{aligned}$$

Then as before if the $H^{s,\theta}$ norm of df is sufficiently small, we obtain

$$\|A_0\|_{L_t^p W_x^{a,q}} \lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}} \tag{3.114}$$

as needed. ☐

Corollary 3.5.4. If $s > \frac{1}{4}$ and the $H^{s,\theta}$ norm of df is sufficiently small, we have in particular $A_0 \in L_t^p L_x^\infty$ for p satisfying

$$1 - 2s < \frac{1}{p} < \frac{1}{2}, \tag{3.115}$$

and we have the following estimate

$$\|A_0\|_{L_t^p L_x^\infty} \lesssim \|\phi\|_{H^{s,\theta}} \|df\|_{H^{s,\theta}}. \tag{3.116}$$

Proof. For each $p \in (1 - 2s, \frac{1}{2})$ we can find some a and q, which satisfy the conditions of Theorem 2, part ii). The corollary then follows from the Sobolev Embedding: $W^{a,q}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$ for $aq > 2$. ☐

3.6 Estimates Needed in Section 3.4.5.

Now we are ready to benefit from the theorems asserted in the previous section. Recall from Section 3.4.5 we would like to establish

$$A_0 \in L_t^{\tilde{p}} L_x^\infty \cap L_t^p \dot{W}_x^{s,q}.$$

By Corollary 3.5.4 we have $A_0 \in L_t^{\tilde{p}} L_x^\infty$. Theorem 3.5.1 part i) with $a = s$ gives $A_0 \in L_t^p \dot{W}_x^{s,q}$ for any p, q satisfying

$$\max\left(\frac{1}{3}(1-2s), \frac{s}{2}\right) < \frac{1}{q} < \frac{1+s}{2}, \tag{3.117}$$

$$1 - \frac{2}{q} - s \leq \frac{1}{p} \leq \frac{1}{2}(1 - \frac{1}{q}) \quad \text{and} \quad \frac{1}{p} \leq \frac{1}{2}(1 - \frac{2}{q} + s). \tag{3.118}$$

The conditions on p, q are further restricted since we would like to use

$$H^{s,\theta} \hookrightarrow H^{1-(\frac{1}{2}-\frac{2}{q})-(\frac{1}{2}-\frac{1}{p}),\theta}(\mathbb{R}^{2+1}) \hookrightarrow L_t^{(1/2-1/p)^{-1}} L_x^{(1/2-1/q)^{-1}}, \tag{3.119}$$

in (3.74). This gives two more inequalities p, q must satisfy

$$1-(\frac{1}{2}-\frac{2}{q})-(\frac{1}{2}-\frac{1}{p}), \quad \text{so the first embedding holds in(3.123)}, \tag{3.120}$$

$$\frac{1}{2}-\frac{1}{p} \leq \frac{1}{2}(1-\frac{1}{2}+\frac{1}{q}), \quad \text{so the second embedding holds in(3.123)}, \tag{3.121}$$

(3.118) and (3.121) force $\frac{1}{p}$ to actually equal $\frac{1}{2}(1-\frac{1}{q})$. As a result (3.120) also gives a stricter upper bound on $\frac{1}{q}$. Namely

$$\frac{1}{q} < \frac{2}{3}s. \tag{3.122}$$

Moreover, we need a specific case of part ii) in Theorem 3.5.1, because we need $A_0 \in L_t^p \dot{W}_x^{s,q}$, where p, q in addition satisfy

$$\frac{1}{q} < \frac{1}{2}, \quad 1-\frac{2}{p} \leq \frac{1}{q}, \quad \text{and} \quad \frac{2}{q}-\frac{1}{2}+\frac{1}{p} \leq s, \tag{3.123}$$

When we put (3.123) together with (3.80) and (3.79) we obtain second line of (3.72)

$$\frac{2}{p} = 1-\frac{1}{q}, \quad \max(\frac{1}{3}(1-2s), \frac{s}{2}) < \frac{1}{q} < s. \tag{3.124}$$

Let $a = s$, and consider

$$\begin{aligned} \|A_0\|_{L_t^p \dot{W}_x^{s,q}} &= \|\triangle^{-1}(d^*[A_0, *df] + d^*[df, \phi])\|_{L_t^p \dot{W}_x^{s,q}} \\ &\lesssim \|D^{-1}(A_0 df)\|_{L_t^p \dot{W}_x^{s,q}} + \|D^{-1}(df\phi)\|_{L_t^p \dot{W}_x^{s,q}} \\ &\lesssim \|D^{s-1}(A_0 df)\|_{L_t^p L_x^q} + \|D^{s-1}(df\phi)\|_{L_t^p L_x^q} \\ &\lesssim \|A_0 df\|_{L_t^p L_x^r} + \|D^{s-1}(df\phi)\|_{L_t^p L_x^q}, \end{aligned} \tag{3.125}$$

where we use the Sobolev embedding with $\frac{1}{q} = \frac{1}{r} - \frac{1-s}{2}$. The latter term is bounded using the Klainerman-Tataru theorem (see Appendix A). For the former we use $\frac{1}{r} = \frac{1}{q} + \frac{1-s}{2} = (\frac{1}{q} - \frac{s}{2}) + \frac{1}{2}$.

$$\|A_0 df\|_{L^p_t L^r_x} \leq \|A_0\|_{L^p_t L_x^{(1/q-s/2)^{-1}}} \|df\|_{L^\infty_t L^2_x} \leq \|A_0\|_{L^p_t \dot{W}^{s,q}_x} \|df\|_{H^{s,\theta}}. \tag{3.126}$$

Then if the $H^{s,\theta}$ norm of df is sufficiently small, we obtain

$$\|A_0\|_{L^p_t \dot{W}^{s,q}_x} \lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{H^{s,\theta}} \tag{3.127}$$

as needed.

For the non-homogeneous estimate with $aq > 2$, $\frac{1}{4} < s < \frac{1}{2}$ and $0 < a < 4s - 1$ and p, q satisfying (3.107), (3.108) we have

$$\begin{aligned} \|A_0\|_{L^p_t W^{a,q}_x} &\lesssim \|D^{-1}(A_0 df)\|_{L^p_t W^{a,q}_x} + \|D^{-1}(df\phi)\|_{L^p_t W^{a,q}_x} \\ &\lesssim \|D^{-1}(A_0 df)\|_{L^p_t L^q_x} + \|D^{-1}(df\phi)\|_{L^p_t L^q_x} \\ &\quad + \|D^{a-1}(A_0 df)\|_{L^p_t L^q_x} + \|D^{a-1}(df\phi)\|_{L^p_t L^q_x} \end{aligned} \tag{3.128}$$

Klainerman-Tataru theorem takes care of the second term and the last term (see Appendix). Consider the first term

$$\begin{aligned} \|D^{-1}(A_0 df)\|_{L^p_t L^q_x} &\lesssim \|A_0 df\|_{L^p_t L^r_x}, \quad \frac{1}{q} = \frac{1}{r} - \frac{1}{2}, \\ &\leq \|A_0\|_{L^p_t L^q_x} \|df\|_{L^\infty_t L^2_x}. \end{aligned} \tag{3.129}$$

For our purposes right now we just need to show the estimate works for one a, so suppose $0 < a \leq s$, then

$$\begin{aligned} \|D^{a-1}(A_0 df)\|_{L^p_t L^q_x} &\lesssim \|A_0 df\|_{L^p_t L^r_x}, \quad \frac{1}{q} = \frac{1}{r} - \frac{1-a}{2} \\ &\lesssim \|A_0\|_{L^p_t L^q_x} \|D^a df\|_{L^\infty_t L^2_x}, \quad \frac{1}{r} = \frac{1}{q} + (\frac{1}{2} - \frac{a}{2}) \\ &\lesssim \|A_0\|_{L^p_t W^{a,q}_x} \|df\|_{H^{s,\theta}}, \end{aligned}$$

as needed.

Chapter 4

Ward Wave Map

4.1 Introduction

In this chapter we would like to prove Main Theorem 2. First we show how one can derive (WWM) from (ASDYM). Second, we show conservation of energy, and then we finally prove our theorem.

4.2 Derivation of the Ward Wave Map

In this section we derive the Ward Wave Map from the (ASDYM). This is done in four steps. First we introduce (ASDYM) connections. Then we give an equivalent formulation called a Lax Pair. The next step is a dimensional reduction followed by a proof of existence of appropriate gauge transformation, which will give us a Lax Pair for the Ward Map. The final step consists of introducing new functions and a change of variables that will allow us to recover (WWM). We closely follow [27] and [4].

4.2.1 Self-Dual and Anti-Self-Dual Yang Mills Connections

Let

$$dx_1^2 + dx_2^2 - dx_3^2 - dx_4^2$$

be a metric on $\mathbb{R}^{2,2}$. Recall from Chapter 3 a connection A is called anti-self-dual Yang Mills (ASDYM) if its curvature form satisfies:

$$F = - * F,$$

which in coordinates is equivalent to

$$F_{13} = -F_{24}, F_{12} = -F_{34}, F_{23} = F_{14}.$$

4.2.2 Lax Pair of ASDYM

Next, we give an equivalent condition for a connection A to be ASDYM.

Proposition 4.2.1. A connection $A = \{D_i\}$ is ASDYM if and only if

$$[\lambda(D_1 - D_3) - (D_4 + D_2), \lambda(D_4 - D_2) - (D_1 + D_3)] = 0 \tag{4.1}$$

for all $\lambda \in \mathbb{C}$.

Proof. First we use the bilinearity of the bracket to expand (4.1), collect the terms corresponding to powers of λ, and observe that equation 4.1 will hold for all $\lambda \in \mathbb{C}$ if and only if the coefficients of $\lambda^2, \lambda, \lambda^0$ are zero. Coefficient of λ^2 is:

$$[D_1 - D_3, D_4 - D_2] = 0.$$

Since, $F_{ij} = [D_i, D_j]$, we obtain

$$F_{14} - F_{12} - F_{34} + F_{32} = 0. \tag{4.2}$$

Coefficient of λ is:

$$[D_1 + D_3, D_1 - D_3] + [D_4 - D_2, D_4 + D_2] = 0.$$

It follows, $F_{13} - F_{31} - F_{42} + F_{24} = 0$, so $F_{13} = -F_{24}$ as needed. Finally, coefficient of $\lambda^0 = 1$ is:

$$[D_4 + D_2, D_1 + D_3] = 0, \tag{4.3}$$

so $-F_{14} + F_{43} + F_{21} + F_{23} = 0$. Adding (4.2) and (4.3) we get $F_{12} = -F_{34}$ as needed. Using it in 4.3, we obtain $F_{14} = F_{23}$ as needed. □

Remark 4.2.1. This condition, writing an equation as a zero curvature of a connection or its portion, is referred to as a Lax pair formulation. The additional parameter λ is called spectral, twistor or Riemann-Hilbert parameter [5].

4.2.3 Dimensional Reduction & Gauge Transformations

Now, having a Lax Pair formulation of ASDYM, we proceed to the dimensional reduction, where we assume A is independent of x_3. We also let $A_3 = \phi$. Set $x = x_1$ and

$$u = \frac{x_2 + x_4}{2}, \qquad v = \frac{x_4 - x_2}{2},$$

$$A_u = A_4 + A_2 \qquad A_v = A_4 - A_2.$$

Then from $\lambda^2, \lambda^1, \lambda^0$ coefficients we will have,

$$[\partial_x + A_1 - \phi, \partial_v + A_v] = 0 \tag{4.4}$$

$$[\partial_x + A_1 - \phi, \partial_x + A_1 + \phi] - [\partial_u + A_u, \partial_v + A_v] = 0$$

$$[\partial_u + A_u, \partial_x + A_1 + \phi] = 0.$$

The equivalent Lax-Pair formulation is:

$$[\lambda(\partial_x + A_1 - \phi) - (\partial_u + A_u), \lambda(\partial_v + A_v) - (\partial_1 + A_1 + \phi)] = 0 \tag{4.5}$$

We will need the following proposition to transform (4.5) into a Lax Pair for the Ward Map.

Proposition 4.2.2. [27] Given smooth maps $A, B : \mathbb{R}^2 \to gl(n, \mathbb{C})$ the following statements are equivalent:

1) the linear system for $g : \mathbb{R}^2 \supset U \to GL(n, \mathbb{C})$

$$g_x = gA, \qquad g_y = gB, \qquad g(0,0) = g_0$$

has a solution.

2) $A_y + BA = B_x + AB$.

3) $[\partial_x + A, \partial_y + B] = 0$.

4) There exists $g : \mathbb{R} \supset U \to GL(n\mathbb{C})$ so that

$$\begin{cases} g(\partial_x + A)g^{-1} = \partial_x \\ g(\partial_y + B)g^{-1} = \partial_y \end{cases}$$

Proof. We will show $4 \Leftrightarrow 1 \Leftrightarrow 2 \Leftrightarrow 3$.

It is easy to see by direct computation that 4 and 1 are equivalent. Start with the first equation in statement 4:

$$g(\partial_x + A)g^{-1} = \partial_x$$

$$g\partial_x g^{-1} + gAg^{-1} = \partial_x$$

$$\partial_x - g_x g^{-1} + gAg^{-1} = \partial_x,$$

where the last line holds iff $g_x = gA$. The proof for the other equation is exactly the same. Also, to go from line 2 to line 3 we used a simple computation that is shown in detail in Section 4.3, equation (10). $1 \Leftrightarrow 2$ follows by differentiating $g_x = gA$ with respect to y and $g_y = gB$ with respect to x, and using the fact that the mixed partials are equal. Lastly, 2 is just another way to write 3. $\square$

By the first equation in (4.4) we have that statement 3 holds. Therefore, we have statement 4, which means that locally we can gauge our connection to be a trivial connection as follows:

$$\begin{cases} g(\partial_x + A_1 - \phi)g^{-1} = \partial_x \\ g(\partial_v + A_v)g^{-1} = \partial_v \end{cases}$$

This will reduce (4.5) to Ward Map Lax Pair given by:

$$[\lambda\partial_x - \partial_u - A, \lambda\partial_v - \partial_x - B] = 0, \tag{4.6}$$

where A and B are to be defined next. First

$$\begin{aligned} 0 &= g[\lambda(\partial_x + A_1 - \phi) - (\partial_u + A_u), \lambda(\partial_v + A_v) - (\partial_x + A_x + \phi)]g^{-1} \\ &= [g\big(\lambda(\partial_x + A_1 - \phi) - (\partial_u + A_u)\big)g^{-1}, g\big(\lambda(\partial_v + A_v) - (\partial_x + A_x - \phi + 2\phi)\big)g^{-1}] \\ &= [\lambda\partial_x - g(\partial_u + A_u)g^{-1}, \lambda\partial_v - 2\partial_x g\phi g^{-1}] \end{aligned}$$

Now we obtain (4.6) when we observe (using (10) in Section 4.3):

$$g(\partial_u)g^{-1} = -g_u g^{-1} + \partial_u,$$

and if we let $A = gA_u g^{-1} - g_u g^{-1}$ and $B = 2g\phi g^{-1}$.

4.2.4 Final Steps

Now we show how to transform (4.6) into (WWM). Let $\Psi : \mathbb{R}^{2,1} \times \Omega \to GL(n, \mathbb{C})$ solve

$$\begin{cases} (\lambda\partial_x - \partial_u)\Psi = A\Psi, \\ (\lambda\partial_v - \partial_x)\Psi = B\Psi, \end{cases}$$

and satisfy $\Psi(x, u, v, \overline{\lambda})^*\Psi(x, u, v, \lambda) = I$, where $\Psi^* = \overline{\Psi}^T$. Let $J(x, u, v) = \Psi(x, u, v, 0)^{-1}$, and $A = J^{-1}J_u$ and $B = J^{-1}J_x$. If we consider coefficient of λ in (4.6), we will obtain

$$B_x = A_v.$$

Now all is left is to plug in for A and B and change to the standard variables: $t = v + u, y = u - v$. The result will be equation (**??**), which we present again for convenience:

$$(J^{-1}J_t)_t - (J^{-1}J_x)_x - (J^{-1}J_y)_y - [J^{-1}J_t, J^{-1}J_y] = 0.$$

4.3 Conservation of Energy

In this section we provide an alternate proof of conservation of energy for (WWM), which was shown in [29] using the energy momentum tensor. Here we show it directly. The precise result is as follows

Theorem 4.3.1. Let

$$E(J(t)) = \frac{1}{2}\int_{\mathbb{R}^2} \|J^{-1}J_t\|^2 + \|J^{-1}J_x\|^2 + \|J^{-1}J_y\|^2 dxdy,$$

where $\|\cdot\|$ is the trace norm

$$\|A\|^2 =< A, A >= \text{tr}\, A^*A.$$

Then if J solves (WWM),

$$E(J(t)) = E(J(0)) \quad \text{for all} \quad t.$$

We need the following lemma for the proof.

Lemma 4.3.2.

i) Let $< \cdot, \cdot >$ and $[\cdot, \cdot]$ denote the inner product induced by the trace norm, and the Lie bracket respectively. Then for any two matrices A, B we have

$$< [A, B], A >= 0 \tag{4.7}$$

ii) Let s be a differentiable map from $\mathbb{R}^{n+1}$ into a Lie group, then

$$\partial_\alpha(s^{-1}) = -s^{-1}(\partial_\alpha s)s^{-1}. \tag{4.8}$$

iii) Let s be a twice differentiable map from $\mathbb{R}^{n+1}$ into a Lie group, then

$$(s^{-1}s_\alpha)_\beta = (s^{-1}s_\beta)_\alpha + [s^{-1}s_\beta, s^{-1}s_\alpha]. \tag{4.9}$$

Proof. For i) using properties of trace

$$\operatorname{tr}(AB - BA)A = \operatorname{tr} ABA - \operatorname{tr} BAA = \operatorname{tr} ABA - \operatorname{tr} ABA = 0.$$

For ii) since

$$s^{-1}s = 1,$$

the product rule gives

$$\partial_\alpha(s^{-1})s + s^{-1}\partial_\alpha s = 0.$$

Solve for $\partial_\alpha s^{-1}$ to obtain (4.8).

Lastly, iii) can be verified by a straightforward computation using (4.8). $\square$

Proof of Theorem 4.3.

$$
\begin{aligned}
&\frac{d}{dt}E(J(t)) \\
&= \int \sum_\alpha < (J^{-1}J_\alpha)_t, J^{-1}J_\alpha > dxdy \\
&= \int \sum_\alpha < (J^{-1}J_t)_\alpha, J^{-1}J_\alpha > dxdy \quad \text{by (4.9) \&(4.7)} \\
&= \int < (J^{-1}J_t)_t, J^{-1}J_t > +\partial_x < J^{-1}J_t, J^{-1}J_x > - < (J^{-1}J_x)_x, J^{-1}J_t > \\
&\qquad + \partial_y < (J^{-1}J_t), J^{-1}J_x > - < (J^{-1}J_y)_y, J^{-1}J_t > dxdy \\
&= \int < [J^{-1}J_t, J^{-1}J_y], J^{-1}J_t > dxdy \\
&= 0 \quad \text{by (4.9)},
\end{aligned}
$$

where we use the divergence theorem to go the line before last together with the assumption that J solves (WWM).

$\square$

4.4 Proof of Main Theorem 2

We recall the statement of the theorem.

Main Theorem 2. *Ward Wave Map (WWM) is locally well-posed for initial data in* $H^s(\mathbb{R}^2) \times H^{s-1}(\mathbb{R}^2)$ *for* $s > \frac{5}{4}$.

First (WWM) can be written as a semilinear wave equation as follows

$$\square J = JW(\partial J, \partial J),$$

where

$$W(\partial J, \partial J) = -J^{-1}\partial^\alpha J J^{-1}\partial_\alpha J + [J^{-1}J_y, J^{-1}J_t].$$

By discussion in Section 2.2 it is enough to establish following estimates

$$\|JW(\partial J, \partial J)\|_{H^{s,\theta-1+\epsilon}} \lesssim \|J\|^2_{\mathcal{H}^{s+1,\theta}}, \tag{4.10}$$

$$\|JW(\partial J, \partial J) - J'W(\partial J', \partial J')\|_{H^{s,\theta-1+\epsilon}} \lesssim \|J - J'\|_{\mathcal{H}^{s+1,\theta}}. \tag{4.11}$$

Write

$$W(\partial J, \partial J) = W_1(\partial J, \partial J) + W_2(\partial J, \partial J),$$

with

$$W_1(\partial J, \partial J) = -J^{-1}\partial^\alpha J J^{-1}\partial_\alpha J \quad \text{and} \quad W_2(\partial J, \partial J) = [J^{-1}J_y, J^{-1}J_2].$$

4.4.1 Proof of (4.10)

Use Theorem 2.3.5 to obtain

$$H^{s+1,\theta} \cdot H^{s,\theta+\epsilon-1} \hookrightarrow H^{s,\theta+\epsilon-1}.$$

Hence

$$\|JW(\partial J, \partial J)\|_{H^{s,\theta-1+\epsilon}} \lesssim \|J\|_{\mathcal{H}^{s+1,\theta}} \| \|W(\partial J, \partial J)\|_{H^{s,\theta-1+\epsilon}} \tag{4.12}$$

Next

$$\|W(\partial J, \partial J)\|_{H^{s,\theta-1+\epsilon}} \lesssim \|W_1(\partial J, \partial J)\|_{H^{s,\theta-1+\epsilon}} + \|W_2(\partial J, \partial J)\|_{H^{s,\theta-1+\epsilon}}. \tag{4.13}$$

For W_1, observe

$$W_1(\partial J, \partial J) = \partial^\alpha(J^{-1})\partial_\alpha J = Q_0(J^{-1}, J).$$

Thus by Theorem 2.5.5 we have

$$\|W_1(\partial J, \partial J)\|_{H^{s,\theta-1+\epsilon}} \lesssim \|J\|_{\mathcal{H}^{s+1,\theta}} \|J\|_{\mathcal{H}^{s+1,\theta}} \tag{4.14}$$

as needed.

For W_2 by part iii) of Lemma 4.3.2 we have

$$[J^{-1}J_2, J^{-1}J_t] = (J^{-1}J_2)_t - (J^{-1}J_t)_2, \tag{4.15}$$

which on Fourier side looks as follows

$$\begin{aligned}(\widehat{J^{-1}J_2})_t(\tau,\xi) - (\widehat{J^{-1}J_t})_2(\tau,\xi) \\ &= i\tau\widehat{J^{-1}J_2}(\tau,\xi) - i\xi_2\widehat{J^{-1}J_t}(\tau,\xi) \\ &= -\iint(\tau\eta_2 - \xi_2\lambda)\widehat{J^{-1}}(\tau-\lambda,\xi-\eta)\widehat{J}(\lambda,\eta)d\lambda d\eta.\end{aligned}$$

Therefore

$$W_2(\partial J, \partial J) = Q_{t2}(J^{-1}, J),$$

and by Corollary 2.5.10 we have

$$\|W_2(\partial J, \partial J)\|_{H^{s,\theta-1+\epsilon}} \lesssim \|J\|_{\mathcal{H}^{s+1,\theta}} \|J\|_{\mathcal{H}^{s+1,\theta}} \tag{4.16}$$

as needed.

4.4.2 Proof of (4.11)

Since W is bilinear there is not much to prove as it mostly follows from previous section. Observe

$$\begin{aligned}JW(\partial J, \partial J) - J'W(\partial J', \partial J') \\ &= \underbrace{(J-J')W(\partial J, \partial J)}_{I} + \underbrace{J'\big(W(J-J', J) + W(J', J-J')\big)}_{II}.\end{aligned}$$

For (I) use Theorem 2.3.5

$$\begin{aligned}\|(J-J')W(\partial J,\partial J)\|_{H^{s,\theta-1+\epsilon}} &\lesssim \|J-J'\|_{\mathcal{H}^{s+1,\theta}}\|W(\partial J,\partial J)\|_{H^{s,\theta-1+\epsilon}}\\ &\lesssim \|J-J'\|_{\mathcal{H}^{s+1,\theta}}\|J^{-1}\|_{\mathcal{H}^{s+1,\theta}}\|J\|_{\mathcal{H}^{s+1,\theta}},\end{aligned}$$

where the last line follows from (4.14) and (4.16).

For (II) again by Theorem 2.3.5 and (4.14) and (4.16) we have

$$\begin{aligned}&\|J'\big(W(J-J',J)+W(J',J-J')\big)\|_{H^{s,\theta-1+\epsilon}}\\ &\qquad\lesssim \|J'\|_{\mathcal{H}^{s+1,\theta}}(\|J\|_{\mathcal{H}^{s+1,\theta}}+\|J'\|_{\mathcal{H}^{s+1,\theta}})\|J-J'\|_{\mathcal{H}^{s+1,\theta}},\end{aligned}$$

which completes the proof.

Appendices

Appendix A

Setting up Klainerman-Tataru and Klainerman-Selberg Theorem

A.1 Elliptic Estimates: Setting up Klainerman-Tataru Theorem

We said that several estimates in the previous theorems follow from the Klainerman-Tataru theorem [18]. We need to check that it is in fact the case. We begin by stating the theorem. We state it for two dimensions only, and as it was given in [17] (the original result holds for $n \geq 2$).

Theorem. Let $1 \leq p \leq \infty$, $1 \leq q < \infty$. Assume that

$$\frac{1}{p} \leq \frac{1}{2}\left(1 - \frac{1}{q}\right), \tag{A.1}$$

$$0 < \sigma < 2\left(1 - \frac{1}{q} - \frac{1}{p}\right), \tag{A.2}$$

$$s_1, s_2 < 1 - \frac{1}{q} - \frac{1}{2p}, \tag{A.3}$$

$$s_1 + s_2 + \sigma = 2(1 - \frac{1}{q} - \frac{1}{2p}). \tag{A.4}$$

Then

$$\|D^{-\sigma}(uv)\|_{L^p_t L^q_x(\mathbb{R}^2)} \lesssim \|u\|_{H^{s_1,\theta}} \|v\|_{H^{s_2,\theta}}, \tag{A.5}$$

provided $\theta > \frac{1}{2}$.

We apply the theorem several times. We examine each case.

Application of Klainerman-Tataru Thm in (3.83) **for** $\|D^{-1}(df\phi)\|_{L_t^p L_x^q}$**,** p **and** q **as in** (3.79) **and** (3.80) **with** $a = 0$**.**

We check that (A.1)-(A.4) hold. Note, $\sigma = 1$ here.

For (A.1) we have by (3.79)

$$\frac{1}{p} \leq \frac{1}{2} - \frac{1}{q},$$

which implies (A.1). For (A.2) we need

$$1 < 2(1 - \frac{1}{q} - \frac{1}{p}),$$

but this is the same as

$$\frac{1}{p} < \frac{1}{2} - \frac{1}{q},$$

which holds by the right hand side of (3.80). Next note $s_1 = s_2$ and that with $\sigma = 1$ (A.4) implies (A.3). So we show (A.4) only, which requires

$$2s_1 + 1 = 2(1 - \frac{1}{q} - \frac{1}{2p}).$$

We are okay as long as $s_1 \leq s$, so after rewriting (A.1) we must have

$$1 - \frac{2}{q} - 2s \leq \frac{1}{p}, \tag{A.6}$$

but that is the left hand side of (3.80).

Application of Klainerman-Tataru Thm in (4.3) **for** $\|D^{a-1}(df\phi)\|_{L_t^p L_x^q}$**,** p **and** q **as in** (3.79) **and** (3.80) **with** $0 < a < 1$**.**

Note $\sigma = 1 - a$ now. (A.1) and (A.2) hold by the right hand side of (3.80). Next (A.4) with $\sigma = 1 - a$ implies (A.3), so we show (A.4) only. We have

$$2s_1 + 1 - a = 2(1 - \frac{1}{q} - \frac{1}{2p}),$$

and again we are okay as long as $s_1 \leq s$. Therefore we must have

$$1 - \frac{2}{q} + a - 2s \leq \frac{1}{p}, \tag{A.7}$$

but as before, this is the left hand side of (3.80).

Application of Klainerman-Tataru Thm in (4.4) **for** $\|D^{-1}(df\phi)\|_{L_t^p L_x^q}$, $\|D^{a-1}(df\phi)\|_{L_t^p L_x^q}$, **where** p **and** q **are as in** (3.107) **and** (3.108) **with** $0 < a < 1$**.**

If we observe that (3.80) ensured above that conditions (A.1)-(A.4) were satisfied, we are done since (3.108) implies (3.80).

A.2 Elliptic Estimates: Setting up Klainerman-Selberg Theorem

We also say that several estimates in Theorems 1 and 2 follow from Theorem D [17]. We need to check that it is in fact the case. We begin by stating the theorem.

Theorem. The embedding

$$H^{\frac{n}{2} - \frac{n}{q} - \frac{1}{p}, \theta}(\mathbb{R}^{n+1}) \hookrightarrow L_t^p L_x^q$$

holds whenever

$$2 \leq p \leq \infty, 2 \leq q < \infty, \frac{2}{p} \leq (n-1)(\frac{1}{2} - \frac{1}{q}) \quad \text{and} \quad \theta > \frac{1}{2}.$$

We use the theorem four times. We discuss each case.

Application of Klainerman-Selberg Thm in (4.5) **for** $\|df\|_{L_t^{2p}L_x^{2q}}\|\phi\|_{L_t^{2p}L_x^{2q}}$**,** p **and** q **as in** (3.79) **and** (3.80) **with** $a = 1$**.**

We need

$$H^{1-\frac{2}{2q}-\frac{1}{2p},\theta} \hookrightarrow L_t^{2p}L_x^{2q}.$$

This requires

$$\frac{2}{2p} \leq \frac{1}{2} - \frac{1}{2q},$$

but this follows from the right hand side of (3.80).

Application of Klainerman-Selberg Thm in (4.6) **for** $\|D^{a-1}df\|_{L_t^{p_1}L_x^{q_1}}$ **and** $\|\phi\|_{L_t^{p_2}L_x^{q_2}}$ **with** $1 < a \leq 1 + s$**.**

We show we can in fact find $p_i, q_i, i = 1, 2$ where

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \tag{A.8}$$

with p and q as in (3.79) and (3.80) and so that Theorem D applies. We consider two cases: $s > 1$ and $s \leq 1$.

Case $s > 1$**.**

With p and q as in (3.79) and (3.80) let

$$(p_1, q_1) = (0, 2) \quad (p_2, q_2) = (p, \frac{2q}{2-q}). \tag{A.9}$$

Then we have

$$\begin{aligned}
\|D^{a-1}df\|_{L_t^{p_1}L_x^{q_1}}\|\phi\|_{L_t^{p_2}L_x^{q_2}} =& \|D^{a-1}df\|_{L_t^\infty L_x^2}\|\phi\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} \\
&\leq \|df\|_{L_t^\infty H_x^s}\|\phi\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} \\
&\lesssim \|df\|_{H^{s,\theta}}\|\phi\|_{L_t^p L_x^{(1/q-1/2)^{-1}}}
\end{aligned} \tag{A.10}$$

So we just need to check the theorem applies to the second term. First, it is clear that $2 \leq p \leq \infty$ and $2 \leq (1/q - 1/2)^{-1} < \infty$. Second, we need $\frac{2}{p} \leq (\frac{1}{2} - (\frac{1}{q} - \frac{1}{2}))$, but this follows from the right hand side of (3.80). Finally we must have

$$1 - 2(1/q - 1/2) - \frac{1}{p} \leq s.$$

Rearrange to obtain

$$2 - 2/q - s \leq \frac{1}{p}.$$

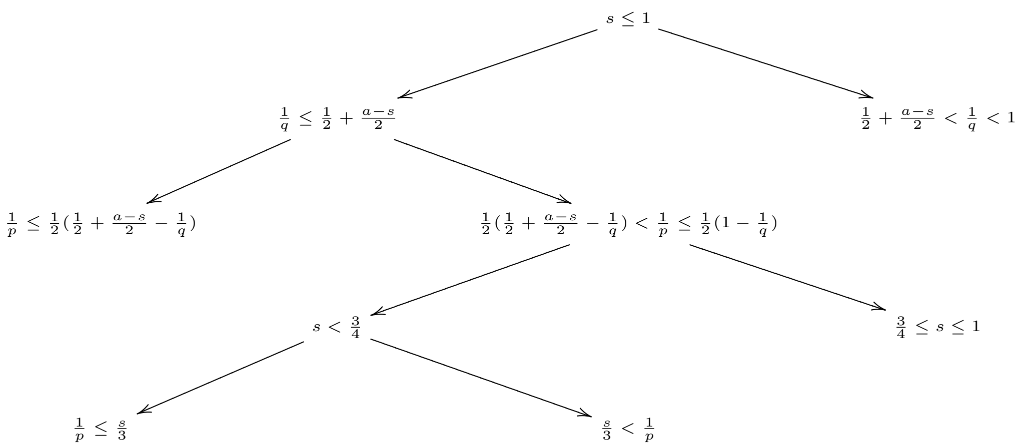
Since for $s > 1$ the left hand side is negative, we win.

Remark A.2.1. Observe that for $s > 1$ we do not need any lower bounds on p, i.e., we could rewrite (3.80) as

$$0 \leq \frac{1}{p} \leq \frac{1}{2}(1 - \frac{1}{q}). \tag{A.11}$$

Case $s \leq 1$.

This case is very convoluted. Within it we distinguish two subcases: $\frac{1}{q} \leq \frac{1}{2} + \frac{a-s}{2}$ and $\frac{1}{2} + \frac{a-s}{2} < \frac{1}{q} < 1$. The latter is straighforward whereas the former involves two subcases, and the second subcase has two subsubcases, where the first subsubcase has two subsubsubcases. This is described in the following diagram.



We begin with the straightforward case and suppose $\frac{1}{2} + \frac{a-s}{2} < \frac{1}{q} < 1$. Write

$$\frac{1}{q} = (\frac{1}{q} - \frac{1}{2}) + \frac{1}{2}$$

Then as for the $s > 1$ case we have

$$\begin{aligned} \|D^{a-1} df \phi\|_{L_t^p L_x^q} &\leq \|D^{a-1} df\|_{L_t^\infty L_x^2} \|\phi\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} \\ &\leq \|D^s df\|_{L_t^\infty L_x^2} \|\phi\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} \\ &\lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{L_t^p L_x^{(1/q-1/2)^{-1}}} \end{aligned} \tag{A.12}$$

And again, we just need to check the theorem applies to the second term. First, it is clear that $2 \leq p \leq \infty$ and $2 \leq (1/q - 1/2)^{-1} < \infty$. Second, we need $\frac{2}{p} \leq (\frac{1}{2} - (\frac{1}{q} - \frac{1}{2}))$, but this follows from the right hand side of (3.80). Finally we must have

$$1 - 2(1/q - 1/2) - \frac{1}{p} \leq s.$$

Rearrange to obtain

$$2 - 2/q - s \leq \frac{1}{p}.$$

We claim the left hand side is negative. This is obvious when we observe that for $a > 1$

$$\frac{2-s}{2} < \frac{1}{2} + \frac{a-s}{2} < \frac{1}{q}.$$

Now suppose $\frac{1}{q} \leq \frac{1}{2} + \frac{a-s}{2}$, and look at the first subcase

$$1 - \frac{2}{q} + a - 2s \leq \frac{1}{p} \leq \frac{1}{2}(\frac{1}{2} + \frac{a-s}{2} - \frac{1}{q}).$$

Let

$$(p_1, p_2) = (0, p) \tag{A.13}$$

and write $\frac{1}{q}$ as

$$\frac{1}{q} = \frac{a-s}{2} + (\frac{1}{q} - \frac{a-s}{2}),$$

Then we have

$$\begin{aligned} \|D^{a-1}df\|_{L_t^{p_1} L_x^{q_1}} \|\phi\|_{L_t^{p_2} L_x^{q_2}} =& \|D^{a-1}df\|_{L_t^{\infty} L_x^{\frac{2}{a-s}}} \|\phi\|_{L_t^p L_x^{(1/q-(a-s)/2)^{-1}}} \\ & \lesssim \|D^{1-a+s+a-1}df\|_{L_t^{\infty} L_x^2} \|\phi\|_{L_t^p L_x^{(1/q-(a-s)/2)^{-1}}} \\ & \lesssim \|df\|_{H^{s,\theta}} \|\phi\|_{L_t^p L_x^{(1/q-(a-s)/2)^{-1}}}, \end{aligned} \tag{A.14}$$

where we use

$$\frac{a-s}{2} = \frac{1}{2} - \frac{1-a+s}{2}.$$

So we just need to check Theorem D applies to the second term. Again, it is clear that $2 \leq p \leq \infty$, and by our choice of q we also have $2 \leq (1/q - (a - s)/2)^{-1} < \infty$. Second, we need

$$\frac{2}{p} \leq (\frac{1}{2} - (\frac{1}{q} - \frac{a-s}{2})) = (\frac{1}{2} + \frac{a-s}{2} - \frac{1}{q}),$$

but this is exactly what we are assuming in this case. Finally we need,

$$H^{s,\theta} \hookrightarrow H^{1-2(\frac{1}{q}-\frac{a-s}{2})-\frac{1}{p},\theta}.$$

This follows from

$$1-2(\frac{1}{q}-\frac{a-s}{2})-\frac{1}{p} \leq s,$$

which is equivalent to left hand side of (3.80).

Now suppose $\frac{1}{2}+\frac{a-s}{2}-\frac{1}{q} < \frac{2}{p} \leq 1-\frac{1}{q}$. We examine the easier situation first: $s \geq 3/4$. Write

$$\frac{1}{p_1}=\frac{1}{p}-\frac{1}{p_2} \quad where \quad \frac{1}{p_2}=\frac{1}{2}(\frac{1}{2}+\frac{a-s}{2}-\frac{1}{q}),$$

and

$$\frac{1}{q_1}=\frac{a-s}{2} \quad \text{and} \quad \frac{1}{q_2}=\frac{1}{q}-\frac{a-s}{2}.$$

It is easy to check $2 \leq p_i \leq \infty, 2 \leq q_i < \infty, i=1,2$. We must check

$$1-\frac{2}{q_1}+a-1-s \leq \frac{1}{p_1} \leq \frac{1}{2}(\frac{1}{2}-\frac{1}{q_1}) \tag{A.15}$$

$$1-\frac{2}{q_2}-s \leq \frac{1}{p_2} \leq \frac{1}{2}(\frac{1}{2}-\frac{1}{q_2}). \tag{A.16}$$

Plug in for $\frac{1}{p_1}$ and for $\frac{1}{q_1}$ in (A.15) to obtain

$$1-a+s+a-1-s \leq \frac{1}{p}-\frac{1}{p_2} \leq \frac{1}{2}(\frac{1}{2}-\frac{a-s}{2})$$
$$\frac{1}{2}(\frac{1}{2}+\frac{a-s}{2}-\frac{1}{q}) \leq \frac{1}{p} \leq \frac{1}{2}(\frac{1}{2}-\frac{a-s}{2})+\frac{1}{2}(\frac{1}{2}+\frac{a-s}{2}-\frac{1}{q})$$
$$\frac{1}{2}(\frac{1}{2}+\frac{a-s}{2}-\frac{1}{q}) \leq \frac{1}{p} \leq \frac{1}{2}(1-\frac{1}{q}),$$

which holds by our assumptions on p. Next plug in for $\frac{1}{p_2}$ and for $\frac{1}{q_2}$ in (A.16) to obtain

$$\begin{aligned} 1 - \frac{2}{q} + a - s - s \leq \frac{1}{2}(\frac{1}{2} + \frac{a-s}{2} - \frac{1}{q}) \leq \frac{1}{2}(\frac{1}{2} + \frac{a-s}{2} - \frac{1}{q}) \\ 1 - \frac{2}{q} + a - 2s \leq \frac{1}{2}(\frac{1}{2} + \frac{a-s}{2} - \frac{1}{q}) \\ \frac{1}{2}(1 + a - \frac{7}{3}s) \leq \frac{1}{q} \end{aligned}$$

Now, if $s \geq \frac{3}{4}$ we have the left hand side must be $\leq \frac{1}{2}$. Next let $s < \frac{3}{4}$, and $\frac{1}{2}(\frac{1}{2} + \frac{a-s}{2} - \frac{1}{q}) < \frac{1}{p} \leq \frac{s}{3}$. Set $p_i, q_i, i = 1, 2$ just like above for the case $s \geq \frac{3}{4}$. Everything works out the same till we come to showing (A.16) and we have to establish

$$\frac{1}{2}(1 + a - \frac{7}{3}s) \leq \frac{1}{q}.$$

or equivalently

$$\frac{1}{2} + \frac{a-s}{2} - \frac{2}{3}s \leq \frac{1}{q}. \tag{A.17}$$

In this case, the lower bound for p can be rewritten with $\frac{1}{q}$ on the right hand side giving

$$\frac{1}{2} + \frac{a-s}{2} - \frac{2}{p} < \frac{1}{q},$$

and since $\frac{1}{p} \leq \frac{s}{3}$, (A.17) follows as needed.

Finally we look at the last part: $s < 3/4$ with $\frac{s}{3} \leq \frac{1}{p} \leq \frac{1}{2}(1 - \frac{1}{q})$. Write

$$\frac{1}{q_1} = \frac{1}{q} - \frac{1}{q_2} \quad where \quad \frac{1}{q_2} = \frac{1-s}{2} - \frac{s}{6},$$

and

$$\frac{1}{p_1} = \frac{1}{p} - \frac{1}{p_2} \quad where \quad \frac{1}{p_2} = \frac{s}{3}.$$

Note $0 < \frac{1}{q_2} < \frac{1}{2}$ since $0 < s < \frac{3}{4}$. We also have $0 < \frac{1}{q_1}$ since $\frac{3-4s}{6} < \frac{1}{2} < \frac{1}{q}$. To show $\frac{1}{q_1} \leq \frac{1}{2}$ we need

$$\frac{1}{q} \leq \frac{1}{2} + \frac{3-4s}{6},$$

or equivalently

$$\frac{s}{3} \leq \frac{1}{2}(1 - \frac{1}{q}),$$

but that is exactly the case we are considering

$$\frac{s}{3} \leq \frac{1}{p} \leq \frac{1}{2}(1 - \frac{1}{q}).$$

It is obvious $0 \leq \frac{1}{p_i} \leq \frac{1}{2}, \quad i = 1, 2.$ Now we check

$$a - s - \frac{2}{q_1} \leq \frac{1}{p_1} \leq \frac{1}{2}(\frac{1}{2} - \frac{1}{q_1}),$$

and

$$1 - s - \frac{2}{q_2} \leq \frac{1}{p_2} \leq \frac{1}{2}(\frac{1}{2} - \frac{1}{q_2}).$$

Plug in for p_1 and q_1 to get

$$\begin{aligned} a - s - \frac{2}{q} + \frac{3-4s}{3} \leq \frac{1}{p} - \frac{s}{3} &\leq \frac{1}{2}(\frac{1}{2} - \frac{1}{q} + \frac{3-4s}{6}) \\ a - s - \frac{2}{q} + 1 - \frac{4s}{3} \leq \frac{1}{p} - \frac{s}{3} &\leq \frac{1}{2}(1 - \frac{1}{q} - \frac{2s}{3}) \qquad (A.18) \\ a - 2s - \frac{2}{q} + 1 \leq \frac{1}{p} &\leq \frac{1}{2}(1 - \frac{1}{q}), \end{aligned}$$

but this follows from left hand side of (3.80). Now we plug in for p_2 and q_2 to obtain

$$1 - s - \frac{3-4s}{3} \leq \frac{s}{3} \leq \frac{1}{2}(\frac{1}{2} - \frac{3-4s}{6}), \qquad (A.19)$$

which simplifies to $\frac{s}{3}$ everywhere, and concludes checking all the cases.

Application of Klainerman-Selberg Thm in (4.4) **for** $\|D^{a-1}(df\phi)\|_{L_t^p L_x^q}$**.**

Now we look at how we can use the theorem in (4.4). For $a = 1$, it works similarily just like in Section 3.1. For $a > 1$, we again consider two cases, but this time $s \geq a$ and $s < a$.

$s \geq a$

We set

$$\frac{1}{p_i} = \frac{1}{2p}, \quad \frac{1}{q_i} = \frac{1}{2q}, \quad i = 1, 2.$$

Clearly $0 \leq p_i \leq 2, \quad 0 < q_i < 1, i = 1, 2$. Then we need

$$a - s - \frac{1}{q} \leq \frac{1}{2p} \leq \frac{1}{2}(\frac{1}{2} - \frac{1}{2q}), \tag{A.20}$$

and

$$1 - s - \frac{1}{q} \leq \frac{1}{2p} \leq \frac{1}{2}(\frac{1}{2} - \frac{1}{2q}), . \tag{A.21}$$

(A.21) follows from (A.20), and the left hand side in (A.20) is negative while we can rewrite the right hand side as

$$\frac{1}{p} \leq \frac{1}{2} - \frac{1}{2q},$$

which holds by (3.108).

$s < a$

Here the steps are exactly the same as in section 3.2.2 except that that we do not have to look at the case $\frac{1}{2} + \frac{a-s}{2} < \frac{1}{q}$ since here $\frac{1}{q} < \frac{1}{2}$.

Appendix B

Bilinear Estimates in the "Easy Region."

This appendix is really just a simple exercise, but since it does not take too much space, and since we refer to it several times, we include it here for completeness. We refer to this region as easy, not only because the math we use is very straightforward, but also, because in this region we do not need any special structure. In fact all we need is that we can bound the symbol in question by

$$(|\tau| + |\xi|)(|\lambda| + |\eta|). \tag{B.1}$$

Plug into (4.3) we obtain

$$\begin{aligned} I &\lesssim \iint \frac{(|\tau| + |\xi|)(|\lambda| + |\eta|)F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)}{w_-^{\frac{1}{2}-s}(\tau+\lambda,\xi+\eta)w_+(\tau,\xi)w_-^{s+\frac{1}{2}}(\tau,\xi)(1+|\eta|)^s w_+(\lambda,\eta)w_-^{s+\frac{1}{2}}(\lambda,\eta)} \\ &\leq \iint \frac{F(\tau,\xi)G(\lambda,\eta)H(\tau+\lambda,\xi+\eta)}{w_-^{\frac{1}{2}-s}(\tau+\lambda,\xi+\eta)w_-^{s+\frac{1}{2}}(\tau,\xi)(1+|\eta|)^s w_-^{s+\frac{1}{2}}(\lambda,\eta)} \end{aligned} \tag{B.2}$$

Next by Cauchy Schwarz, and dropping $w_-^{s-\frac{1}{2}}(\tau+\lambda,\xi+\eta)$ (since it is ≤ 1), we have,

$$I \lesssim \|F\|\|G\|\Big\{\iint \frac{H^2(\tau+\lambda,\xi+\eta)}{w_-^{2(s+\frac{1}{2})}(\tau,\xi)(1+|\eta|)^{2s}w_-^{2(s+\frac{1}{2})}(\lambda,\eta)}d\tau d\xi d\lambda d\eta\Big\}^{\frac{1}{2}} \tag{B.3}$$

Proof for the region: $|\tau| \geq 2|\xi|$. In this region $|\tau| - |\xi| \geq |\xi|$, so we can substitute $(1+|\xi|)^{2(s+\frac{1}{2})}$ for the weight $w_-(\tau,\xi)^{2(s+\frac{1}{2})}$. Consequently,

$$I \leq \|F\|\|G\|\Big\{\iint \frac{H^2(\tau+\lambda,\xi+\eta)}{(1+|\xi|)^{2(s+\frac{1}{2})}(1+|\eta|)^{2s}w_-^{2(s+\frac{1}{2})}(\lambda,\eta)}d\tau d\xi d\lambda d\eta\Big\}^{\frac{1}{2}} \quad \text{(B.4)}$$

Next we perform several changes of variables. First we translate $\tau+\lambda$ to τ, $\xi+\eta$ to ξ, and then finally we let

$$v = |\lambda| - |\eta|, \quad \text{(B.5)}$$

which we substitute for λ. Since $H \in L^2$, we are done if for fixed ξ we can bound:

$$\int\int \frac{1}{(1+|\xi-\eta|)^{2(s+\frac{1}{2})}(1+|\eta|)^{2s}(1+|v|)^{2(s+\frac{1}{2})}}dv d\eta \quad \text{(B.6)}$$

Now, as long as $s > 0$, we have that $2(s+\frac{1}{2}) > 1$, so it is obvious that

$$\int \frac{dv}{(1+|v|)^{2(s+\frac{1}{2})}}$$

is bounded. Therefore we are left with

$$J = \int \frac{1}{(1+|\xi-\eta|)^{2(s+\frac{1}{2})}(1+|\eta|)^{2s}}d\eta. \quad \text{(B.7)}$$

For completeness we show the details for this integral. We break $\mathbb{R}^2$ into two regions:

$$E_1 = \{\eta : |\xi-\eta| > |\eta|\} \quad \text{and} \quad E_2 = \{\eta : |\xi-\eta| \leq |\eta|\}. \quad \text{(B.8)}$$

Then

$$J = J_1 + J_2,$$

where

$$J_i = \int_{E_i} \frac{d\eta}{(1+|\xi-\eta|)^{2(s+\frac{1}{2})}(1+|\eta|)^{2s}}. \tag{B.9}$$

Now

$$\begin{aligned} J_1 &= \int_{E_1} \frac{d\eta}{(1+|\xi-\eta|)^{2(s+\frac{1}{2})}(1+|\eta|)^{2s}} \\ &\leq \int_{\mathbb{R}^2} \frac{d\eta}{(1+|\eta|)^{2(s+\frac{1}{2})}(1+|\eta|)^{2s}} < \infty, \end{aligned} \tag{B.10}$$

as long as $s > \frac{1}{4}$. Next

$$\begin{aligned} J_2 &= \int_{E_2} \frac{d\eta}{(1+|\xi-\eta|)^{2(s+\frac{1}{2})}(1+|\eta|)^{2s}} \\ &\leq \int_{\mathbb{R}^2} \frac{d\eta}{(1+|\xi-\eta|)^{2(s+\frac{1}{2})+2s}} = C, \end{aligned} \tag{B.11}$$

as long as $s > \frac{1}{4}$, and where C is independent of ξ since we can change variables in the integral. This concludes the proof in the case $|\tau| \geq 2|\xi|$.

Proof for the region: $|\lambda| \geq 2|\eta|$. This case is analogous to the previous one, but again for completeness, we show the details. In this case $|\lambda| - |\eta| \geq |\eta|$, so we can substitute $(1+|\eta|)^{2(s+\frac{1}{2})}$ for the weight $w_-(\lambda,\eta)^{2(s+\frac{1}{2})}$. Consequently,

$$I \leq \|F\|\|G\| \Big\{ \int\int \frac{H^2(\tau+\lambda, \xi+\eta)}{w_-^{2(s+\frac{1}{2})}(\tau,\xi)(1+|\eta|)^{2s}(1+|\eta|)^{2(s+\frac{1}{2})}} d\tau d\xi d\lambda d\eta \Big\}^{\frac{1}{2}} \tag{B.12}$$

Next we perform several changes of variables. First we translate $\tau+\lambda$ to λ, $\xi+\eta$ to ξ, and then we let

$$v = |\tau| - |\xi - \eta|. \tag{B.13}$$

Since $H \in L^2$, we are done if we can bound

$$\int\int \frac{dv d\eta}{(1+|v|)^{2(s+\frac{1}{2})}(1+|\eta|)^{2s}(1+|\eta|)^{2(s+\frac{1}{2})}} \tag{B.14}$$

Now as long as $s > 0$, we have that $2(s + \frac{1}{2}) > 1$, so it is obvious that

$$\int \frac{dv}{(1 + |v|)^{2(s+\frac{1}{2})}}$$

is bounded. Therefore we are left with

$$J = \int \frac{d\eta}{(1 + |\eta|)^{2(2s+\frac{1}{2})}}, \tag{B.15}$$

but this is bounded as long as $s > \frac{1}{4}$. This concludes the proof.

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Vita

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