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Realizability of Tropical Lines in the Fan Tropical Plane

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Realizability of Tropical Lines in the Fan Tropical Plane

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I dedicate this dissertation to everyone who made a difference in my life.

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Realizability of Tropical Lines in the Fan Tropical Plane

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In this thesis we construct an analogue in tropical geometry for a class of Schubert varieties from classical geometry. In particular, we look at the collection of tropical lines contained in the fan tropical plane. We call these tropical spaces "tropical Schubert prevarieties", and develop them after creating a tropical analogue for flag varieties that we call the "flag Dressian". Having constructed this tropical analogue of Schubert varieties we then determine that the 2-skeleton of these tropical Schubert prevarieties is realizable. In fact, as long as the lift of the fan tropical plane is in general position, only the 2-skeleton of the tropical Schubert prevariety is realizable.

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Chapter 1

Introduction

Tropical geometry deals with the study of what are called "tropical varieties." Tropical varieties are piecewise linear spaces that sit inside of an ambient Euclidean space and are of interest to geometers because they are analogues of classical varieties from algebraic geometry. In fact, there is a method of "*tropicalization*" which translates algebraic varieties into tropical ones. While tropical varieties are much simpler than their algebraic counterparts, they still retain some geometric information, which is what makes them useful as a tool for study. For example, a classical problem in enumerative algebraic geometry is counting the number of plane algebraic curves of genus g and degree d passing through $3d + g - 1$ points in general position. This number can be computed through a complicated study of moduli spaces of plane curves via intersection theory on Hilbert schemes [3]. Using tropical geometry, Mikhalkin was able to compute the same number by doing a simpler count of tropical curves [11].

However, there are some difficulties that arise when using tropical varieties to get information about algebraic varieties. One important problem is the fact that a given moduli of tropical spaces is generally larger than the corresponding moduli of classical spaces. For example, consider the moduli space of d-dimensional tropical linear spaces inside an ambient n-dimensional tropical projective space, called, the *Dressian*, and denoted $Dr(d, n)$. The Dressian is generally larger than the classical Grassmannian $Gr(d, n)$ for $d > 2$ and sufficiently large n. For example, the dimension of the Dressian $Dr(3, n)$ is of order $\Theta(n^2)$ (Theorem 3.6 of [15]) whereas the dimension of the Grassmannian $Gr(3, n)$ is $3(n-3)$. This means that there are some tropical spaces that cannot be obtained by tropicalizing a classical space of the same type. Tropical spaces that can be"*lifted*", namely, obtained as a tropicalization of a classical space, are called "*realizable*" tropical spaces. The problem of determining when a tropical spaces lifts is called "the lifting problem", and is naturally an important problem as the answer tells us how to obtain correct counts using tropical spaces.

One means of solving the lifting problem is by making use of deformation theory. Nishinou and Siebert use what they call "*toric degenerations*" (see Section 3 of [12]) where they look at deformations of curves inside of an ambient toric variety. The authors develop a correspondence between certain curves called pre-log curves that can be deformed inside of a family, and tropical curves in an ambient tropical projective space and thereby count the number of algebraic curves on an arbitrary complete toric variety [Theorem 8.3 of [12]]. This technique involves using log deformation theory to compute the sizes of obstruction groups and then discount curves where an obstruction exists.

In this thesis, we will be looking at developing a tropical analogue to a class of Schubert varieties. We proceed by first developing a tropical analogue of flag varieties we call "flag Dressians" (Definition 2.2.6), denoted $FD(d_1, d_2, \ldots, d_s; n)$, which parametrize all s-step flags of d_i-dimensional tropical linear spaces in an ambient n-dimensional tropical projective space. We show that the flag Dressian is a tropical prevariety by finding relations that define it (Theorem 2.3.2). Having determined relations for the flag Dressian, we focus in particular on $FD(1, 2; n)$, where $n > 2$, namely, the flag of tropical lines contained in a tropical plane in an ambient n-dimensional tropical projective space. We consider the tropical subspace of $FD(1, 2; n)$ consisting of pairs (C, V) where $C \subset V$ with V the fan tropical plane and C any tropical line contained in it. The resulting subspace is a tropical analogue of a classical Schubert variety, which we refer to as a *tropical Schubert prevariety.*

This problem is different from other work on lifting problems in the literature in that the ambient variety is not toric. To determine if a point of our tropical Schubert prevariety is realizable we consider a tropical line C inside of the fan tropical plane V with $C \subset V$. Our method of determining if a lift of $C \subset V$ exists makes use of the theory of toric degenerations with some modifications. Namely, we first take a subdivision of $C \subset V$ to form a degenerating family of toric varieties, and in particular obtain tropical degenerations $\bar{\mathcal{V}}$ of V and $\bar{\mathcal{C}}$ of C from the special fiber using Theorems 4.1.5 and 4.1.4. We then attempt to construct a pre-log curve for C from the subdivision. If a pre-log curve cannot be constructed then a lift does not exist, and if we can construct a pre-log curve we then need to determine if it can be deformed to a curve inside $\mathcal{V}$. Using some results of log differentials we compute log canonical bundles of $\bar{\mathcal{C}}$ and $\bar{\mathcal{V}}$, which have an easy description when $\mathcal{V}$ and $\bar{\mathcal{C}}$ have normal crossings boundaries, and then use these to compute the cohomology of the log normal bundle of $\bar{\mathcal{C}}$ in $\bar{\mathcal{V}}$. After computing the cohomology we obtain our main result which states:

Theorem 1.0.1. *The 2-skeleton of the tropical Schubert prevariety of all tropical lines in the fan tropical plane is realizable, and for $\mathcal{V}$ in general position, only the 2-skeleton is realizable*

(Theorems 7.1.2 and 7.2.2). This result implies that, in general, the realizable part of the tropical Schubert prevariety is the same dimension as the corresponding classical Schubert variety of lines in a fixed plane. However, the entire tropical Schubert prevariety is generally much larger.

For other work on tropical lines contained in an ambient tropical space (tropical surfaces in this case) see [18] and [17]. For more results on lifting tropical curves see [1] and [2].

Chapter 2

Background on Tropical Geometry

Tropical geometry is a branch of mathematics that deals with certain types of piecewise linear spaces called "tropical varieties". As the name suggests, "tropical varieties" are related to varieties from algebraic geometry, and sometimes can carry information about algebraic varieties. In particular, there is a method of "tropicalizing" algebraic varieties into tropical varieties. This capability makes tropical geometry useful for algebraic geometers, as it provides a means of gaining information about algebraic varieties, which can be quite complicated, by looking at much simpler piecewise linear spaces. On the other hand, tropical varieties, and general tropical spaces are studied in their own right as objects of interest from a purely combinatorial perspective. As piecewise linear spaces, they have shown to have connections to many areas of combinatorics, such as graph theory and matroid theory.

The goal of this thesis is to make use of both these aspects in questions of moduli of linear spaces. In particular, we will attempt to understand the structure of classical varieties and subvarieties of flags of linear spaces by studying their tropical analogues. In order to do so, let us define tropical varieties and the objects of our study.

2.1 Tropical Spaces

Tropical spaces are piecewise linear spaces. More specifically, they are balanced, weighted, polyhedral complexes in $\mathbb{R}^n$. There are a number of ways of obtaining a tropical space. For example, a **plane tropical curve** is a graph in $\mathbb{R}^2$ where every edge has rational slope, has unbounded edges only in directions $(-1,0), (0,-1)$ and $(1,1)$, and every vertex is balanced (the weighted sum of the primitive integral vectors is zero at every vertex) [5]. Tropical curves arise as limits of amoebas, valuations of points of varieties over fields with valuations such as Puiseux series, or as varieties over the max-plus (or min-plus) semiring [5]. For our purposes we will be dealing with tropical spaces mainly through the means of "tropicalization", which we discuss below.

2.2 Tropicalization

Let $K := \bigcup_{n=1}^{\infty} \mathbb{C}((t^{1/n}))$ be the field of **Puiseux series** in the variable t, namely, K is the field of power series of the form

$$p(t) = \sum_{i=1}^{\infty} c_i t^{q_i}$$

where $c_i \in \mathbb{C}$ are nonzero and $q_i \in \mathbb{Q}$ ($q_i \leq q_j$ for $i < j$) with a common denominator. We can define a $\mathbb{Q}$-valued **valuation** of an element $p(t) \in K$, denoted $val(p)$ to be q_1. Using this valuation we can define an **order** map:

$$\begin{array}{rccc} order: & (K^{\times})^n & \to & \mathbb{Q}^n \\ & (p_1, \ldots, p_n) & \mapsto & (val(p_1), \ldots, val(p_n)) \end{array}$$

With this order map we can now define tropical varieties and prevarieties.

Remark 2.2.1. In all of what follows, our tropical spaces will be subspaces of some **tropical projective space**, denoted $\mathbb{TP}^{n-1} := \mathbb{R}^n/\mathbb{R}(1,1,\ldots,1)$.

Definition 2.2.1. A **tropical variety** is a subset of $\mathbb{TP}^{n-1}$ of the form

$$\mathcal{T}(I) := \overline{order(V(I))}/\mathbb{R}(1,1,\ldots,1)$$

where $I \subset K[X_1,\ldots,X_n]$ is a homogeneous ideal and the bar represents the topological closure of $order(V(I))$ in $\mathbb{R}^n$. The space $\mathcal{T}(I)$ is also called the **tropicalization** of the variety $V(I)$ and is also denoted $Trop(V(I))$. If $I = \langle f \rangle$ is generated by a single polynomial f, then we call $Trop(V(f))$ a **tropical hypersurface**, and a **tropical prevariety** is a finite intersection of tropical hypersurfaces.

Convention: We will use uppercase letters, such as X to represent variables before tropicalizing, and lowercase letters, such as x to represent tropicalized variables.

2.2.1 Tropical Linear Spaces and Realizability

Having defined tropical varieties we can define tropical lines, planes, and tropical linear spaces in general. Since we will be looking at collections of tropical linear spaces, we will use Plücker coordinates and embeddings. Let us define these coordinates and use them to provide us with a new way of defining tropical linear spaces.

Definition 2.2.2. Let $\{p_J; J \subset [n],\ |J| = d\}$ be a collection of real numbers satisfying the **three term tropical Plücker relations**:

$$(p_J) \in Trop(P_{S_{ij}}P_{S_{kl}} - P_{S_{ik}}P_{S_{jl}} + P_{S_{il}}P_{S_{jk}})$$

where $S \in \binom{[n]}{d-2}$ and i, j, k and l are distinct elements of $[n]\backslash S$. These are simply the tropicalizations of each of the classical Plücker relations. We call such a set of numbers p_J a **tropical Plücker vector**, and the numbers themselves **tropical Plücker coordinates**. The $(d-1)$**-dimensional tropical linear space with Plücker coordinates** P_J is the subset of $\mathbb{R}^n/(1, \ldots, 1)$ given by

$$\bigcap_{1 \leq j_1 < \cdots < j_{d+1} \leq n} Trop\left(\sum_{r=1}^{d+1} (-1)^r P_{j_1 \cdots \hat{j_r} \cdots j_{d+1}} X_{j_r}\right). \tag{2.1}$$

A **tropical line** is a tropical linear space of dimension 1, and a **tropical plane** is a tropical linear space of dimension 2.

We can use the tropical Plücker relations to define the tropical Grassmannian.

Definition 2.2.3. Let $K[p_J;\ J \in \binom{[n]}{d}]$ be the ring of Plücker coordinates over K, and $I_{d,n}$ be the ideal generated by the Plücker relations of all $d \times d$ minors of $d \times n$ matrices defining $(d-1)$-linear spaces in projective $(n-1)$-space. These relations include the three term Plücker relations before tropicalization.

The **tropical Grassmannian** of $(d-1)$-dimensional tropical linear spaces in $(n-1)$-dimensional tropical space is defined to be $Trop(V(I_{d,n}))$. Note that the tropical Grassmannian is the same as $Trop(Gr(d,n))$, namely, the tropicalization of the classical Grassmannian $Gr(d,n)$.

Just as the classical Grassmannian $Gr(d,n)$ parametrizes $(d-1)$-dimensional linear spaces in $(n-1)$-dimensional projective space, the tropical Grassmannian also parametrizes $(d-1)$-dimensional tropical linear spaces in $(n-1)$-dimensional tropical projective space. However, $Trop(Gr(d,n))$ does not parametrize all $(d-1)$-dimensional tropical linear spaces in $(n-1)$-dimensional tropical projective space, only those that are *realizable.*

Definition 2.2.4. A tropical space T is **realizable** if T can be obtained as the tropicalization of an algebraic space over K. Namely if, in other words, we can write T as $Trop(V(I))$ for some $I \subset K[x_1,\ldots,x_n]$.

Tropical varieties are all, by definition, realizable. Since tropical linear spaces are given by intersections of finitely many hypersurfaces (equation 2.1) they are tropical prevarieties, but need not be tropical varieties. All tropical varieties are tropical prevarieties, but the reverse is not generally true.

Example 1. *A realizable tropical linear space in* $\mathbb{TP}^{n-1}$ *is a tropical variety where the ideal* I *is generated by linear forms*

$$p_1X_1 + p_2X_2 + \cdots + p_nX_n$$

where $p_1, p_2, \ldots, p_n \in K$.

The following is an example of a tropical line in $\mathbb{TP}^2$:

Example 2. *Consider the ideal* $I := \big\langle (t+t^2)X_1 + (t^{1/2}+t^{-1/2})X_2 + X_3) \big\rangle$. *If we let* $w_i := val(X_i)$, $1 \le i \le 3$, *then* $Trop(V(I))$ *consists of points* $(w_1, w_2, w_3) \in \mathbb{TP}^2$ *satisfying*

$$1 + w_1 = -1/2 + w_2 \le w_3,$$

$$1 + w_1 = w_3 \le -1/2 + w_2,$$

$$-1/2 + w_2 = w_3 \le 1 + w_1.$$

Hence the graph of the tropical line consists of three rays in the plane with common vertex $(-1, 1/2, 0)$. *We can graph this in* $\mathbb{R}^2$ *by dehomogenizing and taking* $w_3 = 0$. *We then get the three rays in the directions* $(0, 1)$, $(1, 0)$, *and* $(-1, -1)$ *and vertex* $(-1, 1/2)$

As the tropical Grassmannian only parametrizes those tropical linear spaces that are realizable, we would like another moduli space to parametrize all tropical linear spaces, both realizable and non-realizable. Such a space exists and is called the Dressian.

Definition 2.2.5. The **Dressian**, denoted $Dr(d, n)$, is the tropical prevariety defined by all three term Plücker relations. The Dressian parametrizes all $(d-1)$-dimensional tropical linear spaces contained in $\mathbb{TP}^{n-1}$.

All tropical lines are in fact realizable; in other words $Trop(Gr(2, n)) \cong Dr(2, n)$. However, $Dr(d, n)$ is larger than $Trop(Gr(d, n))$ for values of $d > 2$.

Having defined tropical linear spaces, we can then consider flags of tropical linear spaces. As the name *Dressian* has been suggested for the tropical prevariety parametrizing all tropical linear spaces of a given dimension (See [15]), we propose the name *flag Dressian* to differentiate it from the tropicalization of flag varieties.

Definition 2.2.6. Let the **flag Dressian**, denoted, $FD(d_1, d_2, \ldots, d_s; n)$ be the moduli space of flags of tropical linear spaces in $\mathbb{R}^n/(1, \ldots, 1)$; namely, the set of s-tuples $(X_1, \ldots, X_s))$ such that each X_i is a $(d_i - 1)$-dimensional tropical linear space in $\mathbb{R}^n/(1, \ldots, 1))$ and $X_i \subset X_{i+1}$ for every i. We will show below that the flag Dressian is in fact a tropical prevariety. In the case where $s = 2$ we will also call the flag Dressian a **tropical incidence prevariety**.

Note, however, that $FD(d_1, d_2, \ldots, d_s; n)$ is not the tropicalization of the classical flag variety, $Trop(Fl(d_1, d_2, \ldots, d_s; n))$. In particular, an element $(L_1, \ldots, L_s) \in FD(d_1, d_2, \ldots, d_s; n)$ is a flag of tropical linear spaces where none of the L_i have to be realizable, but for $(K_1, \ldots, K_s) \in Trop(Fl(d_1, d_2, \ldots, d_s; n))$ all of the K_i must be realizable. One may ask if it is possible there is $(X_1, X_2) \in FD(d_1, d_2; n)$ where each X_i is realizable, but for any A, B with $Trop(A) = X_1$ and $Trop(B) = X_2$ we have $(A, B) \notin Fl(d_1, d_2; n)$, namely $A \not\subset B$ (so containment may not be realizable).

2.3 Relations for Tropical Incidence Prevarieties

Having defined the flag Dressian, in order to determine its structure, we need a condition for when one tropical linear space is contained in another. In this case, we will develop relations on the Plücker coordinates that determine containment. One of our main tools for determining these relations will be duality.

2.3.1 Duality

Lemma 2.3.1. *Let $\{x_I\}$ be tropical Plücker coordinates for a tropical linear space X and define $x_J^{\perp} := x_{[n]\setminus J}$. Then $\{x_J^{\perp}\}$ are also tropical Plücker coordinates for a tropical linear space we denote by $X^{\perp}$. We call $X^{\perp}$ the tropical* ***dual linear space****, or* ***orthogonal complement*** *to X. Moreover, $X \mapsto X^{\perp}$ is an inclusion reversing bijection on tropical linear spaces.*

Proof. See the remarks preceding Proposition 2.12 in [13]. □

2.3.2 Tropical Incidence Relations

We will now make use of (2.1) and duality to derive relations for a tropical incidence variety. We will show that in fact, the relations that we derive for tropical incidence prevarieties are the same as the relations for classical incidence varieties where the first subscript is exchanged.

Theorem 2.3.2. *Given tropical linear spaces X and Y ($\dim X \leq \dim Y$), we have $X \subset Y$ if and only if the tropical Plücker coordinates of X and Y*

satisfy the incidence relations:

$$Trop(\sum_{i \in T \setminus S} X_{S \cup \{i\}} Y_{T \setminus \{i\}}),$$

where $S, T \subset [n]$ *with* $|S| = p - 1$ *and* $|T| = q + 1$.

Proof. Let $q = \dim\ Y$, then from (2.1) we know that Y is the intersction over all sequences J of the tropical hyperplane H_J given by

$$Trop\left(\sum_{j \in J} Y_{J \setminus j} W_j\right),$$

(where the W_j represent the variables). Note that we can ignore the signs in front of the terms since we are tropicalizing. We have $X \subset Y$ if and only if $X \subset H_J\ \forall J$ if and only if $H_J^{\perp} \subset X^{\perp}\ \forall J$.

Let $p = \dim\ X$, then we also have $X^{\perp}$ as the intersection of tropical hyperplanes $H_{I'}$ of the form

$$Trop\left(\sum_{r=1}^{n-p+1} X^{\perp}_{i'_1 \cdots \hat{i'_r} \cdots i'_{n-p+1}} W_{i'_r}\right).$$

Let $H_J^{\perp} = Trop(V)$. Using Lemma 2.3.1 we can write the coordinates of V as

$$V_j = \begin{cases} Y_{J \setminus j} & \text{for } j \in J \\ 0 & \text{for } j \notin J. \end{cases}$$

Now $H_J^{\perp} \subset X^{\perp}\ \forall J$ if and only if $\forall J, I\ H_J^{\perp} \subset H_{I'}$, so plugging in the points $H_J^{\perp}$ into the equations for $H_{I'}$ gives

$$Trop\left(\sum_{r=1}^{n-p+1} X^{\perp}_{i'_1 \cdots \hat{i'_r} \cdots i'_{n-p+1}} Y_{J_{i'_r}}\right) \tag{2.2}$$

which, by Lemma 2.3.1, is equivalent to

$$Trop\left(\sum_{r=1}^{n-p+1} X_{[n]\backslash\{i'_1\cdots\hat{i'_r}\cdots i'_{n-p+1}\}} Y_{J_{i'_r}}\right). \tag{2.3}$$

We write the above equation more conveniently as

$$Trop\left(\sum_{r=1}^{n-p+1} X_{\{[n]\backslash I'\}\cup\{i'_r\}} Y_{J_{i'_r}}\right). \tag{2.4}$$

Taken over all possible sequences, tropicalizing and then intersecting the tropicalizations gives us a tropical analog to the classical incidence variety.

Note that we can simplify equation (2.4) by using $S := [n]\backslash I'$ and $T := J$ and writing the above equation as

$$Trop(\sum_{i\in T\backslash S} X_{S\cup\{i\}} Y_{T\backslash\{i\}}). \tag{2.5}$$

These equations are the tropicalization of the classical incidence relations:

$$\sum_{i\in T\backslash S} X_{S\cup\{i\}} Y_{T\backslash\{i\}} \tag{2.6}$$

□

Note that classically the flag varieties are only set-theoretically defined by these relations. The ideal, however, is defined by exchanging k subscripts where $1 \leq k \leq n$.

Example 3. *Consider $FD(1,2;4)$. This is the flag of all arrangements of tropical lines in $\mathbb{TP}^3$ containing a fixed point. Let us use (2.6) to find the polynomials for $FD(1,2;4)$:*

Using $S = \emptyset$ and $T = \{1, 2, 3\}$ we have

$$X_1Y_{2,3} + X_2Y_{1,3} + X_3Y_{1,2}. \tag{2.7}$$

Similarly, using $S = \emptyset$ and $T = \{1, 2, 4\}$ we get

$$X_1Y_{2,4} + X_2Y_{1,4} + X_4Y_{1,2}, \tag{2.8}$$

using $S = \emptyset$ and $T = \{1, 3, 4\}$:

$$X_1Y_{3,4} + X_3Y_{1,4} + X_4Y_{1,3}, \tag{2.9}$$

and using $S = \emptyset$ and $T = \{2, 3, 4\}$:

$$X_2Y_{3,4} + X_3Y_{2,4} + X_4Y_{2,3}. \tag{2.10}$$

Note that if we specify in particular that the point X that the lines Y pass through be $X = [1, 1, 1, 1]$ then equations (2.7), (2.8), (2.9), and (2.10) become

$$Y_{2,3} + Y_{1,3} + Y_{1,2},$$

$$Y_{2,4} + Y_{1,4} + Y_{1,2},$$

$$Y_{3,4} + Y_{1,4} + Y_{1,3},$$

$$Y_{3,4} + Y_{2,4} + Y_{2,3},$$

respectively.

Chapter 3

A Taxonomy of Tropical Lines in the Fan Tropical Plane

3.1 Categorizing Tropical Lines Inside the Fan Tropical Plane

We want to classify all tropical lines contained in the fan tropical plane. We will do this by categorizing tropical lines according to the number of vertices each line has and then describing the conditions such a line must satisfy.

Our goal is to classify all tropical lines contained in a fan tropical plane. Let us start by discussing the objects in question:

Definition 3.1.1. The **fan tropical plane in** $\mathbb{R}^n$, which we will denote by V, is the tropicalization of a classical plane with constant coefficients. Geometrically, this is the 2-fan in $\mathbb{R}^n$ given by the union of all 2-cones generated by coordinate axes and the vector $e_0 := -\sum_{i=1}^{n} e_i$. In other words

$$V := \bigcup_{0 \leq i < j \leq n} PosSpan(e_i, e_j).$$

where $PosSpan(e_i, e_j) := \{ae_i + be_j;\ a, b \in \mathbb{R}_{\geq 0}\}$. In particular, V is a fan.

Remark 3.1.1. The fan tropical plane is a tropical linear space as given by Definition 2.2.2 where the tropical Plücker coordinates are all 0.

While tropical lines are simply 1-dimensional tropical linear spaces, the following characterization of tropical lines as given by [14] will be useful:

Remark 3.1.2. A *tropical line*, denoted C, is a connected tree whose vertices are at least trivalent that has exactly $n+1$ unbounded edges (rays) in the directions $e_0, e_1, \ldots, e_n$ where $e_1, \ldots, e_n$ are the directions of the axes and e_0 is as in Definition 3.1.1. In addition, every vertex must satisfy the balancing condition: namely, at a vertex x of C, if d_i are the primitive vectors outward from x along the edges at x, then we have $\sum d_i = 0$.

Our aim, in other words, is to determine all C such that $C \subset V$.

One way we can do this is by determining what possible configurations a vertex of C can have depending on where it is in relation to V. That is, we would like be able to categorize the lines according to whether or not the vertex lies in the interior of a 2-cone or on the 1-skeleton, etc.

Let us discuss some constructions involving tropical lines. Let C be a tropical line with bounded edges. Let F be a bounded edge of C with vertices v and v'. If we were to remove the edge F then C would split into two **subtrees** we will label C_v and $C_{v'}$, where $v \in C_v$ and $v' \in C_{v'}$. Note that since C is a tropical line its unbounded edges are in directions $e_0, e_1, \ldots, e_n$. This means that the directions of the unbounded edges of C_v are disjoint from $C_{v'}$. In addition, one can obtain a tropical line C_F by removing F. We define it as follows:

Definition 3.1.2. Given a tropical line C with bounded edges, let F be one such edge, then we define the tropical line C_F as

$$C_F := (C_v + (v' - v)) \cup C_{v'}$$

where we identify v and v' in the union. Call this new tropical line, the **contraction of C by F.**

Proof. (That the contraction of a tropical line is well-defined): To show that C_F is in fact a tropical line, we need to show that its unbounded edges consist of $e_0, \ldots, e_n$ and that at every vertex the balancing condition is satisfied. Let E_v be the set of directions of the unbounded edges of the subtree C_v and similarly for $E_{v'}$. The unbounded edges of C_F are $E_v \cup E_{v'}$, but these are just the unbounded edges of C, so this condition is satisfied. Now we need only show that the balancing condition is satisfied. Since C_v and $C_{v'}$ are both subtrees of C, every vertex not including v and v' is balanced. To show that $v = v' \in C_F$ is balanced we need to show that the primitive vectors along all edges out of that vertex sum to zero. In the tropical line C, let d_{F_v} be the primitive direction vector of F outward at v and similarly for $d_{F_{v'}}$. At v in L we have from the balancing condition that

$$d_{F_v} + \sum_{G \neq F,\, G \ni v} d_{G_v} = 0.$$

Similarly we have

$$d_{F_{v'}} + \sum_{G \neq F,\, G \ni v'} d_{G_{v'}} = 0.$$

This means that in C_F the sum of the primitive direction vectors outward at $v = v'$ are

$$\sum_{G \neq F,\, G \ni v} d_{G_v} + \sum_{G \neq F,\, G \ni v'} d_{G_{v'}} = -(d_{F_v} + d_{F_{v'}})$$

but $d_{F_{v'}} = -d_{F_v}$ since they are primitive vectors on the same edge in opposite directions, hence $d_{F_v} + d_{F_{v'}} = 0$ and the edges of C_F at the vertex $v = v'$ are balanced. □

Lemma 3.1.1. *(Edge Direction Lemma): Every edge of a tropical line C is in a direction of the form*

$$\pm \sum_{i \in S \subset \{0,\ldots,n\}} e_i.$$

Moreover, if the edge is bounded, then $|S| \geq 2$. More precisely, let the edge, call it F, have vertices v and v'. Let E_v be the set of directions of the unbounded edges of the subtree L_v and similarly for $E_{v'}$, then we have

$$d_{F_v} = -\sum_{e_i \in E_v} e_i.$$

or equivalently

$$d_{F_{v'}} = -\sum_{e_i \in E_{v'}} e_i.$$

So for F, we have that S is either $\{i;\ e_i \in E_v\}$ or $\{i;\ e_i \in E_{v'}\}$ depending on which direction we take.

Proof. This is, by definition, true of the unbounded edges, so we need only show this for the bounded ones. We will show this by induction on the number of bounded edges of C. Let C have only one bounded edge F, and let v be a vertex of that edge. Let d_{F_v} be as in the statement of the lemma. Since C is at least trivalent, there must be at least two other edges at v and they must be unbounded. Since these other edges are unbounded they must be among the e_i. Let E_v be the set of these unbounded edges, then the balancing condition gives

$$\left(\sum_{e_i \in E_v} e_i\right) + d_{F_v} = 0$$

hence

$$d_{F_v} = -\sum_{e_i \in E_v} e_i.$$

Since C is at least trivalent we have $|E_v| > 1$, so this proves the base case. Let us proceed by induction: Let C now be a tropical line with $k+1$ bounded edges. Let F', with $F' \neq F$, be one of these edges, and label its vertices w and w'. The tropical line $C_{F'}$ then has k bounded edges, so by induction, the direction of every bounded edge of $C_{F'}$ is given by $\pm \sum_{i \in S \subset \{0,\ldots,n\}} e_i$ where $|S| \geq 2$. The conclusion holds regardless of the choice of F'. In addition, since the directions of E_v in $C_{F'}$ and C are the same, the lemma is proven. □

Notice, also, that since every vertex is (at minimum) trivalent, the edges of the vertex must span at least a plane. This is true because if E_a, E_b and E_c are the edges of a vertex, then since C is a tropical line all of the edges must be balanced, hence $E_a + E_b + E_c = 0$, so the edges span a plane generated by any two of the edges. This gives us the following:

Lemma 3.1.2. *If x is a vertex of C and is not in the 1-skeleton of V then x is trivalent with two rays in directions e_i and e_j and the third ray in $-(e_i+e_j)$,* *unless we are in the degenerate case where $V \cong \mathbb{R}^2$.*

Proof. If x is not on in the 1-skeleton of V, then x must be in the relative interior of a 2-cone, say $PosSpan(e_i, e_j)$ for $i \neq j$. Note that every edge of C is of the form $\pm \sum_{i \in S \subset \{0,\ldots,n\}} e_i$ by the Edge Direction Lemma, so the possible edges of x are $\pm e_i, \pm e_j, \pm(e_i + e_j)$. Since the edges are balanced at x, the directions of the edges must be either $\{e_i, e_j, -(e_i + e_j))\}$ or $\{-e_i, -e_j, e_i + e_j\}$. If the directions from x are $\{-e_i, -e_j, e_i + e_j\}$, and the edge in direction $e_i + e_j$ lies entirely in the 2-cone, then it must be an unbounded edge, which is impossible, unless $C \subset \mathbb{R}^2$ and $e_2 = -e_0 - e_1$. □

Basically, this result tells us that if a 2-cone has a vertex in its relative interior, it must be the only vertex in that 2-cone. The following lemma puts restrictions on where we can have more than one vertex:

Lemma 3.1.3. *A tropical line C cannot have more than one vertex on the 1-skeleton of V. Hence, if a vertex of C lies on the 1-skeleton of V, all remaining vertices must lie in the interiors of 2-cones.*

Proof. We first show that two vertices on the 1-skeleton cannot be connected by a bounded edge. Let x be a vertex on the 1-skeleton, say on the axis in the e_i direction. Let y be a vertex on the 1-skeleton of V, say in the e_j direction, then the edge between x and y has direction $be_j - ae_i$ for some $a, b > 0$. However, by the Edge Direction Lemma, every edge in C is of the form $\pm \sum_{i \in S \subset \{0,\ldots,n\}} e_i$, which implies that x and y cannot share an edge. Therefore, we have that any neighbor of x must lie in the interior of some 2-cone.

Hence, in order for C to have vertices on the 1-skeleton of V, they must be connected by a path whose internal vertices are in the interior of a 2-cone. A vertex in the interior of a 2-cone has only one bounded edge by Lemma 3.1.2, which means such a vertex cannot be on an internal edge of a path (as that would imply it has at least two bounded edges to two adjacent vertices), so we are done. $\square$

Lemma 3.1.4. *If a vertex x on the 1-skeleton of V has a neighbor y, that neighbor must be unique, unless x is at the origin.*

Proof. Assume that x has another neighbor z in the interior of a 2-cone, then this 2-cone must be distinct from the one y is in. Say x lies on the e_i axis, $y \in PosSpan(e_i, e_j)$, then we have $z \in PosSpan(e_i, e_k)$ with $j \neq k$. However, $y \in PosSpan(e_i, e_j)$ means that the unbounded ray in the e_i direction has its vertex at y, and $z \in PosSpan(e_i, e_k)$ means that one of the edges at z is the unbounded edge in the e_i direction. This is a contradiction since we then have two rays in the e_i direction. $\square$

The above leads us to the following:

Lemma 3.1.5. *If C is a tropical line, then C meets the 1-skeleton of V in either a vertex, or along a bounded edge. If C has no vertex in the 1-skeleton, then V must be in $\mathbb{R}^2$ or $\mathbb{R}^3$. If $V \subset \mathbb{R}^3$ and C has no vertex on the 1-skeleton, then C has one bounded edge, which goes through the origin.*

If C has only one vertex, that vertex may also lie anywhere on the 1-skeleton.

Proof. Suppose C has more than one vertex, then the bounded edge between two vertices, say x and y either meets the 1-skeleton of V or passes through the origin: The only way a bounded edge cannot meet the 1-skeleton of V is if it lies entirely in the interior of a 2-cone, which is impossible by Lemma 3.1.2 as two distinct vertices would then have common unbounded edges. Assume that we can let the bounded edge pass through the origin with y the other vertex. Let $x \in PosSpan(e_i, e_j)$ and $y \in PosSpan(e_k, e_l)$, so that at x the bounded edge is in the $-e_i - e_j$ direction, and at y in the $-e_k - e_l$ direction. This means that $-e_i - e_j = e_k + e_l$. This can happen if and only if $C \subset \mathbb{R}^3$, because in that case we have exactly four unbounded edges. So, for example, we can have $x \in PosSpan(e_1, e_2)$ and $y \in PosSpan(e_0, e_1)$ and $-e_0 - e_1 = e_1 + e_2$. This is the only case where C meets V along a bounded edge. Such a relation does not exist for $C \subset \mathbb{R}^n$ if $n \neq 3$, and x and y would then have to be vertices for different bounded edges. The bounded edge for x would have to meet a vertex v on the 1-skeleton by Lemma 3.1.4 and the edge for y would also have to meet v by Lemma 3.1.3, as v would then be have to be uniquely on the 1-skeleton.

Assume C has only one vertex, say x. If x lies in the interior of a 2-cone, then that vertex is trivalent, by Lemma 3.1.2. Two of the edges are unbounded, and the remaining edge can only be unbounded if $V \subset \mathbb{R}^2$. If not, then the third edge must be a bounded edge, in which case C no longer has one vertex. Say x now lies on the 1-skeleton, on the axis e_i, then rays of x can go off into $PosSpan(e_i, e_j)$ for any $i \neq j$. Since x is the only vertex, all the edges are the unbounded rays, and therefore x can sit on any axis as the ray in direction e_j lies in the cone $PosSpan(e_i, e_j)$, and we can do this for any ray. Hence, if x is the only vertex, then x can sit anywhere on the 1-skeleton. □

Remark 3.1.3. In general, if C has more than two vertices, the bounded edges are forced to meet at the one skeleton, for any vertex that lies in the interior of a 2-cone has a bounded edge that meets the 1-skeleton. With the exception of $C \subset \mathbb{R}^3$, there must be a vertex where the bounded edge meets the 1-skeleton. Lemma 3.1.3 tells us that this vertex can be the only vertex of C on the 1-skeleton of V.

3.2 The Taxonomy and a Tropical Schubert Prevariety

We now use the above results to obtain the following:

Theorem 3.2.1. *(Taxonomy of tropical lines in the fan plane): All tropical lines C in the fan tropical plane V ($V \subset \mathbb{R}^n$, $n > 2$) are of three types:*
(Type 1): C has only one vertex and this vertex lies anywhere on the 1-skeleton of V.
(Type 2): C has only two vertices, one lies in the relative interior of a 2-cone of V, while the other vertex lies on the boundary of the cone.
(Type 2A): $C \subset \mathbb{R}^3$ and the origin lies in the relative interior of the bounded edge.

(Type 3): C has more than two vertices, with one at the origin, and the remaining vertices lie in the relative interiors of 2-cones, with bounded edges meeting at the origin.

Proof. The cases of Type 1 and 2A are proven by Lemma 3.1.5.

If C is of Type 2, then Lemmas 3.1.3 and 3.1.5 tell us that exactly one vertex of C, call it x, is on the 1-skeleton of V, and the remaining vertex, call it y, is then in the interior of a 2-cone. x must then lie on the boundary of the 2-cone containing y.

To show the remaining ones are of Type 3, assume C has more than two vertices. As before, we know that exactly one vertex of C is on the 1-skeleton of V, call it x. Now pick two other vertices, say y and z of C. Since y and z are in interiors of 2-cones by Lemma 3.1.2 they are on bounded edges that meet the 1-skeleton, and so must meet at x. This means that x is a vertex of all the bounded edges. Moreover, x has more than one neighbor, and by Lemma 3.1.4, x must lie at the origin. This proves the theorem. □

Having classified all tropical lines contained in a fan tropical plane we want to use these results to determine the structure of a "tropical Schubert variety." We define these objects as follows:

Definition 3.2.1. A **tropical Schubert prevariety** is a tropical subprevariety of $FD(d_1, \ldots, d_s; n)$ that parametrizes flags of tropical linear spaces where at least one tropical linear space remains fixed. We will denote by

$\sigma \subset FD(2,3;n)$ the Schubert variety of all tropical lines contained in the fan tropical plane.

Remark 3.2.1. We know that tropical Schubert prevarieties are in fact tropical subprevarieties as they are obtained by tropicalizing Plücker relations where we fix some of the coordinates.

Let us see how the different types of tropical lines as described in Theorem 3.2.1 are parametrized by σ:

We know that Type 1 tropical lines have only one vertex that can be placed anywhere on the 1-skeleton of V, and therefore expect a correspondence between the 1-skeleton of V and a subset of σ. In particular, if we define B_i to be the set of points in σ corresponding to Type 1 tropical lines with the vertex on the e_i ray, then we have a one-to-one correspondence between points of the ray e_i and B_i. Each B_i is then a ray in σ corresponding to the rays of the 1-skeleton of V.

Type 2 tropical lines have two vertices, one on the 1-skeleton of V, and the other inside a 2-cone. Call the vertex on the 1-skeleton x, and the one in the 2-cone y. The bounded edge between x and y is in a 2-cone, say, $Cone(e_i, e_j)$, with x on the e_i ray. We know that for a fixed x the possible choices for y are on a ray in the e_i+e_j direction. Define $B_{i,j} \subset \sigma$ to be the subset parametrizing such tropical lines of Type 2, then points of the ray in the $e_i + e_j$ direction correspond to points of $B_{i,j}$. Note that if we take $x = y$ then our Type 2 tropical line degenerates into a Type 1 tropical line, and as we vary x along the e_i ray we are looking at points of B_i. If we take $x \neq y$, and we vary both x and y, then we are looking at points on the 2-cone $Cone(B_i, B_{i,j})$ generated by the rays B_i and $B_{i,j}$.

Given pairwise disjoint subsets $I_1, \ldots, I_k \subset \{0, 1, \ldots, n\}$, where each $|I_j| = 2$. Type 3 tropical lines are determined by bounded edges sitting in 2-cones that pairwise meet only at the origin. If we consider a bounded edge in the 2-cone $Cone(e_{j_1}, e_{j_2})$, then it must have a vertex at the origin, and another vertex on the ray in the $e_{j_1}+e_{j_2}$ direction from the origin. Any tropical line with only this bounded edge would correspond to a point of $Cone(B_{I_j})$, where $B_{I_j} := B_{j_1,j_2}$. If we let the cones containing the bounded edges of a Type 3 tropical line be indexed by $I_1, \ldots, I_k \subset \{0, 1, \ldots, n\}$, then these tropical lines correspond to points in the k-cone $Cone(B_{I_1}, \ldots, B_{I_k})$. Hence, all Type 3 tropical lines sit inside of cones of σ of dimension 3 or higher.

Chapter 4

Realizability of a Tropical Schubert Prevariety

Our taxonomy of tropical lines in the standard tropical plane lets us study the Schubert subvariety of $FD(2,3;n)$ of tropical lines in a fixed tropical plane. In particular, we want to determine what part of the Schubert variety is realizable. To approach this problem we look at embeddings of $f : C \hookrightarrow V$ of tropical lines in the standard tropical plane. Finding conditions for when f lifts will let us determine what part of the Schubert variety can be realized. To do this, we use the correspondence between pre-log curves and tropical curves. This correspondence was developed using techniques of toric degenerations and deformation theory, which we review here following Nishinou and Siebert [12].

4.1 Toric degenerations

Toric degenerations are degenerations of toric varieties, which correspond to polyhedral fans. The degenerations are given by toric morphisms of toric varieties into $\mathbb{A}^1$, where the general fiber (those over nonzero points of $\mathbb{A}^1$) is isomorphic to the toric variety in question. Let us start with the polyhedral construction related to a toric variety. We will be following Section

3 of [12] in what follows:

For K the field of Puiseux series let $(K^*)^n$ be an algebraic torus, with $n \geq 2$. Let $N := Hom(K^*, (K^*)^n)$ be the one-parameter subgroup lattice and $M := Hom(N, \mathbb{Z})$ the character lattice. Then $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^n$, and similarly for $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$

Definition 4.1.1. We call a **polyhedron** the solution set in $N_{\mathbb{Q}} \cong \mathbb{Q}^n$ of finitely many linear inequalities $\langle m, \cdot \rangle \geq$ constant, $m \in M_{\mathbb{Q}}$. The sets where some of the defining inequalities are equalities are called **faces**, and the **vertices** are zero-dimensional faces. A polyhedron is **strongly convex** if it has at least one vertex.

Definition 4.1.2. A **polyhedral decomposition** of $N_{\mathbb{Q}}$ is a covering $\mathcal{P} = \{P\}$ of $N_{\mathbb{Q}}$ by a finite number of strongly convex polyhedra satisfying the following properties:

(i) If $P \in \mathcal{P}$ and $P' \subset P$ is a face, then $P' \in \mathcal{P}$

(ii) If $P, P' \in \mathcal{P}$, then $P \cap P'$ is a common face of P and P'.

For any polyhedral decomposition $\mathcal{P}$ we can construct a fan $\Sigma_{\mathcal{P}}$ from its unbounded elements by rescaling $\mathcal{P}$ by $a \in \mathbb{Q}_{>0}$ and taking a going to 0. Any bounded $P \in \mathcal{P}$ goes to $0 \in N_{\mathbb{Q}}$. We define

Definition 4.1.3. The **asymptotic fan**, or **recession fan**, of $\mathcal{P}$ is

$$\Sigma_{\mathcal{P}} := \left\{ \lim_{a \to 0} aP \subset N_{\mathbb{Q}};\ P \in \mathcal{P} \right\}.$$

$\Sigma_{\mathcal{P}}$ is in fact a complete fan.

A polyhedral decomposition defines a degenerating family of toric varieties. To see this extend the fan $\Sigma_{\mathcal{P}}$ to the fan $\tilde{\Sigma}_{\mathcal{P}} \subset N_{\mathbb{Q}} \times \mathbb{Q}$ as follows: For each $P \in \mathcal{P}$ let $C(P)$ be the closure of the cone spanned by $P \times \{1\} \subset N_{\mathbb{Q}} \times \mathbb{Q}$, namely

$$C(P) := \overline{\{a \cdot (n, 1);\ a \geq 0,\ n \in P\}}.$$

Since P is given by inequalities $\langle m_i, \cdot \rangle \geq c_i$ we have

$$C(P) = \{(n, b) \in N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0};\ \langle m_i, n\cdot \rangle - b \cdot c_i \geq 0\}.$$

Hence any $C(P)$ is a convex polyhedral cone and

$$\tilde{\Sigma}_{\mathcal{P}} := \{\sigma \subset C(P) \text{ face };\ P \in \mathcal{P}\}$$

is a fan covering $N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0}$. Note that by identifying $N_{\mathbb{Q}}$ with $N_{\mathbb{Q}} \times \{0\} \subset N_{\mathbb{Q}} \times \mathbb{Q}$ we have

$$\Sigma_{\mathcal{P}} = \{\sigma \cap N_{\mathbb{Q}} \times \{0\};\ \sigma \in \tilde{\Sigma}_{\mathcal{P}}\}.$$

The map of polyhedral complexes $N_{\mathbb{Q}} \times \mathbb{Q} \to \mathbb{Q}$ where $N_{\mathbb{Q}} \times \mathbb{Q}$ is mapped to the second factor, and we get an induced map on toric varieties

$$\pi : X(\tilde{\Sigma}_{\mathcal{P}}) \to \mathbb{A}^1.$$

Let us look at the fibers of this map. We have

Lemma 4.1.1. *For any closed point $t \in \mathbb{A}^1 \backslash \{0\}$, the fiber $\pi^{-1}(t) \subset X(\tilde{\Sigma}_{\mathcal{P}})$, with the torus action of $\mathbb{T}(N) \subset \mathbb{T}(N \times \mathbb{Z})$, is torically isomorphic to $X(\Sigma_{\mathcal{P}})$.*

In fact, because $\mathbb{T}(\mathbb{Z})$ acts transitively on the closed points of $\mathbb{A}^1\backslash\{0\}$ the fibers of π over those points must be isomorphic.

The structure of the central fiber is more complex. In the case where $\mathcal{P}$ is **integral**, namely, the vertices of $\mathcal{P}$ are in N, then it is possible to describe the central fiber easily. Integrality is in fact a necessary and sufficient condition for $\pi^{-1}(0)$ to be reduced. Given $P \in \mathcal{P}$ we define the fan

$$\Sigma_P := \{\mathbb{Q}_{\geq 0} \cdot (P' - P) \subset N_{\mathbb{Q}}/L(P);\ P' \in \mathcal{P},\ P \subset P'\}$$

where $L(P)$ is the linear subspace of $N_{\mathbb{Q}}$ spanned by differences $v - w$ where $v, w \in P$. For simplicity we write $X_P := X(\Sigma_P)$. In particular, $X_v = X(\Sigma_v)$ is a toric divisor in $X(\tilde{\Sigma}_{\mathcal{P}})$. We have (Proposition 3.5 of [12])

Proposition 4.1.2. *If $\mathcal{P}$ is integral, then there is a system of closed embeddings $X_P \to \pi^{-1}(0)$, for $P \in \mathcal{P}$, compatible with the directed system inducing an isomorphism $\pi^{-1}(0) \cong \varinjlim_{P\in\mathcal{P}} X_P$. In fact $\pi^{-1}(0) = \bigcup_{v' \in Vert(\mathcal{P})} X_{v'}$.*

From these constructions we have

Definition 4.1.4. Given a polyhedral complex $\mathcal{P}$ the associated **toric degeneration** is the family $X(\tilde{\Sigma}_{\mathcal{P}}) \to$ *Spec* $\mathbb{C}[[t]]$ of toric varieties.

Remark 4.1.1. Note that previously we had described the family $\pi : X(\tilde{\Sigma}_{\mathcal{P}}) \to \mathbb{A}^1$, but for our purposes we will need to work over *Spec* $\mathbb{C}[[t]]$ instead of $\mathbb{A}^1$, so we define toric schemes to be the base change of the family $X(\tilde{\Sigma}_{\mathcal{P}}) \to \mathbb{A}^1$ to *Spec* $\mathbb{C}[[t]]$.

4.1.1 Tropical Degenerations

Here we take the torus T to be over *Spec* $\mathbb{C}[[t]]$, and $\mathcal{P}$ to be a polyhedral complex in $M \otimes \mathbb{R}$, where M is the $\mathbb{Z}$-dual of the character group of T. We will take Y to be a subvariety of the general fiber over T, and examine the closure, $\bar{Y}$ of Y in $X(\tilde{\Sigma}_{\mathcal{P}})$, the toric variety of $\tilde{\Sigma}_{\mathcal{P}}$.

Definition 4.1.5. We call the map

$$\bar{Y} \times T \to X(\tilde{\Sigma}_{\mathcal{P}}),\ (y,t) \mapsto ty$$

the **structure map**.

Definition 4.1.6. If the structure map on Y is faithfully flat and $X(\tilde{\Sigma}_{\mathcal{P}}) \to$ *Spec* $\mathbb{C}[[t]]$ is proper, then we say that $\bar{Y}$ is a **tropical compactification**. The special fiber of $X(\tilde{\Sigma}_{\mathcal{P}}) \to$ *Spec* $\mathbb{C}[[t]]$ is the **tropical degeneration** of Y.

The motivation for the adjective *tropical* for the above compactification and degeneration is the following:

Proposition 4.1.3. *If $\bar{Y}$ is a tropical compactification, then the support of the fan $|\tilde{\Sigma}_{\mathcal{P}}| = Trop(Y)$. (See section 2 of [16] and Theorem 1.5, Proposition 6.5 of [10]).*

We will also need the following definitions:

Definition 4.1.7. The compactification $\bar{Y}$ has **combinatorial normal crossings** if for any collection of irreducible divisors $B_1, \ldots, B_r$ the codimension of $\cup_i B_i$ is r.

Definition 4.1.8. We say that Y is **schün** if $Y \subset T$ has a tropical compactification whose structure map is smooth.

For example, linear spaces in tori are schün [16]. We have the following result for schün varieties (See [16] and [10]):

Theorem 4.1.4. *(Tevelev-Luxton-Qu): If Y is schün, then $\bar{Y} \subset X(\tilde{\Sigma}_{\mathcal{P}})$ is regularly embedded, normal, and has toroidal singularities. If $\tilde{\Sigma}_{\mathcal{P}}$ is strictly simplicial, then $\bar{Y} \backslash Y$ is a divisor with simple normal crossings.*

Theorem 4.1.5. *Let Y be schün, then every subdivsion of $Trop(Y)$ gives rise to a tropical degeneration of Y.*

4.2 Pre-log Curves

Having defined toric and tropical degenerations, we relate them to tropical curves via what are called pre-log curves. In particular, we will look at tropical curves as polyhedral complexes that live in some $\mathcal{P}$, and relate them to stable maps of curves into the corresponding central fiber X_0. We begin with the following:

Definition 4.2.1. Let X be a toric variety over a scheme S, and $\mathcal{C} \subset X$ an algebraic curve. A stable map $\varphi : \mathcal{C} \to X$ is **torically transverse** if for every closed point $s \in S$ we have for $\varphi_s : \mathcal{C}_s \to X$

(1) $\varphi_s^{-1}(int(X)) \subset C$ is dense, and

(2) $\varphi_s(\mathcal{C}_s)$ is disjoint from strata of codimension greater than 1.

Now we are in a position to define pre-log curves:

Definition 4.2.2. Let $X_0 = \cup_{v \in \mathcal{P}} X_v$ be the central fiber of the toric degeneration $X \to \mathbb{A}^1$ defined by an integral polyhedral decomposition $\mathcal{P}$ of $N_{\mathbb{Q}}$. A **pre-log curve** $\mathcal{C}$ on X_0 is a stable map $\varphi : C \to X_0$ with the following properties:

(i) For all vertices v of $\mathcal{P}$ the projection $\mathcal{C} \times_{X_0} X_v \to X_v$ is a torically transverse stable map.

(ii) Let $P \in \mathcal{C}$ map to the singular locus of X_0, then $\mathcal{C}$ has a node at P, and φ maps the two branches $(\mathcal{C}', P)$ and $(\mathcal{C}'', P)$ of $\mathcal{C}$ at P to two different irreducible components $X_{v'}, X_{v''} \subset X_0$. Moreover, if w' is the intersection index with the toric boundary $D' \subset X_{v'}$ of the restriction $(\mathcal{C}', P) \to (X_{v'}, D')$, and w'' that of $(\mathcal{C}'', P) \to (X_{v''}, D'')$, then $w' = w''$.

4.3 Applications to a Tropical Schubert Prevariety

Our goal is to understand the realizability of σ, the tropical Schubert prevariety of tropical lines contained in the fan tropical plane. A point of σ corresponds to a tropical line C so that $C \subset V$, and V is the fan tropical plane. Let $\mathcal{V}$ be a plane with $Trop(\mathcal{V}) = V$. Suppose that C lies in the tropicalization of the classical Schubert variety of lines contained in $\mathcal{V}$, then there must exist a curve $\mathcal{C} \subset \mathcal{V}$ with $Trop(\mathcal{C}) = C$.

Take a strictly simplicial subdivision of V so that it contains C in the 1-skeleton and call it $\mathcal{P}$. This subdivision gives rise to tropical degenerations

$\bar{\mathcal{V}}$ of $\mathcal{V}$ and $\bar{\mathcal{C}}$ of $\mathcal{C}$ via Theorem 4.1.5, which have strictly normal crossings boundary and meet the boundary strata of $X(\tilde{\Sigma}_{\mathcal{P}})$ transversely via Theorem 4.1.4. In particular, $\bar{\mathcal{C}}$ is a union of irreducible components, one for each vertex in C, and two components of $\bar{\mathcal{C}}$ meet if the two corresponding vertices are joined by an edge, in which case they intersect at a single point on the corresponding stratum of $\bar{\mathcal{V}}$. Since C is given by linear equations we have that $\bar{\mathcal{C}}$ is rational, and hence a rational pre-log curve.

Our goal is determine realizability by reversing this process of determining $\bar{\mathcal{C}}$. Namely, we attempt to construct a pre-log curve $\bar{\mathcal{C}}$ from a subdivision of V by attaching a $\mathbb{P}^1$ to each vertex of C. If this is not possible, then by Theorem 4.1.5 that point of σ cannot be realizable. If it is possible, see if a lift can be constructed by deforming $\bar{\mathcal{C}}$ to a curve in the general fiber. In order to do this, we will need to be able to compute sizes of obstruction groups, which we can do using log deformation theory.

4.4 Log Structures and Obstructions to Lifting

For our purposes, log structures are a tool to help us compute our obstruction by adding some extra structure to the spaces we are considering and computing cohomology over that structure. Essentially, log structures allow one to pretend that varieties with mild singularities are smooth and use the appropriate machinery. In particular, we will develop a notion of log differentials, which will lead naturally to log canonical and normal bundles, over which we can compute cohomology. The following information on log schemes is from [8]

We start with

Definition 4.4.1. A **pre-log structure** on a scheme X is a pair $(\mathcal{M}, \alpha)$ where $\mathcal{M}$ is a sheaf of monoids and α is a homomorphism $\alpha : \mathcal{M} \to \mathcal{O}_X$. If α induces an isomorphism $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ then we say $(\mathcal{M}, \alpha)$ is a **log structure**. A **log scheme** $(X, \mathcal{M}, \alpha)$ is a scheme X with a log structure $(\mathcal{M}, \alpha)$. For simplicity, we will sometimes denote $(X, \mathcal{M}, \alpha)$ by $\underline{X}$.

Remark 4.4.1. Note that a pre-log structure $(\mathcal{M}, \alpha)$ canonically induces a log structure $(\mathcal{M}^a, \alpha^a)$, called the **associated** log structure, by adjoining units to $\mathcal{M}$.

Definition 4.4.2. For a given log scheme $(X, \mathcal{M})$ a **chart** of $\mathcal{M}$ is a homomorphism $P \to \mathcal{M}$ from the constant sheaf of a monoid P which induces an isomorphism from the associated log structure P^a to $\mathcal{M}$.

Definition 4.4.3. A log structure $\mathcal{M} \to \mathcal{O}_X$ on a scheme X is said to be **fine** if $\mathcal{M}$ has ütale locally a chart $P \to \mathcal{M}$ with P a finitely generated integral monoid.

We also have a notion of a morphism of log schemes:

Definition 4.4.4. A **morphism of log schemes** $f : (X, \mathcal{M}, \alpha) \to (Y, \mathcal{N}, \beta)$ is a pair (f, ϕ) where f is a morphism of schemes $f : X \to Y$ and ϕ is a homomorphism of sheaves of monoids on X, $\phi : f^{-1}\mathcal{N} \to \mathcal{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} f^{-1}\mathcal{N} & \xrightarrow{\phi} & \mathcal{M} \\ \downarrow & & \downarrow \\ f^{-1}\mathcal{O}_Y & \longrightarrow & \mathcal{O}_X \end{array}$$

Also, like schemes we can glue log schemes similarly. We also have a notion of smoothness for morphisms of log schemes:

Definition 4.4.5. Let $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ be a morphism of log schemes. We say that f is **log smooth** if the following conditions hold:

1) f is locally of finite presentation

2) For any commutative diagram of log schemes

$$\begin{array}{ccc} (T', \mathcal{L}') & \xrightarrow{s'} & (X, \mathcal{M}) \\ {\scriptstyle t}\downarrow & & {\scriptstyle f}\downarrow \\ (T, \mathcal{L}) & \xrightarrow{s} & (Y, \mathcal{N}) \end{array}$$

where t is a thickening of order 1 there exists ütale locally a morphism $g : (T, \mathcal{L}) \to (X, \mathcal{M})$ such that $s' = g \circ t$ and $s = f \circ g$.

4.4.1 Examples of Log Structures

Example 4. *There is always the trivial log structure on X given by taking $\mathcal{M} = \mathcal{O}^*$ and $\alpha = Id_{\mathcal{O}_X}$.*

Example 5. *Given any commutative ring A and a monoid P, there is a natural pre-log structure on Spec $A[P]$ given by $\alpha : P \to A[P]$. The induced log scheme (Spec $A[P], P^a, \alpha^a$) is called the canonical monoid log structure.*

Example 6. *Given a rational fan* Δ *in* $\mathbb{R}^n$*, there is a natural log structure on the toric variety* $X(\Delta)$*: For each cone* $\sigma \in \Delta$ *we have a toric affine open set* $U_\sigma = Spec\ k[M \cap \sigma^\vee]$ *with a monoid log structure. We can glue these log structures on each affine open to get a log structure on* $X(\Delta)$*.*

Example 7. *We can apply the previous example to the toric variety* $X(\tilde{\Sigma}_{\mathcal{P}})$*, where* $\mathcal{P}$ *is a rational polyhedral complex, showing that* $X(\tilde{\Sigma}_{\mathcal{P}})$ *also has a log structure.*

The following example is the one that we will make the most use of and lends itself naturally to a log structure:

Example 8. *Let* X *be a regular scheme with a fixed reduced divisor* D *with normal crossings. Define a log structure* $\mathcal{M}$ *as*

$$\mathcal{M} = \{g \in \mathcal{O}_X;\ g \text{ is invertible outside } D\} \subset \mathcal{O}_X.$$

Note that in the case where X *is a toric variety, this log structure is equivalent to the log structure given in Example 6, where* D *is given by the rays of the fan that defines* X*.*

We will apply the above example to the following:

Example 9. *We have the map* $\mathcal{V} \to \mathbb{A}^1 = Spec\ \mathbb{C}[t]$ *arising from the toric degeneration for the pre-log curve* $\bar{\mathcal{C}}$*. The special fiber over the map is a divisor* D *with normal crossings, so give* $\mathcal{V}$ *the log structure induced by* D *as in Example 8. We can give* $Spec\ \mathbb{C}[t]$ *the log structure induced by the divisor*

$\{0\}$. *The map $\mathcal{V} \to \mathbb{A}^1 = Spec\ \mathbb{C}[t]$ can now be viewed as a morphism of log schemes that is in fact log smooth.*

4.4.2 Log Differentials

Having defined log structures we can use them to construct differentials, which are essentially the usual differentials along with those of the form $dlog(\alpha(a)) = d(\alpha(a))/\alpha(a)$. We have

Definition 4.4.6. Given a morphism of log schemes $f : (X, \mathcal{M}, \alpha) \to (Y, \mathcal{N}, \beta)$ the **sheaf of log differentials** of $\underline{X}$ over $\underline{Y}$, denoted $\Omega_{\underline{X}/\underline{Y}}$ is the quotient of

$$\Omega_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}^{gp})$$

by the $\mathcal{O}_X$-submodule generated locally by local sections of the following forms:

$$(d\alpha(a), 0) - (0, \alpha(a) \otimes a) \text{ where } a \in \mathcal{M}$$

and

$$(0, 1 \otimes a) \text{ with } a \in Im(f^{-1}(\mathcal{N}) \to \mathcal{M}).$$

Let us examine these log differentials in the case of toric varieties:

Example 10. *Let X be a toric variety over a field k with log structure as in Example 6. Then $\underline{X}$ is a toric scheme over Spec k. We then have an isomorphism of $\mathcal{O}_X$-modules*

$$\Omega_{\underline{X}/k} \cong \mathcal{O}_X \otimes_{\mathbb{Z}} M.$$

We can see this from the fact that on an affine local piece $U_\sigma = Spec\ k[M \cap \sigma^\vee]$ *we have the map on ordinary differentials*

$$\Omega_{k[M\cap\sigma^\vee]/k} \to M \otimes_{\mathbb{Z}} k[M \cap \sigma^\vee]$$

given by

$$d\chi^m \mapsto m \otimes \chi^m.$$

On log differentials, this translates to the isomorphism $\Omega_{\underline{X}/k} \cong \mathcal{O}_X \otimes_{\mathbb{Z}} M$ *given by*

$$\frac{d\chi^m}{\chi^m} = m \otimes 1.$$

4.4.3 Log Canonical Sheaf

Just as in the case of ordinary differentials where the **canonical sheaf** $\omega_X := \bigwedge^{\dim\ X} \Omega_{X/k}$ we define the **log canonical sheaf** to be $\omega_{\underline{X}} := \bigwedge^{\dim\ X} \Omega_{\underline{X}/k}$. The following result relates the log canonical sheaf to the canonical sheaf for the case we are interested in and will be very useful for later computations:

Proposition 4.4.1. *Let* X *be a smooth variety with* D *a divisor with simple normal crossings, then if* $\underline{X}$ *has the log structure induced by* D *we have*

$$\omega_{\underline{X}/\mathbb{C}} \cong \omega_{X/\mathbb{C}} \otimes \mathcal{O}(D).$$

Proof. In order to prove this we will need to make use of the following exact sequence (Equation (8.1.6) of [4]): For X and D as in the statement of the proposition we have

$$0 \to \Omega_{X/\mathbb{C}} \to \Omega_{\underline{X}/\mathbb{C}} \to \mathcal{O}_D \to 0.$$

The above sequence is an exact sequence of locally free sheaves so taking wedge products we have

$$\bigwedge^n \Omega_{\underline{X}/\mathbb{C}} \cong \bigwedge^n \Omega_{X/\mathbb{C}} \otimes \mathcal{O}_D.$$

This translates to

$$\omega_{\underline{X}/\mathbb{C}} \cong \omega_{X/\mathbb{C}} \otimes \mathcal{O}(D)$$

on canonical sheaves, which is what we want. □

We will also need the following result (Proposition 5.3 of [8]):

Proposition 4.4.2. *Given morphisms of fine log schemes* $\underline{X} \xrightarrow{f} \underline{Y} \xrightarrow{g} \underline{Z}$, *if* f *is log smooth then we have the following exact sequence*

$$0 \to f^*\Omega_{\underline{Y}/\underline{Z}} \to \Omega_{\underline{X}/\underline{Z}} \to \Omega_{\underline{X}/\underline{Y}} \to 0.$$

We can apply this proposition to Example 9 by letting f be the map $\mathcal{V} \to \mathbb{A}^1 = Spec\ \mathbb{C}[t]$ and adding the map $Spec\ \mathbb{C}[t] \to Spec\ \mathbb{C}$. Giving $Spec\ \mathbb{C}$ the log structure induced by the divisor $\{0\}$ we have

$$0 \to f^*\Omega_{\underline{\mathbb{C}[t]}/\underline{\mathbb{C}}} \to \Omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}}} \to \Omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}[t]}} \to 0 \qquad (4.1)$$

We can use equation 4.1 to obtain the following:

Lemma 4.4.3. *Given $\underline{\mathcal{V}}$ a toric degeneration with log structure given as in Example 9, we have*

$$\omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}}} \cong \omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}[t]}}.$$

Proof. We know that $f^*\Omega_{\underline{\mathbb{C}[t]}/\underline{\mathbb{C}}} \cong \mathcal{O}(\underline{\mathcal{V}})$. Taking top wedge powers of the differential sheaves in equation 4.1 gives

$$\bigwedge^n \to \Omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}}} \cong \bigwedge^n \Omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}[t]}} \otimes_{\mathcal{O}(\underline{\mathcal{V}})} \mathcal{O}(\underline{\mathcal{V}})$$

which becomes

$$\omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}}} \cong \omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}[t]}}.$$

□

We can combine the above results on $\underline{\mathcal{V}}$ to obtain

Proposition 4.4.4. *Let $\underline{\mathcal{V}}$ be the toric degeneration with log structure as given in Example 9. If $\mathcal{V}_i$ is an irreducible component of D then we have that*

$$\omega_{\underline{\mathcal{V}}/\mathbb{C}[t]}|_{\mathcal{V}_i} \cong \left(\omega_{\mathcal{V}_i/\mathbb{C}}\right) \otimes \left(\bigotimes_{i \neq j} \mathcal{O}(\mathcal{V}_i \cap \mathcal{V}_j)\right)$$

Proof. Combining Lemma 4.4.3 with Proposition 4.4.1 we have

$$\omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}[t]}}|_{\mathcal{V}_i} \cong \omega_{\underline{\mathcal{V}}/\underline{\mathbb{C}}}|_{\mathcal{V}_i} \cong \left(\omega_{\mathcal{V}/\mathbb{C}} \otimes \mathcal{O}(D)\right)|_{\mathcal{V}_i}.$$

We know that D is a principal divisor (since it is the special fiber), which means that $\mathcal{O}(D)$ is trivial. Hence

$$\left(\omega_{\mathcal{V}/\mathbb{C}} \otimes \mathcal{O}(D)\right)|_{\mathcal{V}_i} \cong \omega_{\mathcal{V}/\mathbb{C}}|_{\mathcal{V}_i}.$$

Now apply Proposition 8.20 of Chapter II of [6] in the case where $X = \mathcal{V}$, $Y = \mathcal{V}_i$ a divisor of X with associated invertible sheaf $\mathcal{L} = \mathcal{O}(\mathcal{V}_i)$. Proposition 8.20 implies

$$\omega_Y \cong \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_Y,$$

which here translates to

$$\omega_{\mathcal{V}_i} \cong \omega_{\mathcal{V}} \otimes \mathcal{O}(\mathcal{V}_i) \otimes \mathcal{O}_{\mathcal{V}_i} \cong (\omega_{\mathcal{V}}) \otimes \mathcal{O}(\mathcal{V}_i).$$

Hence

$$\omega_{\mathcal{V}/\mathbb{C}} \cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes \mathcal{O}(-\mathcal{V}_i)$$

so that

$$\omega_{\mathcal{V}/\mathbb{C}}|_{\mathcal{V}_i} \cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes \mathcal{O}(-\mathcal{V}_i)|_{\mathcal{V}_i}.$$

Since D is principal we have $D = \mathcal{V}_i + \sum_{j\neq i} \mathcal{V}_j \sim 0$ which implies $-\mathcal{V}i \sim \sum_{j\neq i} \mathcal{V}_j$. We can conclude then that

$$\begin{aligned}
\omega_{\underline{\mathcal{V}}/\mathbb{C}[t]}|_{\mathcal{V}_i} &\cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes \mathcal{O}(\sum_{j\neq i} \mathcal{V}_j)|_{\mathcal{V}_i} \\
&\cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes \mathcal{O}(\sum_{j\neq i} \mathcal{V}_i \cap \mathcal{V}_j) \\
&\cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes \left(\bigotimes_{i\neq j} \mathcal{O}(\mathcal{V}_i \cap \mathcal{V}_j) \right).
\end{aligned}$$

□

Because we will be taking subdivisions that will correspond to blowing-up toric varieties we need the following result for our computations (Exercise 8.5 in Chapter II of [6]):

Lemma 4.4.5. *Given a nonsingular variety X and Y a nonsingular subvariety of codimension $r \geq 2$, let $\pi : \tilde{X} \to X$ be the blow-up of X along Y, and let $Y' = \pi^{-1}(Y)$, then*

$$\omega_{\tilde{X}} \cong \pi^* \omega_X \otimes \mathcal{L}((r-1)Y')$$

4.5 Obstructions to Lifting

Recall that we want to determine the size of the obstruction to having $\bar{\mathcal{C}} \subset \bar{\mathcal{V}}$. We can determine this by computing the cohomology of the log normal bundle of $\bar{\mathcal{C}}$ in $\bar{\mathcal{V}}$. This follows from the following lemma, which is analogous to Lemma 7.2 of [12]:

Lemma 4.5.1. *Let $\bar{\mathcal{V}}$ be log smooth over Spec $\mathbb{C}[[t]]$ and let $\bar{\mathcal{C}}_1$ be a pre-log curve over Spec $\mathbb{C}$. Let*

$$f_1 : \bar{\mathcal{C}}_1 \to \bar{\mathcal{V}} \cong \bar{\mathcal{V}} \times_{\mathbb{C}[[t]]} \mathbb{C}$$

be a strict embedding of log schemes. Suppose that we have a lift $\bar{\mathcal{C}_k}$ of $\bar{\mathcal{C}_1}$ to Spec $\mathbb{C}[[t]]/t^k$ and $f_k : \bar{\mathcal{C}_k} \to \bar{\mathcal{V}_k}$ lifting f_1, then the obstruction to lifting $(\bar{\mathcal{C}_k}, f_k)$ to $(\bar{\mathcal{C}_{k+1}}, f_{k+1})$ lies in $H^1(\bar{\mathcal{C}_1}, \mathcal{N}_{f_1})$. If this class vanishes, then the set of lifts $(\bar{\mathcal{C}_{k+1}}, f_{k+1})$ is a torsor under $H^0(\bar{\mathcal{C}_1}, \mathcal{N}_{f_1})$

Proof. We omit the proof as it is analogous to the proof of Lemma 7.2 of [12]. □

To compute this cohomology, we will need the log canonical bundles of both $\bar{\mathcal{C}}$ and $\bar{\mathcal{V}}$, which can be described more easily when $\bar{\mathcal{C}}$ and $\bar{\mathcal{V}}$ have normal crossings boundaries. We ensure this is the case as follows: Recall that we subdivide V so that it contains C in the 1-skeleton. In particular, we want this subdivision to give us a special fiber that is a divisor with simple normal crossings. We can do this via the following proposition (Proposition 2.3 in [7]):

Proposition 4.5.2. *Given a complete rational polyhedral complex $\mathcal{P}$ in $\mathbb{R}^n$, there exists an integer d and a subdivision $\mathcal{P}'$ of $d\mathcal{P}$ such that the general fiber of $X(\tilde{\Sigma}_{\mathcal{P}'})$ is a smooth toric variety and the special fiber of $X(\tilde{\Sigma}_{\mathcal{P}'})$ is a divisor with simple normal crossings. In addition, if the asymptotic fan $\Sigma_{\mathcal{P}}$ is simplicial and unimodular, then we can choose $\mathcal{P}'$ so that $\Sigma_{\mathcal{P}'} = \Sigma_{\mathcal{P}}$.*

Taking a sufficiently fine subdivision, let us compute the cohomology of the log normal bundle of $\underline{\bar{\mathcal{C}}}$ in $\underline{\bar{\mathcal{V}}}$. We begin with the following exact sequence

$$0 \to \mathcal{T}_{\underline{\bar{\mathcal{C}}}} \to f^*\mathcal{T}_{\underline{\bar{\mathcal{V}}}} \to \mathcal{N}_f \to 0,$$

where $\mathcal{T}_{\underline{\bar{\mathcal{C}}}}$ and $\mathcal{T}_{\underline{\bar{\mathcal{V}}}}$ are the tangent bundles of $\underline{\bar{\mathcal{C}}}$ and $\underline{\bar{\mathcal{V}}}$ respectively, and $\mathcal{N}_f$ is the log normal bundle of $\underline{\bar{\mathcal{C}}}$ in $\underline{\bar{\mathcal{V}}}$. Taking the dual of the sequence gives

$$0 \to \mathcal{N}_f^\vee \to f^*\Omega_{\underline{\bar{\mathcal{V}}}} \to \Omega_{\underline{\bar{\mathcal{C}}}} \to 0.$$

Since this is an exact sequence of locally free sheaves we can take wedge powers to obtain

$$\bigwedge^2 f^*\Omega_{\underline{\bar{\mathcal{V}}}} \cong \bigwedge^1 \mathcal{N}_f^\vee \otimes \bigwedge^1 \Omega_{\underline{\bar{\mathcal{C}}}}$$

which gives us

$$f^*\omega_{\underline{\bar{\mathcal{V}}}} \cong \mathcal{N}_f^\vee \otimes \omega_{\underline{\bar{\mathcal{C}}}}.$$

Solving for $\mathcal{N}_f^\vee$ gives

$$\mathcal{N}_f \cong (f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee \otimes \omega_{\underline{\bar{\mathcal{C}}}}.$$

Hence, we have reduced computing the degree of the log normal bundle to the degrees of the log canonical bundles of $\underline{\bar{\mathcal{C}}}$ and $\underline{\bar{\mathcal{V}}}$, and we can assume these log canonical bundles have normal crossings boundaries.

Chapter 5

Cohomology of Line Bundles

Our goal in this section is to develop some formulas for the dimensions of cohomology groups of line bundles. For convenience, we develop them here to refer to later on when we compute the dimension of the obstruction group.

5.1 Setup

We start with some basic cohomology facts for line bundles over projective curves. Given a **projective curve** C, and a **line bundle** $\mathcal{L}$ on the curve let us take C to have irreducible components C_i that intersect pairwise in nodal singularities. Let $P_{ij} := C_i \cap C_j$ and let $k_{P_{ij}}$ denote the **skyscraper sheaf at** P_{ij}, then we have the following exact sequence:

$$0 \to \mathcal{L} \to \oplus \mathcal{L}|_{C_i} \to \oplus k_{P_{ij}} \to 0.$$

This sequence gives rise to an exact sequence on cohomology:

$$\begin{array}{cccccccc} 0 & \to & H^0(C,\mathcal{L}) & \to & \oplus H^0(C_i,\mathcal{L}|_{C_i}) & \stackrel{f}{\to} & \oplus_{C_i\cap C_j} k \\ & \stackrel{\delta}{\to} & H^1(C,\mathcal{L}) & \stackrel{g}{\to} & \oplus H^1(C_i,\mathcal{L}|_{C_i}) & \to & 0, \end{array}$$

where we have labeled some of the maps for use later on.

5.2 Basic Cohomology of $\mathbb{P}^1$ Over Twists of the Structure Sheaf

For the cases we are considering, we will have $C_i \cong \mathbb{P}^1$ for all i. Moreover, since $\mathcal{L}$ is a line bundle, then on each C_i we have $\mathcal{L}|_{C_i}$ is a twist of the structure sheaf, so let us recall some basic cohomology facts about $\mathbb{P}^1$ over twists of the structure sheaf.

For H^0 we have:

$$\dim(H^0(\mathbb{P}^1, \mathcal{O}(n))) = \begin{cases} n+1, & n \geq 0 \\ 0, & n < 0. \end{cases}$$

We combine this with Serre duality applied to $\mathbb{P}^1$ to get: (since $\omega_{\mathbb{P}^1} \cong \mathcal{O}(-2)$)

$$H^1(\mathbb{P}^1, \mathcal{O}(n)) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2) \otimes \mathcal{O}(-n))^* \cong H^0(\mathbb{P}^1, \mathcal{O}(-n-2))^*$$

which gives us

$$\dim(H^1(\mathbb{P}^1, \mathcal{O}(n))) = \begin{cases} 0, & n \geq 0 \\ -n-1, & n < 0. \end{cases}$$

5.3 Determining $\dim(H^1(C, \mathcal{L}))$

In our case we will want $\dim(H^1(C, \mathcal{L}))$. We do this by examining the long exact sequence above, in combination with what we know about the cohomology of $\mathbb{P}^1$.

With f, δ and g as above we know that $ker(f) \cong H^0(C, \mathcal{L})$ and that $H^1(C, \mathcal{L})$

surjects onto $\oplus H^1(C_i, \mathcal{L}|_{C_i})$. Hence

$$\dim(H^0(C, \mathcal{L})) = \dim(\oplus H^0(C_i, \mathcal{L}|_{C_i})) - \dim(im(f))$$

and

$$\dim(H^1(C, \mathcal{L})) = \dim(\oplus H^1(C_i, \mathcal{L}|_{C_i})) + \dim(im(\delta)).$$

Note that since we have

$$\dim(H^1(C, \mathcal{L})) = \dim(\oplus H^1(C_i, \mathcal{L}|_{C_i})) + \dim(im(\delta)).$$

to find $\dim(H^1(C, \mathcal{L}))$ we need $\dim(\oplus H^0(C_i, \mathcal{L}|_{C_i}))$ and $\dim(im(\delta))$. For simplicity, we introduce the following:

Definition 5.3.1. Given a component C_i of C we define d_i to be the integer such that $\mathcal{L}|_{C_i} \cong \mathcal{O}(d_i)$ (true since $C_i \cong \mathbb{P}^1$). In other words, d_i is the **degree of the twisting** of the structure sheaf that gives $\mathcal{L}|_{C_i}$.

Definition 5.3.2. Let $r_i := \dim(H^0(C_i, \mathcal{L}|_{C_i}))$.

We know that $\dim(\oplus H^1(C_i, \mathcal{L}|_{C_i})) = \sum_{d_i<0} -d_i - 1$, so we only need $\dim(im(\delta))$. Since $im(f) = ker(\delta)$ and the exact sequence on δ gives $\dim(im(\delta))$ $= \dim(\oplus_{C_i \cap C_j} k) - \dim(ker(\delta)) = \dim(\oplus_{C_i \cap C_j} k) - \dim(im(f))$. We can then reduce the above to

$$\dim(H^1(C, \mathcal{L})) = \left(\sum_{d_i<0} -d_i - 1 \right) + \dim(\oplus_{C_i \cap C_j} k) - \dim(im(f)). \qquad (5.1)$$

Equation (5.1) tells us what we need for a formula for $\dim(H^1(C, \mathcal{L}))$. Besides the d_i that are negative, this is determined by $\dim(im(f))$, so if we have a formula for $\dim(im(f))$ then we have one for $\dim(H^1(C, \mathcal{L}))$. Note that in the cases where f is surjective we have $H^1(C, \mathcal{L}) \cong \oplus H^1(C_i, \mathcal{L}|_{C_i})$.

Note that, having reduced the cohomology, we now look at different arrangements of components of C. The following construction will prove useful:

Construction: Start with a curve composed of two disjoint components, C_A and C_B, each of which can be reduced to a collection of $\mathbb{P}^1$s, and let $\mathcal{L}$ be the line bundle on the whole curve. We can combine the components C_A and C_B into one connected curve and extend $\mathcal{L}$ while preserving $\dim(im(f))$ by adding an additional $\mathbb{P}^1$, called C', that intersects only one component in each of C_A and C_B and does so at only one point of each. The only condition we require is that $r' \geq 2$. We call this new curve, with such a line bundle, the **join** of C_A and C_B, denoted $J(C_A, C_B)$, or simply $\tilde{C}$ if there is no confusion.

Proof. (that $\dim(im(f))$ is preserved): Let the additional component C' intersect components C_a in C_A with $x_a := C_a \cap C'$ and C_b in C_B with $x_b := C_b \cap C'$ and call the new curve $\tilde{C}$ with $\tilde{f}$ in place of f. If f_A and f_B are the restrictions of f on the components C_A and C_B, then the codomain of $\tilde{f}$ had dimension two higher than the sum of the dimensions of the codomains of f_A and f_B, due to x_a and x_b. However, the dimension of the domain also increases by two, since C' provides a contribution of $r' \geq 2$, and $\tilde{f}$ maps this component bijectively onto $\oplus_{x_a, x_b} k$. $\square$

Note that if we allow $r' = 0$ and r_a and r_b are both large enough then $\tilde{f}$ maps surjectively onto $\oplus_{x_a,x_b} k$, in which case C' can contribute to $H^1(C, \mathcal{L})$ (by the amount of $-d' - 1$). If r_a and r_b are not large enough, then $\dim(\oplus_{C_i \cap C_j} k)$ provides a contribution of two, and $\dim(H^1(C, \mathcal{L}))$ changes by $2 + (-d' - 1)$.

The arrangements that will show up in the cases we are looking at are a "chain of $\mathbb{P}^1$s", or a "hub with spokes", or some combination of the above. We describe each of these cases below.

5.3.1 A chain of $\mathbb{P}^1$s

Let C be a chain of $\mathbb{P}^1$s. We can arrange the components so that only C_i and C_{i+1} intersect, with $x_i := C_i \cap C_{i+1}$. Note that if we have n components, then there are then $n - 1$ intersection points, giving us

$$\dim(\oplus_{C_i \cap C_j} k) = n - 1.$$

Let us look at an example of a simple chain, one with only three components, hence two intersection points, and determine conditions necessary for f to be surjective. This example will be instructive in what follows:

Convention: We will use c_i to denote a section of $\mathcal{L}_{C_i}$ over C_i. For simplicity, in what follows we will say "assign constant or function" c_i on C_i we mean we can find a section c_i of $\mathcal{L}_{C_i}$ over C_i.

Example 11. *Let C be chain with components C_1, C_2, C_3 and intersection points x_1, x_2. In order for f to be surjective, we need to have:*

Case (a): Any two $r_i \geq 1$

Case (b): $r_2 \geq 2$.

In case (a), say $r_1 = 0$ and $r_2 = r_3 = 1$, then on C_2 and C_3 we can assign constants c_2 and c_3 respectively. In this case the images of f on x_1 and x_2 are c_2 and $c_3 - c_2$ respectively, and f is surjective. If $r_1 = r_3 = 1$ and $r_2 = 0$ then similarly f takes $-c_1$ and c_3 on x_1 and x_2 respectively. If only one of the r_i's had value 1, it would be impossible for f to map to two distinct nonzero values at two different points. This brings us to:

Case (b). In this case, if $r_2 \geq 2$ then on C_2 we can assign a function $c_2(x)$ (c_2 is linear if $r_2 = 2$). We can choose $c_2(x)$ so that whatever values we want to assign to x_1 and x_2 can be obtained from $c_2(x)$.

Proposition 5.3.1. *Let C be a chain of length n and suppose that $r_i \leq 1, \forall i$, then $\dim(im(f)) = min(n-1, \#(C_i;\ r_i = 1))$*

Proof. : If $r_i = 1$ for all i, then the value at x_i is obtained by $c_{i+1} - c_i$ and so f is surjective. So assume that at least one of the r_i is zero, then then proposition states that $\dim(im(f))$ is the number of C_i's with nonzero r_i. Let us prove this by induction on the number of nonzero r_i.

Consider the case where only one r_i is nonzero. If either r_1 or r_n is nonzero, then f maps onto x_1 or x_n. If $r_i = 1$ for $i \neq 1, n$, then we can assign a constant c_i to component C_i, in which case f maps to c_i on both x_{i-1} and

x_i. Hence dim($im(f)$)=1 and the base case holds.

Now let us look at the case where r_i is nonzero for $n-2$ components C_i (we skip the case where we have $m < n-2$ components since similar arguments apply), then $r_j = r_k = 0$ for some $j \neq k$, $j, k \in [n]$. For $r_i = r_{i+1} = 1$ let the value at x_i be $c_{i+1} - c_i$. If $j, k \notin \{1, 2\}$ and $j, k \notin \{n-1, n\}$ then one of C_j, C_k has adjacent components with r_i both nonzero. Assume that it is j, in which case $r_{j-1} = r_{j+1} = 1$. If we now let $r_j = 1$, then we can assign $c_j - c_{j-1}$ to x_j and $c_{j+1} - c_j$ to x_{j+1} and we have that $\dim(im(f)) = n - 1$. If, however, we have $j, k \in \{1, 2\}$ or $j, k \in \{n-1, n\}$, then we must assign values differently. Consider the case $j, k \in \{1, 2\}$, say $j = 1$ and $k = 2$. If we let $r_j = r_1 = 1$, then we can assign c_1 to x_1. If, on the other hand, we have $r_k = r_2 = 1$, then we assign c_2 to x_2 and $c_3 - c_2$ to x_3, and either way we have $\dim(im(f)) = n - 1$. The case where $j, k \in \{n-1, n\}$ is similar. Since f can map to at most $n - 1$ distinct points we are done. □

Proposition 5.3.2. *Given a chain C of $\mathbb{P}^1$s and a line bundle $\mathcal{L}$ over C, let C_{S_j} denote the subchains of C where each component C_i, $i \in S_j$, has $r_i < 2$, and C_T the remaining components then*

$$\dim(H^1(C, \mathcal{L})) = \left(\sum_{d_i < 0} -d_i - 1 \right) + (n-1) - |\{x_i \in C_T\}|$$
$$- \textstyle\sum_j min(|S_j| - 1, \#(C_i;\ r_i = 1, i \in S_j)).$$

Proof. We already know from equation (5.1) that we only need $\dim(im(f))$ to find $\dim(H^1(C, \mathcal{L}))$. On a given subchain C_{S_j} we have by Proposition 5.3.1

that

$$\dim(im(f|_{C_{S_j}})) = min(|S_j| - 1, \#(C_i;\ r_i = 1, i \in S_j).$$

Note that for any point $x_i \in C_T$ f will be surjective since $x_i \in C_j$ with $r_j \geq 2$. We proceed as follows:

If C is a chain, then we can determine $\dim(im(f))$ using the following procedure:

(Step 1) Determine the $r_i's$ such that $r_i \geq 2$.

(Step 2) Split the chain into smaller chains C_{S_j} where we separate the C_i with $r_i \geq 2$ from C_i with $r_i < 2$.

(Step 3) For the x_i that are in the chains with $r_i > 2$, f is surjective here, so the number of such x_i is put toward $\dim(im(f))$. The remaining x_i are in chains with $r_i \leq 1$ and we use the formula in Proposition 5.3.1 to determine f on these chains. The sum of all these gives $\dim(im(f))$.

To be more precise we can write

$$\dim(im(f)) = \#(x_i \in C; r_i \geq 2) + \sum_j min(|S_j| - 1, \#(C_i;\ r_i = 1, i \in S_j)) \tag{5.2}$$

We can combine equations (5.1) and (5.2) to get

$$\dim(H^1(C, \mathcal{L})) = \left(\sum_{d_i < 0} -d_i - 1 \right) + (n-1) - |\{x_i \in C_T\}|$$
$$- \textstyle\sum_j min(|S_j| - 1, \#(C_i;\ r_i = 1, i \in S_j)). \qquad \square$$

5.3.2 Hub with spokes

A hub with spokes setup is where we have one component C_0 meeting all other components C_i, each at only one point x_i, and no other components meet. Say we then have n components, $C_0, C_1, \ldots, C_{n-1}$. We still have $n-1$ intersection points.

In order for f to be surjective, we need:
$r_0 \geq m$ where there are m d_is with $d_i < 0$, $i > 0$.

Basically, whenever d_0 is not large enough, and some d_is are negative, then f is no longer surjective, and we have

Proposition 5.3.3.

$$\dim(im(f)) = min(n-1, r_0 + m')$$

where m' is the number of C_i's with $d_i \geq 0$ and i nonzero.

Proof. If $r_0 = 0$ then we cannot assign a function to C_0. We can only assign functions to those C_i with $d_i \geq 0$, so to each of these assign $c_i(x)$. We then have $c_i(x_i)$ for m' $x_i = C_0 \cap C_i$ and the proposition follows.
If $r_0 > 0$, then with the same assignment as before we can map f to m' points. Only in this case, we can furthermore choose a function $c_0(x)$ of degree $d_0 = r_0 - 1$ on C_0 and assign values to points x_j on C_j with $d_j < 0$. Hence allowing f to map up to r_0 more points, proving the proposition. □

The overall formula here is much simpler: combining equation (5.1) with Proposition 5.3.3 we get:

Proposition 5.3.4. *Given a curve C and a line bundle $\mathcal{L}$ on it, let C have components $C_0, C_1, \ldots, C_n$ where each $C_i \cong \mathbb{P}^1$ and $|C_0 \cap C_i| = 1$ and $|C_i \cap C_j| = 0$ for any nonzero i and j we have*

$$\dim(H^1(C, \mathcal{L})) = \left(\sum_{d_i < 0} -d_i - 1 \right) + (n-1) - min(n-1, r_0 + m').$$

Chapter 6

Computations of The Obstruction

Recall the taxonomy of tropical lines in the fan tropical plane:

Type 1 tropical lines have only one vertex and lie on the 1-skeleton.

Type 2 tropical lines have two vertices: one on the 1-skeleton and the other in the interior of a 2-cone.

Type 2A tropical lines have two vertices, each in the interior of a 2-cone.

Type 3 tropical lines have three or more vertices.

We will compute the size of the obstruction group for pre-log curves given in each type of the taxonomy. What we see is that the normal bundle is trivial on components corresponding to vertices in the relative interior of bounded edges. For the endpoints of the edges, the normal bundle is $\mathcal{O}_{\bar{\mathcal{C}}}(1)$. For vertices on the 1-skeleton or at the origin, the normal bundle is $\mathcal{O}_{\bar{\mathcal{C}}}(1-r)$, where r is the number of bounded edges of the tropical line; one can think of tropical lines of Type 2A as a Type 3 tropical line where the origin subdivides the bounded edge into two edges.

We will start with tropical lines of Type 2A, as this case captures many of the different configurations where we need to create a component of a pre-log curve.

6.1 Tropical Lines of Type 2A in the Fan Tropical Plane

A tropical line with the tropical plane will look like the polyhedral complex in Figure 6.1.

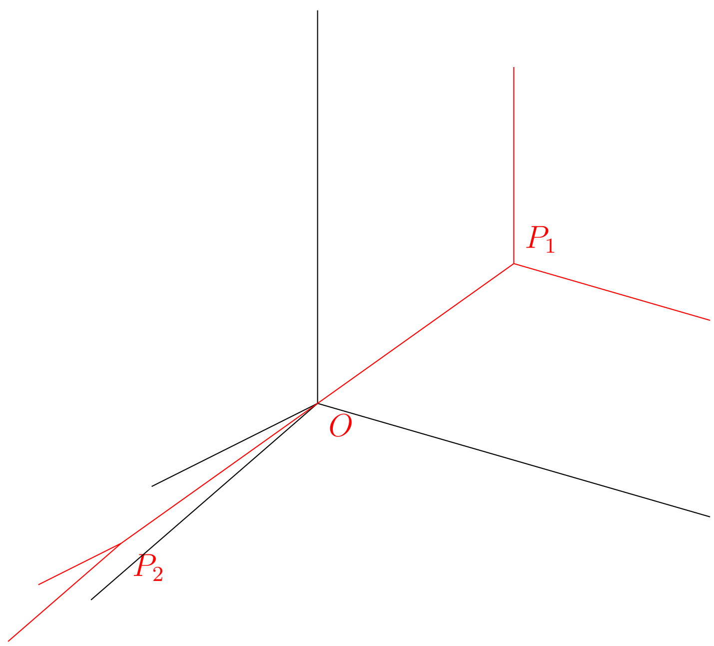

Figure 6.1: The tropical line in red with the coordinate axes of the tropical plane in black.

In order to make use of toric degnerations, and in particular, the results on the special fiber given by [12] we first require that our polyhedral complex, call it $\mathcal{P}$ is integral, namely, that the vertices have integer coordinates. This is the main condition in Proposition 4.1.2 which then tells us that the special fiber is reduced. A priori $\mathcal{P}$ will not be integral, as it simply given by a tropical line sitting inside of a tropical plane. However, the vertices of $\mathcal{P}$ will be rational (that is, will have rational coordinates), and since there are finitely many vertices we can find an integer d large enough so that the complex $d\mathcal{P}$ obtained by scaling $\mathcal{P}$ by d is an integral polyhedral complex. On the scheme level we are adjoining a dth root of t to the base, changing from *Spec* $\mathbb{C}[t]$ to *Spec* $\mathbb{C}[t^{1/d}]$. Proposition 4.5.2 assures us that we can get the same asymptotic fan as the complex $\mathcal{P}$ so long as our new complex, call it $\mathcal{P}'$, is simplicial and unimodular, and the special fiber of this new complex is a divisor with simple normal crossings.

Hence, in the example above, we must rescale the diagram by a factor d so that the vertices P_1 and P_2 have integral coordinates, and then we must further add unbounded edges at integral points on the bounded edge between P_1 and P_2 in order to make sure the fan is simplicial and unimodular. For simplicity, let us assume the new complex $\mathcal{P}'$ adds only two vertices on the edge between P_1 and P_2. This complex will look like the one in Figure 6.2.

We will compute the normal bundle by looking at the toric varieties at each vertex of $\mathcal{P}'$. Given a vertex P, we have the toric variety X_P, which is the ambient toric variety of the plane and the line, $\mathcal{V}_P$ the component of the toric variety corresponding to the degenerated $\mathbb{P}^2$ sitting inside of X_P, and $\mathcal{C}_P$ is the component of the pre-log curve inside of X_P. Let us examine what these varieties look like for each subcomplex and use the information to compute the size of the normal bundle.

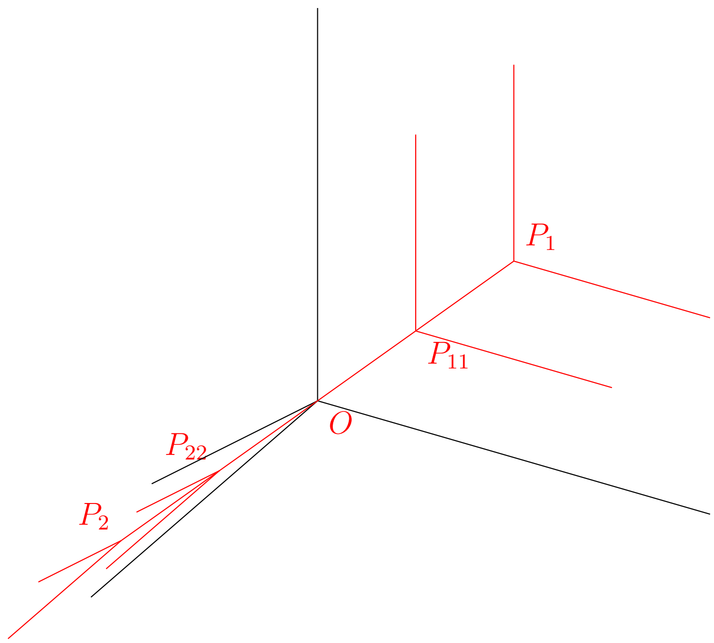

Figure 6.2: The subdivided tropical line in red with the coordinate axes of the tropical plane in black.

6.1.1 Complexes at vertices in interiors of bounded edges

First, we claim that normal bundle at the additional vertices P_{11} and P_{22} are trivial. Let us show this. At either point the complex looks like the one in Figure 6.3.

Here, the ambient toric variety X_P is the same as $\mathcal{V}_P$, namely, $\mathbb{P}^2$ blown-up at a point, as we can see from the fan. Let E denote the exceptional divisor, and D_0, D_1, D_2 the boundary divisors corresponding to rays e_0, e_1, e_2 respectively. If $\mathcal{O}(1)$ is the preimage in $\mathcal{V}_P$ of a line that misses the blow-up point, then we know that $D_1 \sim D_2 \sim \mathcal{O}(1) - E$. Hence, on $\bar{\mathcal{C}}$ we have that $\mathcal{O}(D_1) \cong \mathcal{O}_{\bar{\mathcal{C}}}(0)$. In other words, $\bar{\mathcal{C}}$ misses divisors D_1 and D_2 and only meets

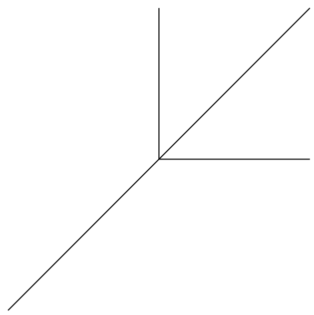

Figure 6.3: A diagram of the complex at P_{11}.

the divisors D_0 and the exceptional divisor E. Using Proposition 4.4.1 we have that

$$\begin{aligned}\omega_{\underline{\bar{\mathcal{C}}}} &\cong \omega_{\bar{\mathcal{C}}} \otimes \mathcal{O}(D)\\ &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}(E) \otimes \mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)\\ &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1)\\ &\cong \mathcal{O}_{\bar{\mathcal{C}}}(0).\end{aligned}$$

Let us now compute $(f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee$. We have, combining Proposition 4.4.4 with Lemma 4.4.5 that

$$\begin{aligned}\omega_{\underline{\mathcal{V}}/\mathbb{C}[t]}|_{\mathcal{V}_i} &\cong \left(\omega_{\mathcal{V}_i/\mathbb{C}}\right) \otimes \left(\bigotimes_{i\neq j} \mathcal{O}(\mathcal{V}_i \cap \mathcal{V}_j)\right)\\ &\cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes (\mathcal{O}(E) \otimes \mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \mathcal{O}(D_2))\\ &\cong \mathcal{O}(-3) \otimes \mathcal{O}(E) \otimes (\mathcal{O}(E) \otimes \mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \mathcal{O}(D_2))\,.\end{aligned}$$

Pulling this back to $\bar{\mathcal{C}}$ we have

$$\begin{aligned} f^*\omega_{\bar{\mathcal{V}}} &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-3) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes (\mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1)) \\ &\cong \mathcal{O}_{\bar{\mathcal{C}}}(0) \end{aligned}$$

Hence we have that $(f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee \cong \mathcal{O}_{\bar{\mathcal{C}}}(0)$ and $\omega_{\underline{\bar{\mathcal{C}}}} \cong \mathcal{O}_{\bar{\mathcal{C}}}(0)$ so that the normal bundle $\mathcal{N}_f$ is trivial at these additional components. We state this as a proposition for easy reference as follows:

Proposition 6.1.1. *The normal bundle $\mathcal{N}_f$ is trivial on components of $\underline{\bar{\mathcal{C}}}$ corresponding to vertices of the subdivision $\mathcal{P}'$ that lie in the relative interior of bounded edges.*

Proof. See the computation above. □

6.1.2 Complexes at the endpoints of bounded edges in a 2-cone

Let us now look at the components of $\bar{\mathcal{C}}$ corresponding to vertices P_1 and P_2. The complex here looks like that of Figure 6.4

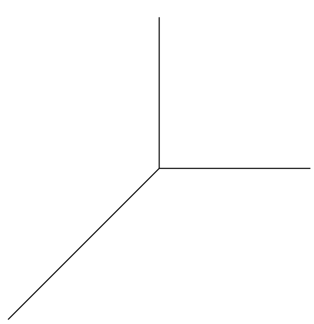

Figure 6.4: A diagram of the complex at P_1.

Here the complex is simply a $\mathbb{P}^2$, and $X_P \cong \mathcal{V}_P$. As before, let the boundary divisors corresponding the rays in directions e_0, e_1, e_2 be represented by D_0, D_1, D_2 respectively. Here $\bar{\mathcal{C}}$ only meets all three boundary divisors D_0, D_1 and D_2, so we have

$$\begin{aligned}\omega_{\underline{\bar{\mathcal{C}}}} &\cong \omega_{\bar{\mathcal{C}}} \otimes \mathcal{O}(D)\\ &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)\\ &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1)\\ &\cong \mathcal{O}_{\bar{\mathcal{C}}}(1).\end{aligned}$$

To compute $(f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee$ note that

$$\begin{aligned}\omega_{\underline{\mathcal{V}}/\mathbb{C}[t]}|_{\mathcal{V}_i} &\cong \left(\omega_{\mathcal{V}_i/\mathbb{C}}\right) \otimes \left(\bigotimes_{i\neq j} \mathcal{O}(\mathcal{V}_i \cap \mathcal{V}_j)\right)\\ &\cong \mathcal{O}(-3) \otimes \left(\mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)\right).\end{aligned}$$

Since $\bar{\mathcal{C}}$ meets D_0, D_1 and D_2 we have $(f^*\omega_{\underline{\bar{\mathcal{V}}}}) \cong \mathcal{O}_{\bar{\mathcal{C}}}(-3) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \cong \mathcal{O}(0)$. Hence $(f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee \cong \mathcal{O}_{\bar{\mathcal{C}}}(0)$.

The normal bundle at this component is then given by $\mathcal{N}_f \cong (f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee \otimes \omega_{\underline{\bar{\mathcal{C}}}} \cong \mathcal{O}_{\bar{\mathcal{C}}}(0) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \cong \mathcal{O}_{\bar{\mathcal{C}}}(1)$. So we have the following:

Proposition 6.1.2. *On components of $\underline{\bar{\mathcal{C}}}$ corresponding to vertices of the subdivision $\mathcal{P}'$ that are the endpoints bounded edges inside the relative interior of a 2-cone of $\mathcal{P}'$, we have the normal bundle $\mathcal{N}_f \cong \mathcal{O}_{\bar{\mathcal{C}}}(1)$.*

Proof. See the computation above. □

6.1.3 The complex at the origin

At the origin, we have the complex shown in Figure 6.5.

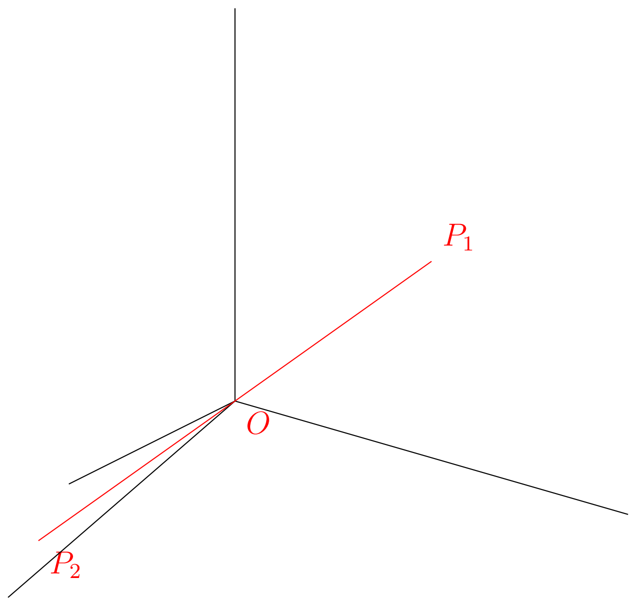

Figure 6.5: The complex at the origin.

Here there is some difference between X_P and $\mathcal{V}_P$. The fan of a blow-up of a $\mathbb{P}^3$ at two skew boundary lines, which is what we have for X_P, so $\mathcal{V}_P$ is a $\mathbb{P}^2$ blown-up at the two points where the $\mathbb{P}^2$ meets the skew boundary lines. In this case $\mathcal{C}_P$ meets only the two exceptional divisors of $\mathcal{V}_P$.

Let D_i be the divisor of $\mathcal{V}_P$ corresponding to the ray in the e_i direction of the fan. Let E_1 be the exceptional divisor of $\mathcal{V}_P$ corresponding to the ray in the interior of $Cone(e_0, e_1)$, and E_2 that of $Cone(e_1, e_2)$. We then have that $D_0 \sim D_1 \sim \mathcal{O}(1) - E_1$ and $D_2 \sim D_3 \sim \mathcal{O} - E_2$. Hence, the curve $\bar{\mathcal{C}}$ misses the boundary divisor altogether. Let us now compute $(f^*\omega_{\bar{\mathcal{V}}})^\vee$. Again making use of Proposition 4.4.4 and Lemma 4.4.5 we have

$$\begin{aligned}
\omega_{\underline{\mathcal{V}}/\mathbb{C}[t]}|_{\mathcal{V}_i} &\cong \big(\omega_{\mathcal{V}_i/\mathbb{C}}\big) \otimes \left(\bigotimes_{i\neq j} \mathcal{O}(\mathcal{V}_i \cap \mathcal{V}_j)\right) \\
&\cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes (\mathcal{O}(E_1) \otimes \mathcal{O}(E_2) \otimes \mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \mathcal{O}(D_2) \otimes \mathcal{O}(D_3)) \\
&\cong \mathcal{O}(-3) \otimes \mathcal{O}(E_1) \otimes \mathcal{O}(E_2) \otimes (\mathcal{O}(E_1) \otimes \mathcal{O}(E_2) \otimes \mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \\
&\otimes \mathcal{O}(D_2) \otimes \mathcal{O}(D_3)).
\end{aligned}$$

Pulling it back to $\bar{\mathcal{C}}$ we have

$$\begin{aligned}
f^*\omega_{\bar{\mathcal{V}}} &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-3) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes (\mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1)) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(1)
\end{aligned}$$

Hence we have that $(f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee \cong \mathcal{O}_{\bar{\mathcal{C}}}(-1)$. We also have

$$\begin{aligned}
\omega_{\underline{\bar{\mathcal{C}}}} &\cong \omega_{\bar{\mathcal{C}}} \otimes \mathcal{O}(D) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}(E_1) \otimes \mathcal{O}(E_2) \otimes \mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \mathcal{O}(D_2) \otimes \mathcal{O}(D_3) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(0).
\end{aligned}$$

Putting these two together we have $\mathcal{N}_f \cong (f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee \otimes \omega_{\underline{\bar{\mathcal{C}}}} \cong \mathcal{O}_{\bar{\mathcal{C}}}(-1) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(0) \cong \mathcal{O}_{\bar{\mathcal{C}}}(-1)$. We summarize this in the following:

Proposition 6.1.3. *The component of the normal bundle corresponding to the origin in the subdivision $\mathcal{P}'$ for a Type 2A tropical line is isomorphic to $\mathcal{O}_{\bar{\mathcal{C}}}(-1)$.*

6.2 Tropical Lines of Type 1 in the Fan Tropical Plane

Recall that all tropical lines of Type 1 have only one vertex and can lie anywhere on the 1-skeleton of V. See Figure 6.6 for an example of a tropical line of Type 1 in $\mathbb{R}^3$.

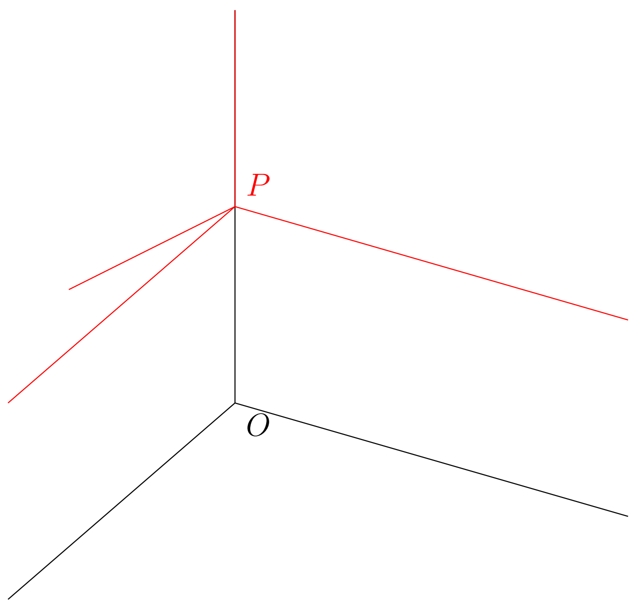

Figure 6.6: The Type 1 tropical line in red with the coordinate axes of the tropical plane in black.

In this case the curve $\bar{\mathcal{C}}$ consists of only one component corresponding to the vertex P on the 1-skeleton of V.

6.2.1 The vertex sitting at the origin

Let us start with the case where P coincides with the origin O. In this case the fan is simply the fan for $\mathbb{P}^n$, so the ambient toric variety $X_P \cong \mathbb{P}^n$ and $\mathcal{V}_P$ is simply a plane $\mathbb{P}^2$ sitting in $\mathbb{P}^n$ in general position. The curve

$\mathcal{C}_P$ meets all of the boundary divisors transversely, so, taking D_i the divisor corresponding to the ray in the e_ith direction as before, we have

$$\begin{aligned}
\omega_{\underline{\bar{\mathcal{C}}}} &\cong \omega_{\bar{\mathcal{C}}} \otimes \mathcal{O}(D) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \cdots \otimes \mathcal{O}(D_n) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \cdots \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2 + (n+1)) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(n-1).
\end{aligned}$$

Using Proposition 4.4.4 we have

$$\begin{aligned}
\omega_{\underline{\mathcal{V}}/\mathbb{C}[t]}|_{\mathcal{V}_i} &\cong \left(\omega_{\mathcal{V}_i/\mathbb{C}}\right) \otimes \left(\bigotimes_{i \neq j} \mathcal{O}(\mathcal{V}_i \cap \mathcal{V}_j)\right) \\
&\cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes \left(\mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \cdots \otimes \mathcal{O}(D_n)\right) \\
&\cong \mathcal{O}(-3) \otimes \left(\mathcal{O}(D_0) \otimes \mathcal{O}(D_1) \otimes \cdots \otimes \mathcal{O}(D_n)\right).
\end{aligned}$$

Pulling it back we have

$$\begin{aligned}
f^*\omega_{\bar{\mathcal{V}}} &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-3) \otimes \left(\mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \cdots \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1)\right) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-3 + (n+1)) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(n-2).
\end{aligned}$$

Hence, the normal bundle $\mathcal{N}_f \cong (f^*\omega_{\underline{\bar{\mathcal{V}}}})^\vee \otimes \omega_{\underline{\bar{\mathcal{C}}}} \cong \mathcal{O}_{\bar{\mathcal{C}}}(2-n) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(n-1) \cong \mathcal{O}_{\bar{\mathcal{C}}}(1)$.

6.2.2 The vertex on an axis

If P sits on the ray in direction e_i, the complex at P is the fan for $\mathbb{P}^n$ with an additional ray in the $-e_i$ direction. Hence the ambient variety X_P is the blow-up of $\mathbb{P}^n$ along a point Q. This point Q must meet all of the boundary divisors D_j for $0 \leq j \leq n$ and $j \neq i$. In particular, Q is the intersection of the n divisors D_j, where $j \neq i$. We know the plane $\mathcal{V}_P$ must meet these boundary divisors as well, so Q and its blow-up is in $\mathcal{V}_P$ which means $\mathcal{V}_P$ is the blow-up of $\mathbb{P}^2$ at a point. Call the exceptional divisor E, then each of the divisors $D_j \sim \mathcal{O}(1) - E$ for $j \neq i$. However, the curve $\bar{\mathcal{C}}$ misses the exceptional divisor E, so despite the fact that E shows up in both Proposition 4.4.4 and Lemma 4.4.5, the computation remains the same. When we take the pull-back $f^*\omega_{\bar{\mathcal{V}}}$, the contribution of E disappears, and similarly does not contribute to $\omega_{\underline{\bar{\mathcal{C}}}}$, while the contribution from the remaining divisors remains the same. In summary, we have:

Proposition 6.2.1. *The curve $\underline{\bar{\mathcal{C}}}$ corresponding to a tropical line of Type 1 has only one component, corresponding to the only vertex of the tropical line in $\mathcal{P}'$, and the normal bundle is $\mathcal{O}_{\bar{\mathcal{C}}}(1)$.*

6.3 Tropical Lines of Type 2

In this case, our tropical line has two vertices, one on the 1-skeleton of $\mathcal{V}$, and another in the interior of a 2-cone. A diagram of an example of this case is provided in Figure 6.7.

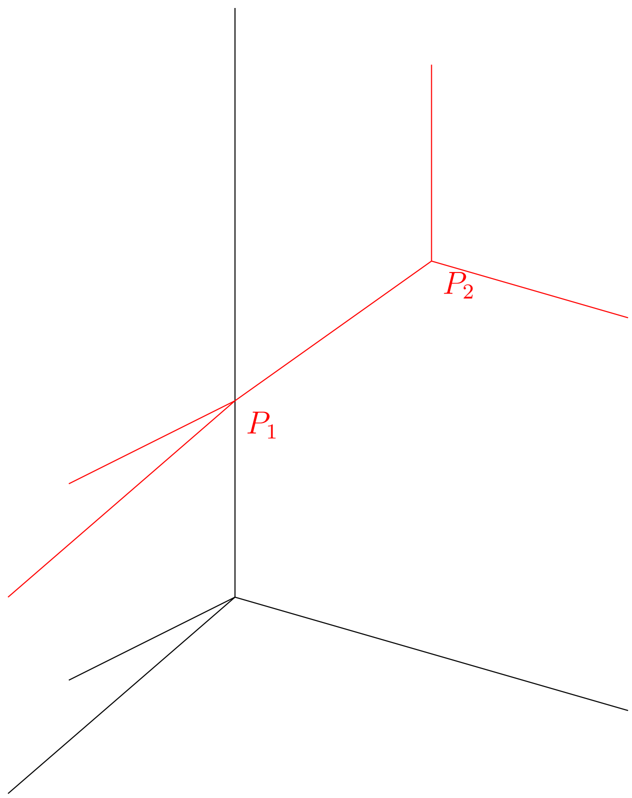

Figure 6.7: The Type 2 tropical line in red with the coordinate axes of the tropical plane in black.

Taking a normal crossings subdivision, the complex can look like that of Figure 6.8.

Note that the computations for the normal bundle corresponding to

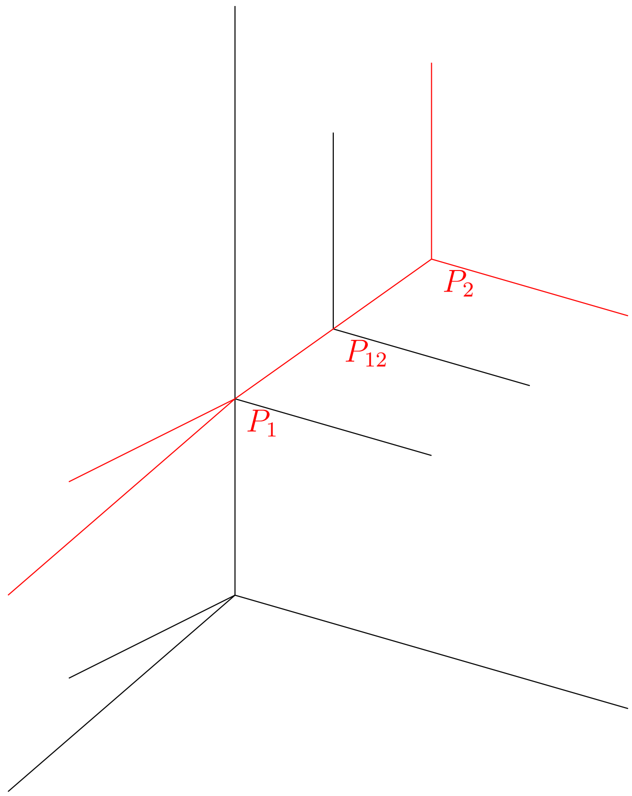

Figure 6.8: The Type 2 tropical line in red with the coordinate axes of the tropical plane, and additional rays, in black.

vertices P_{12} and P_2 will be exactly the same as the case of the equivalent complexes in the case of the configuration of Type 2A. Hence, the normal bundle will be trivial at P_{12} and will be $\mathcal{O}_{\bar{\mathcal{C}}}(1)$ at P_2. So it remains to compute the bundle at the point on the 1-skeleton of $\mathcal{V}$.

6.3.1 The complex at the vertex on the 1-skeleton

Let us compute the normal bundle at the point P_1 in Figure 6.8. The complex here is given by Figure 6.9.

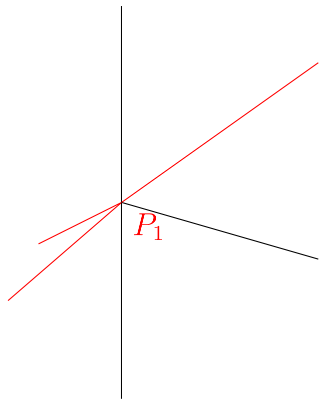

Figure 6.9: The complex at the vertex P_1 on the axis.

The complex here is very similar to that of complex at the vertex P on the 1-skeleton in Figure 6.6, only here we have an additional ray in a 2-cone. Let us take P_1 to be on the axis in the e_i direction, then in addition to the rays in directions $e_0, e_1, \ldots, e_n$ we have a ray in the $-e_i$ direction and a ray in the $e_i + e_j$ direction for some $j \neq i$. Let E_2 be the exceptional divisor corresponding to the ray in the $e_i + e_j$ direction, and E_1 be the exceptional divisor corresponding to the ray in the $-e_i$ direction. We then have that E_2 is the exceptional divisor of a blow-up of $\mathbb{P}^n$ along a codimension 2 subvariety, and E_1 that of a point. Hence our ambient toric variety here corresponds to the blow-up of $\mathbb{P}^n$ at a point and a codimension 2 subvariety. Our toric variety $\mathcal{V}_P$ is then the blow-up of $\mathbb{P}^2$ at two distinct points. We have the following relations among our divisors: $\mathcal{O}(1) \sim D_k + E_1$ for any $k \neq i, j$ and $\mathcal{O}(1) \sim D_i + E_2 + E_1 \sim D_j + E_2$. Let us use these facts to determine the normal bundle for the component corresponding to this complex. We have

$$\begin{aligned}
\omega_{\underline{\mathcal{V}}/\mathbb{C}[t]}|_{\mathcal{V}_i} &\cong \left(\omega_{\mathcal{V}_i/\mathbb{C}}\right) \otimes \left(\bigotimes_{i\neq j} \mathcal{O}(\mathcal{V}_i \cap \mathcal{V}_j)\right) \\
&\cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes \left(\mathcal{O}(E_1) \otimes \mathcal{O}(E_2) \bigotimes_{0\leq k\leq n} \mathcal{O}(D_k)\right) \\
&\cong \mathcal{O}(-3) \otimes \mathcal{O}(E_1) \otimes \mathcal{O}(E_2) \otimes \left(\mathcal{O}(E_1) \otimes \mathcal{O}(E_2) \bigotimes_{0\leq k\leq n} \mathcal{O}(D_k)\right).
\end{aligned}$$

Note that our curve $\underline{\bar{\mathcal{C}}}$ misses the exceptional divisor E_1, but meets the exceptional divisor E_2. Hence the fact that $\mathcal{O}(1) \sim D_k + E_1$ for any $k \neq i, j$ means that the curve meets these D_k. In addition, $\mathcal{O}(1) \sim D_j + E_2$ means that $D_j \sim \mathcal{O}(1) - E_2$, and $\mathcal{O}(1) \sim D_i + E_2 + E_1$ implies $D_i \sim \mathcal{O}(1) - E_2 - E_1$, so $\underline{\bar{\mathcal{C}}}$ misses divisors D_i and D_j. This gives us

$$\begin{aligned}
f^*\omega_{\bar{\mathcal{V}}} &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-3) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \left(\mathcal{O}_{\bar{\mathcal{C}}}(1) \bigotimes_{0\leq k\leq n,\ k\neq i,j} \mathcal{O}_{\bar{\mathcal{C}}}(1)\right) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-3 + 1 + (1 + n - 1)) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(n - 2).
\end{aligned}$$

To compute the log canonical sheaf of $\underline{\bar{\mathcal{C}}}$ we use Propostion 4.4.1 to get

$$\begin{aligned}
\omega_{\underline{\bar{\mathcal{C}}}} &\cong \omega_{\bar{\mathcal{C}}} \otimes \mathcal{O}(D) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}(E_2) \bigotimes_{0 \le k \le n,\ k \ne i,j} \mathcal{O}_{\bar{\mathcal{C}}}(1) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(1) \otimes \cdots \otimes \mathcal{O}_{\bar{\mathcal{C}}}(n-1) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-1 + (n-1)) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(n-2).
\end{aligned}$$

Therefore, the normal bundle $\mathcal{N}_f \cong (f^* \omega_{\underline{\bar{\mathcal{V}}}})^\vee \otimes \omega_{\underline{\bar{\mathcal{C}}}} \cong \mathcal{O}_{\bar{\mathcal{C}}}(2-n) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(n-2) \cong \mathcal{O}_{\bar{\mathcal{C}}}(0)$. We state this result as a proposition:

Proposition 6.3.1. *The normal bundle is trivial at the component of $\underline{\bar{\mathcal{C}}}$ corresponding to the vertex on the 1-skeleton of V of a Type 2 tropical line.*

6.4 Tropical Lines of Type 3

The polyhedral complex here is a mix of Type 1 and Type 2A in the sense that the tropical line may have some edges in the interiors of 2-cones and some on the axes. Let us assume that the tropical line C has r vertices not including the origin. Label the vertices that are not the origin $P_0, P_2, \ldots, P_{2r-2}$. For simplicity, take $P_i \in Cone(e_i, e_{i+1})$. Taking normal crossings subdivisions we will have additional vertices on the interiors of bounded edges as well. Using Proposition 6.1.1 we know that the normal bundle components will be trivial on the vertices in interiors of edges, and by Proposition 6.1.2, will be $\mathcal{O}_{\bar{\mathcal{C}}}(1)$ at each P_i.

The complex at the origin is the fan for $\mathbb{P}^n$ with an additional ray for each 2-cone containing a P_i. Hence, the ambient toric variety X_P at the origin is the blow-up of $\mathbb{P}^n$ along r codimension 2 subvarieties in general position, so that $\mathcal{V}_P$ is the blow-up of $\mathbb{P}^2$ at r distinct points. Let E_i denote the exceptional divisor corresponding to the edge meeting P_i, and D_j the boundary divisor corresponding to the ray in the e_j direction. We then have, as in the Type 2A case, that $D_i \sim D_{i+1} \sim \mathcal{O}(1) - E_i$ for any $i \in \{0, 2, \ldots, 2r-2\}$, hence, the curve $\bar{\mathcal{C}}$ misses these divisors. We then have, using of Proposition 4.4.4 and Lemma 4.4.5, that

$$\begin{aligned}
\omega_{\underline{\mathcal{V}}/\mathbb{C}[t]}|_{\mathcal{V}_i} &\cong \left(\omega_{\mathcal{V}_i/\mathbb{C}}\right) \otimes \left(\bigotimes_{i \neq j} \mathcal{O}(\mathcal{V}_i \cap \mathcal{V}_j)\right) \\
&\cong \omega_{\mathcal{V}_i/\mathbb{C}} \otimes \left(\bigotimes_{i \in \{0,2,\ldots,2r-2\}} \mathcal{O}(E_i) \bigotimes_{0 \leq j \leq n} \mathcal{O}(D_j)\right) \\
&\cong \mathcal{O}(-3) \bigotimes_{i \in \{0,2,\ldots,2r-2\}} \mathcal{O}(E_i) \otimes \left(\bigotimes_{i \in \{0,2,\ldots,2r-2\}} \mathcal{O}(E_i) \bigotimes_{0 \leq j \leq n} \mathcal{O}(D_j)\right).
\end{aligned}$$

Pulling it back to $\bar{\mathcal{C}}$ we have

$$\begin{aligned}
f^*\omega_{\bar{\mathcal{V}}} &\cong \mathcal{O}_{\bar{\mathcal{C}}}(-3) \bigotimes_{i \in \{0,2,\ldots,2r-2\}} \mathcal{O}(1) \otimes \left(\bigotimes_{i \in \{0,2,\ldots,2r-2\}} \mathcal{O}(1) \bigotimes_{2r \leq j \leq n} \mathcal{O}(1)\right) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-3) \otimes \mathcal{O}(r) \otimes (\otimes \mathcal{O}(r) \otimes \mathcal{O}(n-2r)) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(n-3).
\end{aligned}$$

The log canonical sheaf of the curve comes out to

$$\begin{aligned}
\omega_{\underline{\bar{\mathcal{C}}}} &\cong \omega_{\bar{\mathcal{C}}} \otimes \mathcal{O}(D) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \bigotimes_{i\in\{0,2,\ldots,2r-2\}} \mathcal{O}(E_i) \bigotimes_{0\leq j\leq n} \mathcal{O}(D_j) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(-2) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(r) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(n-2r) \\
&\cong \mathcal{O}_{\bar{\mathcal{C}}}(n-r-2).
\end{aligned}$$

Hence, the normal bundle $\mathcal{N}_f \cong (f^*\omega_{\underline{\bar{\mathcal{Y}}}})^\vee \otimes \omega_{\underline{\bar{\mathcal{C}}}} \cong \mathcal{O}_{\bar{\mathcal{C}}}(3-n) \otimes \mathcal{O}_{\bar{\mathcal{C}}}(n-r-2) \cong \mathcal{O}_{\bar{\mathcal{C}}}(1-r)$.

Proposition 6.4.1. *The normal bundle component corresponding to the origin in the case of a Type 3 tropical line is isomorphic to $\mathcal{O}_{\bar{\mathcal{C}}}(1-r)$ where $r \geq 2$ is the number of bounded edges of the tropical line.*

6.5 The Size of the Obstruction

Having computed the normal bundle for all possible configurations of tropical lines in the plane, let us now compute the size of the obstruction. We will proceed by type.

6.5.1 Type 1 Tropical Lines

This is the simplest case since our curve has only one component, and recall, from Proposition 6.2.1, that the normal bundle on that component came out to $\mathcal{O}_{\bar{\mathcal{C}}}(1)$. Hence, we have that

$$H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f) \cong H^1(\mathbb{P}^1, \mathcal{O}(1)),$$

and $\dim(H^1(\mathbb{P}^1, \mathcal{O}(1))) = 0$, so we have no obstructions, as would be expected.

6.5.2 Type 2 Tropical Lines

In this case these is exactly one bounded edge, so we will have two components for the endpoints, and additional components for vertices in the interior of the bounded edge. As a result, our curve is a chain of $\mathbb{P}^1$s, and we can apply Proposition 5.3.2 to determine the size of the obstruction. Assume that there are k vertices, call them Q_i, for $1 \leq i \leq k$, and let $\underline{\bar{\mathcal{C}}}_{Q_i}$ be the curve component for Q_i. Let Q_1 denote the vertex on the 1-skeleton of V and Q_k the end point of the bounded edge in the interior of the 2-cone, with the remaining vertices in the relative interior, and Q_i adjacent to Q_{i+1}.

We know, using Proposition 6.3.1, that $\mathcal{N}_f|_{\underline{\bar{\mathcal{C}}}_{Q_1}}$ is trivial, and the same is true for $\mathcal{N}_f|_{\underline{\bar{\mathcal{C}}}_{Q_i}}$ for any $i < k$. The component $\mathcal{N}_f|_{\underline{\bar{\mathcal{C}}}_{Q_k}} \cong \mathcal{O}_{\bar{\mathcal{C}}}(1)$ by Proposition 6.1.2. Hence, the normal bundle has non-negative degree on all the components. Let us define

$$r_i := H^0(\underline{\bar{\mathcal{C}}}_{Q_i}, \mathcal{N}_f|_{\underline{\bar{\mathcal{C}}}_{Q_i}}).$$

We have that $r_i = 1$ for $1 \leq i < k$ and $r_k = 2$, so in this chain with k components we have a subchain with $k-1$ components with $r_i = 1$, so Proposition 5.3.2 then tells us that

$$\dim(H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f)) = 0 + (k-1) - 1 - (k-2) = 0.$$

Therefore, we have no obstructions to lifting in this case.

6.5.3 Type 2A Tropical Lines

As in the case of Type 2 tropical lines, the components of the corresponding curve reduce to a chain. As before, let P_1 and P_2 be the endpoints of the bounded edge, each of which lies in the relative interior of a 2-cone. We also have additional vertices on the relative interior of the bounded edge including the origin. Assume that there are a total of k vertices in $\mathcal{P}'$, including P_1, P_2 and the origin. Let O denote the origin, and label the vertices on the part of the edge between P_1 and O by P_{1i} for some positive integers i, and those between P_2 and O by P_{2j}, for some positive integers j, and the curve component corresponding to vertex Q is denoted by $\underline{\bar{\mathcal{C}}}_Q$. Define

$$r_Q := H^0(\underline{\bar{\mathcal{C}}}_Q, \mathcal{N}_f|_{\underline{\bar{\mathcal{C}}}_Q}).$$

Applying Propositions 6.1.1, 6.1.2, and 6.1.3 and formulas for cohomology of twists of the structure sheaf give $r_O = 0$, $r_{P_{1i}} = r_{P_{2j}} = 1$, and $r_{P_1} = r_{P_2} = 2$. Applying Proposition 5.3.2 gives

$$\dim(H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f)) = 0 + (k-1) - 2 - (k-3) = 0.$$

As in the case of Type 2 tropical lines, there is no obstruction to lifting.

6.5.4 Type 3 Tropical lines

For these case, the corresponding curve will reduce to a combination of chains and a hub with spokes. Let the tropical line have r bounded edges, and let O denote the origin, P_i denote the endpoints of the bounded edges that are not the origin, with $1 \leq i \leq r$, and P_{ij} denote the vertices on the interior of the bounded edge with vertices O and P_i. Hence, we have a chain of components for the set of vertices on each bounded edge of the tropical line C, and all of these chains have a common component corresponding to the origin, so every component that meets $\underline{\bar{\mathcal{C}}}_O$, along with $\underline{\bar{\mathcal{C}}}_O$ forms a hub with spokes. We will compute the contribution to $H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f)$ by computing the contribution on the hub with spokes, and that of the chains separately, and combining them. Once again, defining r_Q as in the previous case, and applying Propositions 6.1.1 and 6.1.2 we know that $r_{P_{ij}} = 1$ and $r_{P_i} = 2$. Proposition 6.4.1 tells us that the curve component at the origin is isomorphic to $\mathcal{O}_{\bar{\mathcal{C}}}(1-r)$, so $r_O = 0$. Using Proposition 5.3.2 we can see that there is no contribution to $\dim(H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f))$ from any of these chains, so we can focus on the hub with spokes arrangement of components at the origin.

To compute the contribution to $\dim(H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f))$ from the hub with spokes we will use Proposition 5.3.4. For a given i, let P_{i1} denote the vertex on the relative interior of that bounded edge adjacent to O. The hub with spokes has $r+1$ components: one corresponding to O, and r for each P_{i1}. We know that $r_{i1} = 0$ for any i, hence $m' = r$ in Proposition 5.3.4, which combined with Proposition 6.4.1 gives us

$$\dim(H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f)) = -(1-r) - 1 + r - r = r - 2.$$

Hence, the lifting of tropical lines of Type 3 contained in the fan tropical plane is obstructed only when the tropical line contains more than 2 bounded edges.

Chapter 7

Lifting points on the tropical Schubert prevariety

7.1 Lifting points inside the 2-skeleton of the tropical Schubert prevariety

Recall that we are looking at a tropical subprevariety of $FD(2,3;n)$ that parametrizes all tropical lines C in a fixed tropical plane V. We denote this tropical subprevariety by $\sigma \subset FD(2,3;n)$. Points of σ correspond to configurations of $C \subset V$, which are determined by either where C sits on the 1-skeleton (in the cases where C is of Type 1 or 2), or by the size of the bounded edges. Hence, Type 1 lines correspond to some of the rays of σ, while Type 2 lines correspond to 2-cones of σ. Type 3 lines will sit on 3-cones or higher dimensional cones. Hence, geometrically, σ is a collection of cones sitting inside of $FD(2,3;n)$. From the obstruction computations we have the following result on the cones of σ:

Theorem 7.1.1. *Let B be a cone of σ, the tropical Schubert prevariety, of dimension at least 2. If x is a point in the relative interior of B, then over the normal bundle of the tropical line $C \subset V$ we have* $\dim\ H^0 = \dim\ B$ *and* $\dim\ H^1 = \dim\ B - 2$.

Proof. For Type 3 tropical lines, the cohomology result for dim H^1 follows from the fact that, as we computed earlier, if $r \geq 2$ is the number of bounded edges of the tropical line, then $\dim(H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f)) = r - 2$, and dim $B = r$. For Type 2 tropical lines dim $B = 2$ and dim $H^1 = 0$. Using the formulas for H^0 we can get that dim $H^0 =$ dim B. □

Theorem 7.1.2. *Given a cone B of σ with* dim $B = 2$*, then any point $x \in B$ is a realizable point.*

Before we can do this, recall that we need to construct a pre-log curve $\bar{\mathcal{C}}$ and $\bar{\mathcal{V}}$ containing it from $C \subset V$. In order to do this, we take a unimodular and normal crossings subdivision of V which contains C in its 1-skeleton. We can get $V = Trop(\mathcal{V})$, and using Theorem 4.1.4 we can get $\bar{\mathcal{V}}$. We will make use of the following:

Lemma 7.1.3. *Given the setup $C \subset V$ from a point of B as in Theorem 7.1.2, there exists a corresponding pre-log curve with $\bar{\mathcal{C}} \subset \bar{\mathcal{V}}$ such that $\bar{\mathcal{C}}$ meets the boundary strata of $\bar{\mathcal{V}}$ transversely. This pre-log curve satisfies the following properties:*

(a) Components of $\bar{\mathcal{C}}$ meet $\bar{\mathcal{V}}$ where the C meets the 1-skeleton of V.

(b) The curve $\bar{\mathcal{C}}$ meets exactly those boundary strata of $\bar{\mathcal{V}}$ that correspond to cells of V contained in C.

Proof. We know from our cohomology computations that $\dim(H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f)) = 0$, so that there are no obstructions to deforming $\bar{\mathcal{C}}$ to something contained in $\bar{\mathcal{V}}$.

Recall from the discussion following Theorem 3.2.1 that points of B correspond to tropical lines of Type 1, 2, 2A, and 3 where there are only two bounded edges. In each of these cases we will construct the pre-log curve $\bar{\mathcal{C}}$ from this information. To show that $\bar{\mathcal{C}}$ meets the boundary strata of $\bar{\mathcal{V}}$ transversely we need to examine all the cases where components of $\bar{\mathcal{C}}$ meet components of $\bar{\mathcal{V}}$. For simplicity we will use the notation for vertices used during the obstruction computation. Recall that in the subdivision, we construct a pre-log curve by attaching a copy of $\mathbb{P}^1$ to each vertex of the subdivided $C \subset V$.

Type 1 tropical lines:

Using the notation in Section 6.2, our Type 1 tropical line has only one vertex P sitting on the rays in the e_i-directions, or the origin. Except in the case where P is not the origin we have that $\mathcal{V}_P$ is the blow-up of $\mathbb{P}^2$ at a point. However, $\mathcal{C}_P$ misses the exceptional divisor, so $\mathcal{C}_P$ is the strict transform of a general line in $\mathbb{P}^2$ that misses the blow-up point, and therefore will meet the boundary divisors of $\mathcal{V}_P$ transversely. In the case where P is the origin, $\mathcal{V}_P$ is simply a $\mathbb{P}^2$, and $\mathcal{C}_P$ is just then a general line in $\mathbb{P}^2$, which also meets all the boundary divisors transversely.

Type 2 tropical lines:

A normal crossings subdivision of V containing C in the 1-skeleton will look something the complex shown in Figure 6.8. Using the notation of Section 6.3 we have that C has $m+1$ vertices $P_1, P_{12}, P_{13}, \ldots, P_{1m}, P_2$, where P_1 sits on an axis, and $P_{12}, \ldots, P_2$ are in the interior of $Cone(e_i, e_j)$. For Type

2 tropical lines, there are two components that meet the boundary divisors of $\mathcal{V}$. These components correspond to the vertices P_1 and P_2 at the endpoints of the bounded edge, where P_1 sits on one of the rays in the e_i direction, and P_2 sits in the interior of a 2-cone. Recall that at P_1 we have that $\mathcal{V}_{P_1}$ is $\mathbb{P}^2$ blownup at two distinct points, and our line $\mathcal{C}_{P_1}$ meets only one of the blow-up points. Hence, $\mathcal{C}_{P_1}$ is the strict transform of a general line passing through a blow-up point, which therefore meets the boundary divisors transversely. We then have that $\mathcal{C}_{P_{12}}$ meets $\mathcal{C}_{P_1}$ at $\mathcal{C}_{P_1} \cap \mathcal{V}_{P_{12}}$. We have that $\mathcal{V}_{P_{12}}$ is $\mathbb{P}^2$ blownup at a point and that $\mathcal{C}_{P_{12}}$ meets the blow-up point. Hence, we have that $\mathcal{C}_{P_{12}}$ is the strict transform of a line through $\mathcal{C}_{P_1} \cap \mathcal{V}_{P_{12}}$ and the blow-up point, and is therefore fixed. Similarly, $\mathcal{C}_{P_{13}}$ is the strict transform of a line through $\mathcal{C}_{P_{12}} \cap \mathcal{V}_{P_{13}}$ and the blow-up point of $\mathcal{V}_{P_{13}}$. Hence, $\mathcal{C}_{P_{13}}$ is also fixed, and meets $\mathcal{C}_{P_{12}}$ at a point. Hence, for every vertex in the relative interior of the bounded edge, each component of $\bar{\mathcal{C}}$ is fixed. At P_2 we simply need $\mathcal{C}_{P_2}$ to pass through a fixed point and meet the boundary divisors of $\mathbb{P}^2$ transversely. This is true for a general line through a fixed point. Hence, we have constructed $\bar{\mathcal{C}}$.

Type 2A and Type 3 tropical lines:

For Type 2A and Type 3 tropical lines, the situation is similar to the case of Type 2 tropical lines. For the vertices at the endpoints of bounded edges in the relative interiors of 2-cones the setup is the same, as are the vertices in the interiors of the bounded edges (with exception at the origin). For the vertex at the origin, the component of $\bar{\mathcal{C}}$ misses the boundary of the component of $\bar{\mathcal{V}}$ entirely. In this case, we have that $\mathcal{V}_O$ is $\mathbb{P}^2$ blownup at two distinct points, and that $\mathcal{C}_O$ is the strict transform of a line that must meet both of these points. On C we have O is adjacent to both P_{i1} and P_{j1}. Hence, $\mathcal{C}_{P_{i1}}$ meets $\mathcal{C}_O$ at $\mathcal{C}_O \cap \mathcal{V}_{P_{i1}}$, and similarly for $\mathcal{C}_{P_{j1}}$. As both P_{i1} and P_{j1} are interior points of bounded edges, the construction of components of $\bar{\mathcal{C}}$ works the same as in the case of interior points of Type 2 tropical lines, and likewise for endpoints of bounded edges inside 2-cones.

Notice that the components of $\bar{\mathcal{C}}$ meet the boundary of $\bar{\mathcal{V}}$ only when the corresponding cells of V are contained in C. □

We now proceed to use the above lemma in the proof of Theorem 7.1.2:

Proof. (of Theorem 7.1.2):

To prove this, let us start with a point on the 2-skeleton of σ. We have $C \subset V$, so taking a normal crossings subdivision, we can construct a pre-log curve $\bar{\mathcal{C}}$ that meets the boundary strata of $\bar{\mathcal{V}}$ transversely using Lemma 7.1.3. To realize C, we need to deform $\bar{\mathcal{C}}$ to something contained in $\mathcal{V}$, which is possible only when $\bar{\mathcal{C}} \subset \bar{\mathcal{V}}$. Since $\bar{\mathcal{V}}$ is normal crossings, we have that $\mathcal{V}$ is also normal crossings. Since $\dim(H^1(\underline{\bar{\mathcal{C}}}, \mathcal{N}_f)) = 0$ there are no obstructions to deforming $\bar{\mathcal{C}}$ to something contained in $\mathcal{V}$. This means that there exists $\mathcal{C} \subset \mathcal{V}$. Theorem 4.1.3 and Proposition 4.2 of [9] tells us that $\mathcal{C}$ is in fact a line, which concludes the proof. □

7.2 Lifting points outside the 2-skeleton of the tropical Schubert prevariety

While the above shows that points within the 2-skeleton of the Schubert prevariety are realizable, we claim that for general $\mathcal{V}$, no point outside the 2-skeleton is realizable. To show this, assume that there is in fact a line $\mathcal{C}$ realizing a point on the 3-skeleton of the Schubert prevariety. We take a normal crossings subdivion of $V = Trop(\mathcal{V})$ containing $C = Trop(\mathcal{C})$. We then have by Theorem 4.1.4 that $\bar{\mathcal{C}}$ is smooth with simple normal crossings boundary meeting that of $\bar{\mathcal{V}}$.

Example 12. *Consider the case where the ambient toric variety is $\mathbb{P}^5$, and we have a Type 3 tropical line with three bounded edges. In this case one component of $\bar{\mathcal{V}}$ is $\mathbb{P}^2$ blown-up at three points, and the component of $\bar{\mathcal{C}}$ inside of the blown-up $\mathbb{P}^2$ must meet all of the exceptional divisors at the blown-up points, which implies that the blown-up points must then be collinear. These blow-up points must lie on intersections of boundary divisors of $\mathbb{P}^5$. For example, let our three points be P_0, P_1, P_2 with $P_i \in Z(x_{2i}) \cap Z(x_{2i+1})$. Since all three points must also lie in $\mathbb{P}^2$, for $\mathbb{P}^2$ in general position, these points are not collinear.*

We can generalize the example above to state the following:

Lemma 7.2.1. *Let $\mathcal{V} \subset \mathbb{P}^n$ be a 2-dimensional linear subspace that meets the boundary of $\mathbb{P}^n$ transversely. For k a positive integer, let $I_1 := \{i_1, i_2\}, \dots, I_k := \{i_{2k-1}, i_{2k}\}$ be disjoint 2-element subsets of $\{0, \dots, n\}$, and let $P_j := \mathcal{V} \cap Z(x_a) \cap Z(x_b)$ where $a, b \in I_j$, $a \neq b$. If $k \geq 3$ and $\mathcal{V}$ is in general position, then the P_j are not collinear.*

Proof. It suffices to show the lemma for the case where $k = 3$ (for $k > 3$ the additional points are irrelevant). We can also assume that $n = 5$, since we can simply project $\mathcal{V}$ into $\mathbb{P}^5$, and for $\mathcal{V}$ in general position the projection is an isomorphism of $\mathcal{V}$.

Choosing a linear embedding of $\mathcal{V}$ into $\mathbb{P}^5$ is equivalent to choosing six linear functions, $\ell_1, \ldots, \ell_6$, on $\mathcal{V}$. The first four give us $P_1 := Z(\ell_1) \cap Z(\ell_2)$ and $P_2 := Z(\ell_3) \cap Z(\ell_4)$. If we let L be the line through P_1 and P_2 and let $Q := L \cap Z(\ell_5)$, then we simply need $Q \notin Z(\ell_6)$. The last condition, which says that ℓ_6 does not vanish at Q, is an open condition on $\Gamma(\mathcal{V}, \mathcal{O}(1))$, and so is true for general ℓ_6. Therefore, for $\mathcal{V}$ in general position inside $\mathbb{P}^n$, the P_j are not collinear. $\square$

The above lemma, along with our obstruction result implies

Theorem 7.2.2. *For a plane $\mathcal{V}$ in general position, no point in the relative interior of a cone B,* $\dim\ B \geq 3$*, of the tropical Schubert prevariety is realizable.*

Proof. Let us assume that in fact points of $int(B)$ are realizable. Let $x \in int(B)$ be one such point, and let $\mathcal{C}$ be a line realizing x. We then have that $Trop(\mathcal{C}) \subset Trop(\mathcal{V})$, and we can take a normal crossings subdivision of $Trop(\mathcal{V})$ that contains $Trop(\mathcal{C})$ in its 1-skeleton. In particular, Theorem 4.1.4 then tells us that $\bar{\mathcal{C}}$ is a pre-log curve inside $\bar{\mathcal{V}}$ with simple, normal crossings boundary, whose structure is provided by $Trop(\mathcal{C})$.

We know that $Trop(\mathcal{C})$ is a Type 3 tropical line, which means that the component of $\bar{\mathcal{V}}$ corresponding to the origin is $\mathbb{P}^2$ blown-up in at least three points, and the component of $\bar{\mathcal{C}}$ contained in that component of $\bar{\mathcal{V}}$ must meet all of the exceptional divisors. This is only possible if all of the blow-up points are collinear, which Lemma 7.2.1 tells us is not the case for $\mathcal{V}$ in general position. □

Bibliography

[1] Tristam Bogart and Eric Katz. Obstructions to lifting tropical curves in surfaces in 3-space. *SIAM J. Discrete Math.*, 26(3):1050–1067, 2012.

[2] Erwan Brugalle and Kristin Shaw. Obstructions to approximating tropical curves in surfaces via intersection theory. *preprint arXiv:1101.0297*, 2011.

[3] L. Caporaso and J. Harris. Counting plane curves of any genus. *Invent. Math.*, 131:345–392, 1998.

[4] John B. Little David A. Cox and Henry K. Schenck. *Toric Varieties*. American Mathematical Society, 2011.

[5] Andreas Gathmann. Tropical algebraic geometry. *Jahresber. Deutsch. Math.-Verein.*, 108(1):3–32, 2006.

[6] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, 1977.

[7] David Helm and Eric Katz. Monodromy filtrations and the topology of tropical varieties. *Canad. J. Math.*, 64(4):845–868, 2012.

[8] Fumiharu Kato. Log smooth deformation theory. *Tohoku Math. J. (2)*, 48(3):317–354, 1996.

[9] Eric Katz and Sam Payne. Realization spaces for tropical fans. *Combinatorial aspects of commutative algebra and algebraic geometry, Abel Symp.*, 6:73–88, 2011.

[10] Mark Luxton and Zhenhua Qu. Some results on tropical compactifications. *Trans. Amer. Math. Soc.*, 363(9):4853–4876, 2011.

[11] Grigory Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *J. Amer. Math. Soc.*, 18:313–377, 2005.

[12] Takeo Nishinou and Bernd Siebert. Toric degenerations of toric varieties and tropical curves. *Duke Math J.*, 135(1):1–51, 2006.

[13] David E Speyer. Tropical linear spaces. *SIAM J. Discrete Math.*, 22(4):1527–1558, 2008.

[14] David E Speyer and Bernd Sturmfels. The tropical grassmannian. *Adv. Geom.*, 4(3):389–411, 2004.

[15] Michael Joswig Sven Herrmann, Anders Jensen and Bernd Sturmfels. How to draw tropical planes. *Electron. J. Combin.*, 16(2, Special volume in honor of Anders Bjorner, Research Paper):6–26, 2009.

[16] Jenia Tevelev. Compactifications of subvarieties of tori. *Amer. J. Math.*, 129.

[17] Magnus Dehli Vigeland. Tropical lines on smooth tropical surfaces. *preprint arXiv:0708.3847*, 2007.

[18] Magnus Dehli Vigeland. Smooth tropical surfaces with infinitely many tropical lines. *Ark. Mat.*, 48(1):177–206, 2010.

Vita

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