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Quiver gauge theories, chiral rings and random matrix models

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Quiver gauge theories, chiral rings and random matrix models

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Dedicated to my past lovers, unjustly sacrificed on the altar of science.

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Quiver gauge theories, chiral rings and random matrix models

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Dimensional deconstruction of 5D SQCD with general n_c, n_f and $k_{\rm CS}$ gives rise to 4D $\mathcal{N}=1$ gauge theories with large quivers of $\mathrm{SU}(n_c)$ gauge factors. We first describe the spectrum of the model in the deconstructive limit and show its properties. We then construct the chiral rings of such theories, off-shell and on-shell. Anomaly equations for the various resolvents allowed by the model permit us to calculate all the relevant chiral operators. The results are broadly similar to the chiral rings of single $\mathrm{U}(n_c)$ theories with both adjoint and fundamental matter, but there are also some noteworthy differences such as nonlocal meson-like operators where the quark and antiquark fields belong to different nodes of the quiver. And because the analyzed gauge groups are $\mathrm{SU}(n_c)$ rather than $\mathrm{U}(n_c)$, our chiral rings also contain a whole collection of baryonic and antibaryonic operators. We then investigate the random matrix model corresponding to such chiral ring. We find that bifundamental chiral operators correspond to unitary matrices. We derive the loop equations and show that they are in perfect agreement with the anomaly equations of the gauge model. An exact expression for the free energy is found in the large $\hat{N}$ (rank of the matrix) limit. A formula for the effective superpotential is derived and some examples are illustrated.

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Chapter 1

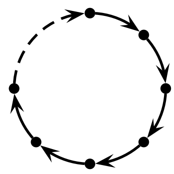
Introduction

1.1 Chiral ring, quivers and matrices

Regardless of their applicability to the weak scale phenomenology, supersymmetric gauge theories are important from the abstract QFT point of view because they allow for exact calculation of some non-perturbative data. In a 4D, $\mathcal{N} = 1$ theory, the exactly-computable non-perturbative data form a mathematical structure known as *the chiral ring.* Physically, this ring contains invariant combinations of the scalar VEVs, gaugino condensates and abelian gauge couplings; usually, these data are sufficient to completely determine the phase structure of the theory and its moduli spaces, if any.

Two recent discoveries excited much interest in the chiral rings of gauge theories with adjoint and fundamental matter fields: first, Dijkgraaf and Vafa [1–3] found that the gaugino condensates and the abelian gauge couplings of 4D, $\mathcal{N} = 1$ gauge theories correspond to perturbative amplitudes of matrix models without any spacetime or SUSY at all. Second, Cachazo, Douglas, Seiberg and Witten [4],[5], [6] evaluated the entire on-shell chiral ring of an $\mathrm{U}(N_c)$ theory with adjoint and fundamental matter using generalized Konishi anomaly equations [7, 8]. In the process, they verified the Dijkgraaf-Vafa conjecture by showing that the loop equation of the matrix model is identical to the anomaly equation for a particular resolvent $R(X)$ summarizing the gaugino condensates. Both approaches — the matrix models and the anomaly equations — are readily extended to the *quiver* theories with multiple gauge group factors "connected" by the bi-fundamental matter fields. Thus far, most work on this subject concerned the non-chiral quivers where the bi-fundamental fields come in $(\bar{\mathbf{n}}, \mathbf{m}) + (\mathbf{n}, \bar{\mathbf{m}})$ conjugate pairs; this is a natural limitation of the matrix correspondence[1] but the anomaly-equations technology has no particular difficulties with 4D chirality [9]–[10].

In this dissertation we analyze the inherently chiral $\widehat{A}_n$ quivers of one-way arrows



which means that all the bi-fundamental fields are chiral $(\mathbf{n}_i, \bar{\mathbf{n}}_{i+1})$ multiplets *not* accompanied by their $(\bar{\mathbf{n}}_i, \mathbf{n}_{i+1})$ conjugates. Also, our quivers do not have any adjoint matter fields — although the cyclic product of all the bi-fundamental fields does act as some kind of a collective adjoint multiplet. Finally, the individual gauge groups corresponding to our quivers' nodes are of the $\mathrm{SU}(n_c)$ rather than $\mathrm{U}(n_c)$ type, and this adds all kinds of *baryonic* generators to the chiral ring of the theory.

[1]An un-constrained complex $n \times m$ matrix corresponds to a whole $\mathcal{N} = 2$ hypermultiplet in the bi-fundamental $(\bar{\mathbf{n}}, \mathbf{m})$ representation of the $\mathrm{SU}(n) \times \mathrm{SU}(m)$, or in $\mathcal{N} = 1$ terms to a conjugate pair $(\bar{\mathbf{n}}, \mathbf{m}) + (\mathbf{n}, \bar{\mathbf{m}})$ of chiral multiplets. A chiral bi-fundamental $(\bar{\mathbf{n}}, \mathbf{m})$ multiplet *not* accompanied by its $(\mathbf{n}, \bar{\mathbf{m}})$ conjugate corresponds to a matrix subject to non-linear constraints, which makes for a much more complicated matrix model. In particular, the chiral $[\mathrm{SU}(n_c)]^N$ quiver theory presented in this dissertation corresponds to a model of N unitary $\mathrm{SU}(n_c)$ matrices. This matrix model — and its implications for the gaugino condensates — will be presented in the second chapter of this dissertation.

Specifically, our chiral quiver theories follow from the *dimensional deconstruction* of the 5D SQCD [11]–[12], hence the name *deconstructive quivers.* In general, dimensional deconstruction [11, 13] relates simple gauge theories in spaces of higher dimension to more complicated theories in fewer dimensions of space: the extra dimensions of space are 'deconstructed' into quiver diagrams of the 'theory space'. For example, starting in $4+1$ dimensions, we deconstruct the extra space dimension by discretizing the x^4 coordinate into a lattice of small but finite spacing a and then interpreting the result as a 4D gauge theory with a large *quiver* of gauge groups. In order to have a finite number of 4D fields, the x^4 is also compactified to a large circle of length $2\pi R = Na$ (hence N lattice points), but eventually one may take the $N \to \infty$ limit and recover the uncompactified 5D physics. In this limit, the lattice spacing a remains finite and serves as UV regulator which breaks part of the 5D Lorentz symmetry as well as 4 out of 8 supercharges but preserves the (latticized) 5D gauge symmetry of the theory.

1.1.1 Deconstructive quiver

The deconstruction of SQCD_5 will be discussed in more detail in a separate paper [12] (see also [14] for the quarkless case). From the 4D point of view, the result is an $\mathcal{N}=1$ supersymmetric gauge theory with a quiver diagram Each green circle of this diagram corresponds to a simple $\mathrm{SU}(n_c)_\ell$

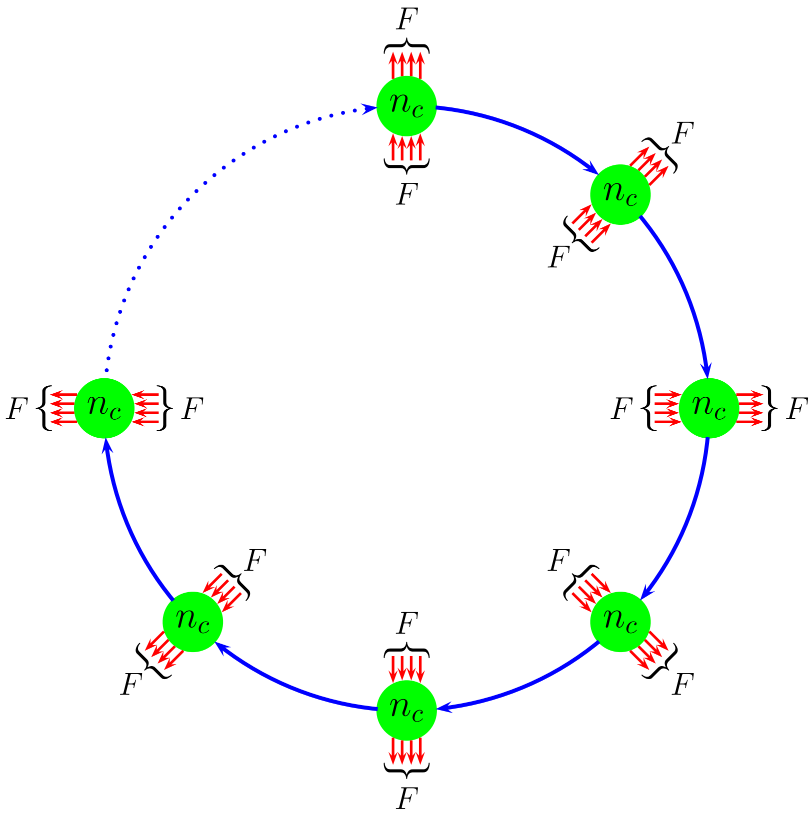


Figure 1.1: Diagram of the chiral quiver

factor of the net 4D gauge group

$$G_{4\text{D}} = \prod_{\ell=1}^{N} [\text{SU}(n_c)]_\ell \tag{1.1}$$

while the red and blue arrows denote the chiral superfields:

$$\begin{aligned} \} \text{ quarks } Q_{\ell,f} &= (\Box_\ell), \\ \} \text{ antiquarks } \widetilde{Q}^f_\ell &= (\overline{\Box}_\ell), \\ \longrightarrow \text{ bifundamental link fields } \Omega_\ell &= (\Box_{\ell+1}, \overline{\Box}_\ell), \end{aligned} \tag{1.2}$$

where $f = 1, 2, \ldots, F$ and $\ell = 1, 2, \ldots, N$ is understood modulo N. From the 4D point of view, N is a fixed parameter of the quiver theory; in our analysis we shall assume N to be large but finite.

Similar to many other deconstructed theories, the quiver (1.1) can be obtained by orbifolding a simple 4D gauge theory with higher SUSY, namely $\mathcal{N} = 2$ SQCD with F flavors and $N \times n_c$ colors: The $\mathbb{Z}_N$ twist removes the extra supercharges and reduces the gauge symmetry from $\mathrm{SU}(N \times n_c)$ down to

$$S([\mathrm{U}(n_c)]^N) = [\mathrm{SU}(n_c)]^N \times [\mathrm{U}(1)]^{N-1} \,. \tag{1.3}$$

However, the abelian photons of the orbifold theory suffer from triangular anomalies and therefore must be removed from the effective low-energy theory. In string theory such removal is usually accomplished via the Green-Schwarz terms [15], but at the field theory level we simply discard the abelian factors of the orbifolded symmetry (1.3) and interpret the nodes (green circles) of the quiver diagram (1.1) as purely non-abelian $\mathrm{SU}(n_c)_\ell$ factors.

In this dissertation we study chiral rings of deconstructed SQCD_5 theories with generic numbers of colors and flavors, Chern-Simons levels, and sizes of the compact fifth dimension. In 4D terms, this means $[\mathrm{SU}(n_c)]^N$ theories with quiver diagrams of the general form (1.1), but with most general numbers n_c, F, N, as well as generic quark masses; for the sake of 4D chirality we assume $N \geq 3$ quiver nodes. We shall see that the chiral rings of such theories resemble the rings of refs. [4]–[6] but also have two novel features: first, our rings have meson-like generators involving the quark and the antiquark fields belonging to different nodes of the quiver. Such operators are non-local from the 5D point of view, but in 4D terms they are legitimate generators of the chiral ring. Second, in the absence of abelian gauge fields, all kinds of baryonic and antibaryonic operators are gauge-invariant and thus also belong to the chiral ring; in fact, there is a whole zoo of such operators.

1.1.2 Random matrix model

To the chiral ring of the gauge model should correspond a random matrix model of some kind. Until now it has been shown in a series of papers [1–4] how a group of non-chiral gauge theories are well described by hermitian random matrix models. Specifically, the great majority of such models contain one or more scalar fields in the adjoint representation. Such fields typically correspond to hermitian matrices or more generally matrices whose eigenvalues are one variable holomorphic functions [16]. Chiral gauge models cannot be described using hermitian matrices but have to rely on a different set of random variables. The natural choice have fallen on the set of unitary matrices. In particular to each bi-fundamental link variable Ω_ℓ correspond a unitary matrix U_ℓ, while to the flavors $Q_{\ell,f}$ and $\widetilde{Q}_\ell^f$ correspond complex matrices B_ℓ and A_ℓ (constrained by the condition $B^\dagger = A$). Such set, considered as a sub-manifold of the set of all $\hat{N} \times \hat{N}$ complex matrices, admit a series of loop equations that match perfectly the anomaly equations of the gauge theory.

Once established the correspondence a functional can be build for the random matrix model using as action the superpotential of the gauge model. First the flavor matrices are integrated out under the assumption that are massive. Then a series of transformation reduce the integration over all the U_ℓ to an integration over a single unitary matrix $\mathcal{U}$. The model now can be solved in the large $\hat{N}$ limit using standard resolvent techniques in complex space. In such limit we recover an expression for the free energy made out of two contribution: the sphere and the disk. Since the eigenvalues of the unitary matrix $\mathcal{U}$ are on the unit circle a general expression for the free energy is derived on the cylinder. From previous works presented by Cachazo et Al. [4],[5],[6] a formula for the derivation of the effective superpotential for the gaugino condensates is used. A particular simple example illustrates the quasi-polynomial nature of such superpotential.

1.2 Presentation of the material

The main part of this dissertation (after this introduction) is organized as follows: in section 2.1 we study the deconstructive $[\mathrm{SU}(n_c)]^N$ quiver theories at the classical level. First, in section 2.1.1 we spell out the superpotentials and explain how the 4D quiver theories deconstruct the 5D SQCD. Next, in section 2.1.2 we survey the classical vacua of the quiver theories and find that they form the same Coulomb, mesonic, and baryonic moduli spaces as the 5D theories compactified on a large circle. And then in section 2.1.3 we break the correspondence by deforming the 4D superpotentials in order to trigger the gaugino condensation at the quantum level. This deformation is similar to the adjoint field's superpotential in the single-$\mathrm{U}(n_c)$ theory and has similar consequences for the classical vacua of a quiver theory: the Coulomb moduli space breaks up into a large discrete set of isolated vacua.

In section 2.2 we study the fully-quantum $[\mathrm{SU}(n_c)]^N$ quiver theories and derive their chiral rings, or rather the non-baryonic sub-rings. In section 2.2.1 we construct the off-shell chiral rings (un-constrained by the anomalous equations of motion). Similar to the single-U(n_c) theory of [4]–[6], the non-baryonic generators of the quivers' off-shell rings combine into several resolvents where the cyclic-ordered product $\Omega_N\Omega_{N-1}\cdots\Omega_2\Omega_1$ of the bi-fundamental link fields plays the role of the adjoint field Φ. The main difference from the single-U(n_c) theory is a much richer set of mesonic generators and hence resolvents: in a quiver theory, the quark and the antiquark fields of a meson-like chiral operator may belong to different quiver nodes as long as they are connected by a chain of link fields which maintain the gauge invariance. From the 5D point of view such operators are non-local and they create / annihilate un-bound $\bar{q}q$ pairs, but in 4D they are local, chiral and gauge invariant and thus do belong to the chiral ring.

In section 2.2.2 we turn to the on-shell chiral rings: we calculate the generalized Konishi anomalies for suitable variations of the quark, antiquark and link fields of the quiver theories and derive the anomalous equations of motions for all the resolvents. We solve the equations in terms of a few polynomials and find that all the on-shell resolvents are single-valued on the same hyperelliptic Riemann surface $\mathbf{\Sigma}$ defined by the quadratic equation (2.103) for the gaugino-bilinear resolvent. Physically, $\mathbf{\Sigma}$ is the Seiberg-Witten curve [17] of the theory, and in section 2.2.3 we use analytic considerations to show that it indeed looks like the SW curve of the SQCD$_5$ compactified on a circle [18, 19]. We also find that this curve is completely determined at the level of *one diagonal instanton*, that is one instanton of the $\mathrm{SU}(n_c)_{\mathrm{diag}} \equiv \mathrm{diag}[\mathrm{SU}(n_c)^N]$ or equivalently one instanton in each and every $\mathrm{SU}(n_c)_\ell$ factor of the total 4D gauge group. Finally, we study the vacua of the quantum quiver theories and show that in the weak coupling limit we have exactly the same Coulomb, Higgs, pseudo-confining, *etc.*, vacua as expected in a semisclassical theory, but in the strong coupling regime all vacua with similar numbers of massless photons are interchangeable by the monodromies in the parameter space of the theory.

In section 2.3 we complete the chiral rings by adding all kinds of baryonic, antibaryonic and other generators with non-trivial $[\mathrm{U}(1)_B]^N$ quantum numbers. In section 2.3.1 we warm up by studying baryon-like generators in a theory with a single $\mathrm{SU}(n_c)$ gauge group, n_f quarks and antiquarks and an adjoint field Φ: Because the gauge group is $\mathrm{SU}(n_c)$ rather than $\mathrm{U}(n_c)$, the chiral baryon operators are gauge invariant, and so are the Φ-baryons comprised of n_c quarks plus any number of adjoint fields Φ. Off-shell, such operators exist for any $n_f \geq 1$, and we summarize them in baryonic resolvents. On-shell however, the Φ-baryons follow from the ordinary baryons, hence no baryonic VEVs whatsoever for $n_f < n_c$, and even for $n_f \geq n_c$ baryonic VEVs exist only for the classical-like branches of the moduli space.

In section 2.3.2 we analyze the baryonic generators of the $[\mathrm{SU}(n_c)]^N$ quiver theory. Off-shell, we find a whole zoo of baryon-like generators comprised of n_c quarks belonging to different quiver nodes and connected to each other by chains of link fields — and each chain may wrap a few times around the whole quiver to emulate the Φ fields of Φ-baryons. Again, there is a big pile of independent baryon-like operators for any $n_f \geq 1$, but only off-shell.

On-shell, we tame the zoo by solving the equations of motions for the baryonic resolvents and showing that all baryon-like VEVs follow from those of ordinary baryonic operators (all quarks at the same node and no link fields). Consequently, all baryonic VEVs of the quantum theory follow the classical rules: they require $n_f \geq n_c$ as well as overdetermined Coulomb moduli of the baryonic branch. However, the precise constraint on the quark masses due to baryonic branch's existence is subject to quantum corrections at the one-diagonal-instanton level.

In section 2.3.3 we calculate quantum corrections to the determinants of link chains,

$$\det\Big(\Omega_{\ell_2}\cdots\Omega_{\ell_1}\Big) = \det(\Omega_{\ell_2}) \times \cdots \times \det(\Omega_{\ell_2}) + \text{corrections}. \qquad (1.4)$$

We find that for chains of less than N links the corrections come from instantons in the individual $\mathrm{SU}(n_c)_\ell$ gauge groups rather than the diagonal instantons, but the determinant $\det(\Omega_N \cdots \Omega_1)$ of the whole quiver is subject to separate individual-instanton and diagonal-instanton corrections. We evaluate the corrections and summarize their effect on the Coulomb moduli space of a deconstructive quiver theory. A particularly technical part of our analysis is removed to the 4.2 of this dissertation. We conclude the chapter with 4.1 where we discuss the open questions related to the present research.

In section 3.1 we explain why the choice of matrix valued random variables falls on the unitary set. We illustrate the guiding principles that lead us to such a choice. The set up of the loop equations in this context is far from trivial since the deformation of a generic matrix U brings us outside the unitary manifold. This apparent problem is overcome forcing the deformation to be holomorphic. The resulting loop equation will be, then, perfectly well defined. We illustrate the general way of proceeding in the simplest case before jumping to the quiver model.

In section 3.2 we derive the set of all loop equations for the quiver. We deform in all possible allowed ways the matrices U, A and B inside the functional $\mathcal{Z}$. The set of equations derived matched perfectly with the anomaly equations calculated in section 2.2.2. This result establish an exact correspondence between the chiral quiver gauge theory and the proposed random matrix model. This is the first case in which such correspondence is verified and gives credibility to a larger duality that is believed to exist between gauge theories and random matrix models.

In section 3.3 we solve for the free energy of the matrix model. We first integrate out the massive degrees of freedom coming from the flavors. Then we show how the functional integral reduces to a simpler one where all the link matrices integrations are reduced to only one $\mathcal{U}$. Since the eigenvalues of the $\mathcal{U}$ lies on the unit circle we introduce a resolvent $R(w)$ defined on it. In the large $\hat{N}$ limit the steepest descent method can be used to derive an integral expression for the free energy. The eigenvalues are then constrained by the saddle point equation that corresponds to the first loop equation for $R(w)$ derived in section 3.2. A general expression for the free energy and the superpotential is given in terms of contour integration on the cylinder.

Finally in section 3.4 a simple example of superpotential is calculated.

Chapter 2

Chiral rings of deconstructive $[\mathrm{SU}(n_c)]^N$

2.1 Deconstructive quiver theories and their classical vacua

Deconstruction of SQCD_5 with general numbers of colors and flavors will be explained in much detail in the companion paper [12]. In this section, we summarize the salient features of the deconstructed theory from the 4D point of view. In the immediately following section 2.1.1 we write down the superpotential of the 4D theory and briefly explain how deconstruction works for the most symmetric vacuum of the 5D theory. Of course, the $[\mathrm{SU}(n_c)]^N$ quiver theory has many other classical vacua, and we describe them in section 2.1.2. Finally, in section 2.1.3 we deform the quiver's superpotential in order to trigger some kind of a gaugino condensation and describe the effects of this deformation on the quiver's vacua. (The gaugino condensates themselves will be discussed in the later section 2.2).

2.1.1 Deconstruction summary

The 4D gauge theory of the deconstructed SQCD_5 is

$$G_{4\mathrm{D}} = \prod_{\ell=1}^{N} [\mathrm{SU}(n_c)]_\ell \tag{2.1}$$

with equal gauge couplings $g_\ell \equiv g$ for all the factors to assure discrete translation invariance in the x^4 direction. The chiral superfields comprise the quarks, the antiquarks, and the bilinear link fields specified in the table (1.2), and also singlets s_ℓ (one for each $\ell = 1, 2, \ldots, N$). The superpotential has two distinct parts, $W = W_{\text{hop}} + W_\Sigma$, where

$$W_{\text{hop}} = \gamma \sum_{\ell=1}^{N} \sum_{f=1}^{F} \Big(\widetilde{Q}^f_{\ell+1} \Omega_\ell Q_{\ell,f} - \mu_f \, \widetilde{Q}^f_\ell Q_{\ell,f} \Big) \tag{2.2}$$

facilitates the propagation of the (anti) quark fields in the x^4 direction, while

$$W_\Sigma = \beta \sum_{\ell=1}^{N} s_\ell \, (\det \Omega_\ell \, - \, v^{n_c}) \tag{2.3}$$

sets up an $\mathrm{SL}(n_c, \mathbb{C})$ (AKA complexified $\mathrm{SU}(n_c)$) linear sigma model at each link of the latticized 5D theory.[1] Disregarding the massive "radial" mode, we have

$$\Omega_\ell(x) = v \times \underset{\substack{\text{Path}\\ \text{ordered}}}{\exp} \left(\int_{a\ell}^{a(\ell+1)} dx^4 \Big(i A_4(x) + \phi(x) \Big) \right) + \text{fermionic terms} \tag{2.4}$$

where $A_\mu(x)$ and $\phi(x)$ are the 5D vector fields and their scalar superpartners. The simplest 5D vacuum with $\phi \equiv A^4 \equiv 0$ corresponds to the 4D field configuration

$$\langle \Omega_\ell \rangle \equiv v \times 1_{n_c \times n_c} \tag{2.5}$$

which Higgses the 4D gauge symmetry $[\mathrm{SU}(n_c)]^N$ down to its diagonal subgroup $\mathrm{SU}(n_c)_{\text{diag}} = \operatorname{diag}(\prod_\ell \mathrm{SU}(n_c)_\ell)$. The rest of the 4D vectors acquire

[1] Although the β coupling is non-renormalizable for $n_c > 2$, all the resulting divergences can be regularized via higher-derivative lagrangian terms for the singlet fields s_ℓ without disturbing the chiral ring of the theory.

masses

$$M^2(k) = 4g^2|v|^2 \sin^2 \frac{\pi k}{N} = \frac{4}{a^2} \sin^2 \frac{aP_4}{2} \tag{2.6}$$

where the second equality follows from identifying the lattice spacing a as

$$a = \frac{1}{g|v|} \tag{2.7}$$

and the lattice momentum P_4 as

$$P_4 = \frac{2\pi k}{Na}\,, \qquad k = 1, 2, \ldots \text{ modulo } N\,. \tag{2.8}$$

In the large-N limit, the bottom end of the spectrum (2.6) becomes a Kaluza-Klein tower

$$M^2 \;\approx\; P_4^2 = \left(\frac{2\pi k}{Na}\right)^2 \tag{2.9}$$

of a massless relativistic 5D vector field (compactified on a circle of length $2\pi R = Na$); this is the momentum-space view of the dimensional deconstruction.

Similarly, the low-energy end of the 4D quark spectrum comprises Kaluza-Klein towers for relativistic 5D hypermultiplets of masses $m_f \ll (1/a)$. Indeed, for a quark flavor of 4D mass $\gamma\mu_f$, the mass matrix for the $Q_{\ell,f}$ and $\widetilde{Q}^f_\ell$ fields ($\ell = 1, 2, \ldots, N$) has eigenvalues

$$M^2(f, k) = |\gamma|^2 \left| ve^{2\pi ik/N} \;-\; \mu_f \right|^2, \tag{2.10}$$

and *for* $|\mu_f| \approx |v|$, the bottom end of this spectrum becomes (in the large-N limit)

$$M^2 \approx m_{5\mathrm{D}}^2 + P_4^2 \tag{2.11}$$

where

$$m_{5\text{D}} = |\gamma| \times \Big(|\mu_f| - |v|\Big) \ll \frac{1}{a}\,, \tag{2.12}$$

$$P_4 = \frac{2\pi k}{Na} + \text{constant (Wilson line)}\,, \tag{2.13}$$

and

$$a = \frac{1}{|\gamma v|}\,. \tag{2.14}$$

The 4D quarks with $|\mu_f| \not\approx |v|$ do not have low-mass $M^2 \ll (1/a)^2$ modes and do not deconstruct any 5D particles. For $\mu_f \gg v$, the 4D quark decouples above the deconstruction threshold $(1/a)$ and has no low-energy effect whatsoever, but quarks with $\mu_f \ll v$ decouple at the threshold itself and modify the Chern-Simons interactions of the deconstructed SQCD_5. Since the Chern-Simons level k_{cs} affects the moduli space geometry and even the phase structure of the 5D theory, it must be deconstructed correctly. Thus, the deconstructive quiver should have

$$F = n_f + \Delta F \geq n_f \tag{2.15}$$

4D flavors, where n_f is the number flavors in 5D,

$$\Delta F = n_c - \frac{n_f}{2} - k_{\text{cs}}\,, \qquad 0 \leq \Delta F \leq 2n_c - n_f\,, \tag{2.16}$$

and

$$\mu_f = v \times \exp(am_f^{5\text{D}}) \sim v \qquad \text{for } f \leq n_f \qquad \text{but} \qquad \mu_f \ll v \qquad \text{for } f > n_f\,. \tag{2.17}$$

However, from the holomorphic, purely-4D point of view, there is no qualitative difference between $\mu_f \sim v$ and $\mu_f \ll v$, but there is a difference between

$\mu_f \neq 0$ and between $\mu_f = 0$. Hence, for the purposes of this article, we shall assume

$$\begin{aligned} &\text{generic } \mu_f \neq 0 \qquad \text{for } f = 1, 2, \ldots, n_f \\ \text{but} \qquad &\mu_f = 0 \qquad \text{for } f = (n_f + 1), \ldots, F\,. \end{aligned} \tag{2.18}$$

Note that consistency between eqs. (2.7) and (2.14) requires equal gauge and Yukawa couplings, $g = |\gamma|$. In a quantum theory, this means equality of the renormalized physical couplings,

$$g^{\text{phys}} = |\gamma|^{\text{phys}}\,, \tag{2.19}$$

or in non-perturbative terms, in the very low energy limit $E \ll (1/Na)$ the effective theory (the diagonal $\mathrm{SU}(n_c)$ with an adjoint field Φ and several quark flavors) should be $\mathcal{N} = 2$ supersymmetric. Without this condition, the deconstructed theory would have quarks and gluons with different effective speed of light in the x^4 direction. This is a common problem in lattice theories with some continuous dimensions (*eg.* hamiltonian lattice theories with continuous time but discrete space), and the common solution is fine-tuning of the lattice parameters. For the deconstructed SQCD_5, the fine tuning involves the Kähler parameters (such as coefficients of the quarks', antiquarks' and links' kinetic-energy lagrangian terms) and does not affect any of the holomorphic properties of the quiver such as its chiral ring. Consequently, *in the present article* we may disregard eq. (2.19) and treat the holomorphic γ and the gauge coupling g (or rather its dimensional transmutant Λ) as free parameters of the quiver theory.

We conclude this section by acknowledging that there are many ways to skin a cat or to deconstruct SQCD_5 with given n_c, n_f and k_{cs}. For example, one can pack several 5D quark flavors into a single 4D flavor with a complicated dispersion relation $M_{\mathrm{4D}}(P_4)$ by generalizing the hopping superpotential (2.2) to allow the quark to hop over several lattice spacing at once. Indeed,

$$W_{\mathrm{hop}} = \sum_{q=0}^{p} \Gamma_p \sum_{\ell=1}^{N} \widetilde{Q}_{\ell+q} \Omega_{\ell+q-1} \cdots \Omega_\ell Q_\ell \tag{2.20}$$

endows a single 4D flavor with p light modes when the polynomial

$$H_p(x) \;=\; \sum_{q=0}^{p} \Gamma_p x^p \tag{2.21}$$

has all p of its roots $\mu_1, \ldots, \mu_p$ located close to the circle $|x| = |v|$. We found however that such p-fold quarks are pretty much equivalent to p ordinary flavors with masses $\mu_1, \ldots, \mu_p$. Consequently, we shall henceforth stick to the $p = 1$ model (2.2) because of its twin virtues of relative simplicity and renormalizability.

2.1.2 The classical vacua

As explained in [12], the classical moduli space of the $[\mathrm{SU}(n_c)]^N$ quiver theory has exactly the same Coulomb, Higgs and mixed branches as the un-deconstructed SQCD_5. The Coulomb branch is distinguished by zero classical values of the quark and antiquark scalars while the link fields have non-zero VEVs subject to D/F term constraints

$$\forall\ell: \quad \langle\Omega_\ell\rangle^\dagger \langle\Omega_\ell\rangle - \langle\Omega_{\ell-1}\rangle \langle\Omega_{\ell-1}\rangle^\dagger \propto 1_{n_c\times n_c}\,, \qquad \mathrm{Det}\,\langle\Omega_\ell\rangle = v^{n_c}\,. \tag{2.22}$$

Consequently, all the $\langle\Omega_\ell\rangle$ matrices are equal and diagonal modulo an ℓ-dependent gauge transform:

$$\forall\ell: \quad \langle\Omega_\ell\rangle = \mathrm{diag}\,(\omega_1, \omega_2, \ldots, \omega_{n_c}) \tag{2.23}$$

for some complex moduli $(\omega_1, \omega_2, \ldots, \omega_{n_c})$ satisfying $\prod_j \omega_j = v^{n_c}$. Note that each ω_j is gauge-equivalent to $\omega_j \times \sqrt[N]{1}$, hence the $(\omega_1^N, \omega_2^N, \ldots, \omega_{n_c}^N)$ makes a better coordinate system for the moduli space, although it's still redundant with respect to permutations of the ω_j^N. Generically, all the ω_j^N are distinct and the 4D gauge symmetry is broken all the way down to the Cartan $(\mathrm{U}(1))^{n_c-1}$ subgroup of the $\mathrm{SU}(n_c)_{\mathrm{diag}} = \mathrm{diag}\Big[\prod_\ell \mathrm{SU}(n_c)_\ell\Big]$, but a non-abelian subgroup $\mathrm{SU}(k) \subset \mathrm{SU}(n_c)_{\mathrm{diag}}$ survives un-Higgsed when k of the ω_j^N happen to coincide.

According to the deconstruction map (2.4),

$$\omega_j = v \times \exp(a(\phi_j + iA_j^4)) \tag{2.24}$$

where ϕ_j are the real 5D moduli scalars and $A_j^4 \times Na$ are the Wilson lines of the diagonal gauge fields around the deconstructed dimension. Of course, the deconstruction works only for $\phi_j \ll (1/a) \implies \omega_j \sim v$, but this restriction does not affect the 4D theory as such.

Unlike the Coulomb branch which exists for any quark masses, the Higgs and the mixed branches require coincidences between the μ_f or rather among the non-zero μ_f^N. For example, for $\mu_1^N = \mu_2^N \neq 0$ there is a mixed mesonic branch where one of the ω_j^N is frozen at the same value. Indeed, let

$$\omega_1 = e^{2\pi i k_1/N}\,\mu_1 = e^{2\pi i k_2/N}\,\mu_2 \tag{2.25}$$

for some integer (k_1, k_2); then the quark mass matrix due to the superpotential (2.2) allows for the squark VEVs

$$\left\langle Q^j_{\ell,f} \right\rangle = e^{2\pi i k_f \ell/N} Q_f\,, \qquad \left\langle \widetilde{Q}^f_{\ell,j} \right\rangle = e^{-2\pi i k_f \ell/N} \widetilde{Q}^f \tag{2.26}$$

for j=1 and f=1,2 only, subject to F-term and D-term constraints

$$Q_1\,\widetilde{Q}^1 + Q_2\,\widetilde{Q}^2 \;=\; 0\,, \tag{2.27}$$

$$(|Q_1|^2 - |\widetilde{Q}^1|^2) + (|Q_2|^2 - |\widetilde{Q}^2|^2) \;=\; 0\,. \tag{2.28}$$

These VEVs Higgs the $(\mathrm{SU}(n_c))^N$ symmetry down to $(\mathrm{SU}(n_c-1))^N$, which is further broken by the link VEVs $\langle\Omega_\ell\rangle$ down to a subgroup of the $\mathrm{SU}(n_c - 1)_{\mathrm{diag}} = \mathrm{diag}\Big[\prod_\ell \mathrm{SU}(n_c-1)_\ell\Big]$. For generic values of the un-frozen Coulomb moduli $(\omega_2^N, \ldots, \omega_{n_c}^N)$, the surviving gauge symmetry is $\mathrm{U}(1)^{n_c-2}$, but coincidences among these moduli allow for un-Higgsing of a non-abelian $\mathrm{SU}(k) \subset \mathrm{SU}(n_c-1)_{\mathrm{diag}}$.

The mesonic branch of the quiver deconstructs the mesonic branch of the SQCD_5 where $\phi_1 = m_1^{5\mathrm{D}} = m_2^{5\mathrm{D}}$. Although the deconstruction requires $\phi_j \ll (1/a)$ and hence $m_{1,2}^{5\mathrm{D}} \ll (1/a)$, this restriction does not affect the 4D theory as such. In 4D, a coincidence $\mu_1^N = \mu_2^N$ gives rise to a a mesonic branch regardless of whether $\mu_{1,2}$ is larger, smaller, or similar to v, *as long as* $\mu_{1,2}^N \neq 0$. On the other hand, having two or more exactly massless 4D flavors (*i.e.*, $\Delta F \geq 2$) does not lead to a mesonic branch of the $[\mathrm{SU}(n_c)]^N$ quiver because the link eigenvalues ω_j cannot vanish. (Note the constraint $\det(\Omega_\ell) = v^{n_c} \neq 0$.)

Further coincidences among the μ_f^N allow for multi-mesonic mixed branches with more squark VEVs, more frozen Coulomb moduli ω_j^N (*eg.*, $\omega_1^N = \mu_1^N = \mu_2^N$, $\omega_2^N = \mu_3^N = \mu_4^N$), and a lower rank of the un-Higgsed gauge symmetry. Such multi-mesonic branches work similarly to the single-meson mixed branch, so we need not discuss them any further. Instead, let us consider the purely-Higgs baryonic branch which exists for $n_f \geq n_c$ when n_c of the 5D quark masses add to zero, or in 4D terms, when the product of n_c of the μ_f^N happens to equal to the $(v^{n_c})^N$. Indeed, for $\mu_1^N \times \mu_2^N \times \cdots \times \mu_{n_c}^N = v^{Nn_c}$ we may freeze *all* of the Coulomb moduli at

$$\omega_j = e^{2\pi i k_j/N} \mu_j \qquad \forall j = 1, 2, \ldots, n_c \,, \qquad k_j \in \mathbb{Z}\,, \tag{2.29}$$

which gives zero modes to all quark colors $j = 1, \ldots, n_c$ for $f = j$ and allows non-zero VEVs

$$\left\langle Q^j_{\ell,f} \right\rangle = \delta^j_f e^{2\pi i k_j \ell/N} Q^j\,, \qquad \left\langle \widetilde{Q}^f_{\ell,j} \right\rangle = \delta^f_j \, e^{-2\pi i k_j \ell/N}\, \widetilde{Q}_j \tag{2.30}$$

subject to the D term constraint

$$\text{same } (|Q^j|^2 \, - \, |\widetilde{Q}_j|^2) \; \forall j \tag{2.31}$$

and the F-term constraint

$$\frac{\partial W}{\partial \Omega^j_{\ell,j}} = \gamma e^{-2\pi i k_j/N}\, Q^j \widetilde{Q}_j + \beta s_\ell \times \frac{v^{n_c}}{\omega_j} = 0\,. \tag{2.32}$$

The simplest solutions to these constraints are either $\widetilde{Q}_j \equiv 0$, same $Q^j \equiv Q \forall j$ (baryonic VEVs only) or *vice verse* $Q^j \equiv 0$, same $\widetilde{Q}_j \equiv \widetilde{Q} \; \forall j$ (antibaryonic VEVs only), but thanks to the singlet fields s_ℓ enforcing the $\det \Omega_\ell = v^{n_c}$

constraints, there are other solutions where both baryonic and antibaryonic VEVs are present at the same time. In the deconstruction limit $\phi_j = m_j^{5\mathrm{D}} \ll (1/a) \Longrightarrow \omega_j \approx v$ (up to a phase) in eq. (2.32), we have $Q^j \equiv Q$, $\widetilde{Q}_j \equiv \widetilde{Q}$ and hence

$$\left\langle Q^j_{\ell,f} \right\rangle = \delta^j_f e^{2\pi i k_j \ell/N} \times Q\,, \qquad \left\langle \widetilde{Q}^f_{\ell,j} \right\rangle = \delta^f_j e^{-2\pi i k_j \ell/N} \times \widetilde{Q} \tag{2.33}$$

for some arbitrary pair $(Q, \widetilde{Q})$ of complex moduli which deconstruct the baryonic hyper-modulus of the SQCD_5. Outside the deconstruction limit, the 4D baryonic branch exists anyway, albeit with more complicated anti/squark VEVs. In any case, there are two complex moduli and the $[\mathrm{SU}(n_c)]^N$ gauge symmetry is completely Higgsed down.

This completes our survey of the classical moduli space of the $[\mathrm{SU}(n_c)]^N$ quiver. The bottom line is, all the classical vacua of this quiver are deconstructive, *i.e.* correspond to the SQCD_5's vacua according to the deconstruction map (2.4) and the zero modes of the massless 5D gauge bosons match all the massless 4D vector fields.

2.1.3 Deforming the superpotential

In the extreme infrared limit $E \ll (1/Na)$, the $\mathcal{N} = 1$ quiver theory reduces to the $\mathcal{N} = 2$ SQCD_4 with several flavors, and thanks to this extra supersymmetry, the gauginos do not form bilinear condensates. According to Cachazo, Douglas, Seiberg and Witten [4–6, 20], the gaugino condensates play a key role in the chiral ring of the $\mathrm{U}(n_c)$ theory with an adjoint chiral field Φ. Indeed, the best way for understanding the $\mathcal{N} = 2$ SQCD_4 involves deforming the theory to $\mathcal{N} = 1$ (via superpotential $\operatorname{tr} \mathcal{W}(\Phi)$ for the adjoint field) in order to turn on the gaugino condensation, although eventually, *after* solving the anomaly equations of the chiral ring of the deformed theory (including the chiral gaugino condensates) one may turn off the deformation and return to $\mathcal{N} = 2$ SUSY.

In the quiver theory, the role of the adjoint field Φ is played by the quiver-ordered product $(\Omega_N \cdots \Omega_2 \Omega_1)$ of the link fields. Hence, to study the chiral ring of the quiver, we need to temporarily deform the superpotential according to

$$W = W_{\text{hop}} + W_{\Sigma} \longrightarrow W = W_{\text{hop}} + W_{\Sigma} + W_{\text{def}} \tag{2.34}$$

where

$$W_{\text{def}} = \operatorname{tr} \mathcal{W}(\Omega_N \Omega_{N-1} \cdots \Omega_2 \Omega_1) \stackrel{\text{def}}{=} \sum_{k=1}^{d} \frac{\nu_k}{k} \operatorname{tr} \Big((\Omega_N \Omega_{N-1} \cdots \Omega_2 \Omega_1)^k \Big) . \tag{2.35}$$

Although this deformation does not make any sense from the 5D point of view — indeed, it is utterly non-local in the x^4 direction — as well as grossly non-renormalizable in 4D, it does lead to non-zero gaugino condensates which will help us later in section 2.2. But before we study such non-perturbative effects, we need to know the effect of the deformation (2.34) on the classical vacua of the theory.

The general effect is similar to the deformed $\mathcal{N} = 2$ SQCD_4. The Coulomb branch collapses to a discrete set of isolated vacua where each Coulomb modulus ω_j^N takes one of d possible values $(\wp_1, \wp_2, \ldots, \wp_d)$, namely the roots of the polynomial

$$\widetilde{\mathcal{W}}(X) = \sum_{k=1}^{d} \nu_k X^k + \beta \langle s \rangle v^{n_c} \tag{2.36}$$

where $\langle s \rangle \equiv \langle s_\ell \rangle$ is the common expectation value of the singlet fields s_ℓ which adjusts itself to assure $\prod_j \omega_j = v^{n_c}$. Indeed, given the Coulomb VEVs (2.23),

$$\frac{\partial W}{\partial \omega_j} = \frac{N}{\omega_j} \widetilde{\mathcal{W}}(\omega_j^N) \qquad \Longrightarrow \qquad \forall j: \quad \widetilde{\mathcal{W}}(\omega_j^N) = 0\,. \tag{2.37}$$

Note that each root $\wp_i$ of the $\widetilde{\mathcal{W}}$ polynomial may capture several moduli ω_j^N, or just one, or even none at all. The individual Coulomb vacua of the quiver are distinguished by the 'occupation numbers' $n_i = \#\{j : \omega_j^N = \wp_i\} = 0, 1, 2, \ldots$ for $i = 1, \ldots d$; altogether, $\sum_i n_i = n_c$. For any given set of the n_i, the surviving gauge symmetry of the $[\mathrm{SU}(n_c)]^N$ quiver theory is

$$G_{\text{unbroken}} \;=\; S\left[\prod_i \mathrm{U}(n_i)\right] \;\subset\; \mathrm{SU}(n_c)_{\text{diag}} \tag{2.38}$$

where each $n_i \geq 2$ gives rise to a nonabelian factor $\mathrm{SU}(n_i)$. In the quantum theory, such nonabelian factors develop mass gaps due to pseudo-confinement[2] and the ultimate low-energy limit of the theory has only the abelian photons. The net number of such surviving photons is

$$n_{\text{Abel}} = \#\{i : n_i \neq 0\} - 1 \leq n_c - 1\,. \tag{2.39}$$

Besides the purely-Coulomb vacua, the deformed quiver also has discrete mesonic vacua where the squark VEVs are determined by the deformation (2.35). In such vacua, some of the Coulomb moduli ω_j^N are frozen at

[2]Following the terminology of ref. [6] we call an $\mathrm{SU}(n_i)$ gauge factor pseudo-confining rather than confining because the overall quiver theory contains fields with anti/fundamental quantum numbers (with respect to the $\mathrm{SU}(n_i)$) which prevent the complete confinement. In this terminology, the ordinary QCD with finite-mass quarks is also pseudo-confining rather than confining.

non-zero, non-degenerate μ_f^N while the rest follow the roots $\wp_i$ of the deformation polynomial (2.36). For example, let $\omega_1^N = \mu_1^N \neq 0$ while the $(\omega_2^N, \ldots, \omega_{n_c}^N)$ are captured by the $\wp_i$. Then at this point, the D/F term constraints *require* non-zero squark and antisquark VEVs for $j = f = 1$ only:

$$\left\langle Q_{\ell,1}^1 \right\rangle = e^{2\pi i k\ell/N} Q\,, \qquad \left\langle \widetilde{Q}_{\ell,1}^1 \right\rangle = e^{-2\pi i k\ell/N} \widetilde{Q}\,,$$
$$|Q|^2 = |\widetilde{Q}|^2\,, \qquad Q\widetilde{Q} = -\frac{\widetilde{\mathcal{W}}(\mu_1^N)}{\gamma\mu_1} \neq 0\,. \tag{2.40}$$

In this vacuum $\sum_i n_i = n_c - 1 < n_c$, which reduces the unbroken gauge symmetry according to eq. (2.38). Likewise, we may have fixed squark VEVs for several distinct (j, f) pairs when the corresponding moduli ω_j^N are trapped at the $\mu_f^N \neq 0$ instead of the roots $\wp_i$; this results in even lower $\sum_i n_i \leq n_c - 2$ and hence further Higgsing down of the gauge symmetry.

In the quantum theory, the Higgs mechanism is complementary to pseudo-confinement and the two types of vacua are continuously connected in the overall parameter/moduli space of the theory. The way this duality works in the deformed $\mathcal{N} = 2$ SQCD_4 is explained in detail in [5, 6], and the same arguments apply to the deconstructive quiver theories under discussion. The bottom line is, all the Higgs and the pseudo-confining Coulomb vacua (of the same theory) which have the same *abelian rank* n_{Abel} are continuously connected to each other in the quantum quiver theory. The purely-abelian Coulomb vacua with no non-abelian factors at all form a separate class because they have higher abelian rank than any Higgs or pseudo-confining vacuum.

Eventually, we shall un-deform the quiver theory by taking the limit $\nu_k \to 0$ in the deformation superpotential (2.35). We should be careful to maintain finite roots $\wp_i$ with $n_i > 0$ and to allow them to move all over the complex plane (subject to the constraint $\prod_i \wp_i^{n_i} = v^{Nn_c}$ in the $[\mathrm{SU}(n_c)]^N$ case) while the overall scale of the polynomial $\widetilde{\mathcal{W}}(X)$ diminishes away. In this limit, the purely-Coulomb vacua where each root $\wp_i$ captures a single modulus ω_i^N span the whole Coulomb moduli space of the un-deformed quiver while each individual vacuum state adiabatically recovers its un-deformed properties. At the same time, the Higgs and the pseudo-confining vacua asymptote to the Coulomb vacua we already have while losing their distinguishing features. For example, the Higgs vacuum (2.40) loses the squark VEVs in the $\widetilde{\mathcal{W}}(X) \to 0$ limit and becomes indistinguishable from a Coulomb vacuum which simply happens to have $\omega_1^N = \mu_1^N$ and hence a massless quark mode with $j = f = 1$. Likewise, the pseudo-confining vacuum with $\omega_1^N = \omega_2^N = \wp_1$ becomes indistinguishable from the ordinary Coulomb vacuum with $\wp_1 \approx \wp_2$ when the SU(2) sector loses its mass gap in the un-deformed limit of the quiver.

Therefore, as far as the un-deformed deconstructive quiver theory is concerned, the pseudo-confining and the isolated-Higgs vacua are artifacts of the deformation and we should focus on the abelian Coulomb vacua with $n_i = 1$ (or 0) only. Nevertheless, the very existence of the pseudo-confining and isolated-Higgs vacua of the deformed theory affects its chiral ring, and so we will take them into consideration in section 2.2.

We conclude this section by discussing the mesonic and baryonic branches of the *deformed* quiver theories, assuming the quark masses allow their existence in the first place. The mesonic branches are of mixed Coulomb+Higgs type, and the two kinds of moduli sub-spaces are affected in two different ways: the Coulomb subspace of a mesonic branch (the ω_j^N which are not frozen by the squark VEVs) becomes discretized similarly to the main Coulomb branch, but the Higgs subspace remains continuous, although its complex structure *may* be deformed. For example, the mesonic Higgs branch (2.26) which exists for $ev_1^N = \mu_1^N = \mu_2^N \neq 0$ has its F-term constraint (2.27) for the squark and antisquark VEVs deformed to

$$Q_1\widetilde{Q}^1 + Q_2\widetilde{Q}^2 = -\frac{\widetilde{\mathcal{W}}(\mu_1^N)}{\gamma\,\mu_1}\,, \tag{2.41}$$

but despite this deformation, we still have continuously variable anti/squark VEVs governed by two independent complex Higgs moduli. On the other hand, the un-frozen Coulomb moduli $(\omega_2^N,\ldots,\omega_{n_c}^N)$ of the deformed quiver are no longer continuously variable but restricted to the discrete set of the $\wp_i$ roots.

The multi-mesonic Coulomb+Higgs branches suffer similar effects: The Coulomb moduli become discretized but the Higgs moduli remain continuously variable, although the complex structure of the mesonic moduli space suffers a deformation. Likewise, a baryonic Higgs branch of the $[\mathrm{SU}(n_c)]^N$ quiver survives as a continuous moduli space with two complex moduli, but its complex structure is deformed as the F-term constraint (2.32) becomes

$$\gamma e^{2\pi i k_j/N}\,Q^j\widetilde{Q}_j + \frac{\beta\,\langle s\rangle\,v^{n_c}}{\omega_j} + \sum_k \nu_k\,\omega_j^{kN-1} = 0 \qquad \forall j\,. \tag{2.42}$$

This completes our summary of the classical quiver theories. The quantum theories and their chiral rings will be addressed in the following sections 2.2 and 2.3.

2.2 The non-baryonic chiral ring

The main subject of this paper is the quantum chiral ring of the $[\mathrm{SU}(n_c)]^N$ quiver theory from the purely $D = 4$, $\mathcal{N} = 1$ point of view. Thus, we consider the size N of the quiver as a fixed, finite parameter of the theory and allow the superpotential deformation (2.35) despite its non-locality in the x^4 direction. Using the techniques of Cachazo, Douglas, Seiberg and Witten [4]–[6], we package the chiral ring's generators into several resolvent functions of an auxiliary complex variable X, and then derive and solve the anomaly equations for these resolvents.

The new aspects of the present work (compared to Cachazo *et al.*) are due to a more complicated object of study: a whole quiver of N gauge groups instead of just one, and each gauge group is $\mathrm{SU}(n_c)$ rather than $\mathrm{U}(n_c)$. In this section we focus on the quiver issues and study the non-baryonic generators of the chiral ring. The baryons and other generators allowed by the $\mathrm{SU}(n_c)$ rather than $\mathrm{U}(n_c)$ symmetries will be addressed in the following section 2.3.

Let us start with a brief review of the chiral ring basics. Most generally, the chiral ring of a 4D $\mathcal{N} = 1$ gauge theory is the $\overline{Q}$ cohomology in the algebra of local *gauge-invariant* operators $\mathcal{O}(x)$ of the theory. That is, we consider chiral operators $[\overline{Q}^{\dot{\alpha}}, \mathcal{O}\} = 0$ and identify them modulo $\overline{Q}$ commutators,

$$\mathcal{O}_1 \stackrel{\mathrm{cr}}{=} \mathcal{O}_2 \iff \mathcal{O}_1 - \mathcal{O}_2 = \left[\overline{Q}^{\dot{\alpha}}, \mathcal{O}'\right\} \tag{2.43}$$

where the operator $\mathcal{O}'(x)$ is also local and gauge invariant. In the superfield formalism we may use the $\overline{D}^{\dot{\alpha}}$ super-derivative instead of the $\overline{Q}^{\dot{\alpha}}$ supercharges, thus chiral operators $\mathcal{O}(z)$ satisfy $\overline{D}^{\dot\alpha}\,\mathcal{O}=0$ and in the chiral ring

$$\mathcal{O}_1 \overset{\text{cr}}{=} \mathcal{O}_2 \iff \mathcal{O}_1 - \mathcal{O}_2 = \overline{D}^{\dot\alpha}\Big(\text{gauge-invariant } \mathcal{O}'\Big)\,. \tag{2.44}$$

In the chiral ring the spacetime location of an operator is irrelevant,

$$\partial_{\alpha\dot\alpha}\mathcal{O} = \frac{1}{2i}\,\overline{D}_{\dot\alpha}(D_\alpha\mathcal{O}) \overset{\text{cr}}{=} 0 \implies \forall x_1, x_2:\ \mathcal{O}(x_1) \ \overset{\text{cr}}{=}\ \mathcal{O}(x_2)\,, \tag{2.45}$$

and therefore all operator products are also position independent,

$$\mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\cdots\mathcal{O}_n(x_n) \ \overset{\text{cr}}{=}\ \text{same } \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n \qquad \forall\, x_1, x_2, \ldots, x_n\,. \tag{2.46}$$

This position independence distinguishes the chiral ring from the more general operator algebra of the quantum theory and makes it exactly solvable. It also makes it a bona-fide ring, which simplifies the analysis: once we construct all the independent generators from the fundamental fields of the theory, the operator products (2.46) follow from the ring structure without any further work.

Finally, note the distinction between the *off-shell* and the *on-shell* chiral rings of the same theory: In the off-shell chiral ring the equivalence relations (2.44) must be operatorial identities of the quantum theory, but in the *on-shell* chiral ring we use both the identities and the equations of motion. Classically

$$\frac{\partial W}{\partial\phi} = \frac{1}{4}\overline{D}^2\left(\frac{\partial K}{\partial\phi}\right) \overset{\text{cr}}{=} 0 \tag{2.47}$$

for any independent chiral field ϕ, but at the quantum level eqs. (2.47) are corrected by the generalized Konishi anomalies, cf. eq. (2.74) on page 38.

2.2.1 Generating the chiral ring

In this subsection we generate (*i. e.*, construct independent generators of) the off-shell chiral rings of the $[\mathrm{SU}(n_c)]^N$ quiver theories. Or rather the *almost* off-shell rings where the operatorial identities of the theory are supplemented by the anomaly-free equations of motion for the singlet fields s_ℓ:

$$\forall\ell: \quad \frac{\partial W}{\partial s_\ell} \overset{\mathrm{cr}}{=} 0 \qquad \Longrightarrow \qquad \det(\Omega_\ell) \overset{\mathrm{cr}}{=} v^{n_c}\,. \tag{2.48}$$

We group these particular equations of motion with the operatorial identities because the s_ℓ fields do not do anything interesting besides imposing the constraints (2.48) on the link fields to set up the $\mathrm{SL}(n_c,\mathbb{C})_\ell$ sigma models.

We begin with the chiral ring generators made from the link fields Ω_ℓ and nothing else. Because of eqs. (2.48) we cannot form chiral gauge invariants from the individual link fields; instead, we have to take traces $\mathrm{tr}(\Omega_N\Omega_{N-1}\cdots\Omega_2\Omega_1)^k$ of whole chains of links wrapped several times around the quiver. Also, thanks to $\det(\Omega_\ell)\neq 0$ the matrix inverses Ω_ℓ^{-1} are well-defined chiral operators; this allows us to takes traces $\mathrm{tr}(\Omega_1^{-1}\Omega_2^{-1}\cdots\Omega_N^{-1})^k$ of the inverse link chains wrapped around the quiver in the opposite direction. Conveniently, both types of traces can be summarized via a single resolvent

$$\begin{aligned} T(X) &= \mathrm{tr}\left(\frac{1}{X-\Omega_N\cdots\Omega_1}\right) \\ &= \sum_{k=0}^{\infty}\frac{1}{X^{k+1}}\times\mathrm{tr}\left(\Omega_N\Omega_{N-1}\cdots\Omega_2\Omega_1\right)^k \\ &= -\sum_{k=1}^{\infty}X^{k-1}\times\mathrm{tr}\left(\Omega_1^{-1}\Omega_2^{-1}\cdots\Omega_{N-1}^{-1}\Omega_N^{-1}\right)^k\,. \end{aligned} \tag{2.49}$$

Classically, this resolvent has simple poles at the Coulomb moduli of the quiver

$$T(X) = \sum_{j=1}^{n_c} \frac{1}{X - \omega_j^N} \tag{2.50}$$

and we may use contour integrals

$$n(\mathcal{C}) \ = \oint_{\mathcal{C}} \frac{dX}{2\pi i}\, T(X) \tag{2.51}$$

as gauge-invariant counts of the Coulomb moduli inside any particular contour $\mathcal{C}$; for example, $n_i = n(\mathcal{C}_i)$ for a sufficiently small contour $\mathcal{C}_i$ surrounding the deformation root $\wp_i$.

In the quantum theory of the quiver, the resolvent (2.49) behaves similarly to its $\operatorname{tr}(\frac{1}{X-\Phi})$ analogue in the deformed $\mathcal{N} = 2$ SQCD$_4$: the poles at $\wp_i$ become $\sqrt[2]{\ }$ branch cuts, but the n_i — defined as $n(\mathcal{C}_i)$ for suitable contours $\mathcal{C}_i$ — remain exactly integer. And since the monodromies in the parameter space of the deformed quiver theory can entangle a $\mathcal{C}_i$ with any other cycle of the Riemann surface of the $T(X)$, it follows that *all closed-contour integrals (2.51) of the link resolvent (2.49) must have integer values.* Indeed, consider the differential

$$T(X)\, dX = d \operatorname{tr} \log(X - \Omega_N \cdots \Omega_1) = d \log \det(X - \Omega_N \cdots \Omega_1)\,. \tag{2.52}$$

Regardless of any quantum corrections to the determinant $\det(X - \Omega_N \cdots \Omega_1)$, its logarithm will always have exactly integer $\times\ 2\pi i$ differences between its branches. Consequently, the quantum quiver theory has exactly integer contour integrals (2.51) for all contours $\mathcal{C}$ which are closed on the Riemann surface of the $T(X)$.

The readers who find the above argument too heuristic are referred to Cachazo *et al* for a rigorous proof; the arguments of ref. [6] apply equally well to the present case and we don't see the need of repeating them here almost verbatim.

Next, let us add the quark and antiquark fields to the picture and form all kinds of chiral operators with mesonic quantum numbers. Besides the true mesons

$$[M_\ell]_f^{f'} = \widetilde{Q}_\ell^{f'} Q_{\ell,f} \tag{2.53}$$

which are local in 5D as well as in 4D, there are other meson-like chiral gauge-invariant operators where the quark and the antiquark are located at different quiver nodes $\ell \neq \ell'$ but are connected to each other by a chain of link fields, *eg.,* $\widetilde{Q}_{\ell'}^{f'}\Omega_{\ell'-1}\cdots\Omega_\ell Q_{\ell,f}$. From the 5D point of view, these are *bi-local* operators which create/annihilate *un-bound* pairs of quarks and antiquarks, while the link chains deconstruct the un-physical Wilson strings which allow for manifest gauge invariance of such bi-local operators:

$$\begin{aligned} [\mathbf{M}(x_2,x_1)]_f^{f'} \quad &= \quad \widetilde{Q}^{f'}(x_2) \times \underset{\substack{\text{Path}\\ \text{ordered}}}{\exp}\left(i\int_{x_1}^{x_2} dx^\mu\, A_\mu(x)\right) \times Q_f(x_1) \\ &\longrightarrow \quad \widetilde{Q}_{\ell'}^{f'}\,\Omega_{\ell'-1}\Omega_{\ell'-2}\cdots\Omega_{\ell+1}\Omega_\ell\, Q_{\ell,f} \\ &\qquad \text{for } x_1^{0,1,2,3} = x_2^{0,1,2,3}\,, \quad x_1^4 = a\ell \text{ and } x_2^4 = a\ell'\,. \end{aligned} \tag{2.54}$$

From the 4D point of view however, these operators are local, chiral and gauge invariant — and therefore belong to the chiral ring of the quiver.[3]

[3]Actually, to make the chiral operator on the second line of eq. (2.54), the Wilson string on the top line must be modified to incorporate the 5D scalar field $\phi(x)$ along with the

Besides the "split mesons" (2.54) where the link chain runs directly from the quark to the antiquark, we may have the chain going several times around the whole quiver, thus $\widetilde{Q}^{f'}_{\ell'}\Omega_{\ell'-1}\cdots\Omega_1(\Omega_N\cdots\Omega_1)^k\Omega_N\cdots\Omega_{\ell+1}\Omega_\ell Q_{\ell,f}$, or in reverse direction (via inverse links), thus $\widetilde{Q}^{f'}_{\ell'}\Omega^{-1}_{\ell'}\Omega^{-1}_{\ell'+1}\cdots\Omega^{-1}_{\ell-2}\Omega^{-1}_{\ell-1}Q_{\ell,f}$ or even $\widetilde{Q}^{f'}_{\ell'}\Omega^{-1}_{\ell'}\Omega^{-1}_{\ell'+1}\cdots\Omega^{-1}_N(\Omega^{-1}_1\cdots\Omega^{-1}_N)^k\Omega^{-1}_1\cdots\Omega^{-1}_{\ell-1}Q_{\ell,f}$. To summarize all these meson-like operators, we define mesonic resolvents

$$\mathcal{M}_{\ell',\ell}(X) = \widetilde{Q}_{\ell'}\,\frac{\Omega_{\ell'-1}\cdots\Omega_\ell}{X-\Omega\cdots\Omega}\,Q_\ell \tag{2.55}$$

where the flavor indices of the quarks and antiquarks are suppressed for notational simplicity (or in other words, each $\mathcal{M}_{\ell',\ell}(X)$ is an $F\times F$ matrix), the quiver indices are understood modulo N, and the $\dfrac{\Omega_{\ell'-1}\cdots\Omega_\ell}{X-\Omega\cdots\Omega}$ is a short-hand for

$$\begin{aligned} &\Omega_{\ell'-1}\cdots\Omega_\ell\times\frac{1}{X-\Omega_{\ell-1}\cdots\Omega_1\Omega_N\cdots\Omega_\ell}\\ &=\quad \frac{1}{X-\Omega_{\ell'-1}\cdots\Omega_1\Omega_N\cdots\Omega_{\ell'}}\times\Omega_{\ell'-1}\cdots\Omega_\ell\,. \end{aligned} \tag{2.56}$$

The resolvents (2.55) with $\ell\le\ell'<\ell+N$ suffice to generate all the meson-like operators; for $\ell'=\ell+N$ there is a periodicity equation

$$\mathcal{M}_{\ell+N,\ell}(X) = X\times\mathcal{M}_{\ell,\ell}(X)\ -\ M_\ell \tag{2.57}$$

where the last (X-independent) term on the right hand side is the matrix of the true mesons (2.53). Classically, the resolvents (2.55) have simple poles at

gauge field $A_4(X)$. Indeed, according to the deconstruction map (2.4),

$$\widetilde{Q}^{f'}_{\ell'}\,\Omega_{\ell'-1}\Omega_{\ell'-2}\cdots\Omega_{\ell+1}\Omega_\ell\,Q_{\ell,f} = \widetilde{Q}^{f'}_{\ell'}\times\exp_{\substack{\text{Path}\\\text{ordered}}}\left(\int_{a\ell}^{a\ell'} dx^4\,(\phi(x)+iA_4(x))\right)\times Q_{\ell,f}\,.$$

Despite this correction, the Wilson string remains un-physical and the quark-antiquark pair remains unbound.

$X = \mu_f^N$, but only for mesonic vacua with $\langle Q_f\rangle, \langle\widetilde{Q}_f\rangle \neq 0$. In the quantum theory, such poles exist for all vacua, but only the mesonic vacua have them on the "physical sheet" of the quiver's spectral curve; we shall explain this issue the following subsection 2.2.2.

Meanwhile, consider the chiral gaugino superfields $W_\ell^\alpha = \lambda_\ell^\alpha + F_\ell^{\alpha\beta}\theta_\beta + \cdots \in \mathrm{Adj}(\mathrm{SU}(n_c)_\ell)$ and their gauge-invariant combinations with the other chiral fields of the quiver. Although there is a great multitude of such combinations, most of them turn out to be total $\overline{D}^2$ super-derivatives and thus do not belong to the chiral ring. This follows from the appearance of the tensor sum of all gaugino superfields

$$\mathbb{W}^\alpha = \bigoplus_{\ell=1}^{N} W_\ell^\alpha \tag{2.58}$$

in the anti/commutation algebra of the gauge-covariant spinor derivatives ∇^α and $\overline{\nabla}^{\dot\beta}$:

$$\begin{aligned} & \left[\overline{\nabla}_{\dot\alpha}, \left\{\overline{\nabla}_{\dot\beta}, \nabla_\gamma\right\}\right] = 4\epsilon_{\dot\alpha\dot\beta}\mathbb{W}_\gamma \\ \Longrightarrow \qquad \forall\,\text{chiral }\Phi: \quad & -\frac{1}{8}\overline{\nabla}^2\nabla^\alpha\Phi = \mathbb{W}^\alpha\Phi, \end{aligned} \tag{2.59}$$

thus

$$\begin{aligned} -\frac{1}{8}\overline{\nabla}^2\nabla^\alpha Q_\ell^f &= W_\ell^\alpha Q_\ell^f\,, \\ -\frac{1}{8}\overline{\nabla}^2\nabla^\alpha \widetilde{Q}_\ell^f &= -\widetilde{Q}_\ell^f W_\ell^\alpha\,, \\ -\frac{1}{8}\overline{\nabla}^2\nabla^\alpha \Omega_\ell &= W_{\ell+1}^\alpha\Omega_\ell - \Omega_\ell W_\ell^\alpha\,, \\ -\frac{1}{8}\overline{\nabla}^2\nabla^\alpha W_\ell^\beta = -\frac{1}{8}\overline{\nabla}^2\nabla^\beta W_\ell^\alpha &= \left\{W_\ell^\alpha, W_\ell^\beta\right\}. \end{aligned} \tag{2.60}$$

Therefore, any gauge invariant combination of chiral superfields which includes both gauginos and quarks or antiquarks is a total $\overline{D}^2$ super-derivative — which does not belong to the chiral ring. For example,

$$\begin{aligned} &\widetilde{Q}_{\ell'}\Omega_{\ell'-1}\cdots\Omega_{\ell''}W^{\alpha}_{\ell''}\Omega_{\ell''-1}\cdots\Omega_{\ell}Q_{\ell} = \\ &= -\frac{1}{8}\,\overline{D}^2\Big((\widetilde{Q}_{\ell'}\Omega_{\ell'-1}\cdots\Omega_{\ell''})\,\nabla^{\alpha}\,(\Omega_{\ell''-1}\cdots\Omega_{\ell}Q_{\ell})\Big) \stackrel{\rm cr}{=} 0\,. \end{aligned} \tag{2.61}$$

Furthermore, for any gauge-covariant combination $\Xi_{\ell,\ell'}$ of gaugino and link fields which transforms as $(\Box_{\ell},\overline{\Box}_{\ell'})$, we have

$$\begin{aligned} &\mathrm{tr}\,(\Xi_{\ell,\ell'}\,W^{\alpha}_{\ell'}\,\Omega_{\ell'-1}\cdots\Omega_{\ell}) - \mathrm{tr}\,(\Xi_{\ell,\ell'}\,\Omega_{\ell'-1}\cdots\Omega_{\ell}W^{\alpha}_{\ell}) = \\ &= -\frac{1}{8}\overline{D}^2\Big(\Xi_{\ell,\ell'}\,\nabla^{\alpha}\,(\Omega_{\ell'-1}\cdots\Omega_{\ell})\Big) \stackrel{\rm cr}{=} 0\,. \end{aligned} \tag{2.62}$$

In particular, for any ℓ and ℓ',

$$\mathrm{tr}\left(W^{\alpha}_{\ell}\,(\Omega_{\ell-1}\cdots\Omega_{\ell+1}\Omega_{\ell})^k\right) \stackrel{\rm cr}{=} \mathrm{tr}\left(W^{\alpha}_{\ell'}\,(\Omega_{\ell'-1}\cdots\Omega_{\ell'+1}\Omega_{\ell'})^k\right) \tag{2.63}$$

and likewise

$$\mathrm{tr}\left(W^{\alpha}_{\ell}\,(\Omega^{-1}_{\ell}\Omega^{-1}_{\ell+1}\cdots\Omega^{-1}_{\ell-1})^k\right) \stackrel{\rm cr}{=} \mathrm{tr}\left(W^{\alpha}_{\ell'}\,(\Omega^{-1}_{\ell'}\Omega^{-1}_{\ell'+1}\cdots\Omega^{-1}_{\ell'-1})^k\right), \tag{2.64}$$

which means that all chiral ring's generators which involve a single gaugino operator are summarized in a single ℓ-independent gaugino resolvent

$$\begin{aligned} \Psi^{\alpha}(X) &\stackrel{\rm def}{=} \frac{1}{4\pi}\,\mathrm{tr}\left(\frac{W^{\alpha}}{X-\Omega\cdots\Omega}\right) \\ &\equiv \frac{1}{4\pi}\,\mathrm{tr}\left(W^{\alpha}_{\ell}\times\frac{1}{X-\Omega_{\ell}\cdots\Omega_{\ell-1}}\right) \qquad \langle\langle\text{same }\forall\ell\rangle\rangle\,. \end{aligned} \tag{2.65}$$

Physically, this resolvent encodes the exactly massless abelian photinos of the $[\mathrm{SU}(n_c)]^N$ quiver. Indeed, contour integrals of the $\Psi(X)$ yield traces of the diagonal gaugino fields of the $\mathrm{SU}(n_c)_{\mathrm{diag}}$ over a subspace where the Coulomb moduli ω_j^N happen to lie inside the integration contour:

$$\oint_{\mathcal{C}} \frac{dX}{2\pi i}\, \Psi^\alpha(X) = \frac{1}{4\pi}\, \mathrm{tr}\left(W^\alpha|_{\omega^N \text{ inside } \mathcal{C}}\right) \tag{2.66}$$

where the W^α can be thought as belonging to the $\mathrm{SU}(n_c)_{\mathrm{diag}}$ because any W_ℓ^α would yield the same generator of the chiral ring regardless of ℓ. In particular, for the $\mathcal{C}_i$ contour surrounding a deformation root $\wp_i$ (or in the fully quantum theory, surrounding the branch cut near an $\wp_i$), the integral (2.66) restricts the gaugino fields W^α to the $\mathrm{U}(n_i)$ subgroup of the unbroken gauge symmetry (2.38) — and then the trace extracts the abelian photino W_i^α in the $\mathrm{U}(1)_i$ center of the $\mathrm{U}(n_i)$:

$$4\pi \oint_{\mathcal{C}_i} \frac{dX}{2\pi i}\, \Psi^\alpha(X) = \mathrm{tr}\left(W^\alpha|_{\mathrm{U}(n_i)}\right) = W_i^\alpha\,. \tag{2.67}$$

Note that for the whole $\mathrm{SU}(n_c)_{\mathrm{diag}}$, $\mathrm{tr}(W^\alpha) = 0$ and hence $\sum_i W_i^\alpha = 0$; in terms of the gaugino resolvent $\Psi^\alpha(X)$, it means no residue at $X = \infty$ and $\Psi^\alpha(X) = O(1/X^2)$ rather than $O(1/X)$.

Next, consider the chiral ring generators involving two gaugino operators $W_{\ell_1}^\alpha$ and $W_{\ell_2}^\beta$ inserted into a closed chain $\mathrm{tr}(\Omega_N \cdots \Omega_1)^k$ of link operators. Again, the specific points of insertion do not matter: For any ℓ_1 and ℓ_2 and any $k_1 + k_2 = k - 1$,

$$\begin{aligned}
\mathrm{tr}\Big(W_{\ell_1}^\alpha\, (\Omega_{\ell_1-1}\cdots\Omega_{\ell_1})^{k_1}\, \Omega_{\ell_1-1}\cdots\Omega_{\ell_2}\, W_{\ell_2}^\beta\, (\Omega_{\ell_2-1}\cdots\Omega_{\ell_2})^{k_2}\, \Omega_{\ell_2-1}\cdots\Omega_{\ell_1}\Big)& \\
\overset{\mathrm{cr}}{=} \mathrm{tr}\Big(W_{\ell_1}^\alpha W_{\ell_1}^\beta (\Omega_{\ell_1-1}\cdots\Omega_{\ell_1})^k\Big)& \\
\overset{\mathrm{cr}}{=} \mathrm{tr}\Big(W_\ell^\alpha W_\ell^\beta (\Omega_{\ell-1}\cdots\Omega_\ell)^k\Big) \qquad \forall \text{ other } \ell& \\
\overset{\mathrm{cr}}{=} -\frac{1}{2}\,\epsilon^{\alpha\beta} \times \mathrm{tr}\Big(W_\ell^2\, (\Omega_{\ell-1}\cdots\Omega_\ell)^k\Big)&
\end{aligned} \tag{2.68}$$

where the last equality follows from the fourth eq. (2.60). Thanks to reversibility of the link matrices Ω_ℓ in the $[\mathrm{SU}(n_c)]^N$ quiver theory, eqs. (2.68) extend to negative $k_{1,2}$; in particular, for $k_1+k_2+1=k=0$ we have

$$\forall\ell_1,\ell_2:\quad \mathrm{tr}\left(W_{\ell_1}^2\right)\stackrel{\mathrm{cr}}{=}\mathrm{tr}\left(W_{\ell_2}^2\right) \tag{2.69}$$

without any link fields being involved at all (except at the intermediate stages).[4] Consequently, all generators involving two gauginos are summarized in a single ℓ-independent "gaugino bilinear" resolvent

$$R(X)\stackrel{\mathrm{def}}{=}\frac{1}{32\pi^2}\,\mathrm{tr}\left(\frac{W^2}{X-\Omega\cdots\Omega}\right)$$
$$\equiv\frac{1}{32\pi^2}\,\mathrm{tr}\left(W_\ell^\alpha W_{\ell,\alpha}\times\frac{1}{X-\Omega_{\ell-1}\cdots\Omega_\ell}\right)\quad\langle\langle\text{same }\forall\ell\rangle\rangle\,. \tag{2.70}$$

The contour integrals of this resolvent encode the "gaugino condensates" of the non-abelian factors $\mathrm{SU}(n_i)$ of the unbroken gauge symmetry (2.38):

$$\mathcal{S}_i\equiv\oint_{\mathcal{C}_i}\frac{dX}{2\pi i}\,R(X)=\frac{1}{32\pi^2}\,\mathrm{tr}\left(W^\alpha W_\alpha|_{\mathrm{U}(n_i)}\right). \tag{2.71}$$

Or rather, the "gaugino condensation" in the $\mathrm{SU}(n_i)$ factor is the leading contribution to the $\mathcal{S}_i$ for $n_i\geq 2$, in which case $\mathcal{S}_i\sim e^{-8\pi^2/n_i g^2_{\mathrm{diag}}}$ develops at the fractional instanton level $1/n_i$. For $n_i=1$ there is no gaugino condensation *per se*, but the $\mathcal{S}_i$ "condensate" develops anyway at the one-whole-instanton level $\mathcal{S}_i\sim e^{-8\pi^2/g^2_{\mathrm{diag}}}$, thanks to coset instantons in the broken $\mathrm{SU}(n_c)_{\mathrm{diag}}/\,\mathrm{U}(1)_i$.

[4]Strictly speaking, the identities (2.69) depend on the on-shell equations (2.48). The off-shell operatorial identities of the quiver's chiral ring have form

$$\det(\Omega_\ell)\times\left(\mathrm{tr}(W_\ell^2)\;-\;\mathrm{tr}(W_{\ell+1}^2)\right)\stackrel{\mathrm{cr}}{=}0$$

and imply eqs. (2.69) if and only if all $\det(\Omega_\ell)\neq 0$.

In any case, we count instanton levels with respect to the *diagonal* $\mathrm{SU}(n_c)$ subgroup of the quiver; in terms of the whole $[\mathrm{SU}(n_c)]^N$ gauge group, *one diagonal instanton means one instanton in* each $\mathrm{SU}(n_c)_\ell$ *factor* according to

$$\exp\left(-\frac{8\pi^2}{g^2_{\mathrm{diag}}}\right) = \prod_\ell \exp\left(-\frac{8\pi^2}{g^2_\ell}\right), \tag{2.72}$$

or in un-deconstructed 5D terms, one euclidean $0^{\mathrm{time}} + 1^{\mathrm{space}}$ instanton brane wrapped around the compactified x^4 dimension. However, apart from this quiver-specific instanton counting, the gaugino bilinear resolvent (2.70) behaves similarly to its $\mathrm{tr}(\frac{W^\alpha W_\alpha}{X-\Phi})$ analogue in the deformed $\mathcal{N} = 2$ SQCD$_4$.

Finally, the quiver's chiral ring does not have any generators involving three or more gaugino fields. Indeed, let us insert $W^\alpha_{\ell'}, W^\beta_{\ell''}, \ldots, W^\gamma_{\ell'''}$ into a closed chain $\mathrm{tr}(\Omega_N \cdots \Omega_1)^k$ of the link operators. Applying eq. (2.62) several times, we can move all the gaugino operators to the same quiver node ℓ, thus $\mathrm{tr}(W^\alpha_\ell W^\beta_\ell \cdots W^\gamma_\ell (\Omega_{\ell-1} \cdots \Omega_\ell)^k)$, but then the fourth eq. (2.60) implies $W^\alpha_\ell W^\beta_\ell \cdots W^\gamma_\ell = \overline{\nabla}^2(\text{something})$ and therefore, the whole shmeer $= \overline{D}^2(\text{something else}) \stackrel{\mathrm{cr}}{=} 0$ and does not belong to the chiral ring.

Altogether, we have constructed all the generators of the $[\mathrm{SU}(n_c)]^N$ quiver's chiral ring, except for the baryons, the antibaryons, and their multi-local cousins comprised of n_c quarks or antiquarks located at different quiver nodes connected by chains of link operators. The $[\mathrm{SU}(n_c)]^N$ quiver has a whole zoo of such multi-local baryon-like chiral operators, and we prefer to discuss them in a separate section 2.3.2.

2.2.2 Anomalous equations of motion and their solutions

Thus far, we generated the (almost) *off-shell* chiral ring of the $[\mathrm{SU}(n_c)]^N$ quiver. In this section, we focus on the *on-shell* chiral ring in which the resolvents $T(X)$, $\mathcal{M}_{\ell',\ell}(X)$, $\Psi^\alpha(X)$ and $R(X)$ satisfy the equations of motion of the quantum quiver theory. Generically, such equations follow from infinitesimal gauge-covariant field-dependent variations of the fundamental chiral fields of the theory

$$\Phi \;\longrightarrow\; \Phi + \delta\Phi \quad (\Phi,\ \text{other chiral operators})\,. \tag{2.73}$$

Classically, $\delta W^{\mathrm{tree}} \equiv \frac{\partial W^{\mathrm{tree}}}{\partial \Phi} \times \delta\Phi \stackrel{\mathrm{cr}}{=} 0$, but in the quantum theory generalized Konishi anomalies change this equation to

$$\frac{\partial W^{\mathrm{tree}}}{\partial \Phi} \times \delta\Phi \stackrel{\mathrm{cr}}{=} \frac{1}{32\pi^2}\,\mathrm{Tr}\left(\frac{\mathbb{W}^\alpha \mathbb{W}_\alpha\, \partial\delta\Phi}{\partial\Phi}\right) \tag{2.74}$$

where the trace on the right hand side is taken with respect to all indices of the field Φ, color and flavor. As a specific example in the quiver context, let Φ be the quark field $Q_{\ell'}$ and consider the variation

$$\delta Q_{\ell'} = \frac{\Omega_{\ell'-1}\Omega_{\ell'-2}\cdots\Omega_\ell}{X - \Omega\cdots\Omega} \times Q_\ell \times \varepsilon \tag{2.75}$$

where ε is an infinitesimal $F \times F$ matrix in the flavor space. For this variation,

$$\begin{aligned} \delta W^{\mathrm{tree}} = \delta W^{\mathrm{hop}} &= \gamma\left[\mathrm{tr}\,(\widetilde{Q}_{\ell'+1}\Omega_{\ell'} \times \delta Q_{\ell'}) - \mathrm{tr}\,(\mu \times \widetilde{Q}_{\ell'} \times \delta Q_{\ell'})\right] \\ &= \gamma\Big[\,\mathrm{tr}\left(\widetilde{Q}_{\ell'+1} \times \frac{\Omega_{\ell'}\Omega_{\ell'-1}\cdots\Omega_\ell}{X - \Omega\cdots\Omega} \times Q_\ell \times \varepsilon\right) - \\ &\qquad - \mathrm{tr}\left(\mu \times \widetilde{Q}_{\ell'} \times \frac{\Omega_{\ell'-1}\Omega_{\ell'-2}\cdots\Omega_\ell}{X - \Omega\cdots\Omega} \times Q_\ell \times \varepsilon\right)\Big] \\ &= \gamma\,\mathrm{tr}\Big((\mathcal{M}_{\ell'+1,\ell}(X) - \mu \times \mathcal{M}_{\ell',\ell}(X)) \times \varepsilon\Big) \end{aligned} \tag{2.76}$$

while the Konishi anomaly exists only for $\ell' = \ell$ (otherwise $\delta Q_{\ell'}$ does not depend on the $Q_{\ell'}$ itself) and amounts to

$$\frac{1}{32\pi^2}\,\mathrm{Tr}\left(W^{\alpha}_{\ell'}W_{\ell',\alpha}\frac{\delta_{\ell',\ell}}{X-\Omega_{\ell'-1}\cdots\Omega_{\ell'}}\otimes\varepsilon\right)\stackrel{\mathrm{cr}}{=}\delta_{\ell',\ell}\,R(X)\times\mathrm{tr}(\varepsilon)\,. \tag{2.77}$$

Substituting these formulæ into the generic eq. (2.74), we arrive at the anomalous equations of motion for the on-shell mesonic resolvents: In $F\times F$ matrix notations,

$$\mathcal{M}_{\ell'+1,\ell}(X)-\mu\times\mathcal{M}_{\ell',\ell}(X)\stackrel{\mathrm{cr}}{=}\delta_{\ell',\ell}\,\gamma^{-1}\,R(X)\times 1_{F\times F}\,. \tag{2.78}$$

Consequently,

$$\mathcal{M}_{\ell',\ell}(X)\ \stackrel{\mathrm{cr}}{=}\ \mu^{\ell'-\ell}\times\mathcal{M}_{\ell,\ell}(X)+\gamma^{-1}\mu^{\ell'-\ell-1}\times R(X) \tag{2.79}$$

for $\ell<\ell'\le\ell+N$ and hence in light of the periodicity equation (2.57),

$$\textbf{on shell:}\quad\begin{cases}\mathcal{M}_{\ell'=\ell}(X)=\dfrac{1}{X-\mu^N}\times\left(\gamma^{-1}\mu^{N-1}\,R(X)+M_\ell\right),\\[2ex]\mathcal{M}_{\ell'>\ell}(X)=\dfrac{\mu^{\ell'-\ell-1}}{X-\mu^N}\times\left(\gamma^{-1}\,XR(X)+\mu\times M_\ell\right).\end{cases} \tag{2.80}$$

Thus, we have solved for *all* of the on-shell mesonic resolvents in terms of the ordinary mesons M_ℓ and the gaugino bilinear resolvent $R(X)$.

Likewise, starting with the infinitesimal antiquark variation

$$\delta\widetilde{Q}_\ell=\varepsilon\times\widetilde{Q}_{\ell'}\times\frac{\Omega_{\ell'-1}\Omega_{\ell'-2}\cdots\Omega_\ell}{X-\Omega\cdots\Omega} \tag{2.81}$$

we also arrive at anomalous equations of motion for the on-shell mesonic resolvents, but this time we have

$$\mathcal{M}_{\ell',\ell-1}(X)-\mathcal{M}_{\ell',\ell}(X)\times\mu\stackrel{\mathrm{cr}}{=}\delta_{\ell',\ell}\,\gamma^{-1}\,R(X)\times 1_{F\times F} \tag{2.82}$$

and consequently

$$\textbf{on shell:}\quad \begin{cases} \mathcal{M}_{\ell'=\ell}(X) = \big(\gamma^{-1}\mu^{N-1}\,R(X) + M_{\ell'}\big) \times \dfrac{1}{X-\mu^N}\,, \\ \mathcal{M}_{\ell'>\ell}(X) = \big(\gamma^{-1}\,XR(X) + M_{\ell'}\times\mu\big) \times \dfrac{\mu^{\ell'-\ell-1}}{X-\mu^N}\,. \end{cases} \tag{2.83}$$

Note that (2.82)–(2.83) are just as valid as (2.78)–(2.80), and to assure mutual consistency of the two equation systems, all meson matrices M_ℓ — and hence all the mesonic resolvent matrices $\mathcal{M}_{\ell',\ell}(X)$ — must commute with the μ^N matrix and therefore must be block-diagonal in its eigenbasis. Furthermore, if $\mu_{f'} = \mu_f$ whenever $\mu_{f'}^N = \mu_f^N$, then all the M_ℓ matrices must be equal to each other, $M_\ell \equiv M$; otherwise $M_\ell = \mu^\ell \times M \times \mu^{-\ell}$. In terms of the matrix elements, we have

$$\begin{aligned} [M_\ell]_f^{f'} &= M_f^{f'} \times \left(\frac{\mu_{f'}}{\mu_f}\right)^\ell \qquad \text{for } \mu_{f'}^N = \mu_f^N \neq 0 \text{ only} \\ [M_\ell]_f^{f'} &= 0 \qquad \text{otherwise.} \end{aligned} \tag{2.84}$$

In particular, the matrix elements with $\mu_{f'} = \mu_f = 0$ must vanish because the on-shell mesonic resolvents of the $[\mathrm{SU}(n_c)]^N$ quiver *cannot* have poles at $X = 0$. Indeed,

$$\text{for } X \to 0\,, \qquad \mathcal{M}_{\ell',\ell}(X) \;\to\; \widetilde{Q}_{\ell'}\,\Omega_{\ell'}^{-1}\Omega_{\ell'+1}^{-1}\cdots\Omega_{\ell-1}^{-1}\,Q_\ell \neq \infty \tag{2.85}$$

because on shell $\det\Omega_\ell \neq 0$ and the inverse links are well-defined chiral operators.

Next, let us vary a link field Ω_ℓ according to

$$\delta\Omega_\ell = \varepsilon\,\frac{\Omega_\ell}{X - \Omega\cdots\Omega} \tag{2.86}$$

where ε is now an infinitesimal number rather than a matrix. For this variation,

$$d\delta\Omega_\ell = \frac{\varepsilon}{X - \Omega_\ell \cdots \Omega_{\ell+1}} \times d\Omega_\ell \times \frac{X}{X - \Omega_{\ell-1} \cdots \Omega_\ell} \tag{2.87}$$

while

$$\mathbb{W}^\alpha \mathbb{W}_\alpha \, d\delta\Omega_\ell = W^\alpha_{\ell+1} W_{\ell+1,\alpha} \, d\delta\Omega_\ell - 2W^\alpha_{\ell+1} \, d\delta\Omega_\ell \, W_{\ell,\alpha} + d\delta\Omega_\ell \, W^\alpha_\ell W_{\ell,\alpha} \,, \tag{2.88}$$

hence the Konishi anomaly comes up to

$$\begin{aligned} &\frac{1}{32\pi^2} \operatorname{Tr}\left(\frac{\mathbb{W}^\alpha \mathbb{W}_\alpha \, d\delta\Omega_\ell}{d\Omega_\ell}\right) = \\ &= \frac{\varepsilon X}{32\pi^2} \operatorname{Tr}\Big(\frac{W^\alpha_{\ell+1} W_{\ell+1,\alpha}}{X - \Omega \cdots \Omega} \otimes \frac{1}{X - \Omega \cdots \Omega} - 2 \frac{W^\alpha_{\ell+1}}{X - \Omega \cdots \Omega} \otimes \\ &\qquad \otimes \frac{W_{\ell,\alpha}}{X - \Omega \cdots \Omega} + \frac{1}{X - \Omega \cdots \Omega} \otimes \frac{W^\alpha_\ell W_{\ell,\alpha}}{X - \Omega \cdots \Omega}\Big) \\ &= \varepsilon X (R(X) \times T(X) - \Psi^\alpha(X) \times \Psi_\alpha(X) + T(X) \times R(X)) \,. \end{aligned} \tag{2.89}$$

At the same time, the classical superpotential varies according to

$$\delta W_{\text{hop}} = \varepsilon\gamma \operatorname{tr}\left(\widetilde{Q}_{\ell+1} \frac{\Omega_\ell}{X - \Omega \cdots \Omega} Q_\ell\right) \equiv \varepsilon\gamma \operatorname{tr}\Big(\mathcal{M}_{\ell+1,\ell}(X)\Big) \,, \tag{2.90}$$

$$\delta W_\Sigma = \varepsilon\beta s_\ell \operatorname{tr}\left(\frac{\det \Omega_\ell}{X - \Omega \cdots \Omega}\right) = \varepsilon\beta s_\ell \Big(\det \Omega_\ell = v^{n_c}\Big) T(X) \,, \tag{2.91}$$

$$\delta W_{\text{def}} = \varepsilon \sum_{k=1}^{d} \nu_k \operatorname{tr}\left(\frac{(\Omega \cdots \Omega)^k}{X - \Omega \cdots \Omega}\right) = \varepsilon \Big[X \mathcal{W}'(X) \, T(X)\Big]_- \,. \tag{2.92}$$

where following the notations of [4], the $[X\mathcal{W}'(X)T(X)]_-$ stands for the negative-power part of the $X\mathcal{W}'(X)T(X)$ with respect to the power series expansion around $X = \infty$. Thus, we arrive at an anomalous equation of motion

$$\Big[\widetilde{\mathcal{W}}(X) \, T(X)\Big]_- + \gamma \operatorname{tr}\Big(\mathcal{M}_{\ell+1,\ell}(X)\Big) \stackrel{\text{cr}}{=} 2X \, R(X) \, T(X) - X \, \Psi^\alpha(X) \, \Psi_\alpha(X) \tag{2.93}$$

where $\widetilde{\mathcal{W}}(X) = X\mathcal{W}'(X) + \beta v^{n_c} s_\ell$ according to eq. (2.36) — and all singlets are equal on shell, $s_\ell \equiv s$. Solving eq. (2.93) for the on-shell link resolvent, we have

$$T(X) = \frac{t(X) \,-\, \gamma\, \mathrm{tr}\, \mathcal{M}_{\ell+1,\ell}(X) \,-\, X\Psi^2(X)}{\widetilde{\mathcal{W}}(X) \,-\, 2XR(X)} \tag{2.94}$$

for some polynomial $t(X) = [\widetilde{\mathcal{W}}(X)\, T(X)]_+$ of degree $\leq$ (d$-$1); we shall derive a more specific formula later in this section.

The anomalous equation of motion for the gaugino resolvent $\Psi^\alpha(X)$ also follows from varying a link field Ω_ℓ, but this time we have

$$\delta\Omega_\ell = \frac{\Omega_\ell}{X - \Omega \cdots \Omega} \times \frac{W_\ell^\alpha\, \varepsilon_\alpha}{4\pi} \tag{2.95}$$

where ε_α is an infinitesimal spinor. Consequently, $\delta W_{\text{hop}} \overset{\text{cr}}{=} 0$ (because of anti/quark and gaugino operators present in the same expression) and

$$\delta W_{\text{tree}} \overset{\text{cr}}{=} \delta W_{\text{def}} + \delta W_\Sigma = \Big[\widetilde{\mathcal{W}}(X)\Psi^\alpha(X)\varepsilon_\alpha\Big]_- \tag{2.96}$$

while the Konishi anomaly is

$$\begin{aligned}
&\frac{1}{32\pi^2}\,\mathrm{Tr}\Big(\frac{\mathbb{W}^\alpha \mathbb{W}_\alpha\, d\delta\Omega_\ell}{d\Omega_\ell}\Big) = \\
&= \quad \frac{X}{128\pi^3}\,\mathrm{Tr}\Big(\frac{W^\beta_{\ell+1} W_{\ell+1,\beta}}{X - \Omega\cdots\Omega} \otimes \frac{W_\ell^\alpha \varepsilon_\alpha}{X - \Omega\cdots\Omega} - 2\frac{W^\beta_{\ell+1}}{X - \Omega\cdots\Omega} \otimes \\
&\qquad\qquad \otimes \frac{W_\ell^\alpha \varepsilon_\alpha W_{\ell,\beta}}{X - \Omega\cdots\Omega} + \frac{1}{X - \Omega\cdots\Omega} \otimes \frac{W_\ell^\alpha \varepsilon_\alpha W_\ell^\beta W_{\ell,\beta}}{X - \Omega\cdots\Omega}\Big) \\
&\overset{\text{cr}}{=} \quad X(R(X) \times \Psi^\alpha(X)\varepsilon_\alpha + \Psi^\beta(X) \times R(X)\varepsilon_\beta + T(X) \times 0)\,,
\end{aligned} \tag{2.97}$$

and hence

$$\Big[\widetilde{\mathcal{W}}(X)\Psi^\alpha(X)\Big]_- \overset{\text{cr}}{=} 2XR(X)\Psi^\alpha(X)\,. \tag{2.98}$$

This anomalous equation of motion has a particularly simple solution for the on-shell gaugino resolvent:

$$\Psi^\alpha(X) = \frac{\zeta^\alpha(X)}{\widetilde{\mathbb{W}}(X) \,-\, 2XR(X)} \tag{2.99}$$

where $\zeta^\alpha(X) = [\widetilde{\mathbb{W}}(X)\Psi^\alpha(X)]_+$ is a spinor-valued polynomial of X of degree $\leq$ (d − 2). (Note $\Psi^\alpha(X) = O(1/X^2)$ for $X \to \infty$ because $\mathrm{tr}(W^\alpha_\ell) \equiv 0$.)

Finally, consider yet another link variation

$$\delta\Omega_\ell = \frac{\Omega_\ell}{X - \Omega\cdots\Omega} \times \frac{W^\alpha_\ell W_{\ell,\alpha} \times \varepsilon}{32\pi^2} \tag{2.100}$$

where ε is once again an infinitesimal c-number. This time, the Konishi anomaly comes up to

$$\frac{1}{32\pi^2}\,\mathrm{Tr}\left(\frac{\mathbb{W}^\alpha\mathbb{W}_\alpha\, d\delta\Omega_\ell}{d\Omega_\ell}\right) \stackrel{\mathrm{cr}}{=} \varepsilon X\Big(R(X)\times R(X) + 0 + 0\Big) \tag{2.101}$$

while

$$\delta W_{\mathrm{tree}} \stackrel{\mathrm{cr}}{=} \Big[\widetilde{\mathbb{W}}(X)R(X)\varepsilon\Big]_-\,, \tag{2.102}$$

and therefore, the on-shell gaugino bilinear resolvent $R(X)$ satisfies the quadratic equation

$$X\,[R(X)]^2 = \Big[\widetilde{\mathbb{W}}(X)R(X)\Big]_- \equiv \widetilde{\mathbb{W}}(X)\times R(X) - F(X) \tag{2.103}$$

where $F(X) = [\widetilde{\mathbb{W}}(X)R(X)]_+$ is yet another polynomial of X of degree $\leq$ (d − 1). Consequently,

$$R(X) = \frac{\widetilde{\mathbb{W}}(X) \mp \sqrt{\widetilde{\mathbb{W}}^2(X) \,-\, 4XF(X)}}{2X} \tag{2.104}$$

and according to eqs. (2.80), (2.83), (2.94) and (2.99), *all the on-shell resolvents of the quiver theory — $\mathcal{M}_{\ell',\ell}(X)$, $T(X)$, $\Psi^\alpha(X)$ and $R(X)$ — are meromorphic functions of the coordinates X and Y of the hyperelliptic Riemann surface $\mathbf{\Sigma}$ of*

$$Y^2 = \widetilde{\mathcal{W}}^2(X) - 4XF(X)\,. \tag{2.105}$$

In particular,

$$\Psi^\alpha(X,Y) = \frac{\zeta^\alpha(X)}{Y} \tag{2.106}$$

which means that $\mathbf{\Sigma}$ is the *Seiberg-Witten spectral curve* encoding the abelian gauge couplings of the quantum quiver theory modulo the $\mathrm{Sp}(n_{\mathrm{Abel}},\mathbb{Z})$ electromagnetic duality group.

The sign choice in eq. (2.106) corresponds to

$$R(X,Y) = \frac{\widetilde{\mathcal{W}}(X) - Y}{2X}\,. \tag{2.107}$$

In the $X \to \infty$ limit $Y \approx \pm\widetilde{\mathcal{W}}(X)$ depending on the sheet of the Riemann surface $\mathbf{\Sigma}$; on the $Y \approx +\widetilde{\mathcal{W}}(X)$ sheet, the gaugino bilinear resolvent behaves physically as $R \approx F(X)/\widetilde{\mathcal{W}}(X) = O(1/X)$ while on the other sheet we have un-physical divergence $R \approx \widetilde{\mathcal{W}}(X)/X = O(X^{d-1})$. Likewise, for $X \to 0$ R is regular on the first sheet but has an unphysical pole on the second sheet,

$$R(X,Y) \underset{X\to 0}{\longrightarrow} \begin{cases} \text{finite on the } Y \approx +\widetilde{\mathcal{W}}(X) \text{ sheet,} \\ \dfrac{\beta v^{n_c} s}{X} \text{ on the } Y \approx -\widetilde{\mathcal{W}}(X) \text{ sheet,} \end{cases} \tag{2.108}$$

so following Cachazo *et al* we shall refer to the two sheets of $\mathbf{\Sigma}$ "the physical sheet" and "the unphysical sheet". As in [5, 6], the distinction between the two sheets is clear over most of the X plane in the weakly coupled regime of the quiver but becomes blurred in the strongly coupled regime (except for $X \to \infty$ or $X \to 0$).

By this point, we have solved the anomalous equations of motion for the whole on-shell chiral ring of the quiver in terms of the three polynomials $t(X)$, $\zeta^\alpha(X)$ and $F(X)$ and one X-independent meson matrix $M_f^{f'}$. However, there are additional constraints on these parameters following from a yet another anomalous equation of motion due to quark-dependent variation of the link field

$$\delta\Omega_\ell = \frac{Q_{\ell+1}\varepsilon\widetilde{Q}_\ell}{X - \Omega\cdots\Omega} \tag{2.109}$$

where ε is an infinitesimal $F \times F$ matrix in the flavor space. This time, the Konishi anomaly is

$$\begin{aligned} &\frac{1}{32\pi^2}\,\mathrm{Tr}\left(\frac{\mathbb{W}^\alpha\mathbb{W}_\alpha\, d\delta\Omega_\ell}{d\Omega_\ell}\right) \overset{\mathrm{cr}}{=} \\ \overset{\mathrm{cr}}{=}\ & \frac{1}{32\pi^2}\,\mathrm{Tr}\left(0 + 0 + \frac{Q_{\ell+1}\varepsilon\widetilde{Q}_\ell\Omega_{\ell-1}\cdots\Omega_{\ell+1}}{X - \Omega\cdots\Omega} \otimes \frac{W_\ell^\alpha W_{\ell,\alpha}}{X - \Omega\cdots\Omega}\right) \\ =\ & \mathrm{tr}\Big(\varepsilon\,\mathcal{M}_{\ell+N,\ell+1}(X)\Big) \times R(X) \end{aligned} \tag{2.110}$$

while the tree-level superpotential varies according to

$$\delta W_{\mathrm{hop}} = \gamma\,\mathrm{tr}(\varepsilon\,\mathcal{M}_{\ell,\ell}(X) \times M_{\ell+1})\,, \tag{2.111}$$

$$\delta W_{\mathrm{def}} = \Big[W'(X)\,\mathrm{tr}\,(\varepsilon\,\mathcal{M}_{\ell+N,\ell+1}(X))\Big]_-\,, \tag{2.112}$$

$$\begin{aligned} \delta W_\Sigma &= \beta v^{n_c} s\,\mathrm{tr}\,(\varepsilon\,\mathcal{M}_{\ell,\ell+1}(X)) \\ &= \frac{\beta v^{n_c} s}{X}\,\mathrm{tr}\left(\varepsilon \times \left(\mathcal{M}_{\ell+N,\ell+1}(X) - \widetilde{Q}_\ell\Omega_\ell^{-1}Q_{\ell+1}\right)\right). \end{aligned} \tag{2.113}$$

This gives us an anomalous equation

$$R(X,Y) \times \mathcal{M}_{\ell+N,\ell+1}(X,Y) \overset{\mathrm{cr}}{=}$$

$$\stackrel{\text{cr}}{=} \gamma \mathcal{M}_{\ell,\ell}(X,Y) \times M_{\ell+1} + \frac{\widetilde{\mathcal{W}}(X)}{X} \times \mathcal{M}_{\ell+N,\ell+1}(X,Y) - \frac{C(X)}{X} \tag{2.114}$$

where $C(X)$ is yet another degree $\leq d-1$ polynomial of X. When combined with the on-shell equations (2.80), (2.83) and (2.104) for the mesonic and gaugino-bilinear resolvents, eq. (2.114) yields a quadratic equation for the meson matrix M, namely

$$(\gamma\mu M)^2 + \frac{\mu^N}{X}\widetilde{\mathcal{W}}(X) \times (\gamma\mu M) + \mu^N F(X) = \gamma\mu^2 \frac{X - \mu^N}{X} C(X) \,. \tag{2.115}$$

In the eigenbasis of the quark mass matrix μ (or rather of the μ^N) we may sequentially substitute $X = \mu_f^N$ and apply all the resulting equations at once since the M matrix does not depend on X. Consequently, the right hand side of eq. (2.115) vanishes regardless of the $C(X)$ polynomial, while the left hand side yields a matrix equation:

$$(\gamma\mu M)^2 + \widetilde{\mathcal{W}}(\mu^N) \times (\gamma\mu M) + \mu^N F(\mu^N) = 0 \,. \tag{2.116}$$

Now consider a quark flavor f with a non-degenerate μ_f^N. According to eq. (2.116), the on-shell value of the meson operator M_f^f satisfies a quadratic equation which has two solutions

$$M_f^f = \frac{-\widetilde{\mathcal{W}}(\mu_f^N) \mp \sqrt{\widetilde{\mathcal{W}}^2(\mu_f^N) - 4\mu_f^N F(\mu_f^N)}}{2\gamma\mu_f} = -\left.\frac{XR(X, \mp Y_{\text{phys}})}{\gamma\mu_f}\right|_{X=\mu_f^N} \tag{2.117}$$

corresponding to two different vacua of the quiver. The physical identities of these vacua become apparent in the weakly coupled regime of the theory where $XF(X) \ll \widetilde{\mathcal{W}}^2(X)$ over most of the complex X plane:[5] For the upper-sign solution of eq. (2.117) we have

$$M_f^f \equiv \widetilde{Q}^f Q_f \approx -\frac{\widetilde{\mathcal{W}}(\mu_f^N)}{\gamma\mu_f} \tag{2.118}$$

precisely as in eq. (2.40) — which strongly suggest that this is the discrete mesonic vacuum with one of the Coulomb moduli frozen at $\omega_j^N = \mu_f^N$ by the squark VEVs eq. (2.118).

To confirm the frozen Coulomb modulus in the fully-quantum language of the chiral ring we turn to the link resolvent $T(X,Y)$ and check its analytic structure near $X = \mu^N$: according to eq. (2.51), a frozen Coulomb modulus will manifest itself via a simple pole of residue exactly +1; more generally, k Coulomb moduli frozen at the same value yield a pole of residue k. According to eq. (2.94), the poles of $T(X,Y)$ at finite X follow from the poles of the mesonic resolvent $\mathcal{M}_{\ell+1,\ell}(X,Y)$ at $X = \mu_f^N$ and have residues

$$\operatorname*{Res}_{X=\mu_f^N}\Big[T(X,Y)\Big] = -\frac{\gamma}{Y} \times \operatorname*{Res}_{X=\mu_f^N}\Big[\operatorname{tr}\mathcal{M}_{\ell+1,\ell}(X)\Big]. \tag{2.119}$$

By eq. (2.80)

$$= -\left.\frac{\gamma\mu_f M_f^f + XR(X,Y)}{Y}\right|_{X=\mu_f^N} \tag{2.120}$$

by eq. (2.117)

$$= \begin{cases} +1 & \text{on the physical sheet of } \Sigma, \\ 0 & \text{on the unphysical sheet.} \end{cases}$$

Note that in terms of the spectral curve $\mathbf{\Sigma}$, $X = \mu^N$ describes two distinct points but only one of them carries a pole of the link resolvent. For the upper-sign solution (2.117) the pole is on the physical sheet, but it moves to the

[5] We shall see later in this section that $F(X) = O(\Lambda^{N(2n_c-F)})$.

unphysical sheet for the lower-sign solution where

$$\operatorname*{Res}_{X=\mu_f^N}\Big[T(X,Y)\Big] = \begin{cases} 0 & \text{on the physical sheet of the } \mathbf{\Sigma}, \\ +1 & \text{on the unphysical sheet.} \end{cases} \tag{2.121}$$

Only the physical sheet of $\mathbf{\Sigma}$ is visible at the classical and perturbative levels of the string theory, so the upper-sign solution (2.117) indeed has a Coulomb modulus frozen at $X = \mu_f^N$ but the lower-sign solution does not have any frozen moduli. Classically, this goes along with the zero meson VEV, but the quantum corrections generate $M_f^f \neq 0$; however, in the weak coupling regime

$$M_f^f \equiv \widetilde{Q}^f Q_f \approx -\frac{\mu_f^{N-1} F(\mu_f^N)}{\gamma \widetilde{\mathcal{W}}(\mu_f^N)} \propto \Lambda^{N(2n_c-F)} \longrightarrow 0\,. \tag{2.122}$$

Now consider a degenerate pair of quark flavors, say $\mu_1^N = \mu_2^N \neq 0$. In this case, the corresponding 2×2 block M_2 of the meson matrix M satisfies the quadratic equation (2.116) as a matrix, so each of the two block's eigenvalue may independently choose either root (2.117). This gives us two discrete solutions plus continuous family:

1. Both eigenvalues pick the bigger root (the upper sign in eq. (2.117)). This is a discrete mesonic vacuum where two Coulomb moduli are frozen at the same value $\omega_1^N = \omega_2^N = \mu_{1,2}^N$ by two quark flavors.

2. One eigenvalue picks the bigger root and the other picks the smaller root — which gives us a continuous family of solutions parameterizing the spontaneous breakdown SU(2) → U(1) of the flavor symmetry. This is the continuous mesonic branch of the quiver where two degenerate flavors freeze one Coulomb modulus at $\omega_1^N = \mu_{1,2}^N$. Classically, the M_2 block

of the meson matrix satisfies rank(M_2) $= 1$ as well as the trace condition (2.41); in the quantum theory, the M_2 has two non-zero eigenvalues but one of them is much smaller than the other (in the weak coupling regime) while the trace condition remains unchanged.

3. Both eigenvalues pick the smaller root (the lower sign in eq. (2.117)). This is a discrete Coulomb vacuum where none of the ω_j^N are frozen at $X = \mu_{1,2}^N$, and in the weakly coupled regime, the whole M_2 meson block becomes small according to eq. (2.122).

To confirm our identification of the above vacua we calculate the poles of the link resolvent $T(X,Y)$ at $X = \mu_{1,2}^N$ and find

$$\operatorname*{Res}_{X=\mu_{1,2}^N}\Big[T(X,Y)\Big] \;=\; -\left.\frac{\gamma\mu_{1,2}\,\mathrm{tr}(M_2)\,+\,2XR(X,Y)}{Y}\right|_{X=\mu_f^N} \tag{2.123}$$

$$= \begin{cases} k & \text{on the physical sheet,} \\ 2-k & \text{on the unphysical sheet,} \end{cases} \tag{2.124}$$

where $k = 0,1,2$ is the number of eigenvalues of M_2 equal to the bigger root of eq. (2.116). Again, the pole count *on the physical sheet* gives us the correct number of frozen Coulomb vacua for each of the three solutions.

Likewise, for $m > 2$ degenerate flavors we have two discrete solutions plus $m-1$ continuous families distinguished by the number $k = 0,1,2.\ldots,m$ of meson eigenvalues which pick the bigger root of the quadratic equation. The residue calculation yields

$$\operatorname*{Res}_{X=\mu_{1,2}^N}\Big[T(X,Y)\Big] = \begin{cases} k & \text{on the physical sheet,} \\ m-k & \text{on the unphysical sheet,} \end{cases} \tag{2.125}$$

which confirms the physical meaning of k as the number of Coulomb moduli frozen at $X = \mu_n$. Classically, up to m moduli can be frozen at this point, and this is exactly what we see in the quantum theory as well.

Note that all poles of the link resolvent $T(X,Y)$ on both sheets of the spectral curve $\mathbf{\Sigma}$ have non-negative integer residues. (cf. (2.121)–(2.125)). For the poles on the physical sheet this follows from the physical meaning of the poles as frozen Coulomb moduli, the residue being the number of such moduli frozen at the same point. The poles on the unphysical sheet do not have a clear physical meaning — indeed the whole unphysical sheet of $\mathbf{\Sigma}$ does not exist at the perturbative level of the quiver theory — but their residues are subject to the same rules as the physical poles by reasons of analytic continuation: as one wanders around the parameter space of the quiver theory — and in particular changes the roots $\wp_i$ of the deformation $\widetilde{\mathcal{W}}(X)$ — the point $X = \mu_f^N$ may cross the branch cut of the Riemann surface $\mathbf{\Sigma}$ and move the pole from the unphysical sheet to the physical sheet or vice verse. Thus any pole is physical *somewhere* in the parameter space of the theory — and that's why in eq. (2.125) the residue $m - k$ on the unphysical sheet has the same spectrum of values as the physical sheet's residue k, namely $m - k = 0, 1, 2, \ldots, m$.

Actually, for the $[\mathrm{SU}(n_c)]^N$ quiver theory with some exactly massless quark flavors (*i.e.*, for $\Delta F > 0$), the link resolvent $T(X,Y)$ has a pole stuck to the unphysical sheet at $X = 0$. Indeed, because of the unphysical pole (2.108) of the $R(X,Y)$ at $(X = 0, Y_{\mathrm{unphys}})$, the massless block of the mesonic resolvent $\mathcal{M}_{\ell+1,\ell}$ also have a pole of residue $(+\beta v^{n_c} s = -Y)\gamma^{-1}\delta_f^{f'}$ at this point (cf.

eqs. (2.80) and (2.83)), and consequently

$$T(X,Y) \underset{X\to 0}{\longrightarrow} \begin{cases} \text{finite on the physical sheet,} \\ \dfrac{\Delta F}{X} \text{on the unphysical sheet.} \end{cases} \tag{2.126}$$

Note that despite its unphysical nature, this pole has a non-negative integer residue anyway. Again, this is related to our ability to move the poles all over the spectral curve Σ — including the physical sheet — by changing the parameters of the theory. Indeed, having ΔF exactly massless quark flavors is simply a choice of parameters $\mu_{n_f+1}, \ldots, \mu_{n_F+\Delta F}$, and once we change these parameters to $\mu' \neq 0$, the pole (2.126) moves away from zero to $X = \mu'^N$, which may end up on the physical sheet when μ'^N crosses a branch cut of the spectral curve.

2.2.3 Analytic considerations

Having learned what we could from the anomaly equations of the quiver, let us now consider the analytic properties of the link resolvent $T(X,Y)$. As in [5, 6], the differential $T(X,Y)dX$ is meromorphic, and has exactly integer periods (in units of $2\pi i$) for all closed contours on the Riemann surface $\mathbf{\Sigma}$, including the little contours around the poles at $X = \mu_f^N$ as well as big contours around the branch cuts of $\mathbf{\Sigma}$ over the X plane. Consequently,

$$T(X,Y)\,dX = \frac{d\,\Xi}{\Xi} \tag{2.127}$$

for some meromorphic function $\Xi(X,Y)$, and since all poles of T at finite X have positive residues, it follows that $\Xi(X,Y)$ has zeros rather than poles at $X \neq \infty$. Furthermore, the product $\Xi(X,+Y) \times \Xi(X,-Y)$ is a single-valued function of X, has a simple zero at each $X = \mu_f^N$ (regardless of the corresponding pole of $T(X,Y)$ being on the physical or the unphysical sheet), and has no essential singularity at $X = \infty$, which immediately implies polynomial behavior

$$\Xi(X,+Y) \times \Xi(X,-Y) = \alpha B(X) \equiv \alpha \prod_{f=1}^{F} (X - \mu_f^N) = \alpha\, X^{\Delta F} \prod_{f=1}^{n_f} (X - \mu_f^N) \tag{2.128}$$

for some constant (i.e. X-independent) α. Also, for $X \to \infty$ on the physical sheet $T(X,Y) \approx n_c X^{-1}$, which translates to $\Xi(X,Y) \propto X^{n_c}$ — and therefore on the unphysical sheet, $\Xi(X,Y) \propto X^{F-n_c}$. For $F \leq 2n_c$ this means that the sum $\Xi(X,+Y) + \Xi(X,-Y)$ — which is also a single-valued holomorphic function of X — grows like X^{n_c} at $X \to \infty$, hence

$$\Xi(X,+Y) + \Xi(X,-Y) = P(X) \equiv \prod_{j=1}^{n_c} (X - \varpi_j) \tag{2.129}$$

is another polynomial of X of degree n_c with some roots $\varpi_1, \ldots, \varpi_{n_c}$ — and therefore

$$\Xi(X,Y) = \frac{1}{2} \left(P(X) \pm \sqrt{P^2(X) \,-\, 4\alpha B(X)} \right). \tag{2.130}$$

Finally, to assure that this function is single-valued on $\mathbf{\Sigma}$, the ratio

$$\frac{\sqrt{P^2(X) - 4\alpha B(X)}}{Y} = \sqrt{\frac{P^2(X) - 4\alpha B(X)}{\widetilde{\mathbb{W}}^2(X) - 4XF(X)}}$$

must be a rational function of X — which means polynomial factorization

$$\widetilde{\mathbb{W}}^2(X) - 4XF(X) \;=\; H(X) \times K^2(X)\,, \tag{2.131}$$

$$P^2(X) - 4\alpha B(X) \;=\; H(X) \times G^2(X)\,, \tag{2.132}$$

where $H(X)$, $K(X)$ and $G(X)$ are three more polynomials of X of respective degrees $2h$, $(d-h)$ and (n_c-h) where $d = \deg(\widetilde{\mathcal{W}})$ and $h = 1, 2, \ldots, \min(d, n_c)$ depending on a particular vacuum of the quantum quiver theory. Specifically, a vacuum which supports $n_{\text{photon}}^{\text{free}}$ free massless photon abelian gauge fields should have $h = n_{\text{photon}}^{\text{free}} + 1$ and the $H(X)$ polynomial should have $2h$ distinct simple roots. Indeed, according to eqs. (2.131) and (2.106), these are conditions for the gaugino resolvent $\Psi(X)$ to have precisely h branch cuts and therefore $h - 1 = n_{\text{photon}}^{\text{free}}$ independent zero modes corresponding to $h - 1$ species of free massless photons.[6]

But regardless of a particular solution, (2.131)–(2.132) allow us to rewrite the Seiberg-Witten curve of the quiver in terms of the parameters of the link resolvent $T(X)$ rather than gaugino condensate resolvent. Specifically, for $\hat{Y} = \frac{G(X)}{2K(X)} Y + \frac{\widetilde{\mathcal{W}}(X)}{2}$ we have

$$\mathbf{\Sigma} = \{(X, \hat{Y}) : \; \hat{Y}^2 - P(X) \times \hat{Y} + \alpha B(X) = 0\} \tag{2.133}$$

which looks exactly like the SW curve [18, 19] of the $\mathcal{N} = 2$ SQCD_4 with n_c colors and F flavors. This similarity is no accident but a reflection of the ultra-low energy limit of the quiver theory: For finite N, energies $E \ll 1/(Na)$ are governed by the dimensional reduction of the deconstructed SQCD_5 to 4D —

[6] In terms of the contour integrals $\oint_{\mathcal{C}} \Psi^\alpha(X) dX$, contours around branch cuts yield free photons while contours around poles (if any) yield photons interacting with massless quarks and antiquarks (or monopoles and antimonopoles, dyons and antidyons, *etc., etc.* In the deformed quiver theory, such massless quarks, *etc.*, generally develop non-zero VEVs, so only the free photons remain exactly massless. In terms of the resolvent $\Psi^\alpha(X)$ it means no poles but only branch cuts, hence in eq. (2.106) the spinor-valued polynomial $\zeta^\alpha(X)$ must factorize as $\zeta^\alpha(X) = K(X) \times \hat{\zeta}^\alpha(X) \implies \Psi^\alpha(X) = \pm \frac{\hat{\zeta}^\alpha(X)}{\sqrt{H(X)}}$ where $\hat{\zeta}^\alpha(X)$ has rank $h - 2$ and hence $h - 1 = n_{\text{photon}}^{\text{free}}$ independent spinor-valued coefficients $\Longrightarrow$ zero modes.

which is nothing but the $\mathcal{N} = 2$ SQCD$_4$ whose gauge symmetry is the diagonal SU(n_c) subgroup of the $[\mathrm{SU}(n_c)]^N$ quiver. In this effective $\mathcal{N} = 2$ SQCD$_4$, the α parameter in eq. (2.133) is generated at the one-instanton level of the diagonal SU(n_c); in terms of the whole quiver, this means one instanton in each and every $\mathrm{SU}(n_c)_\ell$ factor according to eq. (2.72). Indeed, it is easy to see that were we to turn off quantum effects in a single $\mathrm{SU}(n_c)_\ell$ factor (by turning off $\Lambda_\ell \to 0$), then the low-energy limit of the whole quiver theory would become that classical $\mathrm{SU}(n_c)_\ell$ gauge theory with F quark flavors and an adjoint chiral field Φ_ℓ made from $\Omega_{\ell-1} \cdots \Omega_1 \Omega_N \cdots \Omega_\ell$, and without the IR-visible quantum effects we would have $\alpha = 0$. Hence, in the fully-quantum quiver

$$\alpha \sim \exp\left(-\frac{8\pi^2}{g^2_{\mathrm{diag}}}\right) \sim \prod_\ell \Lambda_\ell^{2n_c-F} = \left(\Lambda^{2n_c-F}\right)^N \tag{2.134}$$

where the last equality assumes $\Lambda_\ell \equiv \Lambda$, as required by the discretized translational invariance of the deconstructed SQCD$_5$.

To verify that $\alpha \neq 0$ is indeed generated at the one-diagonal-instanton level — and also to determine the pre-exponential factors in eq. (2.134) — we use the anomalous $\mathrm{U}(1)_A \times \mathrm{U}(1)_\Omega \times \mathrm{U}(1)_R$ symmetry of the quiver which transforms both the chiral fields and the couplings according to the following charge table: In light of these charges, the only allowed formula for α of the type $\alpha = \Lambda^{\mathrm{power}} \times$ something else is

$$\alpha = \left(\Lambda^{2n_c-F}\gamma^F\right)^N \times \text{a numeric constant}, \tag{2.135}$$

so the one-diagonal-instanton level of eq. (2.134) is indeed correct.

FIELDS & PARAMETERS	CHARGES		
	q_A	q_Ω	q_R
Q_ℓ	1	0	0
$\widetilde{Q}_\ell$	1	0	0
Ω_ℓ	0	1	0
W_ℓ^α	0	0	1
μ_f	0	1	0
γ	-2	-1	2
$\Lambda_\ell^{2n_c-F}$	$2F$	$2n_c$	$-2F$
X	0	N	0
$\widetilde{\mathcal{W}}(X)$	0	0	2
$F(X)$	0	$-N$	4
α	0	$(2n_c-F)N$	0

Table 2.1: Charges of fields and parameters

Finally, the numeric factor in eq. (2.135) follows from the decoupling of ultra-heavy quark flavors from the low-energy chiral ring of the quiver. Indeed, suppose one of the n_f massive quark flavors of the quiver theory becomes ultra-heavy, *eg.* $\gamma\mu_1 \to \infty$. In this limit, the chiral ring equations should approximate those of the theory without the ultra-heavy flavor; in particular, in eq. (2.128) we should have

$$\forall \text{ fixed finite } X: \ \ \alpha B(X)\big|_{n_f} \longrightarrow \alpha' B'(X)\big|_{n'_f=n_f-1} \Longrightarrow \alpha' = \alpha \times (-\mu_1^N)\,. \tag{2.136}$$

At the same time, for each $\mathrm{SU}(n_c)_\ell$ gauge factor of the quiver we decouple one ultra-heavy quark flavor of mass $(-\gamma\mu_a) \to \infty$ out of $n_c + F$ effective flavors altogether, hence

$$\forall \ell: \ (\Lambda'_\ell)^{2n_c-F+1} = (\Lambda_\ell)^{2n_c-F} \times (-\gamma\mu_a)\,. \tag{2.137}$$

Consequently, mutual consistency of (2.135)–(2.137) completely determines

$$\alpha = (-1)^F \left((-\gamma)^F \Lambda^{2n_c-F}\right)^N . \tag{2.138}$$

For the $\mathcal{N} = 2$ SQCD$_4$, roots of the $P(X)$ polynomial in eq. (2.133) are the Coulomb moduli of the theory; classically, $P = \det(X - \Phi)$ and the roots are the eigenvalues of the adjoint field. The same holds true for the $[\mathrm{SU}(n_c)]^N$ quiver theory: in the weak coupling limit $\alpha \to 0$, eq. (2.130) yields $\Xi(X,Y) \approx P(X)$ *on the physical sheet of the spectral curve* $\boldsymbol{\Sigma}$ — and therefore

$$T(X)\big|_{\substack{\text{physical}\\ \text{sheet}}} \approx \sum_{j=1}^{n_c} \frac{1}{X - \varpi_j} , \tag{2.139}$$

exactly as in the classical formula (2.50), with the roots ϖ_j of the $P(X)$ polynomial playing the role of the classical Coulomb moduli ω_j^N. In other words, classically $\varpi_j \equiv \omega_j^N$ and $P(X) = \det(X - \Omega_N \cdots \Omega_1)$. In the quantum theory, we need to redefine the Coulomb moduli because the link eigenvalues ω_j are no longer chiral gauge-invariant operators; in light of eq. (2.139), we *define* the Coulomb moduli of the quantum quiver as the roots of the $P(X)$,

$$\omega_j^N \stackrel{\text{def}}{=} \varpi_j \qquad \text{i.e.} \qquad P(X) \equiv \prod_{j=1}^{n_c} (X - \omega_j^N) . \tag{2.140}$$

For $F < n_c$, this is equivalent to identifying $P(X)$ with the $\det(X - \Omega_N \cdots \Omega_1)$ operator in the quantum theory, but for $F \geq n_c$ the relation is more complicated; we shall return to this issue in section 2.3.3.

For a deformed quiver with $\widetilde{\mathcal{W}}(X) \not\equiv 0$, the Coulomb moduli (2.140) are constrained by the need to satisfy both (2.131)–(2.132) at the same time as well as[7]

$$\prod_{j=1}^{n_c} \omega_j^N \equiv (-1)^{n_c} P(X=0) = V^{Nn_c} = [v^{n_c} + \text{quantum corrections}]^N \quad (2.141)$$

which translates the determinant constraint (2.48) into the language of the Coulomb moduli (2.140). Consequently, the continuous moduli space of the un-deformed deconstructive quiver breaks into a discrete set of solutions describing different vacua of the deformed quiver. For example, let us consider the $[\mathrm{SU}(2)]^N$ quiver with one quark flavor and degree $d = 2$ deformation superpotential $\mathcal{W}(X) = \nu_2(\frac{1}{2}X^2 + AX)$. For this quiver, we have one $h = 2$ solution and five $h = 1$ solutions. The $h = 2$ solution

$$\begin{aligned}
P(X) &= X^2 + AX + V^{2N}\,, \\
B(X) &= X - \mu^N\,, \\
G(X) &\equiv 1\,, \qquad K(X) \equiv \nu_2\,, \\
\beta v^{n_c} s &= \nu_2 \sqrt{V^{4N} + 4\alpha\mu^N}\,, \\
&\Downarrow \\
\widetilde{\mathcal{W}}(X) &= \nu_2 \left(X^2 + AX + \sqrt{V^{4N} + 4\alpha\mu^N} \right), \\
F(X) &= \frac{\nu_2^2}{2} \left((\sqrt{V^{4N} + 4\alpha\mu^N} - V^{2N}) \times (X + A) + 2\alpha \right), \quad (2.142)
\end{aligned}$$

describes the unique discrete Coulomb vacuum with unbroken $\mathrm{U}(1) \subset \mathrm{SU}(2)_{\mathrm{diag}}$. Classically, this calls for $n_1 = n_2 = 1$ 'occupation numbers' for the two roots $\wp_{1,2}$ of the $\widetilde{\mathcal{W}}(X)$ polynomial — and indeed, in the weak coupling limit $\alpha \to 0$

[7] According to eq. (2.264) in section 2.3.3, $V^{Nn_c} = V_1^{Nn_c} + V_2^{Nn_c}$, where $V_{1,2}^{n_c}$ are the two roots of eq. (2.230).

we have $\widetilde{\mathcal{W}}(X) \approx \nu_2 P(X)$ and hence $\omega_1^N \approx \wp_1$ and $\omega_2^N \approx \wp_2$. Also, in this limit

$$F(X) \propto \alpha \Longrightarrow \text{ gaugino condensates } \mathcal{S}_{1,2} \propto \alpha \tag{2.143}$$

as expected for a purely abelian Coulomb vacuum where gaugino bilinears are generated by the coset instantons of the $\mathrm{SU}(2)_{\mathrm{diag}}/\,\mathrm{U}(1)$. Outside the weak coupling limit, the Coulomb moduli $\omega_{1,2}^N$ move away from the roots $\wp_{1,2}$ of the $\widetilde{\mathcal{W}}(X)$ polynomial and the occupation numbers $n_{1,2}$ become ill-defined. Nevertheless, me may identify the $h = 2$ solution with the Coulomb vacuum of the quiver simply because $h = 2$ implies a Seiberg-Witten curve of genus $g = 1$ and hence a single exactly massless photon of the unbroken $\mathrm{U}(1) \subset \mathrm{SU}(2)_{\mathrm{diag}}$.

The $h = 1$ solutions describe mesonic and pseudo-confining vacua without any massless photons whatsoever. The general form of these solutions is given by

$$\begin{aligned}
P(X) &= X^2 - (c + 2d)X + V^{2N}\,, \\
B(X) &= X - \mu^N\,, \\
G(X) &= X - (c + d)\,, \\
H(X) &= (X - d)^2 + 4e\,, \\
K(X) &= \nu_2(X + A + d)\,, \\
\beta v^{n_c} s &= -\nu_2(A + d)\sqrt{d^2 + 4e}\,, \\
&\Downarrow \\
\widetilde{\mathcal{W}}(X) &= \nu_2\left[X^2 + AX - (A + d)\sqrt{d^2 + 4e}\right], \\
F(X) &= -\nu_2^2 e\left[\frac{2(A + d)}{d + \sqrt{d^2 + 4e}}\,(X + A) + X + 2(A + d)\right],
\end{aligned} \tag{2.144}$$

where the parameters c, d, e satisfy a quintic equation system

$$c + d - \frac{e}{c} = \mu^N\,, \qquad ce = \alpha\,, \qquad 2e + cd + d^2 = V^{2N}\,, \tag{2.145}$$

hence five solutions. In the weak coupling limit $\alpha \to 0$, four solutions

$$d \approx \pm V^N\,, \qquad c \approx \pm'\sqrt{\frac{\alpha}{\pm V^N - \mu^N}}\,, \qquad e \approx \pm'\sqrt{\alpha(\pm V^N - \mu^N)}\,,$$

$$\Longleftarrow$$

$$\begin{aligned} P(X) &\approx (X \mp V^N)^2 \mp' 4\sqrt{\alpha(\pm V^N - \mu^N)}\,, \\ \widetilde{\mathbb{W}}(X) &\approx \nu_2(X \mp V^N)(X + A \pm V^N)\,, \end{aligned} \tag{2.146}$$

describe two pairs of pseudo-confining vacua (*i.e.*, two classical vacua solutions denoted by the un-primed $\pm$ signs, each giving rise to two quantum vacuum states denoted by the primed signs $\pm'$), while the fifth solution

$$c \approx \frac{\mu^{2N} - V^{2N}}{\mu^N}\,, \qquad d \approx \frac{V^{2N}}{\mu^N}\,, \qquad e \approx \frac{\alpha\mu^N}{\mu^{2N} - V^{2N}}\,,$$

$$\Longleftarrow$$

$$\begin{aligned} P(X) &\approx \Big(X - \frac{V^{2N}}{\mu^N}\Big) \times \Big(X - \mu^N\Big)\,, \\ \widetilde{\mathbb{W}}(X) &\approx \nu_2\Big(X - \frac{V^{2N}}{\mu^N}\Big) \times \Big(X + A + \frac{V^{2N}}{\mu^N}\Big) \end{aligned} \tag{2.147}$$

describes a discrete mesonic vacuum of the quiver. Indeed, the solutions (2.146) has both Coulomb moduli / roots of $P(X)$ captured by the same deformation root, $\varpi_1 \approx \varpi_2 \approx \wp_1 \implies$ classically unbroken pseudo-confining $\mathrm{SU}(2)_{\mathrm{diag}}$, but the (2.147) solution has only one Coulomb modulus $\varpi_1 \approx \wp_1$ captured by the deformation root while the other modulus is frozen at the quark mass, $\varpi_2 \approx \mu^N$ regardless of the deformation superpotential. Also, for the fifth solution (2.147), $T(X, Y)$ has a pole at $X = \mu^N$ on the physical sheet — and hence large mesonic VEV $\langle M \rangle$ according to eq. (2.118) — while the first four solutions (2.146) have this pole on the un-physical sheet — and hence small mesonic VEVs according to eq. (2.122).

For all five $h = 1$ solutions, the $R(X)$ resolvent has a single branch cut near the first deformation root $\wp_1 \approx d$ while the second deformation root $\wp_2$ remains unoccupied. In the weak coupling limit the branch cut becomes a pole, and indeed all five solutions have

$$R(X)\big|_{\substack{\text{physical}\\\text{sheet}}} \approx \frac{F(X)}{\widetilde{\mathcal{W}}(X)} \approx -\frac{\nu_2 e(2 + A/d)}{X - d} \tag{2.148}$$

with a single pole of residue

$$\mathcal{S} = -\nu_2 \left(2 + \frac{A}{d}\right) e = \begin{cases} O(\sqrt{\alpha}) & \text{for the first four solutions } (2.146)\text{, and} \\ O(\alpha) & \text{for the fifth solution } (2.147). \end{cases} \tag{2.149}$$

This gives us yet another way of distinguishing between the mesonic and the pseudo-confining vacua of a weakly coupled quiver: the solutions with $\mathcal{S} \propto \alpha^{1/2}$ describe the pseudo-confining vacua with classically unbroken $\mathrm{SU}(2)_{\mathrm{diag}}$ which produces the gaugino condensate at the half-instanton level, while the solution with $\mathcal{S} \propto \alpha^1$ describes the discrete mesonic vacuum where the $\mathrm{SU}(2)_{\mathrm{diag}}$ is broken at the classical level and it takes a whole (diagonal) instanton to generate $\mathcal{S} \neq 0$.

For the strongly coupled quiver theory, the distinction between the mesonic and the pseudo-confining $h = 1$ solutions becomes moot. Indeed, for large α all five solutions have both Coulomb moduli / roots of $P(X)$ nowhere near deformation roots $\wp_{1,2}$ or the quark mass μ^N, so the occupation numbers are of no use. Likewise, all five solutions have large mesonic VEVs as well as large gaugino condensates $\mathcal{S}$ which depend on α in a complicated way. Furthermore, all five solutions are permuted into each other by monodromies of the (μ^N, α, V^{2N}) parameter space around the discriminant locus of eqs. (2.145), hence it's mathematically impossible to find any global distinction which remains valid for all parameter values. Physically, this means that the difference between the pseudo-confining and the mesonic vacua of the quantum quiver theory is an artifact of the weak coupling approximation. At strong coupling, there are simply five photon-less vacua, without any meaningful way of telling which is mesonic and which is pseudo-confining: this is an example of the confinement $\leftrightarrow$ Higgs duality which shows up in all kinds of supersymmetric gauge theories.

Other quiver theories with more colors and/or flavors exhibit the same general behavior: in the weak coupling regime of the deformed quiver, the solutions of (2.131)–(2.132) and (2.141) approximate the classical vacua discussed in section 2.1.3; in the $\alpha \to 0$ limit, the approximation becomes exact. When the coupling becomes strong we still have exactly the same number of solutions, but their physical identities become blurred and only the net abelian rank $n_{\text{Abel}} = h - 1$ and the number n_{mes} of *continuous* mesonic moduli remain clear. Generally, all vacua of the same n_{Abel} and n_{mes} are connected by the monodromies of the parameter space and thus cannot be physically told apart in the strongly coupled regime.

Finally, consider what happens when we un-deform a deconstructive quiver by removing the deformation superpotential (2.35). Specifically, let us take to zero the leading coefficient ν_d of the polynomial $\widetilde{\mathcal{W}}(X)$ while keeping its roots $\wp_1, \ldots, \wp_d$ finite. In this limit, eq. (2.131) yields $K(X) \propto \nu_d$, $F(X) \propto \nu_d^2$, hence $Y \propto \nu_d$, $R(X) \propto \nu_d$ and therefore all gaugino condensates diminish as $O(\nu_d)$. And according to eq. (2.117), all *discrete* mesonic VEVs also diminish as $O(\nu_d)$. Thus, in the undeformed limit $\nu_d = 0$, the pseudo-confining and the discrete mesonic vacua lose their characteristic VEVs and become indistinguishable from the Coulomb vacua which just happen to have similar moduli $(\omega_1^N, \ldots, \omega_{n_c}^N)$.

Ultimately, for $\nu_d = 0$ eq. (2.131) becomes a trivial $0 = 0$ identity and the roots (2.140) become free moduli of the continuous Coulomb moduli space. For generic values of these moduli, the Seiberg-Witten spectral curve (2.133) has maximal genus $g = n_c - 1$ describing a purely-Coulomb vacuum with a maximal number $n_{\text{Abel}} = n_c - 1$ of massless photons. There are also interesting curves of reduced genus, eg. $g = n_c - 2$ when $B(X)$ has a double zero and $P(X)$ has a simple zero at *exactly* the same point — physically, this corresponds to a mesonic vacuum family with a frozen Coulomb modulus at a degenerate quark mass, $\omega_1^N = \mu_1^N = \mu_2^N$. Finally, there are reduced-genus curves at special loci in the Coulomb moduli space where some roots of the $P^2(X) - 4\alpha B(X)$ just happen to coincide. In this case, the low-energy theory retains it full complement of $n_{\text{Abel}} = n_c - 1$ photons but in addition some electrically or magnetically charged particles become exactly massless, and the spectral curve has a reduced rank $g < n_{\text{Abel}}$ because some photons are no longer free in the infrared limit. This works exactly as in the $\mathcal{N} = 2$ SQCD$_4$ — and indeed the very-low-energy limit of the $[\mathrm{SU}(n_c)]^N$ quiver theory **is** the $\mathcal{N}=2$ SQCD$_4$ with the $\mathrm{SU}(n_c)_{\mathrm{diag}}$ gauge symmetry.[8]

This completes our study of the $[\mathrm{SU}(n_c)]^N$ quiver's chiral ring — or rather its sub-rings generated by the non-baryonic operators. The baryonic and the anti-baryonic generators are presented in the following section 2.3.

2.3 Baryons and other determinants

In this section we complete the chiral ring of the $[\mathrm{SU}(n_c)]^N$ quiver theory by adding all kinds of baryonic and antibaryonic generators. Note that besides the ordinary baryonic operators $B^{(\ell)}_{f_1,\ldots,f_{n_c}} = \epsilon_{j_1,\ldots,j_{n_c}} Q^{j_1}_{\ell,f_1} \cdots Q^{j_{n_c}}_{\ell,f_{n_c}}$ comprised of n_c quarks at the same quiver node ℓ, there is a great multitude of baryon-like gauge-invariant operators where the quarks sit at different nodes $\ell_1, \ell_2, \ldots, \ell_{n_c}$ but are connected to each other via chains of link fields. Or rather each quark is connected by a link chain $\Omega_{\ell-1}\Omega_{\ell-2}\cdots\Omega_{\ell_i}$ to a common node ℓ where n_c quark indices of the $\mathrm{SU}(n_c)_\ell$ are combined into a gauge singlet,

$$\begin{aligned} B_{f_1,f_2,\ldots,f_{n_c}}(\ell;\ell_1,\ell_2,\ldots,\ell_{n_c}) \quad &= \quad \epsilon_{j_1,j_2,\ldots,j_{n_c}} (\Omega_{\ell-1}\cdots\Omega_{\ell_1} Q_{\ell_1,f_1})^{j_1} \times \\ \times (\Omega_{\ell-1}\cdots\Omega_{\ell_2} Q_{\ell_2,f_2})^{j_2} \quad &\cdots \quad (\Omega_{\ell-1}\cdots\Omega_{\ell_{n_c}} Q_{\ell_{n_c},f_{n_c}})^{j_{n_c}} . \end{aligned} \tag{2.150}$$

From the 5D point of view, these are *multi-local* operators which create/annihilate *un-bound* sets of n_c quarks while the link chains deconstructs the un-physical

[8] From the deconstruction point of view, this effective very-low-energy theory is the Kaluza-Klein reduction of the SQCD$_5$ on a circle of size $2\pi R = Na$. and any 4D non-perturbative effects at $E < (Na)^{-1}$ are artifacts of the finite-size compactification. However, as far as a finite-N quiver is concerned, the non-perturbative effects associated with the low-energy $\mathrm{SU}(n_c)_{\mathrm{diag}}$ are just as important as any other non-perturbative effect in the theory.

Wilson strings which allow for manifest gauge invariance of such multi-local operators

$$\begin{aligned}
[B(x;x_1,\ldots,x_{n_c})]_{f_1,\ldots,f_{n_c}} &= \epsilon_{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \underset{\substack{\text{Path}\\ \text{ordered}}}{\exp} \left(i \int_{x_i}^{x} dx^\mu A_\mu(x) \right)^{j_i}_{j'_i} Q^{j'_i}_{\ell_i,f_i} \\
&\longrightarrow \epsilon_{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} (\Omega_{\ell-1}\cdots\Omega_{\ell_i} Q_{\ell_i,f_i})^{j_i} \qquad (2.151) \\
&\quad \text{for } x_i^{0,1,2,3} \equiv x^{0,1,2,3}\,,\ x^4 = a\ell \text{ and } x_i^4 = a\ell_i\,.
\end{aligned}$$

However, just like their mesonic counterparts (2.54), from the 4D point of view these are local chiral gauge-invariant operators and we must include them in the $[\mathrm{SU}(n_c)]^N$ quiver's chiral ring.

Note that each of the n_c link chains $\Omega_{\ell-1}\cdots\Omega_{\ell_i}$ may wrap a few times around the whole quiver $(\Omega_{\ell-1}\cdots\Omega_1(\Omega_N\cdots\Omega_1)^k\Omega_N\cdots\Omega_{\ell_i})$, or go in reverse direction $(\Omega_\ell^{-1}\Omega_{\ell+1}^{-1}\cdots\Omega_{\ell_i+1}^{-1}$ for $\ell_i > \ell)$, or both, and all these possibilities give rise to a whole zoo of baryonic generators. To make sure the readers do not get lost in this big zoo, we would like to begin our presentation with a smaller menagerie of baryonic operators in a single $\mathrm{SU}(n_c)$ gauge theory with an adjoint field Φ as well as quarks and antiquarks. Once we explore this menagerie in section 2.3.1 below, we shall return to the baryonic and antibaryonic operators of the $[\mathrm{SU}(n_c)]^N$ quiver in section 2.3.2. Finally, in section 2.3.3 we shall study quantum corrections to determinants of link chains (as in $\det(\Omega_{\ell'}\cdots\Omega_\ell) = \det(\Omega_{\ell'})\cdots\det(\Omega_\ell) + \text{corrections}$). Of particular importance is the determinant $\det(\Omega_N\cdots\Omega_1)$ of the whole quiver: it governs quantum corrections to $V^{n_c} = v^{n_c} + \cdots$ in eq. (2.141) and affects the origin of the quivers' baryonic Higgs branch in the Coulomb moduli space.

2.3.1 Φ-baryons in the single $\mathrm{SU}(n_c)$ theory

Consider the chiral ring of the $\mathcal{N}=1$ $\mathrm{SU}(n_c)$ gauge theory with n_f chiral quark and antiquark fields Q^f and $\widetilde{Q}_f$, a single adjoint field Φ, and the superpotential

$$W = \mathrm{tr}\Big(\mathcal{W}(\Phi)\Big) + \widetilde{Q}^{f'} m_{f'}^{f}(\Phi) Q_f \qquad (2.152)$$

where $m_{f'}^{f}(\Phi)$ is a matrix-valued polynomial of Φ of degree $p \geq 1$. The non-baryonic generators of this ring are *exactly* as in the $\mathrm{U}(n_c)$ theory with the same chiral fields: according to Cachazo *et al* [4–6, 20], all of these generators are encoded in just four resolvents

$$\begin{aligned} T(X) &= \mathrm{tr}\,\frac{1}{X-\Phi}\,, & \Psi^\alpha(X) &= \frac{1}{4\pi}\,\mathrm{tr}\,\frac{W^\alpha}{X-\Phi}\,, \\ R(X) &= \frac{1}{32\pi^2}\,\mathrm{tr}\,\frac{W^\alpha W_\alpha}{X-\Phi}\,, & \mathcal{M}_f^{f'}(X) &= \widetilde{Q}_{f'}\frac{1}{X-\Phi}Q_f\,. \end{aligned} \qquad (2.153)$$

Un-gauging the $\mathrm{U}(1)$ factor of the $\mathrm{U}(n_c)$ makes all kinds of (anti) baryonic chiral operators gauge-invariant and hence adds them to the chiral ring of the $\mathrm{SU}(n_c)$ theory. Note that besides the ordinary baryons $B_{f_1,\ldots,f_{n_c}} = \epsilon_{j_1,\ldots,j_{n_c}} Q_{f_1}^{j_1}\cdots Q_{f_{n_c}}^{j_{n_c}}$ made of n_c quarks and nothing else, there are baryonic generators comprising n_c quarks plus any number of the adjoint operators Φ:

$$B_{f_1,f_2,\ldots,f_{n_c}}(k_1,k_2,\ldots,k_{n_c}) = \epsilon_{j_1,j_2,\ldots,j_{n_c}} \Big(\Phi^{k_1}Q_{f_1}\Big)^{j_1}\Big(\Phi^{k_2}Q_{f_2}\Big)^{j_2}\Big(\Phi^{k_{n_c}}Q_{f_{n_c}}\Big)^{j_{n_c}}\,; \qquad (2.154)$$

henceforth, we shall call such operators Φ-baryons. Likewise, the anti-baryonic operators include the ordinary antibaryons as well as Φ-antibaryons

$$\tilde{B}^{f_1,f_2,\ldots,f_{n_c}}(k_1,k_2,\ldots,k_{n_c}) = \epsilon^{j_1,j_2,\ldots,j_{n_c}} \Big(\widetilde{Q}^{f_1}\Phi^{k_1}\Big)_{j_1}\Big(\widetilde{Q}^{f_2}\Phi^{k_2}\Big)_{j_2}\Big(\widetilde{Q}^{f_{n_c}}\Phi^{k_{n_c}}\Big)_{j_{n_c}}\,. \qquad (2.155)$$

Note that the Φ-baryons and the Φ-antibaryons are antisymmetric with respect to simultaneous permutations of the flavor indices $f_1, f_2, \cdots, f_{n_c}$ and the Φ-numbers $k_1, k_2, \ldots, k_{n_c}$, but there is no antisymmetry with respect to the flavor indices only. Consequently, the Φ-baryons and the Φ-antibaryons exist as non-trivial generators of the (off-shell) chiral ring for any number of flavors $n_f \geq 1$, unlike the ordinary baryons and antibaryons which exist only when $n_f \geq n_c$. Indeed, even for a single flavor we can build Φ-baryons such as

$$B(0,1,2,\ldots,n_c-1) = \epsilon_{j_1,j_2,\ldots,j_{n_c}} (Q)^{j_1}(\Phi Q)^{j_2}(\Phi^2 Q)^{j_3}\cdots(\Phi^{n_c-1}Q)^{j_{n_c}}\,, \quad \textit{etc.} \tag{2.156}$$

However, we shall see that all the baryonic and antibaryonic operators *vanish on shell* unless $n_f \times p \geq n_c$. For the simplest case of (deformed) $\mathcal{N}=2$ SQCD with linear quark masses $m_f^{f'} = \delta_f^{f'}(\Phi - \mu_f)$, this means *no on-shell baryonic or antibaryonic generators unless* $n_f \geq n_c$, and furthermore, all the on-shell Φ-baryons are completely determined by the ordinary baryons of similar flavors, and ditto for the Φ-antibaryons.

Similarly to the mesonic resolvent $\mathcal{M}_f^{f'}(X)$ encoding both the ordinary mesons and the Φ-mesons $\widetilde{Q}^{f'}\Phi^k Q_f$, we define the baryonic and antibaryonic resolvents

$$\begin{aligned}
\mathcal{B}_{f_1,f_2,\ldots,f_{n_c}}(X_1,X_2,\ldots,X_{n_c}) &= \epsilon_{j_1,j_2,\ldots,j_{n_c}} \prod_{i=1}^{n_c}\left(\frac{1}{X_i-\Phi}\,Q_{f_i}\right)^{j_i} \qquad (2.157)\\
&= \sum_{k_1,k_2,\ldots,k_{n_c}} \frac{B_{f_1,f_2,\ldots,f_{n_c}}(k_1,k_2,\ldots,k_{n_c})}{X_1^{k_1+1}X_2^{k_2+1}\cdots X_{n_c}^{k_{n_c}+1}}\,,\\
\widetilde{\mathcal{B}}^{f_1,f_2,\ldots,f_{n_c}}(X_1,X_2,\ldots,X_{n_c}) &= \epsilon^{j_1,j_2,\ldots,j_{n_c}} \prod_{i=1}^{n_c}\left(\widetilde{Q}^{f_i}\frac{1}{X_i-\Phi}\right)_{j_i} \qquad (2.158)
\end{aligned}$$

$$= \sum_{k_1,k_2,\ldots,k_{n_c}} \frac{\tilde{B}^{f_1,f_2,\ldots,f_{n_c}}(k_1,k_2,\ldots,k_{n_c})}{X_1^{k_1+1} X_2^{k_2+1} \cdots X_{n_c}^{k_{n_c}+1}}$$

encoding all kinds of baryonic and antibaryonic operators. Unlike the mesonic resolvent, the (anti) baryonic resolvents depend on n_c independent complex variables $X_1, X_2, \ldots, X_{n_c}$ — this is necessary to encode the (anti) baryonic operators with arbitrary *independent* numbers $k_1, k_2, \ldots, k_{n_c}$ attached to each (anti) quark. By construction, the (anti) baryonic resolvents (2.158)–(2.159) are antisymmetric with respect to simultaneous permutations of the flavor indices $f_1, f_2, \ldots, f_{n_c}$ and the arguments $X_1, X_2, \ldots, X_{n_c}$ but not under the separate permutations; this allows for the non-trivial *off-shell* (anti) baryonic resolvents for any flavor number $n_f \geq 1$.

The *on-shell* baryonic resolvents are subject to equations of motion stemming from the quark-dependent variations of the antiquark fields, namely

$$\delta\widetilde{Q}_j^f = \left(\frac{1}{X_1 - \Phi}\right)_j^{j_1} \times \varepsilon^{f,f_2,\ldots,f_{n_c}} \epsilon_{j_1,j_2,\ldots,j_{n_c}} \left(\frac{1}{X_2 - \Phi} Q_{f_2}\right)^{j_2} \cdots \left(\frac{1}{X_{n_c} - \Phi} Q_{f_{n_c}}\right)^{j_{n_c}} \tag{2.159}$$

where $\varepsilon^{f,f_2,\ldots,f_{n_c}}$ is an infinitesimal tensor in the flavor space (n_c indices, no particular symmetry). This variation is anomaly free since the $\delta\widetilde{Q}_j^f$ does not depend on the antiquark field $\widetilde{Q}_j^f$ itself, hence the equation is simply

$$\begin{aligned}
\delta W_{\text{tree}} &= \delta\widetilde{Q}_j^f \times [m_f^{f_1}(\Phi)]^j_{j'} Q_{f_1}^{j'} \\
&= \varepsilon^{f,f_2,\ldots,f_{n_c}} \epsilon_{j_1,j_2,\ldots,j_{n_c}} \left(\frac{m_{f'}^{f_1}(\Phi)}{X_1 - \Phi} Q_{f_1}\right)^{j_1} \times \\
&\times \left(\frac{1}{X_2 - \Phi} Q_{f_2}\right)^{j_2} \cdots \left(\frac{1}{X_{n_c} - \Phi} Q_{f_{n_c}}\right)^{j_{n_c}} \\
&= \varepsilon^{f,f_2,\ldots,f_{n_c}} \times \left[m_{f'}^{f_1}(X_1)\, \mathcal{B}_{f_1,f_2,\ldots,f_{n_c}}(X_1,X_2,\ldots,X_{n_c})\right]_{-\text{ wrt } X_1}
\end{aligned}$$

$$\stackrel{\text{cr}}{=} 0 \tag{2.160}$$

where the subscript '$-$ wrt X_1' means 'the negative-power part of the power series with respect to the $X_1 \to \infty$, but only with respect to the X_1'. Hence,

$$\begin{aligned} m_{f'}^{f_1}(X_1) \times \mathcal{B}_{f_1,f_2,\ldots,f_{n_c}}(X_1, X_2, \ldots, X_{n_c}) &\stackrel{\text{cr}}{=} \\ &\stackrel{\text{cr}}{=} [m_{f'}^{f_1}(X_1)\mathcal{B}_{f_1,f_2,\ldots,f_{n_c}}(X_1, X_2, \ldots, X_{n_c})]_{+\,\text{wrt}\,X_1} \\ &= \text{Polynomial}(X_1) \text{ of degree } \le p-1 \\ &\quad\ \text{with coefficients depending on } X_2, \ldots, X_{n_c}\,. \end{aligned} \tag{2.161}$$

Furthermore, the X_2 dependence of these coefficients is restricted in exactly the same way as the X_1 dependence of the baryonic resolvent itself, thus

$$m_{f_1'}^{f_1}(X_1)\, m_{f_2'}^{f_2}(X_2) \times \mathcal{B}_{f_1,f_2,\ldots,f_{n_c}}(X_1, X_2, \ldots, X_{n_c}) \stackrel{\text{cr}}{=} \text{Polynomial}(X_1, X_2) \tag{2.162}$$

with coefficients depending on the $X_3, \ldots, X_{n_c}$. Iterating this argument, we arrive at

$$m_{f_1'}^{f_1}(X_1)\, m_{f_2'}^{f_2}(X_2) \cdots m_{f_{n_c}'}^{f_{n_c}}(X_{n_c}) \times \mathcal{B}_{f_1,f_2,\ldots,f_{n_c}}(X_1, X_2, \ldots, X_{n_c}) \tag{2.163}$$

being a polynomial of degree $\le p-1$ with respect to each of the variables $X_1, X_2, \ldots, X_{n_c}$; in other words, the on-shell baryon resolvent has form

$$\begin{aligned} \mathcal{B}_{f_1,\ldots,f_{n_c}}(X_1, \ldots, X_{n_c}) &\stackrel{\text{cr}}{=} [m^{-1}(X_1)]_{f_1}^{f_1'} \cdots [m^{-1}(X_{n_c})]_{f_{n_c}}^{f_{n_c}'} \times \\ &\quad \times \sum_{k_1=0}^{p-1} \cdots \sum_{k_{n_c}=0}^{p-1} b_{f_1',\ldots,f_{n_c}'}(k_1, \ldots, k_{n_c})\, X_1^{k_1} \cdots X_{n_c}^{k_{n_c}} \end{aligned} \tag{2.164}$$

for some coefficients $b_{f'_1,\ldots,f'_{n_c}}(k_1,\ldots,k_{n_c})$. Likewise, the on-shell antibaryonic resolvent has form

$$\widetilde{\mathcal{B}}^{f_1,\ldots,f_{n_c}}(X_1,\ldots,X_{n_c}) \stackrel{\rm cr}{=} [m^{-1}(X_1)]^{f_1}_{f'_1}\cdots[m^{-1}(X_{n_c})]^{f_{n_c}}_{f'_{n_c}} \times \\ \times\sum_{k_1=0}^{p-1}\cdots\sum_{k_{n_c}=0}^{p-1}\tilde{b}^{f'_1,\ldots,f'_{n_c}}(k_1,\ldots,k_{n_c})\,X_1^{k_1}\cdots X_{n_c}^{k_{n_c}} \tag{2.165}$$

for some coefficients $\tilde{b}^{f'_1,\ldots,f'_{n_c}}(k_1,\ldots,k_{n_c})$.

The total antisymmetry of the baryonic and antibaryonic resolvents under simultaneous permutations of the flavor indices and the variables $X_1,\ldots,X_{n_c}$ translates into the total antisymmetry of the $b_{f'_1,\ldots,f'_{n_c}}(k_1,\ldots,k_{n_c})$ and $\tilde{b}^{f'_1,\ldots,f'_{n_c}}(k_1,\ldots,k_{n_c})$ coefficient under simultaneous permutation of the flavors and the power indices $k_1,\ldots,k_{n_c}$. But each power index k_i has only p allowed values $k_i=0,1,\ldots,(p-1)$ and hence each (k_i,f_i) index combination can take only $p\times n_f$ distinct values, which means we cannot possibly antisymmetrize n_c such index pairs unless $n_c\le p\times n_f$. Thus, *the baryonic and the antibaryonic resolvents — and hence all the baryonic and the antibaryonic generators of the chiral ring — vanish on shell unless $p\times n_f\ge n_c$.* In particular, in the (deformed) $\mathcal{N}=2$ SQCD with linear quark masses we have

$$\mathcal{B}_{f_1,f_2\ldots,f_{n_c}}(X_1,X_2,\ldots,X_{n_c}) \stackrel{\rm cr}{=} \frac{B_{f_1,f_2,\ldots,f_{n_c}}}{(X_1-\mu_{f_1})(X_2-\mu_{f_2})\cdots(X_{n_c}-\mu_{f_{n_c}})} \tag{2.166}$$

$$\widetilde{\mathcal{B}}^{f_1,f_2\ldots,f_{n_c}}(X_1,X_2,\ldots,X_{n_c}) \stackrel{\rm cr}{=} \frac{\tilde{B}^{f_1,f_2,\ldots,f_{n_c}}}{(X_1-\mu_{f_1})(X_2-\mu_{f_2})\cdots(X_{n_c}-\mu_{f_{n_c}})} \tag{2.167}$$

where $B_{f_1,f_2,\ldots,f_{n_c}}$ and $\tilde{B}^{f_1,f_2,\ldots,f_{n_c}}$ are the ordinary, Φ-less baryon and antibaryon. And therefore, all the on-shell baryonic and antibaryonic generators vanish unless $n_f\ge n_c$.

Note that from the baryonic point of view, a theory with non-renormalizable $\widetilde{Q}\Phi^p Q$ interactions of order $p \geq 2$ behaves as an effective $\mathcal{N} = 2$ SQCD with $n_f^{\text{eff}} = n_f \times p$ flavors. Likewise, Cachazo *et al* showed [6] that the Seiberg-Witten curve of a $p \geq 2$ theory —

$$Y^2 - Y \times P(X) + \Lambda^{2n_c - pn_f} B(X) = 0 \tag{2.168}$$

where $P(X)$ is a polynomial of degree n_c and $B(X) = \det(m(X))$ is a polynomial of degree $p \times n_f$ — looks exactly like the $\mathcal{N} = 2$ SQCD curve for the same number n_c of colors but $n_f^{\text{eff}} = p \times n_f$ flavors. Altogether, as far as the chiral ring is concerned, a single quark flavor with $\widetilde{Q}\Phi^p Q$ interactions of order $p > 1$ is physically equivalent to p ordinary quark flavors. And since many formulæ look much more complicated for $p \geq 2$ we shall henceforth limit our presentation to the $p = 1$ case.[9]

Classically, the baryonic branches of the moduli space have overdeter-

[9] In the quiver theory we have a similar situation where a single quark flavor with a p-node hopping superpotential (2.20) acts as p ordinary flavors with nearest-neighbor hopping (2.2). Indeed, classically such a p-node-hopping flavor deconstructs up to p light 5D flavors, and in the fully-quantum chiral ring of the quiver it is physically equivalent to p ordinary flavors, although many formulæ become more complicated. For example, eq. (2.128) for the $B(X)$ becomes

$$\alpha B(X) = \Lambda^{N(2n_c - pF)} \times \text{Det}\left(X^{1/N}\delta_{\ell',\ell}\delta^f_{f'} - (\Gamma_{\ell'-\ell})^f_{f'}\right) \\ \text{with respect to both } \ell', \ell \text{ and } f', f \text{ indices} \tag{2.169}$$

(this is actually a polynomial of X of degree pF because of the translational $\mathbb{Z}_N$ invariance of the quiver), while formulæ for the on-shell mesonic and baryonic resolvents would take pages simply to write down, never mind the derivations. Consequently, we have limited our presentation to the simpler case of the nearest-neighbor hopping.

mined Coulomb moduli:

$$B_{f_1,\ldots,f_{n_c}} \neq 0 \;\; or \;\; \tilde{B}^{f_1,\ldots,f_{n_c}} \neq 0 \qquad requires\ (\varpi_1, \ldots, \varpi_{n_c}) = (\mu_{f_1}, \ldots, \mu_{f_{n_c}})\,, \tag{2.170}$$

which in turn requires $\mu_{f_1} + \cdots + \mu_{f_{n_c}} = 0$ because $\varpi_1 + \cdots + \varpi_{n_c} = \mathrm{tr}(\Phi) = 0$. In the quantum theory, a non-zero on-shell value of a baryonic or an antibaryonic operator also over-determines the Coulomb moduli, although their exact values receive quantum corrections. To see how this works, consider a baryonic resolvent $\mathcal{B}_{f_1,\ldots,f_{n_c}}(X_1, \ldots, X_{n_c})$ at a point where the variables $X_1, \ldots, X_{n_c}$ are all equal to each other. By construction (2.158),

$$\begin{aligned} \mathcal{B}_{f_1,\ldots,f_{n_c}}(X_1 = \cdots = X_{n_c}) &= \epsilon_{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \left(\frac{1}{X - \Phi}\, Q_{f_i} \right)^{j_i} \\ &= \det \left(\frac{1}{X - \Phi} \times Q_{f_1,\ldots,f_{n_c}} \right) \end{aligned} \tag{2.171}$$

where X denotes $X_1 = X_2 = \cdots = X_{n_c}$ and $Q_{f_1,\ldots,f_{n_c}}$ is the $n_c \times n_c$ block of the color $\times$ flavor quark matrix corresponding to flavors $f_1, \ldots, f_{n_c}$. Note $\det(Q_{f_1,\ldots,f_{n_c}}) = B_{f_1,\ldots,f_{n_c}}$ and therefore

$$\begin{aligned} \det \left(\frac{1}{X - \Phi} \times Q_{f_1,\ldots,f_{n_c}} \right) &= \det \left(\frac{1}{X - \Phi} \right) \times B_{f_1,\ldots,f_{n_c}} + \text{inst. corr.} \\ &\overset{\text{cr}}{=} (\text{quantum corr.}) \ \det \left(\frac{1}{X - \Phi} \right) \times B_{f_1,\ldots,f_{n_c}} \end{aligned} \tag{2.172}$$

where the second equality follows from the fact that any instantonic term on the first line must involve a baryonic operator with appropriate flavor indices and all such operators are proportional to the $B_{f_1,\ldots,f_{n_c}}$. Consequently,

$$\mathcal{B}_{f_1,\ldots,f_{n_c}}(X_1 = \cdots = X_{n_c}) \overset{\text{cr}}{=} (\text{quantum corrected}) \ \det \left(\frac{1}{X - \Phi} \right) \times B_{f_1,\ldots,f_{n_c}}\,, \tag{2.173}$$

and comparing this equation with the on-shell formula (2.166) we immediately see that a non-zero on-shell baryon $B_{f_1,\ldots,f_{n_c}} \neq 0$ requires

$$\text{(quantum corrected)}\ \det\left(\frac{1}{X-\Phi}\right) = \prod_{i=1}^{n_c} \frac{1}{X-\mu_{f_i}}\,. \tag{2.174}$$

Likewise, the antibaryonic resolvents satisfy

$$\widetilde{\mathcal{B}}^{f_1,\ldots,f_{n_c}}(X_1=\cdots=X_{n_c}) \stackrel{\text{cr}}{=} \text{(quantum corrected)}\ \det\left(\frac{1}{X-\Phi}\right) \times \tilde{B}^{f_1,\ldots,f_{n_c}}\,, \tag{2.175}$$

and hence in light of eq. (2.167), a non-zero on-shell antibaryon $\tilde{B}^{f_1,\ldots,f_{n_c}}$ requires exactly the same determinant condition (2.174) as a non-zero on-shell baryon $B_{f_1,\ldots,f_{n_c}}$. In other words, eq. (2.174) is the quantum-corrected version of the classical eq. (2.170) for the Coulomb moduli of the baryonic branches.

The trouble with eq. (2.174) is that in the quantum theory a determinant of an operator-valued matrix has several definitions yielding different results. For example, the definition

$$\det_1\left(\frac{1}{X-\Phi}\right) \stackrel{\text{def}}{=} \exp\left(\text{tr}\left(\log\frac{1}{X-\Phi}\right)\right) \tag{2.176}$$

yields on-shell (for the $T(X)$ resolvent given by eqs. (2.127) and (2.130))

$$\det_1\left(\frac{1}{X-\Phi}\right) = \frac{1}{\Xi(X)}\,, \tag{2.177}$$

but another definition

$$\det_2\left(\frac{1}{X-\Phi}\right) \stackrel{\text{def}}{=} \mathcal{D}_{n_c}\left(\text{tr}\left(\frac{1}{X-\Phi}\right), \text{tr}\left(\frac{1}{X-\Phi}\right)^2, \cdots, \text{tr}\left(\frac{1}{X-\Phi}\right)^{n_c}\right) \tag{2.178}$$

(where $\mathcal{D}_n(t_1, t_2, \ldots, t_n)$ is the Newton's polynomial formula for the determinant of an ordinary $n \times n$ matrix A in terms of the traces $t_1 = \mathrm{tr}(A),\ t_2 = \mathrm{tr}(A^2), \ldots, t_n = \mathrm{tr}(A^n)$) yields

$$\det_2\left(\frac{1}{X-\Phi}\right) = \frac{1}{\Xi(X)} \times \frac{1}{n_c!}\,\frac{d^{n_c}\Xi}{dX^{n_c}}\,, \tag{2.179}$$

and yet another definition

$$\begin{aligned}\det_3\left(\frac{1}{X-\Phi}\right) &\stackrel{\mathrm{def}}{=} \frac{1}{\det(X-\Phi)} \\ &\stackrel{\mathrm{def}}{=} \frac{1}{\mathcal{D}_{n_c}\left(\mathrm{tr}(X-\Phi), \mathrm{tr}(X-\Phi)^2, \ldots, \mathrm{tr}(X-\Phi)^{n_c}\right)}\end{aligned} \tag{2.180}$$

yields

$$\det_3\left(\frac{1}{X-\Phi}\right) = \frac{1}{[\Xi(X)]_+}\,. \tag{2.181}$$

However, all such definitions yields exactly the same on-shell determinant when $\Xi(X)$ happens to be a polynomial, and in light of $1/\mathrm{polynomial}(X)$ expression on the right hand side of eq. (2.174) it is clear that

$$B_{f_1,\ldots,f_{n_c}} \neq 0 \ \textit{and/or}\ \tilde{B}^{f_1,\ldots,f_{n_c}} \neq 0 \ \textit{requires}\ \Xi(X) = \prod_{i=1}^{n_c}\left(X - \mu_{f_i}\right). \tag{2.182}$$

Now consider the Seiberg-Witten curve (2.168) of the (deformed) $\mathcal{N} = 2$ SQCD. In a classical baryonic vacuum, the $\mathrm{SU}(n_c)$ is Higgsed down to nothing by the quark and antiquark VEVs and there are no massless gluons or photons whatsoever. In the quantum theory we likewise have no massless photons at all, which means a Seiberg-Witten curve of zero genus. In other words,

$$\Xi(X,Y) = Y = \frac{1}{2}\left(P(X) + \sqrt{P^2(X) - 4\Lambda^{2n_c-n_f}B(X)}\right) \tag{2.183}$$

should have at most one branch cut over the complex X plane. Furthermore, the spectral curves with $h \geq 1$ branch cuts belong to the mesonic vacua with non-zero classical VEVs of $n_c - h$ quark flavors (or to the non-classical pseudo-confining vacua), but in the baryonic vacua all n_c quark flavors develop non-zero classical VEVs. This suggests $h = 0$ for the baryonic vacua, *i.e.* no branch cuts whatsoever; in other words,

$$P^2(X) - 4\Lambda^{2n_c - n_f} B(X) = K^2(X) \tag{2.184}$$

for some polynomial $K(X)$, and $\Xi(X) = \frac{1}{2}(P(X) + K(X))$ is also a polynomial function of X. Of course, this whole argument is rather heuristic, but it serves as a qualitative explanation why eq. (2.182) imposes a polynomial formula on the $\Xi(X)$.

It remains to solve eq. (2.182) for the Coulomb moduli of a baryonic vacuum. Let us factorize

$$B(X) \equiv \prod_{f=1}^{n_f} (X - \mu_f) = B_1(X) \times B_2(X) \tag{2.185}$$

where

$$B_1(X) = \prod_{f = f_1, \ldots, f_{n_c}} (X - \mu_f) \qquad \text{and} \qquad B_2(X) = \prod_{f \neq f_1, \ldots, f_{n_c}} (X - \mu_f)\,. \tag{2.186}$$

Eq. (2.182) amounts to $\Xi(X) = B_1(X)$ while eq. (2.183) implies

$$P^2(X) - 4\Lambda^{2n_c - n_f} B_1(X) B_2(X) = (2\Xi(X) - P(X))^2 = (2B_1(X) - P(X))^2 \tag{2.187}$$

and therefore

$$P(X) = B_1(X) + \Lambda^{2n_c - n_f} B_2(X)\,. \tag{2.188}$$

Thus, in the quantum theory, the Coulomb moduli of a baryonic vacuum with $B_{f_1,\ldots,f_{n_c}} \neq 0$ and/or $\tilde{B}^{f_1,\ldots,f_{n_c}} \neq 0$ are given by

$$\prod_{i=1}^{n_c}(X - \varpi_i) = \prod_{f=f_1,\ldots,f_{n_c}} (X - \mu_f) + \Lambda^{2n_c - n_f} \prod_{f \neq f_1,\ldots,f_{n_c}} (X - \mu_f) \qquad (2.189)$$

where the first term on the right hand side is the classical formula (2.170) while the second term is due to single-instanton quantum effects. Remarkably, there are no higher-order instantonic corrections.

Similar to its classical counterpart (2.170), eq. (2.189) over-determines the Coulomb moduli $(\varpi_i, \ldots, \varpi_{n_c})$ because of the tracelessness constraint

$$\mathrm{tr}(\Phi) = \oint \frac{dX}{2\pi i} \, XT(X) = \oint \frac{X \, d\,\Xi(X)}{2\pi i \, \Xi(X)} = 0\,. \qquad (2.190)$$

The Coulomb moduli (2.189) provide for $\Xi(X) = B_1(X)$ and hence the integral (2.190) evaluates to simply $\sum_i \mu_{f_i}$. In other words, the existence of a baryonic branch with

$$B_{f_1,\ldots,f_{n_c}} \neq 0 \textit{ or } \tilde{B}^{f_1,\ldots,f_{n_c}} \neq 0 \textit{ requires } \mu_{f_1} + \mu_{f_2} + \cdots + \mu_{f_{n_c}} = 0 \qquad (2.191)$$

exactly as in the classical theory.

This completes our study of the baryonic aspects of the single $\mathrm{SU}(n_c)$ theory. In the following section 2.3.2 we shall see that the baryons of the $[\mathrm{SU}(n_c)]^N$ quiver theory behave in a similar way.

2.3.2 Baryonic operators of the $[\mathrm{SU}(n_c)]^N$ Quiver

At this point, we are ready to face the whole zoo of chiral baryonic and antibaryonic operators of the $[\mathrm{SU}(n_c)]^N$ quiver, so let us begin with the zoo's inventory. Most generally, a chiral baryonic operator of the quiver comprises n_c quarks $Q_1, \ldots, Q_{n_c}$ located at arbitrary quiver nodes $\ell_1, \ldots, \ell_{n_c}$ connected by link chains to yet another quiver node ℓ where the color indices of the $\mathrm{SU}(n_c)_\ell$ are combined together into a gauge singlet. Each of the n_c quarks Q_i has an independent flavor index $f_i = 1, \ldots, F$, and each chain connecting the $i^{\underline{\text{th}}}$ quark node ℓ_i to the combination node ℓ can wrap any number of times around the quiver in either direction independently on any other chain. Altogether, this gives us a very large family of chiral baryonic operators

$$\begin{aligned} B_{f_1,\ldots,f_{n_c}}(\ell; \ell_1, \ldots, \ell_{n_c}; k_1, \ldots, k_{n_c}) &= \\ = \epsilon_{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \Big((\Omega_{\ell-1}\Omega_{\ell-2}\cdots\Omega_\ell)^{k_i}\, \Omega_{\ell-1}\Omega_{\ell-2}\cdots\Omega_{\ell_i}\, Q_{\ell_i,f_i} \Big)^{j_i} \end{aligned} \tag{2.192}$$

where each k_i runs from $-\infty$ to $+\infty$: A positive k_i means the link chain from ℓ_i to ℓ wraps k_i times around the quiver in the forward direction while a negative k_i means the chain wraps backwards,

$$\begin{aligned} (\Omega_{\ell-1}\Omega_{\ell-2}\cdots\Omega_\ell)^{(k_i<0)} \times \Omega_{\ell-1}\Omega_{\ell-2}\cdots\Omega_{\ell_i} &= \\ = \left(\Omega_\ell^{-1}\Omega_{\ell+1}^{-1}\cdots\Omega_{\ell-1}^{-1}\right)^{(-1-k_i\geq 0)} \times \Omega_\ell^{-1}\Omega_{\ell+1}^{-1}\cdots\Omega_{\ell_i-1}^{-1}\,. \end{aligned} \tag{2.193}$$

Note that the set of all possible baryonic operators (2.193) is redundant in several ways. First, $B_{f_1,\ldots,f_{n_c}}(\ell; \ell_1, \ldots, \ell_{n_c};\ k_1, \ldots, k_{n_c})$ is totally antisymmetric with respect to permutations of the (f_i, ℓ_i, k_i) index triplets, *i.e.* with respect to *simultaneous* permutations of the flavor, quark-node and chain-wrap indices. This follows from the Bose statistics of the squark and link operators and from the totally antisymmetric contraction of the color indices

$j_1, \ldots, j_{n_c}$. Nevertheless, there is no antisymmetry with respect to permutations of the flavor indices alone, apart from the ℓ_i and k_i indices, and therefore the *off-shell* chiral baryonic operators (2.193) exist for any non-zero flavor number F. However, similarly to the Φ-baryons of the single $\mathrm{SU}(n_c)$ theory, all the baryonic operators (2.193) of the $[\mathrm{SU}(n_c)]^N$ quiver with $n_f < n_c$ vanish *on-shell*; we shall prove this statement later in this section.

The second redundancy is due to quark locations ℓ_i being defined modulo N, hence in a baryonic operator, changing $\ell_i \to \ell_i \pm N$ is equivalent to changing the wrap number $k_i \to k_i \mp 1$. Finally, letting the k_i run from $-\infty$ to $+\infty$ is redundant because any backward-wrapping link chain (2.193) is a linear combination of all the forward-wrapping chains, and *vice verse*, any forward-wrapping chain is a linear combination of the backward-wrapping chains. This follows from the fact that a single meromorphic resolvent

$$\begin{aligned}\frac{\Omega_{\ell-1}\cdots\Omega_{\ell_i}}{X_i - \Omega\cdots\Omega} &= \sum_{k_i\geq 0}\left(\frac{1}{X_i}\right)^{k_i+1} \times \left(\Omega_{\ell-1}\Omega_{\ell-2}\cdots\Omega_{\ell}\right)^{k_i}\Omega_{\ell-1}\Omega_{\ell-2}\cdots\Omega_{\ell_i} \\ &= -\sum_{k_i<0} X_i^{-1-k_i} \times \left(\Omega_{\ell}^{-1}\Omega_{\ell+1}^{-1}\cdots\Omega_{\ell-1}^{-1}\right)^{1-k_i}\Omega_{\ell}^{-1}\Omega_{\ell+1}^{-1}\cdots\Omega_{\ell_i-1}^{-1},\end{aligned}$$

summarizes both types of link chains but can be decomposed into a convergent power series in terms of either only the forward-wrapping chains or else only the backward-wrapping chains.

In light of the above redundancies, we can encode all the chiral baryonic operators of the quiver in a family of baryonic resolvents

$$\mathcal{B}_{f_1,\ldots,f_{n_c}}(\ell;\ell_1,\ldots,\ell_{n_c};X_1,\ldots,X_{n_c}) = \epsilon_{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \left(\frac{\Omega_{\ell-1}\cdots\Omega_{\ell_i}}{X_i - \Omega\cdots\Omega}\, Q_{\ell_i,f_i}\right)^{j_i} \tag{2.194}$$

where $X_1,\ldots,X_{n_c}$ are independent complex numbers, and the resolvents are totally antisymmetric with respect to *simultaneous* permutations of the flavor indices f_i, the quiver indices ℓ_i, and the arguments X_i. Similar to the mesonic resolvents (2.55), the baryonic resolvents (2.194) of the quiver theory trade the wrapping numbers $k_1,\ldots,k_{n_c}$ for the complex arguments $X_1,\ldots,X_{n_c}$, but now we also have to contend with the quiver indices $\ell_1,\ldots,\ell_{n_c}$ and ℓ. The independent resolvents have all ℓ_i running from ℓ down to $\ell-N+1$; for the ℓ_i outside of this range, there is a periodicity condition

$$\begin{aligned}\mathcal{B}(\ell;\ldots,\ell_i=\ell-N,\ldots;X_1,\ldots,X_{n_c}) - X_i \times \mathcal{B}(\ell;\ldots,\ell_i=\ell,\ldots;X_1,\ldots,X_{n_c})\\ = -\epsilon_{j_1,\ldots,j_{n_c}}\, Q^{j_i}_{\ell,f_i} \times \prod_{i'\neq i} \left(\frac{\Omega_{\ell-1}\cdots\Omega_{\ell_{i'}}}{X_{i'} - \Omega\cdots\Omega}\, Q_{\ell_{i'},f_{i'}}\right)^{j_{i'}}\end{aligned} \tag{2.195}$$

which does not depend on the X_i.

The antibaryonic chiral operators of the quiver theory comprise n_c antiquarks located at independent quiver nodes ℓ_i and connected by link chains to yet another quiver node ℓ where the color indices are combined into a gauge singlet. The most general operator of this kind has form

$$\begin{aligned}\widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell;\ell_1,\ldots,\ell_{n_c};k_1,\ldots,k_{n_c}) =\\ = \epsilon^{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \left(\widetilde{Q}^{f_i}_{\ell_i}\, \Omega_{\ell_i-1}\Omega_{\ell_i-2}\cdots\Omega_\ell\, (\Omega_{\ell-1}\Omega_{\ell-2}\cdots\Omega_\ell)^{k_i}\right)_{j_i}\end{aligned} \tag{2.196}$$

for some flavor indices $f_i,\ldots,f_{n_c}$, quiver-node indices $\ell_1,\ldots,\ell_{n_c}$ and ℓ, and wrapping numbers $k_1\ldots,k_{n_c}$. The antibaryonic operators (2.196) are subject to the same redundancy conditions as the baryonic operators (2.193), thus we may encode them in a similar family of the antibaryonic resolvents

$$\widetilde{\mathcal{B}}^{f_1,\ldots,f_{n_c}}(\ell;\ell_1,\ldots,\ell_{n_c};X_1,\ldots,X_{n_c}) = \epsilon^{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \left(\widetilde{Q}^{f_i}_{\ell_i} \frac{\Omega_{\ell-1}\cdots\Omega_{\ell_i}}{X_i - \Omega\cdots\Omega} \right)_{j_i} . \tag{2.197}$$

Again, the $X_1,\ldots,X_{n_c}$ are independent complex numbers, and the resolvents are totally antisymmetric with respect to *simultaneous* permutations of the flavor indices f_i, the quiver indices ℓ_i, and the arguments X_i. The independent antibaryonic resolvents have all the ℓ_i running from ℓ up to $\ell+N-1$; for the ℓ_i outside of this range, there is a periodicity condition

$$\begin{aligned}\widetilde{\mathcal{B}}(\ell;\ldots,\ell_i=\ell+N,\ldots;X_1,\ldots,X_{n_c}) - X_i \times \widetilde{\mathcal{B}}(\ell;\ldots,\ell_i=\ell,\ldots;X_1,\ldots,X_{n_c}) =& \\ = -\epsilon^{j_1,\ldots,j_{n_c}}\, \widetilde{Q}^{f_i}_{\ell,j_i} \times \prod_{\imath'\neq i} \left(\widetilde{Q}^{f_{\imath'}}_{\ell_{\imath'}} \frac{\Omega_{\ell_{\imath'}-1}\cdots\Omega_{\ell}}{X_{\imath'} - \Omega\cdots\Omega} \right)_{j_{\imath'}}&\end{aligned} \tag{2.198}$$

which does not depend on the X_i.

Thus far, we summarized the baryonic and antibaryonic generators of the *off-shell* chiral ring of the $[\mathrm{SU}(n_c)]^N$ quiver. Note that the resolvents (2.194) and (2.197) are totally antisymmetric with respect to simultaneous permutations of the f_i, the ℓ_i and the X_i, but there is no antisymmetry with respect to flavor indices alone. Consequently, there are non-trivial *off-shell* (anti) baryonic generators for any flavor number $F \geq 1$.

The *on-shell* baryonic operators are constrained by equations of motions stemming from quark-dependent variations of the antiquark fields. Generalizing eq. (2.159) from the single $\mathrm{SU}(n_c)$ theory to the $[\mathrm{SU}(n_c)]^N$ quiver, consider

$$\delta\widetilde{Q}^f_{\ell_1,j} = \varepsilon \left(\frac{\Omega_{\ell-1}\cdots\Omega_{\ell_1}}{X_1 - \Omega\cdots\Omega}\right)^{j_1}_{j} \times \epsilon_{j_1,\ldots,j_{n_c}} \prod_{i=2}^{n_c} \left(\frac{\Omega_{\ell-1}\cdots\Omega_{\ell_i}}{X_i - \Omega\cdots\Omega}\, Q_{\ell_i,f_i}\right)^{j_i} \tag{2.199}$$

for an arbitrary combination of (anti) quark flavors f and $f_2,\ldots,f_{n_c}$. This variation is anomaly-free since the $\delta\widetilde{Q}^f_{\ell_1,j}$ does not depend on the antiquark field $\widetilde{Q}^f_{\ell_1,j}$ itself, hence the resulting equation of motion is simply $\delta W_{\text{tree}} \stackrel{\text{cr}}{=} 0$. Or not so simply,

$$\begin{aligned}
\delta W_{\text{tree}} &= \gamma\,\delta\widetilde{Q}^f_{\ell_1,j} \times \left[\left(\Omega_{\ell_1-1}Q_{\ell_1-1,f}\right)^j - \mu_f Q^j_{\ell_1,f}\right] \\
&= \gamma\varepsilon \times \epsilon_{j_1,j_2,\ldots,j_{n_c}} \prod_{i=2}^{n_c} \left(\frac{\Omega_{\ell-1}\cdots\Omega_{\ell_i}}{X_i - \Omega\cdots\Omega}\, Q_{\ell_i,f_i}\right)^{j_i} \times \\
&\quad \times \left[\left(\frac{\Omega_{\ell-1}\cdots\Omega_{\ell_1}}{X_1 - \Omega\cdots\Omega}\Omega_{\ell_1-1}Q_{\ell_1-1,f}\right)^{j_1} - \mu_f \left(\frac{\Omega_{\ell-1}\cdots\Omega_{\ell_1}}{X_1 - \Omega\cdots\Omega}\, Q_{\ell_1,f}\right)^{j_1}\right] \\
&= \gamma\varepsilon \times \Big[\mathcal{B}_{f,f_2,\ldots,f_{n_c}}(\ell;\ell_1-1,\ell_2,\ldots,\ell_{n_c};X_1,X_2,\ldots,X_{n_c}) - \\
&\qquad - \mu_f \mathcal{B}_{f,f_2,\ldots,f_{n_c}}(\ell;\ell_1,\ell_2,\ldots,\ell_{n_c};X_1,X_2,\ldots,X_{n_c})\Big] \\
&\stackrel{\text{cr}}{=} 0\,.
\end{aligned} \tag{2.200}$$

In other words, in the on-shell chiral ring, decreasing the ℓ_1 index of a baryonic resolvent $\mathcal{B}_{f_1,\ldots,f_{n_c}}(\ell;\ell_1,\ldots,\ell_{n_c};X_1,\ldots,X_{n_c})$ by 1 simply multiplies the resolvent by μ_{f_1}, regardless of the $X_1,\ldots,X_{n_c}$ arguments. And since all n_c quarks of a baryon have equal status, decreasing any of the ℓ_i indices by 1 multiplies the resolvent by the $i^{\underline{\text{th}}}$ quark mass, namely μ_{f_i}. Hence, by iteration

$$\mathcal{B}_{f_1,\ldots,f_{n_c}}(\ell;\ell_1,\ldots,\ell_{n_c};X_1,\ldots,X_{n_c}) \stackrel{\text{cr}}{=} \prod_{i=1}^{n_c} \mu_{f_i}^{\ell-\ell_i} \times \mathcal{B}_{\text{same}\,f_i}(\text{all}\,\ell_i=\ell;\,\text{same}\,X_i) \tag{2.201}$$

for any $\ell_i \leq \ell$, and consequently, in light of the periodicity conditions (2.195),

$$\mathcal{B}_{f_1,\ldots,f_{n_c}}(\ell;\ell_1,\ldots,\ell_{n_c};X_1,\ldots,X_{n_c}) \stackrel{\text{cr}}{=} \prod_{i=1}^{n_c} \frac{\mu_{f_i}^{\ell-\ell_i}}{X_i-\mu_{f_i}^N} \times B_{f_1,\ldots,f_{n_c}}(\ell) \tag{2.202}$$

where

$$B_{f_1,\ldots,f_{n_c}}(\ell) = \epsilon_{j_1,\ldots,j_{n_c}} Q^{j_1}_{\ell,f_1} \cdots Q^{j_{n_c}}_{\ell,f_{n_c}} \tag{2.203}$$

is the ordinary, local baryon at the quiver node ℓ. Thus, *on shell, all the baryonic resolvents (2.193) of the quiver — and hence all the non-local baryon-like chiral operators — are proportional to the ordinary local baryons (2.203).*

Likewise, on-shell

$$\widetilde{\mathcal{B}}^{f_1,\ldots,f_{n_c}}(\ell;\ell_1,\ldots,\ell_{n_c};X_1,\ldots,X_{n_c}) \stackrel{\text{cr}}{=} \prod_{i=1}^{n_c} \frac{\mu_{f_i}^{\ell_i-\ell}}{X_i-\mu_{f_i}^N} \times \widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell) \tag{2.204}$$

and all the antibaryonic resolvents (2.196) are proportional to the ordinary antibaryons. And since the ordinary baryons and antibaryons have antisymmetrized flavor indices, it follows that similarly to the deformed $\mathcal{N}=2$ SQCD theory, all the baryonic and antibaryonic generators of the quiver's chiral ring vanish on shell unless $F \geq n_c$. Furthermore, we shall see momentarily that baryonic branches of the quiver theory do not involve massless quark flavors, so *it takes at least $n_f \geq n_c$* massive *quark flavors* (cf. eqs. (2.15) and (2.18) *to get any anti/baryonic VEVs at all.*

Indeed, similarly to the single $\mathrm{SU}(n_c)$ theory of the previous section, baryonic branches of the $[\mathrm{SU}(n_c)]^N$ quiver have overdetermined Coulomb moduli. To derive this result, we again focus on the anti/baryonic resolvents with equal arguments $X_1 = X_2 = \cdots = X_{n_c}$, but this time we also use equal quiver indices $\ell_1 = \ell_2 = \cdots \ell_{n_c} = \ell$. Thus,

$$
\begin{aligned}
&\mathcal{B}_{f_1,\ldots,f_{n_c}}(\text{all}\,\ell_i = \ell;\, \text{all}\, X_i = X) = \\
&\quad = \epsilon_{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \Big(\frac{1}{X - \Omega\cdots\Omega}\, Q_{\ell,f_i} \Big)^{j_i} \quad \text{(cf. eq. (2.194))} \\
&\quad = \det \left(\frac{1}{X - \Omega\cdots\Omega} \times Q_{f_1,\ldots,f_{n_c}}(\ell) \right)
\end{aligned}
$$

(on shell)

$$
\begin{aligned}
&\overset{\text{cr}}{=} \det \left(\frac{1}{X - \Omega\cdots\Omega} \right) \times \Big[\det \Big(Q_{f_1,\ldots,f_{n_c}}(\ell) \Big) \equiv B_{f_1,\ldots,f_{n_c}}(\ell) \Big] + \\
&\qquad + \text{instantonic corrections} \\
&= B_{f_1,\ldots,f_{n_c}}(\ell) \times (\text{quantum corrected}) \det \left(\frac{1}{X - \Omega\cdots\Omega} \right), \qquad (2.205)
\end{aligned}
$$

and comparing this result with the on-shell formula (2.202) for the same baryonic resolvent, we immediately see that a non-trivial baryonic VEV $\langle B_{f_1,\ldots,f_{n_c}}(\ell) \rangle \neq 0$ *requires*

$$
\text{quantum corrected} \det \left(\frac{1}{X - \Omega\cdots\Omega} \right) = \prod_{i=1}^{n_c} \frac{1}{X - \mu^N_{f_i}}\,. \qquad (2.206)
$$

Likewise, for an antibaryonic resolvent with the same indices and arguments we have

$$
\begin{aligned}
&\widetilde{\mathcal{B}}^{f_1,\ldots,f_{n_c}}(\text{all}\,\ell_i = \ell;\, \text{all}\, X_i = X) = \\
&\quad = \epsilon^{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \left(\widetilde{Q}^{f_i}_{\ell}\, \frac{1}{X - \Omega\cdots\Omega} \right)_{j_i} \quad \text{cf. eq. (2.194)} \\
&\quad = \det \left(\widetilde{Q}^{f_1,\ldots,f_{n_c}}(\ell) \times \frac{1}{X - \Omega\cdots\Omega} \right)
\end{aligned}
$$

(on shell)

$$
\overset{\text{cr}}{=} \Big[\det \big(\widetilde{Q}^{f_1,\ldots,f_{n_c}}(\ell) \big) \equiv \widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell) \Big] \times \det \left(\frac{1}{X - \Omega\cdots\Omega} \right) +
$$

+instantoric corrections

$$= \widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell) \times (\text{quantum corrected}) \det\left(\frac{1}{X-\Omega\cdots\Omega}\right), \quad (2.207)$$

and comparing this result with the on-shell formula (2.204) we see that a non-trivial antibaryonic VEV $\left\langle \widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell)\right\rangle \neq 0$ requires exactly the same determinant condition (2.206) as the baryonic VEV $\left\langle B_{f_1,\ldots,f_{n_c}}(\ell)\right\rangle \neq 0$.

From the Coulomb moduli's point of view, the determinant constraint (2.206) of the quiver theory is precisely analogous to the determinant constraint (2.174) of the single SU(n_c) theory, and its solution is precisely analogous to eqs. (2.182) and (2.189). Specifically,

$$B_{f_1,\ldots,f_{n_c}} \neq 0 \textit{ and/or } \tilde{B}^{f_1,\ldots,f_{n_c}} \neq 0 \textit{ requires } \Xi(X) = \prod_{i=1}^{n_c}\left(X - \mu_{f_i}\right), \quad (2.208)$$

which means the Seiberg-Witten curve (2.133) of the quiver should have no branch cuts at all, and the Coulomb moduli $\varpi_i \equiv \omega_i^N$ are given by

$$\prod_{i=1}^{n_c}(X-\varpi_i) = \prod_{f=f_1,\ldots,f_{n_c}}(X-\mu_f^N) + \alpha \prod_{f\neq f_1,\ldots,f_{n_c}}(X-\mu_f^N) \quad (2.209)$$

where the first term on the right hand side describes the classical location of the quiver's baryonic branch ($\omega_1^N = \mu_{f_1}^N, \ldots, \omega_{n_c}^N = \mu_{f_{n_c}}^N$) while the second term is the instantonic correction. Note that all F quark flavors, massive or massless alike, must appear in either product on the right hand side of eq. (2.209). However, eq. (2.208) implies

$$T(X) = \frac{1}{\Xi}\frac{d\Xi}{dX} = \sum_{i=1}^{n_c}\frac{1}{X-\mu_{f_i}^N} \quad (2.210)$$

without any quantum corrections, and whereas the link resolvent of the $[\mathrm{SU}(n_c)]^N$quiver must be regular at $X = 0$ on the physical sheet, it follows that *the quark flavors* $f_1, \ldots, f_{n_c}$ *involved in any anti/baryonic VEV must all be massive*, hence the $n_f \geq n_c$ requirement rather than just $F \geq n_c$.

Eq. (2.209) over-determines the Coulomb moduli of a baryonic branch because only $n_c - 1$ of these moduli are independent while their product is constrained according to eq. (2.141). Consequently, the baryonic branch involving quark flavors $f_1, \ldots, f_{n_c}$ exists if and only if

$$\prod_{f=f_1,\ldots,f_{n_c}} \mu_f^N + \alpha(-1)^F \prod_{f\neq f_1,\ldots,f_{n_c}} \mu_f^N = V^{Nn_c} \equiv V_1^{Nn_c} + V_2^{Nn_c} \tag{2.211}$$

where on the right hand side $V_1^{n_c}$ and $V_2^{n_c}$ are the two roots of eq. (2.230) as we shall explain in the following section 2.3.3. Physically, $V^{n_c} = v^{n_c} +$ quantum corrections, and we shall see that the corrections vanish if the quiver has massless quarks: for $\Delta F > 0$, $V_1^{n_c} = v^{n_c}$ while $V_2 = 0 \implies V^{n_c} = v^{n_c}$ exactly. At the same time, any $\mu_f = 0$ kills the second product on the left hand side of eq. (2.211), which leads us to the un-modified classical condition

$$\prod_{f=f_1,\ldots,f_{n_c}} \mu_f^N = v^{Nn_c}\,. \tag{2.212}$$

For $\Delta F = 0$ (massive flavors only) the situation is more complicated: Both products on the left hand side of eq. (2.211) have non-zero values, and likewise both $V_1 \neq 0$ and $V_2 \neq 0$ on the right hand side. However, taking the second eq. (2.230) to the $N^{\underline{\text{th}}}$ power we have

$$V_1^{Nn_c} \times V_2^{Nn_c} = \left(\Lambda^{2n_c-F}(-\gamma)^F \prod_{\text{all } f} \mu_f\right)^N = \left(\prod_{f=f_1,\ldots,f_{n_c}} \mu_f^N\right) \times \\ \times \left(\alpha(-1)^F \prod_{f\neq f_1,\ldots,f_{n_c}} \mu_f^N\right) \tag{2.213}$$

(cf. eq. (2.138) for the α), and comparing this equation with eq. (2.211) we immediately obtain a much simpler formula

$$\prod_{f=f_1,\ldots,f_{n_c}} \mu_f^N = V_1^{Nn_c} \quad \text{or} \quad V_2^{Nn_c}\,. \tag{2.214}$$

In any case, *the baryonic branch involving quark flavors* $f_1,\ldots,f_{n_c}$ *exists if and only if the masses of these flavors satisfy the classical product condition* (2.212) *for* $\Delta F > 0$ *or the quantum-corrected product condition* (2.214) *for* $\Delta F = 0$.

This almost concludes our presentation of the baryonic aspects of the $[\mathrm{SU}(n_c)]^N$ quiver theory, except for one minor point. Classically, a baryonic branch of the quiver has ℓ-independent VEVs $\langle B_{f_1,\ldots,f_{n_c}}\rangle$ and $\langle \widetilde{B}^{f_1,\ldots,f_{n_c}}\rangle$, modulo an $e^{2\pi ik\ell/N}$ phase factor, but in the quantum theory eqs. (2.202) and (2.204) seem to allow for arbitrary ℓ-dependence of the ordinary baryons and antibaryons. To plug this loophole, consider the baryonic resolvent with equal arguments $X_1 = X_2 = \cdots = X_{n_c}$ and quiver indices $\ell_1 = \ell_2 = \cdots = \ell_{n_c} = \ell - 1$:

$$\begin{aligned}
&\mathcal{B}_{f_1,\ldots,f_{n_c}}(\ell; \text{all}\,\ell_i = \ell-1;\ \text{all}\,X_i = X) = \\
&\quad = \epsilon_{j_1,\ldots,j_{n_c}} \prod_{i=1}^{n_c} \left(\frac{\Omega_{\ell-1}}{X - \Omega\cdots\Omega}\, Q_{\ell-1,f_i} \right)^{j_i} \\
&\quad = \det\left(\Omega_{\ell-1} \times \frac{1}{X - \Omega\cdots\Omega} \times Q_{f_1,\ldots,f_{n_c}}(\ell) \right) \quad \text{(on shell)} \\
&\quad = \det(\Omega_{\ell-1}) \times \det\left(\frac{1}{X - \Omega\cdots\Omega} \times Q_{f_1,\ldots,f_{n_c}}(\ell-1) \right) + \\
&\qquad + \text{instantonic corrections} \\
&\quad = \text{(quantum corrected)} \det(\Omega_{\ell-1}) \times \\
&\qquad \times \mathcal{B}_{f_1,\ldots,f_{n_c}}(\ell-1; \text{all}\,\ell_i = \ell-1;\ \text{all}\,X_i = X)\,,
\end{aligned} \tag{2.215}$$

and comparing this formula with eq. (2.202), we arrive at

$$B_{f_1,\ldots,f_{n_c}}(\ell) = \frac{\mu_{f_1}\times\mu_{f_2}\times\cdots\times\mu_{f_{n_c}}}{(\text{quantum corrected})\det(\Omega_{\ell-1})}\times B_{f_1,\ldots,f_{n_c}}(\ell-1)\,. \tag{2.216}$$

Likewise, for the antibaryons we have

$$\begin{aligned}&\widetilde{\mathfrak{B}}^{f_1,\ldots,f_{n_c}}(\ell;\text{all}\,\ell_i=\ell+1;\ \text{all}\,X_i=X) = \\ &\qquad\widetilde{\mathfrak{B}}^{f_1,\ldots,f_{n_c}}(\ell+1;\text{all}\,\ell_i=\ell+1;\ \text{all}\,X_i=X)\times(\text{quantum corrected})\det(\Omega_\ell)\,,\end{aligned} \tag{2.217}$$

and therefore

$$\widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell) = \frac{\mu_{f_1}\times\mu_{f_2}\times\cdots\times\mu_{f_{n_c}}}{(\text{quantum corrected})\det(\Omega_{\ell})}\times \widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell+1)\,. \tag{2.218}$$

In both eqs. (2.216) and (2.218) we have a quantum-corrected link determinant, and while it may be hard to calculate the quantum corrections here, they obviously do not depend on a particular link Ω_ℓ. Consequently,

$$\begin{aligned}\forall\ell:\qquad & B_{f_1,\ldots,f_{n_c}}(\ell) = C\times B_{f_1,\ldots,f_{n_c}}(\ell-1)\qquad\text{and}\\ & \widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell) = C\times\widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell+1)\end{aligned} \tag{2.219}$$

for some ℓ-independent constant C, and by periodicity of the quiver we must have $C^N=1 \implies C=e^{2\pi ik/N}$. In other words, the anti/baryonic VEVs of the quantum quiver do follow the classical rule of

$$\begin{aligned}& B_{f_1,\ldots,f_{n_c}}(\ell) = e^{+2\pi ik\ell/N}\,B_{f_1,\ldots,f_{n_c}}\qquad\text{and}\\ & \widetilde{B}^{f_1,\ldots,f_{n_c}}(\ell) = e^{-2\pi ik\ell/N}\,\widetilde{B}^{f_1,\ldots,f_{n_c}}\,.\end{aligned} \tag{2.220}$$

And this does complete our presentation of the baryonic issues.

2.3.3 Determinants of link chains

In this section we complete our study of the $[\mathrm{SU}(n_c)]^N$ quiver's chiral ring by calculating the quantum corrections to the determinants of link chains $\det(\Omega_{\ell_2}\cdots\Omega_{\ell_1})$, especially the $\det(\Omega_N\cdots\Omega_1)$ which controls the parameter $V^{n_c} = v^{n_c} + \cdots$ in eqs. (2.141) and (2.211). For simplicity, we focus on the un-deformed deconstructive quivers.

In the quarkless case $F = 0$, the corrections follow from 'local' instantons in the individual $\mathrm{SU}(n_c)_\ell$ gauge factors rather than the 'global' instantons of the whole quiver. Iterating the Seiberg formula [21]

$$\det(\mathcal{M}) = \mathcal{B}\widetilde{\mathcal{B}} + \Lambda^{2n_c} \tag{2.221}$$

for the ordinary SQCD with $n_f = n_c$, Rodríguez [22] and later Chang and Georgi [23] found for an open link chain

$$\det(\Omega_{\ell_2}\cdots\Omega_{\ell_1}) = \mathrm{Poly}\left[\prod_{\ell=\ell_1}^{\ell_2}\det(\Omega_\ell) \times \prod_{\ell=\ell_1+1}^{\ell_2}\left(1 + \frac{\Lambda_\ell^{2n_c}}{\det(\Omega_\ell)\det(\Omega_{\ell-1})}\right)\right] \tag{2.222}$$

where $\mathrm{Poly}[\cdots]$ denotes the polynomial part of the expression in the square brackets, that is, the terms without any net $\det(\Omega_\ell)$ factors in the denominator. For example,

$$\begin{aligned}\det(\Omega_4\Omega_3\Omega_2\Omega_1) \;=\;& \det(\Omega_4)\det(\Omega_3)\det(\Omega_2)\det(\Omega_1) + \det(\Omega_4)\det(\Omega_3)\Lambda_2^{2n_c} + \\ &+ \det(\Omega_4)\Lambda_3^{2n_c}\det(\Omega_1) + \Lambda_4^{2n_c}\det(\Omega_2)\det(\Omega_1) + \\ &+ \Lambda_4^{2n_c}\Lambda_2^{2n_c} + \cancel{\frac{\det(\Omega_4)\Lambda_3^{2n_c}\Lambda_2^{2n_c}}{\det(\Omega_2)}} + \cancel{\frac{\Lambda_4^{2n_c}\Lambda_3^{2n_c}\det(\Omega_1)}{\det(\Omega_3)}} \\ &+ \cancel{\frac{\Lambda_4^{2n_c}\Lambda_3^{2n_c}\Lambda_2^{2n_c}}{\det(\Omega_3)\det(\Omega_2)}}\,.\end{aligned} \tag{2.223}$$

Likewise, the closed link chain wrapped around the whole quiver has determinant

$$\det(\Omega_N \cdots \Omega_1) = \text{Poly}\left[\prod_{\ell=1}^{N} \det(\Omega_\ell) \times \prod_{\ell=1}^{N}\left(1 + \frac{\Lambda_\ell^{2n_c}}{\det(\Omega_\ell)\det(\Omega_{\ell-1})}\right)\right]; \tag{2.224}$$

for the ℓ-independent $\Lambda_\ell \equiv \Lambda$ and $\det(\Omega_\ell) \equiv v^{n_c}$, this expression evaluates to [14]

$$\det(\Omega_N \cdots \Omega_1) = V_1^{Nn_c} + V_2^{Nn_c} \tag{2.225}$$

where $V_1^{n_c}$ and $V_2^{n_c}$ are the two roots of the quadratic equation system

$$V_1^{n_c} \ + \ V_2^{n_c} = v^{n_c}\,, \qquad V_1^{n_c} \times V_2^{n_c} = \Lambda^{2n_c}\,. \tag{2.226}$$

In the deconstruction limit $N \to \infty$ the right hand side of eq. (2.225) is dominated by the larger root, hence the quantum corrections may be summarized as $v \to V = \max(V_1, V_2)$; as discussed in [14], this leads to $(1/g_5^2) \geq 0$ in the deconstructed theory and prevents phase transitions. However, for the present purposes we are interested in fixed-N quivers and exact holomorphic relations in the chiral ring, so both roots of eqs. (2.226) are equally important.

In the quiver theories with quarks, the link chain determinants are affected by both local and global instantonic effects. The local effects follow from integrating out the massive (mass $= \gamma\mu_f$) quark-antiquark pairs from each individual $\text{SU}(n_c)_\ell$ gauge group factor, thus

$$\Lambda_\ell^{2n_c} \to \Lambda_\ell^{2n_c - F} \times \prod_{f=1}^{F}(-\gamma\mu_f)\,, \tag{2.227}$$

and then proceeding exactly as in the quarkless case, hence

$$\det(\Omega_{\ell_2}\cdots\Omega_{\ell_1}) = \\ = \text{Poly}\left[\prod_{\ell=\ell_1}^{\ell_2}\det(\Omega_\ell)\times\prod_{\ell=\ell_1+1}^{\ell_2}\left(1+\frac{\Lambda_\ell^{2n_c}\prod_f(-\gamma\mu_f)}{\det(\Omega_\ell)\det(\Omega_{\ell-1})}\right)\right]+\cdots, \tag{2.228}$$

$$\begin{aligned}\det(\Omega_N\cdots\Omega_1) &= \\ &= \text{Poly}\left[\prod_{\ell=1}^{N}\det(\Omega_\ell)\times\prod_{\ell=1}^{N}\left(1+\frac{\Lambda_\ell^{2n_c}\prod_f(-\gamma\mu_f)}{\det(\Omega_\ell)\det(\Omega_{\ell-1})}\right)\right]+\cdots \\ &= V_1^{Nn_c}+V_2^{Nn_c}+\cdots\end{aligned} \tag{2.229}$$

where

$$V_1^{n_c}+V_2^{n_c}=v^{n_c}\,,\qquad V_1^{n_c}\times V_2^{n_c}=\Lambda^{2n_c-F}\prod_{f=1}^{F}(-\gamma\mu_f) \tag{2.230}$$

and the '$\cdots$' stand for the non-local quantum corrections, if any. Note that the products of quark masses in (2.227)–(2.230) involves all F quark flavors. *When some of the flavors are exactly massless (i.e., $\Delta F>0$) the* local *instanton corrections to the determinants (2.222)–(2.224) go away.*

The non-local effects arise from the quark and the antiquark fields propagating between different quiver nodes according to the hopping superpotential (2.2). In a moment, we shall see that all such effects must be completely global and involve all N quiver nodes at once, thus *eg.*

$$\begin{aligned}\det(\Omega_N\cdots\Omega_1) &= V_1^{Nn_c}+V_2^{Nn_c}+V_{\text{global}}^{Nn_c}\quad\text{where} \\ V_{\text{global}}^{Nn_c} &= O\Big((\Lambda^{2n_c-F}\gamma^F)^N\Big)\equiv O(\alpha)\,.\end{aligned} \tag{2.231}$$

Indeed, let us temporarily promote the gauge and the superpotential couplings of the quiver theory to ℓ-dependent background fields and allow for generic flavor dependence of the quarks' couplings and masses, thus separate Λ_ℓ for each $\mathrm{SU}(n_c)_\ell$ factor and[10]

$$W_{\text{tree}} = \sum_{\ell=1}^{N} \Big(\mathrm{tr}(\widetilde{Q}_{\ell+1} \Omega_\ell Q_\ell \Gamma_\ell) - \mathrm{tr}(m_\ell \widetilde{Q}_\ell Q_\ell) + \beta s_\ell (\det \Omega_\ell - v_\ell^{n_c}) \Big) . \qquad (2.232)$$

The promoted theory has a separate $[\mathrm{U}(F)\times\mathrm{U}(F)\times\mathrm{U}(1)]_\ell$ flavor symmetry for each quiver node: for any $2N$ $\mathrm{U}(F)$ matrices U_ℓ and $\tilde{U}_\ell$ and any N unimodular complex numbers η_ℓ we may transform

$$Q'_\ell = Q_\ell U_\ell \,, \qquad \widetilde{Q}'_\ell = \tilde{U}_\ell \widetilde{Q}_\ell \,, \qquad \Omega'_\ell = \eta_\ell \Omega_\ell \,, \qquad s'_\ell = \eta_\ell^{-n_c} s_\ell \,, \qquad (2.233)$$

$$m'_\ell = \tilde{U}_\ell^{-1} m_\ell U_\ell^{-1} \,, \qquad \Gamma'_\ell = \eta_\ell^{-1} \tilde{U}_{\ell+1}^{-1} \Gamma_\ell U_\ell^{-1} \,, \qquad v'_\ell = \eta_\ell v_\ell \,, \qquad (2.234)$$

$$(\Lambda_\ell^{2n_c - F})' = \Lambda_\ell^{2n_c - F} \times \eta_\ell^{n_c} \eta_{\ell-1}^{n_c} \det(U_\ell) \det(\tilde{U}_\ell) \,. \qquad (2.235)$$

All quantum effects in the promoted theory must transform covariantly under this exact $[\mathrm{U}(F)\times\mathrm{U}(F)\times\mathrm{U}(1)]^N$ symmetry, and this is a very strong constraint on the holomorphic equations of the chiral ring. Indeed, there are only $N + F$ independent holomorphic invariant combinations of the background fields,[11] for example

$$y_\ell = \frac{\Lambda_\ell^{2n_c - F} \det(m_\ell)}{v_\ell^{n_c} v_{\ell-1}^{n_c}} \,, \qquad \ell = 1, \ldots, N \,, \qquad (2.238)$$

---

[10] Note the Yukawa coupling γ being promoted to the matrices $[\Gamma_\ell]^f_{f'}$ and the quark masses $\gamma \mu_f$ to $[m_\ell]^f_{f'}$.

[11] Generally,

$$\#(\text{invariants}) = \#(\text{fields}) - \#(\text{symmetries}) + \#(\text{generically unbroken symmetries}). \quad (2.236)$$

The $[\mathrm{U}(F)\times\mathrm{U}(F)\times\mathrm{U}(1)]^N$ symmetry of the promoted theory is mostly spontaneously broken by the non-zero values of the background fields m_ℓ, Γ_ℓ, v_ℓ and Λ_ℓ. Generically, only the $\mathrm{U}(1)^F \subset \mathrm{U}(F)_{\text{diag}} = \mathrm{diag}\left([\mathrm{U}(F)]^{2N}\right)$ flavor symmetry remains unbroken, hence

$$\#(\text{invariants}) = N(F^2 + F^2 + 1 + 1) - N(F^2 + F^2 + 1) + F = N + F \,. \qquad (2.237)$$

and

$$C_k = \frac{\alpha b_k}{(v_1 \cdots v_N)^{2n_c - k}}\,, \qquad k = 1, \ldots, F\,, \tag{2.239}$$

where

$$\alpha = (-1)^F \prod_{\ell=1}^{N} \Lambda^{2n_c - F} \times \prod_{\ell=1}^{N} \det(-\Gamma_\ell) \tag{2.240}$$

and b_k are coefficient of the characteristic polynomial

$$B(X) \equiv \sum_{k=0}^{F} b_k X^k = \det\left(X - m_1\Gamma_1^{-1} m_2 \Gamma_2^{-1} \cdots m_N \Gamma_N^{-1}\right). \tag{2.241}$$

Eventually, we will let the background fields take their usual values $\Gamma_\ell \equiv \gamma 1_{F\times F}$ and $m_\ell \equiv \gamma\mu$, and consequently eq. (2.240) would reduce to the good old eq. (2.138) while the polynomial (2.241) would become exactly as in eq. (2.128), thus

$$b_F = 1\,, \qquad b_{F-1} = -\sum_{f=1}^{F} \mu_f^N\,, \qquad b_{F-2} = +\sum_{f<f'} \mu_f^N \mu_{f'}^N\,, \qquad \textit{etc., etc.} \tag{2.242}$$

For the moment, however, we need to keep the background fields completely generic, and it is important to note that despite the appearance of Γ_ℓ^{-1} in eq. (2.241), the products αb_k are actually polynomial in all the Γ_ℓ fields as well as in the m_ℓ fields.

In general, the quantum corrections to link chain determinants may depend on the Coulomb moduli ϖ_j of the quiver theory. The symmetry (2.233) acts on the moduli according to $\varpi_j' = \varpi_j \times (\eta_1 \cdots \eta_N)$, hence most generally

$$\det(\Omega_{\ell_2} \cdots \Omega_{\ell_1}) = v_{\ell_2}^{n_c} \cdots v_{\ell_1}^{n_c} \times \left[1 + \mathcal{F}\left(y_\ell, C_k, \frac{\varpi_j}{(v_1 v_2 \cdots v_N)}\right)\right] \tag{2.243}$$

for some analytic function $\mathcal{F}$. Actually, $\mathcal{F}$ has to be a polynomial of its arguments because of the following considerations:

- In the un-deformed quiver theory there is no (pseudo) confinement or gaugino condensation, hence the determinants (2.243) should be single-valued functions of the $\Lambda_\ell^{2n_c-F}$ and the $v_\ell^{n_c}$ as well as of the Γ_ℓ and m_ℓ matrices.

- The determinants should not diverge for any finite values of the background fields m_ℓ, Γ_ℓ, $v_\ell^{n_c}$ and $\Lambda_\ell^{2n_c-F}$ — and this includes regular behavior for $m_\ell \to 0$, $\Gamma_\ell \to 0$, $\Lambda_\ell^{2n_c-F} \to 0$ and especially $v_\ell^{n_c} \to 0$.

- Likewise, the moduli dependence of the determinants (2.243) should be regular
throughout the Coulomb moduli space.

Furthermore, every term in $v_{\ell_2}^{n_c} \cdots v_{\ell_1}^{n_c} \times \mathfrak{F}$ must carry a non-negative integer power of every $v_\ell^{n_c}$ parameter. Consequently, for open link chains of length $\ell_2 - \ell_1 + 1 < N$

$$\det(\Omega_{\ell_2} \cdots \Omega_{\ell_1}) = v_{\ell_2}^{n_c} \cdots v_{\ell_1}^{n_c} \times \Big[1 + \mathfrak{F}\,(y_\ell \text{ only})\Big], \qquad (2.244)$$

while for the closed chain of length N (once around the quiver)

$$\det(\Omega_N \cdots \Omega_1) = (v_1 \cdots v_N)_c^n \times \left[1 + \mathfrak{F}_1(y_\ell) + \mathfrak{F}_2\left(\frac{\alpha b_k}{(v_1 \cdots v_N)^{2n_c-k}}, \frac{\varpi_j}{(v_1 \cdots v_N)}\right)\right]. \qquad (2.245)$$

Physically, the $O(y_\ell)$ effects due to 'local' instantons in the individual $\mathrm{SU}(n_c)_\ell$ factors, while the $O(\alpha)$ effects are due to 'globally coordinated' instanton effects in all the $\mathrm{SU}(n_c)_\ell$ factors at once — or equivalently, due to the 'diagonal' instantons in the $\mathrm{SU}(n_c)_{\mathrm{diag}}$. Thus, all quantum corrections to the determinant (2.244) are solely due to the local instanton effects $\Longrightarrow$ the explicit part of eq. (2.222) (without the '$+\cdots$' part) is actually exact. By comparison, the determinant (2.245) suffers from two completely separate sets of quantum corrections, one purely local and the other purely global. The local corrections should be exactly as in eq. (2.224), hence in terms of eq. (2.231) the V_1 and V_2 are exactly as in eq. (2.230) the global correction is given by

$$\begin{aligned} V^{Nn_c}_{\text{global}} &= (v_1\cdots v_N)^{n_c}\times\mathcal{F}_2\left(\frac{\alpha b_k}{(v_1\cdots v_N)^{2n_c-k}},\frac{\varpi_j}{(v_1\cdots v_N)}\right)\\ &= \alpha\sum_{k\geq n_c} b_k\times\mathcal{H}^{(1)}_k(\varpi)+\alpha^2\sum_{k_1+k_2\geq 3n_c} b_{k_1}b_{k_2}\times\mathcal{H}^{(2)}_{k_1,k_2}(\varpi)-\\ &\quad+\alpha^3\sum_{k_1+k_2+k_3\geq 5n_c} b_{k_1}b_{k_2}b_{k_3}\times\mathcal{H}^{(3)}_{k_1,k_2,k_3}(\varpi)+\cdots \end{aligned} \tag{2.246}$$

where $\mathcal{H}^{(1)}_k$, $\mathcal{H}^{(2)}_{k_1,k_2}$, $\mathcal{H}^{(3)}_{k_1,k_2,k_3}$, *etc.*, are symmetric homogeneous polynomials of the Coulomb moduli $(\varpi_1,\ldots,\varpi_{n_c})$ of respective degrees $(k-n_c)$, $(k_1+k_2-3n_c)$, $(k_1+k_2+k_3-5n_c)$, *etc.* Since the b_k coefficients exist only for $k\leq F$ (cf. (2.241)–(2.242), the flavor number F puts an upper limit on the sums in eq. (2.246), which immediately gives us several general rules:

(A) For $F<n_c$ all sums are empty and $V^{Nn_c}_{\text{global}}=0$, *i.e.* there is no global correction.

(B) For $F=n_c$ there is only one valid term

$$V^{Nn_c}_{\text{global}}=\alpha\times b_F\times\mathcal{H}^{(1)}_{F=n_c}=\alpha\times 1\times\text{a numeric constant},\tag{2.247}$$

hence the global correction exists but does not depend on the moduli ϖ_j.

(C) For $F>n_c$ there are several valid terms and the global correction becomes moduli dependent.

(D) For $n_c \leq F < \frac{3}{2}n_c$ only the first sum (in eq. (2.246)) has valid terms, hence $\Delta V^{Nn_c} \propto \alpha^1 \implies$ the global corrections arises at the one diagonal instanton level only.

(E) For $\frac{3}{2}n_c \leq F < 2n_c$ several instanton levels contribute to the global correction $V^{Nn_c}_{\text{global}}$, up to the maximum of $\left\lfloor \frac{n_c}{2n_c - F} \right\rfloor$ diagonal instantons.

(F) Finally, for $F = 2n_c$ all instanton levels contribute to the global correction and eq. (2.246) becomes an infinite power series rather than a finite polynomial.

To understand the physical significance of these rules we need to take a closer look at the closed link chain $\Omega_N \cdots \Omega_1$. As a composite chiral field, the closed chain has adjoint-like gauge quantum numbers, so the precise definition of its determinant is somewhat ambiguous in the quantum theory. To understand and resolve this ambiguity, consider the characteristic "polynomial"

$$\chi(X) = \det\Big(X - \Omega_N \cdots \Omega_1\Big)\,, \tag{2.248}$$

which may actually be a non-polynomial function of X, depending on the specific definition of the determinant on the right hand side. For example, adapting the definitions (2.176)–(2.181) to the present situation, we have

$$\det{}_1(X - \Omega\cdots\Omega) \stackrel{\text{def}}{=} \exp\Big[\operatorname{tr}\Big(\log(X - \Omega\cdots\Omega)\Big)\Big]\,, \tag{2.249}$$

$$\det{}_2(X - \Omega\cdots\Omega) \stackrel{\text{def}}{=} \mathcal{D}_{n_c}\Big(\operatorname{tr}(X - \Omega\cdots\Omega), \ldots, \operatorname{tr}(X - \Omega\cdots\Omega)^{n_c}\Big)\,, \tag{2.250}$$

$$\begin{aligned}\det{}_3(X - \Omega\cdots\Omega) &\stackrel{\text{def}}{=} \left[\det{}_2\left(\frac{1}{X - \Omega\cdots\Omega}\right)\right]^{-1} \\ &\stackrel{\text{def}}{=} \left[\mathcal{D}_{n_c}\left(\operatorname{tr}\frac{1}{X - \Omega\cdots\Omega}, \ldots, \operatorname{tr}\frac{1}{(X - \Omega\cdots\Omega)^{n_c}}\right)\right]^{-1}\end{aligned} \tag{2.251}$$

which respectively yield on shell

$$\begin{aligned} \text{the non-polynomial} \quad \det_1(X - \Omega\cdots\Omega) &= \Xi(X)\,, & (2.252)\\ \text{the polynomial} \quad \det_2(X - \Omega\cdots\Omega) &= \Big[\Xi(X)\Big]_+\,, & (2.253)\\ \text{and the non-polynomial} \quad \det_3(X - \Omega\cdots\Omega) &= \frac{n_c!\,\Xi(X)}{(d/dX)^{n_c}\Xi(X)} & (2.254) \end{aligned}$$

However, expanding the three functions (2.252)–(2.254) in power series of $X \to \infty$, we obtain exactly the same polynomial parts for all three functions

$$\Big[\det_1(X-\Omega\cdots\Omega)\Big]_+ = \Big[\det_2(X-\Omega\cdots\Omega)\Big]_+ = \Big[\det_3(X-\Omega\cdots\Omega)\Big]_+ = \Big[\Xi(X)\Big]_+ \qquad (2.255)$$

and only the negative-power parts are different. Although its dangerous to generalize from just three data points, we believe eq. (2.255) should work for all sensible definitions of the quantum determinant, and this gives us an unambiguous formula for the polynomial part of the characteristic "polynomial" (2.248), namely

$$\Big[\chi(X)\Big]_+ = \Big[\Xi(X)\Big]_+ = P(X) - \sum_{d\geq 1} \frac{(2d-2)!}{d!(d-1)!}\,\alpha^d \left[\frac{[B(X)]^d}{[P(X)]^{2d-1}}\right]_+ \qquad (2.256)$$

where the second equality comes from expanding eq. (2.130) for the $\Xi(X)$ in powers of α. Physically, the $P(X)$ term is the classical characteristic polynomial of the quiver and the $\sum_d$ adds quantum corrections, the $d^{\underline{\text{th}}}$ term representing the effect of d *diagonal* instantons. Note that for $F < 2n_c$ the sum stops at a finite instanton level $d_{\text{max}} = \left\lfloor \frac{n_c}{2n_c - F} \right\rfloor$, and for $F < n_c$ there are no quantum corrections at all and $\chi(X) = P(X)$.

Classically,

$$(-1)^{n_c} \det(\Omega_N \cdots \Omega_1) = \text{free term of } \chi(X) \stackrel{\text{cl}}{=} \chi(0) \qquad (2.257)$$

but in the quantum theory we should reinterpret the free part of $\chi(X)$ as the free part of the polynomial part $[\chi(X)]_+$ because only the polynomial part is unambiguous. Or equivalently, we may identify the free term of $\chi(X)$ as the coefficient of the X^0 term in the power series expansion around $X \to \infty$, thus

$$(-1)^{n_c} \det(\Omega_N \cdots \Omega_1) = \Big[\chi(X)\Big]_+\Big|_{X=0} = \Big[\chi(X)\Big]_0 = \oint \frac{dX\,\chi(X)}{2\pi i\,X} \tag{2.258}$$

where the integration contour is a very large circle on the physical sheet of the Seiberg-Witten curve (2.133). Hence, following the instanton expansion (2.256), we write

$$\begin{aligned}
&(-1)^{n_c} \det(\Omega_N \cdots \Omega_1) = \\
&= P(0) - \alpha \oint \frac{dX\,B(X)}{2\pi i X P(X)} - \alpha^2 \oint \frac{dX\,B^2(X)}{2\pi i X P^3(X)} - -3\alpha^3 \oint \frac{dX\,B^3(X)}{2\pi i X P^5(X)} \qquad (2.259) \\
&= P(0) - \sum_{k\geq n_c} \alpha b_k \times \widehat{\mathcal{H}}^{(1)}_k(\varpi) - \sum_{k_1+k_2\geq 3n_c} \alpha^2 b_{k_1} b_{k_2} \times \widehat{\mathcal{H}}^{(2)}_{k_1,k_2}(\varpi) - \\
&- \sum_{k_1+k_2+k_3\geq 5n_c} \alpha^3 b_{k_1} b_{k_2} b_{k_3} \times \widehat{\mathcal{H}}^{(3)}_{k_1,k_2,k_3}(\varpi) - \cdots \qquad (2.260)
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\mathcal{H}}^{(1)}_k(\varpi) &= \oint \frac{dX\,X^{k-1}}{2\pi i\,P(X)}\,, \\
\widehat{\mathcal{H}}^{(2)}_{k_1,k_2}(\varpi) &= \oint \frac{dX\,X^{k_1+k_2-1}}{2\pi i\,P^3(X)}\,, \\
\widehat{\mathcal{H}}^{(3)}_{k_1,k_2,k_3}(\varpi) &= 3\oint \frac{dX\,X^{k_1+k_2+k_3-1}}{2\pi i\,P^5(X)}\,, \\
&\dots
\end{aligned} \tag{2.261}$$

are homogeneous symmetric polynomial of the quiver's moduli $(\varpi_1, \ldots, \varpi_{n_c})$ of respective degrees $(k - n_c)$, $(k_1 + k_2 - 3n_c)$, $(k_1 + k_2 + k_3 - 5n_c)$, *etc.* And because these degrees are exactly as for the $\mathcal{H}_k^{(1)}(\varpi)$, $\mathcal{H}_{k_1,k_2}^{(2)}$, *etc.*, polynomials appearing in the expansion (2.246), *the instanton expansion* (2.260) *must satisfy exactly the same general rules* (A) *through* (F). In particular,

(A) For $F < n_c$

$$(-1)^{n_c}\det(\Omega_N\cdots\Omega_1) = P(0) \tag{2.262}$$

without any instantonic corrections whatsoever, and therefore

$$(-1)^{n_c}P(0) = V_1^{Nn_c} + V_2^{Nn_c}\,. \tag{2.263}$$

In other words, *the quantum-corrected constraint on the n_c redundant Coulomb moduli $\omega_i^N \equiv \varpi_i$ of the quiver is*

$$\prod_{i=1}^{n_c}\varpi_i = V_1^{Nn_c} + V_2^{Nn_c}\,, \quad \text{exactly.} \tag{2.264}$$

(B) For $F = n_c$ there is a quantum correction at the one-diagonal-instanton level but this correction is moduli independent,

$$(-1)^{n_c}\det(\Omega_N\cdots\Omega_1) = P(0) - \alpha\,. \tag{2.265}$$

This formula exactly parallels eq. (2.247) up to an unknown numerical constant in the latter, and if that constant happens to be equal to $(-1)^{n_c-1}$ then eq. (2.264) would remain valid for $F = n_c$ as well as for $F < n_c$.

(C) For $F > n_c$ the instantonic corrections become moduli dependent according to the polynomials (2.261). Comparing eqs. (2.246) and (2.260) we find

$$\begin{aligned}(-1)^{n_c} P(0) &= V_1^{Nn_c} + V_2^{Nn_c} + \alpha \sum_{k \geq n_c} b_k \left[\mathcal{H}_k^{(1)}(\varpi) + (-1)^{n_c} \widehat{\mathcal{H}}_k^{(1)}(\varpi) \right] + \\ &+ \alpha^2 \sum_{k_1+k_2 \geq 3n_c} b_{k_1} b_{k_2} \left[\mathcal{H}_{k_1,k_2}^{(2)}(\varpi) + (-1)^{n_c} \widehat{\mathcal{H}}_{k_1,k_2}^{(2)}(\varpi) \right] + \cdots \end{aligned} \tag{2.266}$$

and if we are lucky and

$$\begin{aligned}\text{all} \quad \mathcal{H}_{k_1,\ldots,k_d}^{(d)}(\varpi) &\equiv -(-1)^{n_c} \widehat{\mathcal{H}}_{k_1,\ldots,k_d}^{(d)}(\varpi) \\ &= (-1)^{n_c-1} \frac{(2d-2)!}{d!(d-1)!} \oint \frac{dX}{2\pi i} \frac{X^{(k_1+\cdots+k_d-1)}}{[P(X)]^{(2d-1)}} \end{aligned} \tag{2.267}$$

then eq. (2.264) continues to hold true for $F > n_c$.

We wanted to conclude this section by proving that the $\mathcal{H}_{k_1,\ldots,k_d}^{(d)}(\varpi)$ polynomials are indeed given by eqs. (2.268) and therefore *eq. (2.264) does hold true for any* $F < 2n_c$ and maybe for $F = 2n_c$ as well,[12] but as of this writing our proof is only 95% complete ☹. It is also longer than it ought to be, so we present it in the appendix to this dissertation rather than here.

[12]To be precise, we believe that for $F = 2n_c$ eqs. (2.268) hold for the $\mathcal{H}_k^{(1)}(\varpi)$ polynomials controlling the one-instanton-level corrections, but we are not at al sure about the higher instanton levels $d \geq 2$. The trouble with the $F = 2n_c$ case is that the ultraviolet gauge couplings τ_ℓ of the $[\mathrm{SU}(n_c)]^N$ quiver are asymptotically finite rather than asymptotically free, so we don't really know what happens to the theory beyond the weak coupling approximation. And even in the weak coupling limit, the sub-leading quantum corrections are liable to depend on the ultraviolet completion of the theory.

Chapter 3

The Random Matrix Model

3.1 Unitary matrices: why and how

In this dissertation we will focus on building matrix correspondents of chiral models formed out of fields transforming in the bi-fundamental representation of a $SU(n_c) \times SU(n_c)$ gauge group. Since the gauge theory is anomaly free there will be other chiral fields that we are temporarily leaving out of the picture for sake of simplicity. In particular we want to buld the random matrix model of the deconstructive quiver described in the previous chapter.

Similar correspondences have been established for fields in non-chiral gauge theories. In [1, 4] have been conjectured and demonstrated that fields transforming under the adjoint representation of the gauge group corresponds to hermitian random matrix models. More precisely it has been shown [16] that such models are realized as holomorphic random matrix models obtained as a multidimensional contour integral in the space of complex matrices. In the latest approach the hermitian matrix model is just a special case where the contours are chosen to be lying on the real axis. Similarly it has been shown [2, 24] that matter fields are in correspondence with complex matrices subject to hermiticity constraints relating fields transforming in the fundamental representation with the ones in the anti-fundamental. In building these models two assumptions have been used: the symmetries and the degrees of freedom of the matrix model have to be the same as the one for the fields in the gauge theory. As an illustrative example let us take a $U(n_c)$ Yang-Mills model with a field Φ in the adjoint and n_f fields Q_f, $\widetilde{Q}_f$ respectively in the fundamental and anti-fundamental representation of $U(n_c)$. Φ has n_c^2 real degrees of freedom and real eigenvalues so the natural choice falls on an hermitian matrix M transforming in the $U(\hat{N})$ representation ($\hat{N}$ plays the role of color in the matrix model). Similarly Q_f and $\widetilde{Q}_f$ have $n_c \times n_f$ complex degrees of freedom and correspond to a $\hat{N} \times \hat{F}$ complex matrix B and a $\hat{F} \times \hat{N}$ complex matrix A. Since the two fields are in conjugate representations not all their degrees of freedom are independent; this is easily expressed in the matrix language imposing the constraint $A = B^\dagger$. As we can see these constraints are purely linear in character and their imposition is what defines a sub-manifold $\Gamma \subset Mat_{\hat{N}}(\mathbb{C})$ of the space of complex matrices. Clearly working out which is the integration manifold is not the whole story. Until now we have only illustrated the guiding principles to define the random matrix model corresponding the a specific non-chiral gauge theory. This is not as saying this is the correct one. There is one fundamental requirement to be satisfied by the model besides the symmetry and degrees of freedom assumptions: the loop equations of the matrix model and the anomaly equations of the gauge theory have to match.

We can now start building a unitary one matrix model as a functional integral on a subspace of the space of complex matrices. We use a construction analogous to the one outlined in [16] and rely on the two assumptions just illustrated. A bi-fundamental field Ω can be seen as one set of fields Q_c, $\widetilde{Q}_c$ having half the degrees of freedom or equivalently having n_c^2 real degrees of freedom. This sounds similar to the hermitian case apart from the fact that Ω doesn't necessary has to have real eigenvalues ($\Omega \neq \Omega^\dagger$). The symmetry requirements will define a manifold $\mathcal{M} \subset Mat_{\hat{N}}(\mathbb{C})$ that is invariant under the action of the symmetry group $SU(\hat{N})_L \otimes SU(\hat{N})_R$ and whose matrices have non-zero distinct eigenvalues. In other words if $U \in \mathcal{M}$ then also $LUR^\dagger \in \mathcal{M}$ with $L \in SU(\hat{N})_L$ and $R \in SU(\hat{N})_R$. The manifold so defined is still too big since $dim_{\mathbb{C}}\mathcal{M} = \hat{N}^2$. According to the counting of degrees of freedom we want a manifold $\Gamma \subset \mathcal{M}$ such that $dim_{\mathbb{R}}\Gamma = dim_{\mathbb{C}}\mathcal{M} = \hat{N}^2$. Γ can be constructed as follows: let $\gamma : S^1 \to \mathbb{C}$ be the map of the unit circle in the complex plane. We can define

$$\Gamma = \{U \in \mathcal{M} \,|\, \sigma_i \in \gamma\} \tag{3.1}$$

This construction of Γ is equivalent to say that U is a unitary matrix and has the advantage of making the symmetries of the model manifest. On the other side the way Γ is "carved out" of $\mathcal{M}$ is far from unique, but this shouldn't trouble us since at the end of the day only an integration on eigenvalues will be performed.

Once the integration manifold has been setup we can proceed to verify if our model possess loop equations that match the anomaly equations of the corresponding gauge theory. This operation will be less obvious than expected, for unitary matrices, due to how the equations are derived from the matrix action functional. The solution will be the definition of an holomorphic measure for the functional integral, or in other words an holomorphic deformation of the maps γ used in the definition of Γ.

Let's start with a generic unitary matrix model having an action functional of the form

$$\mathcal{Z} = e^{-\frac{\hat{N}^2}{\hat{S}^2}\mathcal{F}} = \frac{1}{Vol[SU(\hat{N})]} \int d^{\hat{N}^2}\Omega \, e^{-\frac{\hat{N}}{\hat{S}}\mathcal{W}(U)} \tag{3.2}$$

with

$$\begin{aligned} d^{\hat{N}^2}\Omega &= \prod_{i,j=1}^{\hat{N}} U^{-1}_{ik} dU_{kj} = \varepsilon^{a_1 \cdots a_{\hat{N}^2}} \, \omega_{a_1} \cdots \omega_{a_{\hat{N}^2}} \\ \omega_{a_i} &= \mathrm{Tr}\left[iU^{-1} dU \lambda_{a_i} \right] \end{aligned} \tag{3.3}$$

the holomorphic Haar measure for the unitary (λ_{a_i} is a chosen basis of matrices spanning the $SU(\hat{N})_R$ symmetry group) matrix. $\mathcal{W}$ is exactly the superpotential of the chiral gauge Lagrangian where the chiral fields are replaced by their matrix correspondents. For normalization purposes we divide by the volume of the symmetry group. We also divide the coefficient of the exponent of the integrand by $\hat{S}$, a variable ultimately describing gaugino condensates. In doing so we will satisfy a "quantization" condition for the density of the eigenvalues of U to be defined later on. The action is clearly invariant under $SU(\hat{N})_L \otimes SU(\hat{N})_R$ while the measure as a slightly larger invariance group: $SL(\hat{N}, \mathbb{C})_L \otimes SU(\hat{N})_R$. Symmetry considerations become important when we want to derive the loop equations of the model. As the Konishi anomaly equations are the consequence of generic holomorphic variations of the chiral field, loop equations are derived by varying the correspondent unitary matrix within the domain of integration Γ. In particular we want to impose infinitesimal unitary left variations that are holomorphic

$$U \longrightarrow U' = LU = U + f(U). \tag{3.4}$$

with $f(U)$ an infinitesimal function of U. In general this may not be possible since $U' \in \Gamma$ while $U + f(U)$ is not necessarily unitary. The solution to this apparent problem is to deform the integration manifold $\Gamma \to \Gamma' \subset \mathcal{M}$ in a way analog to the deformation of the unit circle in the simple case of a single complex variable of integration. This is possible since the Haar measure is actually $SL(\hat{N}, \mathbb{C})$ invariant and so it doesn't require any modification.

Any generic function $f(U)$ will make the job and since any function can be locally expressed in power series we set $f(U) = \delta U = \epsilon U^k$, with ϵ an infinitesimal coefficient. The loop equation are retrieved imposing such variation to the action and the measure and requiring that $\delta \mathcal{Z} = 0$ when integrated over Γ'. To simplify our notation let's express U parametrically as

$$U = \prod_{j=1}^{\hat{N}} e^{-ix_j \lambda_{a_j}} \tag{3.5}$$

This choice implies that $\omega_{a_i} = dx_i$ so that formally the Jacobian coming from the deformation of the holomorphic measure can be written as

$$J(\omega_{a_i}) = \det \left(\frac{d(x_i + \delta x_i)}{dx_j} \right) \tag{3.6}$$

The formal structure of the loop equation then take the form

$$\langle \mathrm{Tr} \, \frac{d\, \delta x_i}{dx_j} \rangle - \frac{\hat{N}}{\hat{S}} \langle \mathrm{Tr}(\delta U \frac{d\mathcal{W}}{dU}) \rangle = 0. \tag{3.7}$$

We have just written the linear part in ϵ of $\delta\mathcal{Z} = 0$ where the second term stems from the variation of the action $\mathcal{W}$. The equation can be rendered more explicit writing

$$d\,\delta x_i = \sum_{n=1}^{k} \sum_{j=1}^{\hat{N}} \mathrm{Tr}\left[U^n \lambda_{a_i}^{\dagger} U^{k-n} \lambda_{a_j}\right] d\,x_j \tag{3.8}$$

and chosing the specific basis for λ_{a_i}

$$\lambda_a \equiv (\lambda_{\alpha\beta})_{ij} \equiv \delta_{\alpha i}\delta_{\beta j} \tag{3.9}$$

The loop equation then becomes

$$\langle \sum_{n=1}^{k} \mathrm{Tr}\, U^n \,\mathrm{Tr}\, U^{k-n} \rangle - \frac{\hat{N}}{\hat{S}} \langle \mathrm{Tr}(\delta U \frac{d\mathcal{W}}{dU}) \rangle = 0. \tag{3.10}$$

To make this equation independent from the exponent k of the variation δU, we can write it as an equation for the resolvent

$$R(z) = \frac{\hat{S}}{\hat{N}} \sum_{k=0}^{\infty} \frac{\mathrm{Tr}(U^k)}{z^{k+1}}. \tag{3.11}$$

We will eventually obtain an equation for the unitary matrix that can be compared, after having taken the large $\hat{N}$ limit, to the correspondent Konishi anomaly equation. Its detailed form depends on the specifics of the action $\mathcal{W}$ and can be generically expressed as

$$z\,\langle R(z) \rangle^2 + \frac{\hat{S}}{\hat{N}} \langle R(z) \rangle - \frac{\hat{S}}{\hat{N}} \langle \mathrm{Tr}\left[\sum_{k=0}^{\infty} \frac{U^k}{z^{k+1}} \frac{d\mathcal{W}}{dU} \right] \rangle = 0. \tag{3.12}$$

Clearly this is not the only possible loop equation for the unitary matrix U. There are other possible variations depending on both the chiral fields left over fro the chiral gauge model. Typically such variations will be functions of complex matrices like A and B. The procedure, though, is exactly the same and we will see it in detail only for the specific model we are going to analyze in the next section.

3.2 Loop equations for the Chiral Gauged Quiver

As we already mentioned we are considering a matrix model with the same formal symmetries of its correspondent gauge model. Essentially we are considering the closed quiver of chapter 2 whose building blocks are a chain of $SU(n_c)$ gauge symmetries. This is a 4D $\mathcal{N}=1$ theory with n_f flavors Q_f, $\widetilde{Q}_f$ for each block and a chiral bi-fundamental field Ω linking every block with the next (Figure 1.1). The total gauge symmetry is then $G_{4\text{D}} = \prod_{\ell=1}^{K} SU(n_c)_\ell$[1] , where the index ℓ refers to different blocks of the chain. The superpotential of the quiver has three distinct parts $W = W_{Hop} + W_\Sigma + W_{Def}$. The first part $W_{Hop} = \gamma \sum_{\ell=1}^{K} (\widetilde{Q}_{\ell+1}\Omega_\ell Q_\ell - \mu \widetilde{Q}_\ell Q_\ell)$ facilitates the propagation of the (anti)quark fields in the x^4 direction, the second $W_\Sigma = \beta \sum_{\ell=1}^{K} s_\ell(\det \Omega_\ell - v^{n_c})$ is a Lagrange multiplier that implements the gauge symmetry at each site and the third is a term $W_{Def} = \sum_{p=1}^{d} \frac{\nu_p}{p} \operatorname{tr}(\Omega_K \Omega_{K-1} \cdots \Omega_2 \Omega_1)^p$ whose effect is to deform the $\mathcal{N}=2\,\text{SQCD}_4$. This model has both deconstructive and non-deconstructive phases [12, 14] and an interesting chiral ring structure studied in detail in chapter 2. The subject of this and subsequent sections will be the study of the random matrix model associated with it.

First we need to define the random matrix variables accordingly to

[1]In order to not confuse N, number of nodes, with $\hat{N}$, rank of the matrices U_ℓ, we are going to use here K instead of N

the rules derived in the previous section. Accordingly to each hypermultiplet corresponds a set of matrices A_ℓ, B_ℓ, and to each link a unitary matrix U_ℓ. They transforms as

$$\begin{aligned} U_\ell &\rightarrow S_{\ell+1} U S_\ell^{-1} \\ A_\ell &\rightarrow A S_{\ell+1}^{-1} \qquad S_\ell \in SU_\ell(\hat{N}) \\ B_\ell &\rightarrow S_\ell B \end{aligned} \tag{3.13}$$

where in matrix language $\hat{N}$ is the number of colours and $\hat{F}$ is number of flavours. The functional $\mathcal{Z}$ has K copies of (3.2) and the integration measure is opportunely modified to include all the matrix variables

$$\mathcal{Z} = e^{-\frac{\hat{N}^2}{\hat{S}^2}\mathcal{F}} = \frac{1}{Vol[SU(\hat{N})^K]} \int \prod_{\ell=1}^{K} d^{\hat{N}^2}\Omega_\ell \, d^{\hat{F}\hat{N}} A_\ell \, d^{\hat{N}\hat{F}} B_\ell \, e^{-\frac{\hat{N}}{\hat{S}}\mathcal{W}(U,A,B)}. \tag{3.14}$$

The expression for the matrix action has the same functional form of the chiral superpotential of the quiver where field variables have been substituted by their matrix correspondents. We omit the redundant term W_Σ and think of the matrices U_ℓ to be rescaled by the quantity $v^{\hat{N}}$. The functional action is then

$$\begin{aligned} \mathcal{W}(U,A,B,s) &= \mathcal{W}_{Hop} + \mathcal{W}_{Def} \qquad (3.15) \\ \mathcal{W}_{Hop} &= \sum_{\ell=1}^{K} \{\gamma \operatorname{Tr}^{\hat{F}}(A_{\ell+1} U_\ell B_\ell - \mu A_\ell B_\ell)\} \\ \mathcal{W}_{Def} &= \sum_{p=1}^{d} \frac{\nu_p}{p} \operatorname{Tr}^{\hat{N}}(U_K \cdots U_1)^p. \end{aligned}$$

Again $\mathcal{W}_{Def}$ was introduced in the chiral gauge theory according to Dijkgraaf-Vafa recipe in order to break $\mathcal{N} = 2$ to $\mathcal{N} = 1$ supersymmetry. We will see how this term plays a major role in the expression for the loop equations; the elliptic curve generated by one of them will have a topological structure determined by the coefficents ν_p.

In this section we want to derive all the independent loop equations generated by $\mathcal{Z}$ and compare them to the Konishi anomaly equations of section 2.2.2. For the same reasons explained in the previous section we want to have a holomorphic measure for the unitary matrices U_ℓ in order to allow for non-unitary deformation the domain of integration Γ. Generic variations for U_ℓ can be expressed in power series, the difference from the simpler example of one matrix being that we have an ordered product of K unitary matrices. The obvious generalization leads to two different possible deformations, the second one including flavor matrices

$$\delta U_\ell = \epsilon (U_\ell \cdots U_{\ell+1})^k U_\ell \tag{3.16}$$

$$\delta U_\ell = B_{\ell+1}\, \epsilon\, A_\ell (U_{\ell-1} \cdots U_\ell)^k. \tag{3.17}$$

The measure is readily modified using (3.3) for each $d\Omega_\ell$. Using (3.16) it is easy to show that (3.10) becomes

$$\langle \sum_{n=1}^{k} \mathrm{Tr}\, C^n\, \mathrm{Tr}\, C^{k-n} \rangle - \frac{\hat{N}}{\hat{S}} \langle \mathrm{Tr}(\delta U_\ell \frac{d\mathcal{W}}{dU_\ell}) \rangle = 0 \tag{3.18}$$

with $C = U_{\ell-1} \cdots U_\ell$. We can eliminate both the k and the ℓ dependence rewriting the equation in terms the resolvent $R(z) = \frac{\hat{S}}{\hat{N}} \sum_{k=0}^{\infty} \frac{\mathrm{Tr}(C^k)}{z^{k+1}}$ defined out of the matrix valued variable C and using the explicit expression for $\mathcal{W}$.

After some manipulation we have

$$z\,\langle R^2(z)\rangle - \frac{\hat{S}}{\hat{N}}\langle R(z)\rangle - \gamma\frac{\hat{S}}{\hat{N}}\left\langle \sum_{k=0}^{\infty}\frac{\mathrm{Tr}(C^k U_\ell B_\ell A_{\ell+1})}{z^{k+1}}\right\rangle - z\,V'(z)\langle R(z)\rangle + \langle P(z)\rangle = 0. \tag{3.19}$$

where we have used the definitions

$$V(z) = \sum_{p=1}^{d}\frac{\nu_p}{p}z^p \qquad ; \qquad P(z) = \frac{\hat{S}}{\hat{N}}\sum_{p=1}^{d}\nu_p\sum_{k=0}^{p-1}\mathrm{Tr}(C^k)z^{p-k-1}. \tag{3.20}$$

We can keep only dominant terms in the large $\hat{N}$ limit so that in its final form the loop equation for the resolvent of the ordered product C is

$$z\,\langle R(z)\rangle^2 - z\,V'(z)\langle R(z)\rangle + \langle P(z)\rangle = 0. \tag{3.21}$$

This is the first of series of equations. It can be solved in terms of the resolvent $R(z)$ to find that its domain of definition is a Riemannian surface with branch cuts determined by the coefficients of the polynomial $P(z)$.

The second variation (3.17) of U_ℓ generates an equation that impose a constraint on the resolvent. The equation is obtained in exactly the same fashion imposing the total variation of the functional action, $\mathcal{Z}$, to be zero

$$\langle\sum_{n=0}^{k-1}\mathrm{Tr}\left[\epsilon A_\ell C^n U_{\ell-1}\cdots U_{\ell+1}B_{\ell+1}\right]\left(\mathrm{Tr}\,C^{k-n-1} - \mathrm{Tr}\,\mathbb{I}\right)\rangle - \frac{\hat{N}}{\hat{S}}\langle\mathrm{Tr}(\delta U_\ell\frac{d\mathcal{W}}{dU_\ell})\rangle = 0. \tag{3.22}$$

In terms of the resolvents $R(z)$ and $\mathcal{M}(z)$ this equation becomes

$$\langle R(z)\mathcal{M}_{\ell+K,\ell+1}(z)\rangle + \frac{\langle Q(z)\rangle}{z} = \left(\frac{\hat{S} + z\,V'(z)}{z}\right)\langle\mathcal{M}_{\ell+K,\ell+1}(z)\rangle + \gamma\,\langle\mathcal{M}_{\ell,\ell}(z)\mathrm{M}_{\ell+1}\rangle \tag{3.23}$$

where we have defined the following quantities

$$\begin{aligned}\mathcal{M}_{\ell,\ell'}(z) &= \left[\frac{A_\ell U_{\ell-1}\cdots U_{\ell'}B_{\ell'}}{z-C}\right]\\ \mathrm{M}_l &= A_\ell B_\ell \\ Q(z) &= \sum_{p=1}^{d}\nu_p\sum_{k=1}^{p}A_\ell C^{k-1}U_{\ell-1}\cdots U_{\ell+1}B_{\ell+1}\,z^{p-k}-\gamma(\mathrm{M}_\ell\mathrm{M}_{\ell+1}).\end{aligned}\tag{3.24}$$

At first glance we can notice that the potential $V(z)$ has been shifted by an amount equal to the total fraction of the unitary eigenvalues $\hat{S}$ (the gaugino condensate, in the field theory language). We believe that this is nothing than an artifact coming from the deformation of the measure and can be reabsorbed in the value that constraints the determinant of the unitary matrices. Here, as for $P(z)$ in equation (3.21), the polynomial $Q(z)$ is at most of z^{p-1} order. Notice also that we have written the equation in matrix valued form ($\hat{F}\times\hat{F}$), so that it is independent of the variation parameter ϵ.

There are two remaining loop equations generated by variations of the complex matrices A_ℓ and B_ℓ corresponding to the matter sector in field theory language. Variations for these can be built in analogy with what just done for U_ℓ giving the following expressions

$$\delta A_\ell = \epsilon\, A_{\ell'}\, U_{\ell'-1}\cdots U_\ell(U_{\ell-1}\cdots U_\ell)^k \tag{3.25}$$

$$\delta B_\ell = (U_{\ell-1}\cdots U_\ell)^k U_{\ell-1}\cdots U_{\ell'}\, B_{\ell'}\,\epsilon \tag{3.26}$$

where ϵ is just an infinitesimal $\hat{F}\times\hat{F}$ matrix. We can now impose each of this variations independently on the functional integral and require that $\delta\mathcal{Z}=0$. The formal expressions are very similar to (3.7)

$$\langle \mathrm{Tr}\,\frac{d\,\delta A_\ell}{dA_\ell}\rangle - \frac{\hat{N}}{\hat{S}}\langle \mathrm{Tr}(\delta A_\ell\frac{d\mathcal{W}}{dA_\ell})\rangle = 0, \tag{3.27}$$

$$\langle \mathrm{Tr}\,\frac{d\,\delta B_\ell}{dB_\ell}\rangle - \frac{\hat{N}}{\hat{S}}\langle \mathrm{Tr}(\delta B_\ell \frac{d\mathcal{W}}{dB_\ell})\rangle = 0. \tag{3.28}$$

Again the first term in the two expressions originates from the linear part of the Jacobian while the second comes from the variation of the action. We can treat these two equations on parallel grounds. Plugging in the variations (3.25, 3.26)

$$\begin{aligned} \mathrm{Tr}(\epsilon)\,\mathrm{Tr}(C^k)\delta_{\ell,\ell'} &= \gamma\left[\mathrm{Tr}(\epsilon A_{\ell'} C'^k U_{\ell'-1}\cdots U_{\ell-1}B_{\ell-1})\right. \\ &- \left.\mu\,\mathrm{Tr}(\epsilon A_{\ell'} U_{\ell'-1}\cdots U_\ell C^k B_\ell)\right] \qquad (3.29) \\ \mathrm{Tr}(\epsilon)\,\mathrm{Tr}(C^k)\delta_{\ell',\ell} &= \gamma\left[\mathrm{Tr}(\epsilon A_{\ell+1} U_\ell\cdots U_{\ell'} C'^k B_{\ell'})\right. \\ &- \left.\mu\,\mathrm{Tr}(\epsilon A_\ell C^k U_{\ell-1}\cdots U_{\ell'} B_{\ell'})\right]. \qquad (3.30) \end{aligned}$$

Note that C' is defined as C with the index ℓ substituted by ℓ'. Written in terms of the unitary resolvent $R(z)$ and the meson resolvent $\mathcal{M}_{\ell,\ell'}$, defined in (3.24), we arrive at their final form

$$\mathbb{I}_{\hat{F}\times\hat{F}}\delta_{\ell,\ell'}\gamma^{-1}\langle R(z)\rangle = \langle \mathcal{M}_{\ell',\ell-1}(z)\rangle - \mu\langle \mathcal{M}_{\ell',\ell}(z)\rangle \tag{3.31}$$

$$\mathbb{I}_{\hat{F}\times\hat{F}}\delta_{\ell',\ell}\gamma^{-1}\langle R(z)\rangle = \langle \mathcal{M}_{\ell+1,\ell'}(z)\rangle - \mu\langle \mathcal{M}_{\ell,\ell'}(z)\rangle \tag{3.32}$$

We have again written an $\hat{F}\times\hat{F}$ equation factorizing out the ϵ dependence. At first glance these seem to be two distinct equations. Upon a more careful look we see how the LHS of both is symmetric respect to the exchange of ℓ with ℓ'. On the opposite the RHS seem to not respect such property. In fact the two equations looks like they may be exchanged under this symmetry but not quite. Indeed (3.31, 3.32) are two series of equations as ℓ varies for fixed ℓ'. We can summarize them in a more compact way if we write the first equation as a system of equations with ℓ assuming values in the range $\ell' - K \leq \ell < \ell'$. Analogously the second equation can be written as a system of equations with values of ℓ now ranging in the interval $\ell' < \ell \leq \ell' + K$. In doing so we arrive exactly at the same equation

$$\langle \mathcal{M}_{\ell',\ell}(z) \rangle = \gamma^{-1} \mu^{\ell'-\ell-1} \langle R(z) \rangle + \mu^{\ell'-\ell} \langle \mathcal{M}_{\ell',\ell'}(z) \rangle \tag{3.33}$$

valid for any value of $\ell \neq \ell'$. This is the reason why we kept the delta symbol on the RHS of (3.31, 3.32). This result shouldn't be surprising since the degrees of freedom encoded in A_ℓ and B_ℓ are not independent ($A_\ell^\dagger \equiv B_\ell$). Consequently their variations are correlated and lead to the same equation. We can elaborate a little more making use of the recurrence formula (coming directly from definition (3.24))

$$\langle \mathcal{M}_{\ell'+N,\ell'}(z) \rangle = z \langle \mathcal{M}_{\ell',\ell'}(z) \rangle - \mathrm{M}_{\ell'} \tag{3.34}$$

to determine

$$\langle \mathcal{M}_{\ell'=\ell}(z) \rangle = \left(\gamma^{-1} \mu^{K-1} \langle R(z) \rangle + \mathrm{M}_{\ell'} \right) \frac{1}{z - \mu^K} \tag{3.35}$$

$$\langle \mathcal{M}_{\ell'\neq\ell}(z) \rangle = \left(z\, \gamma^{-1} \langle R(z) \rangle + \mu \mathrm{M}_{\ell'} \right) \frac{\mu^{\ell'-\ell-1}}{z - \mu^K} \tag{3.36}$$

These formulae exhaust the possible loop equations of our model.

Comparing them with the anomaly equations derived in section 2.2.2 we see that there is perfect agreement. This fact together with the arguments expressed in section two makes the unitary random matrix model the dual of our quiver chiral gauge model.

3.3 Solving for the Free Energy

In the last section we have proved that the quiver (1.1) corresponds to the random matrix model (3.14). We have shown that the matrix model not only possesses the same symmetry and degrees of freedom of the gauge theory but also satisfy the same equations for various resolvents. In this section we want to analyze this random matrix model further with the final objective of obtaining an expression for the low energy effective potential, typically given in terms of massless gaugino condensates. We are going to proceed in simple steps. We will first integrate out massive matter degrees of freedom; the resulting action will depend only on ordered products of K matrices U_ℓ. The integration over the U_ℓ matrices will factorize into $K-1$ trivial group volume evaluation and and one integral with non-trivial integrand. The application of standard saddle point method in the large $\hat{N}$ limit will lead to sphere and disk contribution for the free energy $\mathcal{F}$. A formal multi-cut solution will be derived from which an expression for the effective superpotential is calculated.

We want to integrate massive modes A_ℓ and B_ℓ out of the (3.14) action functional

$$\mathcal{Z} = e^{-\frac{\hat{N}^2}{\hat{S}^2}\mathcal{F}} = \frac{1}{\Big(Vol[SU(\hat{N})]\Big)^K \eta^{\hat{N}\hat{F}K}} \int \prod_{\ell=1}^{K} d^{\hat{N}^2}\Omega_\ell \, d^{\hat{F}\hat{N}}A_\ell \, d^{\hat{N}\hat{F}}B_\ell \, e^{-\frac{\hat{N}}{\hat{S}}\mathcal{W}(U,A,B)}. \tag{3.37}$$

Notice that we are able to write the volume of the gauge group as a product of volumes since the quiver symmetry groups are mutually independent. In the large $\hat{N}$ limit the leading term of $Vol[SU(\hat{N})]$ is equal to $Vol[U(\hat{N})]$ and can be written as [25]

$$Vol[SU(\hat{N})] = \frac{\hat{N}^2}{2}\left(\ln\frac{2\pi}{\hat{N}} + \frac{3}{2}\right) \tag{3.38}$$

We have also introduced the normalization constant η defined as

$$\int d\,\bar{x}d\,x\,e^{-\frac{\hat{N}}{\hat{S}}\gamma\Lambda\bar{x}x} = \left(\frac{2\pi\hat{S}}{\hat{N}\Lambda\gamma}\right) \doteq \eta, \tag{3.39}$$

where Λ is interpreted as the lightest mass scale of the 5D deconstructed quarks and will serve as a cut-off parameter when expressing the free energy through contour integrals. The only part of $\mathcal{Z}$ that is involved by the integrating out process is

$$\frac{1}{\eta^{\hat{N}\hat{F}K}}\int\prod_{\ell=1}^{K} d^{\hat{F}\hat{N}}A_\ell\, d^{\hat{N}\hat{F}}B_\ell\, e^{-\frac{\hat{N}}{\hat{S}}\mathcal{W}_{Hop}} = [\text{Det}\, D]^{-1}\left(\frac{2\pi\hat{S}}{\hat{N}\gamma\eta}\right)^{\hat{N}\hat{F}K} \tag{3.40}$$

where

$$D = \begin{pmatrix} \mu & 0 & 0 & \cdots & 0 & U_K \\ U_1 & \mu & 0 & \cdots & 0 & 0 \\ 0 & U_2 & \mu & \cdots & 0 & 0 \\ \vdots & \cdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & U_{K-1} & \mu \end{pmatrix}. \tag{3.41}$$

The determinant is implicitly intended to be performed on all indices: quiver, color and flavor. Here we are giving its explicit form only for quiver indices deliberately leaving the rest in general form to be exploited later on

$$[\text{Det}\, D]^{-1}\left(\frac{2\pi\hat{S}}{\hat{N}\gamma\eta}\right)^{\hat{N}\hat{F}K} = \left[\det_{\hat{N}\hat{F}\times\hat{N}\hat{F}}\left(\prod_{\ell=1}^{K} U_\ell - \mu^K\right)\right]^{-1}\Lambda^{\hat{N}\hat{F}K}. \tag{3.42}$$

This last expression depends on U_ℓ matrices only through their ordered product $\prod_{\ell=1}^{K} U_\ell$. The part of the action $\mathcal{W}$ of the matrix functional left untouched by the integration (namely $\mathcal{W}_{Def}$) depends on the ordered product of K U_ℓ

matrices as well. Consequently the entire matrix functional $\mathcal{Z}$ depends only on $\prod_{\ell=1}^{K} U_\ell$ and it naively seems that we can further integrate out $K-1$ of the U_ℓ factors. This can be made precise making use of the large symmetry group of the model to build a series of cascading changes of variables. Consider the following chain of $\bigotimes_\ell SU_\ell(\hat{N})_R$ transformations

$$
\begin{array}{lll}
(1) & \left\{ \begin{array}{ll} U_K^{(1)} = U_K V_K & \\ U_\ell^{(1)} = U_\ell & \ell \neq K \end{array} \right. & V_K = U_{K-1} \\
\vdots & & \vdots \\
(i) & \left\{ \begin{array}{ll} U_K^{(i)} = U_K^{(i-1)} V_K^{(i-1)} & \\ U_\ell^{(i)} = U_\ell & \ell \neq K \end{array} \right. & V_K^{(i-1)} = U_{K-i} \ . \\
\vdots & & \vdots \\
(K-1) & \left\{ \begin{array}{ll} U_K^{(K-1)} = U_K^{(K-2)} V_K^{(K-2)} & \\ U_\ell^{(K-1)} = U_\ell & \ell \neq K \end{array} \right. & V_K^{(K-2)} = U_1
\end{array}
$$

As a consequence of the chain of transformation we arrive at the result

$$
\begin{aligned}
\prod_{\ell=1}^{K} U_\ell = U_K^{(1)} U_{K-2} \cdots U_1 = \cdots = U_K^{(i)} U_{K-i-1} \cdots U_1 \\
= \cdots = U_K^{(K-2)} U_1 = U_K^{(K-1)} \equiv \mathcal{U}.
\end{aligned} \tag{3.43}
$$

Then it is clear that their net effect is to have the action $\mathcal{W}$ depending only on the variable $\mathcal{U}$ and completely independent of the other $K-1$ unitary matrices $U_\ell, \ \ell \neq K$. In the same fashion the measure remains invariant under the listed transformations. Take for example the $(i+1)$ transformation. The measure transforms linearly

$$
(i+1) \qquad \left\{ \begin{array}{ll} d\,\Omega_K^{(i+1)} = \left[V_K^{(i)}\right]^{-1} \left(d\,\Omega_K^{(i+1)} - d\,\Omega_{K-i-1}\right) V_K^{(i)} & \\ d\,\Omega_\ell^{(i+1)} = d\,\Omega_\ell & \ell \neq K \end{array} \right.
$$

so that the Jacobian matrix is lower triangular and its determinant is trivially one. We can then integrate over all the non dynamical U_ℓ and eliminate $K-1$ powers of the volume of the gauge group that normalizes the functional integral. What remains of $\mathcal{Z}$ expressed in the limit of large $\hat{N}$ (since that is what matters when the saddle point method is used to find the free energy) is

$$\begin{aligned}
\mathcal{Z} &= e^{-\frac{\hat{N}^2}{\hat{S}^2}\mathcal{F}} = \int \mathcal{U}^{-1} d\,\mathcal{U}\, e^{-\frac{\hat{N}}{\hat{S}}\mathcal{W}(\mathcal{U})} \\
\mathcal{W}(\mathcal{U}) = \mathcal{W}_{Def}(\mathcal{U}) &+ \frac{\hat{S}}{\hat{N}} \ln\left[\det\left(\mathcal{U} - \mu^K\right)\right] \\
&+ \hat{S}\left[\frac{\hat{N}}{2}\left(\ln\frac{2\pi}{\hat{N}} + \frac{3}{2}\right) - K\hat{F}\ln\Lambda\right]
\end{aligned} \tag{3.44}$$

We are now reduced to the integration of a single unitary matrix variable $\mathcal{U}$. It is customary [26, 27] to use the residual $SU(\hat{N})$ symmetry to express the generic Haar measure in terms of more convenient variables. The new set is given by the eigenvalues ${\lambda_k}^2$ of $\mathcal{U}$ and its normalized off-diagonal components θ_{ij}. The cost of such transformation is the introduction of a Jacobian solely dependent on λ's and usually addressed as Vandermonde determinant

$$\mathcal{U}^{-1} d\,\mathcal{U} \longrightarrow 4 \prod_{k=1}^{\hat{N}} d\,\lambda_k \prod_{i<j} \sin^2\left(\frac{\lambda_i - \lambda_j}{2}\right) \prod_{i,j} d\,\theta_{ij}. \tag{3.45}$$

The action $\mathcal{W}$ depends on $\mathcal{U}$ only through its eigenvalues so that we can integrate out the angular variables θ_{ij} and get just a constant factor up front that doesn't contain any dynamical information. This factor can be readily ignored since it doesn't enter in any way in our analysis. Since the exponent of

[2] Actually the eigenvalues of $\mathcal{U}$ are $e^{i\lambda_k}$ but due to the form of the Haar measure its expression in terms of the new variables depends directly on λ's. Consequently the eigenvalues assume values in the interval $[0, 2\pi]$

the integrand is proportional to $\hat{N}$ we can use the saddle point approximation [26]. The eigenvalues are then constrained by the saddle point equation

$$\frac{\partial \mathcal{W}}{\partial \lambda_k} = 0, \tag{3.46}$$

and in the large $\hat{N}$ limit the integral (3.44) is given by the dominant contribution in the $\hat{N}^2$ expansion. This is equivalent to the standard genus g expansion $\hat{N}^{2-2g}$ that we are familiar with from 2D gravity [28]: the first contribution comes from the sphere meanwhile the sub-dominant term is given by the disk. At the same time the eigenvalues assume continuum spectrum and can be expressed more elegantly introducing a density of eigenvalues $\rho(\lambda)$. We end up with an exact expression for the free energy $\mathcal{F}$

$$\begin{aligned} e^{-\frac{\hat{N}^2}{\hat{S}^2}\mathcal{F}} &= \exp\left\{-\frac{\hat{N}^2}{\hat{S}^2}\left[\sum_{p=1}^{d}\int d\,\lambda\,\rho(\lambda)\,\frac{\nu_p}{p}\,e^{i\lambda p} - \int d\,\lambda\,\rho(\lambda)\,d\,\lambda'\,\rho(\lambda')\ln\sin^2\left(\frac{\lambda-\lambda'}{2}\right)\right.\right. \\ &+ \left.\left.\frac{\hat{S}^2}{2}\left(\ln\frac{2\pi}{\hat{N}}+\frac{3}{2}\right)\right] - \frac{\hat{N}}{\hat{S}}\int d\,\lambda\,\rho(\lambda)\,\ln\left[\frac{\det(e^{i\lambda}-\mu^K)}{\Lambda^{\hat{F}K}}\right]\right\}. \end{aligned} \tag{3.47}$$

Following the prescription of [20] we identify the two contributions to $\mathcal{F}$. The sphere, coming from $\mathcal{W}_{Def}$ and the Vandermonde determinant, is defined as $\mathcal{F}_S = \lim_{\hat{N}\to\infty}\mathcal{F}$

$$\mathcal{F}_S = \int d\,\lambda\,\rho(\lambda)\,V(\lambda) - \int d\,\lambda\,\rho(\lambda)\,d\,\lambda'\,\rho(\lambda')\ln\left[\sin^2\left(\frac{\lambda-\lambda'}{2}\right)\right]. \tag{3.48}$$

On the other end the disk contribution arises from the matter sector and is defined as $\mathcal{F}_D = \lim_{\hat{N}\to\infty}\frac{\hat{N}}{\hat{S}}(\mathcal{F}-\mathcal{F}_S)$

$$\mathcal{F}_D = \int d\,\lambda\,\rho(\lambda)\,\ln\left[\frac{\det(e^{i\lambda}-\mu^K)}{\Lambda^{\hat{F}K}}\right]. \tag{3.49}$$

Notice that we have discarded from (3.47) any constant factor that doesn't depend on the eigenvalues density and ultimately won't contribute to the effective potential.

The expressions (3.48)and (3.49) are still uncomplete solutions for the free energy. In fact we have buried the crucial information in the density of eigenvalues of which we have given just a very informal definition. Solving for $\rho(\lambda)$ means to solve for the saddle point equation (3.46) rewritten for a continuum spectrum of eigenvalues $\lim_{\hat{N}\to\infty} \frac{\partial}{\partial\lambda}\frac{\partial\mathcal{F}}{\partial\rho(\lambda)} = 0$. The limit procedure is introduced so that we consider only dominant contributions, consistently with the expression given for the free energy. The saddle point equation takes, then, the simple form

$$V'(\lambda) - 2\int d\,\lambda'\,\rho(\lambda')\,\cot\left(\frac{\lambda-\lambda'}{2}\right) = 0. \tag{3.50}$$

This is just the loop equation (3.21) in disguised form. Multiplying both sides by $\rho(\lambda)\,\cot\left(\frac{w-\lambda}{2}\right)$ and integrating over λ we recover, after some algebra, the loop equation for the resolvent $\hat{R}$ on the cylinder [3]

$$\hat{R}^2(w) - \hat{R}(w)V'(w) + F(w) + \hat{S}^2 = 0 \tag{3.51}$$

with

$$\hat{R}(w) = \int_0^{2\pi} d\,\lambda\,\rho(\lambda)\,\cot\left(\frac{w-\lambda}{2}\right) \tag{3.52}$$

$$F(w) = \int_0^{2\pi} d\,\lambda\,\rho(\lambda)\,\left(V'(w) - V'(\lambda)\right)\cot\left(\frac{w-\lambda}{2}\right). \tag{3.53}$$

[3]The resolvent defined on the cylinder is related to the one on the complex plane through the relation $\hat{R}(w) = \frac{\hat{N}}{\hat{S}} z\,R(z) - i\hat{S}$

The differences between (3.21) and (3.51) are a natural consequence stemming from the dependence of all the expressions on the real variable λ and not $e^{i\lambda}$. We are then naturally lead to use a complex variable w that is related to the complex plane through the cylinder map $z = e^{iw}$. So the differences between $R(z)$ and $\hat{R}(w)$ are just an artifact of such map. Moreover the eigenvalue's density has to be periodic of 2π to ensure single valuedness of the resolvent. For analogour reasons the function $F(w)$ is not the same as $P(z)$ but can be rewritten in such a way that in the large $\hat{N}$ limit the two loop equations are the same. In some sense the unitary matrix case can be seen as a special case of hermitian model with compact, periodic support for its eigenvalues [2]. We have just reduced our model to solving a specific case of unitary circular ensemble. Equation (3.51) directly determines the form of $\rho(\lambda)$

$$\hat{S} = \sum_{i=1}^{n} \hat{S}_i = \sum_{i=1}^{n} \int_{a_i^-}^{a_i^+} d\,\lambda \rho(\lambda) \tag{3.54}$$

where $[a_i^-, a_i^+]$ are cuts on the fundamental interval $[0, 2\pi]$ where eigenvalues are condensed. These play exactly the same role of branch cuts on the real axis in the hermitian matrix case. In order to see why is this so, let's solve (3.51) for $\hat{R}$

$$\hat{R}(w) = \frac{V'(w) \pm Y(w)}{2} \qquad ; \qquad Y^2(w) = V'^2(w) - 4(F(w) + \hat{S}^2). \tag{3.55}$$

This equations shows a two sheeted Riemann surface with branch cuts defined by Y; crossing a cut brings from one sheet to the other. It is now standard procedure to show, with a simple contour integration argument, that

$$-2\pi i \rho(\lambda) = \hat{R}(\lambda + i\zeta) - \hat{R}(\lambda - i\zeta). \tag{3.56}$$

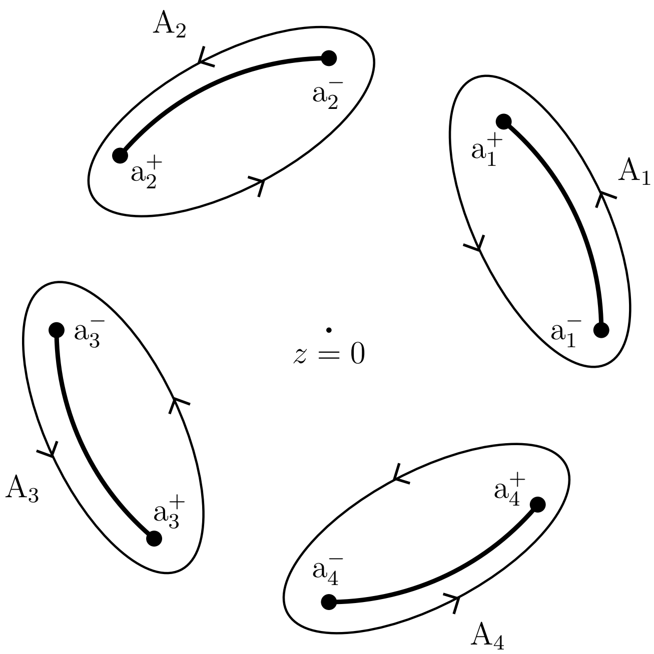


Figure 3.1: Branch cuts on the unit circle

Thus the density of eigenvalues has support exactly on the branch cuts defined by Y (3.1). Moreover a generic function $f(\lambda)$, integrated along the cuts, is equal to contour integrals A_i around the cuts

$$\int_0^{2\pi} d\,\lambda\,\rho(\lambda)\,f(\lambda) = -\frac{1}{2\pi i}\sum_{i=1}^{n}\oint_{A_i}\hat{R}(w)\,f(w)d\,w. \tag{3.57}$$

A general formula expressing the effective superpotential W_{eff} can be recovered from the free energy adapting the procedure outlined by Cachazo et Al. [5, 6] to the cylinder. According to their generic formula the effective potential for $\hat{S}$ is (up to terms that involve two derivatives of the free energy)

$$W_{eff} = \sum_{i=1}^{n}\hat{N}_i\frac{\partial\mathcal{F}_S}{\partial\hat{S}_i} + 2\pi i\sum_{i=1}^{n-1}b_i\hat{S}_i + \mathcal{F}_D. \tag{3.58}$$

Here the b_i are integers numbers arising from ambiguities in defining the Veneziano-Yankielowicz superpotential and indicating a rotation from confinement to oblique confinement (equivalent to a shift in the theta angle for the gluino condensate $\hat{S}_i$). The general expression for the sphere and disk contributions can be recovered using the equations (3.57) and modifying carefully the integration contours A_i. Performing contour integration on the cylinder is essentially the same as doing it on the complex plane with one exception: there may be potential problem arising from the undefined point $z = 0$ generated by the exponential map. These are avoided if we are careful to define the integration contour. As an illustrative example we can derive an expression for the disk contribution $\mathcal{F}_D$. Let us define $w = \sigma_1 - i\sigma_2$ and write the determinant in the logarithm of (3.49) as $\prod_{I=1}^{L} (e^{iw} - e^{iw_I})$. Then

$$\mathcal{F}_D = -\frac{1}{2\pi i} \sum_{I=1}^{L} \sum_{i=1}^{n} \oint_{A_i} d\,w\, R(w) \ln \left(\frac{e^{iw} - e^{iw_I}}{\Lambda^{K\hat{F}}} \right) \tag{3.59}$$

Modifying opportunely the contours this integral will give contributions along the branch cuts originating from the points w_I and extending to very large σ_2. We have to be careful to evaluate contributions coming from the contours connecting very large positive values with very large negative values of σ_2 running on both sides of the branch cut starting from $\sigma_2 = -\infty$. Fortunately due to the single value definition of $\hat{R}$ the eigenvalue density is periodic of 2π so that contribution from both contours cancel each other. The remaining contours are simple integration along σ_1 at very large $|\sigma_2|$ along

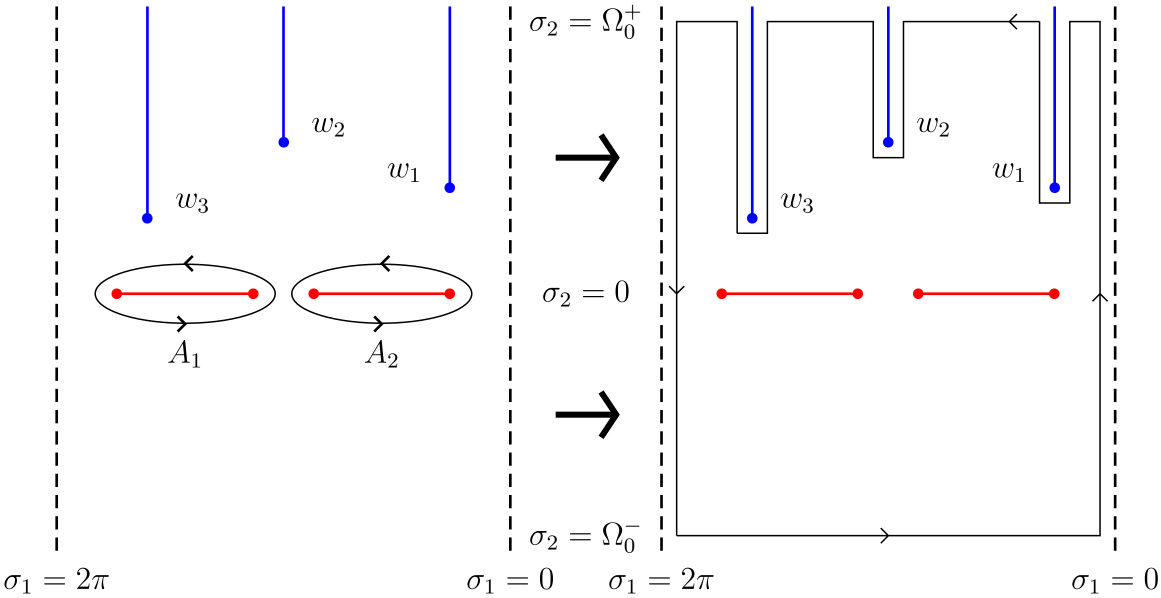


Figure 3.2: Integration contour for $\mathcal{F}_D$ on the cylinder

with integrations along the cuts stemming from the points w_I

$$\mathcal{F}_D = -\sum_{I=1}^{L} \int_{w_I}^{\Omega_0^+} d\,w\,R(w) - \hat{S}\left(L\Omega_0^+ + 2K\Omega - i\sum_{I=1}^{L} w_I\right). \tag{3.60}$$

Here Ω_0^+ is just a cut-off imposed for large positive values of σ_2 meanwhile $\Omega = \ln \Lambda^{\hat{F}}$. Following the same procedure and noticing that $\frac{\partial \mathcal{F}_S}{\partial \hat{S}_i} = \frac{\partial \mathcal{F}}{\partial \rho(\lambda)}$ we can start from a similar expression as (3.59) for the sphere free energy contribution

$$\frac{\partial \mathcal{F}_S}{\partial \hat{S}_i} = V(a_i^+) + \frac{1}{\pi i}\sum_{j=1}^{n} \oint_{A_j} d\,w\,R(w) \ln\left[\sin^2\left(\frac{w-a_i^+}{2}\right)\right]. \tag{3.61}$$

The integrand has clearly a branch cut starting at $w = a_i^+$ and extending until $\sigma_2 = \infty$ but there is not essential singularity. We can modify the contour again and arrive at an expression similar to (3.60) for the sphere contribution

$$\frac{\partial \mathcal{F}_S}{\partial \hat{S}_i} = V(a_i^+) + 2\int_{a_i^+}^{\Omega_0^+} d\,w\,R(w) + 2\hat{S}\left(\Omega_0^+ + \Omega_0^-\right). \tag{3.62}$$

Ω_0^- is a similar cut-off imposed on very large negative values of σ_2. Taking $\Omega_0^- = -\Omega_0^+$ the last term vanishes as we would expect since it is an artifact of the singularity of the point $z = 0$. We can then plug in a solution for $\hat{R}$ taking care of choosing the physical sheet where $\lim_{\sigma_2 \to \infty} \hat{R} = 1$ Thus we select the minus sign in (3.55) and substitute it in the expressions for $\frac{\partial \mathcal{F}_S}{\partial \hat{S}_i}$ and $\mathcal{F}_D$. Assembling everything in the effective potential we arrive at the final expression

$$\begin{aligned} W_{eff} \;=\; & -\sum_{i=1}^{n} \hat{N}_i \int_{a_i^+}^{\Omega_0^+} d\,w\,Y(w) + \frac{1}{2}\sum_{I=1}^{L}\int_{w_I}^{\Omega_0^+} d\,w\,Y(w) + \frac{1}{2}(2\hat{N} - L)V(\Omega_0^+) \\ & + \sum_{I=1}^{L} V(w_I) + \left(i\sum_{I=1}^{L} w_I + 2K\Omega - L\Omega_0^+ \right)\hat{S} + 2\pi i \sum_{i=1}^{n-1} b_i \hat{S}_i. \end{aligned} \tag{3.63}$$

The potential divergences limited by the cut-off Ω_0^+ are balanced by the normalization constant Ω and define the bare coupling constant $2\pi i\tau$. Since $Y(w)$ vanishes at $w = a_i^+$ we can write $\int_{a_i^+}^{\Omega_0^+} = \frac{1}{2}\int_{B_i}$ where B_i is a contour that goes from Ω_0^+, cross the i^{th} cut and goes back to $\widetilde{\Omega}_0^+$ on the second sheet.

$$\begin{aligned} W_{eff} \;=\; & -\frac{1}{2}\sum_{i=1}^{n} \hat{N}_i \int_{B_i} d\,w\,Y(w) + \frac{1}{2}\sum_{I=1}^{L}\int_{w_I}^{\Omega_0^+} d\,w\,Y(w) + \frac{1}{2}(2\hat{N} - L)V(\Omega_0^+) \\ & + \sum_{I=1}^{L} V(w_I) + \left(i\sum_{I=1}^{L} w_I + 2\pi i\tau \right)\hat{S} + 2\pi i \sum_{i=1}^{n-1} b_i \hat{S}_i. \end{aligned} \tag{3.64}$$

Notice that this is very similar to the result obtained in [6]. The only difference being that the the bare coupling constant τ is shifted by an amount equal to the eigenvalues of the matrix D (3.41). This is exactly what we would expect since the introduction of extra 4D hypermultiplets change the Chern-Simons level of the 5D coupling constant [12].

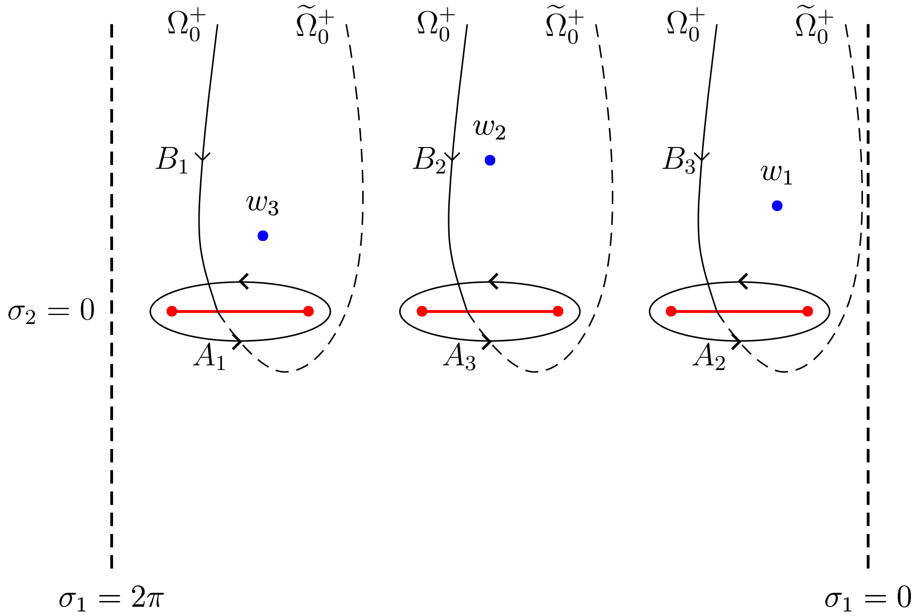


Figure 3.3: B_i and A_i contours on the cylinder

3.4 Examples of Effective Superpotentials

At the end of last section we have given an explicit formula (3.64) expressing the effective superpotential. The only necessary ingredient needed is the form of the tree superpotential of the gauge theory underlying the random matrix model. Here we want to give some examples of such expression starting from one of the simplest case possible. Due to the relative simplicity of the cases considered we don't use the (3.64) but a far direct even if less general procedure [29]. The result of section 4 standing, we prefer to solve directly for the free energy using an exact expression for the density $\rho(\lambda)$. Such an expression is derived from knowledge of the tree superpotential $\mathcal{W}$ and the asymptotic behavior of the resolvent $R(z)$ as well as from the number s of branch cuts of Y. In general we can express this last quantity as

$$Y(z) = \frac{Q(z)}{z}\sqrt{\sigma(z)} \tag{3.65}$$

where $Q(z)$ is an unknown polynomial. Generically $Q(z)$ is determined by inspection knowing that the resolvent $R(z) \approx \frac{\hat{S}}{z}$ for large z.

$$Q(z) = \mathrm{Pol}\,\frac{V'(z)}{z\,\sqrt{\sigma(z)}} \tag{3.66}$$

Such condition determines also part of the unknowns of $\sigma(z)$, namely the endpoints of the branch cuts. If $V(z)$ is of degree d then $Q(z)$ is of degree $d-s$ ($s \leq d$) and equation (3.66) gives $d - s + s + 1 = d + 1$ equations. There remain $s+1$ constants that are determined requiring the free energy is minimal. In other words the system of cuts has to be stable and no eigenvalues is permitted to migrate from one cut to the other. This is directly written in integral form as

$$\int_{a_i^+}^{a_{i+1}^-} \frac{Q(z)}{z}\sqrt{\sigma(z)} = 0 \tag{3.67}$$

where the integral is performed between the endpoints of distinct branch cuts. There are $s-1$ independent integrals of this kind, giving the remaining conditions required. Obviously this methods becomes a little cumbersome in the case of several disjoint cuts, but it is easily applied to our simple examples.

Once $Q(z)$ and all the endpoints are determined in terms of the coupling constants of $V(z)$ and the filling fractions (gluino condensates) $\hat{S}_i$ we can express the density of eigenvalues

$$\rho(x) = \frac{1}{2\pi i} Y(x), \tag{3.68}$$

plug it in the expression for the free energy and perform the integrations along the branch cuts. In order to simplify such task we are going to rewrite the expression for the sphere free energy $\mathcal{F}_S$ so to avoid an overly complicated double integration. Let's take equation (3.50), integrate it between the two intervals $[\theta_i^-, \lambda]$ and $[\lambda, \theta_i^+]$ (with $[\theta_i^+, \theta_i^-]$ a branch cut), and combine the two results

$$\begin{aligned} \int_{\theta_i^-}^{\theta_i^+} d\,\lambda'\,\rho(\lambda')\ln\left[\sin^2\left(\frac{\lambda-\lambda'}{2}\right)\right] &= \frac{V(\lambda)}{2} - \frac{1}{4}\left(V(\theta_i^-) + V(\theta_i^+)\right) \\ &+ \frac{1}{2}\int_{\theta_i^-}^{\theta_i^+} d\,\lambda'\,\rho(\lambda')\ln\left[\sin^2\left(\frac{\theta_i^+-\lambda'}{2}\right)\sin^2\left(\frac{\theta_i^- - \lambda'}{2}\right)\right]. \end{aligned} \tag{3.69}$$

We can then take this expression and plug it in (3.48) to obtain a simplified formula for the sphere free energy

$$\begin{aligned} \mathcal{F}_S &= \frac{1}{2}\sum_i \int_{\theta_i^-}^{\theta_i^+} d\,\lambda'\,\rho(\lambda')\left\{V(\lambda') - \hat{S}_i \ln\left[\sin^2\left(\frac{\theta_i^+-\lambda'}{2}\right)\sin^2\left(\frac{\theta_i^- - \lambda'}{2}\right)\right]\right\} \\ &+ \frac{\hat{S}_i}{4}\left(V(\theta_i^-) + V(\theta_i^+)\right). \end{aligned} \tag{3.70}$$

Let's now rewrite both $\mathcal{F}_S$ and $\mathcal{F}_D$ in terms of the complex integration variable $x = e^{i\lambda}$. This is just a technical stratagem that allows us to perform the integral on the unit circle in a more readable fashion

$$\mathcal{F}_S = \frac{1}{2}\sum_i \int_{a_i^-}^{a_i^+} \frac{d\,x}{x}\rho(x)\left\{V(x) - \hat{S}_i\left[\ln\left|\frac{a_i^+ - x}{2}\right|^2 + \ln\left|\frac{a_i^- - x}{2}\right|^2\right]\right\}$$

$$
\begin{aligned}
& + \quad \frac{\hat{S}_i}{4}\left(V(a_i^-) + V(a_i^+)\right) && (3.71)\\
\mathcal{F}_D \quad &= \quad \sum_i \int_{a_i^-}^{a_i^+} \frac{d\,x}{x}\rho(x)\left[\sum_{flavors} \ln\left(x-\mu^K\right) - \hat{F}K \ln\ \Lambda\right], && (3.72)
\end{aligned}
$$

where $a_i^{\pm} = e^{\theta_i^{\pm}}$.

3.4.1 One cut: $SU(\hat{N})$ unbroken

We want to consider the same example used in section 2.2.2 with $d = 2$ and $\hat{F} = 1$

$$
V(z) = \alpha\, z + \frac{\beta}{2} z^2 = \beta z(z-a) \tag{3.73}
$$

where $a = -\frac{\alpha}{\beta}$ and $|a| = 1$, since we are on the unit circle. Classically since $zV'(z)$ has only one zero, the gauge theory is described by an unbroken $SU(n_c)$ gauge group. The quantum theory becomes interesting when this zero splits to a cut on the unit circle. In this case $Y(z)$ can be written as

$$
Y(z) = \frac{Q(z)}{z}\sqrt{(z-a^+)(z-a^-)} \tag{3.74}
$$

and $a^+ = a + 2C \quad a^- = a - 2C$. The function $Q(z)$ is determined as previously described. If we write $Q(z) = Az + B$ the fact that $R(z)$ does not have terms of $\mathcal{O}(z)$ and $\mathcal{O}(1)$ give $A = \beta$ and $B = \alpha + \frac{\beta}{2}(a^+ + a^-)$. The last constraint is though redundant since $\frac{(a^+ + a^-)}{2} = a$ implies $B = 0$. There is a third condition coming from the $\mathcal{O}(\frac{1}{z})$ term

$$
(a^+ - a^-)^2 = \frac{16}{\beta}\hat{S} - \frac{3}{4}a^2. \tag{3.75}
$$

Through these conditions we can trade any dependence on the cut endpoints for the other two independent variables: $\hat{S}$ and a. We will do this at the end of the integration process for the free energy. We can now take the expression for the density of eigenvalues (3.68)

$$\rho(x) = \frac{\beta}{2\pi i}\sqrt{(x-a^+)(x-a^-)} \tag{3.76}$$

and use it in (3.71),(3.72). There are essentially four integrals to be performed

$$\begin{aligned}
I_1 &= -\frac{\beta}{4\pi}\int_{a^-}^{a^+} d\,x\,\sqrt{(x-a^+)(x-a^-)}(\alpha+\frac{\beta}{2}x) \\
I_2^{\pm} &= \hat{S}\frac{\beta}{4\pi}\int_{a^-}^{a^+}\frac{d\,x}{x}\sqrt{(x-a^+)(x-a^-)}\ln\left|\frac{a_i^{\pm}-x}{2}\right|^2 \\
I_3 &= -\frac{\beta}{2\pi}\int_{a^-}^{a^+}\frac{d\,x}{x}\sqrt{(x-a^+)(x-a^-)}\ln\left(x-\mu^K\right) \\
I_4 &= K\ln\Lambda\frac{\beta}{2\pi}\int_{a^-}^{a^+}\frac{d\,x}{x}\sqrt{(x-a^+)(x-a^-)}
\end{aligned} \tag{3.77}$$

$I_2^{\pm}$ and I_3 cannot be straightforwardly calculated since the integrand contains the function $\frac{\ln(c+x)}{x}$. This is substantially different from the hermitian case where there is no $\frac{1}{x}$ coefficient leading to a much simpler calculation. The only way to perform the integration is to expand part of the integrand in an exact power series. In this way these integrals can be reduced to a combination of a series of terms originating from the logarithm asymptotic formula

$$\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k}. \tag{3.78}$$

We can then define the k-index dependent integrals

$$J_k = \int_{a^-}^{a^+} d\,x\,\sqrt{(x-a^+)(x-a^-)}\,x^{k-1}, \tag{3.79}$$

$$\begin{aligned}
H_k &= \int_{a^-}^{a^+}\frac{d\,x}{x}\sqrt{(x-a^+)(x-a^-)} \\
&\times \left[\cos^k(i\ln(x)-i\ln(a^+))+\cos^k(i\ln(x)-i\ln(a^-))\right],
\end{aligned} \tag{3.80}$$

and show that I_1, I_3 and I_4 are given by a combination of J_k (for example is fairly easy to see that I_1 is given by J_1 plus J_2), while $I_2^{\pm}$ are given by an infinite sum of H_k. It is convenient to introduce the following functions of $\hat{S}$ and a

$$\begin{aligned} h(\hat{S},a) &= (a^+ - a^-)^2 = \frac{16}{\beta}\hat{S} - \frac{3}{4}a^2 \\ g(\hat{S},a) &= \sqrt{a^+a^-} = \sqrt{-\frac{4\hat{S}}{\beta} + \frac{13}{2^4}a^2} \end{aligned} \tag{3.81}$$

in order to express the integrals J_k and H_k only in terms of the specified invariants With these conventions the first integrals of the series J_k and H_k are

$$\begin{aligned} J_0 &= i\pi(g(\hat{S},a) + a) \\ J_1 &= \frac{i\pi}{2^3}h(\hat{S},a) \\ J_2 &= \frac{i\pi}{2^3}h(\hat{S},a)\,a \\ J_3 &= \frac{i\pi}{2^7}h(\hat{S},a)[2^4a^2 + h(\hat{S},a)] \\ J_4 &= \frac{i\pi}{2^7}h(\hat{S},a)[2^4a^3 + 3\,a\,h(\hat{S},a)] \\ H_1 &= \frac{i\pi}{2^4g^2(\hat{S},a)}[-2^4a^3 + 3\,a\,h(\hat{S},a) + a^2g(\hat{S},a)] \\ H_2 &= \frac{i\pi}{2^9g^4(S,a)}[17\cdot 2^5a^5 - 13\cdot 2^4a^3h(\hat{S},a) + 17\cdot 2\,ah^2(\hat{S},a) \\ &+ 2^9a^4g(\hat{S},a) + 9\cdot 2^5a^2h(\hat{S},a)g(\hat{S},a) - 3\cdot 2^3h^2(\hat{S},a)g(\hat{S},a)]. \end{aligned} \tag{3.82}$$

The first important fact is that all the integrals are polynomials of the invariants a and S. This is not at all obvious and since the superpotential is directly derived from the free energy through (3.58) it has to depend on S and a only

$$\begin{aligned}
\mathcal{F}_S &= -\frac{\beta}{4\pi}\left\{\alpha J_1 + \frac{\beta}{2}J_2 + \hat{S}\left[2\ln\frac{1}{2}\,J_0 - \sum_{k=1}^{\infty}\frac{H_k}{k}\right]\right\} \\
&+ \frac{\hat{S}}{4}\left(V(a^-) + V(a^+)\right) \qquad (3.83) \\
\mathcal{F}_D &= -\frac{\beta}{2\pi}\left[\left(\pi i + K\ln\frac{\mu}{\Lambda}\right)J_0 - \sum_{k=1}^{\infty}\frac{J_k}{k(\mu^K)^k}\right]. \qquad (3.84)
\end{aligned}$$

We can now use (3.58) adapted to the one cut case and derive the expression for the effective superpotential

$$\begin{aligned}
\mathcal{W}_{eff} &= -\frac{\beta}{2\pi}\left\{-n_c\frac{2\pi i\hat{S}}{\beta g(\hat{S},a)} + J_0\left(n_c\ln\frac{1}{2} + \pi i + K\ln\frac{\mu}{\Lambda}\right) - \sum_{k=1}^{\infty}\frac{1}{k}\left[\frac{1}{2}n_c H_k\right.\right. \\
&+ \left.\left.\frac{J_k}{(\mu^K)^k}\right] - \frac{1}{2}n_c\hat{S}\frac{\partial}{\partial\hat{S}}\left(\sum_{k=1}^{\infty}\frac{H_k}{k}\right)\right\} + \frac{n_c}{4}\left(V(a^-) + V(a^+)\right). \qquad (3.85)
\end{aligned}$$

Notice that we have discarded constant terms (indipendent of β) that can always be absorbed.

Chapter 4

Open questions and conclusions

4.1 Chiral ring of the quiver theory

Having analyzed the chiral rings of deconstructive $[\mathrm{SU}(n_c)]^N$ quivers in much detail, we would like to conclude this dissertation by discussing the implications of the present research and the open questions it raises.

The most immediate implication of our rings concerns the quantum effects in deconstructed SQCD_5; this will be discussed at length in ref. [12], but we would like to present a few highlights here. From the deconstruction point of view, our most important results are the Seiberg-Witten curve (2.133) and the quantum-corrected constraint (2.264) on the Coulomb moduli of the quiver. In the large quiver size limit $N \to \infty$ governed by the 4D $\leftrightarrow$ 5D map of moduli and parameters[1]

$$\begin{aligned} \varpi_i &= V^N \times \exp(Na\phi_i)\,, && (4.1) \\ \mu_f &= V \times \exp(am_f)\,, && (4.2) \\ \alpha &= \frac{V^{2n_cN}}{\prod_f \max(\mu_f^N, V^N)} \times \exp\left(-Na\,\frac{8\pi^2}{g_{5\mathrm{D}}^2}\right) && (4.3) \end{aligned}$$

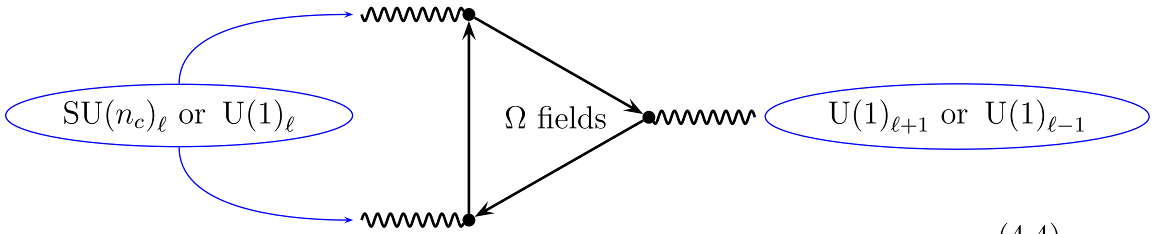
the 4D abelian gauge couplings encoded in the Seiberg-Witten curve (2.133)

[1] In the following (4.1)–(4.3), the ϕ_i denote the Coulomb moduli of the 5D SQCD, the m_i are the 5D quark masses, and the $g_{5\mathrm{D}}^2$ is the 5D gauge coupling *at the origin of the Coulomb moduli space.*

behave precisely as in a 5D gauge theory compactified on a large circle of size $2\pi R = Na$. The 5D abelian couplings implied by this procedure — using dimensional deconstruction as a UV completion of SQCD_5 — are in perfect agreement with those of SQCD_5 embedded in string or M theory, which means they are intrinsic properties of the 5D theory.

Furthermore, thanks to eq. (2.264), the V parameter in (4.1)–(4.3) is the greater of V_1, V_2 roots of eq. (2.230). Consequently, for $\Delta F = 0$ there is a lower limit on the V/Λ ratio and hence according to eq. (4.3) the inverse 5D gauge coupling also has a finite lower limit; on the other hand, for $\Delta F > 0$ there are no limits and the $g_{5\mathrm{D}}^{-2}$ ranges all the way from $+\infty$ to $-\infty$. In 5D terms this means that SQCD_5 theories with maximal Chern-Simons levels $k_{\mathrm{cs}} = n_c - \frac{1}{2}n_f$ have only the positive-coupling phase (see ref. [14] for the $n_f = 0$, $k_{\mathrm{cs}} = n_c$ case), but for lower Chern-Simons levels there are both positive-coupling and negative-coupling phases.

We would like to extend our techniques from the deconstructed SQCD_5 to other deconstructed 5D gauge theories, but this remains an open question. Naively, the simplest extension is promoting the $\mathrm{SU}(n_c)$ gauge theory to the $\mathrm{U}(n_c)$: Classically, all one has to do in 4D is to promote each $[\mathrm{SU}(n_c)]_\ell$ factor to a $[\mathrm{U}(n_c)]_\ell$, dispense with the s_ℓ singlet fields and the W_Σ part of the superpotential (cf. eq. (2.3)), and expand the theory around a vacuum with non-zero link VEVs $\langle\Omega_\ell\rangle \propto \mathbf{1}_{n_c\times n_c}$. Unfortunately, the abelian gauge fields of such a $[\mathrm{U}(n_c)]^N$ quiver suffer from triangular anomalies



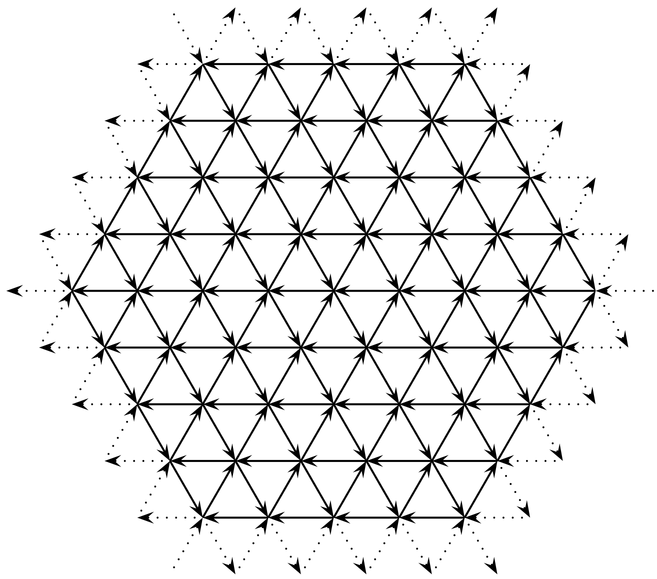
(4.4)

and the quantum theory does not work. To cancel the anomalies we need additional chiral superfields (cf. [30, 31] for the $[\mathrm{U}(1)]^N$ quiver) with non-trivial $\mathrm{SU}(n_c)_\ell$, $\mathrm{U}(1)_\ell$ and $\mathrm{U}(1)_{\ell\pm1}$ quantum numbers, for example

$$\left.\begin{aligned} A_\ell &= \Big(\square_\ell\,,\,(\mathbf{1}^{-n_c})_{\ell+1}\Big)\\ B_\ell &= \Big(\overline{\square}_\ell\,,\,(\mathbf{1}^{+n_c})_{\ell-1}\Big)\\ C_\ell &= \Big((\mathbf{1}^{-n_c})_\ell\,,\,(\mathbf{1}^{+n_c})_{\ell+1}\Big)\end{aligned}\right\}\quad \text{for all } \ell=1,\ldots,N\,. \tag{4.5}$$

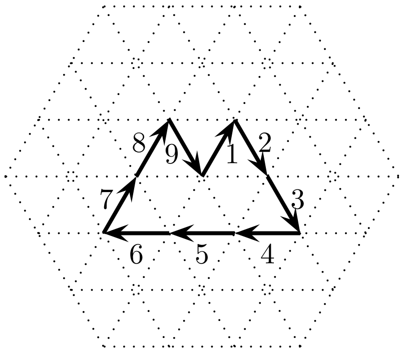
Then we need to endow all these fields with suitable superpotential couplings and find a vacuum state of the theory where all the light particles correspond to Kaluza-Klein modes of the 5D $\mathrm{U}(n_c)$ theory compactified on a circle, *and nothing else*. There will be of course all kinds of particles with $O(V)$ masses, and we need to find their effects on the Chern-Simons interactions of the 5D theory and perhaps adjust the number ΔF of massless quark flavors. At this point we have a complicated 4D $[\mathrm{U}(n_c)]^N$ theory which is no longer described by a simple quiver diagram (1.1), and now the real work begins: evaluating the chiral ring of the theory and its implications for the quantum deconstruction.

A bigger open question concerns quantum deconstruction of other types of 5D gauge theories, for example $\mathrm{SO}(n)$ or $\mathrm{Sp}(n)$. Again, one must first deconstruct the 5D theory at the classical level and verify the quantum consistency of the resulting 4D quiver theory, and then one must study the chiral ring in all its glorious details. But the really big challenge comes from deconstructing 2 extra dimensions at once: start with a 6D SYM theory with 16 supercharges, discretize the x^4 and x^5 coordinates into a 2D lattice, and interpret the result as a 4D, $\mathcal{N} = 1$ gauge theories with a complicated quiver. Classically, the procedure is well known [32]; for example the SU(n) SYM theory in 6D deconstructs to into the $[\mathrm{SU}(n)]^{N^2}$ theory in 6D with a quiver diagram forming a 2D triangular lattice



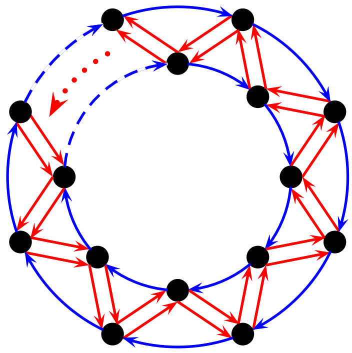
(4.6)

But at the quantum level, the chiral ring of this quiver poses a formidable challenge because every closed loop on the lattice gives rise to an independent generator of the ring, *eg.* $\mathrm{Tr}(\Omega_9\Omega_8\cdots\Omega_2\Omega_1)$ for the loop of 9 links on the following picture:



(4.7)

Another open question is finding a *simple* string model of the deconstructed $SQCD_5$ and investigating the string origin of the quantum effects discussed in this paper at the 4D field theory level. Ironically, while some complicated quiver theories do have simple string models — for example the lattice quiver (4.6) deconstructing the 6D SYM obtains via $n_c \times N^2$ fractional D3 branes at a $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold point [32] — the simplest known string model of the $[\mathrm{SU}(n_c)]^N$ quiver (1.1) involves *brane webs* on an orbifold [33]. The issue is not 5D versus 6D, and other, more complicated deconstructed 5D gauge theories do have web-less string models. For example, $2N \times n$ fractional D3 branes at the $\mathbb{C}^3/\mathbb{Z}_{2N}[1, 1, -2]$ orbifold point give rise to the 4D theory with the



(4.8)

quiver diagram which deconstructs an $\mathrm{SU}(n) \times \mathrm{SU}(n)$ theory in 5D. But this theory has a very different chiral ring than the rings discussed in section 2.2–section 2.3 of this dissertation, and it needs to be worked out in detail before we can analyze its implications for the string theory.

4.2 Random matrix model

The random matrix portion of the model has shed light on the otherwise uneasy task of calculating the effective superpotential of the gauge theory in terms of gaugino condensates. In achieving this task we had to establish a set of rules on how to realize the duality between bi-fundamental sets of fields and bosonic random variables. In more precise words we had to identify which integration manifold we had to choose for the corresponding matrices so that the duality could be set in place. We have then, identified bi-fundamentals with unitary matrices and confirmed the identifications of hyper-multiplets with pairs of complex conjugate matrices.

In the process we have established similar results to the hermitian random matrix case and have shown that the integration manifold has to be holomorphically modified in order to accommodate for non-unitary variation of the matrices representing bi-fundamentals of the quiver. We have shown that the loop equations correspond to the anomaly equations of the quiver theory and so lead to the same solutions. In the matrix model we do not have the whole set of anomaly equations reproduced but a subset big enough to guarantee the validity of the duality.

Having set the duality we have proceeded to calculate the effective superpotential of the model in the large $\hat{N}$ limit using the saddle point approximation. Such a calculation revealed that the matrix integral can be reduced to a one-matrix integral only whose eigenvalues are concentrated on intervals set on the unit circle. The overall expression of the free energy, from which the superpotential is derived, is calculated as contour integrals on the cylinder and support conclusions regarding the semiclassical limit of the superpotentials.

It is still unclear how to reproduce a fairly simple superpotential from the example we have analyzed even if it should in principle possible. Our expression for the simple $SU(n_c)$ case is a little too clouded and needs some refining but nonetheless is expressed in powers of the gaugino condensate and so seems to reproduce a proper superpotential. This results conclude our analysis of the quiver gauge theory, its chiral ring and its exact effective superpotential.

Appendix

An (almost) proof of the quantum corrections of the determinants of link chains

In this appendix we (almost) prove that the $\mathcal{H}^{(d)}_{k_1,\ldots,k_d}(\varpi)$ polynomials in eq. (2.246) are indeed given by eqs. (2.268) and therefore eq. (2.264) does hold true for any $F < 2n_c$ and maybe for $F = 2n_c$. Specifically we shall prove that (1) the polynomials $\mathcal{H}^{(d)}_{k_1,\ldots,k_d}(n_c;\varpi_1,\ldots,\varpi_{n_c})$ have the same form for all quivers with the same color number n_c regardless of the quiver's size N or the flavor number F (as long as $k_1,\ldots,k_d \le F \le 2n_c$ and $N \ge 2$), and (2) they are recursively related to the $\mathcal{H}^{(d)}_{k'_1,\ldots,k'_d}(n'_c;\varpi_1,\ldots,\varpi_{n'_c})$ polynomials for smaller values of the $k'_1,\ldots,k'_d$ indices and/or color numbers n'_c. These relations are consistent with eqs. (2.268) and could be used to prove them all by mathematical induction from a few special cases. Alas, verifying those special cases remains a loophole; we hope to close it soon, but we have not done it yet.

We begin by proving the N independence of the $\mathcal{H}^{(d)}_{k_1,\ldots,k_d}$ polynomials. Consider what happens when one of the v_ℓ parameters of the promoted quiver theory becomes very large, say $v_1 \gg$ any other mass scale of the theory. Physically, in the $v_1 \to \infty$ limit we have a high-energy threshold due to the large semiclassical VEV $\langle \Omega_1 \rangle = v_1 \times 1_{n_c \times n_c}$ (modulo gauge transforms): the $\mathrm{SU}(n_c)_1 \times \mathrm{SU}(n_c)_2$ gauge symmetry is Higgsed down to a single $\mathrm{SU}(n_c)_{1+2}$ factor, the Ω_1 chiral field is eaten up, and the Q_1 quarks and $\widetilde{Q}_2$ antiquarks field become massive. Integrating out these heavy fields we see that the effective low-energy theory has the same quiver structure as the high-energy theory — except for the $N^{\rm low} = N^{\rm high} - 1$ — and the same coupling parameters except for the[2]

$$\Lambda_{1+2}^{2n_c-F} = \Lambda_1^{2n_c-F}\Lambda_2^{2n_c-F} \times \frac{\det(v_1\Gamma_1)}{v_1^{2n_c}} \qquad \text{and} \qquad m_{1+2} = m_1 \times \frac{1}{v_1\Gamma_1} \times m_2\,. \quad (9)$$

In terms of the α and b_k parameters this means

$$\alpha^{\rm low} = \frac{\alpha^{\rm high}}{v_1^{2n_c-F}} \qquad \text{and} \qquad b_k^{\rm low} = \frac{b_k^{\rm high}}{v_1^{F-k}} \tag{10}$$

while the Coulomb moduli of the low-energy and the high-energy theories are related according to

$$\varpi_j^{\rm high} = v_1 \times \varpi_j^{\rm low}\,. \tag{11}$$

Consequently, eq. (2.246) of the low-energy theory becomes

$$\begin{aligned}
\left[V_{\rm global}^{Nn_c}\right]^{\rm low} &= \sum_d (\alpha^{\rm low})^d \sum_{k_1,\ldots,k_d} b_{k_1}^{\rm low}\cdots b_{k_d}^{\rm low} \times \mathcal{H}_{k_1,\ldots,k_d}^{(d)\,\rm low}(\varpi_j^{\rm low}) \\
&= \sum_d (\alpha^{\rm high})^d \sum_{k_1,\ldots,k_d} b_{k_1}^{\rm high}\cdots b_{k_d}^{\rm high} \times v_1^{\sum k - 2dn_c} \times \mathcal{H}_{k_1,\ldots,k_d}^{(d)\,\mathbf{low}}(v_1^{-1}\,\varpi_j^{\rm high}) \\
&= \frac{1}{v_1^{n_c}} \times \sum_d (\alpha^{\rm high})^d \sum_{k_1,\ldots,k_d} b_{k_1}^{\rm high}\cdots b_{k_d}^{\rm high} \times \mathcal{H}_{k_1,\ldots,k_d}^{(d)\,\mathbf{low}}(\varpi_j^{\rm high})
\end{aligned} \tag{12}$$

where the last equality follows from the $\mathcal{H}_{k_1,\ldots,k_d}^{(d)}$ being homogeneous polynomials of respective degrees $\sum k - (2d-1)n_c$. By comparison, the high energy theory has

$$\left[V_{\rm global}^{Nn_c}\right]^{\rm high} = \sum_d (\alpha^{\rm high})^d \sum_{k_1,\ldots,k_d} b_{k_1}^{\rm high}\cdots b_{k_d}^{\rm high} \times \mathcal{H}_{k_1,\ldots,k_d}^{(d)\,\mathbf{high}}(\varpi_j^{\rm high}) \tag{13}$$

[2]The quarks $Q_{1+2,f} \approx Q_{2,f}$ and the antiquarks $\widetilde{Q}_{1+2}^{f'} \approx \widetilde{Q}_1^{f'}$ of the low energy theory acquire small masses $[m_{1+2}]_f^{f'}$ via the see-saw mechanism when the heavy fields $Q_{1,f}$ and $\widetilde{Q}_2^{f'}$ are integrated out.

regardless of the v_1 parameter being large or small because the global correction is completely independent of any of the v_ℓ. On the other hand, in the $v_1 \to \infty$ limit quantum effects associated with the Ω_1 link field become small because of asymptotic freedom (assuming $F < 2n_c$), hence

$$\det(\Omega_N \cdots \Omega_2\Omega_1) \;\to\; \det(\Omega_N \cdots \Omega_2) \times v_1^{n_c} \tag{14}$$

and therefore the global corrections to these determinants should obey the same scaling law

$$\left[V_{\text{global}}^{Nn_c}\right]^{\text{high}} = \left[V_{\text{global}}^{Nn_c}\right]^{\text{low}} \times v_1^{n_c}\,. \tag{15}$$

Consequently, substituting eqs. (12) and (13) into this formula and comparing the moduli-dependent coefficients of similar $(\alpha^{\text{high}})^d b_{k_1}^{\text{high}} \cdots b_{k_d}^{\text{high}}$ we see that we must have

$$\mathcal{H}_{k_1,\ldots,k_d}^{(d)\,\text{high}}(\varpi_j) = \mathcal{H}_{k_1,\ldots,k_d}^{(d)\,\text{low}}(\varpi_j) \quad \forall k_1 + \cdots + k_d \geq (2d-1)n_c\,. \tag{16}$$

And therefore by induction, the polynomials $\mathcal{H}_{k_1,\ldots,k_d}^{(d)}(\varpi)$ have exactly the same form for all quiver sizes $N \geq 2$. $\mathfrak{Q}.\ \mathfrak{E}.\ \mathfrak{D}.$

Next, let us show that the flavor number F also does not affect the form of the $\mathcal{H}_{k_1,\ldots,k_d}^{(d)}(\varpi)$ polynomials (as long as $F \geq k_1,\ldots,k_d$). For this purpose we no longer need the ℓ-dependent matrices of quark masses and Yukawa couplings, so let $[\Gamma_\ell]_f^{f'} \equiv \gamma\delta_f^{f'}$ and $[m_\ell]_f^{f'} \equiv \gamma\mu_f\delta_f^{f'}$, and consider the limit in which one of the quark flavors becomes very heavy, say $\mu_1 \to \infty$. Again, we integrate out the heavy fields and derive the low-energy effective theory which has $F^{\text{low}} = F^{\text{high}} - 1$ and

$$\left[\Lambda_\ell^{2n_c-F}\right]^{\text{low}} = \left[\Lambda_\ell^{2n_c-F}\right]^{\text{high}} \times (-\gamma\mu_1) \implies \alpha^{\text{low}} = -\mu_1^N \times \alpha^{\text{high}}\,. \tag{17}$$

Let us keep the low-energy physics fixed while $\mu_1 \to \infty$, thus fixed $\alpha^{\rm low}$ and fixed $\mu_2, \ldots, \mu_F \implies$ fixed $B^{\rm low}(X)$. In terms of the high-energy theory, this means

$$[\alpha B(X)]^{\rm high} = \frac{X - \mu_1^N}{-\mu_1^N} \times [\alpha B(X)]^{\rm low} \implies [\alpha b_k]^{\rm high} \underset{\mu_1\to\infty}{\longrightarrow} [\alpha b_k]^{\rm low} \tag{18}$$

and therefore

$$\left[V^{Nn_c}_{\rm global}\right]^{\rm high} \underset{\mu_1\to\infty}{\longrightarrow} \sum_d (\alpha^{\rm low})^d \sum_{k_1,\ldots,k_d} b^{\rm low}_{k_1} \cdots b^{\rm low}_{k_d} \times \mathcal{H}^{(d)\,\mathbf{high}}_{k_1,\ldots,k_d}(\varpi^{\rm high}_j)\,. \tag{19}$$

On the other hand, decoupling of the heavy quark flavor implies

$$\begin{aligned} \left[V^{Nn_c}_{\rm global}\right]^{\rm high} &\underset{\mu_1\to\infty}{\longrightarrow} \left[V^{Nn_c}_{\rm global}\right]^{\rm low} \quad \text{for } \varpi^{\rm low}_j \equiv \varpi^{\rm high}_j \\ &= \sum_d (\alpha^{\rm low})^d \sum_{k_1,\ldots,k_d} b^{\rm low}_{k_1} \cdots b^{\rm low}_{k_d} \times \mathcal{H}^{(d)\,\mathbf{low}}_{k_1,\ldots,k_d}(\varpi^{\rm high}_j)\,. \end{aligned} \tag{20}$$

Comparing (19)–(20) we immediately see that we should have

$$\mathcal{H}^{(d)\,\rm high}_{k_1,\ldots,k_d}(\varpi_j) = \mathcal{H}^{(d)\,\rm low}_{k_1,\ldots,k_d}(\varpi_j) \qquad \forall F \geq k_1, \ldots, k_d \tag{21}$$

and therefore by induction, the polynomials $\mathcal{H}^{(d)}_{k_1,\ldots,k_d}(\varpi)$ have exactly the same form for all sufficiently large flavor numbers. $\mathcal{Q}.\ \mathcal{E}.\ \mathcal{D}.$

Now let us relate quiver theories with different color numbers. To integrate out a color we need each link field Ω_ℓ to have one very large eigenvalue $\omega_1 \to \infty$, or in gauge invariant terms we need a very large Coulomb modulus $\varpi_1 = \omega_1^N \to \infty$. To make sure this modulus stays large, we trap it on a mesonic branch of the moduli space where $\varpi_1 = \mu_1^N = \mu_2^N$ and then take the degenerate mass to infinity. Integrating out all fields which become superheavy in this limit, we arrive at the effective low-energy theory which now has

$$F^{\rm low} = F^{\rm high} - 2\,, \qquad n_c^{\rm low} = n_c^{\rm high} - 1\,, \qquad \text{and} \qquad \alpha^{\rm low} = \alpha^{\rm high}\,. \tag{22}$$

Classically

$$\det(\Omega_N \cdots \Omega_1)^{\rm high} \stackrel{\rm cl}{=} \omega_1^N \times \det(\Omega_N \cdots \Omega_1)^{\rm low}\,, \tag{23}$$

but the quantum corrections may also have sub-leading contributions, thus we look for

$$\left[V^{Nn_c}_{\rm global}\right]^{\rm high} = \varpi_1 \times \left[V^{Nn_c}_{\rm global}\right]^{\rm low} + \cdots \tag{24}$$

where the '$\cdots$' denote terms which do not grow in the $\varpi_1 = \mu_1^N = \mu_2^N \to \infty$ limit. To be precise, eq. (24) holds when we identify $(\varpi_2, \varpi_2, \ldots, \varpi_{n_c})^{\rm high} = (\varpi_1, \varpi^{\rm low})$ and

$$B^{\rm high}(X) = (X - \varpi_1)^2 \times B^{\rm low}(X) \implies b_k^{\rm high} = b_{k-2}^{\rm low} - 2\varpi_1 b_{k-1}^{\rm low} + \varpi^2 b_k^{\rm low}\,. \tag{25}$$

Let us plug these identifications and the respective eqs. (2.246) for both the high-energy and the low energy theories into eq. (24). The result looks rather messy, but grinding through the algebra and matching similar powers of $\alpha^{\rm high} = \alpha^{\rm low}$ and similar products $b_{k_1}^{\rm low} \cdots b_{k_d}^{\rm low}$ on both sides we arrive at

$$\mathcal{H}_k^{(1)\rm high}(\varpi_1, \varpi^{\rm low}) - 2\varpi_1 \mathcal{H}_{k-1}^{(1)\rm high}(\varpi_1, \varpi^{\rm low}) + \varpi_1^2 \mathcal{H}_{k-2}^{(1)\rm high}(\varpi_1, \varpi^{\rm low}) = \varpi_1 \times \mathcal{H}_{k-2}^{(1)\,\rm low}(\varpi^{\rm low}) + \cdots \tag{26}$$

for the one-diagonal-instanton level, and more generally

$$\sum_{q_1,\ldots,q_d=0,1,2} \binom{2}{q_1} \cdots \binom{2}{q_d} (-\varpi_1)^{q_1+\cdots+q_d} \times \mathcal{H}^{(d)\,\rm high}_{k_1-q_1,\ldots,k_d-q_d}(\varpi_1, \varpi^{\rm low}) = = \varpi_1 \times \mathcal{H}^{(d)\,\rm low}_{k_1-2,\ldots,k_d-2}(\varpi^{\rm low}) + \cdots \tag{27}$$

where the '$\cdots$' denotes terms independent of the ϖ_1 modulus.

Eqs. (27) give us recursive relations between the $\mathcal{H}$ polynomials of quiver theories with different color numbers. It is easy to see that these relations are consistent with eqs. (2.268): Since the $\widehat{\mathcal{H}}^{(d)}_{k_1,\ldots,k_d}$ polynomials do not depend on how the index sum $K = k_1 + \cdots + k_d$ is partitioned into individual $k_1, \ldots, k_d$ indices, the left hand side of eq. (27) becomes

$$\sum_{Q=0}^{2d} \binom{2d}{Q} (-\varpi_1)^Q \times \left[\mathcal{H}_{K-Q}^{(d)\,\text{high}}(\varpi_1, \varpi^{\text{low}}) = (-1)^{n_c-1} c_d \oint \frac{dX}{2\pi i} \frac{X^{K-Q-1}}{[P^{\text{high}}(X)]^{(2d-1)}} \right] =$$

$$\begin{aligned} &= (-1)^{n_c-1} c_d \oint \frac{dX}{2\pi i} \frac{X^{K-2d-1} \times (X - \varpi_1)^{2d}}{[P^{\text{high}}(X)]^{(2d-1)}} \\ &= (-1)^{n_c-2} c_d \oint \frac{dX}{2\pi i} \frac{(\varpi_{\cdot} X^{K-2d-1} - X^{k-21})}{[P^{\text{low}}(X)]^{(2d-1)}} \\ &= \varpi_1 \times \mathcal{H}_{K-2d}^{(d)\,\text{low}}(\varpi^{\text{low}}) - \mathcal{H}_{K-2d+1}^{(d)\,\text{low}}(\varpi^{\text{low}}) \end{aligned} \tag{28}$$

where $c_d = \frac{(2d-2)!}{d!(d-1)!}$, the third line follows from $P^{\text{high}}(X) = (X - \varpi_1) \times P^{\text{low}}(X)$, and the second term on the last line does not depend on the ϖ_1 modulus in perfect agreement with the right hand side of eq. (27).

Working in the other direction, the recursive formulæ (27) allow us to completely determine all of the $\mathcal{H}_{k_1,\ldots,k_d}^{(d)}(n_c; \varpi)$ polynomials for all color numbers n_c *provided we already know a few of these polynomials.* Indeed, consider the one-instanton level and suppose we already know the $\mathcal{H}_k^{(1)}(n_c - 1, \varpi)$ polynomials for the $n_c - 1$ colors. Then for the n_c colors, eqs. (26) consecutively determine all but one of the $\mathcal{H}_k^{(1)}(n_c; \varpi)$ polynomials according to

$$\begin{aligned} \mathcal{H}_{n_c+1}^{(1)}(n_c; \varpi_1, \varpi_2, \ldots, \varpi_{n_c}) &= 2\varpi_1 \times \mathcal{H}_{n_c}^{(1)}(n_c; \cancel{\varpi_1}, \cancel{\varpi_2}, \ldots, \cancel{\varpi_{n_c}}) + \\ &\quad + \varpi_1 \times \mathcal{H}_{n_c-1}^{(1)}(n_c - 1; \cancel{\varpi_2}, \ldots, \cancel{\varpi_{n_c}}) \\ &\quad + \varpi_1\text{-independent}\,, \end{aligned} \tag{29}$$

$$\begin{aligned} \mathcal{H}_{n_c+2}^{(1)}(n_c; \varpi_1, \varpi_2, \ldots, \varpi_{n_c}) &= 2\varpi_1 \times \mathcal{H}_{n_c+1}^{(1)}(n_c; \varpi_1, \varpi_2, \ldots, \varpi_{n_c}) - \\ &\quad - \varpi_1^2 \times \mathcal{H}_{n_c}^{(1)}(n_c - 1; \cancel{\varpi_1}, \cancel{\varpi_2}, \ldots, \cancel{\varpi_{n_c}}) \\ &\quad + \varpi_1 \times \mathcal{H}_{n_c}^{(1)}(n_c - 1; \varpi_2, \ldots, \varpi_{n_c}) \end{aligned} \tag{30}$$

$$+\varpi_1\text{-indep}\,,$$

$$\begin{aligned}
\mathcal{H}^{(1)}_{n_c+3}(n_c;\varpi_1,\varpi_2,\ldots,\varpi_{n_c}) &= 2\varpi_1 \times \mathcal{H}^{(1)}_{n_c+2}(n_c;\varpi_1,\varpi_2,\ldots,\varpi_{n_c}) - \qquad (31)\\
&\quad -\varpi_1^2 \times \mathcal{H}^{(1)}_{n_c+1}(n_c-1;\varpi_1,\varpi_2,\ldots,\varpi_{n_c}) +\\
&\quad +\varpi_1 \times \mathcal{H}^{(1)}_{n_c+1}(n_c-1;\varpi_2,\ldots,\varpi_{n_c})\\
&\quad +\varpi_1\text{-indep}\,,\\
\mathcal{H}^{(1)}_{2n_c}(n_c;\varpi_1,\varpi_2,\ldots,\varpi_{n_c}) &= 2\varpi_1 \times \mathcal{H}^{(1)}_{2n_c-1}(n_c;\varpi_1,\varpi_2,\ldots,\varpi_{n_c}) - \qquad (32)\\
&\quad -\varpi_1^2 \times \mathcal{H}^{(1)}_{2n_c-2}(n_c-1;\varpi_1,\varpi_2,\ldots,\varpi_{n_c}) +\\
&\quad +\varpi_1 \times \mathcal{H}^{(1)}_{2n_c-2}(n_c-1;\varpi_2,\ldots,\varpi_{n_c})\\
&\quad +\varpi_1\text{-indep}\,, \qquad (33)
\end{aligned}$$

where the ϖ_1-independent term on the right hand side of each equation is uniquely determined by the requirement that the polynomial on the left hand side is homogeneous and totally symmetric in all of the Coulomb moduli $(\varpi_1, \varpi_2, \ldots, \varpi_{n_c})$. For example, suppose we already know that

$$\mathcal{H}^{(1)}_3(n_c=2) = -(\varpi_2+\varpi_3)\,,\quad \mathcal{H}^{(1)}_3(n_c=3) = +1\,,\quad \mathcal{H}^{(1)}_4(n_c=3) = +(\varpi_1+\varpi_2+\varpi_3)\,; \qquad (34)$$

then eq. (30) tells us

$$\mathcal{H}^{(1)}_5(n_c=3) = \varpi_1^2 + \varpi_2(\varpi_2+\varpi_3) + \varpi_1\text{-independent} \qquad (35)$$

and the only homogeneous symmetric polynomial of this form is

$$\mathcal{H}^{(1)}_5(n_c=3) = (\varpi_1^2+\varpi_2^2+\varpi_4^2) + (\varpi_1\varpi_2+\varpi_1\varpi_3+\varpi_2\varpi_3)\,. \qquad (36)$$

The one exception to this method is the zero-degree case of $k = n_c$

where eq. (26) reduces to a triviality

$$\mathcal{H}^{(1)}_{n_c}(n_c; \cancel{\varpi_1}, \cancel{\varpi_2}, \ldots, \cancel{\varpi_{n_c}}) = \varpi_1\text{-independent} \tag{37}$$

and there is no symmetry argument to determine the numerical constant $\mathcal{H}^{(1)}_{n_c}$. To plug this hole we need a separate recursive relation

$$\mathcal{H}^{(1)}_{n_c}(n_c) = -\mathcal{H}^{(1)}_{n_c-1}(n_c - 1)\,, \tag{38}$$

so let us cook up yet another integrating-out scheme. Consider the quiver theory with $F = n_c$ in the limit of $\varpi_1 = \mu_1^N \to \infty$ while the remaining moduli and quark masses remain finite.[3] Integrating out the fields which become superheavy in this limit we arrive at the low-energy theory with

$$F^{\text{low}} = n_c^{\text{low}} = F^{\text{high}} - 1 = n_c^{\text{high}} - 1 \qquad \text{and} \qquad \alpha^{\text{low}} = \frac{-1}{\mu_1^N} \times \alpha^{\text{high}}\,. \tag{39}$$

Consequently, according to eq. (2.247) for the low energy theory

$$\left[V^{Nn_c}_{\text{global}}\right]^{\text{low}} = \alpha^{\text{low}} \times \left[\mathcal{H}^{(1)}_{n_c}(n_c)\right]^{\text{low}} \tag{40}$$

and therefore

$$\left[V^{Nn_c}_{\text{global}}\right]^{\text{high}} = \varpi_1 \times \left[V^{Nn_c}_{\text{global}}\right]^{\text{low}} + O(1) = -\alpha^{\textbf{high}} \times \left[\mathcal{H}^{(1)}_{n_c}(n_c)\right]^{\text{low}} + O(\varpi_1^{-1})\,. \tag{41}$$

On the other hand, we may apply eq. (2.247) to the high-energy theory itself since it also has $F = n_c$, thus

$$\left[V^{Nn_c}_{\text{global}}\right]^{\text{high}} = \alpha^{\text{high}} \times \left[\mathcal{H}^{(1)}_{n_c}(n_c)\right]^{\text{high}}, \tag{42}$$

[3]We may force $\varpi_1 \equiv \mu_1^N$ while $\mu_1 \to \infty$ by working with discrete Higgs vacua (2.40) of the theory with a slightly deformed superpotential. We should allow for generic roots of the deformation polynomial $\widetilde{\mathcal{W}}(X)$ in order to keep the remaining Coulomb moduli $\varpi_2, \ldots, \varpi_{n_c}$ generic, but the overall magnitude of the deformation should be kept infinitesimal to assure that the quantum corrections to the link chain determinants remains as in the undeformed theory.

and comparing this formula to eq. (41) we immediately see that the $\mathcal{H}^{(1)}_{n_c}$ constants of the two theories must be related according to eq. (38).

Altogether, eqs. (38) and (29)–(32) allow us to derive all of the $\mathcal{H}^{(1)}_{k}(\varpi)$ polynomials for n_c colors from the similar polynomials for $n_c - 1$ colors. Hence by induction in n_c, *once we verify the induction base*

$$\mathcal{H}^{(1)}_2(n_c = 2) = -1\,, \quad \mathcal{H}^{(1)}_3(n_c = 2) = -(\varpi_1 + \varpi_2)\,, \quad \mathcal{H}^{(1)}_4(n_c = 2) = -(\varpi_1^2 + \varpi_2^2 + \varpi_1\varpi_2) \tag{43}$$

then all of the $\mathcal{H}^{(1)}_k$ polynomials for all color numbers must be exactly as in eqs. (2.268).

At the higher instanton levels $d \geq 2$ we have a similar situation: Once we know the $\mathcal{H}^{(d)}_{k_1 \ldots, k_d}(n_c - 1, \varpi)$ polynomials for $n_c - 1$ colors, eqs. (27) let us sequentially construct most of the $\mathcal{H}^{(d)}_{k_1 \ldots, k_d}(n_c, \varpi)$ polynomials for n_c colors according to formulæ similar to (29)–(32). Again, the ϖ_1-independent terms can be uniquely determined by the total symmetry of the polynomials in all n_c Coulomb moduli $(\varpi_1, \varpi_2, \ldots, \varpi_{n_c})$. And once again, in the zero-degree cases of $k_1 + \cdots + k_d = (2d-1)n_c$ the symmetry argument does not work and we need an additional recursive formula like (38) to obtain the numerical constants

$$\mathcal{H}^{(d)}_{k_1, \ldots, k_d}(n_c; \cancel{\varpi_1}, \cancel{\varpi_2}, \ldots, \cancel{\varpi_{n_c}}) = (-1)^{n_c - 1}\,\frac{(2d-2)!}{d!(d-1)!} \qquad \forall k_1 + \cdots + k_d = (2d-1)n_c\,. \tag{44}$$

Once we verify these constants we may use induction in n_c, and given the induction base

$$\mathcal{H}^{(d)}_{3,4,\ldots,4}(n_c = 2) \quad = \quad -\frac{(2d-1)!}{d!(d-1)!}\,(\varpi_1 + \varpi_2)\,,$$

$$\mathcal{H}^{(d)}_{4,4,\ldots,4}(n_c=2) \;=\; -\frac{(2d-1)!}{d!(d-1)!}\Big(d(\varpi_1+\varpi_2)^2-\varpi_1\varpi_2\Big)\,, \qquad (45)$$

all of the $\mathcal{H}^{(d)}_{k_1\ldots,k_d}$ *for all* n_c *must be exactly as in eqs. (2.268) and eq. (2.264) must hold true for all numbers of colors and flavors.*

This completes our argument. To complete the proof we would need to close the remaining loopholes, namely (A) verify the zero-degree formulæ (44) for $d \geq 2$ instantons, and (B) verify the induction base eqs. (43) and (45). This work is in progress, but we have not yet finished it.

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Vita

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