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Type II Flux Compactifications

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Type II Flux Compactifications

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Orientifolds of type II string theory offer a promising toolkit for model builders, especially when one includes not only the usual fluxes from NSNS and RR field strengths, but also fluxes that are T-dual to the NSNS three-form flux. These additional ingredients can help stabilize moduli and lead to D-term contributions to the effective scalar potential. We describe in general how these fluxes appear as parameters of an effective $\mathcal{N} = 1$ supergravity theory in four dimensions for type IIA and type IIB string theory. We also show how these fluxes arise from compactifications on six-dimensional spaces that can be described by toroidal fibers twisted over a toroidal base. This approach leads us to a more subtle treatment of the quantization of the general NSNS fluxes. We illustrate these phenomena with examples of certain orientifolds of $T^6/\mathbb{Z}_4$.

Contents

Chapter 1

Introduction

String theory is our best candidate for a theory of quantum gravity. It is based on the simple idea that the fundamental objects in our universe are not zero-dimensional i.e., point-like, but rather one-dimensional strings. For this theory to be anomaly free one has to demand that the strings live in a ten-dimensional spacetime. To make contact with our universe which seems to have one time and three space dimensions we need to curl up six space dimensions and make them so small that they would not have been detected by experiments conducted so far. This idea of curling up extra dimensions and the problem that comes with it are rather old. In the 1920s Kałuża and Klein [1, 2] proposed to unify general relativity and electromagnetism by compactifying five-dimensional general relativity on a circle. This leads to general relativity and electromagnetism in four dimensions but also to one scalar field which corresponds to fluctuations around a given value for the radius of the compact circle. This scalar field is massless since there is no particular preferred value for the radius. A similar though more complicated story is true if we compactify string theory on a six-dimensional space. We find generically hundreds of massless scalar fields which are normally called moduli. Up to date, however, no scalar fields have been observed and the existence of such a large number of moduli is not consistent with observations[1]. Therefore, we need a mechanism that gives large masses to these scalar fields which would make them invisible to the experiments that have been conducted so far. In this dissertation which is based on the papers [3, 4, 5] we explain how this can be accomplished in the case of type II string theory by turning on fluxes through the compact six-dimensional space.

One major focus of recent research activity has been devoted to understanding the space of string theory vacua which can be described using the formalism of four-dimensional $\mathcal{N} = 1$ supergravity since this case is the most likely to lead to (semi) realistic models. As different constructions and compactifications have been explored, the number of tools in the model builder's kit has grown, even as our understanding of how they can be used and combined has sometimes diluted. For instance string theory admits field strengths of various cohomological degree, and turning on fluxes of these field strengths through compact cycles of the internal space can often help stabilize the moduli of the compactification as we will see below. The fields in the bosonic sector of type II string theory come from two sectors which are labeled Neveu-Schwarz-Neveu-Schwarz (NSNS) and Ramond-Ramond (RR). The bosonic fields in the NSNS sector are the metric, a two-form B and a scalar field called the dilaton. The bosonic fields in the RR sector are forms C of odd (even) degree for type IIA (IIB). The field strength which we can turn on through cycles of the compact manifold include RR fluxes $F = dC$ and fluxes of the NSNS three form field strength $H = dB$. This situation is fairly well understood. The physics described by string theory compactified on a torus does not change if we invert (in certain units) the radii of arbitrary subtori. Under the inversion of the radius of a circle which is threaded with H-flux, the H-flux is converted into twists of the internal space metric, which are known as geometric fluxes. Further inversions can introduce so-called non-geometric fluxes, which ruin the global geometric description of the

[1] There are different bounds on the masses of the scalar fields depending on how they interact with the standard model sector.

internal space, but still seem to give a consistent picture in the four-dimensional effective theory. In fact, in the effective theory, one can in principle combine all of these fluxes, up to certain constraints and consistency conditions, but it has not been demonstrated that a ten-dimensional construction can necessarily always be found.

Our goal in this dissertation is to carefully explore how all of these ingredients can be combined in the context of $\mathcal{N} = 1$ toroidal orientifolds of type II supergravity, though we believe that several of our methods can be applied in broader contexts. To this end we will follow two different approaches, examining these constructions from the effective field theory point of view and also trying to present honest ten-dimensional constructions of as broad a class as possible. Throughout the paper we will illustrate each method by referring to examples of certain orientifolds of $T^6/\mathbb{Z}_4$, whose structure is rich enough to illustrate many of the phenomena and techniques that we will describe.

In the effective field theory approach our primary goal is to classify the possible (untwisted sector) fluxes and translate them into the 4D $\mathcal{N} = 1$ language. We will find that the general NSNS fluxes are most naturally parametrized by their action on the untwisted cohomology, along the lines described in [6, 7, 8, 9]. So just as one can replace a discussion of the individual components H_{ijk} of H-flux with coefficients p_K in the expansion $H = p_K b^K$, where b^K are the untwisted three-forms which are anti-invariant under the orientifold action, one can also replace metric flux components f^i_{jk} by coefficients r_{aK} and $\widehat{r}^K_\alpha$, where K again runs over three forms, and a (α) runs over invariant two-forms which are odd (even) under the orientifold involution. Similarly, the nongeometric flux components Q^{ij}_k and R^{ijk} can be replaced by q^a_K, $\widehat{q}^{\alpha K}$, and s_K. In terms of these parameters it is then straightforward to describe the data of the four-dimensional theory, and in particular we find the Kähler potential, the superpotential, and the holomorphic gauge couplings and D-terms. There are additional consistency constraints that such general fluxes must satisfy; on the one hand we have the RR tadpole conditions (to which the orientifold planes and D-branes contribute), and on the other hand there are Bianchi identities, which are a set of constraints, quadratic in the NSNS fluxes. The tadpole condition can be elegantly expressed in terms of our cohomological flux parameters, but unfortunately the Bianchi identities only seem to be cleanly expressed using the original flux components. In any given example, however, we may certainly express the Bianchi identities in our cohomological parameters, but the structure seems complicated and ad-hoc.

The presence of D-terms arising from general NSNS fluxes is a phenomenon that has not, to our knowledge, been previously discussed in the literature. In the type IIA case we describe how adding certain metric fluxes (which are never simply T-duals of H-flux) can lead to electric charges for some of the four-dimensional scalar fields. It will also turn out that certain non-geometric fluxes correspond to magnetic charges for the same fields, making them electric-magnetic dyons in general. However, making use of the Bianchi identities one can show that the dyonic charges are necessarily mutually local, and there is always a consistent Lagrangian description of the effective theory. In the type IIB case we find that NSNS H-flux can give a charge to certain RR axions which leads to D-terms in agreement with [10]. We further explain how this fits into the picture of general NSNS fluxes.

As we introduce more general types of fluxes into our story, we will see how they enter in particular examples of $T^6/\mathbb{Z}_4$ with a certain orientifold action. In the type IIA case we will look for supersymmetric solutions with as many moduli as possible stabilized. For some simple cases, such as having only H-flux, or including certain classes of metric fluxes, we are able to find all supersymmetric solutions, but are unable to stabilize all moduli in these contexts. For generic fluxes, subject to a naive quantization condition, we are able to numerically find supersymmetric solutions with all moduli stabilized. Unfortunately, we will later learn that the naive quantization condition was, in fact, naive. Using the correct quantization we can still stabilize all moduli, but are unable to satisfy the tadpole condition. It seems likely, however, that this is not a result of a fundamental obstacle, but simply relates to a lack of understanding of the correct quantization of RR fluxes in the presence of general NSNS fluxes (or at least from not using the correct representatives for the K-theory or integral cohomology when using the twisted torus language). We will also prove that in the type IIA case fully stabilized supersymmetric Minkowski vacua (as opposed to AdS) require us to at least turn on non-geometric fluxes. In the type IIB we will leave a similar analysis of the vacua for future work and restrict ourselves to writing down the Kähler potential, the superpotential, the holomorphic gauge couplings and D-terms. We will also derive the Bianchi identities and the tadpole conditions that must be satisfied by the fluxes.

After exhausting ourselves in the playground of effective field theory, we then attempt to directly construct some of these models starting from ten dimensions, following the approach of [11]. We do this by splitting our T^6 into a base and a fiber, and then allowing the fiber to vary over the base. The NSNS fluxes are then encoded as twists of the fiber theory as we go around closed, non-contractible loops in the base. We outline how to classify such splittings and twists for a given orientifold action, we show that consistency of our picture implies the Bianchi identities, and we also see clearly how to determine the correct quantization conditions on the NSNS fluxes. Simple integral quantization of the flux components or cohomological parameters turns out to be correct only in a sub-class of cases (which of course includes all situations with only H-flux, and all cases T-dual to those ones).

These constructions enjoy certain advantages; the action of the T-duality group is quite transparent. This approach should easily generalize to many other interesting situations where a well-understood fiber theory is twisted over a toroidal base. Of particular interest, we note that the flux combinations which occur in the low-energy effective theory are naturally described (as noted in [12, 8]) as a sort of covariant derivative on the spin-bundle whose sections are RR fields (see also the interesting discussion in [13]).

In the context of our specific type IIA example, we will classify all base-fiber splittings and all fluxes which can be obtained in these constructions. We will find that all H-fluxes and almost all metric fluxes can be turned on and a sub-class of non-geometric Q-fluxes can also be turned on. Among the metric flux configurations that we can build are some which are not T-duals of H-flux alone, and in particular we can turn on D-terms and cases with non-standard quantization. We cannot turn on any R-flux, which is not surprising, since there are arguments [12] that any construction giving rise to R-flux cannot have even a locally geometric description in ten dimensions.

The outline of the dissertation is as follows. In chapters 2 and 3 we discuss respectively the type IIA and IIB cases. We start in section 2.1 by describing the compactification of type IIA supergravity on an orientifold of a Calabi-Yau manifold in the presence of RR fluxes and NSNS H-flux. Then in section 2.2 we introduce metric fluxes and discuss Scherk-Schwarz reductions. In section 2.3 we introduce non-geometric fluxes. Throughout these sections we demonstrate the general ideas with the example of an $T^6/\mathbb{Z}_4$ orientifold in the subsections 2.1.4, 2.2.3 and 2.3.2. Section 2.4 compares our results for geometric compactifications with the literature on SU(3)-structure and torsion cycles, providing a nice check on our formulae, as well as a purely geometric interpretation of the D-term constraints (they are equivalent to demanding that the manifold be half-flat). In 2.5 we summarize this approach.

In chapter 3 we discuss the type IIB case. In section 3.1 we address compactifications with O3/O7-planes and in section 3.2 compactification with O5/O9-planes. After explaining the general story we discuss as example an orientifold of $T^6/\mathbb{Z}_4$ in section 3.3. This chapter as well as section 2.3 and parts of section 2.2 rely on T-duality arguments and discuss the effective four-dimensional field theory rather than giving an honest ten-dimensional construction.

To see how we can at least obtain a certain subset of the generalized NSNS fluxes we introduced in chapters 2 and 3 we turn to the base-fiber constructions in chapter 4. We introduce the T-duality group in section 4.1 and discuss the orientifold action in the language of $\mathrm{O}(6,6)$. In 4.2 we describe how to encode the NSNS fluxes as twists of our fibers, starting with a particular example for illustration before moving on to the general case. Section 4.3 is devoted to exploring these techniques in the $T^6/\mathbb{Z}_4$ example we used to illustrate the type IIA case. It includes a complete classification of all twists possible with these constructions. Some discussion is presented in 4.4.

In chapter 5 we conclude by briefly summarizing our results and discussing open questions and future research projects.

Finally, Appendix A provides two different derivations of the Bianchi identities for the generalized NSNS fluxes, using the Jacobi identity for a certain Lie algebra, or alternatively by demanding that the covariant derivative, which encodes the action of the fluxes on the spin-bundle of the RR fields, squares to zero.

Chapter 2

Type IIA Flux Compactifications

In this chapter we show how the compactification of type IIA supergravity on a Calabi-Yau manifold together with an orientifold projection leads to an $\mathcal{N} = 1$ supergravity theory in four dimensions. We write down the explicit superpotential, Kähler potential and the gauge-kinetic couplings as functions of the fluxes and moduli. We also show how generalized NSNS fluxes give a charge to RR axions and lead to D-terms. To find supersymmetric vacua one has to solve the F-term and D-term equations given in this chapter.

We illustrate the details in an example of an orbifold of $T^6/\mathbb{Z}_4$ and solve the F-term and D-term equations for certain subsets of the generalized NSNS fluxes. In particular we numerically find a vacuum in which all the moduli are stabilized at tree level.

2.1 IIA Orientifolds with RR Fluxes and H-Flux

2.1.1 The Moduli and Fluxes

Let us first establish some conventions for the IIA orientifolds that we will be discussing. Let X be a Calabi-Yau three-fold, and let σ be an anti-holomorphic involution of X. The cohomology of X then splits into even and odd parts, depending upon the behavior of each class under σ. We will take the following basis of representative forms:

- The zero-form 1,
- a set of odd two-forms ω_a, $a = 1, \ldots, h^{1,1}_-$,
- a set of even two-forms μ_α, $\alpha = 1, \ldots, h^{1,1}_+$,
- a set of even four-forms $\widetilde{\omega}^a$, $a = 1, \ldots, h^{1,1}_-$,
- a set of odd four-forms $\widetilde{\mu}^\alpha$, $\alpha = 1, \ldots, h^{1,1}_+$,
- a six form φ, odd under σ,
- a set of even three-forms a_K, $K = 1, \ldots, h^{2,1} + 1$,
- and a set of odd three-forms b^K, $K = 1, \ldots, h^{2,1} + 1$.

Additionally, it turns out that we can always choose the a_K and b^K to form a symplectic basis such that the only non-vanishing intersections are

$$\int_X a_K \wedge b^J = \delta^J_K. \tag{2.1}$$

For the even-degree forms we will allow ourselves a bit more freedom of scaling, in order to simplify some explicit computations in the case of toroidal orientifold examples. We will take the intersections to be

$$\int_X \varphi = c, \qquad \int_X \omega_a \wedge \omega_b \wedge \omega_c = \kappa_{abc}, \qquad \int_X \omega_a \wedge \mu_\alpha \wedge \mu_\beta = \widehat{\kappa}_{a\,\alpha\beta},$$

$$\int_X \omega_a \wedge \widetilde{\omega}^b = d_a{}^b, \qquad \int_X \mu_\alpha \wedge \widetilde{\mu}^\beta = \widehat{d}_\alpha{}^\beta. \tag{2.2}$$

If we chose the four-forms to be a basis dual to the two forms, then we would of course set $d_a{}^b = \delta_a^b$, $\widehat{d}_\alpha{}^\beta = \delta_\alpha^\beta$, but we will prefer instead to leave things here more general[1].

Now let us describe the four-dimensional fields of this class of compactifications, restricting ourselves, for simplicity, to the bosonic sector. First we have the Kähler moduli, parametrized by complex scalar fields $t^a = u^a + iv^a$ coming from the expansion

$$B + iJ = J_c = t^a \omega_a, \tag{2.3}$$

where the complexified Kähler form J_c must be odd under σ. Note that the Kähler form $J = v^a \omega_a$ determines the compactification volume (in string frame) via

$$\mathcal{V}_6 = \frac{1}{3!} \int_X J \wedge J \wedge J = \frac{1}{6} \kappa_{abc} v^a v^b v^c. \tag{2.4}$$

To describe the complex moduli, let us write the holomorphic three-form as

$$\Omega = \mathcal{Z}^K a_K - \mathcal{F}_K b^K. \tag{2.5}$$

We will use conventions in which

$$i \int_X \Omega \wedge \bar{\Omega} = 1, \qquad \sigma^* \Omega = \bar{\Omega}, \tag{2.6}$$

[1] Note however that Poincaré duality implies in this case that d and $\widehat{d}$ are both invertible matrices. Indeed we will need to use this fact to write explicit expressions below.

so that the $\mathcal{Z}^K$ are real functions of the complex moduli and $\mathcal{F}_K$ are pure imaginary, and together they satisfy the constraint $\mathcal{Z}^K\mathcal{F}_K = -i/2$. We can now define a complexified version [14]

$$\Omega_c = C_3 + 2ie^{-D}\,\mathrm{Re}\,\Omega = \left(\xi^K + 2ie^{-D}\mathcal{Z}^K\right)a_K, \tag{2.7}$$

where $e^{-D} = \mathcal{V}_6^{1/2}e^{-\phi}$ contains the dilaton and we expand the periods of C_3 (which must be even under σ in order to survive the orientifold projection) as $C_3 = \xi^K a_K$. Note that we abuse notation somewhat here as we ignore other pieces which contribute to the ten-dimensional RR three-form potential C_3, namely pieces that give rise to four-dimensional vectors and (local) pieces that give the four-form RR flux, both of which will be discussed below. The complex moduli $N^K = \frac{1}{2}\xi^K + ie^{-D}\mathcal{Z}^K$ are then simply given by the expansion

$$\Omega_c = 2N^K a_K, \tag{2.8}$$

and include the complex structure moduli of the metric, the dilaton, and the RR three-form periods.

Next we turn to the four-dimensional vectors that come from reducing C_3 against the forms μ_α, so that the total field C_3 (before turning on fluxes) is

$$C_3 = \xi^K a_K + A^\alpha \wedge \mu_\alpha, \tag{2.9}$$

with the A^α being one-form gauge potentials in four dimensions. We will associate these potentials to electric U(1) gauge groups in the four-dimensional effective theory, but we will also later be interested in the dual magnetic U(1)s. These are associated to dual one-forms obtained by reducing C_5 against odd four-forms,

$$C_5 = \widetilde{A}_\alpha \wedge \widetilde{\mu}^\alpha. \tag{2.10}$$

Note that there are no vectors arising from C_1 or C_7, because these are projected out by the orientifold.

These account for our bosonic fields in four dimensions. We would also like to include fluxes from RR field strengths and from the NSNS field strength H (we will include more general NSNS fluxes below). Expanding in our cohomological basis, we have

$$F_0 = m_0, \qquad F_2 = m^a \omega_a, \qquad F_4 = e_a \widetilde{\omega}^a, \qquad F_6 = e_0 \varphi, \tag{2.11}$$

and

$$H = p_K b^K. \tag{2.12}$$

Having listed all the moduli and fluxes we will now write down the ten-dimensional supergravity action for type IIA and then reduce it to four dimensions.

2.1.2 The Ten-Dimensional Action for Type IIA Supergravity and the Tadpole Condition

The action for type IIA supergravity in string frame is

$$\begin{aligned} S^{(10)} \;=\; & -\frac{1}{2}\int_{M^4\times X}\Bigg\{ e^{-2\phi}\left(-R*1 - 4d\phi\wedge *d\phi + \frac{1}{2}\tilde{H}\wedge *\tilde{H}\right) \\ & +\tilde{F}_0 * \tilde{F}_0 + \tilde{F}_2\wedge *\tilde{F}_2 + \tilde{F}_4\wedge *\tilde{F}_4 + \tilde{F}_6\wedge *\tilde{F}_6 \\ & +B_2\wedge dC_3\wedge dC_3 + 2B_2\wedge dC_3\wedge F_4 + C_3\wedge H\wedge dC_3 \\ & -\frac{1}{3}\tilde{F}_0\, B_2\wedge B_2\wedge B_2\wedge dC_3 + \frac{1}{20}\tilde{F}_0^2 B_2\wedge B_2\wedge B_2\wedge B_2\wedge B_2 \\ & +\sum_a^{O6/D6}\left(-T^a\int_{\pi^a} \mathrm{d}^7\xi\sqrt{-g}\, e^{-\phi} + \sqrt{2}\, T^a\int_{\pi^a} C_7\right)\Bigg\}, \end{aligned} \tag{2.13}$$

where we take all the RR fluxes to be supported on the compact manifold, i.e. instead of a spacetime filling $\tilde{F}_4$ flux we use its dual $\tilde{F}_6$ that is proportional to the volume form on the compact space. Explicitly the RR fields are

$$\begin{aligned}
\tilde{F}_0 &= F_0, \\
\tilde{F}_2 &= dC_1 + F_2 + F_0\, B, \\
\tilde{F}_4 &= dC_3 + F_4 + F_2 \wedge B + \frac{F_0}{2} B \wedge B, \\
\tilde{F}_6 &= F_6 + F_4 \wedge B + \frac{1}{2} F_2 \wedge B \wedge B + \frac{F_0}{6} B \wedge B \wedge B,
\end{aligned}$$

with F_p being the background fluxes given in (2.11). Similar we take $\tilde{H} = dB + H$ with H given in (2.12).

We have also included the contributions from localized sources in the action (2.13). Note that we have set $\alpha' = 4\pi^2$ so that in our conventions $T^a = \frac{1}{2}$ for D6-branes and $T^a = -1$ for the O6-planes. The unusual factors of $\sqrt{2}$ are a consequence of our normalizations for the RR fields, which follow [14, 15].

The ten-dimensional action includes the piece

$$\int_{\mathbb{R}^4 \times X} \left\{ -\frac{1}{2} \left(F_2 + m_0 B_2 \right) \wedge * \left(F_2 + m_0 B_2 \right) + C_7 \wedge \left[\frac{1}{\sqrt{2}} \delta_{D6} - \sqrt{2} \delta_{O6} \right] \right\}. \quad (2.14)$$

Since $*(F_2 + m_0 B_2) = dC_7 + \cdots$, the vanishing of the C_7 tadpole then implies the constraint

$$- m_0 p_K b^K + \frac{1}{\sqrt{2}} \left[\delta_{D6} \right] = \sqrt{2} \left[\delta_{O6} \right], \quad (2.15)$$

Plugging in our expansion for the fields from the previous section into the ten-dimensional action (2.13) and performing a Kałuża-Klein reduction to four dimensions leads to a four-dimensional action which we discuss in the next subsection.

2.1.3 The $\mathcal{N} = 1$ Four-Dimensional Supergravity Action

If we are in a regime with small string coupling and a large (in string units) volume of the compact space then a four-dimensional effective description is expected to be valid. In this case it is useful to assemble the data just described into an $\mathcal{N} = 1$ four-dimensional effective supergravity theory. Such a theory consists of one gravity multiplet, some number of chiral multiplets, including complex scalars ϕ^I, and some number of vector multiplets including vectors A^α. The theory is then specified by giving three functions which will depend on the complex scalars, namely a Kähler potential K, a holomorphic superpotential W, and holomorphic gauge-kinetic couplings $f_{\alpha\beta}$. The bosonic part of the effective action is

$$\begin{aligned} S^{(4)} &= -\int_{M_4} \left\{ -\frac{1}{2} R * 1 + K_{I\bar{J}} d\phi^I \wedge * d\bar{\phi}^{\bar{J}} + V * 1 \right. \\ &\quad \left. + \frac{1}{2} \left(\mathrm{Re}\, f_{\alpha\beta}\right) F^\alpha \wedge * F^\beta + \frac{1}{2} \left(\mathrm{Im}\, f_{\alpha\beta}\right) F^\alpha \wedge F^\beta \right\}, \end{aligned} \tag{2.16}$$

where the scalar potential is

$$V = e^K \left(K^{I\bar{J}} D_I W \overline{D_J W} - 3|W|^2 \right) + \frac{1}{2} \left(\mathrm{Re}\, f\right)^{-1\, \alpha\beta} D_\alpha D_\beta. \tag{2.17}$$

Here, $*$ is the four-dimensional Hodge star, $K_{I\bar{J}} = \partial_I \bar{\partial}_{\bar{J}} K$, $K^{I\bar{J}}$ is its (transpose) inverse, $F^\alpha = dA^\alpha$, and $D_I W = \partial_I W + (\partial_I K) W$. D_α is the D-term for the U(1) gauge group corresponding to A^α, which in four-dimensional $\mathcal{N} = 1$ SUGRA is given by [16, 17] (for field configurations with $W \neq 0$)

$$D_\alpha = \frac{i}{W} \delta_\alpha \phi^I D_I W = i \partial_I K \delta_\alpha \phi^I + i \frac{\delta_\alpha W}{W}, \tag{2.18}$$

where $\lambda^\alpha \delta_\alpha \phi^I$ is the variation of the field ϕ^I under an infinitesimal gauge transformation $A^\alpha \to A^\alpha + d\lambda^\alpha$. The second term above, proportional to the gauge variation of the superpotential, is to be interpreted as a Fayet-Iliopoulos (F-I) term. It occurs, for instance, when we have gauged an R-symmetry. It will turn out that in our constructions, the superpotential will always remain gauge neutral, and hence we will not generate any F-I terms, and we will always be able to write (even if $W = 0$)

$$D_\alpha = i\partial_I K \delta_\alpha \phi^I. \tag{2.19}$$

Now we plug in the fields and fluxes above into the ten-dimensional SUGRA action (2.13), perform a Kałuża-Klein reduction to four dimensions, and compare to the action (2.16), following [14]. From the kinetic terms we find

$$f_{\alpha\beta} = i\widehat{\kappa}_{a\,\alpha\beta} t^a, \tag{2.20}$$

and

$$K = 4D - \ln\left(\frac{4}{3}\kappa_{abc} v^a v^b v^c\right). \tag{2.21}$$

From the potential terms we then find that

$$\begin{aligned} W =& \int_X \Omega_c \wedge H + \int_X e^{J_c} \wedge F_{RR}, \\ =& 2N^K p_K + c e_0 + d_a{}^b e_b t^a + \frac{1}{2}\kappa_{abc} m^a t^b t^c + \frac{1}{6} m_0 \kappa_{abc} t^a t^b t^c. \end{aligned} \tag{2.22}$$

Here $F_{RR} = F_0 + F_2 + F_4 + F_6$ is the formal sum of RR fluxes, and

$$e^{J_c} = 1 + J_c + \frac{1}{2} J_c \wedge J_c + \frac{1}{6} J_c \wedge J_c \wedge J_c. \tag{2.23}$$

Also, the D-terms in this setup vanish, $D_\alpha = 0$.

To find supersymmetric vacua, we need to solve the F-term equations $D_a W = 0$ and $D_K W = 0$. From the results above, and using the useful fact that

$$\partial_K D = -e^D \mathcal{F}_K, \tag{2.24}$$

we find the real and imaginary parts of these equations to be

$$0 = \operatorname{Re} D_a W = d_a{}^b e_b + \kappa_{abc} u^b m^c + \frac{m_0}{2}\kappa_{abc}\left(u^b u^c - v^b v^c\right) - \frac{3}{2}\frac{\kappa_{abc} v^b v^c}{\kappa_{def} v^d v^e v^f} \operatorname{Im} W, \tag{2.25a}$$

$$0 = \operatorname{Im} D_a W = \kappa_{abc} v^b m^c + m_0 \kappa_{abc} u^b v^c + \frac{3}{2}\frac{\kappa_{abc} v^b v^c}{\kappa_{def} v^d v^e v^f} \operatorname{Re} W, \tag{2.25b}$$

$$0 = \operatorname{Re} D_K W = 2p_K - 4ie^D \mathcal{F}_K \operatorname{Im} W, \tag{2.25c}$$

$$0 = \operatorname{Im} D_K W = 4ie^D \mathcal{F}_K \operatorname{Re} W. \tag{2.25d}$$

Since not all of the $\mathcal{F}_K$ can vanish (recall the condition $i\int_X \Omega \wedge \bar{\Omega} = 1$), (2.25d) requires $\operatorname{Re} W = 0$. One can quickly check that a Minkowski solution (one in which $\operatorname{Im} W$ also vanishes) will force all of the fluxes (RR and NSNS) to vanish; in this case the superpotential vanishes, no moduli are stabilized, and the tadpole must be saturated by adding D6-branes.

Suppose now that we are not in a Minkowski, but rather an AdS solution, in which $\operatorname{Im} W \neq 0$. In order for the metric to be positive definite, the matrix $(\kappa v)_{ab} = \kappa_{abc} v^c$ should be invertible, and so equation (2.25b) tells us that either $m_0 = 0$ and $m^a = 0$, or

$$u^a = -\frac{m^a}{m_0}. \tag{2.26}$$

The former case reduces to the unstabilized Minkowski vacuum mentioned above.

Also, if the F-term equations hold, then one can subtract $e^{-D}\mathcal{Z}^K \operatorname{Re} D_K W + v^a \operatorname{Re} D_a W$ from the imaginary part of the right hand side of (2.22) to show that

$$\operatorname{Im} W = -\frac{2m_0}{15}\kappa_{abc} v^a v^b v^c. \tag{2.27}$$

One can proceed somewhat further in the general case, but since we would like to add more ingredients to our construction, we will refer the reader to [15], and restrict ourselves instead to a specific example.

2.1.4 The Example of $T^6/\mathbb{Z}_4$

Our primary example in this dissertation will be the toroidal orientifold $T^6/\mathbb{Z}_4$ described below. Before we dive into detailing this example and our conventions, the reader may be interested to know why we focus on this compactification, rather than one of the other orientifolds in the literature, such as $T^6/\mathbb{Z}_2^2$ or $T^6/\mathbb{Z}_3^2$ which have been more extensively studied and which are in some sense simpler. We certainly believe that our approach here can be applied to these models. One reason to look at the $T^6/\mathbb{Z}_4$ example is that it, unlike the two other examples mentioned above, admits untwisted two-forms which are *even* under the orientifold involution for the standard orientifold involution in type IIA. By reducing the RR potential C_3 along these forms we find four-dimensional vectors with associated U(1) gauge groups. Later, in subsection 2.2.2, we will see that with certain metric fluxes turned on, some moduli become charged under the U(1)s, and this gives rise to D-terms in the four-dimensional effective potential.

One other interesting property of this particular example that we do not make use of in the present work, is the existence of twisted sector three-forms, which again do not occur in the more well-studied models. In principle these could lead to interesting possibilities for metric and non-geometric twisted-sector fluxes, in the spirit of [9].

Setup

We take the model from [3, 18], namely a certain orientifold of $(T^2)^3$, but several of our conventions will differ, so we review everything here. Let $z^1 = x^1 + ix^2$, $z^2 = x^3 + ix^4$, and $z^3 = x^5 + (\frac{1}{2} + iU)x^6$ be complex coordinates on the tori (we will see below that U is a real modulus parametrizing the complex structure of the third torus), and the torus identifications are given by integer shifts in each x^i. The orientifold group is generated by a $\mathbb{Z}_4$ rotation

$$\Theta : \left(z^1, z^2, z^3\right) \longrightarrow \left(iz^1, iz^2, -z^3\right), \tag{2.28}$$

and the orientifold action is $\Omega_p(-1)^{F_L}\sigma$, where the antiholomorphic involution σ acts as[2]

$$\sigma : \left(z^1, z^2, z^3\right) \longrightarrow \left(\bar{z}^1, i\bar{z}^2, \bar{z}^3\right). \tag{2.29}$$

Note that $\Theta\sigma = \sigma\Theta^3$, so the full orientifold group is in fact isomorphic to the dihedral group D_4. This model is frequently referred to as an orientifold of the orbifold $T^6/\mathbb{Z}_4$ even though it is not a $\mathbb{Z}_2$ quotient of the orbifold. Rather, the precise statement is that the full orientifold group is a $\mathbb{Z}_2$ extension of the $\mathbb{Z}_4$ orbifold group. We emphasize this point now partly as a warning to the reader, since at times we will likely be guilty of sloppy language.

We will begin by describing the untwisted cohomology of $T^6/\mathbb{Z}_4$, dividing further into subspaces which are even or odd under the involution σ. The bases we will present will consist of elements of $H^*(T^6;\mathbb{Z})$ with the correct symmetry properties. In this way we get bases for the untwisted cohomology of the orbifold *over the rationals.* The correct quantization conditions for fluxes in the orientifold are subtle, and should in principle require an understanding of the correct K-theory analog for our model which would go beyond the scope of this dissertation [20]. Instead we will point out where such information would be relevant, and explain why we do not believe that it will affect our results significantly.

We start with the even cohomology, implicitly equating classes with their harmonic form representatives. There is one zero form, namely the unit function 1. For two-forms, there are five independent $(1,1)$-forms invariant under the rotations:

[2]This orientifold is the **ABB** model in the classification of [19].

four odd forms,

$$\begin{aligned}
\omega_1 &= \frac{i}{2} dz^1 \wedge d\bar{z}^1 = dx^1 \wedge dx^2, \\
\omega_2 &= \frac{i}{2} dz^2 \wedge d\bar{z}^2 = dx^3 \wedge dx^4, \\
\omega_3 &= \frac{i}{2U} dz^3 \wedge d\bar{z}^3 = dx^5 \wedge dx^6, \\
\omega_4 &= \frac{1-i}{2} \left(dz^1 \wedge d\bar{z}^2 - i dz^2 \wedge d\bar{z}^1 \right) \\
&= dx^1 \wedge dx^3 - dx^1 \wedge dx^4 + dx^2 \wedge dx^3 + dx^2 \wedge dx^4,
\end{aligned} \tag{2.30}$$

and one even form

$$\begin{aligned}
\mu &= \frac{1+i}{2} \left(dz^1 \wedge d\bar{z}^2 + i dz^2 \wedge d\bar{z}^1 \right) \\
&= dx^1 \wedge dx^3 + dx^1 \wedge dx^4 - dx^2 \wedge dx^3 + dx^2 \wedge dx^4.
\end{aligned} \tag{2.31}$$

Similarly, for four-forms we have four even $(2,2)$-forms

$$\widetilde{\omega}^1 = \omega_2 \wedge \omega_3, \qquad \widetilde{\omega}^2 = \omega_1 \wedge \omega_3, \qquad \widetilde{\omega}^3 = \omega_1 \wedge \omega_2, \qquad \widetilde{\omega}^4 = \omega_3 \wedge \omega_4, \tag{2.32}$$

and one odd $(2,2)$-form,

$$\widetilde{\mu} = \omega_3 \wedge \mu. \tag{2.33}$$

Finally there is one six-form, which is odd under the involution,

$$\varphi = \omega_1 \wedge \omega_2 \wedge \omega_3 = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6. \tag{2.34}$$

The nonzero integrals involving these forms over $X = T^6/\mathbb{Z}_4$ are (wedge products are implicit)

$$\int_X \varphi = \int_X \omega_1\, \omega_2\, \omega_3 = \frac{1}{4}, \qquad \int_X \omega_3\, \omega_4^2 = \int_X \omega_3\, \mu^2 = -1,$$

$$\int_X \omega_1\,\widetilde{\omega}_1 = \int_X \omega_2\,\widetilde{\omega}_2 = \int_X \omega_3\,\widetilde{\omega}_3 = \frac{1}{4}, \qquad \int_X \omega_4\,\widetilde{\omega}_4 = \int_X \mu\,\widetilde{\mu} = -1. \tag{2.35}$$

So explicitly we have $c = \frac{1}{4}$, ${d_a}^b = diag(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -1)$ and $\widehat{d} = -1$. The only non-vanishing components of the totally symmetric triple intersections are $\kappa_{123} = \frac{1}{4}$, $\kappa_{344} = -1$ and $\widehat{\kappa}_{311} = -1$.

The Kähler form will be given by $J = v^1\,\omega_1 + v^2\,\omega_2 + v^3\,\omega_3 + v^4\,\omega_4$, and the corresponding line element is

$$\begin{aligned} ds^2 &= v^1\left((dx^1)^2 + (dx^2)^2\right) + v^2\left((dx^3)^2 + (dx^4)^2\right) \\ &\quad + \frac{v^3}{U}\left((dx^5)^2 + dx^5 dx^6 + \left(\frac{1}{4} + U^2\right)(dx^6)^2\right) \\ &\quad - 2v^4\left(dx^1 dx^3 + dx^1 dx^4 - dx^2 dx^3 + dx^2 dx^4\right). \end{aligned} \tag{2.36}$$

In order for the metric to have the correct (Euclidean) signature, we must have

$$v^1 > 0, \qquad v^2 > 0, \qquad v^3 > 0, \qquad \text{and} \qquad v^1 v^2 - 2(v^4)^2 > 0. \tag{2.37}$$

The volume is given by

$$\mathcal{V}_6 = \frac{1}{3!}\int_X J^3 = \frac{1}{4} v^3\left(v^1 v^2 - 2(v^4)^2\right). \tag{2.38}$$

Note also that having J odd under the anti-holomorphic involution σ implies that the metric is invariant under σ, as required for the orientifold projection. This is why there is no allowed metric deformation corresponding to the even two-form μ.

Next we have the odd cohomology. It turns out that $H^1(X)$ and $H^5(X)$ are empty, so we only need to describe the three-forms. The basis we shall use is

$$\begin{aligned}
a_1 &= \chi^{135} + \chi^{136} + \chi^{145} + \chi^{235} - \chi^{245} - \chi^{246}, \\
a_2 &= \chi^{135} + \chi^{145} + \chi^{146} + \chi^{235} + \chi^{236} - \chi^{245}, \\
b^1 &= -\chi^{135} + \chi^{145} + \chi^{146} + \chi^{235} + \chi^{236} + \chi^{245}, \\
b^2 &= \chi^{135} + \chi^{136} - \chi^{145} - \chi^{235} - \chi^{245} - \chi^{246}.
\end{aligned} \tag{2.39}$$

Here we use notation where $\chi^{145} = dx^1 \wedge dx^4 \wedge dx^5$, etc. The forms a_K are even under the involution σ, while b^K are odd. The nonzero integrals are simply $\int_X a_K \wedge b^L = \delta_K^L$.

The holomorphic three form Ω is

$$\begin{aligned}
\Omega &= \frac{1-i}{2\sqrt{U}} dz^1 \wedge dz^2 \wedge dz^3 \\
&= \frac{1}{2\sqrt{U}} \left[\left(\frac{1}{2} + U\right) a_1 + \left(\frac{1}{2} - U\right) a_2 + i \left(\frac{1}{2} + U\right) b^1 - i \left(\frac{1}{2} - U\right) b^2 \right].
\end{aligned} \tag{2.40}$$

Explicitly we have

$$\mathcal{Z}^1 = \frac{1}{2\sqrt{U}} \left(\frac{1}{2} + U\right), \qquad \mathcal{Z}^2 = \frac{1}{2\sqrt{U}} \left(\frac{1}{2} - U\right),$$

$$\mathcal{F}_1 = \frac{-i}{2\sqrt{U}} \left(\frac{1}{2} + U\right), \qquad \mathcal{F}_2 = \frac{i}{2\sqrt{U}} \left(\frac{1}{2} - U\right). \tag{2.41}$$

In this expression, U is the unique untwisted complex structure modulus, and is a real variable in the range $0 < U < \infty$.

Let us now quickly summarize the twisted sector moduli. The fixed locus of Θ or Θ^3 consists of sixteen points, eight of which are fixed by σ, plus four pairs of points that get swapped by σ. The four pairs will give rise to four even and four odd $(1,1)$-forms and equal numbers of even and odd $(2,2)$-forms. The remaining eight points will each contribute an odd $(1,1)$-form and an even $(2,2)$-form. All together, then, these twisted sectors contribute twelve new complex Kähler moduli, with four moduli of the orbifold being projected out by the orientifold.

The fixed locus of Θ^2 consists of sixteen two-tori. Θ invariance gives four copies of $T^2/\mathbb{Z}_2$, and six pairs of T^2 that get interchanged. Each of these six pairs automatically contributes one even and one odd form. Of the six pairs, two pairs are at σ fixed points and each contributes an odd $(1,1)$-form and an even $(2,2)$-form[3], two more pairs have σ act the same as Θ and hence also act as if they were σ fixed points, and the final two pairs are interchanged by σ, leading to one odd and one even of each $(1,1)$- and $(2,2)$-forms. Similarly the remaining four $T^2/\mathbb{Z}_2$ are each at fixed points of σ, and each contribute an odd $(1,1)$-form and an even $(2,2)$-form. In total then, this sector contains ten two-forms, nine of which are odd, twelve three-forms, which split into six odd and six even, and ten four-forms, nine of which are even.

Fluxes

Finally, we turn to the allowed fluxes which we can turn on in our model. As mentioned above, the correct classification of RR fluxes in this model would involve a careful discussion of K-theory in this setting, and would go beyond the scope of this dissertation. We will instead stick to cohomology. Moreover, we will be interested in so-called "bulk fluxes" - fluxes whose image in rational cohomology has a nonzero projection onto the untwisted sector. For this reason, we will write our fluxes as (recalling that F_0 and F_4 need to be even under σ, while F_2 and F_6 need to be odd)

$$F_0 = m_0, \quad F_2 = m^a\,\omega_a, \quad F_4 = e_a\,\widetilde{\omega}^a, \quad F_6 = e_0\varphi,$$

[3]To see that a σ fixed point gives rise to an odd $(1,1)$-form, note that locally it looks like $(\mathbb{C}^2/\mathbb{Z}_2)\times T^2$ which resolves to $(\mathcal{O}_{\mathbb{P}^1}(-2))\times T^2$. An explicit Kähler metric can be written down for the latter geometry and one can check that the unique normalizable $(1,1)$-form is odd under an antiholomorphic involution.

where quantization conditions say that (remember that we have set $\alpha' = 1/4\pi^2$)

$$\sqrt{2} \int F_p \in \mathbb{Z}, \tag{2.42}$$

with the integral taken over any p-cycle in X. This is of course not completely correct; proper quantization requires combining untwisted and twisted sector fluxes, but it is not quite as bad as one might fear. Indeed, one can argue (see e.g. [21, 15]) that any bulk flux can be written as one of the above, plus twisted sector contributions which correspond to fractional fluxes at fixed points (modulo again certain K-theoretic subtleties).

The NSNS three-form flux is in some sense simpler. It must simply lie in $H^3(X;\mathbb{Z}) \cap H^*_{\mathrm{odd}}(X)$. In principle this can be completely worked out - the difficult step of working out the integral cohomology has already been done in [18] - but again we will not need the full detail and we shall again project onto the untwisted sector cohomology. This allows us to write

$$H_3 = p_K \, b^K, \tag{2.43}$$

and impose the simple quantization $p_K \in \mathbb{Z}$.

At any rate, we will not make too much use of the underlying quantizations of the RR fluxes (though we will encounter a puzzle related to this later on), and the quantization of NSNS fluxes will be treated much more carefully in section 4.

Orientifold Planes

Orientifold planes will lie at the fixed locus of each orientation-reversing element of the orientifold group. For instance, the fixed locus of the involution σ is described by the set

$$\left(T^6\right)^\sigma = \left\{0 \le x^1 \le 1, x^2 \in \{0, \frac{1}{2}\}; 0 \le x^3 = x^4 \le 1; 0 \le x^5 \le 1, x^6 = 0\right\} \subset T^6. \tag{2.44}$$

In homology, this cycle can be written as

$$\left[\left(T^6\right)^\sigma\right] = 2\pi_{135} + 2\pi_{145}, \tag{2.45}$$

where π_{135} is the cycle represented by x^{2i} fixed, x^{2i-1} variable and winding once. We also have, e.g.,

$$\int_{\pi_{135} \subset T^6} \chi^{135} = 1, \tag{2.46}$$

with other choices of integrand giving vanishing results. Using this, we have also picked the orientation of this cycle to be such that it is positively calibrated by $\mathrm{Re}\,\Omega$.

Similar consideration for the other anti-holomorphic involutions of the orientifold group give

$$\begin{aligned} \left[\left(T^6\right)^{\Theta\sigma}\right] &= 2\pi_{145} - 4\pi_{146} + 2\pi_{245} - 4\pi_{246}, \\ \left[\left(T^6\right)^{\Theta^2\sigma}\right] &= 2\pi_{235} - 2\pi_{245}, \\ \left[\left(T^6\right)^{\Theta^3\sigma}\right] &= -2\pi_{135} + 4\pi_{136} + 2\pi_{235} - 4\pi_{236}. \end{aligned} \tag{2.47}$$

The total class of the O6-plane is thus

$$[O6] = 4\left(\pi_{136} + \pi_{145} - \pi_{146} + \pi_{235} - \pi_{236} - \pi_{246}\right). \tag{2.48}$$

It is then easy to verify that

$$\int_{[O6] \subset X} a_1 = 4, \qquad \int_{[O6] \subset X} a_2 = \int_{[O6] \subset X} b^1 = \int_{[O6] \subset X} b^2 = 0, \tag{2.49}$$

where we must be careful to divide by four relative to the result on T_6 (this does not make sense for individual cycles like π_{135} which are not invariant under the orbifold action, but does make sense for the total class $[O6]$).

The reason that it is important to know where the orientifold plane lies is that it contributes to the tadpole for the seven-form RR potential, C_7 as we saw in (2.15). Explicitly, the equation of motion for C_7 includes a contribution proportional to $\delta_{\rm O6}$, the delta-three form supported on the orientifold plane. From the computations above, we see that in cohomology,

$$[\delta_{\rm O6}] = 4b^1. \tag{2.50}$$

Because of the freedom to use D-branes (or anti-D-branes if necessary) to satisfy the tadpole condition, we will attempt first to find vacua without worrying about the tadpole condition and then see what, if anything, we need to add.

AdS Solutions to the F-Term Equations

Now we look for AdS solutions to the F-term equations (2.25) in our $T^6/\mathbb{Z}_4$ example. Since we must have $\mathcal{F}_1 \neq 0$ for a non-degenerate solution, equation (2.25c) tell us that $p_1 \neq 0$ and that

$$\frac{\mathcal{F}_2}{\mathcal{F}_1} = \frac{p_2}{p_1} \qquad \Longrightarrow \qquad U = \frac{1}{2}\frac{p_1+p_2}{p_1-p_2}, \tag{2.51}$$

From this we see that a physical solution requires $|p_1| > |p_2|$.

Next, we use (2.25a) and (2.27) to obtain a set of four quadratic equations for the v^a. The equations are simplest if we write them in terms of quantities

$$\widehat{e}_1 = e_1 - \frac{m^2m^3}{m_0}, \quad \widehat{e}_2 = e_2 - \frac{m^1m^3}{m_0}, \quad \widehat{e}_3 = e_3 - \frac{m^1m^2-2(m^4)^2}{m_0}, \quad \widehat{e}_4 = e_4 - \frac{m^3m^4}{m_0}. \tag{2.52}$$

It turns out that a sensible solution (i.e. one in which v^1, v^2, v^3 are all positive and $v^1 v^2 > 2(v^4)^2$) exists if and only if m_0, $\widehat{e}_1$, $\widehat{e}_2$, $\widehat{e}_3$ are all the same sign and if $\widehat{e}_1\widehat{e}_2 > 2\widehat{e}_4^2$. If these conditions are met, then we have a sensible, physical solution given by

$$v^1 = |\widehat{e}_2| \sqrt{\frac{5}{3m_0}\frac{\widehat{e}_3}{\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2}}, \qquad v^2 = |\widehat{e}_1| \sqrt{\frac{5}{3m_0}\frac{\widehat{e}_3}{\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2}}, \tag{2.53}$$

$$v^3 = \sqrt{\frac{5}{3m_0}\frac{\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2}{\widehat{e}_3}}, \qquad v^4 = \widehat{e}_4 \sqrt{\frac{5}{3m_0}\frac{\widehat{e}_3}{\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2}}\,(\mathrm{sign}\, m_0)\,.$$

Next we can solve for the dilaton. It turns out that $e^D > 0$ implies that p_1 must have the opposite sign of m_0, and then

$$e^D = \left[\frac{27m_0}{10}\frac{p_1^2 - p_2^2}{\widehat{e}_3\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}\right]^{1/2}, \text{ or } e^{\phi} = \frac{3}{2}\sqrt{p_1^2 - p_2^2}\left[\frac{12}{5}m_0\widehat{e}_3\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)\right]^{-1/4}. \tag{2.54}$$

Finally, we can use $\mathrm{Re}\, W = 0$ to solve for one linear combination of the axions, $p_K \xi^K$; the scalar potential is independent of the other linear combination and so this other combination remains a flat direction perturbatively.

$$p_1\xi^1 + p_2\xi^2 = -\frac{1}{4}e_0 + \frac{1}{4m_0}\left(m^1 e_1 + m^2 e_2 + m^3 e_3 - 4m^4 e_4\right) - \frac{m^3\left(m^1 m^2 - 2(m^4)^2\right)}{2m_0^2}. \tag{2.55}$$

Thus, for a given general set of fluxes (satisfying certain inequalities) we have found the unique solution to the F-term equations and have found that all but one of the moduli are fixed. We still, however, need to satisfy the tadpole constraint. Indeed, since $[\delta_{O6}] = 4b^1$, we find

$$-\sqrt{2}m_0 p_1 + N_1 = 8, \qquad -\sqrt{2}m_0 p_2 + N_2 = 0, \tag{2.56}$$

where N_1 and N_2 are the number of D-branes wrapping the cycle dual to b^1 or b^2

respectively. Actually one needs to be a bit careful here; a supersymmetric D-brane should have a positive volume as calibrated by $\mathrm{Re}\,\Omega$, but for the cycle dual to b^2 the orientation picked out by this condition depends on whether U is less than or greater than one half. If $U < \frac{1}{2}$, as is the case when $m_0 p_2 > 0$, then we should have $N_2 > 0$ D6 branes, in agreement with the above. On the other hand, if $U > \frac{1}{2}$, then the cycle dual to b^2 is negatively calibrated and N_2 counts the number of anti-D6 branes. In this case $N_2 < 0$ for a SUSY solution, but we also have $m_0 p_2 < 0$, so the tadpole condition can still be satisfied.

Note that to find a physical solution above, we required that $m_0 p_1 < 0$, and hence immediately $N_1 < 8$. In fact, since we also needed $|p_1| \geq |p_2|$, we have that $N_1 + |N_2| < 8$; the total number of D6-branes is bounded. Hence, we see that in some sense the fluxes here contribute to the tadpole with the same sign as the D-branes. We are not allowed to add as many D-branes as we like to saturate the tadpole, but rather (within SUSY) our gauge groups have bounded rank.

Before moving on, let us note that we could have worked directly with the scalar potential of (2.17) and looked for extrema of the potential [3]. Recall that if we have an extremum at which the value of the potential is negative, so that we have AdS_4, then stability does not require that the extremum be an actual minimum. It is enough that each field Φ^I has a mass squared that is greater than the Breitenlohner-Freedman (BF) bound,

$$m_I^2 > -\frac{3}{4}\left|V_{\mathrm{extremum}}\right|, \tag{2.57}$$

where we assume that Φ^I has a canonically normalized kinetic term. Indeed, for the supersymmetric solution above, it turns out that there is one mode with a negative mass squared, but it is above the BF bound.

2.2 Scherk-Schwarz Reduction of Type IIA

2.2.1 Metric Fluxes

The next ingredient we will be adding is known as *metric flux*. Let us restrict for the moment to the case of toroidal orientifolds. It is well known that by T-dualizing one circle of a torus with H-flux, one can swap some components of the H-flux for some non-constant metric components. The new geometry that results is called a twisted torus, and the one-forms dx^i are no longer globally defined. Instead, they should be replaced by one-forms η^i which are globally defined[4], but which are no longer necessarily closed, satisfying instead

$$d\eta^i = -\frac{1}{2} f^i_{jk} \eta^j \wedge \eta^k, \tag{2.58}$$

where f^i_{jk} are constant coefficients, antisymmetric in the lower two indices. These coefficients are known as metric (or sometimes geometric) fluxes, and arise, like H-flux, from the NSNS sector of the theory.

In fact, one needs not necessarily obtain such solutions by T-duality, but rather one can start from (2.58) directly. An effective four-dimensional theory can still be obtained by performing a generalized Scherk-Schwarz reduction. This has been done in some other models in [22, 23, 12], and some general work has also been done [24, 25, 26], but we will point out a couple of novel features, such as the appearance of nonvanishing D-terms, which have not been explored in these models before.

By taking the exterior derivative of (2.58), we find a consistency condition

$$f^m_{[ij} f^n_{k]m} = 0, \qquad \forall n, i, j, k. \tag{2.59}$$

[4]In fact, all of $\Omega^*(X)$, where X is the twisted torus, is generated by wedge products of the η^i with coefficients being globally defined functions.

In fact, rather than proceeding by T-duality, we can take (2.58) and (2.59) as a starting point for defining a twisted torus X [24, 26]. We will also impose the additional constraint of tracelessness,

$$f^i_{ij} = 0, \qquad \forall j, \tag{2.60}$$

but we will occasionally point out how relaxing this condition would modify our results (relaxing this condition would for example have the effect that the naive volume form of the twisted torus would be exact [27], but it is not immediately obvious that this is contradictory).

Another perspective on these fluxes which is sometimes useful is that if we write

$$\eta^i = N^i_j(x) dx^j, \tag{2.61}$$

then we can construct vector fields

$$Z_i = \left(N^{-1}\right)^j_i \frac{\partial}{\partial x^j}. \tag{2.62}$$

These turn out to be Killing vectors of the twisted torus, and they form a Lie algebra,

$$[Z_i, Z_j] = f^k_{ij} Z_k. \tag{2.63}$$

The Jacobi identity for the Lie algebra simply reproduces (2.59). This algebra is somewhat useful to keep in mind and we can sometimes relate properties of the system with metric fluxes to properties of the algebra. We will discuss these matters in sections 2.5 and 4.

It is natural to consider also H-flux on X, which should be a globally defined three-form

$$H = \frac{1}{6} H_{ijk} \eta^i \wedge \eta^j \wedge \eta^k, \tag{2.64}$$

and must still be closed, leading to the identity

$$f^i_{[jk}H_{\ell m]i} = 0. \tag{2.65}$$

We are assuming here that the coefficients H_{ijk} are constant. In some specific twisted torus cases it can be checked explicitly that each cohomology class has a representative with this property (i.e. constant coefficients in an η^i expansion), and we believe that this will hold in general. Together, (2.59) and (2.65) are known as Bianchi identities.

On a toroidal orientifold we should have both H_{ijk} and f^i_{jk} invariant under the orbifold group. Under the involution σ we should have $\sigma^* H = -H$, since B_2 is odd under world-sheet parity Ω_p. The metric is even under Ω_p and hence should be invariant under σ, and since the f^i_{jk} essentially appear as coefficients in the metric, they too should be even under σ (for a more convincing explanation see [4], or references therein).

Recall that if we applied our discussion of the four-dimensional effective theory above to the toroidal case, then it makes much more sense to describe H-flux not in terms of components H_{ijk}, but rather by coefficients p_K, i.e. $H = p_K b^K$, where b^K are a basis for the odd untwisted three-forms of the toroidal orientifold. A similar choice is convenient for the metric fluxes. Consider a general p-form

$$A = \frac{1}{p!} A_{i_1 \cdots i_p} \eta^{i_1} \wedge \ldots \wedge \eta^{i_p}. \tag{2.66}$$

Let us assume for now that the $A_{i_1 \cdots i_p}$ are constants. In that case, we can define a $(p+1)$-form $f \cdot A = -dA$, which in components reads

$$(f \cdot A)_{i_1 \cdots i_{p+1}} = \binom{p+1}{2} f^j_{[i_1 i_2} A_{|j| i_3 \cdots i_{p+1}]}, \tag{2.67}$$

and where we use the convention that $\binom{n}{m} = 0$ unless $0 \leq m \leq n$.

As a brief aside, note that we can take (2.67) as a definition of the $(p+1)$-form $f \cdot A$ even when the original components $A_{i_1 \cdots i_p}$ are not constant[5]. In this case we can write $dA = d'A - f \cdot A$, where d' is understood to act only on the coefficients $A_{i_1 \cdots i_p}$. This then inspires an approach that will be useful later when we will add non-geometric fluxes as well. Rather than work on the twisted torus with forms expanded in η^i and exterior derivative d, we can work on the flat torus with forms dx^i and replace the exterior derivative by

$$d_f = d - f \cdot . \tag{2.68}$$

In fact, in the presence of H-flux, the natural derivative acting on RR forms is $d_H = d + H\wedge$. In the language above we can either work with the twisted torus and forms η^i, with derivative d_H, where H is also expanded in the η^i, or we can work on the flat torus with forms dx^i and exterior derivative

$$d_{H,f} = d + H \wedge - f \cdot . \tag{2.69}$$

This latter approach will be the one which naturally generalizes to the "non-geometric" case. Note that the requirement $d_{H,f}^2 = 0$ reproduces both of our Bianchi identities above.

Taking either of these two perspectives, we are now ready to define a cohomological parametrization for the metric fluxes, in analogy with the p_K. We simply take a basis for the untwisted two-forms of the toroidal orientifold, with ω_a being

[5]The appropriate generalization when f^i_{jk} are not traceless is

$$(f \cdot A)_{i_1 \cdots i_{p+1}} = \binom{p+1}{2} f^j_{[i_1 i_2} A_{|j| i_3 \cdots i_{p+1}]} + \frac{1}{2} \binom{p+1}{1} f^j_{j[i_1} A_{i_2 \cdots i_{p+1}]}.$$

odd and μ_α being even. Then we expand

$$f \cdot \omega_a = r_{aK} b^K, \qquad f \cdot \mu_\alpha = \widehat{r}_\alpha^K a_K. \tag{2.70}$$

Integration by parts then also furnishes the expansions

$$f \cdot a_K = \left(d^{-1}\right)_a{}^b r_{bK} \widetilde{\omega}^a, \qquad f \cdot b^K = -\left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{r}_\beta^K \widetilde{\mu}^\alpha. \tag{2.71}$$

These coefficients r_{aK} and $\widehat{r}_\alpha^K$ are the analogues of the p_K. Indeed, in the case of H-flux the corresponding expressions would be

$$H \wedge 1 = p_K b^K, \qquad H \wedge a_K = -c^{-1} p_K \varphi. \tag{2.72}$$

The great promise of these cohomological parametrizations of the NSNS fluxes is that they can be generalized beyond the toroidal case; since the p_K, r_{aK} and $\widehat{r}_\alpha^K$ are defined only in terms of maps between representatives of the untwisted cohomology of the toroidal orbifold, we can try to define similar maps between cohomological representatives on any Calabi-Yau space which admits an orientifold involution. In the case of p_K, this is of course completely standard for parametrizing possible H-flux. In general the matrices r_{aK} and $\widehat{r}_\alpha^K$ will be $h_-^{1,1} \times (h^{2,1}+1)$ and $h_+^{1,1} \times (h^{2,1}+1)$ matrices, respectively.

By requiring $d_{H,f}^2$ to vanish on the invariant forms, we learn that the Bianchi identities imply some relations among these coefficients. In particular,

$$p_K \widehat{r}_\alpha^K = 0, \quad \forall \alpha, \qquad r_{aK} \widehat{r}_\alpha^K = 0, \quad \forall a, \alpha. \tag{2.73}$$

Unfortunately, it turns out that these are not the complete set of Bianchi identities; the requirement that $d_{H,f}^2 = 0$ also on non-invariant forms is stronger. This is especially vexing in that it is not clear what these extra Bianchi constraints should be once one moves beyond toroidal examples.

There is one more caveat worth noting in this approach. For H-flux it is automatically true that the odd invariant combinations of flux components H_{ijk} are in a bijective correspondence with the odd invariant untwisted three-forms, so the p_K really do describe all the possible H-fluxes we would like to turn on. This is no longer the case with the metric fluxes; it is not necessarily true that the number of invariant combinations of f^i_{jk} is equal to the number of r_{aK} and $\hat{r}^K_\alpha$. The count of the latter coefficients is given by $\frac{1}{2}b^2 b^3 = h^{1,1}(h^{2,1}+1)$, with the Betti and Hodge numbers here referring to the untwisted sector of the orbifold. In many examples the bijective correspondence does hold. For instance in orientifolds built from orbifolds T^6/Γ, where Γ is any of the crystallographic actions $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ (as in, e.g. [15]), $\mathbb{Z}_4$ (as in [4]), or $\mathbb{Z}_{6-I}$, the cohomological parameters capture all of the possible metric fluxes. Note that the nature of the involution here is irrelevant for the counting. However, in some other examples, like $\mathbb{Z}_{6-II}$, there are more possible combinations of f^i_{jk} than there are components of r_{aK} and $\hat{r}^K_\alpha$ (in th $\mathbb{Z}_{6-II}$ case there are seven invariant combinations of metric flux, but only $\frac{1}{2}3 \cdot 4 = 6$ cohomological parameters).

These "extra" fluxes do not, however, seem to appear in the four-dimensional effective action. The metric fluxes will contribute to the superpotential only through r_{aK}, and contribute to the D-terms only through $\hat{r}^K_\alpha$.

The lack of invariant one- and five-forms ensures that the fluxes associated to F_0, F_4, F_6 remain closed, and the Bianchi identity ensures that H is closed but we now see that the flux $m^a\omega_a$ corresponding to F_2 is not. Looking at (2.14) we see that this results in a new contribution to the C_7 tadpole,

$$-\sqrt{2}\,(m_0 p_K - m^a r_{aK})\,b^K + [\delta_{D6}] = 2\,[\delta_{O6}]\,. \tag{2.74}$$

Actually, this is most naturally expressed by noting that the flux contributions to the tadpole are naturally proportional to

$$d_{H,f}F_{RR}|_{3-\text{form}} = HF_0 - f \cdot F_2. \tag{2.75}$$

2.2.2 Superpotential with Metric Fluxes and D-Terms

There are two avenues towards understanding the effect of these metric fluxes on the four-dimensional effective theory. One can either use T-duality to deduce the way in which the metric fluxes appear in quantities like the superpotential [28], or one can explicitly perform a Kałuża-Klein reduction on a twisted torus [22]. Either method will reveal that the addition of metric fluxes has two effects on the four-dimensional effective theory. First of all, the superpotential (2.22) gets modified by the addition of a term $2N^K r_{aK} t^a$, so that it can be written

$$\begin{aligned} W &= \int_X \Omega_c \wedge d_{H,f}\left(e^{-J_c}\right) + \int_X e^{J_c} \wedge F_{RR} \\ &= 2N^K\left(p_K + r_{aK}t^a\right) + ce_0 + d_a{}^b e_b t^a + \frac{1}{2}\kappa_{abc}m^a t^b t^c + \frac{1}{6}m_0\kappa_{abc}t^a t^b t^c. \end{aligned} \tag{2.76}$$

The second effect is to charge some of the moduli under the electric gauge groups $U(1)^\alpha$. Indeed, recall that the gauge vectors descended from the three-form potential which had the expansion

$$C_3 = A^\alpha \wedge \mu_\alpha + \xi^K a_K, \tag{2.77}$$

where we ignore the (local) parts of C_3 which contribute to the four-form flux $e_a\widetilde{\omega}^a$. In the case without metric fluxes, the four-dimensional gauge transformations $A^\alpha \to A^\alpha + d\lambda^\alpha$ are the descendants of the ten-dimensional three-form gauge transformations

$$C_3 \longrightarrow C_3 + d\left(\lambda^\alpha \mu_\alpha\right) = \left(A^\alpha + d\lambda^\alpha\right) \wedge \mu_\alpha + \xi^K a_K. \tag{2.78}$$

In particular, this ten-dimensional transformation can be done without modifying any of the four-dimensional fields; all the scalars are neutral under these gauge groups.

However, in the presence of metric fluxes $\widehat{r}_\alpha^K$, μ_α is no longer closed, and the transformation above gets modified to

$$C_3 \longrightarrow C_3 + d\left(\lambda^\alpha \mu_\alpha\right) = \left(A^\alpha + d\lambda^\alpha\right) \wedge \mu_\alpha + \left(\xi^K - \lambda^\alpha \widehat{r}_\alpha^K\right) a_K, \tag{2.79}$$

and we see that the four-dimensional field ξ^K is no longer left invariant under this electric gauge transformation.

To calculate the resulting D-terms from (2.19) we need one relation, namely that

$$\frac{\partial}{\partial N^K} D = -e^D \mathcal{F}_K. \tag{2.80}$$

We can then compute

$$D_\alpha = 2ie^D \widehat{r}_\alpha^K \mathcal{F}_K. \tag{2.81}$$

Recall that $\mathcal{F}_K$ in our conventions is pure imaginary, so that D_α is real.

Thus, in a supersymmetric vacuum, we will have to solve not only the F-term equations from the superpotential (2.76), but also the D-term equations,

$$D_\alpha = 0 \qquad \Longrightarrow \qquad \widehat{r}_\alpha^K \mathcal{F}_K = 0. \tag{2.82}$$

If we are in a non-supersymmetric vacuum, then the scalar potential now has a D-term piece,

$$V_D = \frac{1}{2} \left(\mathrm{Re}\, f\right)^{-1\,\alpha\beta} D_\alpha D_\beta = 2e^{2D} \left(\widehat{\kappa} v\right)^{-1\,\alpha\beta} \left(\mathcal{F}_K \widehat{r}_\alpha^K\right) \left(\mathcal{F}_J \widehat{r}_\beta^J\right), \tag{2.83}$$

where we have used

$$\mathrm{Re}\, f_{\alpha\beta} = -\widehat{\kappa}_{a\,\alpha\beta} v^a = -\left(\widehat{\kappa} v\right)_{\alpha\beta}. \tag{2.84}$$

In section 2.4 we give a geometric interpretation for the non-vanishing of the D-terms.

Such D-term contributions have been the subject of much phenomenological interest, as a possible means to uplift the potential to a metastable deSitter vacuum [29, 30, 31, 32, 33], or as a mechanism for generating inflationary potentials [34, 35]. The latter possibility is usually done in a context with F-I parameters turned on, but with a minimal holomorphic coupling $f_{\alpha\beta} = \delta_{\alpha\beta}$; it would be interesting to see if the class of models we are discussing in this paper could lead to phenomenologically useful potentials.

Finally, we verify that the F-I terms are zero by calculating the variation of the superpotential W under gauge transformations. Indeed, we find that under (2.78) we have

$$\delta W = -\lambda^{\alpha}\hat{r}_{\alpha}^{K}\left(p_K + r_{aK}t^a\right) = 0, \tag{2.85}$$

where we have used the cohomological Bianchi identities (2.73).

We turn now to the F-term equations that result from the superpotential.

$$\begin{aligned} 0 = \operatorname{Re} D_a W &= r_{aK}\xi^K + d_a{}^b e_b + \kappa_{abc}u^b m^c + \frac{m_0}{2}\kappa_{abc}\left(u^b u^c - v^b v^c\right) \\ &\quad -\frac{3}{2}\frac{\kappa_{abc}v^b v^c}{\kappa_{def}v^d v^e v^f}\operatorname{Im} W, \end{aligned} \tag{2.86}$$

$$0 = \operatorname{Im} D_a W = 2e^{-D}r_{aK}\mathcal{Z}^K + \kappa_{abc}v^b m^c + m_0\kappa_{abc}u^b v^c + \frac{3}{2}\frac{\kappa_{abc}v^b v^c}{\kappa_{def}v^d v^e v^f}\operatorname{Re} W, \tag{2.87}$$

$$0 = \operatorname{Re} D_K W = 2p_K + 2r_{aK}u^a - 4ie^D\mathcal{F}_K \operatorname{Im} W, \tag{2.88}$$

$$0 = \operatorname{Im} D_K W = 2r_{aK}v^a + 4ie^D\mathcal{F}_K \operatorname{Re} W. \tag{2.89}$$

It is once again true in this case that one can use the F-term equations to show

$$\operatorname{Im} W = -\frac{2m_0}{15}\kappa_{abc}v^a v^b v^c. \tag{2.90}$$

Thus, if we would like to find a supersymmetric Minkowski vacuum, we must have $m_0 = 0$. In such a vacuum, (2.89) says that $r_{aK}v^a = 0$, and then contracting (2.87) with v^a we learn that $\kappa_{abc}v^a v^b m^c = 0$. With a couple more manipulations one can then show that the F-term equations now reduce to

$$M \cdot \begin{pmatrix} u^a \\ \xi^K \end{pmatrix} + \begin{pmatrix} (de)^a \\ p_K \end{pmatrix} = 0, \qquad M \cdot \begin{pmatrix} \widetilde{v}^a \\ 2\mathcal{Z}^K \end{pmatrix} = 0, \tag{2.91}$$

where

$$M = \begin{pmatrix} (\kappa m)_{ab} & r_{aK} \\ r^T_{Jb} & 0 \end{pmatrix}, \tag{2.92}$$

and where $\widetilde{v}^a = e^D v^a$. Two more equations must also be satisfied - there is one relation among the $\mathcal{Z}^K$ ($(\mathcal{Z}^1)^2 - (\mathcal{Z}^2)^2 = \frac{1}{2}$ in our example), and from $\mathrm{Re}\, W = 0$ we have

$$c\, e_0 + d_a{}^b u^a e_b + \frac{1}{2}\kappa_{abc}u^a u^b m^c = 0. \tag{2.93}$$

Since the v^a and D only occur in the combination $\widetilde{v}^a$, there will always be one combination which remains unfixed (this result was also derived by [8]). Explicitly, the mode which scales $e^\phi = g_s \to \lambda g_s$ and $v^a \to \lambda^2 v^a$ leaves $\widetilde{v}^a$ unchanged, so this mode will remain massless. The scaling here is unfortunate; it means that as we go far out along this flat direction, either the string coupling blows up or the volume becomes very small, and our whole framework is expected to break down. This means that we cannot expect parametric control of such an example.

The general situation for supersymmetric AdS vacua, or even more generally for extrema of the full scalar potential, is quite complicated, and we do not have much to say about it here. We do believe that with metric fluxes it should be possible to stabilize all moduli in a supersymmetric AdS vacuum, though a puzzle regarding RR quantization will interfere with our attempts to provide a fully consistent example here.

Let us now examine the situation in our specific example more closely.

2.2.3 Example

Imposing invariance under the orientifold group (2.28), (2.29) for our $T^6/\mathbb{Z}_4$ example, we find that we are left with ten independent metric fluxes,

$$
\begin{aligned}
2\ f^1_{16} &= f^1_{15} = -f^2_{25} = -2\, f^2_{26}, \\
f^1_{26} &= f^2_{16}, \\
f^1_{36} &= -f^2_{46}, \\
f^1_{46} &= f^2_{36}, \\
& \quad f^1_{35} = f^1_{45} = f^2_{35} = -f^2_{45} = f^1_{36} + f^1_{46}, \\
f^3_{16} &= -f^4_{26}, \\
f^3_{26} &= f^4_{16}, \\
& \quad f^3_{15} = f^3_{25} = f^4_{15} = -f^4_{25} = f^3_{16} + f^3_{26}, \\
f^3_{36} &= -f^4_{46}, \\
2\ f^3_{46} &= f^3_{45} = f^4_{35} = 2\, f^4_{36}, \\
f^5_{13} &= -f^5_{24}, \\
f^6_{13} &= -f^6_{14} = -f^6_{23} = -f^6_{24}, \\
& \quad f^5_{14} = f^5_{23} = f^5_{13} + f^6_{13},
\end{aligned}
\tag{2.94}
$$

where we can use the ten fluxes in the left-hand column as representatives.

In terms of r-matrices, we find

$$r_{aK} = \begin{pmatrix} f^1_{36} & -f^1_{46} \\ -f^3_{16} & f^3_{26} \\ f^5_{13} + f^6_{13} & f^5_{13} \\ f^1_{16} - f^1_{26} - f^3_{36} - f^3_{46} & -f^1_{16} - f^1_{26} - f^3_{36} + f^3_{46} \end{pmatrix}, \tag{2.95}$$

$$\widehat{r}^K = \begin{pmatrix} -f^1_{16} + f^1_{26} - f^3_{36} - f^3_{46} & \quad -f^1_{16} - f^1_{26} + f^3_{36} - f^3_{46} \end{pmatrix}. \tag{2.96}$$

Note that there is a one-to-one correspondence between the independent fluxes f^i_{jk} and the entries of r and $\widehat{r}$.

Let us now impose the Bianchi identities $f^m_{[jk}f^i_{\ell]m} = 0$ and $f^m_{[ij}H_{kl]m} = 0$. It turns out that the general solution can be divided into four cases.

$$
\begin{aligned}
&(a) \quad r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & \beta \\ 0 & 0 \end{pmatrix}, \quad \widehat{r} = 0, \qquad \forall p_1, p_2 \\
&(a') \quad r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & -\alpha \\ 0 & 0 \end{pmatrix}, \quad \widehat{r} = \begin{pmatrix} \beta & \beta \end{pmatrix}, \quad p_1 + p_2 = 0, \\
&(a'') \quad r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & \alpha \\ 0 & 0 \end{pmatrix}, \quad \widehat{r} = \begin{pmatrix} \beta & -\beta \end{pmatrix}, \quad p_1 - p_2 = 0, \\
&(b) \quad r = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \\ 0 & 0 \\ \varepsilon & \varphi \end{pmatrix}, \quad \widehat{r} = \begin{pmatrix} \chi & \kappa \end{pmatrix}, \quad \chi p_1 + \kappa p_2 = 0,
\end{aligned}
\tag{2.97}
$$

and where case (b) must additionally satisfy the equations

$$
\alpha\chi + \beta\kappa = \gamma\chi + \delta\kappa = \varepsilon\chi + \varphi\kappa = 0, \qquad 8\alpha\gamma - \varepsilon^2 - \chi^2 = 8\beta\delta - \varphi^2 - \kappa^2. \tag{2.98}
$$

The first set of these equations is simply $r_{aK}\widehat{r}^K = 0$, as we derived above in (2.73). The one remaining equation, however, cannot be obtained from acting $d^2_{H,f}$ on any element of our orbifold-invariant cohomology, though it can be derived by demanding $d^2_{H,f} = 0$ even on non-invariant forms.

Let us try to find supersymmetric solutions to these models. First of all, note that in cases (a') and (a'') the D-term is proportional to

$$\beta\left(\mathcal{F}_1 \pm \mathcal{F}_2\right), \tag{2.99}$$

which is always nonvanishing since $|\mathcal{F}_1| > |\mathcal{F}_2|$ in non-degenerate $(0 < U < \infty)$ vacua. Hence, these two cases can never be supersymmetric. So we shall instead examine case (a) more carefully. Here the D-term equations are automatically satisfied since $\widehat{r} = 0$.

Now using (2.89), and assuming that at least one of α and β is nonzero, we find that $\alpha\mathcal{F}_2 = \beta\mathcal{F}_1$, so

$$U = \frac{1}{2}\frac{\alpha+\beta}{\alpha-\beta}. \tag{2.100}$$

For a physical solution, we need $|\alpha| > |\beta|$. Then from (2.88) we learn that in order to solve the F-term equations we have an extra condition on the fluxes, namely that

$$\alpha p_2 = \beta p_1. \tag{2.101}$$

There are a couple of immediate consequences of this. Firstly, observe that if $p_1 \neq 0$, then U actually has the same form (2.51) as before, and we again have that $|p_1| > |p_2|$. Also, note that the axions ξ^1 and ξ^2 appear in the F-term equations only in the combinations $r_{aK}\xi^K$ and $p_K\xi^K$, but thanks to (2.101), both of these are proportional to $(\alpha\xi^1 + \beta\xi^2)$; the equations are independent of the other linear combination, and hence one of the axions remains unfixed. Below, we will argue that this will happen generically if the rank of the matrix r_{aK} (one, for case (a)) is less than the number of axions in the problem (two).

We can now express the general solution to the F-term equations. First we define some useful quantities,

$$\widehat{e}_1 = e_1 - \frac{m^2m^3}{m_0}, \qquad \widehat{e}_2 = e_2 - \frac{m^1m^3}{m_0}, \qquad \widehat{e}_3 = e_3 - \frac{m^1m^2 - 2(m^4)^2}{m_0},$$

$$\widehat{e}_4 = e_4 - \frac{m^3 m^4}{m^0}, \qquad \widehat{e}_0 = e_0 - \frac{e_1 e_2 - 2e_4^2}{m^3}. \tag{2.102}$$

Then we find that in addition to U given above, we have

$$\begin{aligned}
u^1 &= -\frac{m^1}{m_0} - \frac{\alpha \widehat{e}_2}{\alpha m^3 - p_1 m_0}, \\
u^2 &= -\frac{m^2}{m_0} - \frac{\alpha \widehat{e}_1}{\alpha m^3 - p_1 m_0}, \\
u^3 &= -\frac{m^3}{m_0} + \frac{5\alpha}{m_0} \frac{\left(\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2\right)\left(\alpha m^3 - p_1 m_0\right)}{3\left(\alpha m^3 - p_1 m_0\right)\left(\alpha \widehat{e}_0 - p_1 \widehat{e}_3\right) + \alpha\left(5\alpha - 3p_1 \frac{m_0}{m^3}\right)\left(\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2\right)}, \\
u^4 &= -\frac{m^4}{m_0} - \frac{\alpha \widehat{e}_4}{\alpha m^3 - p_1 m_0}.
\end{aligned} \tag{2.103}$$

In terms of u^3 above, we then have

$$v^3 = \sqrt{-\frac{1}{\alpha m_0}\left(m^3 + m_0 u^3\right)\left(p_1 + \alpha u^3\right)}, \tag{2.104}$$

and

$$\begin{aligned}
\alpha \xi^1 + \beta \xi^2 &= -\frac{\alpha}{4}\left[\frac{m^3 \widehat{e}_3 - m_0 \widehat{e}_0}{\alpha m^3 - p_1 m_0} + \frac{m_0\left(\alpha m^3 + p_1 m_0\right)\left(\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2\right)}{m^3\left(\alpha m^3 - p_1 m_0\right)^2}\right], \\
v^1 &= -\frac{5}{3v^3} \frac{p_1 + \alpha u^3}{\alpha m^3 - p_1 m_0} \widehat{e}_2, \\
v^2 &= -\frac{5}{3v^3} \frac{p_1 + \alpha u^3}{\alpha m^3 - p_1 m_0} \widehat{e}_1, \\
v^4 &= -\frac{5}{3v^3} \frac{p_1 + \alpha u^3}{\alpha m^3 - p_1 m_0} \widehat{e}_4,
\end{aligned} \tag{2.105}$$

$$e^\phi = \frac{3\sqrt{\alpha^2 - \beta^2}\left(\alpha m^3 - p_1 m_0\right)}{2\sqrt{2}\,|\alpha m_0|} \sqrt{\frac{v^3}{\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2}}. \tag{2.106}$$

A good solution requires a number of inequalities and conditions to hold; $\alpha m^3 > p_1 m_0$, $\widehat{e}_1 \widehat{e}_2 > 2\widehat{e}_4^2$, and $(\alpha m^3 - p_1 m_0)(\alpha \widehat{e}_0 - p_1 \widehat{e}_3) > \alpha p_1 m_0 (\widehat{e}_1 \widehat{e}_1 - 2\widehat{e}_4^2)/m^3$, and the quantities $\widehat{e}_1$, $\widehat{e}_2$, and m_0 must have the same sign.

As long as these conditions are respected, we can take various limits of the above solution. For instance, one can check that taking the limit $\alpha, \beta \to 0$ (and using (2.101)) recovers the solution from section 2.1.4. For future reference, let us list also the limit $p_1, p_2 \to 0$. In this case the conditions are that α, m^3, $\widehat{e}_0$ must have the same sign, $\widehat{e}_1$, $\widehat{e}_2$, and m_0 must have the same sign, and $\widehat{e}_1\widehat{e}_2 > 2\widehat{e}_4^2$.

$$u^1 = -\frac{e_2}{m^3}, \quad u^2 = -\frac{e_1}{m^3}, \quad u^3 = -\frac{3(m^3)^2\widehat{e}_0}{m_0\left(3m^3\widehat{e}_0 + 5\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)\right)}, \quad u^4 = -\frac{e_4}{m^3},$$

$$v^1 = |\widehat{e}_2|\sqrt{\frac{5\widehat{e}_0}{3m^3\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}}, \qquad v^2 = |\widehat{e}_1|\sqrt{\frac{5\widehat{e}_0}{3m^3\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}},$$

$$v^3 = \frac{\sqrt{\frac{15(m^3)^3}{m_0^2}\widehat{e}_0\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}}{3m^3\widehat{e}_0 + 5\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}, \qquad v^4 = \widehat{e}_4\sqrt{\frac{5\widehat{e}_0}{3m^3\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}}\,(\mathrm{sign}\, m_0)\,, \quad (2.107)$$

$$\alpha\xi^1 + \beta\xi^2 = -\frac{1}{4}\left[\widehat{e}_3 - \frac{m_0\widehat{e}_0}{m^3} + \frac{m_0}{(m^3)^2}\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)\right],$$

$$e^{\phi} = \frac{3\sqrt{\alpha^2 - \beta^2}}{2\sqrt{2}}\left|\frac{m^3}{m_0}\right|\frac{\left[\frac{15(m^3)^3}{m_0^2}\frac{\widehat{e}_0}{\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2}\right]^{1/4}}{\sqrt{3m^3\widehat{e}_0 + 5\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}}.$$

We will see later in subsection 4.1.1 that these two limits are T-duals of each other.

Returning to the general case, note that the tadpole condition is now

$$\sqrt{2}\left(\alpha m^3 - p_1 m_0\right) + N_1 = 8, \qquad \sqrt{2}\left(\beta m^3 - p_2 m_0\right) + N_2 = 0. \qquad (2.108)$$

If we are looking for a supersymmetric solution, then we want N_1 to be greater than zero, and the sign of N_2 to be fixed by U (as discussed in section 2.1.4), and from solving the F-term equations we have $\alpha m^3 > p_1 m_0$, $\alpha p_2 = \beta p_1$, and $|\alpha| > |\beta|$, so we find, as before, that $N_1 + |N_2| < 8$.

To help understand why we were unable to find solutions with all moduli stabilized in these examples, note that in the general situation the $h^{2,1} + 1$ real fields ξ^K appear in the F-term equations only through the combinations $p_K\xi^K$ and

$r_{aK}\xi^K$. We see immediately that these provide at most $\text{rk}(r)+1$ independent combinations, so that we only have a chance to stabilize all of the axions in the case that $\text{rk}(r) \geq h^{2,1}$ (see also the discussion in [8]). In fact, if $\text{Re}\, W \neq 0$, then we can do even better. In this case we can use (2.89) to show that

$$\mathcal{F}_K = \frac{i}{2} e^{-D} \frac{r_{aK} v^a}{\text{Re}\, W}, \tag{2.109}$$

and then (2.88) implies that

$$p_K = \left(-u^a - \frac{\text{Im}\, W}{\text{Re}\, W} v^a \right) r_{aK}, \tag{2.110}$$

thus reducing us to just $\text{rk}(r)$ independent combinations of axions. In our example, this means that if $\text{Re}\, W \neq 0$, we need an r-matrix of rank two, which was obviously impossible in the context of case (a) above. On the other hand, trying to set $\text{Re}\, W = 0$ seems to typically lead to degenerate solutions, where either the complex structure modulus or a Kähler modulus runs off to the edge of physically allowed values.

So finally, let us turn to case (b), with the hopes of finding an $\mathcal{N}=1$ vacuum with all moduli fixed. Suppose first that $\widehat{r} \neq 0$. In this case, the D-term equations require $|\chi| < |\kappa|$ and fix $\mathcal{F}_2/\mathcal{F}_1 = -\chi/\kappa$. Then the various Bianchi identities enforce $r_{a2} = -(\chi/\kappa) r_{a1}$ and $p_2 = -(\chi/\kappa) p_1$. It is immediately clear in this case that one combination of the ξ's again remains unfixed.

Hence, let $\widehat{r} = 0$. By the argument above, we should look for solutions in which $\text{rk}(r) = 2$. As we will see in section 4, the quantization conditions on metric fluxes is in general not the naive quantization in terms of (even) integers, but is somewhat more complicated. In fact, we will see that we cannot find a correctly quantized set of metric fluxes which both give $\text{rk}(r) = 2$ and make it possible to satisfy the tadpole condition, however we think that this is a reflection of our ignorance of the correct RR quantization conditions under these circumstances. For now, let us willfully ignore these subtleties and pick a set of NSNS fluxes with the naive quantization, namely

$$p_1 = p_2 = 0, \qquad r = \begin{pmatrix} 4 & 2 \\ 2 & 0 \\ 0 & 0 \\ 8 & 0 \end{pmatrix}. \tag{2.111}$$

This choice respects the Bianchi identities. Let us then also choose RR fluxes

$$m_0 = m^1 = m^2 = 0, \qquad e_0 = e_1 = e_2 = -e_3 = e_4 = \sqrt{2}, \qquad -m^3 = m^4 = \frac{1}{\sqrt{2}}, \tag{2.112}$$

which all satisfy that they are in $\mathbb{Z}/\sqrt{2}$. One can check that these choices satisfy the tadpole conditions with no extra branes. Then one can solve the F-term equations with these fluxes, first finding exact solutions for the u^a and ξ^K

$$u^1 = 0, \qquad u^2 = 2, \qquad u^3 = -\frac{9}{2}, \qquad u^4 = -\frac{1}{2},$$

$$\xi^1 = -\frac{1}{4\sqrt{2}}, \qquad \xi^2 = \frac{1}{2\sqrt{2}}, \tag{2.113}$$

and then solving numerically for the rest of the moduli,

$$v^1 \approx 2.58227, \qquad v^2 \approx 4.26420, \qquad v^3 \approx 3.46108, \qquad v^4 \approx -0.50562,$$

$$U \approx 1.03530, \qquad e^{\phi} \approx 11.956. \tag{2.114}$$

Note that the volumes here are not particularly large, though all the cycle volumes are positive, and the string coupling is definitely not small. This solution should be viewed more as an in principle proof that the F-term equations can stabilize all of the moduli at physical values.

2.3 Non-Geometric Fluxes

As we shall see when we consider T-dualities below, by T-dualizing twice on a torus with H-flux, one can find oneself in a *non-geometric* situation, where there is a local geometric description, but globally, one must patch torus fibers together with non-geometric elements of the T-duality group. All of this will hopefully be elucidated more cleanly in the chapter 4, but for now note that at least some such considerations are really forced upon us by T-duality. To this end we will introduce objects Q_k^{ij}, analogous to H_{ijk} and f^i_{jk}.

At the level of the effective four-dimensional theory, it is natural to also introduce the totally antisymmetric R^{ijk}, which one can formally imagine as resulting from T-dualizing all three legs of toroidal H-flux [23]. From a ten-dimensional perspective, it is not clear how to construct such a thing (which would not admit even a local geometric interpretation [12]) since the need to choose an initial trivialization for the H-flux breaks one of the three necessary isometries[6]. However, it does not inconvenience us in our current context to include all these possible non-geometric fluxes, so we shall. Like H_{ijk} and f^i_{jk}, they all arise from the NSNS sector.

2.3.1 General Approach

As mentioned, we introduce fluxes Q_k^{ij} and R^{ijk}. The Q-fluxes, being two T-dualities from H-flux, should be invariant under the orbifold group and odd under the orientifold involution, while the R-fluxes, being two T-dualities away from metric fluxes, should be invariant under the full orientifold group (a pair of T-dualities should preserve the eigenvalue under world-sheet parity; alternatively, for Q-flux we will

[6]In fact, there are also examples of Q-fluxes and metric fluxes f^i_{jk} which seem very difficult to construct from a ten-dimensional viewpoint. For a subset which can be constructed, see chapter 4 below.

see this requirement emerge from the base-fiber approach). As with our previous examples, it is natural to define actions of these fluxes on (components of) differential forms. First, the Q-fluxes allow a map from p-forms to $(p-1)$-forms[7],

$$(Q\cdot A)_{i_1\cdots i_{p-1}} = \frac{1}{2}\binom{p-1}{1} Q^{jk}_{[i_1} A_{|jk|i_2\cdots i_{p-1}]}, \tag{2.115}$$

while the R-fluxes map p-forms to $(p-3)$-forms,

$$(R\cdot A)_{i_1\cdots i_{p-3}} = \frac{1}{6}\binom{p-3}{0} R^{jk\ell} A_{jk\ell i_1\cdots i_{p-3}}. \tag{2.116}$$

The inclusion of the somewhat trivial binomial coefficients is simply to make it clear that Q kills forms below degree two, while R kills forms below degree three.

With these actions, it is convenient to define a differential operator which acts on forms [12, 8],

$$\mathcal{D} = d + H\wedge - f\cdot + Q\cdot - R\cdot. \tag{2.117}$$

Requiring that $\mathcal{D}^2 = 0$ leads to a set of Bianchi identities,

$$\begin{aligned}
H_{m[ij}f^m_{k\ell]} &= 0,\\
H_{m[ij}Q^{m\ell}_{k]} - f^m_{[ij}f^\ell_{k]m} &= 0,\\
H_{ijm}R^{k\ell m} + f^m_{ij}Q^{k\ell}_m - 4f^{[k}_{m[i}Q^{\ell]m}_{j]} &= 0,\\
f^{[j}_{mi}R^{k\ell]m} - Q^{[jk}_m Q^{\ell]m}_i &= 0,\\
Q^{[ij}_m R^{k\ell]m} &= 0,
\end{aligned} \tag{2.118}$$

along with the additional requirement that $H_{ijk}R^{ijk} = f^i_{jk}Q^{jk}_i = 0$, which is trivially

[7] This is actually assuming a tracelessness condition, $Q^{jk}_j = 0$. If this condition is dropped, then the correct generalization would be

$$(Q\cdot A)_{i_1\cdots i_{p-1}} = \frac{1}{2}\binom{p-1}{1} Q^{jk}_{[i_1} A_{|jk|i_2\cdots i_{p-1}]} + \frac{1}{2}\binom{p-1}{0} Q^{jk}_j A_{ki_1\cdots i_{p-1}}.$$

satisfied on orientifolds since there are no odd invariant scalars.

It is again natural to introduce cohomological parameters via the expansions

$$Q \cdot \widetilde{\omega}^a = q_K^a b^K, \qquad Q \cdot \widetilde{\mu}^\alpha = \widehat{q}^{\alpha K} a_K, \tag{2.119}$$

$$R \cdot \varphi = s_K b^K. \tag{2.120}$$

We then have also

$$Q \cdot a_K = -\left(d^{-1}\right)_a{}^b q_K^a \omega_b, \qquad Q \cdot b^K = \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{q}^{\alpha K} \mu_\beta, \tag{2.121}$$

$$R \cdot a_K = c^{-1} s_K 1. \tag{2.122}$$

Again, some, but not all, of the Bianchi identities follow from demanding that $\mathcal{D}^2$ vanish on our cohomological basis, namely

$$\begin{aligned} \widehat{r}_\alpha^K p_K = \widehat{r}_\alpha^K s_K = \widehat{q}^{\alpha K} p_K = \widehat{q}^{\alpha K} s_K &= 0, \quad \forall \alpha, \\ \widehat{r}_\alpha^K r_{bK} = \widehat{r}_\alpha^K q_K^b = \widehat{q}^{\alpha K} r_{bK} = \widehat{q}^{\alpha K} q_K^b &= 0, \quad \forall \alpha, b, \\ c^{-1} p_{[K} s_{J]} + \left(d^{-1}\right)_a{}^b r_{b[K} q_{J]}^a = \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{q}^{\alpha[K} \widehat{r}_\beta^{J]} &= 0, \quad \forall K, J. \end{aligned} \tag{2.123}$$

Also, we still have the possibility that there is not a bijective mapping between flux parameters Q_k^{ij} and cohomological parameters q_K^a and $\widehat{q}^{\alpha K}$, but the same remarks apply as for metric fluxes. There is always a bijective correspondence for the R-fluxes.

The modifications to the tadpole condition can be obtained by T-duality arguments,

$$-\sqrt{2}\mathcal{D}F_{RR} + [\delta_{D6}] = 2\,[\delta_{O6}]\,, \tag{2.124}$$

or

$$-\sqrt{2}\left(p_K m_0 - r_{aK} m^a + q_K^a e_a - s_K e_0\right) + N_K^{(D6)} = 2N_K^{(O6)}. \tag{2.125}$$

Similar arguments can also be used to obtain the superpotential [7, 8],

$$
\begin{aligned}
W &= \int_X e^{J_c} \wedge F_{RR} + \int_X \Omega_c \wedge \mathcal{D}\left(e^{-J_c}\right) \\
&= ce_0 + d_a{}^b t^a e_b + \frac{1}{2}\kappa_{abc} t^a t^b m^c + \frac{1}{6} m_0 \kappa_{abc} t^a t^b t^c \\
&\quad + 2N^K \left(p_K + r_{aK} t^a + \frac{1}{2}\kappa_{abc}\left(d^{-1}\right)_d{}^a q_K^d t^b t^c + \frac{1}{6} c^{-1} s_K \kappa_{abc} t^a t^b t^c \right).
\end{aligned}
\tag{2.126}
$$

The corresponding F-terms are

$$
\begin{aligned}
D_a W =& d_a{}^b e_b + \kappa_{abc} t^b m^c + \frac{1}{2} m_0 \kappa_{abc} t^b t^c \\
&+ 2N^K \left(r_{aK} + \frac{1}{2}\kappa_{abc}\left(d^{-1}\right)_d{}^c q_K^d t^b + \frac{1}{2} c^{-1} s_K \kappa_{abc} t^b t^c \right) + \frac{3i}{2} \frac{\kappa_{abc} v^b v^c}{\kappa_{def} v^d v^e v^f} W.
\end{aligned}
\tag{2.127}
$$

$$
D_K W = 2p_K + 2r_{aK} t^a + \kappa_{abc}\left(d^{-1}\right)_d{}^a q_K^d t^b t^c + \frac{1}{3} c^{-1} s_K \kappa_{abc} t^a t^b t^c - 4e^D \mathcal{F}_K W.
\tag{2.128}
$$

Note that the superpotential only depends on the NS-NS fluxes p_K, r_{aK}, q_K^a, and s_K, and not on the hatted fluxes $\widehat{r}_\alpha^K$ and $\widehat{q}^{\alpha K}$. These appear only in the D-terms, which we turn to next.

Using the newly defined exterior derivative $\mathcal{D}$, we propose that the proper RR gauge transformations should be encoded as

$$
C_{RR} \longrightarrow C_{RR} + \mathcal{D}\Lambda,
\tag{2.129}
$$

where Λ is a formal sum of even forms. Explicitly, if Λ is a four-dimensional scalar times a set of internal forms, then the orientifold projection actually forces

$$
\Lambda = \lambda^\alpha \mu_\alpha + \widetilde{\lambda}_\alpha \widetilde{\mu}^\alpha, \qquad \mathcal{D}\Lambda = d\lambda^\alpha \wedge \mu_\alpha + d\widetilde{\lambda}_\alpha \wedge \widetilde{\mu}^\alpha + \left(\widehat{q}^{\alpha K} \widetilde{\lambda}_\alpha - \widehat{r}_\alpha^K \lambda^\alpha \right) a_K.
\tag{2.130}
$$

From this we can see our earlier conclusion that λ^α generates gauge transformations in the electric gauge groups $U(1)^\alpha$, and that the $\widehat{r}_\alpha^K$ correspond to electric charges.

But we also see that $\widetilde{\lambda}_\alpha$ generates gauge transformations in the corresponding *magnetic* gauge groups $\widetilde{\mathrm{U}(1)}_\alpha$ (whose vectors come from C_5 reduced against $\widetilde{\mu}^\alpha$) and that the non-geometric fluxes $\widehat{q}^{\alpha K}$ correspond to magnetic charges.

We should again quickly check whether the superpotential remains neutral under these magnetic gauge transformations as claimed above. Indeed,

$$\delta W = \widetilde{\lambda}_\alpha \widehat{q}^{\alpha K} \left(p_K + r_{aK} t^a + \frac{1}{2}\kappa_{abc} \left(d^{-1}\right)_d{}^a q_K^d t^b t^c + \frac{1}{6} c^{-1} s_K \kappa_{abc} t^a t^b t^c \right) = 0, \tag{2.131}$$

where we have used the first two lines of (2.123), so there are no F-I terms generated.

Since we have now electric and magnetic charges and dyonic fields (i.e. carrying potentially both electric and magnetic charges), it is interesting to ask if the collection of charged scalars remains mutually local. The condition that two charged scalars be mutually local is

$$\left(\widehat{d}^{-1}\right)_\alpha{}^\beta \left(\widehat{q}^{\alpha K} \widehat{r}^J_\beta - \widehat{q}^{\alpha J} \widehat{r}^K_\beta\right) = 0. \tag{2.132}$$

Note that under the usual normalization for the dual gauge groups, the fields $\widetilde{A}_\alpha$ should be rescaled by the matrix $\widehat{d}_\beta{}^\alpha$, or equivalently the magnetic charges $\widehat{q}^{\alpha K}$ should be rescaled by $\widehat{d}^{-1}$, so that in our conventions the correct mutual locality condition is as above. But (2.132) is simply the final equation of (2.123) and thus is guaranteed by the Bianchi identities.

The mutual locality in turn implies that there always exists a $\mathrm{Sp}(2h_+^{1,1};\mathbb{Z})$ transformation which can rotate all the charges to be electric charges[8]. The resulting electric gauge groups after rotation will have associated D-terms which must vanish in any supersymmetric solution. However, for the moment it will be more convenient to use our original basis of gauge groups, but include also magnetic contributions. Recall that the holomorphic gauge kinetic couplings for the electric groups were

[8]This statement actually relies also on charge quantization, which we have not demonstrated here. A more detailed discussion of the subtleties can be found in section 3.

given by

$$f_{\alpha\beta} = i\,(\widehat{\kappa}t)_{\alpha\beta}\,. \tag{2.133}$$

Similar calculations (by reducing the piece of the ten-dimensional action which is quadratic in C_5) give the holomorphic magnetic gauge kinetic couplings,

$$\widetilde{f}^{\alpha\beta} = -i\,(\widehat{\kappa}t)^{-1\,\gamma\delta}\,\widehat{d}_{\gamma}{}^{\alpha}\widehat{d}_{\delta}{}^{\beta}. \tag{2.134}$$

The magnetic analogs of our previous electric D-terms are

$$\widetilde{D}^{\alpha} = -2ie^{D}\widehat{q}^{\alpha K}\mathcal{F}_K, \tag{2.135}$$

and the resulting D-term contribution to the scalar potential is

$$\begin{aligned} V_D &= \frac{1}{2}\,(\mathrm{Re}\,f)^{-1\,\alpha\beta}\,D_\alpha D_\beta + \frac{1}{2}\left(\mathrm{Re}\,\widetilde{f}\right)^{-1}_{\alpha\beta}\widetilde{D}^{\alpha}\widetilde{D}^{\beta} \\ &= 2e^{2D}\left[(\mathrm{Re}\,f)^{-1\,\alpha\beta}\,\widehat{r}^{K}_{\alpha}\widehat{r}^{J}_{\beta} + \left(\mathrm{Re}\,\widetilde{f}\right)^{-1}_{\alpha\beta}\widehat{q}^{\alpha K}\widehat{q}^{\beta J}\right]\mathrm{Im}(\mathcal{F}_K)\,\mathrm{Im}(\mathcal{F}_J). \end{aligned} \tag{2.136}$$

Though not immediately apparent in this form, this expression is positive semi-definite (the gauge kinetic couplings are positive definite and the D-terms are real), and must vanish in a supersymmetric vacuum. Note that this piece of the potential can have reasonably complicated dependence on all of the scalar fields.

2.3.2 Example

Let us briefly see how some of these results work in our example. Unlike in previous subsections, we will not expend much effort trying to solve the F-term equations, but will content ourselves simply with classifying the fluxes permitted by the orientifold action, and stating the equations that we would like to solve.

The Q-fluxes which survive the orientifold projection are

$$
\begin{aligned}
Q_5^{13} &= -Q_5^{14} = -Q_5^{23} = -Q_5^{24}, \\
Q_6^{13} &= -Q_6^{24}, \\
&\quad Q_6^{14} = Q_6^{23} = -Q_5^{13} + Q_6^{13}, \\
2 \quad Q_1^{15} &= -Q_1^{16} = -2\, Q_2^{25} = Q_2^{26}, \\
Q_2^{15} &= Q_1^{25}, \\
Q_3^{15} &= -Q_4^{25}, \\
Q_3^{16} &= Q_4^{16} = Q_3^{26} = -Q_4^{26}, \\
&\quad Q_4^{15} = Q_3^{25} = -Q_3^{15} - Q_3^{16}, \\
Q_1^{35} &= -Q_2^{45}, \\
Q_1^{36} &= Q_2^{36} = Q_1^{46} = -Q_2^{46}, \\
&\quad Q_2^{35} = Q_1^{45} = -Q_1^{35} - Q_1^{36}, \\
Q_3^{35} &= -Q_4^{45}, \\
2 \quad Q_4^{35} &= -Q_4^{36} = 2\, Q_3^{45} = -Q_3^{46},
\end{aligned}
\tag{2.137}
$$

where we take the ten fluxes in the left-hand column as independent. Similarly there are two independent R-fluxes,

$$
\begin{aligned}
R^{135} &= -R^{245}, \\
R^{136} &= -R^{146} = -R^{236} = -R^{246}, \\
&\quad R^{145} = R^{235} = R^{135} + R^{136}.
\end{aligned}
\tag{2.138}
$$

In more succinct terms,

$$q = \begin{pmatrix} -Q_1^{35} & -Q_1^{35} - Q_1^{36} \\ Q_3^{15} & Q_3^{15} + Q_3^{16} \\ -Q_5^{13} + Q_6^{13} & Q_6^{13} \\ Q_1^{15} - Q_2^{15} - Q_3^{35} - Q_4^{35} & -Q_1^{15} - Q_2^{15} - Q_3^{35} + Q_4^{35} \end{pmatrix}, \tag{2.139}$$

$$\widehat{q} = \begin{pmatrix} -Q_1^{15} + Q_2^{15} - Q_3^{35} - Q_4^{35} & -Q_1^{15} - Q_2^{15} + Q_3^{35} - Q_4^{35} \end{pmatrix}, \tag{2.140}$$

$$s = \begin{pmatrix} R^{135} + R^{136} & R^{135} \end{pmatrix}. \tag{2.141}$$

The Bianchi identities are unfortunately quite complicated and unenlightening. In addition to the identities from (2.123), we have the following extra conditions

$$\begin{gathered}
-8r_{31}s_1 + 8q_1^1 q_1^2 - \left(q_1^4\right)^2 - \left(\widehat{q}^1\right)^2 = -8r_{32}s_2 + 8q_2^1 q_2^2 - \left(q_2^4\right)^2 - \left(\widehat{q}^2\right)^2, \\
\sum_b \kappa_{3ab} \left(d^{-1}\right)_b{}^c r_{c(K} s_{J)} = q^3_{(K} q^a_{J)}, \qquad \text{for } a = 1,2,4;\ \forall K, J, \\
s_1 \widehat{r}^2 + s_2 \widehat{r}^1 = \widehat{q}^1 q_2^3 + \widehat{q}^2 q_1^3, \\
4q_1^1 r_{11} + 4q_1^2 r_{21} - q_1^4 r_{41} - \widehat{q}^1 \widehat{r}^1 - 8q_1^3 r_{31} = 4q_2^1 r_{12} + 4q_2^2 r_{22} - q_2^4 r_{42} - \widehat{q}^2 \widehat{r}^2 - 8q_2^3 r_{32}, \\
r_{a(K} \widehat{q}^{J)} = \kappa_{3ab} \left(d^{-1}\right)_c{}^b q^c_{(K} \widehat{r}^{J)}, \qquad \text{for } a = 1,2,4;\ \forall K \neq J, \\
\kappa_{3ac} \left(d^{-1}\right)_d{}^c q^d_{(K} r_{|b|J)} = \kappa_{3bc} \left(d^{-1}\right)_d{}^c q^d_{(K} r_{|a|J)}, \qquad \text{for } a, b \in \{1,2,4\},\ \forall K, J, \\
q^3_{(K} r_{|3|J)} + p_{(K} s_{J)} = 0, \\
-8p_1 q_1^3 + 8r_{11} r_{21} - \left(r_{41}\right)^2 - \left(\widehat{r}^1\right)^2 = -8p_2 q_2^3 + 8r_{12} r_{22} - \left(r_{42}\right)^2 - \left(\widehat{r}^2\right)^2, \\
\kappa_{3ab} \left(d^{-1}\right)_c{}^b q^c_{(K} p_{J)} = r_{3(K} r_{|a|J)}, \qquad \text{for } a = 1,2,4;\ \forall K, J, \\
p_1 \widehat{q}^2 + p_2 \widehat{q}^1 = \widehat{r}^1 r_{32} + \widehat{r}^2 r_{31}.
\end{gathered} \tag{2.142}$$

The tadpole conditions are just as listed in (2.125), the D-term equations require (2.136) to vanish, and the F-term equations are as given in (2.127).

2.4 $SU(3)$ Structure with Metric Fluxes

Before we discuss type IIB compactifications we analyze the SU(3) structure of the torus with metric fluxes, as was done in a specific case in [28]. For a general discussion of SU(3) structures, see for example [36]. The results fit nicely with the more general discussions of type IIA compactifications to four dimensions [37], [38].

We start with the Kähler 2-form J and the holomorphic three-form Ω of the geometry, given by

$$J = v^a \omega_a, \tag{2.143}$$

$$\Omega = \mathcal{Z}^K a_K - \mathcal{F}_K b^K, \tag{2.144}$$

where $\mathcal{Z}^K$ are real and $\mathcal{F}_K$ are imaginary. These forms satisfy $J \wedge \Omega = 0$, $J \wedge J \wedge J = 6\mathrm{i}\mathcal{V}_6 \Omega \wedge \overline{\Omega}$ and define an SU(3) structure on the twisted torus. The torsion classes are defined by

$$\mathrm{d}J = -12\mathcal{V}_6 \mathrm{Im}(\mathcal{W}_1 \overline{\Omega}) + \mathcal{W}_4 \wedge J + \mathcal{W}_3, \tag{2.145}$$

$$\mathrm{d}\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \mathcal{W}_5^* \wedge \Omega. \tag{2.146}$$

$\mathcal{W}_1$ is a complex scalar, $\mathcal{W}_2$ is a complex primitive (1,1)-form i.e., $\mathcal{W}_2 \wedge J \wedge J = 0$, $\mathcal{W}_3$ is a real primitive (2,1)+(1,2)-form i.e., $\mathcal{W}_3 \wedge J = \mathcal{W}_3 \wedge \Omega = 0$, $\mathcal{W}_4$ is a real 1-form and $\mathcal{W}_5$ is a complex (1,0)-form. The prefactor in the first term of $\mathrm{d}J$ is needed to have $\mathrm{d}(J \wedge \Omega) = 0$.

The torsion classes can be read off from

$$\mathrm{d}J = -r_{aK}v^a b^K, \tag{2.147}$$

$$\mathrm{d}\Omega = -\mathcal{Z}^K \left(d^{-1}\right)_a{}^b r_{bK}\widetilde{\omega}^a - \mathcal{F}_K \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{r}_\beta^K \widetilde{\mu}^\alpha. \tag{2.148}$$

Since there are no $\mathbb{Z}_4$ invariant 1-forms we have immediately $\mathcal{W}_4 = \mathcal{W}_5 = 0$. To determine $\mathcal{W}_1$ we use the fact that $\mathcal{W}_2$ is primitive and that $\int_X \omega_a \wedge \tilde{\omega}^b = d_a{}^b$.

$$\int_X \mathrm{d}\Omega \wedge J = -\mathcal{Z}^K r_{aK} v^a = \int_X \mathcal{W}_1 J \wedge J \wedge J = \mathcal{W}_1 6\mathcal{V}_6 \tag{2.149}$$

$$\Rightarrow \mathcal{W}_1 = -\frac{\mathcal{Z}^K r_{aK} v^a}{6\mathcal{V}_6}. \tag{2.150}$$

Now we can read off $\mathcal{W}_3 = \left(2i\mathcal{Z}^L r_{aL} v^a \mathcal{F}_K - r_{aK}v^a\right) b^K$. It is straight forward to calculate $\mathcal{W}_2$. The torsion classes for a generic choice of metric fluxes are

$$\mathcal{W}_1 = -\frac{\mathcal{Z}^K r_{aK} v^a}{6\mathcal{V}_6}, \tag{2.151}$$

$$\mathcal{W}_2 = -\left(\mathcal{W}_1 v^a + (\kappa v)^{-1\,ab}\,\mathcal{Z}^K r_{bK}\right)\omega_a - (\widehat{\kappa} v)^{-1\,\alpha\beta}\,\mathcal{F}_K \widehat{r}_\beta^K \mu_\alpha, \tag{2.152}$$

$$\mathcal{W}_3 = \left(2i\mathcal{Z}^L r_{aL} v^a \mathcal{F}_K - r_{aK} v^a\right) b^K, \tag{2.153}$$

$$\mathcal{W}_4 = \mathcal{W}_5 = 0, \tag{2.154}$$

where $(\kappa v)^{-1}$ is the inverse of the matrix $(\kappa v)_{ab} = \kappa_{abc}v^c$, and similarly $(\widehat{\kappa} v)^{-1}$ is the inverse of the matrix $(\widehat{\kappa} v)_{\alpha\beta} = \widehat{\kappa}_{a\alpha\beta} v^a$.

Note that the twisted torus is generically not half-flat. For the twisted torus to be half-flat we would have to demand that $\mathrm{Im}(\mathcal{W}_1) = \mathrm{Im}(\mathcal{W}_2) = \mathcal{W}_4 = \mathcal{W}_5 = 0$. This is equivalent to $\mathrm{d}J \wedge J = \mathrm{d}(\mathrm{Im}(\Omega)) = 0$ or that $\mathrm{Im}(\mathcal{F}_K)\widehat{r}_\alpha^K = 0$. But these are precisely the D-term equations that we derived in subsection 2.2.2. So solving the D-term equations is precisely equivalent to demanding that our manifold be half-flat.

Supersymmetric solutions should also have $\mathcal{W}_3 = 0$ [37]. And indeed we can show this using the F-term equations (2.89). We have

$$0 = \mathcal{Z}^K \left(\operatorname{Im} D_K W\right) = 2\mathcal{Z}^K r_{aK} v^a + 2e^D \operatorname{Re} W, \quad \Longrightarrow \quad \operatorname{Re} W = -e^{-D} \mathcal{Z}^K r_{aK} v^a, \tag{2.155}$$

and then plugging this back in to (2.89) we find that each component of $\mathcal{W}_3$ must vanish in a supersymmetric solution.

Note also that in [38] it was shown that Minkowski vacua of type IIA require $\mathcal{W}_1 = 0$. This fits nicely with the observation that $\mathcal{W}_1 = e^D \operatorname{Re} W/(6\mathcal{V}_6)$.

So we see that our results agree very nicely with the language of SU(3) structure and torsion classes.

2.5 Summary and Puzzles

We have laid out an approach to studying a class of four-dimensional $\mathcal{N} = 1$ effective theories. Starting from toroidal orientifolds of IIA string theory with NSNS H-flux turned on, we followed in the footsteps of many authors before us and argued for a more general class of NSNS fluxes. The arguments proceed roughly by showing at each step that a T-duality induces the possibility of a new type of flux, and then we generalize to a framework capable of accommodating these new fluxes as well as the old ones (and thus allowing configurations that are not simply T-dual to previous ones). In this way we introduced metric fluxes f^i_{jk}, then non-geometric fluxes Q^{ij}_k, and finally R^{ijk}.

However, these arguments were really made at the level of the effective field theory. In terms of ten-dimensional constructions, there would seem to be some obstacles to this program. For instance, beginning with H-flux on a torus, say $h\, dx \wedge dy \wedge dz$, to perform a ten-dimensional T-duality, one first picks a trivialization of the B-field such as $B = hx\, dy \wedge dz$. Then the Buscher rules [39] allow one to

T-dualize along either the y or z directions, resulting in metric flux f^y_{xz} or f^z_{xy}, or T-dualize in y and z, resulting in Q^{yz}_x, but it is not obvious how to perform the third T-duality here to get R^{xyz}; our trivialization broke the third isometry, and the Buscher rules no longer apply. Indeed, there are general arguments that any ten-dimensional origin for R-flux cannot even have a local description [12, 40]. So it is very much of interest to ask which configurations can be constructed from ten dimensions. We will discuss this in chapter 4 below.

We have also tried to formulate everything in a language that moves away from the toroidal context. So, instead of phrasing everything in terms of flux components, H_{ijk}, f^i_{jk}, Q^{ij}_k and R^{ijk}, we rewrite our formulae (thereby serendipitously simplifying the $\mathcal{N}=1$ expressions at the same time) in terms of matrices p_K, r_{aK}, $\widehat{r}_{\alpha K}$, q^a_K, $\widehat{q}^{\alpha K}$ and s_K which referred only to the (untwisted) cohomology of the orientifold. Our hope is that this language will also allow the study of general NSNS fluxes on arbitrary type IIA Calabi-Yau orientifold constructions, resulting in a greatly enriched tool-kit for model building. The major flaw right now in this plan is the Bianchi identities, which we were unable, in general, to recast in terms of the cohomological structure alone. Our hope, however, is that this difficulty can be overcome by studying explicit examples.

The most obvious extension in this direction would be to address another fairly prominent gap in our analysis, namely the incorporation of the twisted sectors. We have ignored twisted sector fluxes and moduli throughout our analysis, since we are more interested, in the present work, in elucidating the general structures that one encounters. In other contexts [15], it has been shown that, at least in specific models, it should be possible to stabilize the twisted sector moduli in such a way as to maintain a separation of scales with the bulk physics, but still trust the analysis. We hope that such considerations will still hold in many of the models discussed here. It would be very interesting to incorporate the twisted-sector cohomology into our general flux analysis (indeed if our approach is valid beyond toroidal examples, then it should be able to treat all of the cohomology democratically), possibly along the lines of [9]. In section 4.4 we will mention some ideas in this direction.

Another key point to emphasize here is the quantization of general NSNS fluxes. For H-fluxes alone, the situation is well understood; the H-flux should be understood as an element of $H^3(X;\mathbb{Z})$, or in our terms, the p_K should be integers[9]. In situations related to these by T-duality the answers are just as straightforward; all of the fluxes p_K, r_{aK}, etc. must be integers. It is natural to assume that this is generally the correct condition, especially when we are describing our fluxes in terms of integral cohomology. However, as we shall see in chapter 4, this naive quantization is not generally correct. There will be examples we can construct (which are not simply T-dual to H-flux) where the quantization condition is much more complicated (though still simple from the point of view of our constructions). This still leaves the question of how fluxes are quantized in those models that we will not succeed in constructing from a ten-dimensional point of view. In that case we do not know what the correct quantization conditions should be. It is possible that those models simply have no legitimate ten-dimensional origin. If they do, we see no route to determining the correct quantization conditions without actual constructions.

In section 2.2.3 we presented one example of a model where all moduli were stabilized at a supersymmetric AdS vacuum and the tadpole condition was saturated without the need for extra D-branes. Unfortunately, this example used only our naive quantization conditions. Using the correct quantization on NSNS fluxes which we will derive below we will find that it is no longer possible to stabilize all moduli while also satisfying both the F-term equations and the tadpole, the latter because

[9]Actually, related to our willful ignorance of the twisted sectors, we have glossed over the fact that in our example, p_K should in fact be even integers [18]; our b^K alone are not elements of the integral cohomology, but rather we must take either $nb^1 + mb^2$ with $n + m$ even, or we may take b^K + (twisted). It would be interesting to provide a more complete analysis.

the flux contributions in this case appear to be non-integral! We suspect that this problem with the tadpole is simply an artifact of our not understanding how the generalized NSNS fluxes affect the correct quantization of RR fluxes. It would be extremely gratifying to have a better grasp of these issues so as to be able to construct fully realized stable $\mathcal{N} = 1$ vacua[10].

Finally, let us turn to the issue of the regime of validity of this effective field theory. As in [15], we are able to find models (by taking some of our RR fluxes to be parametrically large, for instance $\widehat{e}_a$ in our solutions with H-flux only) in which the string coupling is small, and in which the compact directions are large enough to trust supergravity, but still much smaller than the AdS radius (which also characterizes the masses of the stabilized moduli), so that the solution would seem to be effectively four-dimensional[11]. However, just as in that situation, our models generally suffer from the concerns expressed by Banks and van den Broek [41]. Namely, due to the presence of the orientifold singularity, there are regions of our compact manifold in which the string coupling diverges (but see also [42]) and we should turn to eleven-dimensional supergravity instead. In this picture, the large flux integers translate into a large stack of M2-branes at the orientifold locus, and so the larger the flux integers, the more backreaction one has to deal with (and is ignoring in the effective description). We have not repeated this analysis in detail in our models, partly because the ten-dimensional (or eleven-dimensional) physics becomes more obscure for us, but the issue undoubtedly persists. We hope however, that our richer structure of fluxes might provide more corners in which to hide.

[10]Of course in section 2.2.3, since we have only turned on H-flux and metric flux, we do still have a global geometric description, and there should be nothing exotic about the quantization of RR fluxes. Our suspicion, however is that we have run into trouble by trying to use the language of the twisted torus, i.e. in using fluxes defined by forms inherited from T^6. For the types of metric flux used here (and similar examples in the literature), the resulting space is quite different from the original T^6, and so the quantization conditions in our chosen basis will seem quite non-standard.

[11]Note that these conditions are not preserved by T-duality.

Chapter 3

Generalized NSNS Fluxes and D-Terms in IIB

We will now follow a very similar procedure in the case of IIB. In the context of IIB, a Calabi-Yau orientifold which does not explicitly break supersymmetry (though the inclusion of fluxes will allow for spontaneous breaking) must be paired with an involution σ which is a holomorphic isometry of the Calabi-Yau three-fold X. It turns out that this still leaves two broad classes of orientifolds, differentiated by their action on the holomorphic $(3,0)$-form Ω. If $\sigma^*\Omega = -\Omega$, then the fixed point set of σ can have complex codimension three or one, so we call these O3/O7 orientifolds, while if $\sigma^*\Omega = \Omega$, then the codimension is two or zero, and we refer to O5/O9 orientifolds. The full orientifold $\mathbb{Z}_2$ action is generated by $(-1)^{F_L}\Omega_p\sigma$ in the former case and $\Omega_p\sigma$ in the latter case, where F_L is the spacetime fermion number in the left-moving sector, and Ω_p is the world-sheet parity operator.

We will treat the two cases separately, but we can use a common cohomological basis. In even degree we have

- The zero-form 1,

- a set of even two-forms μ_α, $\alpha = 1, \ldots, h^{1,1}_+$,
- a set of odd two-forms ω_a, $a = 1, \ldots, h^{1,1}_-$,
- a set of even four-forms $\widetilde{\mu}^\alpha$, $\alpha = 1, \ldots, h^{1,1}_+$,
- a set of odd four-forms $\widetilde{\omega}^a$, $a = 1, \ldots, h^{1,1}_-$,
- a six form φ, even under σ,

with intersections

$$\int_X \varphi = c, \qquad \int_X \mu_\alpha \wedge \mu_\beta \wedge \mu_\gamma = \kappa_{\alpha\beta\gamma}, \qquad \int_X \mu_\alpha \wedge \omega_a \wedge \omega_b = \widehat{\kappa}_{\alpha\, ab},$$

$$\int_X \mu_\alpha \wedge \widetilde{\mu}^\beta = \widehat{d}_\alpha{}^\beta, \qquad \int_X \omega_a \wedge \widetilde{\omega}^b = d_a{}^b. \tag{3.1}$$

In odd degree we will have both odd and even forms, and, since the volume form is even, we can construct a symplectic basis for each of $H^3_+(X)$ and $H^3_-(X)$. For $H^3_+(X)$, we will have a_K, b^K, and for $H^3_-(X)$ we will have $\mathcal{A}_k$, $\mathcal{B}^k$. The nonvanishing intersections are

$$\int_X a_K \wedge b^J = \delta^J_K, \qquad \int_X \mathcal{A}_k \wedge \mathcal{B}^j = \delta^j_k. \tag{3.2}$$

For the O3/O7 case, the index K can take values $1 \leq K \leq h^{2,1}_+$ and k can run over $0 \leq k \leq h^{2,1}_-$, with the extra index accounting for the fact that $H^{(3,0)}(X) \oplus H^{(0,3)}(X)$ is odd, while similarly for O5/O9 we have $0 \leq K \leq h^{2,1}_+$, $1 \leq k \leq h^{2,1}_-$.

3.1 The O3/O7 Case

In this case the orientifold action requires that the holomorphic three form be odd and the Kähler form be even under σ. Also, the B-field should be odd, as should the RR fields C_2 and C_6, while the RR fields C_0 and C_4 should be even. With these projections we have the expansions

$$\Omega = \mathcal{Z}^k \mathcal{A}_k - \mathcal{F}_k \mathcal{B}^k, \qquad J = v^\alpha \mu_\alpha, \qquad B = u^a \omega_a, \tag{3.3}$$

$$C_0, \qquad C_2 = c^a \omega_a, \qquad C_4 = \rho_\alpha \widetilde{\mu}^\alpha + A^K \wedge a_K, \qquad C_6 = 0.$$

Here we have not included any fluxes, nor any fields related to these by the self-duality of the RR five-form field strength in IIB. In fact it is easy to account for the latter; we would just add an extra piece to C_4,

$$C_4' = \chi^\alpha \wedge \mu_\alpha + \widetilde{A}_K \wedge b^K, \tag{3.4}$$

where χ^α is a two-form potential in spacetime which is dual to the scalar field ρ_α, and where $\widetilde{A}_K$ is the magnetic dual gauge field to A^K.

It turns out that the most convenient way to express these moduli is to follow [43][1] and define

$$\begin{aligned}
\Phi_c^{ev} &= e^B \wedge C_{RR}^{(0)} + ie^{-\phi}\,\mathrm{Re}\left(e^{B+iJ}\right) \\
&= \left(C_0 + ie^{-\phi}\right) + \left(C_2 + \left(C_0 + ie^{-\phi}\right) B\right) && (3.5)\\
&\quad + \left(C_4^{(0)} + C_2 \wedge B + \frac{1}{2}\left(C_0 + ie^{-\phi}\right) B \wedge B - \frac{i}{2} e^{-\phi} J \wedge J\right) \\
&= \tau + G^a \omega_a + T_\alpha \widetilde{\mu}^\alpha. && (3.6)
\end{aligned}$$

In this expression a superscript (0) means that only the spacetime scalar part of an

[1] Note that our convention differs from [43] in the sign of the B-field.

expansion is taken, and $C_{RR} = C_0 + C_2 + C_4$. The expansion coefficients,

$$\begin{aligned}
\tau &= C_0 + ie^{-\phi}, \\
G^a &= c^a + \tau u^a, \\
T_\alpha &= \rho_\alpha + \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \left(-\frac{i}{2} e^{-\phi} \kappa_{\beta\gamma\delta} v^\gamma v^\delta + \widehat{\kappa}_{\beta\, ab}\left(c^a u^b + \frac{1}{2}\tau u^a u^b\right)\right),
\end{aligned} \tag{3.7}$$

turn out to be a nice basis for some of the complex scalar fields in four dimensions. The remaining complex scalars are obtained from the fields $\mathcal{Z}^k$. In fact the $\mathcal{Z}^k$ form a good projective basis, and we can use $z^k = \mathcal{Z}^k/\mathcal{Z}^0$, $1 \leq k \leq h_-^{2,1}$, as a basis for the actual complex structure moduli.

The Kähler potential for these fields is then given by

$$K = -\ln\left[i \int_X \Omega \wedge \bar{\Omega}\right] - 4\ln\left[-i\left(\tau - \bar{\tau}\right)\right] - 2\ln\left[2\mathcal{V}_6\right], \tag{3.8}$$

where the volume

$$\mathcal{V}_6 = \frac{1}{6}\int_X J^3 = \frac{1}{6}\kappa_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma \tag{3.9}$$

is implicitly viewed as a function of T_α, τ, and G^a.

The holomorphic gauge kinetic couplings can also be calculated [10], though not as explicitly as in the IIA case. The procedure is to consider the expansion of the holomorphic three-form before the orientifold projection

$$\Omega^{(0)} = \mathcal{Z}^k \mathcal{A}_k - \mathcal{F}_k \mathcal{B}^k + \mathcal{X}^K a_K - \mathcal{G}_K b^K, \tag{3.10}$$

where $\mathcal{F}_k$ and $\mathcal{G}_K$ are both considered to be functions of $\mathcal{Z}^k$ and $\mathcal{X}^K$. Then the electric gauge kinetic couplings are given by

$$f_{KJ} = -\frac{i}{2}\frac{\partial}{\partial \mathcal{X}^K}\mathcal{G}_J|_{\mathcal{X}^K=0}. \tag{3.11}$$

It can be shown [10] that f_{KJ} are holomorphic functions of the complex structure moduli z^k.

The magnetic gauge kinetic couplings can also be computed by simply interchanging a_K and b^K (by a symplectic rotation) in the computation above.

Next we would like to include also a general set of fluxes. In the RR sector we can have

$$F_3 = m^k \mathcal{A}_k + e_k \mathcal{B}^k. \tag{3.12}$$

In the NSNS sector we again introduce the fluxes H_{ijk}, f^i_{jk}, Q^{ij}_k, and R^{ijk} (with the same caveats as before regarding which fluxes can be obtained from known ten-dimensional constructions). The Bianchi identities are still as given in (2.118). It is again convenient to define cohomological parameters, which in our new basis are

$$\begin{gathered} H = p^k \mathcal{A}_k + p_k \mathcal{B}^k, \\ f \cdot \mu_\alpha = \widehat{r}^K_\alpha a_K + \widehat{r}_{\alpha K} b^K, \qquad f \cdot \omega_a = r^k_a \mathcal{A}_k + r_{ak} \mathcal{B}^k, \\ Q \cdot \widetilde{\mu}^\alpha = \widehat{q}^{\alpha k} \mathcal{A}_k + \widehat{q}^\alpha_k \mathcal{B}^k, \qquad Q \cdot \widetilde{\omega}^a = q^{aK} a_K + q^a_K b^K, \\ R \cdot \varphi = s^K a_K + s_K b^K. \end{gathered} \tag{3.13}$$

Note the abuse of notation here; fluxes with upper $H^3(X)$ indices (i.e. K or k) are distinct from and independent of fluxes with lower $H^3(X)$ indices. In other words we can turn on either, both, or neither of p^k and p_k. The discouraged reader should rest assured that this situation will not propagate throughout our entire analysis; shortly we will argue that all of the fluxes with upper $H^3(X)$ indices can consistently be set to zero.

We also have the nonvanishing actions

$$H \wedge \mathcal{A}_k = -c^{-1} p_k \varphi, \qquad H \wedge \mathcal{B}^k = c^{-1} p^k \varphi,$$

$$
\begin{aligned}
f \cdot a_K &= \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{r}_{\beta K} \widetilde{\mu}^\alpha, & f \cdot b^K &= -\left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{r}_\beta^K \widetilde{\mu}^\alpha, \\
f \cdot \mathcal{A}_k &= \left(d^{-1}\right)_a{}^b r_{bk} \widetilde{\omega}^a, & f \cdot \mathcal{B}^k &= -\left(d^{-1}\right)_a{}^b r_b^k \widetilde{\omega}^a, \\
Q \cdot a_K &= -\left(d^{-1}\right)_a{}^b q_K^a \omega_b, & Q \cdot b^K &= \left(d^{-1}\right)_a{}^b q^{aK} \omega_b, \\
Q \cdot \mathcal{A}_k &= -\left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{q}_k^\alpha \mu_\beta, & Q \cdot \mathcal{B}^k &= \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{q}^{\alpha k} \mu_\beta, \\
R \cdot a_K &= c^{-1} s_K 1, & R \cdot b^K &= -c^{-1} s^K 1.
\end{aligned}
\tag{3.14}
$$

Once again, we can define an operator $\mathcal{D}$, as in (2.117), by using the same component-wise action of $H\wedge$ and the actions of the remaining fluxes from equations (2.67), (2.115), and (2.116). The Bianchi identities can still be derived by enforcing $\mathcal{D}^2 = 0$, and by demanding that $\mathcal{D}^2$ vanish on our cohomological basis we get a subset of the Bianchi identities, but naturally expressed in terms of the parameters defined above as

$$
\begin{aligned}
p^k \widehat{q}_k^\alpha - p_k \widehat{q}^{\alpha k} = \widehat{r}_\alpha^K s_K - \widehat{r}_{\alpha K} s^K &= 0, & \forall \alpha, \\
p^k r_{ak} - p_k r_a^k = q^{aK} s_K - q_K^a s^K &= 0, & \forall a, \\
\widehat{r}_{[\alpha}^K \widehat{r}_{\beta]K} = \widehat{q}_k^{[\alpha} \widehat{q}^{\beta]k} &= 0, & \forall \alpha, \beta, \\
\widehat{r}_\alpha^K q_{bK} - \widehat{r}_{\alpha K} q_b^K = \widehat{q}^{\alpha k} r_{bk} - \widehat{q}_k^\alpha r_b^k &= 0, & \forall \alpha, b, \\
r_{[a}^k r_{b]k} = q_K^{[a} q^{b]K} &= 0, & \forall a, b,
\end{aligned}
\tag{3.15}
$$

and

$$
c^{-1} p_k s_J - \left(d^{-1}\right)_a{}^b r_{bk} q_J^a + \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{q}_k^\alpha \widehat{r}_{\beta J} = 0, \qquad \forall k, J, \tag{3.16}
$$

and where (3.16) also holds with either or both of the indices k and J raised.

The equations (3.15) have a very useful interpretation. They tell us that the vectors $(\widehat{r}_\alpha^K, \widehat{r}_{\alpha K})$, (q^{aK}, q_K^a), and (s^K, s_K), are a symplectically orthogonal set with respect to the symplectic basis (a_K, b^K), and that (p_k, p^k), (r_{ak}, r_a^k), and $(\widehat{q}_k^\alpha, \widehat{q}^{\alpha k})$, are a symplectically orthogonal set with respect to $(\mathcal{A}_k, \mathcal{B}^k)$. But given any collection of symplectically orthogonal vectors there exists a symplectic transformation which rotates them so that they all lie within a canonical Lagrangian subspace. In other words, we can rotate our symplectic basis so that all vector components with an upper index vanish, and we are left with only the components carrying a lower index. This procedure is used, for example, in the case of dyonic charge vectors. The symplectic orthogonality conditions are then called mutual locality of the different charged fields, and when they are satisfied we may rotate our electric and magnetic gauge fields so that all charges are purely electric. Thus we are free to assume that all of our fluxes have only lower $H^3(X)$ indices and that all components with upper indices vanish.

Note that we are glossing over an important point, namely that if we want to map our integral symplectic basis into another integral basis, then our rotation should sit inside of $\mathrm{Sp}(n;\mathbb{Z})$. In this case our procedure is only possible if for each charge vector $q_A^{(i)} = (q_k^{(i)}; q^{k\,(i)})$, the ratios of all components are rational, i.e. if there exists some real number $g^{(i)} \geq 0$ and integers $n_A^{(i)}$ such that $q_A^{(i)} = g^{(i)} n_A^{(i)}$ (note that we do not require any relations here between different charge vectors). For instance, for a single such vector, there exists a rotation in $\mathrm{Sp}(n;\mathbb{Z})$ sending q_A to $q'_A = (g_{\max}, 0, \ldots, 0; 0, \ldots, 0)$, where $g_{\max}$ is the largest real number g with the above property (if all the original q_A were integers, then $g_{\max} = \gcd(q_A)$). If some vector does not have this property, there is no $\mathrm{Sp}(n;\mathbb{Z})$ rotation to do what we need. We can still, however, use a $\mathrm{Sp}(n)$ rotation to eliminate the unwanted fluxes, derive any formulae in the simplified situation, and then rotate back to the integral symplectic basis.

For some of the fluxes these issues are merely technicalities (because, as mentioned, we can always undo our rotation at the end), but below we will argue that some of these vectors are in fact physical charges (under the electric and dual magnetic fields of the four-dimensional effective theory) and thus should, for quantum consistency, be quantized, at least when everything has been correctly normalized. Unfortunately, to settle this question one needs to understand the correct quantization condition for these generalized NSNS fluxes. In chapter 4 it is shown how to do this in a broad class of examples with the focus on type IIA and the resulting quantization conditions are found to be quite nontrivial. The construction presented there can easily be used for IIB toroidal orientifolds, and the same conclusions will hold, but a proof of charge quantization in even this class of examples eludes us, though it holds in all cases that we have checked. In the general situation, outside of this class, it is not clear to us how to even check the result.

Neglecting these issues, this simplification means that equations (3.15) are automatically satisfied, and (3.16) reduces to just one set of equations, rather than four.

Let us turn now to potential tadpoles for space-filling RR form fields. By T-dualizing the type IIA tadpole constraint (2.124) we arrive at the IIB tadpole constraint in the presence of general fluxes,

$$\mathcal{D}F_3|_{(9-p)-form} + \left[\delta_{\mathrm{D_p-branes}}\right] = 2^{p-5}\left[\delta_{\mathrm{O_p-planes}}\right]. \tag{3.17}$$

In section 3.3.1, we will briefly discuss how one computes the O-plane contributions above. For now, let us consider this equation degree by degree.

First we have the C_4 tadpole. Since we are in the O3/O7 class of orientifolds, there is certainly the possibility of an orientifold group element with a real codimension six fixed locus, i.e. an O3-plane. Additionally, we can have spacetime-filling D3-branes sitting at points on the internal manifold. With a change of orientation, we can also have anti-D3-branes, but these will break supersymmetry. In total, the constraint reads

$$H \wedge F_3 + [\delta_{D3}] = \frac{1}{4}[\delta_{O3}], \tag{3.18}$$

or in components (integrating over X),

$$-p_k m^k + N_{D3} = \frac{1}{4} N_{O3}. \tag{3.19}$$

Next we can consider the potential C_6 tadpole. Since C_6 needs to be odd under the orientifold projection, we can only get contributions proportional to odd four-forms, i.e. the $\widetilde{\omega}^a$. There can be no contribution to this tadpole from O-planes, since O5-planes are not consistent with the O3/O7 class of orientifolds. In principle we can have D5-branes contributing, but they will necessarily break supersymmetry. To see this, recall that a supersymmetric two-cycle in our compactification manifold should be one that is calibrated by the Kähler form J. But since the orientifold projection picks out an odd two-cycle and forces J to be an even two-form, J clearly vanishes when pulled back to the D5 worldvolume (equivalently, $J \wedge \widetilde{\omega}^a = 0$). Our condition is hence,

$$-f \cdot F_3 + [\delta_{D5}] = 0, \tag{3.20}$$

where any localized contribution breaks SUSY. Thus, in a supersymmetric vacuum, we have, in components,

$$r_{ak} m^k = 0. \tag{3.21}$$

We move on to C_8, and find the result

$$Q \cdot F_3 + [\delta_{D7}] = 4\,[\delta_{O7}]\,, \tag{3.22}$$

or

$$-\widehat{q}_k^\alpha m^k + N_{D7}^\alpha = 4 N_{O7}^\alpha. \tag{3.23}$$

And finally, the C_{10} tadpole is absent, since it must be odd under the orientifold projection, but there is no odd six-cycle on the internal manifold, or equivalently, no odd zero-form.

Let us remark briefly on a special case of the above constraints. If there are no localized sources (we will present such an example in section 3.3) or if the localized sources are engineered to cancel amongst themselves (i.e. any O-plane charge is cancelled by adding D-branes), then the tadpole constraints above make the simple statement that the RR charge vector (e_k, m^k) is again symplectically orthogonal to our various NSNS vectors, which after our earlier rotations are simply $(p_k, 0)$, $(r_{ak}, 0)$, and $(\widehat{q}_k^\alpha, 0)$. In this case there will again be a symplectic rotation which will eliminate the m^k components of F_3, leaving only the e_k. The quantization issues discussed above will still be present, but we will not repeat the details. If the flux contribution to the tadpoles does not vanish however, but rather is required to cancel local source contributions, then this argument does not apply.

Next, we turn to the superpotential. Known results from solutions with H-flux and torsion allow us to use T-duality to write down the superpotential with general NSNS fluxes. We find [43]

$$W = \int_X (F_3 + \mathcal{D}\Phi_c^{ev}) \wedge \Omega. \tag{3.24}$$

Computing explicitly, we find that

$$\mathcal{D}\Phi_c^{ev} = (p_k \tau + r_{ak} G^a + \widehat{q}_k^\alpha T_\alpha)\, \mathcal{B}^k, \tag{3.25}$$

so doing the integration, we find our superpotential to be

$$W = -m^k \mathcal{F}_k - [e_k + p_k \tau + r_{ak} G^a + \widehat{q}_k^\alpha T_\alpha]\, \mathcal{Z}^k. \tag{3.26}$$

Note particularly that W is linear in the moduli τ, G^a, and T_α. Also, observe that in the presence of the nongeometric $\widehat{q}$ fluxes, the superpotential does depend on the volume moduli of the compactification, meaning that there is at least a chance to stabilize everything at tree level.

Finally, we turn to the D-terms. Proceeding as in the IIA case, we note that gauge transformations of the electric gauge fields A^K and the magnetic gauge fields $\widetilde{A}_K$ are generated by

$$\begin{aligned} C_{RR} \quad \longrightarrow \quad & C_{RR} + \mathcal{D}\left(\lambda^K a_K + \widetilde{\lambda}_K b^K\right) \\ & = \left(C_0 - c^{-1} s_K \lambda^K\right) + \left(c^a - \left(d^{-1}\right)_b{}^a q_K^b \lambda^K\right)\omega_a \\ & \quad + \left(\rho_\alpha - \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{r}_{\beta K}\lambda^K\right)\widetilde{\mu}^\alpha \\ & \quad\quad + \left(A^K + d\lambda^K\right)\wedge a_K + \left(\widetilde{A}_K + d\widetilde{\lambda}_K\right)\wedge b^K. \end{aligned} \tag{3.27}$$

Thus the fields τ, G^a, and T_α can all potentially get variations under electric gauge transformations by turning on our general fluxes. Observe that if we had not performed a symplectic rotation of the general fluxes, then both electric and magnetic charges would have been possible, and that indeed the symplectic vectors discussed above would be precisely the dyonic charge vectors, as promised. Note also that the fluxes which contribute to charges of these fields are a complementary set to the fluxes which can appear in the superpotential and tadpole constraints.

The D-terms which result from these variations are

$$\begin{aligned} D_K &= -i\left[c^{-1} s_K \partial_\tau K + \left(d^{-1}\right)_b{}^a q_K^b \partial_a K + \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{r}_{\beta K}\partial^\alpha K\right] \qquad (3.28) \\ &= \frac{e^\phi}{2\mathcal{V}_6}\left[\left(\mathcal{V}_6 - \frac{1}{2}\left(\widehat{\kappa} v u^2\right)\right)c^{-1} s_K + \left(d^{-1}\right)_a{}^b \widehat{\kappa}_{\alpha bc} v^\alpha u^c q_K^a - v^\alpha \widehat{r}_{\alpha K}\right]. \end{aligned}$$

We will see how this works in a specific example below.

3.2 The O5/O9 Case

This case is quite similar to the previous case, so we shall be fairly brief in our description. The holomorphic involution σ now satisfies $\sigma^*\Omega = \Omega$, and the projection on the RR sector is reversed relative to the O3/O7 case (because the projection is no longer accompanied by a factor of $(-1)^{F_L}$), so we are left with the expansions

$$\Omega = \mathcal{Z}^K a_K - \mathcal{F}_K b^K, \qquad J = v^\alpha \mu_\alpha, \qquad B = u^a \omega_a, \tag{3.29}$$

$$C_0 = 0, \qquad C_2 = c^\alpha \mu_\alpha, \qquad C_4 = \rho_a \widetilde{\omega}^a + A^k \wedge \mathcal{A}_k, \qquad C_6 = \gamma \varphi.$$

There are of course also the dual pieces of C_4. Also, the field γ in C_6 is dual to a spacetime two-form field from C_2, but we prefer to work with spacetime scalars in our description.

It is again convenient to introduce the formal sum of forms [43]

$$\begin{aligned}
\Phi_c^{ev} &= e^B \wedge C_{RR}^{(0)} + ie^{-\phi}\,\mathrm{Im}\left(e^{B+iJ}\right) \\
&= \Big(C_2 + ie^{-\phi} J\Big) + \Big(C_4^{(0)} + C_2 \wedge B + ie^{-\phi} B \wedge J\Big) \qquad (3.30) \\
&\quad + \left(C_6 + C_4^{(0)} \wedge B + \frac{1}{2} C_2 \wedge B \wedge B + ie^{-\phi}\left(-\frac{1}{6} J \wedge J \wedge J + \frac{1}{2} J \wedge B \wedge B\right)\right) \\
&= t^\alpha \mu_\alpha + L_a \widetilde{\omega}^a + S\varphi, \qquad (3.31)
\end{aligned}$$

with

$$\begin{aligned}
t^\alpha &= c^\alpha + ie^{-\phi} v^\alpha, \\
L_a &= \rho_a + \left(d^{-1}\right)_a{}^b \widehat{\kappa}_{\alpha\, bc} t^\alpha u^c, \qquad (3.32) \\
S &= \gamma + c^{-1}\left[d_a{}^b \rho_b u^a + \frac{1}{2}\widehat{\kappa}_{\alpha\, ab} t^\alpha u^a u^b - \frac{i}{6} e^{-\phi} \kappa_{\alpha\beta\gamma} v^\alpha v^\beta v^\gamma\right].
\end{aligned}$$

These fields, t^α, L_a, and S, are good holomorphic coordinates on the moduli space, and should be combined with projective coordinates for the complex structure deformations,

$$z^K = \mathcal{Z}^K / \mathcal{Z}^0, \qquad 1 \leq K \leq h_K^{2,1}. \tag{3.33}$$

The Kähler potential is given by the same expression as before,

$$K = -\ln\left[i\int_X \Omega\wedge\bar{\Omega}\right] - 4\ln\left[e^{-\phi}\right] - 2\ln\left[8\mathcal{V}_6\right], \qquad (3.34)$$

but should now be viewed as an implicit function of z^K, t^α, L_a, and S.

To compute the gauge kinetic couplings we follow the same procedure as before, constructing the three-form before the orientifold projection

$$\Omega^{(0)} = \mathcal{Z}^K a_K - \mathcal{F}_K b^K + \mathcal{X}^k \mathcal{A}_k - \mathcal{G}_k \mathcal{B}^k, \qquad (3.35)$$

with $\mathcal{F}_K$ and $\mathcal{G}_k$ considered as functions of $\mathcal{Z}^K$ and $\mathcal{X}^k$. We then have

$$f_{kj} = -\frac{i}{2}\frac{\partial}{\partial\mathcal{X}^k}\mathcal{G}_j|_{\mathcal{X}^k=0}. \qquad (3.36)$$

Next we turn to fluxes. In the RR sector, we have simply

$$F_3 = m^K a_K + e_K b^K. \qquad (3.37)$$

In the NSNS sector we have precisely the same expansion (3.13) as before, with the same Bianchi identities (3.15) and (3.16), and where again we can rotate our symplectic basis so that only "electric" fluxes remain.

We again expect the tadpole to be given by (3.17), but now the degree-by-degree comparison will be different.

There is no possible C_4 tadpole, since a space-filling C_4 field is projected out by the orientifold.

There are possible C_6 tadpoles,

$$-f\cdot F_3 + [\delta_{D5}] = [\delta_{O5}], \qquad (3.38)$$

or

$$-\,\widehat{r}_{\alpha K} m^K + N_\alpha^{D5} = N_\alpha^{O5}. \tag{3.39}$$

For C_8 we have

$$Q \cdot F_3 + [\delta_{D7}] = 0, \tag{3.40}$$

with the caveat that any D7-branes surviving the orientifold projection are necessarily non-supersymmetric. In components, in the supersymmetric case, we find

$$q_K^a m^K = 0. \tag{3.41}$$

Finally, there is a potential C_{10} tadpole

$$-\,R \cdot F_3 + [\delta_{D9}] = 16\,[\delta_{O9}]\,, \tag{3.42}$$

or

$$-\,s_K m^K + N_{D9} = 16 N_{O9}. \tag{3.43}$$

Once again, in the absence of localized sources, a further symplectic rotation can also eliminate the m^K components of F^3.

The superpotential is actually given by the same general expression (3.24), but where the expansion now reads

$$W = -m^K \mathcal{F}_K - \left[e_K + t^\alpha \widehat{r}_{\alpha K} + q_K^a L_a + s_K S\right] \mathcal{Z}^K. \tag{3.44}$$

It remains only to compute the D-terms. We find that under the standard gauge variation,

$$
\begin{aligned}
C_{RR} \quad \longrightarrow \quad & C_{RR} + \mathcal{D}\left(\lambda^k \mathcal{A}_k + \widetilde{\lambda}_k \mathcal{B}^k\right) \\
& = \left(c^\alpha - \left(\widehat{d}^{-1}\right)_\beta{}^\alpha \widehat{q}_k^\beta \lambda^k\right)\mu_\alpha + \left(\rho_a - \left(d^{-1}\right)_a{}^b r_{bk}\lambda^k\right)\widetilde{\omega}^a \\
& \quad + \left(\gamma - c^{-1}p_k\lambda^k\right)\varphi + \left(A^k + d\lambda^k\right)\wedge \mathcal{A}_k + \left(\widetilde{A}_k + d\widetilde{\lambda}_k\right)\wedge \mathcal{B}^k.
\end{aligned} \tag{3.45}
$$

Finally, we can compute the D-terms,

$$
\begin{aligned}
D_k &= -i\left[\left(\widehat{d}^{-1}\right)_\beta{}^\alpha \widehat{q}_k^\beta \partial_\alpha K + \left(d^{-1}\right)_a{}^b r_{bk}\partial^a K + c^{-1}p_k \partial_S K\right] \\
&= \frac{e^\phi}{2\mathcal{V}_6}\left\{\frac{1}{2}\left(\widehat{d}^{-1}\right)_\beta{}^\alpha \widehat{q}_k^\beta \left(\kappa_{\alpha\gamma\delta}v^\gamma v^\delta - \widehat{\kappa}_{\alpha\, ab}u^a u^b\right) + r_{ak}u^a - p_k\right\}.
\end{aligned} \tag{3.46}
$$

Note that the last term above, proportional to p_k, matches the result found in [10].

3.3 Examples

In this section we will work out explicitly the example of D-terms arising in type IIB supergravity compactified on the orbifold $T^6/\mathbb{Z}_4$ with an $O3/O7$ orientifold. A similar example of an $O5/O9$ orientifold can be obtain by slightly modifying the holomorphic involution as explained below.

Before we launch into a description of the example we have in mind, it is worth briefly commenting about why IIB examples that exhibit D-terms are somewhat difficult to find. Consider the O3/O7 case. In order to have a possibility for D-terms we need to have $h_+^{2,1} > 0$, so that we have four-dimensional vectors, and we also need either nongeometric s- or q-fluxes, or else we need metric $\widehat{r}$-fluxes to act as charges. In fact, since most studies of generalized NSNS fluxes (present work included) have really focused on the untwisted sectors of a toroidal orientifold[2], we actually want $h^{2,1}_{+\,\text{untwisted}} > 0$. But nearly all standard examples of O3/O7 toroidal orientifolds use $\sigma = \mathcal{I}_6$, a reflection of all internal coordinates. Under such an involution, of course all untwisted three-forms are odd. Other common examples start with a factorized orbifold of $(T^2)^3$, and take an involution which reflects one of the two-tori, but here too one can show (assuming the orbifold did not enjoy enhanced supersymmetry) that all untwisted three-forms are odd. So we need to look for a slightly more involved example, which we will describe below.

3.3.1 O3/O7 on $T^6/\mathbb{Z}_4$

We start by explicitly spelling out the orbifold and orientifold action and the resulting cohomology. Then we discuss the H, metric and non-geometric fluxes and how they map to cohomological parameters. Finally we write down explicitly the D-terms and the superpotential.

Let $z^1 = x^1 + ix^2 + e^{\pi \mathrm{i}/4}(x^3 + \mathrm{i}x^4)$, $z^2 = x^3 + ix^4 + e^{3\pi \mathrm{i}/4}(x^1 + \mathrm{i}x^2)$, and $z^3 = x^5 + \mathrm{i}x^6$ be complex coordinates on the tori with the identifications $x^i = x^i + 1$. The orientifold group is generated by a $\mathbb{Z}_4$ rotation

$$\Theta : \left(z^1, z^2, z^3\right) \longrightarrow \left(\mathrm{i}z^1, \mathrm{i}z^2, -z^3\right), \tag{3.47}$$

and the orbifold action is $\Omega_p(-1)^{F_L}\sigma$, where the holomorphic involution σ acts as

$$\sigma : \left(z^1, z^2, z^3\right) \longrightarrow \left(-e^{\pi \mathrm{i}/4} z^1, e^{\pi \mathrm{i}/4} z^2, -\mathrm{i}z^3\right). \tag{3.48}$$

Note that $\sigma^2 = \Theta$, so the full orientifold group is in fact $\mathbb{Z}_8$. More specifically, for those familiar with classifications of crystallographic actions on T^6, if we pair σ with a reflection in all six coordinates, the element $\sigma\mathcal{I}_6$ generates the crystallographic

[2]For a way of including part of the untwisted sector see [9].

group $\mathbb{Z}_{8-I}$ (see for instance the review [44]). This particular orientifold is discussed by [45].

We can now write down the untwisted cohomology of $T^6/\mathbb{Z}_4$, dividing further into subspaces which are even or odd under the involution σ. We start with the even cohomology, implicitly equating classes with their harmonic form representatives. There is one even zero form, namely the unit function 1. For two-forms, there are five independent $(1,1)$-forms invariant under the rotations: three even forms,

$$
\begin{aligned}
\mu_1 &= \frac{i}{4}\left(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2\right) = dx^1 \wedge dx^2 + dx^3 \wedge dx^4, \\
\mu_2 &= \frac{i}{2\sqrt{2}}\left(dz^1 \wedge d\bar{z}^1 - dz^2 \wedge d\bar{z}^2\right) \\
&= dx^1 \wedge dx^3 + dx^1 \wedge dx^4 - dx^2 \wedge dx^3 + dx^2 \wedge dx^4, \\
\mu_3 &= \frac{i}{2} dz^3 \wedge d\bar{z}^3 = dx^5 \wedge dx^6,
\end{aligned}
$$

and two odd forms

$$
\begin{aligned}
\omega_1 &= \frac{1-\mathrm{i}}{4}\left(dz^1 \wedge d\bar{z}^2 + \mathrm{i} d\bar{z}^1 \wedge dz^2\right) \\
&= dx^1 \wedge dx^3 - dx^1 \wedge dx^4 + dx^2 \wedge dx^3 + dx^2 \wedge dx^4, \\
\omega_2 &= -\frac{e^{-\pi \mathrm{i}/4}}{4}\left(dz^1 \wedge d\bar{z}^2 - \mathrm{i} d\bar{z}^1 \wedge dz^2\right) = dx^1 \wedge dx^2 - dx^3 \wedge dx^4.
\end{aligned}
$$

Similarly, for four-forms we have three even $(2,2)$-forms

$$
\begin{aligned}
\widetilde{\mu}^1 &= \mu_1 \wedge \mu_3, \qquad \widetilde{\mu}^2 = \mu_2 \wedge \mu_3, \\
\widetilde{\mu}^3 &= dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 = \frac{1}{2}\,\mu_1 \wedge \mu_1 = -\frac{1}{4}\,\mu_2 \wedge \mu_2 \\
&\qquad = -\frac{1}{4}\,\omega_1 \wedge \omega_1 = -\frac{1}{2}\,\omega_2 \wedge \omega_2,
\end{aligned}
$$

and two odd $(2,2)$-form,

$$\widetilde{\omega}^1 = \omega_1 \wedge \mu_3 \qquad \widetilde{\omega}^2 = \omega_2 \wedge \mu_3. \tag{3.49}$$

Finally there is one six-form, which is even under the involution,

$$\varphi = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6. \tag{3.50}$$

For the intersection numbers we find $c = \frac{1}{4}$, $\widehat{d}_\alpha{}^\beta = diag\left(\frac{1}{2}, -1, \frac{1}{4}\right)$ and $d_a{}^b = diag\left(-1, -\frac{1}{2}\right)$. The only non vanishing components of the totally symmetric triple intersections are $\kappa_{113} = \frac{1}{2}$, $\kappa_{223} = -1$ and $\widehat{\kappa}_{311} = -1$, $\widehat{\kappa}_{322} = -\frac{1}{2}$.

In particular, the Kähler form will be given by $J = v^1\mu_1 + v^2\mu_2 + v^3\mu_3$, and the corresponding metric (in the absence of fluxes) is

$$\begin{aligned} ds^2 &= v^1\left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\right) \\ &\quad +2v^2\left(dx^1dx^3 - dx^1dx^4 + dx^2dx^3 + dx^2dx^4\right) \\ &\quad +v^3\left((dx^5)^2 + (dx^6)^2\right). \end{aligned} \tag{3.51}$$

The conditions that the metric be Euclidean signature are that $v^1 > 0$, $v^3 > 0$, and that $(v^1)^2 > 2(v^2)^2$. The volume is

$$\mathcal{V}_6 = \frac{1}{4}v^3\left(\left(v^1\right)^2 - 2\left(v^2\right)^2\right). \tag{3.52}$$

Next we have the odd cohomology. It turns out that $H^1(X)$ and $H^5(X)$ are empty, so we need only describe the three-forms. Since there are only four we drop the index and simply write

$$\begin{aligned}
a &= -\frac{\mathrm{i}}{2}\left(dz^1\wedge dz^2\wedge d\bar{z}^3 - d\bar{z}^1\wedge d\bar{z}^2\wedge dz^3\right) = -\chi^{136}+\chi^{145}+\chi^{235}+\chi^{246},\\
b &= \frac{1}{2}\left(dz^1\wedge dz^2\wedge d\bar{z}^3 + d\bar{z}^1\wedge d\bar{z}^2\wedge dz^3\right) = \chi^{135}-\chi^{245}+\chi^{146}+\chi^{236},\\
\mathcal{A} &= \frac{1}{2}\left(dz^1\wedge dz^2\wedge dz^3 + d\bar{z}^1\wedge d\bar{z}^2\wedge d\bar{z}^3\right) = \chi^{135}-\chi^{245}-\chi^{146}-\chi^{236},\\
\mathcal{B} &= -\frac{\mathrm{i}}{2}\left(dz^1\wedge dz^2\wedge dz^3 - d\bar{z}^1\wedge d\bar{z}^2\wedge d\bar{z}^3\right) = \chi^{136}+\chi^{145}+\chi^{235}-\chi^{246}.
\end{aligned}$$

Here we use notation where $\chi^{145} = dx^1\wedge dx^4\wedge dx^5$, etc. The holomorphic three form

$$\Omega = \frac{1}{\sqrt{2}}\,dz^1\wedge dz^2\wedge dz^3 = \mathcal{Z}\mathcal{A}-\mathcal{F}\mathcal{B} = \frac{1}{\sqrt{2}}(\mathcal{A}+\mathrm{i}\mathcal{B}) \tag{3.53}$$

is odd under σ so that we have the O3/O7 case. The normalization has been chosen so that $i\int_X \Omega\wedge\bar{\Omega} = 1$, and the phase chosen so that $\mathcal{Z}^0$ is real and positive, but these are arbitrary choices. Note that there are no complex structure moduli in this example.

We will now enumerate the general (untwisted) NSNS fluxes that are consistent with the orientifold action. First we expand $H = p^1\mathcal{A} + p_1\mathcal{B}$ where the parameters are $p^1 = H_{135} = -H_{245} = -H_{146} = -H_{236}$ and $p_1 = H_{136} = H_{145} = H_{235} = -H_{246}$.

Now proceed analogously for the other fluxes arising in the NSNS sector. Imposing invariance under the orientifold group, we find that we are left with ten independent metric fluxes,

$$
\begin{aligned}
f^1_{15} &= -f^2_{25} = -f^3_{36} = f^4_{46}, \\
f^1_{16} &= -f^2_{26} = f^3_{35} = -f^4_{45}, \\
f^1_{25} &= f^2_{15} = -f^3_{46} = -f^4_{36}, \\
f^1_{26} &= f^2_{16} = f^3_{45} = f^4_{35}, \\
f^1_{35} &= -f^2_{45} = -f^3_{26} = -f^4_{16}, \\
f^1_{36} &= -f^2_{46} = f^3_{25} = f^4_{15}, \\
f^1_{45} &= f^2_{35} = f^3_{16} = -f^4_{26}, \\
f^1_{46} &= f^2_{36} = -f^3_{15} = f^4_{25}, \\
f^5_{13} &= -f^5_{24} = f^6_{14} = f^6_{23}, \\
f^5_{14} &= f^5_{23} = -f^6_{13} = f^6_{24},
\end{aligned}
$$

where we can use the ten fluxes in the left-hand column as representatives.

In terms of r-matrices, we find

$$
r^1_a = \begin{pmatrix} -f^1_{15} - f^1_{16} - f^1_{25} + f^1_{26} \\ -f^1_{36} - f^1_{45} \end{pmatrix}, \qquad r_{a1} = \begin{pmatrix} f^1_{15} - f^1_{16} - f^1_{25} - f^1_{26} \\ f^1_{35} - f^1_{46} \end{pmatrix}, \tag{3.54}
$$

$$
\widehat{r}^1_\alpha = \begin{pmatrix} f^1_{35} + f^1_{46} \\ -f^1_{15} + f^1_{16} - f^1_{25} - f^1_{26} \\ -f^5_{13} \end{pmatrix}, \qquad \widehat{r}_{\alpha 1} = \begin{pmatrix} f^1_{36} - f^1_{45} \\ -f^1_{15} - f^1_{16} + f^1_{25} - f^1_{26} \\ f^5_{14} \end{pmatrix}. \tag{3.55}
$$

Note that there is a one-to-one correspondence between the independent fluxes f^i_{jk} and the entries of r and $\widehat{r}$. If we consider only these metric fluxes and set $r^1_a = r_{a1} = 0$

we are left with the following Bianchi identities

$$\widehat{r}_{\gamma 1}\,\widehat{r}_{31} + \widehat{r}^{1}_{\gamma}\,\widehat{r}^{1}_{3} = 0,\ \gamma = 1,2, \quad (\widehat{r}_{11})^2 + (\widehat{r}^{1}_{1})^2 - \frac{(\widehat{r}_{21})^2}{2} - \frac{(\widehat{r}^{1}_{2})^2}{2} = 0, \quad \widehat{r}^{1}_{[\alpha}\,\widehat{r}_{\beta]1} = 0. \tag{3.56}$$

Note that only the last Bianchi identity arises from demanding that $\mathcal{D}^2$ vanishes when acting on the invariant forms given above (cf. (3.15)). One solution to (3.56) which gives a D-term is for example to turn on only the components $\widehat{r}^{1}_{3}$ and $\widehat{r}_{31}$.

The Q-fluxes which survive the orientifold projection are

$$\begin{aligned}
Q^{13}_{5} &= -Q^{14}_{6} = -Q^{23}_{6} = -Q^{24}_{5},\\
Q^{13}_{6} &= Q^{14}_{5} = Q^{23}_{5} = -Q^{24}_{6},\\
Q^{15}_{1} &= -Q^{25}_{2} = Q^{36}_{3} = -Q^{46}_{4},\\
Q^{15}_{2} &= Q^{25}_{1} = Q^{36}_{4} = Q^{46}_{3},\\
Q^{15}_{3} &= -Q^{25}_{4} = Q^{36}_{2} = Q^{46}_{1},\\
Q^{15}_{4} &= Q^{25}_{3} = -Q^{36}_{1} = Q^{46}_{2},\\
Q^{16}_{1} &= -Q^{26}_{2} = -Q^{35}_{3} = Q^{45}_{4},\\
Q^{16}_{2} &= Q^{26}_{1} = -Q^{35}_{4} = -Q^{45}_{3},\\
Q^{16}_{3} &= -Q^{26}_{4} = -Q^{35}_{2} = -Q^{45}_{1},\\
Q^{16}_{4} &= Q^{26}_{3} = Q^{35}_{1} = -Q^{45}_{2},
\end{aligned}$$

where we take the ten fluxes in the left-hand column as representatives. In terms of q-matrices, we find

$$q^{a1} = \begin{pmatrix} -Q^{15}_{1} - Q^{15}_{2} + Q^{16}_{1} - Q^{16}_{2} \\ -Q^{15}_{4} + Q^{16}_{3} \end{pmatrix}, \qquad q^{a}_{1} = \begin{pmatrix} Q^{15}_{1} - Q^{15}_{2} + Q^{16}_{1} + Q^{16}_{2} \\ Q^{15}_{3} + Q^{16}_{4} \end{pmatrix}, \tag{3.57}$$

$$\widehat{q}^{\alpha 1} = \begin{pmatrix} -Q_3^{15} + Q_4^{16} \\ Q_1^{15} + Q_2^{15} + Q_1^{16} - Q_2^{16} \\ Q_5^{13} \end{pmatrix}, \qquad \widehat{q}_1^{\alpha} = \begin{pmatrix} -Q_4^{15} - Q_3^{16} \\ -Q_1^{15} + Q_2^{15} + Q_1^{16} + Q_2^{16} \\ Q_6^{13} \end{pmatrix}. \quad (3.58)$$

Note that there is a one-to-one correspondence between the independent fluxes Q^i_{jk} and the entries of q and $\widehat{q}$.

Finally we find $s^1 = -R^{135} = R^{245} = -R^{146} = -R^{236}$ and $s_1 = -R^{136} = R^{145} = R^{235} = R^{246}$.

For a way to understand from a ten-dimensional point which of the NSNS fluxes discussed so far can be turned on see chapter 4. The base-fiber constructions described there can easily be adapted to the IIB case to give a large class of ten-dimensional constructions with H-flux, metric flux, and Q-flux.

If we demand that $\mathcal{D}^2$ vanishes we find the Bianchi identities derived above in equation (2.118). This simplifies if we do a symplectic rotation so that only electric fluxes with lower k, K indices are non-zero. Then we have to satisfy

$$p_1 q_1^c = 0,\ p_1 \widehat{q}_1^{\gamma} = 0,\ r_{c1}\widehat{r}_{31} = 0,\ \widehat{r}_{\gamma 1}\widehat{r}_{31} = 0,\ c, \gamma = 1, 2, \quad (3.59)$$

$$4 p_1 \widehat{q}_1^3 + \frac{(r_{11})^2}{2} + (r_{21})^2 + (\widehat{r}_{11})^2 - \frac{(\widehat{r}_{21})^2}{2} = 0, \quad (3.60)$$

$$s_1 \widehat{r}_{\alpha 1} = 0,\ \alpha = 1, 2, 3,\ \widehat{q}_1^{\gamma}\widehat{q}_1^3 = 0, s_1 r_{c1} - q_1^c \widehat{q}_1^3 = 0,\ \ c, \gamma = 1, 2, \quad (3.61)$$

$$\frac{(q_1^1)^2}{2} + (q_1^2)^2 + (\widehat{q}_1^1)^2 - \frac{(\widehat{q}_1^2)^2}{2} = 0, \quad (3.62)$$

$$r_{c1}\widehat{q}_1^{\gamma} = 0,\ \widehat{r}_{\gamma 1} q_1^c = 0, q_1^c \widehat{r}_{31} = 0\, c, \gamma = 1, 2,\ p_1 s_1 = 0,\ \widehat{r}_{31}\widehat{q}_1^3 = 0, \quad (3.63)$$

$$r_{11} q_1^1 + 2 q_1^2 r_{21} = 0,\ r_{11} q_1^2 - q_1^1 r_{21} = 0, \quad (3.64)$$

$$2\widehat{r}_{11}\widehat{q}_1^1 - \widehat{r}_{21}\widehat{q}_1^2 = 0, \widehat{r}_{11}\widehat{q}_1^2 - \widehat{r}_{21}\widehat{q}_1^1 = 0. \quad (3.65)$$

To determine the holomorphic gauge kinetic coupling we need to consider the expansion of the holomorphic three-form Ω before the orientifold projection. Therefore we write $z^3 = x^5 + \tau x^6$ where τ is the complex structure modulus that will get fixed by the orientifold projection to $\tau = \mathrm{i}$. If we keep the real three forms as defined above in term of the dx^i then we find (here we are using our freedom to not choose a normalization so that $\mathcal{Z}^0$ is as before)

$$\Omega = \frac{\sqrt{2}}{(1-\mathrm{i}\tau)} dz^1 \wedge dz^2 \wedge dz^3 = \mathcal{Z}\mathcal{A} - \mathcal{F}\mathcal{B} + \mathcal{X}a - \mathcal{G}b \tag{3.66}$$

$$= \frac{1}{\sqrt{2}}\left((\mathcal{A} + \mathrm{i}\mathcal{B}) + \frac{(\mathrm{i}-\tau)}{(1-\mathrm{i}\tau)} (a - \mathrm{i}b) \right). \tag{3.67}$$

There is thus a very simple relation $\mathcal{G} = i\mathcal{X}$ for all τ, and the electric gauge kinetic coupling is given by

$$f = -\frac{i}{2} \partial_{\mathcal{X}} \mathcal{G}|_{\mathcal{X}=0} = \frac{1}{2}. \tag{3.68}$$

The D-term in our example in the gauge where all charges are electric i.e., have lower indices, is

$$D = \frac{e^{\phi}}{2\mathcal{V}_6} \left(\left(4\mathcal{V}_6 + v^3 \left(2(u^1)^2 + (u^2)^2 \right) \right) s_1 + v^3 \left(u^1 q_1^1 + u^2 q_1^2 \right) - v^1 r_{11} - v^2 r_{21} - v^3 r_{31} \right). \tag{3.69}$$

So the contribution from the D-term to the potential is

$$V_D = \frac{1}{2} (\mathrm{Re}\, f)^{-1} D^2 \tag{3.70}$$

$$= \frac{e^{2\phi}}{4(\mathcal{V}_6)^2} \left[\left(4\mathcal{V}_6 + v^3 \left(2(u^1)^2 + (u^2)^2 \right) \right) s_1 + v^3 \left(u^1 q_1^1 + u^2 q_1^2 \right) - v^1 r_{11} - v^2 r_{21} - v^3 r_{31} \right]^2 .$$

Finally we need to consider the tadpole constraints in this model. We will start with a more general (but very brief) discussion of orientifold tadpoles. Recall that the elements of a general orientifold group G is a $\mathbb{Z}_2$ extension of an orbifold group H,

$$1 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{Z}_2 \longrightarrow 1, \tag{3.71}$$

where the elements of G that are not in the image of H are to be paired with the worldsheet parity operator Ω_p (and possibly a factor of $(-1)^{F_L}$). Put more simply, we can find a spacetime symmetry σ such that

$$G = H \cup (H\sigma\Omega_p)\,, \tag{3.72}$$

and we require that H be a group of spacetime symmetries, and that $(H\sigma)^2 \subseteq H$. There is a twisted sector of states for each element $h \in H$, but no twisted sectors corresponding to the elements $h\sigma\Omega_p$. For the example at hand, $H = \langle\Theta\rangle = \mathbb{Z}_4$, and σ is as given in equation (3.48), with $\sigma^2 = \Theta$. Each element $h\sigma\Omega_p$, $h \in H$, generates a tadpole, via a crosscap diagram, for a RR field in the $(h\sigma)^2$-twisted sector, localized at $h\sigma$-fixed points.

Consider now the potential tadpoles in our example. There are tadpole contributions to the Θ-twisted sector from $\sigma\Omega_p$ and $\sigma\Theta^2\Omega_p$, and contributions to the Θ^3-twisted sector from $\sigma\Theta\Omega_p$ and $\sigma\Theta^3\Omega_p$. There are no untwisted-sector tadpoles, so we only need to worry about possible twisted-sector (or fractional) O-planes. In fact, the relevant crosscap diagrams are computed in [45, 46], and for this particular model it is shown that the two contributions to each twisted sector cancel. There are no localized tadpoles in this model to worry about, and in particular no need to add any D-branes. In this case we can choose a symplectic basis $(\mathcal{A}, \mathcal{B})$ which preserves the form of the NSNS fluxes above and in which we have the simple relation $F_3 = e\mathcal{B}$ and the tadpole constraints are automatically satisfied.

With this we can very simply write down the superpotential (dropping the redundant subscript 1)

$$W = -\frac{1}{\sqrt{2}}\left[e + p\tau + r_a G^a + \widehat{q}^{\alpha} T_{\alpha}\right]. \tag{3.73}$$

3.3.2 An O5/O9 Example

There is a closely related example of the O5/O9 type which also exhibits D-terms. The construction is the same as above except that we take our involution to be $\sigma' = \sigma \mathcal{I}_6$, with σ as in (3.48). In this case the full orientifold group is $\mathbb{Z}_{8-I}$.

Of course there really is not any new physics; this O5/O9 construction is in fact precisely T-dual to the O3/O7 construction above, by dualizing the x^5 and x^6 coordinates. For this reason the tadpoles also continue to cancel.

Chapter 4

Base-Fiber Approach

In this section we will attempt to put a subset of our class of models on firmer ground by presenting ten-dimensional constructions. These constructions are very much in the spirit of [11] (see also [47, 48, 49, 50, 51, 52]) and are built by allowing a torus fiber to vary over a torus base, but in a way that still admits a generalized Scherk-Schwarz reduction. The NSNS fluxes will be represented by the global twists in the fibers as one transports them around non-contractible cycles in the base. We will find that the Bianchi identities come out naturally, that dualities are implemented very easily, and that the correct quantization conditions are both obvious in this context, and also much more subtle than one would have guessed.

4.1 The T-duality group $O(6,6;\mathbb{Z})$

The T-duality group of type II superstring theory compactified on a d-dimensional torus T^d is denoted $\mathrm{O}(d,d;\mathbb{Z})$ and is defined as follows

$$\mathrm{O}(d,d;\mathbb{Z}) = \left\{ M \in \mathrm{Mat}_{2d\times 2d}(\mathbb{Z}) | MLM^T = L \right\}, \tag{4.1}$$

where

$$L = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix} . \tag{4.2}$$

We will in fact focus primarily on elements with determinant one, which correspond to dualities from IIA to itself (or IIB to itself); elements with determinant minus one interchange solutions of IIA and IIB.

To understand the action of this group on the NSNS sector, it is convenient to combine the torus metric and B-field into a single $d \times d$ matrix $E = G + B$. We assume implicitly here that our coordinate basis is chosen such that each coordinate is periodic with unit period. Let us take an $\mathrm{O}(d, d; \mathbb{Z})$ matrix M and write it in terms of $d \times d$ blocks,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} . \tag{4.3}$$

Then the action of M on the NSNS sector is

$$E \mapsto E' = (aE + b)\,(cE + d)^{-1}\,, \qquad e^{\phi} \mapsto e^{\phi'} = e^{\phi} \left(\frac{\det G'}{\det G} \right)^{1/4} . \tag{4.4}$$

There is a useful alternative phrasing of this transformation. From G and B we can define a symmetric $2d \times 2d$ matrix

$$H = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} . \tag{4.5}$$

Then an element $M \in \mathrm{O}(d, d; \mathbb{Z})$ simply acts by

$$H \mapsto H' = M^T H M. \tag{4.6}$$

From this we can identify certain important elements of $\mathrm{O}(d, d; \mathbb{Z})$. For instance, it includes changes of basis for the lattice which defines T^d. These basis changes lie in the subgroup $\mathrm{GL}(d;\mathbb{Z}) \subset \mathrm{O}(d,d;\mathbb{Z})$ of matrices with the form

$$\hat{g} = \begin{pmatrix} \left(g^T\right)^{-1} & 0 \\ 0 & g \end{pmatrix}, \qquad g \in \mathrm{GL}(d;\mathbb{Z}). \tag{4.7}$$

Similarly, we also have constant integral shifts in the periods of the B-field given by matrices

$$\begin{pmatrix} \mathbf{1}_d & b \\ 0 & \mathbf{1}_d \end{pmatrix}, \qquad b^T = -b. \tag{4.8}$$

Finally there is one more type of element which will be of interest to us, corresponding simply to T-dualizing a sub-torus T^k of T^d, for example that corresponding to the first k coordinates. Then the relevant M is

$$M_k = \begin{pmatrix} 0 & 0 & \mathbf{1}_k & 0 \\ 0 & \mathbf{1}_{d-k} & 0 & 0 \\ \mathbf{1}_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{d-k} \end{pmatrix}. \tag{4.9}$$

From this and the transformation rules (4.4) above, one can compute the usual Buscher rules.

It will also be useful to know how elements of $\mathrm{O}(d,d)$ act on the RR fluxes and potentials, which can be thought of as sections of the spin bundle $\mathrm{Spin}(d,d)$. The action in this case cannot be expressed as simply as in the cases above [53], but it is not hard to write down for certain simple cases which can then be used to generate all of $\mathrm{O}(d,d)$ [54]. In particular, we have three cases.

If B_{ij} is an antisymmetric $d \times d$ matrix, so that $B = \frac{1}{2}B_{ij}dx^i \wedge dx^j$ is a two-form on the torus, then the element

$$g = \begin{pmatrix} \mathbf{1}_d & B \\ 0 & \mathbf{1}_d \end{pmatrix} \tag{4.10}$$

acts as

$$g \cdot F_{RR} = \exp(B) \wedge F_{RR} = F_{RR} + B \wedge F_{RR} + \frac{1}{2} B \wedge B \wedge F_{RR} + \cdots , \tag{4.11}$$

where $F_{RR} = \sum_a F_a$ is the sum of RR fluxes of various degrees.

Similarly, if β^{ij} is antisymmetric, $\beta = \frac{1}{2}\beta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$ an antisymmetric bivector, then

$$\begin{pmatrix} \mathbf{1}_d & 0 \\ \beta & \mathbf{1}_d \end{pmatrix} \cdot F_{RR} = \exp(\iota_\beta) \cdot F_{RR}, \tag{4.12}$$

where $\iota_\beta = \frac{1}{2}\beta^{ij}\iota_{\partial_i}\iota_{\partial_j}$ and ι_v acts by contracting a form with a vector v.

Finally, if $g \in \mathrm{GL}(d)$, then

$$\begin{pmatrix} \left(g^T\right)^{-1} & 0 \\ 0 & g \end{pmatrix} \cdot F_{RR} = |\det g|^{1/2} \left(g^{-1}\right)^* F_{RR}, \tag{4.13}$$

where $(g^{-1})^* F_{RR}$ denotes the pullback of F_{RR} by the map g^{-1}.

We are primarily interested in studying toroidal orientifolds, so it is important to understand how to discuss the orientifold group action in this language. For elements of the orbifold group, this is fairly clear; for any lattice preserving diffeomorphism $g \in \mathrm{GL}(d;\mathbb{Z})$ which acts on our torus, such as a rotation, we simply need to construct the corresponding element $\hat{g} \in \mathrm{O}(d,d;\mathbb{Z})$ as in (4.7) above. To describe the full orientifold action, we also need to know how the world-sheet parity operator

Ω acts. It turns out that Ω can also be expressed as a $2d \times 2d$ matrix,

$$\Omega = \begin{pmatrix} -\mathbf{1}_d & 0 \\ 0 & \mathbf{1}_d \end{pmatrix}, \tag{4.14}$$

with the understanding that this operator also acts on the remaining $10-d$ coordinates (so that, e.g. it does not exchange IIA and IIB, even if d is odd). This is not an element of $\mathrm{O}(d,d;\mathbb{Z})$, since it satisfies that $\Omega L \Omega^T = -L$ rather than (4.1), but it can be thought of as an element of $\mathrm{Spin}(d,d;\mathbb{Z})$ and we can understand its action on NSNS moduli simply by following (4.6), i.e. $\Omega \cdot G = G$, $\Omega \cdot B = -B$. We can also work out the action of Ω on RR fields by following [54], but we in fact know the answer; C_3 and C_7, as well as F_0 and F_4 should be even, while C_1 and C_5, as well as F_2 and F_6, should be odd. In this way, we see that the entire orientifold group can be understood as a finite subgroup of $\mathrm{Spin}(d,d;\mathbb{Z})$ (for instance in our example this subgroup would be generated in this notation by $\hat{\Theta}$ and $\Omega\hat{\sigma}$).

In this notation, the untwisted moduli are simply those which are fixed by the orientifold subgroup $\widehat{\Gamma} \subset \mathrm{Spin}(d,d;\mathbb{Z})$. Note that so far we have not discussed any NSNS fluxes. RR fluxes can be accommodated, and both the RR fluxes and RR potentials are understood to transform according to the rules described above.

Now given any element $h \in \mathrm{SO}(d,d;\mathbb{Z}) \subset \mathrm{Spin}(d,d;\mathbb{Z})$,[1] we can relate a given orientifold with subgroup $\widehat{\Gamma}$ and moduli given by H, etc., to a dual orientifold with subgroup $h\widehat{\Gamma}h^{-1}$ and moduli given by $h^T H h$, etc. Note that in general the elements in $h\widehat{\Gamma}h^{-1}$ need not be block diagonal; the dual orientifold group can be an asymmetric orientifold.

We would like to actually use dualities as a solution generating technique. In this case we focus on elements which do not modify the orientifold group, i.e. the

[1]Though we will not use them here, we can certainly in general consider dualities h which lie in $\mathrm{O}(d,d;\mathbb{Z}) \subset \mathrm{Pin}(d,d;\mathbb{Z})$ and take us from IIA to IIB and vice versa. It will still be true that $h\widehat{\Gamma}h^{-1} \subset \mathrm{Spin}(d,d;\mathbb{Z})$.

set of $h \in \mathrm{SO}(d,d;\mathbb{Z})$ that satisfy $h\widehat{\Gamma} = \widehat{\Gamma}h$. We can consider such h as simply a map on the moduli and fluxes.

4.1.1 Example

Let us see how this works in our example. There our orientifold is generated by $\hat{\Theta}$ and $\Omega\hat{\sigma}$, where

$$\Theta = \begin{pmatrix} {\scriptstyle\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}} & 0 & 0 \\ 0 & {\scriptstyle\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}} & 0 \\ 0 & 0 & -\mathbf{1}_2 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} {\scriptstyle\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}} & 0 & 0 \\ 0 & {\scriptstyle\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}} & 0 \\ 0 & 0 & {\scriptstyle\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}} \end{pmatrix}, \tag{4.15}$$

are both elements of $\mathrm{GL}(6;\mathbb{Z})$.

We would like to identify how to perform a T-duality on (for example) the third two-torus. Unfortunately, by just using (4.9), one finds that $\hat{\Theta}$ is invariant, but $\Omega\hat{\sigma}$ is not. However, one can repair this by combining the standard T-duality with a further rotation $(x^5, x^6) \mapsto (x^6, -x^5)$, defining instead the element

$$M_{T(3)} = \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & {\scriptstyle\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} \\ 0 & 0 & \mathbf{1}_4 & 0 \\ 0 & {\scriptstyle\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & 0 & 0 \end{pmatrix}. \tag{4.16}$$

This version of T-duality does indeed preserve the full orientifold group.

On the NSNS moduli one can check that it acts by sending $t^3 \mapsto -1/t^3$, $e^{\phi} \mapsto e^{\phi}/|t^3|$, and all other moduli remain fixed. To get the action on the RR fields, it is useful to decompose $M_{T(3)}$ as

$$M_{T(3)} = \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_4 & 0 \\ 0 & {\scriptstyle \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 & 0 & {\scriptstyle \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} \\ 0 & 0 & \mathbf{1}_4 & 0 \\ 0 & 0 & 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_4 & 0 \\ 0 & {\scriptstyle \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & 0 & \mathbf{1}_2 \end{pmatrix} . \tag{4.17}$$

Now if one writes, for example,

$$F_{RR} = F^{\perp} + dx^5 \wedge F^5 + dx^6 \wedge F^6 + dx^5 \wedge dx^6 \wedge F_{\parallel}, \tag{4.18}$$

one finds the T-duality relation,

$$M_{T(3)} \cdot F_{RR} = -F^{\parallel} + dx^5 \wedge F^5 + dx^6 \wedge F^6 + dx^5 \wedge dx^6 \wedge F^{\perp}. \tag{4.19}$$

As a consequence, we find that the ξ^K are invariant, while the RR fluxes map according to

$$m_0' = -m^3, \qquad m^{1\prime} = -e_2, \qquad m^{2\prime} = -e_1, \qquad m^{3\prime} = m_0, \qquad m^{4\prime} = -e_4$$

$$e_1' = m^2, \qquad e_2' = m^1, \qquad e_3' = -e_0, \qquad e_4' = m^4, \qquad e_0' = e_3, \tag{4.20}$$

where primed quantities represent the fluxes in the T-dual solution and unprimed ones are from the original solution. We will discuss duality of NSNS fluxes after introducing them in the next section.

We can now similarly introduce T-dualities $M_{T(1)}$ and $M_{T(2)}$, corresponding to dualizing either the first or second two-torus, by simply permuting the two-by-two blocks of $M_{T(3)}$. Under $M_{T(1)}$ we have $t^1 \mapsto -1/t^1$, $t^2 \mapsto t^2 - 2(t^4)^2/t^1$, $t^4 \mapsto -t^4/t^1$, and $e^{\phi} \mapsto e^{\phi}/|t^1|$, while

$$m_0' = -m^1, \qquad m^{1\prime} = m_0, \qquad m^{2\prime} = -e_3, \qquad m^{3\prime} = -e_2, \qquad m^{4\prime} = m^4,$$

$$e_1' = -e_0, \qquad e_2' = m^3, \qquad e_3' = m^2, \qquad e_4' = e_4, \qquad e_0' = e_1, \tag{4.21}$$

with all other moduli left invariant. The action of $M_{T(2)}$ can be obtained from $M_{T(1)}$ by interchanging one and two throughout.

4.2 NSNS Fluxes

In order to get a feeling for how we would like to encode general NSNS fluxes, let us start with the example of $T^6/\mathbb{Z}_4$ with only H-flux, from section 2.1.4.

4.2.1 Example

Here we have $H = p_1 b^1 + p_2 b^2$. In order to represent this flux, let us first pick a trivialization that depends only on the coordinates x^1 and x^2 (these coordinates will then be our *base*). Our B-field is thus

$$\begin{aligned} B = p_1 & \left[-\left(x^1 - x^2\right) dx^3 \wedge dx^5 + x^2 dx^3 \wedge dx^6 + \left(x^1 + x^2\right) dx^4 \wedge dx^5 + x^1 dx^4 \wedge dx^6 \right] \\ & + p_2 \left[\left(x^1 - x^2\right) dx^3 \wedge dx^5 + x^1 dx^3 \wedge dx^6 - \left(x^1 + x^2\right) dx^4 \wedge dx^5 - x^2 dx^4 \wedge dx^6 \right] . \end{aligned} \tag{4.22}$$

If we let E_0 be the combination of the metric and B-field at the point $x^1 = x^2 = 0$ (including values of the moduli t^a), then we can write $E(x^1, x^2) = g(x^1, x^2) \cdot E_0$, where $g(x^1, x^2)$ is a map of the base T^2 into $\mathrm{O}(4,4) \subset \mathrm{O}(6,6)$ given explicitly by

$$
\begin{aligned}
g(x^1,x^2) &= \begin{pmatrix} \mathbf{1}_4 & \begin{smallmatrix} 0 & 0 & (p_2-p_1)(x^1-x^2) & p_1x^2+p_2x^1 \\ 0 & 0 & (p_1-p_2)(x^1+x^2) & p_1x^1-p_2x^2 \\ (p_1-p_2)(x^1-x^2) & (p_2-p_1)(x^1+x^2) & 0 & 0 \\ -p_1x^2-p_2x^1 & -p_1x^1+p_2x^2 & 0 & 0 \end{smallmatrix} \\ 0 & \mathbf{1}_4 \end{pmatrix} \\
&= \exp\left[x^1 M_{x^1} + x^2 M_{x^2}\right],
\end{aligned}
\tag{4.23}
$$

where in the final step we have defined

$$
M_{x^1} = \begin{pmatrix} 0 & \begin{smallmatrix} 0 & 0 & p_2-p_1 & p_2 \\ 0 & 0 & p_1-p_2 & p_1 \\ p_1-p_2 & p_2-p_1 & 0 & 0 \\ -p_2 & -p_1 & 0 & 0 \end{smallmatrix} \\ 0 & 0 \end{pmatrix}, \qquad M_{x^2} = \begin{pmatrix} 0 & \begin{smallmatrix} 0 & 0 & p_1-p_2 & p_1 \\ 0 & 0 & p_1-p_2 & -p_2 \\ p_2-p_1 & p_2-p_1 & 0 & 0 \\ -p_1 & p_2 & 0 & 0 \end{smallmatrix} \\ 0 & 0 \end{pmatrix},
\tag{4.24}
$$

which are mutually commuting constant elements of the Lie algebra $\mathfrak{so}(4,4)$. Note that the map g is not single-valued, but that upon going around a closed cycle in the base the transformation needs to be a symmetry, i.e. we must have

$$
g(n,m) \in \mathrm{O}(4,4;\mathbb{Z}), \forall n,m \in \mathbb{Z} \quad \Leftrightarrow \quad \exp\left(M_{x^1}\right) \in \mathrm{O}(4,4;\mathbb{Z}),\ \exp\left(M_{x^2}\right) \in \mathrm{O}(4,4;\mathbb{Z}).
\tag{4.25}
$$

This is indeed satisfied by these matrices for integer values of p^K (both satisfy $M^2 = 0$ and hence $\exp(M) = 1 + M$).

We see that $g(x^1,x^2)$, or equivalently M_{x^1} and M_{x^2}, encodes our H-fluxes. What about metric fluxes? We would like to see how these fluxes map when we T-dualize on the third two-torus. Since we know how the metric and B-field transform, we have

$$
M_{T(3)} \cdot E(x^1,x^2) = \left(M_{T(3)} g(x^1,x^2) M_{T(3)}^{-1}\right) \cdot \left(M_{T(3)} \cdot E_0\right).
\tag{4.26}
$$

So we see that we should replace our twist $g(x^1,x^2)$ by a new twist in $\mathrm{O}(4,4) \subset$

O(6, 6),

$$g'(x^1,x^2) = M_{T(3)} g M_{T(3)}^{-1}$$
$$= \begin{pmatrix} \mathbf{1}_2 & \begin{smallmatrix} p_1x^2+p_2x^1 & (p_1-p_2)(x^1-x^2) \\ p_1x^1-p_2x^2 & (p_2-p_1)(x^1+x^2) \end{smallmatrix} & 0 & 0 \\ 0 & \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_2 & 0 \\ 0 & 0 & \begin{smallmatrix} -p_1x^2-p_2x^1 & -p_1x^1+p_2x^2 \\ (p_2-p_1)(x^1-x^2) & (p_1-p_2)(x^1+x^2) \end{smallmatrix} & \mathbf{1}_2 \end{pmatrix}, \tag{4.27}$$

or equivalently,

$$M'_{x^1} = \begin{pmatrix} 0 & \begin{smallmatrix} p_2 & p_1-p_2 \\ p_1 & p_2-p_1 \end{smallmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{smallmatrix} -p_2 & -p_1 \\ p_2-p_1 & p_1-p_2 \end{smallmatrix} & 0 \end{pmatrix}, \quad M'_{x^2} = \begin{pmatrix} 0 & \begin{smallmatrix} p_1 & p_2-p_1 \\ -p_2 & p_2-p_1 \end{smallmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{smallmatrix} -p_1 & p_2 \\ p_1-p_2 & p_1-p_2 \end{smallmatrix} & 0 \end{pmatrix}. \tag{4.28}$$

Again these are two commuting elements of $\mathfrak{so}(4,4)$ which exponentiate to elements of $\mathrm{O}(4,4;\mathbb{Z})$, though this time rather than shifting the B-field, they act as diffeomorphisms of the fibered T^4. Note that if we write

$$g'(x^1,x^2) = \begin{pmatrix} \left(h^T\right)^{-1} & 0 \\ 0 & h \end{pmatrix}, \qquad h \in \mathrm{SL}(4), \tag{4.29}$$

then we have

$$\eta^i = \left(h^{-1}\right)^i_j dx^j = \left(h^{-1}\right)^* dx^i. \tag{4.30}$$

These are the proper, globally-defined one-forms, since as we traverse the base, we are forced to transport our fiber one-forms by the map g'.

In the case just described, we can then compute the metric flux components, namely

$$f^5_{13} = -f^5_{24} = -p_2, \quad f^5_{14} = f^5_{23} = -p_1, \quad f^6_{13} = -f^6_{14} = -f^6_{23} = -f^6_{24} = p_2 - p_1, \tag{4.31}$$

or in terms of an r-matrix (compare with (2.95) and (2.96))

$$r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -p_1 & -p_2 \\ 0 & 0 \end{pmatrix}, \qquad \widehat{r} = 0. \tag{4.32}$$

Thus we see that this particular T-duality has simply sent $p_K \mapsto -r_{3K}$ (also there is no H-flux in this new solution described by g').

By combining this map with the map of moduli and RR fluxes under T-duality from the previous subsection, one can verify that the solution of section 2.1.4 and that of (2.107) are precisely T-dual to each other.

Let us now perform one more T-duality using $M_{T(2)}$. This sends us to

$$M''_{x^1} = \begin{pmatrix} 0 & 0 \\ \begin{smallmatrix} 0 & 0 & p_1 & p_2-p_1 \\ 0 & 0 & -p_2 & p_2-p_1 \\ -p_1 & p_2 & 0 & 0 \\ p_1-p_2 & p_1-p_2 & 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}, \qquad M''_{x^2} = \begin{pmatrix} 0 & 0 \\ \begin{smallmatrix} 0 & 0 & -p_2 & p_2-p_1 \\ 0 & 0 & -p_1 & p_1-p_2 \\ p_2 & p_1 & 0 & 0 \\ p_1-p_2 & p_2-p_1 & 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}. \tag{4.33}$$

This will correspond to nongeometric Q-flux. We will argue below in the general case how one should convert these to particular components of Q-flux; for now we merely state the results.

$$Q^{35}_1 = -Q^{45}_2 = -p_1, \qquad Q^{36}_1 = Q^{36}_2 = Q^{46}_1 = -Q^{46}_2 = p_1 - p_2, \qquad Q^{35}_2 = Q^{45}_1 = p_2, \tag{4.34}$$

or

$$q = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \widehat{q} = 0. \tag{4.35}$$

By applying the $M_{T(2)}$ map to the moduli and fluxes of (2.107) we can generate a new solution with only Q-flux (no H-flux or metric flux). It is straightforward to check that the F-term equations are satisfied.

4.2.2 General Situation

Let us attempt to generalize this situation. Let $\widehat{\Gamma}$ be the subgroup of $\mathrm{Spin}(6,6;\mathbb{Z})$ which generates our orientifold group, and suppose we have a splitting of our T^6 into a base of dimension n and a fiber of dimension $6-n$ such that $\widehat{\Gamma}$ acts block diagonally (i.e. such that both the base and fiber form real representations of the orientifold group, which is D_4 in our example). We will also assume that $\widehat{\Gamma}$ acts symmetrically on the base, with each element giving rise to a $\mathrm{GL}(n;\mathbb{Z})$ action. Then for every element $h \in \widehat{\Gamma}$ we can decompose

$$h = h_b \oplus h_f \in \mathrm{GL}(n;\mathbb{Z}) \times \mathrm{Spin}(6-n, 6-n;\mathbb{Z}) \subset \mathrm{Spin}(6,6;\mathbb{Z}). \tag{4.36}$$

We would like to classify the elements $g(\vec{x}_b) \in \mathrm{SO}(6-n, 6-n) \subset \mathrm{SO}(6,6)$, depending on the base coordinates $\vec{x}_b$, by which we can twist our fibers. Such twists will have to satisfy a number of conditions.

First of all, they need to be invariant under the orientifold group action, i.e. we require

$$h_f g(\vec{x}_b) h_f^{-1} = g(h_b \cdot \vec{x}_b), \qquad \forall h \in \widehat{\Gamma}. \tag{4.37}$$

Secondly, we require path independence, in the sense that moving in different directions in the base should commute, i.e.

$$[\partial_i g(\vec{x}_b), \partial_j g(\vec{x}_b)], \qquad \forall i, j \in \{1, \ldots, n\}. \tag{4.38}$$

And finally, moving around a closed path must correspond to an element of the duality group, i.e.

$$g(\vec{x}_b + \vec{\lambda}) g(\vec{x}_b)^{-1} \in \mathrm{SO}(6,6;\mathbb{Z}), \qquad \forall \vec{\lambda} \in \mathbb{Z}^6,\ \vec{x}_b \in \mathbb{R}^n \subset \mathbb{R}^6. \tag{4.39}$$

A very natural simplifying ansatz for the form of $g(\vec{x}_b)$ is to take

$$g(\vec{x}_b) = \exp\left[\vec{x}_b \cdot \vec{M}\right], \tag{4.40}$$

where each component of $\vec{M}$ is an element of the Lie algebra $\mathfrak{so}(6-n, 6-n)$. With this ansatz, the conditions above become

$$h_f M_i h_f^{-1} = (h_b)^j{}_i M_j, \qquad \forall i, \quad \forall h \in \widehat{\Gamma}, \tag{4.41}$$

$$[M_i, M_j] = 0, \qquad \forall i, j, \tag{4.42}$$

and a quantization condition

$$\exp\left[\lambda^i M_i\right] \in \mathrm{SO}(6,6;\mathbb{Z}), \qquad \forall \vec{\lambda} \in \mathbb{Z}^6, \tag{4.43}$$

which can in general be a bit subtle if our base-fiber splitting is not a good splitting of the lattice. Even in those cases however, the correct quantization condition can be worked out without too much trouble. In the simpler case where the splitting does respect the lattice identifications, the quantization condition is simply that

$$\exp M_i \in \mathrm{SO}(6-n, 6-n;\mathbb{Z}), \qquad \forall i. \tag{4.44}$$

To understand how the matrices M_i translate into general NSNS flux components, we will consider RR fields which can be thought of as sections of the spin bundle Spin$(6-n, 6-n)$. The map $g(\vec{x}_b)$ tells us how to transport sections of this bundle as we move around the base, providing us with the correct globally defined RR fields. For instance, in the case with only H-flux, where $g(\vec{x}_b)$ consists solely of linear shifts in the B-field, we saw that the globally defined RR fluxes are given by $F_{RR} = \exp(B) \wedge F_{RR}^{(0)}$. In this case we have

$$dF_{RR} = \exp(B) \wedge \left(dF_{RR}^{(0)} + H \wedge F_{RR} \right) = \exp(B) \wedge d_H F_{RR}^{(0)}. \qquad (4.45)$$

In other words, d_H is a covariant derivative for this bundle [8], and by differentiating our globally defined sections we can deduce the form of d_H and hence the components of H-flux. We will now show that the same story is true more generally. The globally defined RR fluxes are given by $g(\vec{x}_b) \cdot F_{RR}^{(0)}$, and

$$dF_{RR} = g(\vec{x}_b) \cdot \mathcal{D} F_{RR}^{(0)}. \qquad (4.46)$$

This observation is what allows us to compute the flux components from the M_i.

To get a better feeling for these matters, let us look at some basic cases. Note that a general matrix in $\mathfrak{so}(6-n, 6-n)$ has the form

$$M = \begin{pmatrix} -A^T & B \\ C & A \end{pmatrix}, \qquad (4.47)$$

where A is a general $(6-n) \times (6-n)$ matrix, and B and C are antisymmetric $(6-n) \times (6-n)$ matrices.

Suppose first that all of our M_i are nonvanishing only in the top-right block

B above, say[2]

$$M_i = \begin{pmatrix} 0 & (B_i)_{ab} \\ 0 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad g(\vec{x}_b) = \begin{pmatrix} \mathbf{1} & \vec{x}_b \cdot \vec{B} \\ 0 & \mathbf{1} \end{pmatrix}. \tag{4.48}$$

Then

$$\begin{aligned} dF_{RR} &= d\left(\exp\left[\vec{x}_b \cdot \vec{B}\right] \wedge F_{RR}^{(0)}\right) \\ &= \exp\left[\vec{x}_b \cdot \vec{B}\right] \wedge \left(dF_{RR}^{(0)} + \frac{1}{2}\left(B_i\right)_{ab} dx^i \wedge dx^a \wedge dx^b \wedge F_{RR}^{(0)}\right), \end{aligned} \tag{4.49}$$

so

$$H_{iab} = \left(B_i\right)_{ab}. \tag{4.50}$$

Note that for a given base-fiber splitting we can only obtain H-flux with precisely one leg on the base.

Similarly, suppose that the M_i are all block diagonal,

$$M_i = \begin{pmatrix} -A_i^T & 0 \\ 0 & A_i \end{pmatrix} \qquad \Longrightarrow \qquad g(\vec{x}_b) = \begin{pmatrix} e^{-\vec{x}_b \cdot \vec{A}^T} & 0 \\ 0 & e^{\vec{x}_b \cdot \vec{A}} \end{pmatrix}. \tag{4.51}$$

Then

$$\begin{aligned} dF_{RR} \;&=\; d\left(\exp\left[\frac{1}{2}\mathrm{Tr}\left(\vec{x}_b \cdot \vec{A}\right)\right]\left(e^{-\vec{x}_b \cdot \vec{A}}\right)^* F_{RR}^{(0)}\right) \qquad (4.52) \\ &=\; e^{\frac{1}{2}\mathrm{Tr}\left(\vec{x}_b \cdot \vec{A}\right)}\left(e^{-\vec{x}_b \cdot \vec{A}}\right)^*\left(dF_{RR}^{(0)} + \frac{1}{2}\mathrm{Tr}\left(\vec{A}\right) \cdot d\vec{x}_b \wedge F_{RR}^{(0)} - dx^i \wedge \left(A_i \cdot F_{RR}^{(0)}\right)\right), \end{aligned}$$

where A_i acts on a p-form via

$$A_i \cdot \zeta^{(p)} = \binom{p}{1}\left(A_i\right)^b{}_{[a_1}\zeta_{|b|a_2 \cdots a_p]}\frac{1}{p!}dx^{a_1} \wedge \cdots dx^{a_p}. \tag{4.53}$$

[2]We now start using conventions where i, j, etc. refer to base coordinates, while a, b, etc. refer to fiber coordinates.

Comparing with (A.3)[3], we deduce that

$$f^a_{ib} = (A_i)^a{}_b. \tag{4.54}$$

Again we find that f must have exactly one lower index along the base, with the other two indices along the fiber. Note that here we do not require A_i to be traceless, though any nonvanishing trace piece would require a base one-form $\mathrm{Tr}(A_i)dx^i$ which would have to be invariant under the orientifold group.

And also,

$$M_i = \begin{pmatrix} 0 & 0 \\ (C_i)^{ab} & 0 \end{pmatrix} \qquad \Longrightarrow \qquad g(\vec{x}_b) = \begin{pmatrix} \mathbf{1} & 0 \\ \vec{x}_b \cdot \vec{C} & \mathbf{1} \end{pmatrix}. \tag{4.55}$$

So from

$$\begin{aligned} dF_{RR} &= d\left(\exp\left[\frac{1}{2}x^i C^{ab}_i \iota_a \iota_b\right] \cdot F^{(0)}_{RR}\right) \\ &= \exp\left[\frac{1}{2}x^i C^{ab}_i \iota_a \iota_b\right] \cdot \left(dF^{(0)}_{RR} + \frac{1}{2}C^{ab}_i dx^i \wedge \left(\iota_a \iota_b F^{(0)}_{RR}\right)\right), \end{aligned} \tag{4.56}$$

we find

$$Q^{ab}_i = -\left(C_i\right)^{ab}. \tag{4.57}$$

Once again, the lower index must be on the base, while the other two (upper) indices lie along the fiber.

Finally, since the exponent of g is linear in the base coordinates, these derivatives simply add, and we find that the map between the matrices M_i and the fluxes is simply,

$$M_i = \begin{pmatrix} -f^b_{ia} & H_{iab} \\ -Q^{ab}_i & f^a_{ib} \end{pmatrix}. \tag{4.58}$$

[3]In doing such a comparison, we may assume that the components of $F^{(0)}_{RR}$ are constant, so $dF^{(0)}_{RR} = 0$.

Let us see what we can learn from the constraints (4.41) and (4.42). Consider an element of $\widehat{\Gamma}$ of the form

$$h = \Omega\hat{\sigma} = \begin{pmatrix} -\left(\sigma^T\right)^{-1} & 0 \\ 0 & \sigma \end{pmatrix} = \sigma_b \oplus \begin{pmatrix} -\left(\sigma_f^T\right)^{-1} & 0 \\ 0 & \sigma_f \end{pmatrix}. \tag{4.59}$$

Then substituting (4.58) into (4.41) leads to

$$-\left(\sigma_f^T\right)^{-1} H_i \sigma_f^{-1} = (\sigma_b)^j{}_i H_j, \qquad \sigma_f Q_i \sigma_f^T = -(\sigma_b)^j{}_i Q_j, \qquad \sigma_f f_i \sigma_f^{-1} = (\sigma_b)^j{}_i f_j, \tag{4.60}$$

which can be rephrased as the statement that the metric fluxes f should be even under the involution σ, while the H- and Q-fluxes should both be odd under σ.

Now substituting (4.58) into (4.42) leads to the conditions

$$f^a_{c[i} f^c_{j]b} + Q^{ac}_{[i} H_{j]cb} = 0, \qquad H_{ac[i} f^c_{j]b} - H_{bc[i} f^c_{j]a} = 0, \qquad Q^{c[a}_{[i} f^{b]}_{j]c} = 0. \tag{4.61}$$

But it is easy to check that these are precisely the Bianchi identities (A.2) for the situation at hand, namely when each flux has exactly one lower index on the base and all other indices lie along the fiber.

We would like to discuss the quantization condition (4.43), but it is quite complicated in the general case, so let us first see how these base-fiber constructions work in our favorite example.

4.3 Example

To classify the possible base-fiber splittings of our $T^6/\mathbb{Z}_4$ orientifold, we need to know how the coordinates of the T^6 split into representations of the orientifold group D_4. As a real vector space (i.e. forgetting the shift identifications of the torus), it can be checked that this $\mathbb{R}^6$ splits into two isomorphic two-dimensional irreducible real representations and two one-dimensional real representations which are not isomorphic. The latter two are given by the span of x^6 and the span of $\hat{x}^5 = x^5 + \frac{1}{2}x^6$. Because of the isomorphism between the two-dimensional representations, there is a two real parameter family of ways to split up the first four coordinates into irreducible real representations. Indeed, if we define

$$\begin{aligned} \hat{x}^1 &= x^1 + a\left(x^3 + x^4\right), & \hat{x}^2 &= x^2 + a\left(-x^3 + x^4\right), \\ \hat{x}^3 &= b\left(x^1 - x^2\right) + x^3, & \hat{x}^4 &= b\left(x^1 + x^2\right) + x^4, \end{aligned} \tag{4.62}$$

then $\{\hat{x}^1, \hat{x}^2\}$ can be taken to span one invariant subspace, while $\{\hat{x}^3, \hat{x}^4\}$ span the other. The only constraint is that $2ab \neq 1$, so that this change of basis is invertible.

We can now classify all of the possible bases, dimension by dimension.

4.3.1 One-Dimensional Bases

Here there are two cases; either the base is parametrized by x^6, or by $\hat{x}^5 = x^5 + \frac{1}{2}x^6$. Suppose that the base is x^6. Invariance under Θ^2 ensures that M_{x^6} has the form

$$M_{x^6} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & a & 0 & 0 \\ C & 0 & -A^T & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}, \tag{4.63}$$

for 4×4 matrices A, B, C, and a real number a. But now invariance under Θ implies that $a = 0$, and that M_{x^6} in fact lies in $\mathfrak{so}(4,4)$. But then this one-dimensional case is really a restriction of the case with two-dimensional base T^2_3 where only x^6 dependence is allowed (i.e. $M_{\hat{x}^5} = 0$). This case is treated below without restriction.

Since we did not use the action of σ in the argument above, and since this action is the only difference between x^6 and $\hat{x}^5$, we conclude that an $\hat{x}^5$ base also gives nothing new.

4.3.2 Two-Dimensional Bases

Here we consider four-dimensional fibers and two-dimensional bases. There are three possibilities.

1) $\underline{T_1^2 \text{ base}}$

In this case the base is spanned by $\{\hat{x}^1, \hat{x}^2\}$, and the fiber is spanned by $\{\hat{x}^3, \hat{x}^4, x^5, x^6\}$. Our orientifold group is represented as

$$\Theta_b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4.64}$$

$$\hat{\Theta}_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix} \end{pmatrix}, \qquad (\Omega\hat{\sigma})_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{smallmatrix} \end{pmatrix}. \tag{4.65}$$

The constraints (4.41) imply that

$$\hat{\Theta}_f^2 M_{\hat{x}^1} \hat{\Theta}_f^2 = -M_{\hat{x}^1} \qquad \Longrightarrow \qquad M_{\hat{x}^1} = \begin{pmatrix} 0 & a & 0 & c \\ b & 0 & -c^T & 0 \\ 0 & d & 0 & -b^T \\ -d^T & 0 & -a^T & 0 \end{pmatrix}, \tag{4.66}$$

where a, b, c, and d are two-by-two matrices. Next,

$$(\Omega\hat{\sigma})_f \, M_{\hat{x}^1} \, (\Omega\hat{\sigma})_f = M_{\hat{x}^1} \quad \Longrightarrow \quad M_{\hat{x}^1} = \begin{pmatrix} 0 & \begin{smallmatrix} \alpha & \beta \\ \alpha+\beta & -\beta \end{smallmatrix} & 0 & \begin{smallmatrix} \varepsilon & \varphi \\ -\varepsilon & \varphi-\varepsilon \end{smallmatrix} \\ \begin{smallmatrix} \gamma & \gamma \\ \delta & \gamma-\delta \end{smallmatrix} & 0 & \begin{smallmatrix} -\varepsilon & \varepsilon \\ -\varphi & \varepsilon-\varphi \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} \chi & \kappa \\ -\chi-\kappa & \kappa \end{smallmatrix} & 0 & \begin{smallmatrix} -\gamma & -\delta \\ -\gamma & \delta-\gamma \end{smallmatrix} \\ \begin{smallmatrix} -\chi & \chi+\kappa \\ -\kappa & -\kappa \end{smallmatrix} & 0 & \begin{smallmatrix} -\alpha & -\alpha-\beta \\ -\beta & \beta \end{smallmatrix} & 0 \end{pmatrix}. \tag{4.67}$$

Then all of the constraints (4.41) are satisfied if we define

$$M_{\hat{x}^2} = \hat{\Theta}_f M_{\hat{x}^1} \hat{\Theta}_f^{-1} = \begin{pmatrix} 0 & \begin{smallmatrix} \alpha+\beta & -\beta \\ -\alpha & -\beta \end{smallmatrix} & 0 & \begin{smallmatrix} -\varepsilon & \varphi-\varepsilon \\ -\varepsilon & -\varphi \end{smallmatrix} \\ \begin{smallmatrix} \gamma & -\gamma \\ \gamma-\delta & -\delta \end{smallmatrix} & 0 & \begin{smallmatrix} \varepsilon & \varepsilon \\ \varepsilon-\varphi & \varphi \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} -\chi-\kappa & \kappa \\ -\chi & -\kappa \end{smallmatrix} & 0 & \begin{smallmatrix} -\gamma & \delta-\gamma \\ \gamma & \delta \end{smallmatrix} \\ \begin{smallmatrix} \chi+\kappa & \chi \\ -\kappa & \kappa \end{smallmatrix} & 0 & \begin{smallmatrix} -\alpha-\beta & \alpha \\ \beta & \beta \end{smallmatrix} & 0 \end{pmatrix}. \tag{4.68}$$

By imposing the requirement that these matrices commute, we find three extra conditions, namely

$$\beta\gamma + \varepsilon\kappa = 0, \qquad \alpha\gamma + \beta\delta = \varepsilon\chi + \varphi\kappa, \qquad (\alpha+\beta)\,\delta + (\varphi-\varepsilon)\,\chi = 0. \tag{4.69}$$

From the entries of $M_{\hat{x}^1}$ and $M_{\hat{x}^2}$ one can read off the flux components in that basis. One then uses the transformation (4.62) to convert these fluxes back to the lattice compatible basis from before. The resulting fluxes are

$$p = \Delta \begin{pmatrix} \varphi-\varepsilon & \varphi \end{pmatrix},$$

$$r = \begin{pmatrix} 2a^2\Delta^{-1}\delta & 2a^2\Delta^{-1}(\delta-\gamma) \\ \Delta^{-1}\delta & \Delta^{-1}(\delta-\gamma) \\ -\Delta(\alpha+\beta) & -\Delta\alpha \\ 4a\Delta^{-1}\delta & 4a\Delta^{-1}(\delta-\gamma) \end{pmatrix}, \qquad q = \Delta^{-1} \begin{pmatrix} \chi & \chi+\kappa \\ 2a^2\chi & 2a^2(\chi+\kappa) \\ 0 & 0 \\ 4a\chi & 4a(\chi+\kappa) \end{pmatrix}, \tag{4.70}$$

with $\widehat{r} = \widehat{q} = s = 0$, and where a and $\Delta = 1 - 2ab$ are the parameters of the basis transformation. With these definitions, one can check that the constraints (4.69) precisely reproduce the Bianchi identities (2.142) for this case.

Now unless a, b, and Δ are integers, the basis in which the matrices above are expressed is not a basis for our lattice, and so generally the quantization condition is not just that $\exp[M_{\hat{x}^1}]$ and $\exp[M_{\hat{x}^2}]$ are integers. Instead, what we should do is embed these matrices into $\mathfrak{so}(6,6)$, undo the transformation (4.62), and then exponentiate. Following this procedure we find 12×12 matrices M_1, M_2, as well as $M_3 = a(M_1 - M_2)$ and $M_4 = a(M_1 + M_2)$. All four of these matrices turn out to be (three-step) nilpotent, and hence the quantization conditions $\exp[M_i] \in \mathrm{SO}(6,6;\mathbb{Z})$ are simply that the entries of the M_i be integers. Translating back into the matrices above, we learn that the correct quantization condition for this case is nearly the naive one (in fact it is the naive quantization condition in terms of the flux components, f^i_{jk}, q^{ij}_k, etc.); we must have p_K, r_{cK} and q^c_K to be integers, but in addition we require $2ar_{3K}$, $2ap_K$, $a(r_{31} - r_{32})$, and $a(p_1 - p_2)$ to be integers. In particular, if a is an integer (for instance if the transformed basis is a lattice basis), then the naive integer quantization is correct.

Let us take a moment and consider the types of solutions that we get if we restrict to the case $q = 0$ ($\chi = \kappa = 0$). Then the Bianchi identities (4.69) force either $\alpha = \beta = 0$, or $\gamma = \delta = 0$. Either way, we are stuck with an r-matrix of rank one, and, following our discussion in section 2.2.3, we cannot stabilize all of the moduli.

2) T^2_2 base

Here we take our base to be spanned by $\{\hat{x}^3, \hat{x}^4\}$. This case works out almost identically to the case described in detail above. In fact, the expression for $M_{\hat{x}^3}$ is precisely the same as that for $M_{\hat{x}^1}$ in (4.67), while $M_{\hat{x}^4}$ is the same as $M_{\hat{x}^2}$ in (4.68). As such, the Bianchi identities are again simply the three equations in (4.69). What does change slightly is the map back to our flux matrices. For this case we have

$$p = \Delta \begin{pmatrix} \varepsilon - \varphi & -\varphi \end{pmatrix},$$

$$r = \begin{pmatrix} -\Delta^{-1}\delta & \Delta^{-1}(\gamma - \delta) \\ -2b^2\Delta^{-1}\delta & 2b^2\Delta^{-1}(\gamma - \delta) \\ \Delta(\alpha + \beta) & \Delta\alpha \\ -4b\Delta^{-1}\delta & 4b\Delta^{-1}(\gamma - \delta) \end{pmatrix}, \qquad q = -\Delta^{-1} \begin{pmatrix} 2b^2\chi & 2b^2(\chi + \kappa) \\ \chi & \chi + \kappa \\ 0 & 0 \\ 4b\chi & 4b(\chi + \kappa) \end{pmatrix}, \quad (4.71)$$

and $\widehat{r} = \widehat{q} = s = 0$. The quantization conditions are exactly as before but with a replaced by b wherever it occurs.

3) <u>T_3^2 base</u>

Finally there is the case in which our base is spanned by $\{x^5, x^6\}$. This case turns out to be richer than the previous cases. The representation of the orientifold group is

$$\Theta_b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_b = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \tag{4.72}$$

$$\hat{\Theta}_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{smallmatrix} \end{pmatrix}, \qquad (\Omega\hat{\sigma})_f = \begin{pmatrix} \begin{smallmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{smallmatrix} \end{pmatrix}. \tag{4.73}$$

Solving the constraints

$$\begin{aligned} \hat{\Theta}_f M_{x^5} \hat{\Theta}_f^{-1} &= -M_{x^5}, \quad (\Omega\hat{\sigma})_f \, M_{x^5} \, (\Omega\hat{\sigma})_f = M_{x^5}, \\ \hat{\Theta}_f M_{x^6} \hat{\Theta}_f^{-1} &= -M_{x^6}, \quad (\Omega\hat{\sigma})_f \, M_{x^6} \, (\Omega\hat{\sigma})_f = M_{x^5} - M_{x^6}, \end{aligned} \tag{4.74}$$

leads to a twelve-parameter family of solutions,

$$M_{x^5} = \begin{pmatrix} \begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha \end{smallmatrix} & \begin{smallmatrix} \gamma & \gamma \\ \gamma & -\gamma \end{smallmatrix} & 0 & \begin{smallmatrix} \varepsilon & -\varepsilon \\ -\varepsilon & -\varepsilon \end{smallmatrix} \\ \begin{smallmatrix} \beta & \beta \\ \beta & -\beta \end{smallmatrix} & \begin{smallmatrix} 0 & \delta \\ \delta & 0 \end{smallmatrix} & \begin{smallmatrix} -\varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} \nu & -\nu \\ -\nu & -\nu \end{smallmatrix} & \begin{smallmatrix} -\alpha & 0 \\ 0 & \alpha \end{smallmatrix} & \begin{smallmatrix} -\beta & -\beta \\ -\beta & \beta \end{smallmatrix} \\ \begin{smallmatrix} -\nu & \nu \\ \nu & \nu \end{smallmatrix} & 0 & \begin{smallmatrix} -\gamma & -\gamma \\ -\gamma & \gamma \end{smallmatrix} & \begin{smallmatrix} 0 & -\delta \\ -\delta & 0 \end{smallmatrix} \end{pmatrix}, \tag{4.75}$$

$$M_{x^6} = \begin{pmatrix} \begin{smallmatrix} \alpha/2 & \varphi \\ \varphi & -\alpha/2 \end{smallmatrix} & \begin{smallmatrix} \kappa+\gamma/2 & -\kappa+\gamma/2 \\ -\kappa+\gamma/2 & -\kappa-\gamma/2 \end{smallmatrix} & 0 & \begin{smallmatrix} \mu+\varepsilon/2 & \mu-\varepsilon/2 \\ \mu-\varepsilon/2 & -\mu-\varepsilon/2 \end{smallmatrix} \\ \begin{smallmatrix} \chi+\beta/2 & -\chi+\beta/2 \\ -\chi+\beta/2 & -\chi-\beta/2 \end{smallmatrix} & \begin{smallmatrix} \lambda & \delta/2 \\ \delta/2 & -\lambda \end{smallmatrix} & \begin{smallmatrix} -\mu-\varepsilon/2 & -\mu+\varepsilon/2 \\ -\mu+\varepsilon/2 & \mu+\varepsilon/2 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} \pi+\nu/2 & \pi-\nu/2 \\ \pi-\nu/2 & -\pi-\nu/2 \end{smallmatrix} & \begin{smallmatrix} -\alpha/2 & -\varphi \\ -\varphi & \alpha/2 \end{smallmatrix} & \begin{smallmatrix} -\chi-\beta/2 & \chi-\beta/2 \\ \chi-\beta/2 & \chi+\beta/2 \end{smallmatrix} \\ \begin{smallmatrix} -\pi-\nu/2 & -\pi+\nu/2 \\ -\pi+\nu/2 & \pi+\nu/2 \end{smallmatrix} & 0 & \begin{smallmatrix} -\kappa-\gamma/2 & \kappa-\gamma/2 \\ \kappa-\gamma/2 & \kappa+\gamma/2 \end{smallmatrix} & \begin{smallmatrix} -\lambda & -\delta/2 \\ -\delta/2 & \lambda \end{smallmatrix} \end{pmatrix}.$$

Note that the solution is much simpler in terms of the matrix $M_{\hat{x}^6} = M_{x^6} - \frac{1}{2} M_{x^5}$, but that the solution as given corresponds to the basis for the lattice.

Enforcing $[M_{x^5}, M_{x^6}] = 0$ gives six equations

$$
\begin{aligned}
&(\alpha+\delta)\,\chi - (\varphi-\lambda)\,\beta = (\alpha+\delta)\,\pi + (\varphi-\lambda)\,\nu = 0, \\
&(\alpha+\delta)\,\kappa - (\varphi-\lambda)\,\gamma = (\alpha+\delta)\,\mu + (\varphi-\lambda)\,\varepsilon = 0, \\
&\alpha\varphi - \gamma\chi - \beta\kappa - \mu\nu - \varepsilon\pi = 0, \qquad \alpha\varphi + \delta\lambda = 0.
\end{aligned} \tag{4.76}
$$

Translating into flux matrices, we find

$$
p = \begin{pmatrix} \mu - \varepsilon/2 & \mu + \varepsilon/2 \end{pmatrix}, \tag{4.77}
$$

$$
r = \begin{pmatrix} \chi + \beta/2 & \chi - \beta/2 \\ -\kappa - \gamma/2 & -\kappa + \gamma/2 \\ 0 & 0 \\ -\varphi - \lambda + \frac{1}{2}(\alpha - \delta) & -\varphi - \lambda - \frac{1}{2}(\alpha - \delta) \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -\pi + \nu/2 & -\pi - \nu/2 \\ 0 & 0 \end{pmatrix},
$$

$$
\widehat{r} = \begin{pmatrix} \varphi - \lambda - \frac{1}{2}(\alpha + \delta) & -\varphi + \lambda - \frac{1}{2}(\alpha + \delta) \end{pmatrix}, \qquad \widehat{q} = s = 0.
$$

In the case $q = 0$ ($\pi = \nu = 0$), this provides the complete case (b) of section 2.2.3.

One can easily verify that (4.76) gives the correct set of Bianchi identities for this case. In fact the solution to these equations can be broken into four cases,

$$\begin{aligned}
&\text{(i)} && M_{x^5} = 0, \\
&\text{(ii)} && M_{x^6} = \tfrac{1}{2} M_{x^5}, \\
&\text{(iii)} && \alpha + \delta = \varphi - \lambda = 0, \\
& && \alpha\varphi - \gamma\chi - \beta\kappa - \mu\nu - \varepsilon\pi = 0, \\
&\text{(iv)} && \alpha + \delta \neq 0,\ \varphi - \lambda \neq 0, \\
& && \chi = \left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)\beta,\ \pi = -\left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)\nu,\ \kappa = \left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)\gamma,\ \mu = -\left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)\varepsilon, \\
& && \alpha - \delta = \pm\left[(\alpha+\delta)^2 - 8\beta\gamma + 8\varepsilon\nu\right]^{1/2},\ \varphi + \lambda = -\left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)(\alpha - \delta).
\end{aligned} \tag{4.78}$$

In terms of flux matrices, case (i) has $p_1 = p_2$, $r_{a1} = r_{a2}$, $q_1^3 = q_2^3$, and $\widehat{r}^1 = -\widehat{r}^2$. Case (ii) corresponds to $p_1 = -p_2$, $r_{a1} = -r_{a2}$, $q_1^3 = -q_2^3$, and $\widehat{r}^1 = \widehat{r}^2$. Case (iii) is simply $\widehat{r} = 0$, with the other components arbitrary (up to one additional Bianchi identity). And case (iv) is the case with arbitrary $\widehat{r}$, but where the conditions $\widehat{r}^K p_K = \widehat{r}^K r_{aK} = \widehat{r}^K q_K^3 = 0$ put constraints on the other fluxes.

In every case we must finally solve the quantization conditions $\exp[M_{x^5}]$, $\exp[M_{x^6}] \in SO(4,4;\mathbb{Z})$. In each of the four cases this condition is potentially non-trivial because at least one of the two matrices may not be nilpotent. For example, the general expression for the exponentiated version of M_{x^5} includes entries such as

$$e^{\frac{1}{2}(\alpha+\delta)}\left[\cosh\frac{C}{2} + \frac{\alpha-\delta}{C}\sinh\frac{C}{2}\right], \quad \text{or} \quad \frac{2\beta}{C} e^{\frac{1}{2}(\alpha+\delta)}\sinh\frac{C}{2}, \tag{4.79}$$

and many others, where

$$C = \sqrt{(\alpha-\delta)^2 + 8\beta\gamma - 8\varepsilon\nu}. \tag{4.80}$$

Finding the generic situation in which all of these entries are integers is quite difficult. Let us specialize somewhat.

To make contact with the work we did in section 2.2.3, we will focus on case (iii), where $\widehat{r} = 0$, and assume also that $q = 0$. Under what conditions would the naive, integral quantization be correct? The requirement would be that both M_{x^5} and M_{x^6} would have to be nilpotent, and this in turn requires that two expressions vanish,

$$\alpha^2 + 2\beta\gamma = 0, \qquad \varphi^2 + 2\kappa\chi = 0. \tag{4.81}$$

And it turns out that these equations, along with the extra Bianchi identity $\alpha\varphi - \beta\kappa - \gamma\chi = 0$, imply that the rank of r is one. Thus, by arguments in section 2.2.3, we cannot hope to stabilize all moduli. In particular, the numerical solution we presented at the end of section 2.2.3 is not correctly quantized. In fact, it is possible to find solutions to the quantization conditions which do give rise to an r-matrix of rank two and a superpotential which stabilizes all moduli. However, we have argued that such cases are not nilpotent, so the entries of the r-matrix are not integers and in fact are irrational numbers. But now we have a puzzle, since if all the NSNS fluxes are irrational numbers, then it is clearly impossible to satisfy the tadpole condition with RR flux integers!

One plausible solution is that we do not correctly understand the quantization of RR fluxes in the presence of general NS-NS fluxes, and in particular in non-nilpotent cases where the NSNS flux quantization is not the naive one. One approach to this problem would involve viewing both NSNS and RR fluxes as twists in a U-duality group of the fiber, in which case understanding the full quantization conditions would simply reduce to understanding the structure of the duality group, e.g. $E_{7(7)}(\mathbb{Z})$. This is an avenue of ongoing investigation.

4.3.3 Three-Dimensional Bases

There are four possible bases in this case, but it will turn out that they are all contained in previously considered examples, so we will focus just on the case with base $\{\hat{x}^1, \hat{x}^2, \hat{x}^5\}$. The other three cases (with either or both of $\{\hat{x}^3, \hat{x}^4\}$ or $\hat{x}^6$) are similar.

Here we have

$$\Theta_b = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \sigma_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.82}$$

$$\hat{\Theta}_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & & \\ & -1 & & \\ & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & \\ & & & -1 \end{pmatrix}, \qquad (\Omega\hat{\sigma})_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix} & & & \\ & 1 & & \\ & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & & -1 \end{pmatrix}. \tag{4.83}$$

Solving our constraints, we find

$$M_{\hat{x}^1} = \begin{pmatrix} 0 & \begin{smallmatrix} \gamma \\ -\gamma \end{smallmatrix} & 0 & \begin{smallmatrix} -\delta \\ -\delta \end{smallmatrix} \\ \begin{smallmatrix} \alpha & -\alpha \end{smallmatrix} & 0 & \begin{smallmatrix} \delta & \delta \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} -\beta \\ -\beta \end{smallmatrix} & 0 & \begin{smallmatrix} -\alpha \\ \alpha \end{smallmatrix} \\ \begin{smallmatrix} \beta & \beta \end{smallmatrix} & 0 & \begin{smallmatrix} -\gamma & \gamma \end{smallmatrix} & 0 \end{pmatrix}, \qquad M_{\hat{x}^2} = \begin{pmatrix} 0 & \begin{smallmatrix} -\gamma \\ -\gamma \end{smallmatrix} & 0 & \begin{smallmatrix} -\delta \\ \delta \end{smallmatrix} \\ \begin{smallmatrix} -\alpha & -\alpha \end{smallmatrix} & 0 & \begin{smallmatrix} \delta & -\delta \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} -\beta \\ \beta \end{smallmatrix} & 0 & \begin{smallmatrix} \alpha \\ \alpha \end{smallmatrix} \\ \begin{smallmatrix} \beta & -\beta \end{smallmatrix} & 0 & \begin{smallmatrix} \gamma & \gamma \end{smallmatrix} & 0 \end{pmatrix},$$

$$M_{\hat{x}^5} = \begin{pmatrix} \begin{smallmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{smallmatrix} & & & \\ & 0 & & \\ & & \begin{smallmatrix} 0 & -\varepsilon \\ -\varepsilon & 0 \end{smallmatrix} & \\ & & & 0 \end{pmatrix}. \tag{4.84}$$

Now let us check the Bianchi identities. It turns out that they give

$$\alpha\gamma = \beta\delta = 0, \quad \text{and} \quad \varepsilon\alpha = \varepsilon\beta = \varepsilon\gamma = \varepsilon\delta = 0. \tag{4.85}$$

But these then imply that either $M_{\hat{x}^5} = 0$, and we are in a special case of two-dimensional bases, or that $M_{\hat{x}^1} = M_{\hat{x}^2} = 0$, and we are in a special case of a one-dimensional base. Either way, we find nothing new.

The other three possible bases lead to the same conclusions.

4.3.4 Four-Dimensional Bases

Here there are three possibilities. If our base is given by $\{\hat{x}^1, \hat{x}^2, x^5, x^6\}$, then we would have

$$(-\mathbf{1})M_{\hat{x}^1}(-\mathbf{1}) = \hat{\Theta}_f^2 M_{\hat{x}^1}\hat{\Theta}_f^2 = \left(\Theta_b^2 \cdot M\right)_{\hat{x}^1} = -M_{\hat{x}^1} \qquad \Longrightarrow \qquad M_{\hat{x}^1} = 0, \quad (4.86)$$

and similarly $M_{\hat{x}^2} = 0$, so our base is equivalent to just having $\{x^5, x^6\}$.

By the same argument, a base of $\{\hat{x}^3, \hat{x}^4, x^5, x^6\}$ would reduce to a previously considered case. This leaves only the possibility of $\{x^1, x^2, x^3, x^4\}$. But then we have, e.g.

$$(\mathbf{1})M_{x^1}(\mathbf{1}) = \hat{\Theta}_f^2 M_{x^1}\hat{\Theta}_f^2 = -M_{x^1}, \quad (4.87)$$

and this case in fact forces all $M_i = 0$.

4.3.5 Five-Dimensional Bases

This final case is also trivial for the same reasons discussed above. Invariance under Θ^2 forces four of the M_i to vanish, and invariance under Θ takes care of the fifth one.

4.4 Advantages and Puzzles

In this section we have presented a ten-dimensional construction of II toroidal orientifold models with some general NS-NS fluxes. The class of models which we can construct in this way is a sub-class of all the models discussed in the previous chapters. It is not clear how representative a sample this sub-class is. For instance, in the case with only metric flux, the models we can construct do not necessarily correspond to nilpotent algebras, and hence are more general than those obtained by T-dualizing H-flux alone, but they are still a restricted set of algebras, and in particular are all solvable algebras. It would be interesting to compare these properties with the geometric properties evidenced, for example, in the classification of twisted tori in [55]. For our $T^6/\mathbb{Z}_4$ example in the type IIA case we can construct nearly all possible metric fluxes (all of cases (a) and (b), but not the cases (a') or (a'') in (2.97)), but if non-geometric fluxes are included then only a fairly small fraction of possible models can be built in this way.

Having any kind of ten-dimensional construction, however, is obviously a huge advantage, as it gives us a great deal more confidence that our models can really arise from string theory compactifications (though we are certainly not claiming that other models cannot be obtained from string theory, just not using the methods we have explored in this paper). It can also highlight important subtleties that were not so readily apparent from the effective theory approach. As we have seen, the quantization of general NSNS fluxes is one such subtlety. In cases where our matrices M_i are all two-step nilpotent (which also implies that the underlying Lie algebra, as described in appendix A, is nilpotent) then the quantization condition is simply the naive one, with all flux components (which correspond to entries of the M_i) being integers. Matrices that are nilpotent after more than two-steps must have entries which are rationals (with denominator no larger than the number of steps minus one). More generally, however the condition is that certain exponentials of matrices be integral. These conditions often can be solved, giving irrational flux components (see also related discussions in [11]).

This in turn leads to a puzzle, since the integral tadpole contribution is given by a bilinear pairing of the NSNS fluxes with the RR fluxes. If the former are forced by quantization to be irrational numbers, then the latter cannot be integers or rationals, as was presumed. Either such setups are inconsistent or (more likely, in our belief) we have not correctly understood the quantization of RR fluxes in general NSNS flux backgrounds (see also the discussion in footnote 10). Presumably there should be some analog of twisted K-theory for the general flat fiber theories that we are studying, including also the non-geometric fluxes. Matching this onto the work of Mathai and collaborators [56, 57, 58, 59, 60, 40, 61, 62] would be very interesting. Similarly, exploiting the connections between the base-fiber approach described here and spaces with generalized complex structure (see e.g. [54]) could potentially lead to a better understanding of these more general classes of string compactifications. We are currently investigating these directions.

One advantage to following this base-fiber approach is that the constructions should be easy to generalize to any situation with a flat fiber over a flat base (and some aspects of the approach should be applicable to more general smooth bases). In particular, we should certainly be able to accommodate type IIB as well as IIA within the framework presented, and we can also work with orientifold actions which are asymmetric on the fiber (compare for instance with the models of [63]). Furthermore, heterotic string theory on a torus or type II on K3 fibers can also be covered in this framework, since the duality groups in those situations are well-understood. The K3 fibered case in particular could be very interesting, and could be compared to the work of [9]. Finally, one can expand the analysis to include U-duality groups, such as that of M-theory on a torus (see [64, 11, 65, 49, 50, 66, 67, 7]). It would be very interesting to understand how far one could push such a program, and whether one could find interesting solutions with a controlled low energy theory.

We have not yet incorporated any of the twisted sector physics into this story. Beyond getting possible hints by studying K3 fibers at their orbifold points, it would be extremely gratifying to have a more complete picture for how to deal with the twisted sector physics in the presence of fiber twists.

Chapter 5

Conclusions

In this dissertation we have tried to illustrate how generalized NSNS fluxes in type II orientifold compactifications can enrich the structure of the four-dimensional effective field theory. In particular we have shown how they can lead to a stabilization of all moduli in the type IIA case. Furthermore, we have shown how these fluxes can act as electric and magnetic charges for the RR axion fields in four dimensions, thus giving rise to D-term contributions to the scalar potential. The hope is that these extra contributions to the potential will make it more likely to find regions in moduli space that allow for slow roll inflation. It would be very interesting, for example, to repeat the study of [68, 69] with these extra ingredients added. One can also try to use the D-terms to find de Sitter minima of the potential. Unfortunately, because of the relationship (2.18) between D-terms and F-terms, we can not use D-terms to uplift an otherwise supersymmetric vacuum, at least perturbatively (but see [31] for a suggested nonperturbative effect).

The two chapters 2,3 of this work have been at the level of effective field theory, so it is very difficult to know which of these models can really be obtained from ten-dimensional string theory constructions, and in which regimes we can trust the approximations that we have been making, i.e. that the supergravity analysis (perhaps augmented by dualities) holds, that Kałuża-Klein modes are heavy enough to be ignored, and that backreaction of fluxes and localized sources, especially orientifold planes, can be kept under control. These issues deserve a much more detailed exploration. One approach to answering the question of which models can be obtained from well-defined ten-dimensional constructions, at least for toroidal orientifolds, is the base-fiber approach described in chapter 4, and following the spirit of [11]. Although we focused on type IIA, these techniques can easily be carried over to type IIB (or the heterotic string, for that matter), and for a given toroidal orientifold, one can identify which classes of fluxes can be constructed using these methods. These constructions have the advantage of revealing the correct quantization conditions for the generalized NSNS fluxes, which turn out to be non-trivial in general. For configurations of fluxes which are not constructible in this way, it is not clear what quantization conditions are correct, or indeed even if the configurations themselves have a ten-dimensional origin. There seem to be however certain examples of well understood compactification that cannot be obtained from the base-fiber approach [70]. This means that the base-fiber approach as presented in chapter 4 does not describe all fluxes that can be obtained from ten dimensions and it is in principle possible that all the fluxes we can turn on in an effective four-dimensional field theory can be obtained from a ten-dimensional string theory construction. However, in general we do not know what the compact space would be. Only the base-fiber approach allows for an explicit construction and furthermore for a proper quantization of the NSNS fluxes. Nevertheless, even when we do understand the NSNS quantization, there is still some mystery about the RR quantization conditions, which would require a better understanding of the relevant K-theory for these spaces [71].

This is a vexing situation since, as we have seen above, the effective theory structure actually fits together very nicely, and looks as though it could be applied to general Calabi-Yau orientifolds, rather than just toroidal examples. Unfortunately, besides the confusions about quantization conditions, it also seems to be difficult to get the full set of Bianchi identities from geometric data in the general case. Between the quantization conditions (which can sometimes have no nontrivial solutions) and the extra Bianchi identities, it seems likely that these general models will be much more constrained than they might naively appear. However, it is our opinion that this should not necessarily discourage attempts to use these effective theories to construct phenomenologically interesting scenarios. In particular the question of moduli stabilization i.e., the question of whether the potential depends on all moduli or not is independent of the way the moduli are quantized. One might hope that the same is true for example for the existence of de Sitter vacua or regions in moduli space that allow for inflation.

Appendix A

Two Different Derivations of the Bianchi Identities

Here we present two derivations of the Bianchi identities.

The usual derivation [7] is to note that upon reducing on a d-dimensional torus we have (ignoring for now any orientifold group) d vectors from reducing the metric and d vectors from reducing the B-field. Let Z_i and X^i respectively generate the gauge transformations for these two groups of vectors. One then argues, by T-duality or otherwise, that the NSNS fluxes must appear in the Lie brackets as

$$\begin{aligned} [Z_i, Z_j] &= f^k_{ij} Z_k - H_{ijk} X^k, \\ \left[Z_i, X^j\right] &= -f^j_{ik} X^k + Q^{jk}_i Z_k, \\ \left[X^i, X^j\right] &= Q^{ij}_k X^k - R^{ijk} Z_k. \end{aligned} \tag{A.1}$$

The Jacobi identities for this Lie algebra then give us the NSNS Bianchi identities:

$$\begin{aligned} H_{k[i_1 i_2} f^k_{i_3 i_4]} &= 0, \\ H_{k[i_1 i_2} Q^{k\,j}_{i_3]} + f^j_{k[i_1} f^k_{i_2 i_3]} &= 0, \\ H_{k\,i_1 i_2} R^{k\,j_1 j_2} + f^k_{i_1 i_2} Q^{j_1 j_2}_k - 4 f^{[j_1}_{k[i_1} Q^{j_2]k}_{i_2]} &= 0, \\ f^{[j_1}_{k\,i} R^{j_2 j_3]k} + Q^{k[j_1}_i Q^{j_2 j_3]}_k &= 0, \\ Q^{[j_1 j_2}_k R^{j_3 j_4]k} &= 0, \end{aligned} \tag{A.2}$$

Let us present an alternative derivation, as suggested by [12]. We have seen that it is natural to replace the exterior derivative d acting on RR forms by a covariant derivative $\mathcal{D}$. We saw in section 2.3 that such an object was what appeared in the tadpole condition and superpotential, and argued that it should also be used in finding the correct gauge transformations. And in section 4 we saw that it could be understood as a covariant derivative for the spin bundle of which RR fields formed sections. By combining these considerations[1], it is natural to define the general $\mathcal{D}$ by its action on a p-form as

$$\begin{aligned} &\mathcal{D}A^{(p)} = \\ &\binom{p+3}{3} H_{[i_1 i_2 i_3} A_{i_4 \cdots i_{p+3}]} \frac{1}{(p+3)!} dx^{i_1} \wedge \cdots \wedge dx^{i_{p+3}} \\ &- \left\{ \binom{p+1}{2} f^j_{[i_1 i_2} A_{|j| i_3 \cdots i_{p+1}]} + \frac{p+1}{2} f^j_{j[i_1} A_{i_2 \cdots i_{p+1}]} \right\} \frac{1}{(p+1)!} dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}} \\ &+ \frac{1}{2} \left\{ \binom{p-1}{1} Q^{jk}_{[i_1} A_{|jk| i_2 \cdots i_{p-1}]} + \binom{p-1}{0} Q^{jk}_j A_{k\, i_1 \cdots i_{p-1}} \right\} \frac{1}{(p-1)!} dx^{i_1} \wedge \cdots \wedge dx^{i_{p-1}} \\ &- \frac{1}{6} \binom{p-3}{0} R^{jk\ell} A_{jk\ell\, i_1 \cdots i_{p-3}} \frac{1}{(p-3)!} dx^{i_1} \wedge \cdots \wedge dx^{i_{p-3}}. \end{aligned} \tag{A.3}$$

For consistency, we need $\mathcal{D}$ to share a key property with the exterior deriva-

[1] From the base-fiber approach we did not require $f^a_{ia} = 0$, and can argue for the dependence on this trace, and from the effective field theory approach we can deduce how R must appear.

tive that it is replacing, namely that $\mathcal{D}^2 = 0$ on all forms. Computing,

$$
\begin{aligned}
&\mathcal{D}^2 A^{(p)} = \\
&-6\binom{p+4}{4} H_{k\,i_1i_2} f^k_{i_3i_4} A_{i_5\cdots i_{p+4}} \frac{1}{(p+4)!} dx^{i_1}\wedge\cdots\wedge dx^{i_{p+4}} \\
&+\left\{-\binom{p+2}{2}\left(H_{k\ell\,i_1} Q^{k\ell}_{i_2} - \frac{1}{2} Q^{k\ell}_k H_{\ell\,i_1i_2} - \frac{1}{2} f^k_{k\ell} f^\ell_{i_1i_2}\right) A_{i_3\cdots i_{p+2}}\right. \\
&\quad\left.+3\binom{p+2}{3}\left(H_{k\,i_1i_2} Q^{kj}_{i_3} + f^k_{i_1i_2} f^j_{k\,i_3}\right) A_{j\,i_4\cdots i_{p+2}}\right\} \frac{1}{(p+2)!} dx^{i_1}\wedge\cdots\wedge dx^{i_{p+2}} \\
&+\left\{\binom{p}{0}\left(-\frac{1}{6} H_{k\ell m} R^{k\ell m} + \frac{1}{4} f^k_{k\ell} Q^{\ell m}_m\right) A_{i_1\cdots i_p}\right. \\
&\quad+\frac{1}{2}\binom{p}{1}\left(H_{k\ell\,i_1} R^{k\ell j} - Q^{k\ell}_{i_1} f^j_{k\ell} - f^k_{k\ell} Q^{\ell j}_{i_1} - Q^{k\ell}_k f^j_{\ell\,i_1}\right) A_{j\,i_2\cdots i_p} \qquad \text{(A.4)} \\
&\quad\left.-\frac{1}{2}\binom{p}{2}\left(H_{k\,i_1i_2} R^{k\,j_1j_2} + 4 f^{j_1}_{k\,i_1} Q^{k\,j_2}_{i_2} + Q^{j_1j_2}_k f^k_{i_1i_2}\right) A_{j_1j_2\,i_3\cdots i_p}\right\} \frac{1}{p!} dx^{i_1}\wedge\cdots\wedge dx^{i_p} \\
&+\left\{\frac{1}{2}\binom{p-2}{0}\left(f^{j_1}_{k\ell} R^{k\ell\,j_2} + \frac{1}{2} f^k_{k\ell} R^{\ell\,j_1j_2} + \frac{1}{2} Q^{k\ell}_k Q^{j_1j_2}_\ell\right) A_{j_1j_2\,i_1\cdots i_{p-2}}\right. \\
&\quad\left.+\frac{1}{2}\binom{p-2}{1}\left(R^{k\,j_1j_2} f^{j_3}_{k\,i_1} + Q^{j_1j_2}_k Q^{k\,j_3}_{i_1}\right) A_{j_1j_2j_3\,i_2\cdots i_{p-2}}\right\} \frac{1}{(p-2)!} dx^{i_1}\wedge\cdots\wedge dx^{i_{p-2}} \\
&-\frac{1}{4}\binom{p-4}{0} Q^{j_1j_2}_k R^{k\,j_3j_4} A_{j_1j_2j_3j_4\,i_1\cdots i_{p-4}} \frac{1}{(p-4)!} dx^{i_1}\wedge\cdots\wedge dx^{i_{p-4}}.
\end{aligned}
$$

From this we find that in order to ensure that $\mathcal{D}^2 = 0$ on all forms, we need precisely the Bianchi identities found above, and one additional one which does not follow by contraction,

$$2H_{k\ell m} R^{k\ell m} + 3 f^k_{k\ell} Q^{m\ell}_m = 0. \qquad \text{(A.5)}$$

Note that this final identity is satisfied on *any* type II orientifold, because there are generally no scalars (zero-forms) which are odd under the involution.

The coefficients of the two trace terms in (A.3) can be argued from the spin bundle transformations of RR fields, as in section 4.2.2, along with a T-duality argument to get the TrQ term in terms of the Trf term, but there is another nice check as well. If the coefficients of the trace terms were at all different, then $\mathcal{D}^2 = 0$

would lead to more constraints beyond the single extra constraint we found above. Though not inconsistent, these additional requirements seem surprising and ad-hoc. With the given coefficients, however, these additional constraints follow simply from the traces of constraints with more free indices.

Bibliography

[1] T. Kaluza, "On The Problem Of Unity In Physics," Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966 (1921).

[2] O. Klein, "Quantum theory and five-dimensional theory of relativity," Z. Phys. **37**, 895 (1926) [Surveys High Energ. Phys. **5**, 241 (1986)].

[3] M. Ihl and T. Wrase, "Towards a realistic type IIA T**6/Z(4) orientifold model with background fluxes. I: Moduli stabilization," *JHEP* **07** (2006) 027, `hep-th/0604087`.

[4] M. Ihl, D. Robbins, and T. Wrase, "Toroidal Orientifolds in IIA with General NS-NS Fluxes," *JHEP* **08** (2007) 043, `arXiv:0705.3410 [hep-th]`.

[5] D. Robbins and T. Wrase, "D-Terms from Generalized NS-NS Fluxes in Type II," *JHEP* **12** (2007) 058, `arXiv:0709.2186 [hep-th]`.

[6] A. Tomasiello, "Topological mirror symmetry with fluxes," *JHEP* **06** (2005) 067, `hep-th/0502148`.

[7] G. Aldazabal, P. G. Camara, A. Font, and L. E. Ibanez, "More dual fluxes and moduli fixing," *JHEP* **05** (2006) 070, `hep-th/0602089`.

[8] A. Micu, E. Palti, and G. Tasinato, "Towards Minkowski vacua in type II string compactifications," *JHEP* **03** (2007) 104, `hep-th/0701173`.

[9] M. Cvetic, T. Liu, and M. B. Schulz, "Twisting K3 x T**2 orbifolds," hep-th/0701204.

[10] T. W. Grimm and J. Louis, "The effective action of N = 1 Calabi-Yau orientifolds," *Nucl. Phys.* **B699** (2004) 387–426, hep-th/0403067.

[11] A. Dabholkar and C. Hull, "Duality twists, orbifolds, and fluxes," *JHEP* **09** (2003) 054, hep-th/0210209.

[12] J. Shelton, W. Taylor, and B. Wecht, "Generalized flux vacua," *JHEP* **02** (2007) 095, hep-th/0607015.

[13] I. T. Ellwood, "NS-NS fluxes in Hitchin's generalized geometry," hep-th/0612100.

[14] T. W. Grimm and J. Louis, "The effective action of type IIA Calabi-Yau orientifolds," *Nucl. Phys.* **B718** (2005) 153–202, hep-th/0412277.

[15] O. DeWolfe, A. Giryavets, S. Kachru, and W. Taylor, "Type IIA moduli stabilization," *JHEP* **07** (2005) 066, hep-th/0505160.

[16] P. Binetruy, G. Dvali, R. Kallosh, and A. Van Proeyen, "Fayet-Iliopoulos terms in supergravity and cosmology," *Class. Quant. Grav.* **21** (2004) 3137–3170, hep-th/0402046.

[17] K. Choi, A. Falkowski, H. P. Nilles, and M. Olechowski, "Soft supersymmetry breaking in KKLT flux compactification," *Nucl. Phys.* **B718** (2005) 113–133, hep-th/0503216.

[18] R. Blumenhagen, L. Gorlich, and T. Ott, "Supersymmetric intersecting branes on the type IIA T**6/Z(4) orientifold," *JHEP* **01** (2003) 021, hep-th/0211059.

[19] R. Blumenhagen, L. Gorlich, and B. Kors, "Supersymmetric 4D orientifolds of type IIA with D6-branes at angles," *JHEP* **01** (2000) 040, `hep-th/9912204`.

[20] J. Distler, D. S. Freed, and G. W. Moore, to appear.

[21] A. R. Frey and J. Polchinski, "N = 3 warped compactifications," *Phys. Rev.* **D65** (2002) 126009, `hep-th/0201029`.

[22] G. Villadoro and F. Zwirner, "N = 1 effective potential from dual type-IIA D6/O6 orientifolds with general fluxes," *JHEP* **06** (2005) 047, `hep-th/0503169`.

[23] J. Shelton, W. Taylor, and B. Wecht, "Nongeometric flux compactifications," *JHEP* **10** (2005) 085, `hep-th/0508133`.

[24] J. Scherk and J. H. Schwarz, "How to get masses from extra dimensions," *Nucl. Phys.* **B153** (1979) 61–88.

[25] N. Kaloper, R. R. Khuri, and R. C. Myers, "On generalized axion reductions," *Phys. Lett.* **B428** (1998) 297–302, `hep-th/9803066`.

[26] N. Kaloper and R. C. Myers, "The O(dd) story of massive supergravity," *JHEP* **05** (1999) 010, `hep-th/9901045`.

[27] B. Wecht, "Lectures on Nongeometric Flux Compactifications," `arXiv:0708.3984 [hep-th]`.

[28] P. G. Camara, A. Font, and L. E. Ibanez, "Fluxes, moduli fixing and MSSM-like vacua in a simple IIA orientifold," *JHEP* **09** (2005) 013, `hep-th/0506066`.

[29] C. P. Burgess, R. Kallosh, and F. Quevedo, "de Sitter string vacua from supersymmetric D-terms," *JHEP* **10** (2003) 056, `hep-th/0309187`.

[30] G. Villadoro and F. Zwirner, "de Sitter vacua via consistent D-terms," *Phys. Rev. Lett.* **95** (2005) 231602, hep-th/0508167.

[31] A. Achucarro, B. de Carlos, J. A. Casas, and L. Doplicher, "de Sitter vacua from uplifting D-terms in effective supergravities from realistic strings," *JHEP* **06** (2006) 014, hep-th/0601190.

[32] S. L. Parameswaran and A. Westphal, "de Sitter string vacua from perturbative Kaehler corrections and consistent D-terms," *JHEP* **10** (2006) 079, hep-th/0602253.

[33] S. L. Parameswaran and A. Westphal, "Consistent de Sitter string vacua from Kaehler stabilization and D-term uplifting," hep-th/0701215.

[34] P. Binetruy and G. R. Dvali, "D-term inflation," *Phys. Lett.* **B388** (1996) 241–246, hep-ph/9606342.

[35] E. Halyo, "Hybrid inflation from supergravity D-terms," *Phys. Lett.* **B387** (1996) 43–47, hep-ph/9606423.

[36] M. Grana, "Flux compactifications in string theory: A comprehensive review," *Phys. Rept.* **423** (2006) 91–158, hep-th/0509003.

[37] D. Lust and D. Tsimpis, "Supersymmetric AdS(4) compactifications of IIA supergravity," *JHEP* **02** (2005) 027, hep-th/0412250.

[38] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds," *JHEP* **08** (2004) 046, hep-th/0406137.

[39] T. H. Buscher, "A symmetry of the string background field equations," *Phys. Lett.* **B194** (1987) 59.

[40] P. Bouwknegt, K. Hannabuss, and V. Mathai, "Nonassociative tori and applications to T-duality," *Commun. Math. Phys.* **264** (2006) 41–69, `hep-th/0412092`.

[41] T. Banks and K. van den Broek, "Massive IIA flux compactifications and U-dualities," *JHEP* **03** (2007) 068, `hep-th/0611185`.

[42] B. S. Acharya, F. Benini, and R. Valandro, "Fixing moduli in exact type IIA flux vacua," *JHEP* **02** (2007) 018, `hep-th/0607223`.

[43] I. Benmachiche and T. W. Grimm, "Generalized N = 1 orientifold compactifications and the Hitchin functionals," *Nucl. Phys.* **B748** (2006) 200–252, `hep-th/0602241`.

[44] S. Reffert, "The Geometer's Toolkit to String Compactifications," `arXiv:0706.1310 [hep-th]`.

[45] R. Rabadan and A. M. Uranga, "Type IIB orientifolds without untwisted tadpoles, and non- BPS D-branes," *JHEP* **01** (2001) 029, `hep-th/0009135`.

[46] G. Aldazabal, A. Font, L. E. Ibanez, and G. Violero, "D = 4, N = 1, type IIB orientifolds," *Nucl. Phys.* **B536** (1998) 29–68, `hep-th/9804026`.

[47] S. Hellerman, J. McGreevy, and B. Williams, "Geometric constructions of nongeometric string theories," *JHEP* **01** (2004) 024, `hep-th/0208174`.

[48] A. Flournoy, B. Wecht, and B. Williams, "Constructing nongeometric vacua in string theory," *Nucl. Phys.* **B706** (2005) 127–149, `hep-th/0404217`.

[49] C. M. Hull, "A geometry for non-geometric string backgrounds," *JHEP* **10** (2005) 065, `hep-th/0406102`.

[50] A. Dabholkar and C. Hull, "Generalised T-duality and non-geometric backgrounds," *JHEP* **05** (2006) 009, `hep-th/0512005`.

[51] C. M. Hull, "Global aspects of T-duality, gauged sigma models and T- folds," hep-th/0604178.

[52] C. M. Hull, "Doubled geometry and T-folds," hep-th/0605149.

[53] S. F. Hassan, "SO(d,d) transformations of Ramond-Ramond fields and space-time spinors," *Nucl. Phys.* **B583** (2000) 431–453, hep-th/9912236.

[54] M. Gualtieri, "Generalized complex geometry," math.dg/0401221.

[55] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "A scan for new N=1 vacua on twisted tori," hep-th/0609124.

[56] P. Bouwknegt, J. Evslin, and V. Mathai, "T-duality: Topology change from H-flux," *Commun. Math. Phys.* **249** (2004) 383–415, hep-th/0306062.

[57] P. Bouwknegt, J. Evslin, and V. Mathai, "On the topology and H-flux of T-dual manifolds," *Phys. Rev. Lett.* **92** (2004) 181601, hep-th/0312052.

[58] P. Bouwknegt, K. Hannabuss, and V. Mathai, "T-duality for principal torus bundles," *JHEP* **03** (2004) 018, hep-th/0312284.

[59] V. Mathai and J. M. Rosenberg, "T-duality for torus bundles via noncommutative topology," *Commun. Math. Phys.* **253** (2004) 705–721, hep-th/0401168.

[60] V. Mathai and J. M. Rosenberg, "On mysteriously missing T-duals, H-flux and the T-duality group," hep-th/0409073.

[61] P. Bouwknegt, K. Hannabuss, and V. Mathai, "T-duality for principal torus bundles and dimensionally reduced gysin sequences," *Adv. Theor. Math. Phys.* **9** (2005) 749–773, hep-th/0412268.

[62] V. Mathai and J. Rosenberg, "T-duality for torus bundles with H-fluxes via noncommutative topology. II: The high-dimensional case and the T-duality group," *Adv. Theor. Math. Phys.* **10** (2006) 123–158, `hep-th/0508084`.

[63] K. Becker, M. Becker, C. Vafa, and J. Walcher, "Moduli stabilization in non-geometric backgrounds," *Nucl. Phys.* **B770** (2007) 1–46, `hep-th/0611001`.

[64] A. Kumar and C. Vafa, "U-manifolds," *Phys. Lett.* **B396** (1997) 85–90, `hep-th/9611007`.

[65] C. M. Hull and A. Catal-Ozer, "Compactifications with S-duality twists," *JHEP* **10** (2003) 034, `hep-th/0308133`.

[66] C. M. Hull and R. A. Reid-Edwards, "Flux compactifications of M-theory on twisted tori," *JHEP* **10** (2006) 086, `hep-th/0603094`.

[67] C. M. Hull, "Generalised geometry for M-theory," `hep-th/0701203`.

[68] M. P. Hertzberg, M. Tegmark, S. Kachru, J. Shelton, and O. Ozcan, "Searching for Inflation in Simple String Theory Models: An Astrophysical Perspective," `arXiv:0709.0002 [astro-ph]`.

[69] M. P. Hertzberg, S. Kachru, W. Taylor and M. Tegmark, "Inflationary Constraints on Type IIA String Theory," *JHEP* **12** (2007) 095, `arXiv:0711.2512 [hep-th]`.

[70] G. Aldazabal and A. Font, "A second look at N=1 supersymmetric AdS(4) vacua of type IIA supergravity," `arXiv:0712.1021 [hep-th]`.

[71] A. Bergman and D. Robbins, to appear.

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