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Aspects of Relativistic Hamiltonian Physics

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Aspects of Relativistic Hamiltonian Physics

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Dedicated to my grandmother, Ann D'Avignon, and my father, William Richardson.

If I had graduated a year earlier, they would have been there to see it.

Aspects of Relativistic Hamiltonian Physics

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This dissertation presents various new results in relativistic Hamiltonian plasma physics. It begins with an overview of Hamiltonian physics, with an emphasis on noncanonical brackets, and presents various nonrelativistic systems to be generalized later on. There then follows an exposition on action principles for Hall and Extended MHD, which allow the derivation of the noncanonical Hamiltonian brackets for those systems. I next discuss the transition to relativistic Hamiltonian systems, and the special difficulties that arise in this step. A detailed exploration of relativistic Hamiltonian MHD follows, using a novel bracket formulation. This chapter also investigates alternative brackets, gauge degeneracies, and Casimir invariants. Next I lay out the connection between Lagrangian and Eulerian MHD (both in Hamiltonian forms), and present some early work on a bracket-based formulation of the relativistic Navier-Stokes equation. The next chapters develop various results using an antisymmetric relativistic spin tensor, and several unexpected and intriguing

physical consequences of the Jacobi identity. I conclude with a program of future research and several useful appendices.

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Chapter 1

Introduction and Acknowledgments

Physics pushes towards ever greater generalization and abstraction; even as its theories and models grow more complex and rich in consequences, new formalisms and insights allow us to express these theories in more compact and elegant forms. I see this dissertation as a small advance in that push towards abstraction, repeatedly taking systems of multiple, fairly complex equations of motion and recasting them in the compact form $\{f, S\} = 0$ (the meanings of the various parts of this equation will be explained later). I do not think I am alone in a sense of niggling dissatisfaction, appearing irregularly as I learn the results of existing physics: a sense that, even when correct, there is something missing from the common explanations. I regard this dissatisfaction as an important motivation behind the push towards greater abstraction and elegance. I am happy and proud to say that I have been able to satisfy this sense of dissatisfaction, though only in a few, very narrow cases, and I hope that some measure of this joy comes through in my exposition.

Among those who have helped me find these moments of joy, I must single out Philip Morrison, my advisor. Toward his students, he is supportive, kind, patient and engaged; toward his field, he is passionate, resourceful,

thoroughly read, and inventive. We have shared many fun (and often even productive!) discussions about the topics to be found in the chapters to come, and his opinions in plasma physics have shaped not only the topics but the methods, approaches and guiding insights that shaped my research. Our conversations have ranged far beyond the relatively complete research found here, into topics too speculative and embryonic to put into print yet, and I will consider him a vital resource in my research to come. I hope that I will someday live up to the confidence he has shown in me. I also thank Gennady Shvets, my previous advisor, for being patient with my fickleness; Larry Shepley, for many enriching conversations and impish questions that have helped me shore up the relativity in this relativistic plasma physics; Wolfgang Rindler, for more such conversations, offering helpful perspective and unexpected connections; Swadesh Mahajan and Richard Hazeltine, with whom I should have conversed more, but who were invariably helpful when I did.

I thank the members of Dr. Morrison's Friday group, for raising many interesting questions and concerns on my research, and for helping me to develop my still-amateurish public speaking skills. Manasvi Lingam has been particularly helpful, and I admire his sterling qualities as a physicist-to-be; Ioannis Keramidas and George Miloshevich have also been particularly helpful. Cathy Rapinett, of the Institute of Fusion Physics, has been invariable helpful, cheerful and friendly. Among the friends who have provided good company and warm support through my many years in graduate school, I will single out Josef Rickets, Jacob Williamson, Mark Roy, and Rick Niess, but there have been many more who lended me strength during this period. I thank my father, Bill, and my grandmother, Ann, who supported an endeavour whose end they would not see; my mother, MT, who was always up for a weekend visit and respite; and my uncle, Chuck, who provided a few fun adventures I could not have afforded myself. I also owe gratitude to Francesco Pegoraro, Michel Vittot, and Yohei Kawazura, fruitful collaborators and collaborators-to-be.

Most of the material in this dissertation is original, albeit incomplete. The exceptions are Chapters 2, an overview of nonrelativistic Hamiltonian physics, and 4, about the transition to relativity, plus a few labelled overview sections in other chapters. Chapter 5, on Hamiltonian relativistic MHD, represents the most complete portion, and it has been published in Phys. Rev. D [8]. Chapter 8, on physical consequences of the Jacobi Identity, has, in a somewhat different form, been submitted. Chapters 3, on the derivation of the Hall and Extended MHD brackets, and 6, on the relation between Lagrangian and Eulerian relativistic MHD, represent the current states of papers still in the draft stage – the former chapter, developed only this last semester, belies the title of this dissertation, but I do hope to make it relativistic at a future stage. The latter portions of Chapters 2, 4 and 5 provide early, speculative work that I hope to develop in much greater detail in the future. Chapter 7, on relativistic Hamiltonian (and classical) spin, summarizes some foundered research from an earlier stage in my time with Dr. Morrison, and it is, sadly, not likely be developed beyond what I have provided here. Chapter 9 presents the further research I hope to soon accomplish, and it is followed by a few appendices containing useful mathematics.

There are a few idiosyncratic conventions I use in this dissertation. Brackets play a central role, so I use different typographical ones to denote specific kinds of bracket: square brackets $[f, g]$ for finite-dimensional brackets, $\{f, g\}$ for either infinite-dimensional brackets or the combination of Poisson and metriplectic brackets, and (f, g) for the symmetric metriplectic bracket. As for the term itself, sometimes "Poisson bracket" refers to any bracket use-able in a Hamiltonian formalism, and sometimes it refers specifically to the canonical bracket. For this reason, and to foster concision, I will simply call these "brackets", without modification. Hopefully Poisson won't mind.

I hope that you will find something of use in the words to come.

Chapter 2

Overview of non-relativistic Hamiltonian systems

My long journey begins with a single step:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = [f, H] \,. \tag{2.1}$$

Let us discuss this equation.

2.1 General structure (finite- and infinite-dimensional)

Hamilton's equations (2.1) are the culmination of the Hamiltonian formalism, but this formalism requires a great deal of preparation before use. I'll start by enumerating their ingredients, beginning with the arbitrary function f of the dynamical variables. Said variables introduce one major distinction between Hamiltonian systems, that between finite-dimensional and infinite-dimensional systems. In the former case, I have a finite-dimensional vector space called the phase space, typified by a system of N particles with $6N$ variables $(\mathbf{x}_{(i)}, \mathbf{p}_{(i)})$; in the latter case, I have a number of field variables, which themselves might be scalars, vectors or any other tensorial object, defined at each point of my domain. A basic example of an infinite-dimensional system would be Euler's fluid equations using two scalar fields, density n and specific entropy s, and one vector field, the fluid velocity $\mathbf{v}$.

Moreover, nothing stops the two types of systems from mixing: for instance, the same unified Hamiltonian formalism can describe both the $6N$ variables of a collection of particles, and at the same time the infinite-dimensional vector fields representing the electric and magnetic fields governing the particles' motion. To evolve a system, you need equations of motion for each of its basic dynamical variables, and in practice you may need equations of motion for quite a few more; here, Hamilton's equations (2.1) show one of their chief advantages, for every single one of those equations of motions are contained in that little package. The function or functional f can be whatever you please, provided it is a differentiable function of the dynamical variables alone.

Typically the Hamiltonian H is the energy of the system, but there are sufficient exceptions to be wary of this rule. In most of the examples found in this dissertation, there are in fact infinitely many quantities that can take the role of H, due to the existence of special invariants called Casimirs. I do, however, require that the Hamiltonian be a scalar, thus invariant under rotations and translations; in this dissertation, I will also require that it be a true scalar rather than a pseudoscalar, and thus invariant under inversions of the coordinate system. Similarly, I require that it be invariant under time reversals, although this particular symmetry is more for illustration than for practical purposes. Having said all this, what is the Hamiltonian H, exactly? Well, (2.1) shows that it is the generator of time derivatives – and since this "generation" of derivatives is done by the bracket $[f, g]$, I had better explain that object.

Let f, g and h be any functions of the dynamical variables; usually, they are scalar or vector functions, but they may be of any tensorial rank. Let α and β be mere real numbers. The bracket is antisymmetric:

$$[f, g] = -[g, f] \tag{2.2}$$

It is also linear:

$$[\alpha f + \beta g, h] = \alpha[f, h] + \beta[g, h] \tag{2.3}$$

It is in fact bilinear, for using (2.2) and (2.3) together shows that the linearity property holds on both sides of the bracket. Next up we have the Jacobi identity:

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0 \tag{2.4}$$

The bracket also possesses the Leibniz property:

$$[fg, h] = f[g, h] + [f, h]g \tag{2.5}$$

The next, seldom-mentioned property I call "preservation of type", for lack of a better term. Note that (2.1) does not specify what type of object (i.e. scalar, vector, tensorial) the function f is – but whatever type it is, $\mathrm{d}f/\mathrm{d}t$ is the same type, so $[f, H]$ must be that type as well. This fact is a special case of a broader property, which may be stated thus: if f is a tensor of rank r_1, and g is a tensor of rank r_2, then $[f, g]$ is a tensor of rank $r_1 + r_2$. Thus

H has rank zero, since df/dt must have the same rank as f. This property becomes even stronger in relativity: if f is a tensor of covariant rank r_1 and contravariant rank s_1, and g is a tensor of covariant rank r_2 and contravariant rank s_2, then $[f, g]$ is a tensor of covariant rank $r_1 + r_2$ and contravariant rank $s_1 + s_2$. As a specific example from relativity, early on in Chapter 8 I produce a bracket containing two terms of a particle's 4-velocity U^μ:

$$[U^\mu, U^\nu] = F^{\mu\nu}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. From examples like this I also think of the property as a "preservation of indices".

The final, also seldom-mentioned property concerns the behavior of a bracket under coordinate transformations. Let Ψ denote some coordinate transformation, and f_Ψ the new value of f when subjected to that transformation. For instance, if f is a field variable and Ψ is a translation by displacement $\mathbf{\Delta x}$, then $f_\Psi(\mathbf{x}) = f(\mathbf{x} - \mathbf{\Delta x})$, and if Ψ is an inversion of the coordinate system (i.e. a parity transformation) then $f_\Psi(\mathbf{x}) = f(-\mathbf{x})$. My final property states that, under such a transformation, the bracket transforms in a homomorphic manner:

$$([f, g])_\Psi = [f_\Psi, g_\Psi]_\Psi \tag{2.6}$$

Under a transformation, the bracket $[\,,]$ will change to a new form $[\,,]_\Psi$. Equation (2.6) states how this transformation occurs. In the typical case where the bracket is defined in terms of derivatives of the dynamical variables, you can also insert the transformed derivatives into the bracket to obtain the new one, an easier procedure overall.

As I've already mentioned, one can interpret the bracket as a generator of derivatives – in a more sophisticated text, it might be called a derivation. When a function is placed in one side of the bracket, with the other left unspecified (making the overall object an operator), the bracket becomes some kind of derivative. To offer a concrete example of this interpretation, consider the archetypal bracket, namely the canonical bracket for a single particle:

$$[f, g] = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i} \tag{2.7}$$

It is a simple matter to generate partial derivatives over the phase space:

$$[x^i, \cdot\,] = \frac{\partial\,\cdot}{\partial p_i} \qquad\qquad [p_i, \cdot\,] = -\frac{\partial\,\cdot}{\partial x^i}$$

Perhaps this does not impress you. In that case, change the plane coordinates from (x, y, p_x, p_y) to $r = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}(y/x)$, $l_z = xp_y - yp_x$, and $p_r = (xp_x + yp_y)/\sqrt{x^2 + y^2}$. The polar bracket $[\,,]_p$ is still canonical, and I can now generate the following useful derivatives:

$$[\phi, \cdot\,]_p = \frac{\partial\,\cdot}{\partial l_z} \qquad\qquad [l_z, \cdot\,]_p = -\frac{\partial\,\cdot}{\partial \phi}$$

$$[r, \cdot\,]_p = \frac{\partial\,\cdot}{\partial p_r} \qquad\qquad [p_r, \cdot\,]_p = -\frac{\partial\,\cdot}{\partial r}$$

It would be remiss of me not to mention the most important generator of all, an expression which holds for either bracket:

$$[H, \cdot\,] = -\frac{\mathrm{d}\,\cdot}{\mathrm{d}t}$$

This expression is my preferred answer to the question of what H is.

In quantum mechanics, one says that $\mathbf{p}$ (now an operator) generates space translations, $\mathbf{L}$ generates rotations, and H generates time translations. The generation of derivatives shown above is not a quirk of finite-dimensional canonical brackets, but happens for any bracket, though the specific expressions will change. For another example, see Ref.[26], which gives generators for translations, rotations, and center-of-mass motion using the nonrelativistic MHD bracket.

The bracket generalizes the derivative, and so its properties may be interpreted as generalizations of the properties of derivatives. The linearity property (2.3) echoes the linearity property of derivatives (full or partial), and similarly for the Leibniz property (2.5). The Jacobi identity (2.4) generalizes the commutation of partial derivatives, e.g. $\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x$; indeed, for the canonical bracket, proving the Jacobi identity only requires said commutation.

The antisymmetry property proves more difficult to interpret. However, assuming a cooperative Hamiltonian, Hamilton's equations are time-reversible, so I will apply a time reversal transformation to (2.1) (with f a scalar) and use (2.6):

$$\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right)_T = -\frac{\mathrm{d}f_T}{\mathrm{d}t} = -[f_T, H] = ([f, H])_T = [f_T, H_T]_T = [f_T, H]_T$$

From this expression I infer that

$$[f, g]_T = -[f, g] = [g, f]$$

The bracket's antisymmetry expresses its behavior under time reversal. Take this to be a property of a broader class of brackets, so that you could have other objects with $(f, g)_T = (g, f)$. Then, if you want to build an irreversible system, you can divide the equations of motion into reversible and irreversible parts; under a time reversal transformation, the irreversible parts will change by a sign. Looking at the derivation above, if a separate bracket generates this irreversible motion, it would obey $(f, g)_T = (f, g)$; using the hypothetical time-reversal property, this would imply $(f, g) = (g, f)$, giving a symmetric bracket. This idea is at the core of the metriplectic formalism developed by Dr. Morrison[27], which will be investigated at several points in this dissertation. I note that, for every bracket presented in this dissertation, whether purely Hamiltonian or mixed, the hypothetical property $(f, g)_T = (g, f)$ does indeed hold.

I conclude this section by discussing two important distinctions between broad classes of Hamiltonian systems; that between finite- and infinite-dimensional ones, and that between canonical and noncanonical ones. Finite-dimensional systems have already been encountered, and describe particle or other motion involving discrete elements, whereas infinite-dimensional ones describe field theories. The latter will have brackets over functions of functions (called functionals), and functional (or Frechet) derivatives instead of partial ones. Some of the subtleties of this change, as well as a demonstration of the chain rule for functional derivatives, are presented in Appendix 1.1. However, the distinction between canonical and noncanonical systems is the more salient one, because almost every example presented in this text will be the latter.

While most physicists are well-versed in the former, far fewer take the latter seriously. Indeed, in the finite-dimensional case, the Darboux theorem guarantees that a Hamiltonian system can be reduced into quasi-canonical form: if one writes its bracket as

$$[f, g] = J_{ij}(z) \frac{\partial f}{\partial z^i} \frac{\partial g}{\partial z^j} \tag{2.8}$$

then a suitable coordinate change will transform the matrix $J_{ij}(z)$, of rank $2m$ and dimension $N = 2m + n$, into the form

$$\begin{bmatrix} 0_m & I_m & 0_{m\times n} \\ -I_m & 0_m & 0_{m\times n} \\ 0_{n\times m} & 0_{n\times m} & 0_n \end{bmatrix}$$

with I and 0 denoting the identity and zero matrices. (The rank $2m$ must be even because J_{ij} is antisymmetric.) This new matrix is constant except for places where its rank changes. Comparison to (2.7) shows that J has a canonical part, and a degenerate part. However, transferring to the Darboux coordinates may obscure the underlying physics. More, there is no equivalent of the Darboux theorem for infinite-dimensional systems, where noncanonical brackets are the norm rather than the exception. Rather than taking this to be unfortunate, I tend to regard the richer structure of noncanonical systems as a part of their allure, that structure having physical consequences in its own right; see, for example, Chapter 8 for physical consequences of the Jacobi identity for noncanonical systems, an identity which is automatically satisfied for canonical systems.

2.2 Examples of non-canonical Hamiltonian systems

Now I will look at various ways of arriving at noncanonical Hamiltonian systems. One can, for example, notice that the properties (2.2) - (2.4) are also defining features of the well-studied Lie algebras. The most important Lie algebras are those corresponding to the Lie groups of transformations of vector spaces. These are linear algebras, in the sense that if the s^i, with $i \in \{i, 2, ..., n\}$, are the infinitesimal generators of transformations, then their algebra will be $[s^i, s^j] = C^{ij}_{\ \ k} s^k$ for some "structure constants" $C^{ij}_{\ \ k}$.

Not surprisingly, there are many noncanonical brackets corresponding to Lie algebras. For example, suppose a neutral particle has an intrinsic magnetic moment $\mathbf{s}$, which couples to a magnetic field via an interaction energy $\mathbf{s} \cdot \mathbf{B}$. Then the Hamiltonian $H = p^2/2m + U(x) + \mathbf{s} \cdot \mathbf{B}$, along with the noncanonical bracket

$$[f, g] = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p_i} - \mathbf{s} \cdot \left(\frac{\partial f}{\partial \mathbf{s}} \times \frac{\partial g}{\partial \mathbf{s}}\right)$$

will give the correct equations of motion

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \frac{\mathbf{p}}{2m} \\ \frac{d\mathbf{p}}{dt} &= -\nabla U - \nabla\left(\mathbf{m} \cdot \mathbf{B}\right) \\ \frac{d\mathbf{s}}{dt} &= \mathbf{s} \times \mathbf{B} \end{aligned}$$

In this case the structure constants are the Levi-Civita symbols ϵ^{ijk}, given that one can raise and lower indices freely, and they correspond to the Lie group SO(3).

This bracket also demonstrates the existence of Casimir invariants, which are special constants C such that $[f, C] = 0$ for all functions f. In particular, $[C, H] = 0$, so they are constants of motion. For the spin bracket, any function of the squared magnitude of the spin $\sum_i s_i^2$ is a Casimir invariant. The existence of Casimirs is a trait common to noncanonical brackets; in the finite-dimensional case, the number of algebraically independent Casimirs is equal to the nullity of the matrix J_{ij} in (2.8).

The Lie algebra for SO(3) comes into play in any other problem involving rotations in three dimensions. For example, if you have a rigid body with moments of inertia I_i and angular momenta l_i, then the Hamiltonian and bracket

$$H = \sum_i \frac{l_i^2}{2I_i}$$
$$[f, g] = -\mathbf{l} \cdot \left(\frac{\partial f}{\partial \mathbf{l}} \times \frac{\partial g}{\partial \mathbf{l}} \right)$$

will produce the equations

$$\frac{dl_1}{dt} = l_2 l_3 \left(\frac{1}{I_3} - \frac{1}{I_2} \right)$$
$$\frac{dl_2}{dt} = l_3 l_1 \left(\frac{1}{I_3} - \frac{1}{I_1} \right)$$
$$\frac{dl_3}{dt} = l_1 l_2 \left(\frac{1}{I_1} - \frac{1}{I_2} \right)$$

When written in terms of $\omega_i = l_i/I_i$, these are Euler's equations for rigid body motion. By adding back the canonical bracket (2.7) and introducing a suitable potential energy function, one can also account for translational degrees of freedom and some kinds of external torque. Other kinds of Lie algebras also have physical applications; for example, every one of the three-dimensional Lie algebras corresponds to a specific type of anisotropic, homogeneous cosmology [13]. I will investigate the relativistic equivalents of the above spin systems in Chapter 7, finding that they pose some new difficulties.

Another way to come across noncanonical Hamiltonian systems is to simply attempt to invent such systems from thin air, in an attempt to match some already-known physical system. While this process can be unsatisfying, my experience is that one tends to first find brackets this way, and only later discover how to derive them from more fundamental brackets. For example, for a charged particle in special relativity, one has the position X^μ, and the 4-velocity U^μ, under the influence of a Lorentz force $F^{\mu\nu}U_\nu$, where $F^{\mu\nu}$ is the electromagnetic field tensor. Trying to construct a Hamiltonian and bracket using only these quantities (plus the mass m and charge e of the particle), one arrives at a Hamiltonian $H = (1/2)mU^\mu U_\mu$ and a noncanonical bracket

$$[f, g] = \frac{g^{\mu\nu}}{m}\left(\frac{\partial f}{\partial X^\mu}\frac{\partial g}{\partial U^\nu} - \frac{\partial g}{\partial X^\mu}\frac{\partial f}{\partial U^\nu}\right) + \frac{e}{m^2}F^{\mu\nu}\frac{\partial f}{\partial U^\mu}\frac{\partial g}{\partial U^\nu} \tag{2.9}$$

Similarly, if one were only trying to account for the geodesic law of motion using the same variables, one would find the same Hamiltonian $H = (1/2)mg_{\mu\nu}U^\mu U^\nu$ and a different noncanonical bracket

$$[f, g] = \frac{g^{\mu\nu}}{m}\left(\frac{\partial f}{\partial X^\mu}\frac{\partial g}{\partial U^\nu} - \frac{\partial g}{\partial X^\mu}\frac{\partial f}{\partial U^\nu}\right) + \frac{1}{m}\left(g^{\mu\sigma}{}_{,\alpha}g^{\alpha\nu}U_\sigma - g^{\nu\sigma}{}_{,\alpha}g^{\alpha\mu}U_\sigma\right)\frac{\partial f}{\partial U^\mu}\frac{\partial g}{\partial U^\nu}$$

This approach may seem frivolous, but I show in Chapter 8 that it can produce some interesting insights.

For finite-dimensional systems, Darboux's theorem limits how interesting noncanonical systems can be. In effect, Hamiltonian systems are only distinguished by their rank and the global properties of their canonical parts. However, things do become more interesting when you move to infinite-dimensional systems, because then their rank and nullity can change in unexpected ways. For example, finite-dimensional brackets will, when subjected to a differentiable coordinate change, preserve the number of Casimir invariants they have; by contrast, an infinite-dimensional bracket with no Casimirs can obtain an infinite number of Casimirs under such a change.

The typical infinite-dimensional system is a limit of a finite-dimensional system (usually a particle one) as its number of degrees of freedom approaches infinity. A canonical bracket for N particles will be a sum of N copies of (2.7), so it's natural to posit that it will acquire the form

$$\{f, g\} = \int \left(\frac{\delta f}{\delta q^i} \frac{\delta g}{\delta p_i} - \frac{\delta g}{\delta q^i} \frac{\delta f}{\delta p_i} \right) d^3x \tag{2.10}$$

Note that functional derivatives must now be used, because functions will depend on the basic variables via an integral expression. Fluid brackets (including those for various plasma models) often start from this canonical point, with the bracket describing the structure of the Lagrangian (or material, as opposed to Eulerian, or spatial) variables. One then applies a coordinate change to get to the more convenient Eulerian variables, in the process typically introducing new infinite classes of Casimirs. More detail on the Euler-Lagrange map can be found in the next two sections. This procedure will also be used, in Section 6.1, to get the noncanonical bracket for relativistic MHD, and in Chapter 3 to obtain brackets for Hall and Extended MHD.

Equation (2.10) can be taken to express a weighted sum over canonical brackets, with the weighting given by d^3x. More weighted sums occur in kinetic theory, where the state of a fluid is described by a distribution function $f(x,p,t)$ over both position and momentum. In this case, one can write the weighted canonical bracket as

$$\{F,G\}_c = \int f\left(\frac{\partial F_f}{\partial x^i}\frac{\partial G_f}{\partial p_i} - \frac{\partial G_f}{\partial x^i}\frac{\partial F_f}{\partial p_i}\right) d^6z \tag{2.11}$$

where $z = (x,p)$ and $F_f \equiv \delta F/\delta f$. This so-called Lie-Poisson bracket is, one can argue, the simplest noncanonical bracket. This bracket describes a single species of a fluid, but introducing more species is as simple as duplicating the bracket.

The electric and magnetic fields provide, sensibly enough, an infinite-dimensional field theory. They also have their own noncanonical bracket

$$\{F,G\}_f = 4\pi c\int \frac{\delta F}{\delta \mathbf{E}}\cdot\left(\nabla\times\frac{\delta G}{\delta \mathbf{B}}\right) - \frac{\delta G}{\delta \mathbf{E}}\cdot\left(\nabla\times\frac{\delta F}{\delta \mathbf{B}}\right) d^3x \tag{2.12}$$

which will, given the Hamiltonian $H = \int(E^2/2 + B^2/2)d^3x$, produce the two vectorial Maxwell equations without sources. The bracket (2.12) is actually a canonical one in disguise. Switching to $\mathbf{A}$ from $\mathbf{B} = \nabla\times\mathbf{A}$ will, via a chain rule calculation covered in the Appendix 1.1, give $\delta F/\delta\mathbf{A} = \nabla\times(\delta F/\delta\mathbf{B})$.

Thus (2.12) can be rewritten

$$\{F,G\}_f = 4\pi c \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \frac{\delta G}{\delta \mathbf{A}} - \frac{\delta G}{\delta \mathbf{E}} \cdot \frac{\delta F}{\delta \mathbf{A}} \right) d^3x \tag{2.13}$$

showing that **A** and **E** are canonically conjugate to each other, an idea also used in the ADM formalism of Hamiltonian gravity. However, owing to the noninvertibility of the transformation between **A** and **B**, the bracket (2.13) is less general than the bracket (2.12), applying only to gauges in which $\nabla\phi = 0$.

The interaction terms between matter and EM fields appear in two additional bracket components. The first, magnetic one, is:

$$\{F,G\}_B = \int -\frac{e\mathbf{B}}{m^2 c} \cdot \left(\frac{\partial F_f}{\partial \mathbf{v}} \times \frac{\partial G_f}{\partial \mathbf{v}} \right) d^6z \tag{2.14}$$

The magnetic bracket (2.14) has a similar origin as the particle one (2.9); that is, it comes from using $\mathbf{v}$ instead of a canonical momentum $\mathbf{p} = m\mathbf{v} - e\mathbf{A}$. The second interaction term involves the electric field:

$$\{F,G\}_E = \frac{4\pi e}{m} \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \frac{\partial G_f}{\partial \mathbf{v}} - \frac{\delta G}{\delta \mathbf{E}} \cdot \frac{\partial F_f}{\partial \mathbf{v}} \right) d^6z \tag{2.15}$$

This one also results from the use of $\mathbf{v}$, as the switch from $\mathbf{p}$ and $\mathbf{A}$ to $\mathbf{v}$ and $\mathbf{A}$ will alter $\delta F/\delta\mathbf{A}$ in (2.13).

Combining all four brackets (2.11) - (2.15) and using the straightforward Hamiltonian

$$H = \int f m \frac{v^2}{2}\, d^6z + \frac{1}{4\pi} \int \left(\frac{B^2}{2} + \frac{E^2}{2} \right)\, d^3x$$

will produce the following equations of motion, together constituting the Vlasov equation with the dynamical Maxwell equations:

$$\frac{\partial f}{\partial t} = -\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{x}} + \frac{e}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial f}{\partial \mathbf{v}}$$
$$\frac{\partial \mathbf{E}}{\partial t} = c\nabla\times\mathbf{B} + 4\pi e\int f\mathbf{v}\, d^3v$$
$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla\times\mathbf{E}$$

Unfortunately, the relativistic equivalent poses some new difficulties, so far only partially solved. These will be covered in Section 4.4.

2.3 Hamiltonian fluids

2.3.1 Euler's equations, fluid action

My aim in this section is to demonstrate how one important noncanonical Hamiltonian system, that of Eulerian fluids, can be obtained as a reduction of an infinite-dimensional canonical system. To do so I must introduce the distinction between Lagrangian and Eulerian coordinates. In Eulerian coordinates, one chooses a fixed point and watches various quantities (density and velocity being most pertinent) as they develop and change at that point. By contrast, in Lagrangian coordinates one gives each distinct element of fluid its own label, subject only to continuity (in the sense of being homeomorphic with $\mathbb{R}^3$), and that label remains fixed as the fluid element moves about. In Eulerian coordinates (denoted by x and t), with a fluid described by density ρ, pressure p and velocity $\mathbf{v}$, I have Euler's equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p$$

This form disguises its origin as a transformation of Newton's Second Law. In Lagrangian coordinates, a fluid element labelled by a three-dimensional label space a will have a position $q(a, t)$, under the condition that this function be 1-1 at fixed t – that is, two distinct elements of fluid do not end up in the same place at once. With dot denoting time derivative, its equation of motion will be

$$\rho \ddot{\mathbf{q}} = -\nabla p$$

Apparently the Lagrangian coordinates are an easier place to start, and their Hamiltonian structure will reflect this fact.

I also need to know what equations other quantities, such as mass density and specific entropy, will obey. Broadly speaking, one will derive these equations of motion from expressions which show how a quantity is "carried along" by a fluid, conserving some quantity. For example, there is no dissipation introduced yet, so specific entropy (entropy per unit mass) will simply be conserved: $s(a, t) = s(a, 0) \equiv s_0$. The mass of a given fluid element must also be conserved, and expressing this mass as the product of density times an infinitesimal volume gives

$$\rho \, d^3x = \rho_0 \, d^3a$$

with $\rho_0 = \rho(a, 0)$ as before. I can write $d^3x = \mathcal{J} d^3a$, where $\mathcal{J} \equiv |\partial q/\partial a|$ is the Jacobian of the transformation from a to q, as defined in Appendix 1.2. This Jacobian must be nonzero, from the earlier condition that the transformation be injective.

Using that quantity, I can write $\rho = \rho_0/\mathcal{J}$. The Euler-Lagrange map for ρ is thus

$$\rho(x,t) = \int \rho(q)\,\delta(x-q)\,d^3q = \int \rho_0(a)\,\delta(x-q(a,t))\,d^3a \qquad (2.16)$$

From here I can derive the continuity equation

$$\frac{\partial \rho}{\partial t} = \int \rho_0(a)\,\dot{q}_i\,\delta_i^{'}(x-q(a,t))d^3a = \int \rho\,\dot{q}_i\,\delta_i^{'}(x-q)\,d^3q = -\nabla\cdot(\rho\mathbf{v})$$

where I have used the chain rule in the first step, a coordinate change from a to q in the second, and an integration by parts in the third. An advective equation of motion for s may also be derived; in fact, if one changes to the entropy density $\sigma = s\rho$, one acquires another version of the continuity equation.

To write down an action principle for the Lagrangian fluid, I need an analogy of the particle action $S = \int(T-V)dt$. Thus I define an internal energy function $U(\rho,s)$ which obeys the thermodynamic relations

$$T = \left.\frac{\partial U}{\partial s}\right|_\rho \qquad p = \rho^2\left.\frac{\partial U}{\partial \rho}\right|_s \qquad (2.17)$$

as may easily be checked. The fluid action is then

$$\begin{aligned} S = \int L\,dt &= \int\int \left(\frac{1}{2}\rho\dot{q}^2 - \rho U(\rho,s)\right) d^3q\,dt \\ &= \int\int \left(\frac{1}{2}\rho_0\dot{q}^2 - \rho_0 U(\frac{\rho_0}{\mathcal{J}},s_0)\right) d^3a\,dt \equiv \int\int \mathcal{L}\,d^3a\,dt \qquad (2.18) \end{aligned}$$

The variational principle on the Lagrangian $L(q,\dot{q},\partial q/\partial a)$ is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) + \frac{\partial}{\partial a^j}\left(\frac{\partial \mathcal{L}}{\partial q^i_{\ ,j}}\right) - \frac{\partial \mathcal{L}}{\partial q^i}$$

provided the surface integral vanishes. As it happens, there is no contribution from the third term, because all spatial dependence has been incorporated into the Jacobian $|\partial q^i/\partial a^j|$. For that matter, the only contribution from the second term comes from the factor of $\mathcal{J}$ in the energy function of (2.18). Using $\partial\mathcal{J}/\partial q^i_{,j} = A^j_i$ and $\partial A^j_i/\partial a^j = 0$ from Appendix 1.2, the variational principle yields

$$\rho_0 \ddot{q}^i + A^i_{\ j}\frac{\partial}{\partial a^j}\left(\frac{\rho_0^2}{\mathcal{J}^2}\frac{\partial U}{\partial \rho}\right) = 0$$

Dividing by $\mathcal{J}$, then using both the definition of pressure (2.17) and the expression $\partial/\partial q^i = (A^i_{\ j}/\mathcal{J})\partial/\partial a^i$ from the Appendix, I get the desired expression

$$\rho\ddot{\mathbf{q}} + \nabla p = 0$$

One can devise an action principle using Eulerian variables ρ and $\mathbf{v}$, but at the cost of introducing the "Clebsch potentials" as extra variables. On the other hand, the Hamiltonian formulation of Eulerian fluid physics follows directly from the Lagrangian Hamiltonian description, and requires no extra variables to be introduced, so I turn to that now.

2.3.2 The Euler-Lagrange map and the fluid bracket

The Lagrangian action (2.18) has canonical momentum $\pi = \partial\mathcal{L}/\partial\dot{\mathbf{q}} = \rho_0\dot{\mathbf{q}}$ and Hamiltonian density

$$\mathcal{H} = \pi_i \dot{q}^i - \mathcal{L} = \frac{\pi_i \pi^i}{2\rho_0} + \rho_0 U(\frac{\rho_0}{\mathcal{J}}, s_0)$$

which is, of course, the fluid energy. Accompanying it is a canonical (albeit infinite-dimensional) bracket

$$\{f, g\} = \int \left(\frac{\delta f}{\delta q^i} \frac{\delta g}{\delta \pi_i} - \frac{\delta g}{\delta q^i} \frac{\delta f}{\delta \pi_i} \right) d^3a \tag{2.19}$$

From the Hamiltonian $H = \int \mathcal{H}\, d^3a$ I can derive Hamilton's equations

$$\frac{\partial f}{\partial t} = \{f, H\}$$

which work out to be $\dot{q}^i = \pi^i/\rho_0$ and $\dot{\pi}_i = -\nabla_i p$. My next goal is to convert the canonical bracket (2.19) into a noncanonical one using Eulerian variables. To do so I need the following Euler-Lagrange transformations, written in analogy with (2.16):

$$\begin{aligned} \rho(x,t) &= \int \rho_0(a)\, \delta(x - q(a,t))\, d^3a \\ \sigma(x,t) &= \int \rho_0(a)\, s_0(a)\, \delta(x - q(a,t))\, d^3a \\ m^i(x,t) &= \int \pi^i\, \delta(x - q(a,t))\, d^3a \end{aligned} \tag{2.20}$$

I can use variations in q and π to induce variations in ρ, σ and $\mathbf{m}$, using a procedure outlined in the Appendix. These induced variations are

$$\begin{aligned} \delta\rho &= -\int \rho_0\, \delta_i'(x - q)\, \delta q^i\, d^3a \\ \delta\sigma &= -\int \rho_0\, s_0\, \delta_i'(x - q)\, \delta q^i\, d^3a \\ \delta m^i &= \int \delta\pi^i\, \delta(x - q) - \pi^i\, \delta_j'(x - q)\, \delta q^j\, d^3a \end{aligned} \tag{2.21}$$

The variation in an arbitrary functional δf will be the same whether expressed in terms of the Lagrangian or the Eulerian variables, which is to say

$$\delta f = \int \left(\frac{\delta f}{\delta q^i} \delta q^i + \frac{\delta f}{\delta \pi^i} \delta \pi^i \right) \, d^3x = \int \left(\frac{\delta f}{\delta \rho} \delta \rho + \frac{\delta f}{\delta \sigma} \delta \sigma + \frac{\delta f}{\delta m^i} \delta m^i \right) \, d^3a$$

Inserting the variations (2.21) into the right hand side gives

$$\begin{aligned}
\delta f = \int \int & - \left(\rho_0 \frac{\delta f}{\delta \rho} + \rho_0 s_0 \frac{\delta f}{\delta \sigma} + \pi^j \frac{\delta f}{\delta m^j} \right) \delta_i^{'}(x - q) \, \delta q^i \\
& + \frac{\delta f}{\delta m^i} \, \delta \pi^i \, \delta(x - q) \, d^3a \, d^3x \\
= \int \int & \left(\rho_0 \frac{\partial}{\partial x^i} \frac{\delta f}{\delta \rho} + \rho_0 s_0 \frac{\partial}{\partial x^i} \frac{\delta f}{\delta \sigma} + \pi^j \frac{\partial}{\partial x^i} \frac{\delta f}{\delta m^j} \right) \, \delta q^i \, \delta(x - q) \\
& + \frac{\delta f}{\delta m^i} \, \delta \pi^i \, \delta(x - q) \, d^3a \, d^3x
\end{aligned}$$

Thus the Lagrangian functional derivatives can be rewritten as

$$\begin{aligned}
\frac{\delta f}{\delta q^i} &= \int \left(\rho_0 \frac{\partial}{\partial x^i} \frac{\delta f}{\delta \rho} + \rho_0 s_0 \frac{\partial}{\partial x^i} \frac{\delta f}{\delta \sigma} + \pi^j \frac{\partial}{\partial x^i} \frac{\delta f}{\delta m^j} \right) \, \delta(x - q) \, d^3a \\
\frac{\delta f}{\delta \pi^i} &= \int \frac{\delta f}{\delta m^i} \, \delta(x - q) \, d^3a = \left. \frac{\delta f}{\delta m^i} \right|_{x = q(a,t)}
\end{aligned}$$

These can be inserted into the canonical bracket (2.19) to produce (after the remaining delta function eliminates the a integration) the noncanonical Eulerian fluid bracket

$$\begin{aligned}
\{f, g\} = \int & \left[\rho \frac{\partial}{\partial x^i} \left(\frac{\delta f}{\delta \rho} \right) \frac{\delta g}{\delta m_i} - \rho \frac{\partial}{\partial x^i} \left(\frac{\delta g}{\delta \rho} \right) \frac{\delta f}{\delta m_i} \right] \\
& + \left[\sigma \frac{\partial}{\partial x^i} \left(\frac{\delta f}{\delta \sigma} \right) \frac{\delta g}{\delta m_i} - \sigma \frac{\partial}{\partial x^i} \left(\frac{\delta g}{\delta \sigma} \right) \frac{\delta f}{\delta m_i} \right] \\
& + \left[m_j \frac{\partial}{\partial x^i} \left(\frac{\delta f}{\delta m_j} \right) \frac{\delta g}{\delta m_i} - m_j \frac{\partial}{\partial x^i} \left(\frac{\delta g}{\delta m_j} \right) \frac{\delta f}{\delta m_i} \right] d^3x
\end{aligned}$$

This is the simplest example of a procedure which can be used to generate a wide class of Hamiltonian, Eulerian matter models. For instance, adding a Lie-dragged two-form (the magnetic field) produces MHD, and the resulting conversion of the canonical bracket proceeds identically in relativistic MHD. This calculation is covered in Section 6.1. Introducing two such dragged forms allows one to generate brackets for Hall MHD, Extended MHD, and inertial MHD, as covered in Chapter 3.

2.4 Hamiltonian MHD

2.4.1 Overview of MHD

First I give the equations of ideal nonrelativisitic ideal MHD, with the force law and Faraday's law expressed in two alternative ways:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\frac{\nabla p}{\rho} + \frac{1}{4\pi\rho}\big[\,(\nabla\times\mathbf{B})\times\mathbf{B}\big] \tag{2.22}$$

$$= -\frac{\nabla p}{\rho} + \frac{1}{4\pi\rho}\nabla\cdot\big(\mathcal{I}\,B^2/2 - \mathbf{B}\otimes\mathbf{B}\big) \tag{2.23}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) \tag{2.24}$$

$$= -\mathbf{B}\,\nabla\cdot\mathbf{v} + \mathbf{B}\cdot\nabla\mathbf{v} - \mathbf{v}\cdot\nabla\mathbf{B} \tag{2.25}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \tag{2.26}$$

$$\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s = 0\,. \tag{2.27}$$

Here ρ is the fluid density, p its pressure, s its specific entropy, $\mathbf{v}$ the velocity field, and $\mathbf{B}$ the magnetic field. In (2.23) the symbol $\mathcal{I}$ represents the identity tensor. The current $\mathbf{j}$ and electric field $\mathbf{E}$ have been eliminated from these equations, but they can be recovered from the ideal conductor Ohm's Law, $\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B} = 0$, and Ampére's Law, $\mathbf{j} = (c/4\pi)\nabla \times \mathbf{B}$.

Observe the alternative versions of (2.22) and (2.24) given in (2.23) and (2.25), respectively. These equations differ by terms involving $\nabla \cdot \mathbf{B}$, and both Eqs. (2.24) and (2.25) preserve the initial condition $\nabla \cdot \mathbf{B} = 0$, which can be seen by rewriting (2.25):

$$\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{B}\,\nabla \cdot \mathbf{v} + \mathbf{B} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \mathbf{v}\,\nabla \cdot \mathbf{B}\,. \tag{2.28}$$

Upon taking the divergence,

$$\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = -\nabla \cdot (\mathbf{v}\,\nabla \cdot \mathbf{B})\,. \tag{2.29}$$

Consequently, if $\nabla \cdot \mathbf{B}$ is initially identically zero it remains so as well. Equation (2.28) shows that forms (2.24) and (2.25) are equivalent when the magnetic field is divergenceless, although the former reveals its Faraday law origin, while the latter show an advected magnetic flux pointing to the MHD frozen-in constraint. Geometrically (2.25) is $\partial \mathbf{B}/\partial t + \pounds_{\mathbf{v}}\mathbf{B} = 0$, where $\pounds_{\mathbf{v}}\mathbf{B}$ is the Lie derivative of $\mathbf{B}$, a vector density dual to a 2-form. Similarly, Eqs. (2.22) and (2.23) differ by a $\nabla \cdot \mathbf{B}$ term, with the former revealing its Lorentz force origin via a clearly identified current, while the latter takes the form of a conservation law, which Gudunov [10] showed to be superior for numerical computation.

I have distinguished these two forms because they possess different Hamiltonian structures. In Ref. [30] a Poisson bracket was given for the form with (2.22) and (2.24), but this structure required building in the initial condition $\nabla \cdot \mathbf{B} = 0$. However, an alternative and more natural form was first given in Refs. [31, 26], which is entirely free from $\nabla \cdot \mathbf{B} = 0$, it being only a possible choice for an initial condition. Later in the paper I will demonstrate relativistic equivalents of both structures, and the two will also differ by the divergence of a 4-vectorial quantity; to be equivalent, said divergence must vanish, which will motivate my use of the new magnetic quantity h^μ.

Should one wish to add displacement current back into MHD, as is done in the most prevalent version of relativistic MHD, the momentum equation would have to be altered as follows:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{v} = -\frac{\nabla p}{\rho} + \frac{1}{4\pi\rho}\left[\left(\nabla \times \mathbf{B} + \frac{\partial}{\partial t}\left(\frac{\mathbf{v}}{c^2} \times \mathbf{B}\right)\right) \times \mathbf{B}\right]. \qquad (2.30)$$

However, the new term, when compared to $\partial\mathbf{v}/\partial t$, scales as

$$\frac{B^2}{4\pi\rho c^2} = \left(\frac{v_A}{c}\right)^2,$$

where v_A is the Alfvén velocity. In the nonrelativistic limit, waves involving disturbances of the matter must also travel much slower than the speed of light, allowing one to drop the displacement current. This also means that relativistic MHD is free to add said displacement current back in (albeit constrained by Ohm's Law), while still reducing to conventional MHD in the nonrelativistic limit: one simply needs to keep in mind that said limit goes beyond just setting $v/c \to 0$.

2.4.2 Hamiltonian and bracket

Now I cast nonrelativistic MHD into Hamiltonian form. As usual, the two ingredients will be a Hamiltonian and a bracket, with all equations of motion given by

$$\frac{\partial \xi}{\partial t} = [\xi, H] \tag{2.31}$$

for any ξ that is a function of the field variables. Because this is a field theory, both the Hamiltonian and the bracket $[\,,\,]$ will involve an integration over the whole space, and instead of partial derivatives the bracket will use functional derivatives, which I describe in Appendix 1.1. As is typical, the Hamiltonian is simply the energy:

$$H = \int d^3x \left(\frac{1}{2}\rho v^2 + \rho U(\rho, s) + \frac{1}{8\pi} B^2 \right)$$

The scalar field $U(\rho, s)$ is an internal energy function, from which one can derive a temperature $T = \partial U/\partial s$ and a pressure $p = \rho^2 \partial U/\partial \rho$. The gradient of pressure is particularly important:

$$\nabla p = 2\rho \frac{\partial U}{\partial \rho} \nabla \rho + \rho^2 \frac{\partial^2 U}{\partial \rho^2} \nabla \rho + \rho^2 \frac{\partial^2 U}{\partial s \partial \rho} \nabla s \tag{2.32}$$

The Hamiltonian's various functional derivatives are:

$$\frac{\delta H}{\delta \rho} = \frac{1}{2}\mathbf{v} \cdot \mathbf{v} + U + \frac{p}{\rho} \qquad \frac{\delta H}{\delta s} = \rho \frac{\partial U}{\partial s} \qquad \frac{\delta H}{\delta \mathbf{v}} = \rho \mathbf{v} \qquad \frac{\delta H}{\delta \mathbf{B}} = \mathbf{B}$$

Now for the bracket, which (in standard variables) is quite a beast:

$$
\begin{aligned}
[F,G] = -\int d^3x \Bigg(& \left[\frac{\delta F}{\delta \rho}\nabla \cdot \frac{\delta G}{\delta \mathbf{v}} - \frac{\delta G}{\delta \rho}\nabla \cdot \frac{\delta F}{\delta \mathbf{v}}\right] + \left[\frac{\delta F}{\delta \mathbf{v}} \cdot \left(\frac{(\nabla \times \mathbf{v})}{\rho} \times \frac{\delta G}{\delta \mathbf{v}}\right)\right] \\
& + \left[\frac{\nabla s}{\rho} \cdot \left(\frac{\delta F}{\delta s}\frac{\delta G}{\delta \mathbf{v}} - \frac{\delta G}{\delta s}\frac{\delta F}{\delta \mathbf{v}}\right)\right] \\
& + \Bigg[\mathbf{B} \cdot \left(\frac{1}{\rho}\frac{\delta F}{\delta \mathbf{v}} \cdot \nabla \frac{\delta G}{\delta \mathbf{B}} - \frac{1}{\rho}\frac{\delta G}{\delta \mathbf{v}} \cdot \nabla \frac{\delta F}{\delta \mathbf{B}}\right) \\
& + \mathbf{B} \cdot \left(\left(\nabla \frac{1}{\rho}\frac{\delta F}{\delta \mathbf{v}}\right) \cdot \frac{\delta G}{\delta \mathbf{B}} - \left(\nabla \frac{1}{\rho}\frac{\delta G}{\delta \mathbf{v}}\right) \cdot \frac{\delta F}{\delta \mathbf{B}}\right)\Bigg]\Bigg)
\end{aligned}
\tag{2.33}
$$

The bracket's linearity and antisymmetry are apparent, and its Jacobi identity follows from the proof given in Appendix 1.3. While the proof is conducted in four dimensions and uses momentum and entropy density rather than velocity and specific entropy, neither alteration affects the identity.

I will take some care in deriving the equations of motion from the bracket; later in the dissertation, more steps will be skimmed over. To derive the continuity equation at a specific location $\mathbf{x_0}$, use the test function

$$
\xi = \int d^3x \, \rho \, \delta(\mathbf{x} - \mathbf{x_0}) \, .
$$

The delta function eliminates the integral on both sides of (2.31), and (since $\delta F / \delta \rho$ appears only once in (2.33)), I have

$$
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})
$$

evaluated implicitly at x_0; this point is arbitrary, so the continuity equation (2.26) holds over the whole space. Every equation of motion requires such delta test functions, but they are all eliminated in identical fashions, so going forward I will not mention that nicety.

The entropy equation follows a very similar procedure, with only one term and no integrations by part to worry about:

$$\frac{\partial s}{\partial t} = -\nabla s \cdot \mathbf{v}$$

reproducing (2.27).

The magnetic equation will pull down two terms from (2.33), the first of which will require an integration by parts:

$$\mathbf{B} \cdot \left(\frac{1}{\rho} \frac{\delta G}{\delta \mathbf{v}} \cdot \nabla \frac{\delta F}{\delta \mathbf{B}} \right) = B_i \frac{1}{\rho} \frac{\delta G}{\delta v_j} \nabla_j \frac{\delta F}{\delta B_i}$$

$$\Rightarrow -\frac{\delta F}{\delta B_i} \nabla_j \left(B_i \frac{1}{\rho} \frac{\delta G}{\delta v_j} \right) = -\frac{\delta F}{\delta \mathbf{B}} \cdot \left(\frac{1}{\rho} \frac{\delta G}{\delta \mathbf{v}} \cdot \nabla \right) \mathbf{B} - \left(\frac{\delta F}{\delta \mathbf{B}} \cdot \mathbf{B} \right) \nabla \cdot \left(\frac{1}{\rho} \frac{\delta G}{\delta \mathbf{v}} \right)$$

The other term does not require an integration by parts, but it does merit some caution with indices:

$$\mathbf{B} \cdot \left(\left(\nabla \frac{1}{\rho} \frac{\delta G}{\delta \mathbf{v}} \right) \cdot \frac{\delta F}{\delta \mathbf{B}} \right) = B_i \left(\left(\nabla_i \frac{1}{\rho} \frac{\delta G}{\delta v_j} \right) \frac{\delta F}{\delta B_j} \right)$$

So, using the functional derivative $\delta H / \delta \mathbf{v} = \rho \mathbf{v}$, I have

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= -\mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{v} + \mathbf{B} \cdot \nabla \mathbf{v} \\ &= -\mathbf{v} \cdot \nabla \mathbf{B} - \mathbf{B} \nabla \cdot \mathbf{v} + \mathbf{B} \cdot \nabla \mathbf{v} + \mathbf{v} \nabla \cdot \mathbf{B} \\ &= \nabla \times (\mathbf{v} \times \mathbf{B}) \end{aligned}$$

recalling the discussion surrounding (2.25).

The momentum equation is most complicated. First off, I'll consider the terms that give derivatives of the potential energy function U, from the first and second lines of (2.33). Together they give

$$\begin{aligned}
&-\left[\nabla\left(U+\rho\frac{\partial U}{\partial\rho}\right)-\frac{1}{\rho}\nabla s\left(\rho\frac{\partial U}{\partial s}\right)\right]\\
=&-\left[\frac{\partial U}{\partial\rho}\nabla\rho+\frac{\partial U}{\partial s}\nabla s+\frac{\partial U}{\partial\rho}\nabla\rho+\rho\frac{\partial^2 U}{\partial\rho^2}\nabla\rho+\rho\frac{\partial^2 U}{\partial s\partial\rho}\nabla s-\frac{\partial U}{\partial s}\nabla s\right]\\
=&-\frac{1}{\rho}\left[\left(2\rho\frac{\partial U}{\partial\rho}+\rho^2\frac{\partial^2 U}{\partial\rho^2}\right)\nabla\rho+\rho^2\frac{\partial^2 U}{\partial s\partial\rho}\nabla s\right]=-\frac{\nabla p}{\rho}
\end{aligned}$$

using (2.32). So far so good. Two terms involving $\mathbf{v}$ appear, both from the first line of (2.33). They give

$$-\nabla\left(\frac{v^2}{2}\right)+\mathbf{v}\times(\nabla\times\mathbf{v})=-\mathbf{v}\cdot\nabla\mathbf{v}$$

using a standard vector calculus identity. Finally, two terms, one each from the third and fourth lines of (2.33), involve the magnetic field:

$$\frac{1}{4\pi\rho}\nabla_i\left(\frac{B_jB_j}{2}-B_iB_j\right)$$

So, overall, I have the velocity equation in the form (2.23), as desired.

Note that the bracket (2.33) lends itself to expressing the velocity equation in conservation form, i.e. with most quantities written as the divergence of some tensor. The magnetic parts are already in such a form, and the velocity parts partially so. If one switches to momentum $\mathbf{m}=\rho\mathbf{v}$ and entropy density $\sigma=\rho s$, this form becomes even more pronounced, as now the momentum terms in the momentum equation are also in divergence form. More importantly, the bracket simplifies considerably:

$$[F,G] = -\int d^3x \Bigg(\rho \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\delta \rho} \right) \tag{2.34}$$

$$+ \sigma \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) + \mathbf{m} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\delta \mathbf{m}} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\delta \mathbf{m}} \right)$$

$$+ \mathbf{B} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\delta \mathbf{B}} + \left(\nabla \frac{\delta F}{\delta \mathbf{m}} \right) \cdot \frac{\delta G}{\delta \mathbf{B}} - \left(\nabla \frac{\delta G}{\delta \mathbf{m}} \right) \cdot \frac{\delta F}{\delta \mathbf{B}} \right) \Bigg)$$

The previous bracket's many stray factors of ρ are now gone, and every part of the bracket is in semidirect form, unlike (2.33) in which the velocity portion was a bit odd. In fact, while the relativistic Hamiltonian MHD uses a close equivalent of (2.34), it has no equivalent to (2.33), because of the more complex coordinate change: there $m^\mu = (\rho + p - h_\mu h^\mu)u^\mu + \alpha h^\mu$ rather than just $m^\mu = \rho u^\mu$.

2.4.3 Casimirs and alternative brackets

With an integration by parts on the last two magnetic terms of bracket (2.33), one alters its magnetic portion to this form:

$$[F,G] = -\int d^3x \Bigg(\mathbf{B} \cdot \left(\frac{1}{\rho} \frac{\delta F}{\delta \mathbf{v}} \cdot \nabla \frac{\delta G}{\delta \mathbf{B}} - \frac{1}{\rho} \frac{\delta G}{\delta \mathbf{v}} \cdot \nabla \frac{\delta F}{\delta \mathbf{B}} \right)$$

$$+ \mathbf{B} \cdot \left(\left(\nabla \frac{\delta F}{\delta \mathbf{B}} \right) \cdot \frac{1}{\rho} \frac{\delta G}{\delta \mathbf{v}} - \left(\nabla \frac{\delta G}{\delta \mathbf{B}} \right) \cdot \frac{1}{\rho} \frac{\delta F}{\delta \mathbf{v}} \right) \Bigg)$$

or, in terms of momentum,

$$[F,G] = -\int d^3x \Bigg(\mathbf{B} \cdot \left(\frac{\delta F}{\delta \mathbf{m}} \cdot \nabla \frac{\delta G}{\mathbf{B}} - \frac{\delta G}{\delta \mathbf{m}} \cdot \nabla \frac{\delta F}{\mathbf{B}} \right)$$

$$= + \mathbf{B} \cdot \left(\left(\nabla \frac{\delta F}{\delta \mathbf{B}} \right) \cdot \frac{\delta G}{\delta \mathbf{m}} - \left(\nabla \frac{\delta G}{\delta \mathbf{B}} \right) \cdot \frac{\delta F}{\delta \mathbf{m}} \right) \Bigg)$$

These forms bring the advantage that the magnetic equation is immediately in Maxwellian form, without having to add a term proportional to $\nabla \cdot \mathbf{B}$; however, the momentum equation also changes by a term proportional to the same. However, the Jacobi identity no longer holds unconditionally, but now requires the initial condition $\nabla \cdot \mathbf{B}$. Thus, this bracket is considerably less general: the previous ones (2.33) and (2.34) represent a larger class of dynamical systems, only some of which (the divergenceless ones) correspond to physical systems. This distinction between the two brackets will be discussed further in the context of relativistic MHD.

Casimir invariants are functionals C such that $[F, C] = 0$ for all functionals F of the field variables. One class are the entropy Casimirs:

$$C_1 = \int \rho f\left(\frac{\sigma}{\rho}\right) d^3x$$

where f is an arbitrary function of one real number. Another important Casimir is the cross-helicity:

$$C_2 = \int \frac{\mathbf{m} \cdot \mathbf{B}}{\rho} d^3x$$

although in this case it is only a Casimir for functionals F not depending on the entropy density σ. Finally, there is the magnetic helicity

$$C_3 = \int \mathbf{A} \cdot \mathbf{B}\, d^3x$$

The first two Casimirs C_1 and C_2 have natural equivalents in the relativistic theory, but an equivalent to C_3 remains undiscovered, since the calculation that establishes $[F, C_3] = 0$ relies on a vector identity that only holds in three dimensions.

2.5 Metriplectic systems

2.5.1 Overview, examples and properties

Like Hamiltonian systems, metriplectic systems derive all equations of motion from a generating function and a bracket:

$$\frac{\partial f}{\partial t} = \{f, G\}$$

The generator G and the bracket both divide cleanly into Hamiltonian and dissipative parts, written

$$G = H + \lambda S \tag{2.35}$$

$$\{f, g\} = [f, g] + (f, g)$$

Here λ is a constant included to provide consistent units. Let f, g, and h be functions of the dynamical or field variables, and α, β constants. The anti-symmetric bracket has the following familiar properties, which were discussed at length in Section 2.1:

$$[\alpha f + \beta g, h] = \alpha[f, h] + \beta[g, h]$$

$$[fg, h] = f[g, h] + [f, h]g$$

$$[f, g] = -[g, f]$$

$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0$$

$$[f, S] = 0$$

The last equation says that the function S in the generator, usually the entropy, is a Casimir of the antisymmetric bracket. Meanwhile, the symmetric bracket has a related set of properties:

$$(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$$
$$(fg, h) = f(g, h) + (f, h)g$$
$$(f, g) = (g, f)$$
$$(f, f) \geq 0 \tag{2.36}$$
$$(f, H) = 0$$

There is a second derivative expression related to the Jacobi identity, which will be discussed in a later section.

As my first example, I put the damped simple harmonic oscillator into metriplectic form. To the normal phase space for a single particle, I add an entropy variable s, and I suppose that the energy dissipated from the oscillating particle goes to a reservoir with internal energy $U(s)$. Then my equations of motion are (using $dS = dQ/T$ to get the second one)

$$\dot{\mathbf{p}} = -k\mathbf{x} - \frac{b}{m}\mathbf{p}$$
$$\dot{s} = \frac{b}{m^2 T} p^2$$

The generating functions are

$$G = H + \lambda S \qquad H = \frac{p^2}{2m} + \frac{kx^2}{2} + U(s) \qquad S = s$$

Any needed equations of motion are generated in the usual fashion by

$$\frac{\partial f}{\partial t} = \{f, G\} = [f, H] + (f, S)$$

with brackets obeying $[f, S] = (f, H) = 0$ for all f. These brackets are

$$[f, g] = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p^i}$$
$$(f, g) = \frac{bT}{\lambda}\left(\frac{\partial f}{\partial p^i} - \frac{p^i}{mT}\frac{\partial f}{\partial s}\right)\left(\frac{\partial g}{\partial p^i} - \frac{p^i}{mT}\frac{\partial g}{\partial s}\right) \tag{2.37}$$

This is about as trivial as metriplectic systems get, but it's useful to have another example in the bag.

The system can be generalized to nonisotropic, coupled oscillators with differing masses, as well. The index i will now run over $3N$ indices, and the mass m, spring constant k and dissipative coefficient b will all be replaced by $3N \times 3N$-dimensional matrices M, K, and B. I require that M be positive definite, to avoid negative or zero masses and to allow its invertibility. I also require that K be positive semidefinite (only restoring forces are considered), as well as B (energy is only dissipated, not added). The equations of motion are now

$$\dot{p}_i = -K_{ij}x_j - B_{ij}M^{-1}_{jk}p_k$$
$$\dot{s} = \frac{1}{T}B_{ij}(M^{-1}_{ik}p_k)(M^{-1}_{jl}p_l)$$

These equations are simpler when written in terms of velocity, since momentum only appears in the form $v_i = M^{-1}_{ij}p_k$, but I'm keeping the form that gives me a canonical antisymmetric bracket. The generating functions are

$$G = H + \lambda S \qquad H = \frac{1}{2}M^{-1}_{ij}p_ip_j + \frac{1}{2}K_{ij}x_ix_j + U(s) \qquad S = s$$

Finally, the brackets are

$$[f, g] = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial x^i}\frac{\partial f}{\partial p^i}$$

$$(f, g) = \frac{B_{ij}T}{\lambda}\left(\frac{\partial f}{\partial p^i} - \frac{M_{il}^{-1}p^l}{T}\frac{\partial f}{\partial s}\right)\left(\frac{\partial g}{\partial p^j} - \frac{M_{jk}^{-1}p^k}{T}\frac{\partial g}{\partial s}\right) \tag{2.38}$$

The positivity property (2.36) follows if the dissipation matrix B is positive semidefinite.

As a slightly more sophisticated example, consider the rigid body system detailed earlier in Section 2.2. It has Casimirs of the total angular momentum squared, $C = \omega^2$. I can create a symmetric bracket which leaves H invariant by a direct method:

$$(f, g) = -\left[\left(\frac{\partial H}{\partial \omega_i}\frac{\partial H}{\partial \omega_j} - \delta_{ij}\frac{\partial H}{\partial \omega_l}\frac{\partial H}{\partial \omega_l}\right)\frac{\partial f}{\partial \omega_i}\frac{\partial g}{\partial \omega_j}\right] \tag{2.39}$$

The exact behavior of this system will depend on my choice of entropy S, but I can outline some qualitative features of the dynamics. Because the system is still conservative, the dynamics take place along the ellipsoid of constant energy. If entropy increases monotonically with ω^2, then the system will relax to an orientation along the longest axis of the ellipsoid, which corresponds to the smallest principal axis of inertia; if entropy decreases monotonically with ω^2, the system relaxes to the smallest axis of the ellipsoid, corresponding to the largest axis. This echoes a result from textbook classical mechanics which states that the middle axis will not be stable.

2.5.2 Metriplectic Navier-Stokes

Next I look at dissipative fluids. This is a field theory, with the fields being density ρ, specific entropy s (or, later on, entropy density $\sigma = \rho s$), and fluid velocity $\mathbf{v}$ or momentum density $\mathbf{m} = \rho\mathbf{v}$. Additionally, I have the derived quantities: pressure p, heat flux $\mathbf{q}$, and stress tensor σ_{ij}. The system is governed by the Navier-Stokes equation plus the entropy and continuity equations:

$$\begin{aligned}
\frac{\partial v_i}{\partial t} &= v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k} \\
\frac{\partial s}{\partial t} &= -v_k \frac{\partial s}{\partial x_k} + \frac{\sigma_{ik}}{\rho T}\frac{\partial v_i}{\partial x_k} - \frac{1}{\rho T}\frac{\partial q_k}{\partial x_k} \\
\frac{\partial \rho}{\partial t} &= -\frac{\partial}{\partial x_k}\left(\rho v_k\right)
\end{aligned}$$

To close the system there is an equation of state, for my purposes an internal energy function $U(\rho, s)$, plus the following constitutive relations:

$$q_k = -\kappa \frac{\partial T}{\partial x_k}$$

$$\sigma_{ik} = \Lambda_{ikmn} \frac{\partial v_m}{\partial x_n}$$

$$\Lambda_{ikmn} = \eta\left(\delta_{im}\delta_{kn} + \delta_{in}\delta_{km} - \frac{2}{3}\delta_{ik}\delta_{mn}\right) + \zeta\delta_{ik}\delta_{mn} \tag{2.40}$$

Now to put this in metriplectic form. The generator (2.35) has components

$$H = \int \left(\frac{m^2}{2\rho} + \rho U(\rho, s)\right) d^3x \qquad S = \int n\, s\, d^3x$$

The antisymmetric bracket is

$$\begin{aligned}[f,g] = \quad & \int \rho \left(\left(\frac{\delta g}{\delta \mathbf{m}} \cdot \nabla \right) \frac{\delta f}{\delta \rho} - \left(\frac{\delta f}{\delta \mathbf{m}} \cdot \nabla \right) \frac{\delta g}{\delta \rho} \right) \\ & + \mathbf{m} \cdot \left(\left(\frac{\delta g}{\delta \mathbf{m}} \cdot \nabla \right) \frac{\delta f}{\delta \mathbf{m}} - \left(\frac{\delta f}{\delta \mathbf{m}} \cdot \nabla \right) \frac{\delta g}{\delta \mathbf{m}} \right) \\ & + \sigma \left(\left(\frac{\delta g}{\delta \mathbf{m}} \cdot \nabla \right) \frac{\delta f}{\delta \sigma} - \left(\frac{\delta f}{\delta \mathbf{m}} \cdot \nabla \right) \frac{\delta g}{\delta \sigma} \right) d^3x \end{aligned}$$

Meanwhile, the symmetric bracket is

$$\begin{aligned}(f,g) = \frac{1}{\lambda} \int T \Lambda_{ikmn} & \left[\frac{\partial}{\partial x_i} \frac{\delta f}{\delta m_k} - \frac{1}{T} \frac{\partial v_i}{\partial x_k} \frac{\delta f}{\delta \sigma} \right] \left[\frac{\partial}{\partial x_i} \frac{\delta g}{\delta m_k} - \frac{1}{T} \frac{\partial v_i}{\partial x_k} \frac{\delta g}{\delta \sigma} \right] \\ & + \kappa T^2 \frac{\partial}{\partial x_k} \left[\frac{1}{T} \frac{\delta f}{\delta \sigma} \right] \frac{\partial}{\partial x_k} \left[\frac{1}{T} \frac{\delta g}{\delta \sigma} \right] d^3x \end{aligned} \tag{2.41}$$

In Section 6.4 these brackets will be generalized to the relativistic case. This is highly nontrivial, as the relativistic equivalent of Γ involves adding many projection operators.

2.5.3 General form of metriplectic brackets

Having presented a fair number of examples, I was able to observe a pattern in the form of the symmetric brackets. To illustrate this form, let me rewrite each such bracket in a more revealing manner. The anisotropic damped oscillator bracket (2.38) can be rewritten

$$(f,g) = \frac{B_{ij}}{\lambda T} \left(\frac{\partial H}{\partial s} \frac{\partial f}{\partial p^i} - \frac{\partial f}{\partial s} \frac{\partial H}{\partial p^i} \right) \left(\frac{\partial H}{\partial s} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial s} \frac{\partial H}{\partial p^i} \right)$$

The rigid body one (2.39) becomes

$$(f,g) = \frac{1}{2} \left(\frac{\partial H}{\partial \omega_i} \frac{\partial f}{\partial \omega_j} - \frac{\partial f}{\partial \omega_i} \frac{\partial H}{\partial \omega_j} \right) \left(\frac{\partial H}{\partial \omega_i} \frac{\partial g}{\partial \omega_j} - \frac{\partial g}{\partial \omega_i} \frac{\partial H}{\partial \omega_j} \right)$$

Finally, the Navier-Stokes bracket (2.41) can be modified to read

$$(f,g) = \frac{1}{\lambda T}\int \Lambda_{ikmn}\left[\frac{\delta H}{\delta\sigma}\frac{\partial}{\partial x_i}\frac{\delta f}{\delta m_k} - \frac{\delta f}{\delta\sigma}\frac{\partial}{\partial x_i}\frac{\delta H}{\delta m_k}\right]\left[\frac{\delta H}{\delta\sigma}\frac{\partial}{\partial x_m}\frac{\delta g}{\delta m_n} - \frac{\delta g}{\delta\sigma}\frac{\partial}{\partial x_m}\frac{\delta H}{\delta m_n}\right]$$
$$+\kappa\left[\frac{1}{T}\frac{\delta H}{\delta\sigma}\frac{\partial}{\partial x_k}\frac{1}{T}\frac{\delta f}{\delta\sigma}\right]\left[\frac{1}{T}\frac{\delta H}{\delta\sigma}\frac{\partial}{\partial x_k}\frac{1}{T}\frac{\delta g}{\delta\sigma}\right]d^3x$$

It should be clear that all the metriplectic brackets fall into a common form, containing symmetric parts, antisymmetric ones, and two derivatives of the Hamiltonian. Schematically, one can write

$$(f,g) = ([f,H]_i,[g,H]_i)_o$$

with antisymmetric "inner" bracket and symmetric "outer" bracket. These are not actual brackets, because the inner one will tend to add tensorial indices while the outer one contracts them; however, they are bilinear and can be expressed in terms of structure coefficients. Even better, these structure coefficients tend to be constant, leaving the entire nonconstant portion of the symmetric bracket a result of the derivatives of the Hamiltonian. This may be related to Dr. Morrison's discovery of "triple brackets" in [6].

2.5.4 Metriplectic speculations: QM, Dirac and Jacobi

In standard nonrelativistic quantum mechanics, imagine that you use a more general operator in place of the Hamiltonian:

$$i\hbar\frac{\partial}{\partial t}|\Psi> = G|\Psi>$$

where G has a Hermitian and anti-Hermitian part:

$$G = H + S \qquad H = H^\dagger \qquad S = -S^\dagger$$

Then if you work through the equivalent of the proof of Ehrenfest's theorem, which can be found in many QM textbooks, you will now find

$$\frac{\partial}{\partial t} < f >= \frac{i}{h} (< [f, H] > + < (f, S) >)$$

where square brackets denote the commutator and round brackets the anti-commutator. This expression is pleasantly similar to the metriplectic equation of motion

$$\frac{\partial f}{\partial t} = [f, H] + (f, S)$$

However, the commutators and anticommutators do not factor as cleanly as in the metriplectic case; for one thing, you must either have $(H, S) \neq 0$ or $[H, S] \neq 0$, and the probability density will not be conserved. Perhaps a sort of Dirac brackets for metriplectic systems is the way out of this problem; even if not, it's interesting in its own right. Let a quantity A not belong to the nullspace of the symmetric bracket (f, g), so that $(A, A) \geq 0$. Then the new bracket

$$(f, g)_D = (f, g) - \frac{(f, A)(A, g)}{(A, A)}$$

now satisfies $(f, A) = 0$ for all f. This bracket retains its bilinearity, symmetry, and non-negativity properties. The last can be seen by working in the normalized eigenbasis of the original bracket (guaranteed to exist by its symmetry), then splitting ∇f into components parallel and orthogonal to ∇H, and noticing that what remains of $(f, f)_D$ is the squared norm of the orthogonal part. This "Dirac bracket" is much more convenient than its antisymmetric brother, because you can use an odd number of new invariants and you don't have to worry about the singularity of a matrix. To fix the problems with the quantum mechanical speculation, imagine using H as one such invariant (making it now a conserved quantity, since of course it commutes with itself), and the identity operator I as another (so that now probability density is conserved). I'm still looking for a concrete example to work with, though.

I'm also looking for an equivalent to the Jacobi identity

$$[[f,g],h] + [[g,h],f] + [[h,f],g] = 0$$

for finite-dimensional metriplectic systems. While I haven't found anything so far that would merit a new axiom for the symmetric brackets, I have found that the Jacobi equivalent can be expressed in a compact form. In finite dimensions, the brackets are written as bilinear forms with antisymmetric and symmetric parts:

$$\{f,g\} = b^{\mu\nu} f_{,\mu} g_{,\nu} = [f,g] + (f,g) = j^{\mu\nu} f_{,\mu} g_{,\nu} + g^{\mu\nu} f_{,\mu} g_{,\nu}$$

Let's start taking the "metric" part of metriplectic seriously, and see what geometrical structures can tell us. If the bilinear form $b^{\mu\nu}$ has null vectors, then restrict each form, and any gradients, to the subspace that's orthogonal to the nullspace of $b^{\mu\nu}$. (Properly speaking I should stick a tilde or something on all quantities to indicate projection, but I'll omit them.) Use the full bilinear form $b^{\mu\nu}$, which is hopefully invertible on this subspace, to raise and lower indices. (I've tried this using $g^{\mu\nu}$ to raise and lower indices, and it doesn't seem to work.) Define the torsionless (symmetric) and torsional (antisymmetric) parts of the affine connection as follows:

$$S_{\lambda\mu\nu} = \frac{1}{2}\left(g_{\mu\lambda,\nu} + g_{\lambda\nu,\mu} - g_{\nu\mu,\lambda}\right)$$

$$T_{\lambda\mu\nu} = \frac{1}{2}\left(j_{\mu\lambda,\nu} + j_{\lambda\nu,\mu} - j_{\nu\mu,\lambda}\right)$$

$$\Gamma_{\lambda\mu\nu} = S_{\lambda\mu\nu} + T_{\lambda\mu\nu} = \frac{1}{2}\left(b_{\mu\lambda,\nu} + b_{\lambda\nu,\mu} - b_{\nu\mu,\lambda}\right)$$

These expressions can be inverted:

$$g_{\mu\nu,\lambda} = S_{\nu\mu\lambda} + S_{\mu\lambda\nu}$$

$$j_{\mu\nu,\lambda} = T_{\nu\mu\lambda} + T_{\mu\lambda\nu}$$

$$b_{\mu\nu,\lambda} = \Gamma_{\nu\mu\lambda} + \Gamma_{\mu\lambda\nu}$$

Use that affine connection to define a covariant derivative in the standard manner:

$$f_{;\mu} = f_{,\mu}$$

$$f_{;\mu\nu} = \Gamma^{\alpha}{}_{\mu\nu} f_{;\alpha} + f_{;\mu,\nu} = \Gamma^{\alpha}{}_{\mu\nu} f_{,\alpha} + f_{,\mu\nu}$$

Now look at a term in the Jacobi expression.

$$\{\{f,g\},h\} = \{b^{\mu\nu} f_{,\mu} g_{,\nu}, h\}$$

$$= b^{\mu\nu}{}_{,\lambda} b^{\lambda\sigma} f_{,\mu} g_{,\nu} h_{,\sigma} + b^{\mu\nu} b^{\lambda\sigma} f_{,\mu\lambda} g_{,\nu} h_{,\sigma} + b^{\mu\nu} b^{\lambda\sigma} f_{,\mu} g_{,\nu\lambda} h_{,\sigma}$$

$$= b_{\alpha\beta,\lambda} b^{\alpha\mu} b^{\beta\nu} b^{\lambda\sigma} f_{,\mu} g_{,\nu} h_{,\sigma} + b^{\mu\nu} b^{\lambda\sigma} f_{,\mu\lambda} g_{,\nu} h_{,\sigma} + b^{\mu\nu} b^{\lambda\sigma} f_{,\mu} g_{,\nu\lambda} h_{,\sigma}$$

$$= \Gamma_{\beta\alpha\lambda} b^{\alpha\mu} b^{\beta\nu} b^{\lambda\sigma} f_{,\mu} g_{,\nu} h_{,\sigma} + \Gamma_{\alpha\lambda\beta} b^{\alpha\mu} b^{\beta\nu} b^{\lambda\sigma} f_{,\mu} g_{,\nu} h_{,\sigma} + b^{\mu\nu} b^{\lambda\sigma} f_{,\mu\lambda} g_{,\nu} h_{,\sigma} + b^{\mu\nu} b^{\lambda\sigma} f_{,\mu} g_{,\nu\lambda} h_{,\sigma}$$

$$= \Gamma^{\alpha}{}_{\mu\lambda} f_{,\alpha} b^{\lambda\nu} b^{\mu\sigma} g_{,\nu} h_{,\sigma} + \Gamma^{\alpha}{}_{\mu\nu} g_{,\alpha} b^{\mu\lambda} b^{\nu\sigma} f_{,\lambda} h_{,\sigma} + b^{\mu\nu} b^{\lambda\sigma} f_{,\mu\lambda} g_{,\nu} h_{,\sigma} + b^{\mu\nu} b^{\lambda\sigma} f_{,\mu} g_{,\nu\lambda} h_{,\sigma}$$

$$= f_{;\mu\lambda} g_{;\nu} h_{;\sigma} b^{\lambda\nu} b^{\mu\sigma} + g_{;\mu\nu} f_{\lambda} h_{\sigma} b^{\mu\lambda} b^{\nu\sigma}$$

In all,

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\}$$

$$= (f_{;\mu\nu} g_{;\lambda} h_{;\sigma} + g_{;\mu\nu} h_{;\lambda} f_{;\sigma} + h_{;\mu\nu} f_{;\lambda} g_{;\sigma}) \left(b^{\mu\lambda} b^{\nu\sigma} + b^{\mu\sigma} b^{\nu\lambda} \right)$$

I'd still like to find something more restrictive than this, particularly if it lets me raise/lower indices with $g^{\mu\nu}$, instead.

Chapter 3

Derivation of the Hall and Extended MHD Brackets

This chapter will outline a derivation of the Hall MHD bracket, first stated in Abdelhamid et al. [1]. In doing so, they will draw on recent work by Lingam et al. [17] and Keramidas et al [7], along with seminal work by Mahajan [20]. In particular, these lines of work have emphasized the fact that various plasma models – ordinary MHD, Hall MHD, inertial MHD, and extended MHD – are distinguished by their individual variants of the magnetic flux conservation law.

All four models now have Hamiltonian form. The bracket for ordinary MHD was derived in [26]; in fact, this bracket serves for inertial MHD as well [19]. Abdelhamid et al [1] stated, but did not derive, a bracket that applies to extended MHD and all three of its submodels. These Hamiltonian forms use the standard Eulerian variables $(\rho, s, \mathbf{m}, \mathbf{B})$, with a modified $\mathbf{B}^*$ replacing $\mathbf{B}$ in extended and inertial MHD. In principle, all four Hamiltonian models should from distinct descriptions using Lagrangian variables; while Keramidas et al [7] recently derived action principles for each, starting from a two-fluid model, these action principles mix Eulerian and Lagrangian variables.

3.1 Flux Conservation Laws

The essential difference between the four models is the form of their flux conservation laws, of which each has a different version. Mahajan [20] pointed out that a composite fluid will have a number of conserved magnetic helicities equal to the number of species in the fluid. This attribute persists into the various MHD models, even though they are single-fluid models. The archetypal flux conservation law is that of ordinary MHD, $\mathbf{B} \cdot \mathbf{d^2q} = \mathbf{B_0} \cdot \mathbf{d^2a}$. The a variables denote a label space, whose continuous values identify fluid elements at $t = 0$, while the coordinates $q(a, t)$ describe the point to which a specific element flows; thus, $q(a, 0) = a$. More explicitly, I write the flux conservation law as

$$\epsilon_{ijk} B^i(q, t) dq^j dq^k = \epsilon_{ijk} B_0^i(a) da^j da^k \tag{3.1}$$

This expression can be manipulated into a transformation rule for the magnetic field:

$$B^i = \frac{B_0^j}{\mathcal{J}} \frac{\partial q^i}{\partial a^j} \tag{3.2}$$

where $\mathcal{J}$ is the Jacobian determinant of the invertible transformation from a to q.

There are two distinct ways one can modify the flux conservation law (3.1). First, one can advect a flux different from that of $\mathbf{B}$; with an appropriate choice of this flux, one gets inertial MHD. Second, the same flux can be advected, but along a path distinct from that of the fluid. This second approach gives Hall MHD. Specifically, while the fluid itself flows from a to a point $q(a,t)$, the flux element moves from a to a distinct point $q_f(a,t)$, as illustrated by an illuminating figure which does not exist yet. Flux conservation is now

$$\epsilon_{ijk} B^i dq_f^j dq_f^k = \epsilon_{ijk} B_0^i \, da^j \, da^k$$

which gives rise to the transformation rule

$$B^i = \frac{B_0^j}{\mathcal{J}_f} \frac{\partial q_f^i}{\partial a^j} \tag{3.3}$$

The flux Jacobian $\mathcal{J}_f$ is also invertible, and can be written

$$\mathcal{J}_f = \epsilon_{ijk} \epsilon^{lmn} \frac{\partial q_f^i}{\partial a^l} \frac{\partial q_f^j}{\partial a^m} \frac{\partial q_f^k}{\partial a^n}$$

from which one can derive the expression $d\mathcal{J}_f/dt = \mathcal{J} \partial \dot{q}_f^i / \partial q_f^i$.

Taking a full time derivative of $\mathbf{B}(q_f, t)$ in equation (3.2) gives

$$\begin{aligned}
\frac{dB^i}{dt} = \frac{\partial B^i}{\partial t} + \dot{q}_f^j \frac{\partial B^i}{\partial q^j} &= \frac{B_0^j}{\mathcal{J}_f} \frac{\partial \dot{q}_f^i}{\partial a^j} - \frac{B_0^j}{\mathcal{J}_f} \frac{\partial q_f^i}{\partial a^j} \frac{\partial \dot{q}_f^k}{\partial q_f^k} \\
&= \frac{B_0^k}{\mathcal{J}_f} \frac{\partial q_f^l}{\partial a^k} \frac{\partial a^m}{\partial q_f^l} \frac{\partial \dot{q}_f^i}{\partial a^m} - \frac{B_0^j}{\mathcal{J}_f} \frac{\partial q_f^i}{\partial a^j} \frac{\partial \dot{q}_f^k}{\partial q_f^k} \\
&= B^j \frac{\partial \dot{q}_f^i}{\partial q^j} - B^i \frac{\partial \dot{q}_f^k}{\partial q_f^k}
\end{aligned}$$

This equation shows $\mathbf{B}$ advected along q_f, as desired. Since $\mathbf{B}$ is divergenceless, I can add a term proportional to $(\nabla \cdot \mathbf{B})\dot{\mathbf{q}}$ and put the equation in the more familiar Faraday form

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla_q \times (\dot{\mathbf{q}}_f \times \mathbf{B})$$

So far, so good. However, complications arise when you look for the other equations of motion. Some fluid attributes (density, specific entropy) are transported along the flow lines $q(a,t)$, not q_f: mass conservation is described by $n(q,t)d^3q = n_0(a)d^3a$, and entropy conservation (the system has no dissipation) by $s(q,t) = s_0(a,t)$. As a result, the label corresponding to the magnetic field will differ from the label on the other quantities. This situation is shown in an even more illuminating figure, which also does not exist yet. In this nonexistent figure, the fluid element labelled by a flows to $q(a,t)$, while a different label a' shows the origin of the flux element that has been advected to $q(a,t) = q_f(a',t)$. For future use I will need two additional quantities: the point $q(a',t)$, to which the a' element flows, and the difference $q_d(a',t)$ between $q_f(a',t)$ and $q(a',t)$. All these quantities are related via

$$q(a,t) = q_f(a',t) = q(a',t) + q_d(a',t)$$

More relations are available, for example $a'(a,t) = q_f^{-1}(q(a,t),t)$, and in principle I could eliminate all but two of these quantities, but it is simpler to keep the extras around.

3.2 Hall MHD Action

Every point corresponds to two labels; in Hall MHD, these turn out to correspond to ion and electron quantities. Thus q, for example, will appear as both $q(a,t)$ and $q(a',t)$. To simply following expressions I write $q' \equiv q(a',t)$, $q'_d \equiv q_d(a',t)$, $(q')^i{}_{,j} \equiv \partial(q')^i/\partial(a')^j$ and $(q'_d)^i{}_{,j} \equiv \partial(q'_d)^i/\partial(a')^j$, with unprimed expressions such as q denoting unprimed quantities like $q(a,t)$. If I treat primed and unprimed quantities separately, then the full Euler-Lagrange equations, using Lagrangian density $\mathcal{L}$, will be

$$\begin{aligned}\Bigg[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}^i}\right)+\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(\dot{q}')^i}\right)+\frac{\partial}{\partial a^j}\left(\frac{\partial\mathcal{L}}{\partial q^i_{\ ,j}}\right)\\ +\frac{\partial}{\partial(a')^j}\left(\frac{\partial\mathcal{L}}{\partial(q')^i_{\ ,j}}\right)-\frac{\partial\mathcal{L}}{\partial q^i}-\frac{\partial\mathcal{L}}{\partial(q')^i}\Bigg]_{a'=q_f^{-1}(q(a,t))}&=0\end{aligned} \tag{3.4}$$

with a similar expression for q_d. I think these Euler-Lagrange equations can be obtained via Dirac delta function manipulations on a six-dimensional label space, but I'm not sure how. Anyway, many of the terms in the Euler-Lagrange equations are superfluous: only the first four terms will contribute in the q variation, and only the second and fourth terms in the q_d one.

If it were written in terms of ion and electron velocities q_i and q_e, the Lagrangian density would be standard:

$$\begin{aligned}\mathcal{L}=&\frac{1}{2}m_in_0\dot{q}_i^2+\frac{en_0}{c}\dot{\mathbf{q}}_i\cdot\mathbf{A}(q,t)-en_0\phi(q,t)-n_0U_i\left(\frac{n_0}{\mathcal{J}},s_0\right)\\ &+\frac{1}{2}m_in_0\dot{q}_e^2-\frac{en_0}{c}\dot{\mathbf{q}}_e\cdot\mathbf{A}(q'+q'_d,t)+en_0\phi(q'+q'_d,t)-n_0U_e\left(\frac{n_0}{\mathcal{J}_f},s_0\right)\end{aligned} \tag{3.5}$$

In Hall MHD, I treat electron velocity as being different from ion velocity (unlike in regular MHD), but nonetheless neglect terms of order m_e/m_i. The variables used will be center-of-mass velocity $\dot{q}$, and the drift velocity of electrons relative to ions, $\dot{q}_d$. In terms of ion and electron velocity I have

$$\dot{\mathbf{q}}=\frac{m_i\dot{\mathbf{q}}_i+m_e\dot{\mathbf{q}}_e}{m_i+m_e}\qquad\dot{\mathbf{q}}_d=\dot{\mathbf{q}}_e-\dot{\mathbf{q}}_i$$

Inverting these equations and neglecting terms of the order of the mass ratio, I have

$$\begin{aligned}\dot{\mathbf{q}}_i &= \dot{\mathbf{q}} - \frac{m_e}{m_i + m_e}\dot{\mathbf{q}}_d \approx \dot{\mathbf{q}} \\ \dot{\mathbf{q}}_e &= \dot{\mathbf{q}} + \frac{m_i}{m_i + m_e}\dot{\mathbf{q}}_d \approx \dot{\mathbf{q}} + \dot{\mathbf{q}}_d\end{aligned}$$

Thus, rewriting (3.5), setting $m = m_i + m_e \approx m_i$, and remembering the distinction between primed and unprimed labels, the Lagrangian density becomes

$$\begin{aligned}\mathcal{L} = \frac{1}{2} m n_0 \dot{q}^2 &+ \frac{e n_0}{c}\left[\dot{\mathbf{q}} \cdot \mathbf{A}(q,t) - \dot{\mathbf{q}}' \cdot \mathbf{A}(q' + q_d', t) - \dot{\mathbf{q}}_d' \cdot \mathbf{A}(q' + q_d', t)\right] \\ &- e n_0\left[\phi(q,t) - \phi(q' + q_d', t)\right] - n_0\left[U_i\left(\frac{n_0}{\mathcal{J}}, s_0\right) + U_e\left(\frac{n_0}{\mathcal{J}_f}, s_0\right)\right]\end{aligned}$$

In the q equation of motion, the terms arising from $\phi(q,t) - \phi(q' + q_d', t)$ cancel, and most of the terms coming from $\dot{\mathbf{q}} \cdot \mathbf{A}(q,t) - \dot{\mathbf{q}}' \cdot \mathbf{A}(q' + q_d', t)$, cancel in pairs after the $q = q' + q_d'$ evaluation. The only surviving term comes from the advective parts of $d\mathbf{A}/dt$, which are different for the two terms. Setting $p_e = n^2 \partial U_e/\partial n$, $p_i = n^2 \partial U_i/\partial n$, and $p = p_e + p_i$, I have, for the q equation of motion,

$$\left[m n_0 \ddot{q}_i - \frac{e n_0}{c}\left(\dot{q}_d'\right)^j A_{i,j}(q' + q_d', t) + \frac{e n_0}{c}\left(\dot{q}_d'\right)^j A_{j,i}(q' + q_d', t) + \mathcal{J}\nabla p\right]_{a' = q_f^{-1}(q(a,t))}$$

which can, multiplying by $1/\mathcal{J}$ and using $\mathbf{j} = -en\mathbf{q}_d$, be simplified to

$$\rho\ddot{\mathbf{q}} = -\nabla p + \frac{1}{c}\mathbf{j} \times \mathbf{B} \tag{3.6}$$

In the q_d equation of motion, the three final terms come from the full derivative $d\mathbf{A}(q' + q'_d, t)/dt$, and the pressure term comes from the q'_d dependence of $\mathcal{J}_f$:

$$\frac{en_0}{c}\Big((\dot{q}')^j\, A_{j,i}(q' + q'_d, t) + (\dot{q}'_d)^j\, A_{j,i}(q' + q'_d, t)\Big) - en_0\phi_{,i}(q' + q'_d, t)$$
$$+\mathcal{J}\nabla p_e - \frac{en_0}{c}\big((q')^j A_{i,j}(q' + q'_d, t) - (q'_d)^j A_{i,j}(q' + q'_d, t)\big) - \frac{\partial A_i}{\partial t}$$

with the whole thing evaluated at $q(a, t) = q(a', t) + q_d(a', t)$ as usual. Reordering and simplifying, one finds

$$\mathbf{E} + \frac{\dot{\mathbf{q}} \times \mathbf{B}}{c} = \frac{\mathbf{j} \times \mathbf{B}}{nec} - \frac{\nabla p_e}{ne} \tag{3.7}$$

which is Ohm's Law for Hall MHD.

However, the theory is not yet complete, because I am left with no way to find the evolution of $\dot{\mathbf{q}}_d$. I can also not perform the usual Legendre transform, because I have no expression $\dot{q}_d(q, q_d, \pi, \pi_d)$. Nonetheless, one can write down a phase space action whose four variations give all the needed equations. The corresponding density is

$$\begin{aligned}\pi \cdot \dot{\mathbf{q}} + \pi_d \cdot \dot{\mathbf{q}}_d - \frac{1}{2mn_0}\pi^2 + \frac{e}{mc}\left(\pi \cdot \mathbf{A}(q, t) - \pi \cdot \mathbf{A}(q' + q'_d, t)\right) \\ -\frac{1}{2\mathcal{J}_f}\left(\frac{c}{n_0 e}\right)^2 (\nabla_a \times \pi_d)^i\, (\nabla_a \times \pi_d)^j \left(\frac{\partial q^k}{\partial a^i} + \frac{\partial q_d^k}{\partial a^i}\right)\left(\frac{\partial q_k}{\partial a^i} + \frac{\partial q_{(d)k}}{\partial a^i}\right) \\ +n_0 e\left(\phi(q' + q'_d, t) - \phi(q, t)\right) - n_0\left[U_i\left(\frac{n_0}{\mathcal{J}}, s_0\right) + U_e\left(\frac{n_0}{\mathcal{J}_f}, s_0\right)\right]\end{aligned} \tag{3.8}$$

The middle term, note, is simply $B^2/2$, expanded using (3.2) and $\pi_d = -(en_0/c)\mathbf{A}(q, t)$.

There are four phase space variations; as when using (3.4), one sets $q = q' + q'_d$ after taking variations. (And, once again, I hope to fill in this step using a delta-function argument.) Thus the π variation gives

$$\dot{\mathbf{q}} = \frac{\pi}{mn_0}$$

The π_d variation involves an integration by parts on the middle term of the density (3.8), giving (maybe add details later)

$$\dot{\mathbf{q}}_d = -\frac{c}{n_0 e}\nabla \times \mathbf{B}$$

i.e. $\mathbf{j} = c\nabla \times \mathbf{B}$, the missing piece of the earlier tangent space action.

In the q variation, most of the terms once again vanish. The $\partial q/\partial a$ terms in the middle term of (3.8) give two factors of $(B^i B^j/2)_{,i}$, and the $\mathbf{J}_f$ in the same term gives a factor of $(B^2/2)_{,i}$. The remaining terms proceed similarly as in the tangent space calculation. The overall result is

$$-\dot{\pi}^i - C^k_j \frac{\partial}{\partial a^k}\left[B^i B^j - \frac{B^2}{2}\delta^{ij}\right] + \mathcal{J}\nabla^i p = 0$$

which, given $\mathbf{j} = c\nabla \times \mathbf{B}$ and the ϵ-ϵ identity, is the same as (5.10). Finally, the q_d variation gives

$$-\dot{\pi}^i_d - \frac{e}{mc}\pi^j \nabla^i A_j + C^k_j \frac{\partial}{\partial a^k}\left[B^i B^j - \frac{B^2}{2}\delta^{ij}\right] + n_0 e \nabla^i \phi + \mathcal{J}\nabla^i p_e = 0$$

Considering that $\pi_d = -(en_0/c)\mathbf{A}(q, t)$, and $\dot{\pi}_d$ will thus have two terms, this equation is identical to (3.7).

3.3 The Hall MHD Euler-Lagrange map and bracket

The Eulerian quantities ρ, σ, and m^i are defined via standard Euler-Lagrange maps:

$$\begin{aligned}\rho(x,t) = \int \rho(a,t)\,\delta(x-q(a,t))\,d^3q = \int \rho_0(a)\,\delta(x-q(a,t))\,d^3a \\ \sigma(x,t) = \int \rho_0(a)s_0(a)\,\delta(x-q(a,t))\,d^3a \\ m^i(x,t) = \int \pi^i(a,t)\,\delta(x-q(a,t))\,d^3a\end{aligned} \tag{3.9}$$

Apparently, when I induce variations later on, the first two quantities will only have δq variations from the delta functions, while $\mathbf{m}$ will have a δq and $\delta\pi$ variation. The odd one is the magnetic Euler-Lagrange map:

$$\begin{aligned}B^i(x,t) &= \int B_0^j(a')\frac{\partial q_f^i}{\partial (a')^j}\delta(x-q_f(a',t))\,d^3a' \\ &= \int B_0^j(a')\left(\frac{\partial q^i}{\partial (a')^j}+\frac{\partial q_d^i}{\partial (a')^j}\right)\delta(x-q(a',t)-q_d(a',t))\,d^3a'\end{aligned} \tag{3.10}$$

This will have q and q_d dependence via q_f, and π_d dependence via $\pi_d = -(en_0/c)\mathbf{A}$.

Because $\mathbf{B}$ is a Lie-dragged two-form, a suitable choice of gauge will make the vector potential $\mathbf{A}$ a Lie-dragged one-form, so that $A_i\,dq_f^i = A_{(0)i}\,da^i$. As a result,

$$A_j\frac{\partial q^j}{\partial a^i} = A_{(0)} \;\;\Rightarrow\;\; A_i = \frac{A_{(0)j}C^j_{\ i}}{\mathcal{J}_f} \tag{3.11}$$

where $C^j_{\ i}$ is the cofactor matrix of the coordinate transformation $\partial q_f^j/\partial a^i$. Thus, something peculiar occurs when one tries to take phase space variations: because it is related to a Lie-dragged quantity, $\pi_d(t)$ is determined by $\pi_d(0)$, its value at $t = 0$, i.e. on the label space. Inserting $\mathbf{A} = -(c/en_0)\pi_d$ into (3.11), one finds the similar transformation rule

$$\tilde{\pi}_i = \tilde{\pi}_{(0)j} \frac{C_i^j}{\mathcal{J}_f} \tag{3.12}$$

which will be important later. Note that I will have to be more careful with boundary conditions than before: I set δq and δq_d to zero at $t = 0$, but $\delta \pi_d$ is nonzero on the boundary.

I can now show how the Eulerian variables change under variations in the Lagrangian phase-space ones, using (3.9) and (3.10):

$$\begin{aligned}
\delta\rho &= \int \rho_0(a)\, \delta_i'(x - q(a,t))\, \delta q^i \, d^3a \\
\delta\sigma &= \int \sigma_0(a)\, \delta_i'(x - q(a,t))\, \delta q^i \, d^3a \\
\delta m^i &= \int \pi^i \, \delta_j'(x - q(a,t))\, \delta q^j + \delta(x - q)\, \delta\pi^i \, d^3a \\
\delta B^i &= \int B_0^j(a') \left(\frac{\partial q^i}{\partial (a')^j} + \frac{\partial q_d^i}{\partial (a')^j} \right) \delta_k'(x - q(a',t) - q_d(a',t)) \left(\delta q^k + \delta q_d^k\right) \\
&\quad - B_0^j \delta_k' \left(x - q(a',t) - q_d(a',t)\right) \left(\frac{\partial q^k}{\partial (a')^j} + \frac{\partial q_d^k}{\partial (a')^j} \right) \left(\delta q^i + \delta q_d^i\right) \\
&\quad + \frac{\partial B_0^j}{\partial \pi_d^k} \delta\pi_d^k \left(\frac{\partial q^k}{\partial (a')^j} + \frac{\partial q_d^k}{\partial (a')^j} \right) \delta(x - q' - q_d')\, d^3a'
\end{aligned} \tag{3.13}$$

Note that the introduction of q_d and π_d, which do not appear in regular MHD, nonetheless do not require me to add any new Eulerian variables.

The variation induced by an arbitrary function f, in both Lagrangian and Eulerian variables, is

$$\begin{aligned}\delta f &= \int \frac{\delta f}{\delta \rho}\delta\rho + \frac{\delta f}{\delta \sigma}\delta\sigma + \frac{\delta f}{\delta m^i}\delta m^i + \frac{\delta f}{\delta B^i}\delta B^i \, d^3x \\ &= \int \frac{\delta f}{\delta q^i}\delta q^i + \frac{\delta f}{\delta \pi^i}\delta\pi^i + \frac{\delta f}{\delta q_d^i}\delta q_d^i + \frac{\delta f}{\delta \pi_d^i}\delta\pi_d^i \, d^3a \end{aligned} \tag{3.14}$$

Substituting the various (3.13), except for the one term involving $\delta\pi_d$ (which will require more careful attention), into the left side of (3.14) gives the expression

$$\begin{aligned}\iint \Bigg[&\left(\frac{\delta f}{\delta \rho}\rho_0(a) + \frac{\delta f}{\delta \sigma}\sigma_0(a) + \frac{\delta f}{\delta m^i}\pi^i\right)\delta'_j\left(x - q(a,t)\right)\delta q^j \\ &+ \frac{\delta f}{\delta B^i}\left(B_0^j(a)\frac{\partial q_f^i}{\partial a^j}\delta q^k - B_0^j(a)\frac{\partial q_f^k}{\partial a^j}\delta q^i\right)\delta'_k\left(x - q(a,t) - q_d(a,t)\right) \\ &+ \frac{\delta f}{\delta B^i}\left(B_0^j(a)\frac{\partial q_f^i}{\partial a^j}\delta q_d^k - B_0^j(a)\frac{\partial q_f^k}{\partial a^j}\delta q_d^i\right)\delta'_k\left(x - q(a,t) - q_d(a,t)\right) \\ &+ \left(\frac{\delta f}{\delta m^i}\delta\left(x - q(a,t)\right)\right)\delta\pi_i\Bigg] d^3x \, d^3a\end{aligned}$$

In this expression, the disappearance of a' is rather startling, but is still there implicitly via the delta functions, for at a fixed x they will pick out values of a for the magnetic terms distinct from those of the other terms.

Meanwhile, the term that I omitted is, using $\mathbf{B}_0 = \nabla_a \times \mathbf{A}_0$,

$$\begin{aligned} &-\iint \frac{\delta f}{\delta B^i}\epsilon^{jkl}\frac{\partial}{\partial a^k}\left(\frac{c}{n_0 e}\delta\pi_{(d,0)l}\right)\frac{\partial q_f^i}{\partial a^j}\,\delta(x - q - q_d)\, d^3a\, d^3x \\ &= \iint \frac{c}{n_0 e}\frac{\delta f}{\delta B^i}\epsilon^{jkl}\;\delta\pi_{(d,0)l}\frac{\partial q_f^i}{\partial a^j}\frac{\partial q_f^m}{\partial a^k}\delta'_m(x - q - q_d)\, d^3a\, d^3x\end{aligned}$$

Here the $\partial^2 q_f/\partial a\partial a$ term in the integration by parts vanishes because it is a symmetric object contracted with an antisymmetric one, and the second factor of $\partial q_f/\partial a$ appears because I want the delta-function derivative to give a derivative with respect to q (and thus x). These factors may be eliminated in the following manner:

$$\begin{aligned}
\epsilon^{jkl}\frac{\partial q_f^i}{\partial a^j}\frac{\partial q_f^m}{\partial a^k} &= \frac{1}{2}\epsilon^{jkl}\left(\frac{\partial q_f^i}{\partial a^j}\frac{\partial q_f^m}{\partial a^k} - \frac{\partial q_f^i}{\partial a^k}\frac{\partial q_f^m}{\partial a^j}\right) \\
&= \frac{1}{2}\epsilon^{jkl}\frac{\partial q_f^a}{\partial a^j}\frac{\partial q_f^b}{\partial a^k}\delta^{im}_{ab} = \ \frac{1}{2}\epsilon^{jkl}\frac{\partial q_f^a}{\partial a^j}\frac{\partial q_f^b}{\partial a^k}\epsilon^{nim}\epsilon_{nab} \\
&= \frac{1}{2}C^l{}_n\epsilon^{nim}
\end{aligned}$$

Thus, using (3.12), that portion of the δf variation becomes

$$\iint \frac{c}{2n_0 e}\frac{\delta f}{\delta B^i}\mathcal{J}_f\ \delta\pi_{(d)j}\ \epsilon^{jik}\delta_k'(x-q-q_d)\ d^3a\ d^3x$$

Comparison of the expanded Eulerian δf with the right side of (3.14) then gives expressions for the Lagrangian functional derivatives in terms of the Eulerian ones:

$$
\begin{aligned}
\frac{\delta f}{\delta \pi^i} &= \int \frac{\delta f}{\delta m^i} \delta\left(x - q(a,t)\right) d^3x = \left.\frac{\delta f}{\delta m^j}\right|_{x=q(a,t)} \\
\frac{\delta f}{\delta q^i} &= \int \left(\frac{\delta f}{\delta \rho}\rho_0 + \frac{\delta f}{\delta \sigma}\sigma_0 + \frac{\delta f}{\delta m^i}\pi \right) \delta_i'\left(x - q\right) \\
&\quad + \frac{\delta f}{\delta B^j} B_0^k \frac{\partial q_f^j}{\partial a^k} \delta_i'\left(x - q - q_d\right) - \frac{\delta f}{\delta B^i} B_0^k \frac{\partial q_f^j}{\partial a^k} \delta_j'\left(x - q - q_d\right) d^3x \\
&\quad = - \int \left[\rho_0 \frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta \rho}\right) + \sigma_0 \frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta \sigma}\right) + \pi^j \frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta m^j}\right) \right] \delta(x - q) \\
&\quad + \mathcal{J}_f \left[B^j \frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta B^j}\right) - B^j \frac{\partial}{\partial x^j}\left(\frac{\delta f}{\delta B^i}\right) \right] \delta(x - q - q_d)\, d^3x \\
\frac{\delta f}{\delta q_d^i} &= \int \frac{\delta f}{\delta B^j} B_0^k \frac{\partial q_f^j}{\partial a^k} \delta_i'\left(x - q - q_d\right) - \frac{\delta f}{\delta B^i} B_0^k \frac{\partial q_f^j}{\partial a^k} \delta_j'\left(x - q - q_d\right) d^3x \\
&= - \int \mathcal{J}_f \left[B^j \frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta B^j}\right) - B^j \frac{\partial}{\partial x^j}\left(\frac{\delta f}{\delta B^i}\right) \right] \delta(x - q - q_d)\, d^3x \\
\frac{\delta f}{\delta \pi_{(d)i}} &= \int \frac{\delta f}{\delta B^j} \frac{c}{2n_0 e} \mathcal{J}_f\, \epsilon^{ijk} \delta_k'(x - q - q_d)\, d^3x \\
&= \frac{c}{2ne} \int \left(\nabla \times \frac{\delta f}{\delta \mathbf{B}} \right)_i \delta(x - q - q_d)\, d^3x = - \frac{c}{2ne} \left.\left(\nabla \times \frac{\delta f}{\delta \mathbf{B}} \right)_i \right|_{x=q(a,t)+q_d(a,t)}
\end{aligned}
$$

Finally, I can use these functional derivatives to convert the canonical Lagrangian bracket into a generalization of the noncanonical Hall MHD one, where the delta function introduces a factor of $\mathcal{J}^{-1}$ or $\mathcal{J}_f^{-1}$, eliminates the a

integral and converts the remaining Lagrangian quantities into Eulerian ones:

$$
\begin{aligned}
\{f,g\} =& \int \left(\frac{\delta f}{\delta q^i}\frac{\delta g}{\delta \pi^i} - \frac{\delta g}{\delta q^i}\frac{\delta f}{\delta \pi^i} \right) + \left(\frac{\delta f}{\delta q^i_d}\frac{\delta g}{\delta \pi^i_d} - \frac{\delta g}{\delta q^i_d}\frac{\delta f}{\delta \pi^i_d} \right) \, d^3a \\
=& \int \left(\rho \frac{\delta f}{\delta m_i}\frac{\partial}{\partial x^i}\left(\frac{\delta g}{\delta \rho}\right) - \rho \frac{\delta g}{\delta m_i}\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta \rho}\right)\right) \\
&+ \left(\sigma \frac{\delta f}{\delta m_i}\frac{\partial}{\partial x^i}\left(\frac{\delta g}{\delta \sigma}\right) - \sigma \frac{\delta g}{\delta m_i}\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta \sigma}\right)\right) \\
&+ \left(m_j \frac{\delta f}{\delta m_i}\frac{\partial}{\partial x^i}\left(\frac{\delta g}{\delta m_j}\right) - m_j \frac{\delta g}{\delta m_i}\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta m_j}\right)\right) \\
&+ \left(B^j \frac{\delta f}{\delta m_i}\frac{\partial}{\partial x^i}\left(\frac{\delta g}{\delta B^j}\right) - B^j \frac{\delta g}{\delta m_i}\frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta B^j}\right)\right) \\
&+ \left(B^j \frac{\partial}{\partial x^j}\left(\frac{\delta f}{\delta B^i}\right)\frac{\delta g}{\delta m_i} - B^j \frac{\partial}{\partial x^j}\left(\frac{\delta g}{\delta B^i}\right)\frac{\delta f}{\delta m_i}\right) \\
&+ \frac{c}{2ne}\Bigg[B^j \left(\nabla \times \frac{\delta f}{\delta \mathbf{B}}\right)^i \frac{\partial}{\partial x^i}\left(\frac{\delta g}{\delta B^j}\right) - B^j \left(\nabla \times \frac{\delta g}{\delta \mathbf{B}}\right)^i \frac{\partial}{\partial x^i}\left(\frac{\delta f}{\delta B^j}\right) \\
&+ B^j \frac{\partial}{\partial x^j}\left(\frac{\delta f}{\delta B^i}\right)\left(\nabla \times \frac{\delta g}{\delta \mathbf{B}}\right)^i - B^j \frac{\partial}{\partial x^j}\left(\frac{\delta g}{\delta B^i}\right)\left(\nabla \times \frac{\delta f}{\delta \mathbf{B}}\right)^i \Bigg] d^3x \\
\equiv& \quad \{f,g\}_{MHD} + \{f,g\}_{Hall}
\end{aligned}
$$

Here the $\{f,g\}_{Hall}$ terms are those in the square bracket, and the remaining $\{f,g\}_{MHD}$ terms are familiar from ordinary MHD.

The Hall portion of the bracket can be greatly simplified. Take the two terms involving the curl of $\delta f/\delta \mathbf{B}$. They become

$$
\begin{aligned}
& \frac{c}{2ne}\left[B^j \left(\nabla \times \frac{\delta f}{\delta \mathbf{B}}\right)^i \delta^{kl}_{ij} \frac{\partial}{\partial x^k}\left(\frac{\delta g}{\delta B^l}\right)\right] \\
=& \frac{c}{2ne}\left[B^j \left(\nabla \times \frac{\delta f}{\delta \mathbf{B}}\right)^i \epsilon_{mij}\epsilon^{mkl} \frac{\partial}{\partial x^k}\left(\frac{\delta g}{\delta B^l}\right)\right] \\
=& -\frac{c}{2ne}\mathbf{B}\cdot\left[\left(\nabla \times \frac{\delta f}{\delta \mathbf{B}}\right) \times \left(\nabla \times \frac{\delta g}{\delta \mathbf{B}}\right)\right]
\end{aligned}
$$

The other two terms give an identical expressions; together, they eliminate the factor of 1/2 and reproduce the Abdelhamid bracket.

3.4 Extended MHD

I'll move up to extended MHD using a brute-force method. The field variable $\mathbf{B}^*$ will be written as a linear combination of the two-forms $\mathbf{B}_\pm$, each of which is advected along a linear combination of q and $\tilde{q}$. Thus (3.10) will be rewritten as

$$B^{*,i}(x,t) = \int \beta_+ B^j_{0,+}(a)\left(\frac{\partial q^i}{\partial a^j} + \alpha_+ \frac{\partial q^i_d}{\partial a^j}\right)\delta(x - q(a,t) - \alpha_+ q_d(a,t)) + \beta_- B^j_{0,-}(a)\left(\frac{\partial q^i}{\partial a^j} + \alpha_- \frac{\partial q^i_d}{\partial a^j}\right)\delta(x - q(a,t) - \alpha_- q_d(a,t))\, d^3a$$

Presumably I could add two more coefficients, so that I'd have $\delta(x - \gamma_+ q - \alpha_+ q_d)$ for instance, but then I'd end up with an underdetermined linear system, with two superfluous variables. In addition, I assume that $\mathbf{B}_{0,\pm}$ are both identically affine to the canonical momentum $\tilde{\pi}$, so that $\delta\mathbf{B}_{0,\pm} = -d_i \delta\pi_d/\rho_0$ as in the Hall case. Hopefully this assumption can be justified or amended later.

The following changes then appear in the previous calculation: (i) All functional derivatives with respect to $\mathbf{B}$ are now done with respect to $\mathbf{B}^*$; (ii) in the magnetic portion of $\{f,g\}_{MHD}$, $\mathbf{B}$ is replaced by $\beta_+\mathbf{B}_+ + \beta_-\mathbf{B}_- = \mathbf{B}^*$; (iii) in $\{f,g\}_{Hall}$, $\mathbf{B}$ is replaced by $(d_i/\rho)(\beta_+\alpha_+\mathbf{B}_+ + \beta_-\alpha_-\mathbf{B}_-)$. In the Abdelhamid bracket, this last quantity works out to be $(1/\rho)(\mathbf{B}^* - d_e^2\nabla\times\mathbf{V})$. According to [17], the advected quantities are

$$\mathbf{B}_\pm = \mathbf{B}^* + \gamma_\pm \nabla \times \mathbf{V} \tag{3.15}$$

where

$$\frac{1}{\gamma_\pm} = \lambda_\pm = \frac{-d_i \pm \sqrt{d_i^2 + 4d_e^2}}{2d_e^2}$$

are the coefficients from [18]. So I can invert (3.15) to get

$$\mathbf{B}^* = \frac{\gamma_+ \mathbf{B}_- - \gamma_- \mathbf{B}_+}{\Delta\gamma} \qquad \nabla \times \mathbf{V} = \frac{\mathbf{B}_+ - \mathbf{B}_-}{\Delta\gamma}$$

where $\Delta\gamma = \gamma_+ - \gamma_-$. The various coefficients then turn out to be

$$\beta_+ = -\frac{\gamma_-}{\Delta\gamma} \qquad \beta_- = \frac{\gamma_+}{\Delta\gamma}$$

$$\alpha_+ = -\frac{d_e^2}{d_i}\lambda_+ \qquad \alpha_- = -\frac{d_e^2}{d_i}\lambda_-$$

So far this is just done by brute force, but a couple of things are worth noticing. First, the values $\beta_+ + \beta_- = 1$, and both are positive, so $\mathbf{B}^*$ is a weighted average of $\mathbf{B}_+$ and $\mathbf{B}_-$. In addition $\alpha_+ + \alpha_- = 1$. In future work I hope to show how these, like their analogues in the Hall MHD bracket, originate in an action principle for extended MHD.

Chapter 4

The transition to relativistic Hamiltonian systems

4.1 Systems with proper time

When trying to generalize Hamilton's equations

$$\frac{\partial f}{\partial t} = \{f, h\}$$

to relativity, two problems present themselves. The first problem is that the bracket $\{f, g\}$ is expressed in terms of 3-vectorial quantities, implicitly assuming a choice of local reference frame. The second problem is the time derivative occurring on the left hand side, also implying a favored choice of reference frame. Most treatments, including the ADM formalism [25] and the Hamiltonian theory developed in QFT [40], develop what is called a $3+1$ split, retaining the nonrelativistic form of Hamilton's equations at the cost of losing frame-indepedence. I will outline alternatives, in my opinion superior ones due to their frame independence.

The first problem just mentioned turns out to be easily solved; in almost all cases where the bracket uses a 3-vectorial quantity, one can instead use the 4-vectorial equivalent. In this dissertation, the only exception to this rule is the metriplectic form of relativistic Navier-Stokes, which also requires a projection operator, but even in that case the Hamiltonian bracket only requires one to swap in 4-vectors. However, the 4-vectorial equivalents used in the bracket sometimes look much different than their 3-vectorial counterparts; for instance, in relativistic MHD, the momentum is not simply ρv^μ, but instead involves both pressure and magnetic energy density.

The second problem, concerning the time derivative, poses a more subtle challenge. In the case of particle motion, each particle has a well-defined proper time τ, so one can use that derivative. For the particle subject only to the Lorentz force, for instance, I have

$$\frac{df}{d\tau} = \{f, H\} \tag{4.1}$$

where the Hamiltonian is

$$H = \frac{(P^\mu - eA^\mu)(P_\mu - eA_\mu)}{2m}$$

and the bracket is a canonical one,

$$\{f, g\} = \frac{\partial f}{\partial X^\mu}\frac{\partial g}{\partial P_\mu} - \frac{\partial g}{\partial X^\mu}\frac{\partial f}{\partial P_\mu} \; .$$

In this case the 4-velocity is $U^\mu = (P^\mu - eA^\mu)/m$. Note a peculiarity of this Hamiltonian system, in that both sides of Hamilton's equations (4.1) involve time derivatives, a proper time derivative on the LHS and a partial time derivative on the RHS.

Most of the systems I deal with are infinite-dimensional, noncanonical ones. However, in such cases one of the fields will often turn out to be a velocity field. Using this field, one can define a proper time along streamlines by integrating the coordinate time of an observer moving with 4-velocity defined by that field; for example, in fluids, proper time is the coordinate time measured by an observer moving with the fluid. In this case, one can define the proper time derivative in Hamilton's equations as the following, where U^μ is the velocity field:

$$\frac{df}{d\tau} = U^\mu \frac{\partial f}{\partial X^\mu} = \{f, H\}$$

This is the form that the fluid and MHD equations take when using Lagrangian (as opposed to Eulerian) coordinates, for example in Section 6.1.

However, in the general case one will not be able to extract a proper time derivative. In this case the bracket takes on a new role, in a generalized version of Hamilton's Principle.

4.2 Systems without proper time; the phase space action principle

In Lagrangian action principles the equations of motion are derived as the extremization of

$$S = \int L \, dt$$

One switches to the Hamiltonian via

$$H = \mathbf{p} \cdot \dot{\mathbf{q}} - L \tag{4.2}$$

with $\dot{\mathbf{q}}$ written in terms of q and p; then, one obtains Hamilton's equations as usual.

What often goes unmentioned is that the original action principle $\delta S = 0$ can be restated in terms of the Hamiltonian and the phase space variables. Reverse the Legendre transform and write

$$S = \int (\mathbf{p} \cdot \dot{\mathbf{q}} - H) \; dt$$

In this case, $\dot{\mathbf{q}}$ is not written in terms of q and p but is an explicit time derivative. Whereas in the Lagrangian action principle there was only the δq variation, this one (the "phase space action principle") has two, the δq variation and the δp variation. The δp variation gives half of Hamilton's equations,

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}$$

and the δq variations gives, after an integration by parts, the other half:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}$$

The phase space action principle has three attributes which may make it more advantageous than the Lagrangian (or "tangent space") action principle. First, constraints are sometimes more easily implemented in terms of phase space variables than in terms of tangent space ones. Second, the Legendre transform (4.2) may fail to be convex, leaving one able to write the Hamiltonian, but not the Lagrangian, explicitly. Third, the phase space variational principle $\delta S = 0$ can also be written

$$\int [f, S] \, dt = 0$$

for an arbitrary function $f(q,p)$. Considering how I have emphasized the role of the bracket in Hamiltonian physics, it is no wonder that I prefer the phase space action principle, for then variations in terms of arbitrary coordinates can be found by transforming the bracket. In addition, some transformations will eliminate any explicit time derivative inside S, in which case the time integration becomes vestigial, and can be dropped:

$$[f, S] = 0$$

This is a pleasingly compact way of both writing the variational principle $\delta S = 0$ and instructing one how to perform this variation.

This is unimportant in the case of finite-dimensional systems, particularly those with brackets derived from canonical brackets, because in finite dimensions all differentiable, bijective transformations preserve the number of Casimirs. Thus, for a canonical-equivalent system with no Casimirs, δS always means $\partial S/\partial z^i = 0$ for all the degrees of freedom z^i. However, for an infinite-dimensional system, a well-behaved transformation may nonetheless change the number of Casimirs at any point in the phase space; then, you may have $\partial S/\partial z^i = 0$ in some coordinates but not others. For a more detailed explanation of this phenomenon in the context of relativistic MHD, see [43].

Thankfully, it is a straightforward matter to recast the preceding work into its infinite-dimensional equivalent. The action is now

$$S = \int\int \mathcal{L} \, d^3x \; dt$$

The Legendre transform is now

$$\mathcal{H} = \mathbf{p} \cdot \dot{\mathbf{q}} - \mathcal{L}$$

and the phase space action is now

$$S = \iint (\mathbf{p} \cdot \dot{\mathbf{q}} - \mathcal{H}) \; d^3x \, dt$$

The phase space action principle becomes

$$\int \{f, S\} \, dt = 0$$

Or, in cases where there is no explicit proper time derivative, or where the phase space action was not derived from a tangent space action in the first place,

$$\{f, S\} = 0$$

for f, an arbitrary function of the field variables. This is the archetypal action principle for infinite-dimensional relativistic systems, singling out neither a proper time derivative nor a partial one, and you will be seeing quite a bit more of it in the coming pages.

4.3 Relativistic metriplectic systems

Relativistic metriplectic systems are very similar to the nonrelativistic metriplectic ones covered in Section 2.5. As before, the generator of motion and the bracket are now both composite objects. However, the positivity attribute now depends on a sign convention used in the antisymmetric part, and there is now no time derivative singled out. Thus the equations of motion are given by $\{f, G\} = 0$ for arbitrary f, with

$$G = S + \lambda C$$
$$\{f, g\} = [f, g] + (f, g)$$

where λ is a constant used to ensure dimensional consistency. The antisymmetric part of the bracket has the normal attributes defining a Poisson bracket, with the additional condition that C is a Casimir invariant:

$$[\alpha f + \beta g, h] = \alpha[f, h] + \beta[g, h]$$
$$[fg, h] = f[g, h] + [f, h]g$$
$$[f, g] = -[g, f]$$
$$[[f, g], h] + [[g, h], f] + [[h, f], g] = 0$$
$$[f, C] = 0$$

The symmetric bracket has the same properties as before, too:

$$(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$$
$$(fg, h) = f(g, h) + (f, h)g$$
$$(f, g) = (g, f)$$
$$(f, f) \geq 0 \ \ \forall f \quad or \quad (f, f) \leq 0 \ \ \forall f$$
$$(f, S) = 0$$

The difference in the positivity property comes about because one can change the overall sign of $\{f, G\} = 0$ and still get the same equations of motion, which would be the same as flipping each bracket by an overall sign. So the positivity property concerns the sign of the symmetric bracket relative to the sign of the antisymmetric part.

In a canonical bracket, one can perform what is called a $(3+1)$ split to pull out a time derivative. If V^μ is a timelike vector defining an observer, then the split is done by replacing $g^{\mu\nu}$ by its decomposition $P^{\mu\nu} + V^\mu V^\nu$, where $P^{\mu\nu}$ projects 4-vectors onto the spacelike submanifold defined by V^μ, and $V^\mu V^\nu$ is an operator that projects out the part of a vector parallel to V^μ. The part of the bracket corresponding to $P^{\mu\nu}$ will now be, in essence, a three-dimensional bracket, while the part corresponding to $V^\mu V^\nu$ will be a directional derivative; a simple partial time derivative, in fact, for particles when the 4-velocity U^μ is used for V^μ. The sign of this time derivative determines which version of the positivity axiom, above, that one uses. The $(3 + 1)$ split is more complicated for general brackets, and is one of my upcoming topics of research.

As it happens, the symmetric part of the bracket already comes "factored"; it has the three-dimensional part, but nothing corresponding to the part produced by $V^\mu V^\nu$, at least in the cases that I have investigated so far.

4.4 Relativistic Hamiltonian Maxwell-Vlasov and its difficulties

The Maxwell-Vlasov description of Section 2.2 can be made relativistic, although the field parts introduce some difficulties. In this situation, the distribution function $f(z)$ is defined on the eight-dimensional phase space $z = (x, p)$, with position x^μ and momentum p^μ both 4-vectors. The Lie-Poisson bracket is a weighted canonical one:

$$\{F, G\}_c = \int f \left(\frac{\partial F_f}{\partial x^\mu} \frac{\partial G_f}{\partial p_\mu} - \frac{\partial G_f}{\partial x^\mu} \frac{\partial F_f}{\partial p_\mu} \right) d^8 z \tag{4.3}$$

where $F_f = \delta F / \delta f$.

Write the action S as

$$\begin{aligned} S &= \iint f \left(\frac{m}{2} u_\mu u^\mu \right) d^4x \, d^4p \\ &= \iint f \frac{1}{2m} \left(p_\mu - \frac{e}{c} A_\mu \right) \left(p^\mu - \frac{e}{c} A^\mu \right) d^4x \, d^4p \end{aligned} \tag{4.4}$$

Keeping in mind that p_μ is the canonical, not kinetic, momentum, the action principle $\{F(f), S\} = 0$ yields the Vlasov equation

$$u^\mu \frac{\partial f}{\partial x^\mu} + \frac{e}{c} \frac{\partial A^\mu}{\partial x^\nu} u_\mu \frac{\partial f}{\partial p_\nu} = 0$$

The difficulties start to arise when you try to take into account the dynamics of the fields. In (2.13), the vector potential A^μ and the electric field $\mathbf{E}$ are canonically conjugate to each other. However, in relativity the closest such quantities are the 4-potential A^μ and the field tensor $F^{\mu\nu}$, which are of a different type both tensorially (having different numbers of indices) and geometrically ($F^{\mu\nu}$ is a closed two-form, while A^μ is a one-form). Marsden and Morrison [22] solve this by introducing a constant 4-vector V^μ, which can be taken to represent an observer in a (3+1) split of the bracket. I have some ideas for eliminating it, but as of this dissertation they are not well-developed enough to include. Using the observer 4-vector, the quasi-canonical field bracket is

$$\{F, G\}_{EM} = \int \left(\frac{\delta F}{\delta A_\mu} \frac{\delta G}{\delta F^{\mu\nu}} - \frac{\delta G}{\delta A_\mu} \frac{\delta F}{\delta F^{\mu\nu}} \right) V^\nu \, d^4x$$

By adding to the Vlasov action (4.4) the expression

$$\frac{1}{4\pi} \int \frac{1}{2} F^{\mu\nu} \left(A_{\mu,\nu} - A_{\nu,\mu} \right) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} d^4x$$

one finds that the $F^{\mu\nu}$ and A^μ variations give, respectively,

$$\begin{aligned} 0 &= F_{\mu\nu} - \left(A_{\mu,\nu} - A_{\nu,\mu} \right) \\ 0 &= F^{\mu\nu}{}_{,\nu} - \frac{4\pi e}{c} \int f u^\mu \, d^4p \end{aligned} \tag{4.5}$$

The next difficulty arises when trying to switch from x^μ, p^μ, A^μ and $F^{\mu\nu}$, to x^μ, u^μ, and $F^{\mu\nu}$, in analogy with the brackets (2.11) - (2.15). The magnetic interaction bracket (2.14) has a natural analogue

$$\{F, G\}_B = \int F^{\mu\nu} \left(\frac{\partial F_f}{\partial u^\mu} \frac{\partial G_f}{\partial u^\nu} - \frac{\partial G_f}{\partial u^\mu} \frac{\partial F_f}{\partial u^\nu} \right) d^8z$$

while the electric interaction bracket, if you grant the use of V^μ, has a natural analogue

$$\{F, G\}_E = \int V^\mu \left(\frac{\delta F}{\delta F^{\mu\nu}} \frac{\partial}{\partial u_\nu} \left(\frac{\delta G}{\delta f} \right) - \frac{\delta G}{\delta F^{\mu\nu}} \frac{\partial}{\partial u_\nu} \left(\frac{\delta F}{\delta f} \right) \right) d^8z$$

Finally, the purely field part becomes

$$\{F,G\}_{EM} = \int V^{\nu}\left(\frac{\delta F}{\delta F^{\mu\nu}}\frac{\partial}{\partial x^{\lambda}}\frac{\delta G}{\delta F_{\lambda\mu}} - \frac{\delta G}{\delta F^{\mu\nu}}\frac{\partial}{\partial x^{\lambda}}\frac{\delta F}{\delta F_{\lambda\mu}}\right) d^4x$$

Using these three brackets along with (4.3), the $F^{\mu\nu}$ variation once more gives (4.5). However, the f variation gives the wrong equation

$$u^{\mu}\frac{\partial f}{\partial x^{\mu}} + \frac{e}{c}F^{\mu\nu}\left(u_{\nu} + V_{\nu}\right)\frac{\partial f}{\partial u_{\nu}} = 0$$

Dropping $\{F,G\}_E$ fixes the Vlasov equation but breaks Maxwell's equations. One could also try dropping $\{F,G\}_B$ and setting $V^{\mu} = u^{\mu}$ in the field brackets, but that breaks the Jacobi identity for the set of brackets. Michel Vittot and I are actively working on this problem.

Chapter 5

Relativistic MHD

5.1 Overview of relativistic MHD

Turning now to the description of relativistic MHD, I use signature and units such that 4-velocities have positive unit norms $u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu = 1$, where the Minkowski metric $g_{\mu\nu}$ is given by dia$(1, -1, -1, -1)$. The 4-vector field u^μ will denote the plasma's 4-velocity at each point in spacetime; at each such point, this quantity will define a reference frame with locally vanishing 3-velocity, helpful for some purposes. The fluid density is now $\rho = mn(1 + \epsilon)$, where n is the baryon number density, m is the fluid rest mass per baryon (including both proton and electron, for the typical case), and ϵ is the internal energy per baryon, normalized to m. The specific entropy s is unchanged, though later on it will prove more convenient to use the entropy density $\sigma = ns$. I will suppose that the energy can be written $\epsilon(n, \sigma)$, hence $\rho(n, \sigma)$, in which case the pressure is given by

$$p = n\frac{\partial \rho}{\partial n} + \sigma \frac{\partial \rho}{\partial \sigma} - \rho \,, \tag{5.1}$$

which is just the first law of thermodynamics written in terms of these variables; see e.g. [25] pg. 560.

In electromagnetism, having chosen a specific reference frame, one extracts the electric field 3-vector from the field tensor $F^{\mu\nu}$ by $\mathbf{E}^i = -F^{i0}$, $i = 1, 2, 3$, while the magnetic field 3-vector $\mathbf{B}^i = \mathcal{F}_{i0}$, where $\mathcal{F}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}/2$ is the dual of $F^{\mu\nu}$. Given u^μ, one can also define the two 4-vectors $B^\mu \equiv \mathcal{F}^{\mu\nu}u_\nu = \gamma(\mathbf{v}\cdot\mathbf{B}, \mathbf{B} - \mathbf{v}\times\mathbf{E})$ and $E^\mu \equiv F^{\mu\nu}u_\nu = \gamma(\mathbf{v}\cdot\mathbf{E}, \mathbf{E} + \mathbf{v}\times\mathbf{B})$. Note that $\mathbf{B}^i = B^i$ and $\mathbf{E}^i = E^i$ in the reference frame defined by u^μ. In terms of the 4-vectors B^μ and E^μ the field tensor has the decomposition

$$F^{\mu\nu} = \epsilon^{\mu\nu\lambda\sigma}B_\lambda u_\sigma + (u^\mu E^\nu - u^\nu E^\mu) \ , \tag{5.2}$$

a form valid for any timelike 4-vector u^μ. One can also reverse this process by taking B^μ and E^μ to be fundamental, and then defining the field tensor $F^{\mu\nu}$ via (5.2). In this case, different values of B^μ and E^μ can give the same field tensor, for one can add any quantity proportional to u^μ to either 4-vector while leaving the field tensor unchanged; however, if the constraints $E^\lambda u_\lambda = B^\lambda u_\lambda = 0$ are imposed, then this representation is unique. This multiplicity of representations of the field tensor will prove important later.

In MHD ones eliminates the electric field from the theory, if necessary using Ohm's Law to express it in terms of the fluid velocity and magnetic field. In a relativistic context, this is done by setting $E^\mu = F^{\mu\lambda}u_\lambda = 0$, which gives $\mathbf{E} + \mathbf{v}\times\mathbf{B} = 0$ (i.e. Ohm's Law) and, in a specific reference frame,

$$B^\mu = \gamma\left(\mathbf{v}\cdot\mathbf{B},\ \frac{\mathbf{B}}{\gamma^2} + \mathbf{v}\ (\mathbf{v}\cdot\mathbf{B})\right) \tag{5.3}$$

For convenience $b^\mu \equiv B^\mu/\sqrt{4\pi}$ will be used, in which case the MHD field tensor and its dual have the forms

$$F^{\mu\nu} = \sqrt{4\pi}\,\epsilon^{\mu\nu\lambda\sigma} b_\lambda u_\sigma \quad \text{and} \quad \mathcal{F}^{\mu\nu} = \sqrt{4\pi}\,(b^\mu u^\nu - u^\mu b^\nu)\,. \tag{5.4}$$

Although (5.3) satisfies the restriction $b^\lambda u_\lambda = 0$, I noted earlier that this condition is not needed for a representation of the form of (5.2). One can, in fact, construct a family of vectors

$$h^\mu = b^\mu + \alpha u^\mu \tag{5.5}$$

where α is an arbitrary scalar field and now, in general, $h^\mu u_\mu = \alpha \neq 0$. The field tensor $F^{\mu\nu}$ and its dual $\mathcal{F}^{\mu\nu}$ are unchanged when written in terms of h^μ, i.e.

$$\begin{aligned} F^{\mu\nu}/\sqrt{4\pi} &= \epsilon^{\mu\nu\lambda\sigma} b_\lambda u_\sigma = \epsilon^{\mu\nu\lambda\sigma} h_\lambda u_\sigma \\ \mathcal{F}^{\mu\nu}/\sqrt{4\pi} &= b^\mu u^\nu - u^\mu b^\nu = h^\mu u^\nu - u^\mu h^\nu\,. \end{aligned} \tag{5.6}$$

Because b^μ only appears in the equations of relativistic MHD via the form (5.4), one can just as easily use the quantity h^μ, choosing α in order to give it some useful property. When constructing an Eulerian action principle (with covariant Poisson bracket) for relativistic MHD it will prove fruitful to do so. The quantity $b_\mu b^\mu$, which appears in the stress-energy tensor and will be seen in a later section to appear in the action, evaluates to

$$b_\mu b^\mu = \frac{1}{4\pi}\left(E^2 - B^2\right) = -\frac{1}{4\pi}\left(\frac{\mathbf{B}\cdot\mathbf{B}}{\gamma^2} + (\mathbf{v}\cdot\mathbf{B})^2\right) = -\frac{1}{4\pi}B^2_{\text{rest}}$$

where 'rest' indicates a rest frame quantity. Thus the 4-vector b^μ is spacelike. However, since $h_\mu h^\mu = b_\mu b^\mu + \alpha^2$, the status of h^μ will depend on α, remaining spacelike for small α.

Each equation of relativistic MHD can be written as the vanishing of a divergence:

$$\partial_\mu(nu^\mu) = 0 \tag{5.7}$$

$$\partial_\mu(\sigma u^\mu) = 0 \tag{5.8}$$

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \tag{5.9}$$

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{5.10}$$

Equations (5.7) and (5.8) express conservation of particles and entropy, respectively. In addition, (5.9) provides the equivalent of the homogeneous Maxwell's equations; however, one cannot call them Maxwell's equations without qualification, as the constraint $F^{\mu\nu}u_\nu = 0$ is already built in when one expresses $F^{\mu\nu}$ in terms of b^μ or h^μ:

$$\partial_\nu(b^\mu u^\nu - u^\mu b^\nu) = \partial_\nu(h^\mu u^\nu - u^\mu h^\nu) = 0\,.$$

This expression, of course, is the same whether b^μ or h^μ is used, as the quantity α cancels out. Equation (5.10) gives conservation of stress-energy, where the stress-energy tensor $T^{\mu\nu}$ is considerably more complex when written in terms of h^μ rather than b^μ:

$$T^{\mu\nu} = T^{\mu\nu}_{fl} + T^{\mu\nu}_{EM}\,, \tag{5.11}$$

where the fluid and field parts are

$$T^{\mu\nu}_{fl} = (\rho + p)\, u^\mu u^\nu - p\, g^{\mu\nu}\,, \tag{5.12}$$

$$\begin{aligned} T^{\mu\nu}_{EM} &= \frac{1}{4\pi}\left(F^{\mu\lambda}F_\lambda{}^\nu + \frac{1}{4}g^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right) \\ &= -b^\mu b^\nu - \left(b_\lambda b^\lambda\right) u^\mu u^\nu + \frac{1}{2}g^{\mu\nu}b_\lambda b^\lambda \qquad (5.13) \\ &= -h^\mu h^\nu - \left(h_\lambda h^\lambda\right) u^\mu u^\nu + \left(h_\lambda u^\lambda\right)\left(h^\mu u^\nu + u^\mu h^\nu\right) \\ &\quad + \frac{1}{2}g^{\mu\nu}\left(h_\lambda h^\lambda - \left(h_\lambda u^\lambda\right)^2\right), \qquad (5.14) \end{aligned}$$

respectively. Equation (5.13) is obtained by substitution of the first of Eqs. (5.4) and making use of the orthogonality condition $b^\lambda u_\lambda = 0$, while (5.14) follows from (5.6) without orthogonality. I emphasize that, despite appearances, $T^{\mu\nu}_{EM}$ does not depend on one's choice of α. The field part $T^{\mu\nu}_{EM}$ depends on b^μ or h^μ only through the tensor $\mathcal{F}^{\mu\nu}$, in which, as previously noted, α cancels out. Lastly, I note it can be shown that this system preserves $b^\mu u_\mu = 0$ and $u^\mu u_\mu = 1$. I next turn to the problem of devising an action principle for this system.

5.2 Relativistic MHD in Hamiltonian form

The covariant Poisson bracket formalism of Ref. [22] requires two parts: i) an action S that is a covariant functional of the field variables and ii) a covariant Poisson bracket $\{\,,\,\}$ defined on functionals of the fields. Instead the usual extremization $\delta S = 0$, the theory arises from setting $\{F, S\} = 0$ for all functionals F, which is in effect a constrained extremization.

One writes the general Poisson bracket for fields Ψ in the form

$$\{F, G\} = \int dz \, \frac{\delta F}{\delta \Psi} \mathcal{J} \frac{\delta G}{\delta \Psi} \,,$$

where $\delta F/\delta\Psi$ is the functional derivative, dz is an appropriate spacetime measure, and $\mathcal{J}$ is a cosymplectic operator that provides $\{F, G\}$ with the properties of antisymmetry and the Jacobi identity. Thus

$$\{F, S\} = 0 \quad \forall \, F \quad \Rightarrow \quad \mathcal{J} \frac{\delta S}{\delta \Psi} = 0 \,. \tag{5.15}$$

If $\mathcal{J}$ is nondegenerate, i.e., has no null space, then (5.15) is equivalent to $\delta S/\delta\Psi = 0$ and the covariant Poisson bracket formalism reproduces the conventional variational principle.

However, of interest here are matter models like MHD, which when written in terms of Eulerian variables possess nonstandard or noncanonical Poisson brackets (see e.g. Ref. [28]), for which $\mathcal{J}$ possess degeneracy that is reflected in the existence of so-called Casimirs. For such systems the covariant Poisson bracket naturally enforces constraints. For field theories that describe matter, understanding the null space of $\mathcal{J}$ may be a formidable exercise[42], and finding nondegenerate coordinates, which are expected to exist because of the Jacobi identity, may only serve to obscure the structure of the theory. A variation that preserves the constraints, referred to as a dynamically accessible variation in Ref. [32] (see also [28]), can be represented as

$$\delta \Psi_{DA} = \{\psi, G\} \,, \tag{5.16}$$

for some G, whence

$$\delta S = \int dz \, \frac{\delta S}{\delta \Psi} \, \delta \Psi_{DA} = \int dz \, \frac{\delta S}{\delta \Psi} \, \{\psi, G\} = \int dz \, \{S, G\} = 0$$

which shows directly how the Poisson bracket effects the constraints without them being explicitly known.

5.2.1 Action and functional derivatives

I construct the action S in a straightforward fashion:

$$S[n,\sigma,u,F] = \int d^4x \left(\frac{1}{2}\big(p+\rho\big)u_\lambda u^\lambda + \frac{1}{2}\big(p-\rho\big) - \frac{1}{16\pi}F_{\lambda\sigma}F^{\lambda\sigma} \right) \quad (5.17)$$

$$S[n,\sigma,u,b] = \frac{1}{2}\int d^4x \left(\big(p+\rho-b_\lambda b^\lambda\big)u_\lambda u^\lambda + p - \rho \right) \quad (5.18)$$

$$S[n,\sigma,u,h] = \frac{1}{2}\int d^4x \left(\big(p+\rho-h_\sigma h^\sigma\big)u_\lambda u^\lambda + \big(h_\lambda u^\lambda\big)^2 + p - \rho \right) \quad (5.19)$$

Equation (5.17) is the sum of the fluid action of Ref. [22], where thermodynamic variables p and ρ are considered to be functions of n and σ, together with a standard expression for the electromagnetic action.

In (5.18) the MHD expression of (5.4) has been substituted into $F_{\lambda\sigma}F^{\lambda\sigma}$ and finally in (5.19) I obtain the desired form in terms of h^μ. Observe that the integrand of (5.18) when evaluated on the constraint $u_\lambda u^\lambda = 1$ is the total pressure, fluid plus magnetic, $p + |b_\lambda b^\lambda|/2$. This choice of action will be seen to give the desired field equations when inserted into the covariant Poisson bracket.

From the action of (5.19) one derives a momentum m_μ by functional differentiation,

$$m_\mu = \frac{\delta S}{\delta u^\mu} = (p + \rho - h_\sigma h^\sigma)\, u_\mu + \left(h_\lambda u^\lambda\right) h_\mu \equiv \mu u_\mu + \alpha h_\mu \,. \tag{5.20}$$

The quantity

$$\mu = p + \rho - h_\lambda h^\lambda \tag{5.21}$$

is a modified enthalpy density. If αu^μ is small compared to b^μ, h^μ will be spacelike, leaving μ always positive.

Since u^μ and b^μ are independent of α, expressions for them solely in terms of m^μ and h^μ can be obtained. Using $\alpha = h_\lambda u^\lambda$, which follows from (5.5), and $u^\mu = (m^\mu - \alpha h^\mu)/\mu$, which follows from (5.20), I have

$$\alpha = h_\lambda u^\lambda = \frac{1}{\mu}\left(h_\lambda m^\lambda - \alpha h_\lambda h^\lambda\right) \,.$$

Then, solving for α gives

$$\alpha = \frac{h_\lambda m^\lambda}{\mu + h_\sigma h^\sigma} \,. \tag{5.22}$$

Equation (5.22), incidentally, shows that α can be written entirely in terms of the field variables m^μ and h^μ. Thus, one can also write the variables b^μ and u^μ entirely in terms of the new ones:

$$\begin{aligned} u^\mu &= \frac{m^\mu}{\mu} - \frac{h_\lambda m^\lambda}{\mu(\mu + h_\sigma h^\sigma)} h^\mu \\ b^\mu &= h^\mu \left(1 + \frac{(h_\lambda m^\lambda)^2}{\mu(\mu + h_\sigma h^\sigma)^2}\right) - \frac{h_\lambda m^\lambda}{\mu(\mu + h_\sigma h^\sigma)} m^\mu \,. \end{aligned} \tag{5.23}$$

Equations (5.23) are not invertible. This is made evident by considering a frame in which $\mathbf{v} = 0$, i.e., one where $u^\mu = (1, \mathbf{0})$ and $b^\mu = (0, \mathbf{B})/\sqrt{4\pi}$. In this frame $h^\mu = (\alpha, \mathbf{B}/\sqrt{4\pi})$ and $m_\mu = (p+\rho+B^2/4\pi, \alpha\mathbf{B}/\sqrt{4\pi})$. Given any value of α these equations are compatible with (5.22), but produce the same rest frame values of b^μ and u^μ. Thus, Eqs. (5.23) are not one-one. I will explore this degeneracy, which is a kind of gauge condition, more fully in a later section.

Now I am in a position to obtain an action functional in terms of the variables m^μ and h^μ, which are the appropriate ones for the present covariant action principle:

$$S[n,\sigma,m,h] = \frac{1}{2}\int d^4x \left(\frac{m_\lambda m^\lambda}{\mu} - \frac{\left(h_\lambda m^\lambda\right)^2}{\mu(\mu + h_\sigma h^\sigma)} + p - \rho \right) . \tag{5.24}$$

Upon introducing the "mass" matrix

$$\mathcal{M} \equiv \begin{pmatrix} \mu+\alpha^2 & \alpha \\ \alpha & 1 \end{pmatrix} , \tag{5.25}$$

(5.24) can be written compactly as

$$\begin{aligned} S &= \frac{1}{2}\int d^4x \, \left(\Psi_\lambda \cdot \mathcal{M}^{-1} \cdot \Psi^\lambda + h_\lambda h^\lambda - \alpha^2 + p - \rho \right) \\ &= \frac{1}{2}\int d^4x \, \left(u^\lambda m_\lambda + b^\lambda h_\lambda + h_\lambda h^\lambda - \alpha^2 + p - \rho \right) \\ &= \frac{1}{2}\int d^4x \, \left(\Phi_\lambda \cdot \mathcal{M} \cdot \Phi^\lambda + b_\lambda b^\lambda + p - \rho \right) \\ &= \frac{1}{2}\int d^4x \, \left(u^\lambda m_\lambda + b^\lambda h_\lambda + b_\lambda b^\lambda + p - \rho \right) \end{aligned} \tag{5.26}$$

where $\Psi^\lambda \equiv (m^\lambda, h^\lambda)$, $\Phi^\lambda \equiv (u^\lambda, b^\lambda)$ and $\cdot$ indicates summation over the 2×2 matrix $\mathcal{M}$. However, because the mass matrix (5.25) depends on the field variables via μ and α, as given by (5.21) and (5.22), the expression (5.24) is superior for calculations; in addition, the mass matrix is inconsistent in units, so it would have to be normalized before, say, eigenvalue and eigenvector calculations could be done. One possible normalization is given in (5.54) below.

After taking variations of the action, one may impose the constraint $u_\lambda u^\lambda = 1$. In terms of the momentum m^μ, this constraint becomes

$$1 = u_\lambda u^\lambda = \frac{1}{\mu^2}\left(m_\lambda m^\lambda - 2\frac{\left(h_\lambda m^\lambda\right)^2}{\mu + h_\sigma h^\sigma} + \frac{\left(h_\lambda m^\lambda\right)^2}{\left(\mu + h_\sigma h^\sigma\right)^2}\left(h_\tau h^\tau\right)\right) . \qquad (5.27)$$

Thanks to the relations (5.23) and (5.27), all functional derivatives of the action of (5.24) can be reduced to simple expressions, provided (5.27) is applied only after functional differentiation. To start with,

$$\begin{aligned} \frac{\delta S}{\delta n} &= \left(-\frac{m_\lambda m^\lambda}{2\mu^2} + \frac{\left(h_\lambda m^\lambda\right)^2}{2\mu^2(\mu + h_\sigma h^\sigma)} + \frac{\left(h_\lambda m^\lambda\right)^2}{2\mu\left(\mu + h_\sigma h^\sigma\right)^2}\right)\frac{\partial \mu}{\partial n} + \frac{1}{2}\frac{\partial p}{\partial n} - \frac{1}{2}\frac{\partial \rho}{\partial n} \\ &= -\frac{\partial \rho}{\partial n} . \end{aligned} \qquad (5.28)$$

Similarly,

$$\frac{\delta S}{\delta \sigma} = -\frac{\partial \rho}{\partial \sigma} . \qquad (5.29)$$

The remaining functional derivatives are

$$\frac{\delta S}{\delta m^\nu} = \frac{m_\nu}{\mu} - \frac{(h_\lambda m^\lambda)}{\mu(\mu + h_\tau h^\tau)} h_\nu = u_\nu \, , \tag{5.30}$$

$$\begin{aligned}
\frac{\delta S}{\delta h^\nu} &= \frac{m_\lambda m^\lambda}{\mu^2} h_\nu - \frac{\left(h_\lambda m^\lambda\right)^2}{\mu^2(\mu + h_\sigma h^\sigma)} h_\nu - \frac{(h_\lambda m^\lambda)}{\mu(\mu + h_\sigma h^\sigma)} m_\nu \\
&= \left(1 + 2\frac{\left(h_\lambda m^\lambda\right)^2}{\mu^2(\mu + h_\sigma h^\sigma)} - \frac{\left(h_\lambda m^\lambda\right)^2}{\mu^2 \left(\mu + h_\sigma h^\sigma\right)^2} \left(h_\tau h^\tau\right)\right) h_\nu \\
&\qquad - \frac{\left(h_\lambda m^\lambda\right)^2}{\mu^2(\mu + h_\sigma h^\sigma)} h_\nu - \frac{(h_\lambda m^\lambda)}{\mu(\mu + h_\sigma h^\sigma)} m_\nu \\
&= \left(1 + \frac{\left(h_\lambda m^\lambda\right)^2}{\mu(\mu + h_\sigma h^\sigma)^2}\right) h_\nu - \frac{(h_\lambda m^\lambda)}{\mu(\mu + h_\sigma h^\sigma)} m_\nu \\
&= b_\nu \, .
\end{aligned} \tag{5.31}$$

The compact result $\delta S/\delta h^\nu = b_\nu$ gives a meaning to h^ν: it is a conjugate momentum to b^ν, just as m^ν is to u^ν.

5.2.2 Bracket and field equations

The covariant Poisson bracket for relativistic MHD is obtained by extending the nonrelativistic bracket of Refs. [31, 26] to spacetime. This is done by merely summing over the four spacetime indices instead of the three spatial ones and altering a few signs. However, a difficulty arises in choosing an appropriate equivalent of the nonrelativistic momentum and field, because the 4-vectorial equivalents of $\mathbf{M} = \rho\mathbf{v}$ and $\mathbf{B}$ will no longer produce the correct equations. Instead, the 4-vectors m^ν and h^ν provide the appropriate replacements, giving the relativistic MHD bracket

$$\{F,G\} = \int d^4x \left(n \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta n} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta n} \right) + \sigma \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta \sigma} \right) \right.$$
$$+ m_\nu \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta m_\nu} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta m_\nu} \right) + h^\nu \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta h^\nu} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta h^\nu} \right)$$
$$\left. + h^\mu \left[\left(\partial_\mu \frac{\delta F}{\delta m_\nu} \right) \frac{\delta G}{\delta h^\nu} - \left(\partial_\mu \frac{\delta G}{\delta m_\nu} \right) \frac{\delta F}{\delta h^\nu} \right] \right) \tag{5.32}$$

The bracket is complicated, but one can derive the equations of motion fairly quickly, thanks to the simple functional derivatives, as obtained in Eqs. (5.28), (5.29), (5.30), and (5.31), for the action of (5.24):

$$\frac{\delta S}{\delta n} = -\frac{\partial \rho}{\partial n}\,; \qquad \frac{\delta S}{\delta \sigma} = -\frac{\partial \rho}{\partial \sigma}\,; \qquad \frac{\delta S}{\delta m_\nu} = u^\nu\,; \qquad \frac{\delta S}{\delta h_\nu} = b^\nu\,,$$

where u^μ and b^μ here are shorthands for their expressions in terms of the fields m^μ and h^μ as given by (5.23).

Using $F = \int d^4x\, n(x) \delta^4(x - x_0)$ in $\{F, S\} = 0$ gives, after an integration by parts,

$$\partial_\mu (n u^\mu) = 0$$

which is the continuity equation (5.7), evaluated implicitly at x_0; however, since that point is arbitrary, the result holds for the entire spacetime. Going forward such niceties will be skimmed over. In the same manner one also finds the adiabaticity equation (5.8) from a σ variation.

The h^μ variation gives

$$\partial_\nu (h^\mu u^\nu) - h^\nu \partial_\nu u^\mu = 0\,. \tag{5.33}$$

The above equations are not Maxwell's equations, although they are analogous to the nonrelativistic equation (2.25), since they correspond to $\pounds_u h^\mu = 0$, the Lie-dragging of the four-dimensional vector density h^μ by u^μ. The theory obtained from the variational principle can be viewed as a family of theories, only some of which correspond to physical systems. However, if $\partial_\mu h^\mu = 0$, then the usual form of relativistic MHD may be obtained. The situation is exactly analogous to that in non-relativistic Hamiltonian MHD, which can describe systems with $\nabla \cdot \mathbf{B} \neq 0$: in both cases, the physical systems are a subset of the full class of systems described by the formalism. In the nonrelativistic case the condition $\nabla \cdot \mathbf{B} = 0$ is maintained by the dynamics and the similar situation that arises for h^μ will be shown in a later section. There also exists an alternative bracket that builds in $\partial_\mu h^\mu = 0$, given in a later section, where the constraint is enforced by the bracket's Jacobi identity. In any event, with h^μ thus specified, I can subtract a term $u^\mu \partial_\nu h^\nu$ from (5.33), giving the usual equivalent of Maxwell's equations

$$0 = \partial_\mu (h^\mu u^\nu - u^\mu h^\nu) \ .$$

Finally, the m^λ variation gives, after some work,

$$\begin{aligned} 0 &= -n\partial^\mu \left(-\frac{\partial p}{\partial n}\right) - \sigma\partial^\mu \left(-\frac{\partial p}{\partial \sigma}\right) + m_\nu \partial^\mu (u^\nu) + \partial_\nu (m^\mu u^\nu) \\ &\qquad + h_\nu \partial^\mu (b^\nu) - \partial_\nu (h^\nu b^\mu) , \\ &= -\partial^\mu p + \left(\mu u_\nu + \left(h_\lambda u^\lambda\right) h_\nu\right) \partial^\mu u^\nu + \partial_\nu \left(\mu u^\mu u^\nu + \left(h_\lambda u^\lambda\right) h^\mu u^\nu\right) \\ &\qquad + h_\nu \partial^\mu \left(h^\nu - \left(h_\lambda u^\lambda\right) u^\nu\right) - \partial_\nu \left(h^\nu h^\mu - \left(h_\lambda u^\lambda\right) h^\nu u^\mu\right) \\ &= \partial_\nu \Bigg(\left(\rho + p - \left(h_\lambda h^\lambda\right)\right) u^\mu u^\nu + g^{\mu\nu} \left[- p + \frac{1}{2}\left(h_\lambda h^\lambda - \left(h_\lambda u^\lambda\right)^2\right) \right] \\ &\qquad - h^\mu h^\nu + \left(h_\lambda u^\lambda\right) (h^\mu u^\nu + u^\mu h^\nu) \Bigg) , \end{aligned}$$

which is the momentum equation (5.10). Having been derived, it can be replaced with the much simpler, equivalent version involving b^μ.

Now it is clear that the covariant Poisson bracket formalism produces field equations compatible with the usual ones of relativistic MHD. Before probing more deeply the correspondence between that two, which I do in later sections, exploring in particular how one might use the field equations in practice, I discuss some alternative Poisson brackets in the next section.

5.3 Alternative MHD brackets

In this section I present additional Poisson brackets, three of which represent the content of the Poisson bracket of (5.32) with different representations of the magnetic field is represented, one of which possesses an arbitrary metric that represents a background gravitational field.

5.3.1 Constrained bracket

Consider the magnetic field part of the bracket of (5.32),

$$\{F,G\}_h : \;=\; \int d^4x \left(h^\nu \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta h^\nu} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta h^\nu} \right) + h^\mu \left[\left(\partial_\mu \frac{\delta F}{\delta m_\nu} \right) \frac{\delta G}{\delta h^\nu} - \left(\partial_\mu \frac{\delta G}{\delta m_\nu} \right) \frac{\delta F}{\delta h^\nu} \right] \right) \tag{5.34}$$

Just as the nonrelativistic bracket of Ref. [31, 26] has a counterpart in Ref. [30] the terms (5.34) have a relativistic counterpart analogous to Ref. [30] that requires functionals be defined on divergencee-free magnetic fields, which here would be h^μs such that $\partial_\mu h^\mu = 0$. This relativistic counterpart is simply given by an integration by parts of (5.34) and making use of $\partial_\mu h^\mu = 0$, i.e.,

$$\{F,G\}_{\partial h=0} : \;=\; \int d^4x \left(h^\nu \left(\frac{\delta F}{\delta m_\mu} \partial_\mu \frac{\delta G}{\delta h^\nu} - \frac{\delta G}{\delta m_\mu} \partial_\mu \frac{\delta F}{\delta h^\nu} \right) + h^\mu \left[\left(\partial_\mu \frac{\delta F}{\delta h^\nu} \right) \frac{\delta G}{\delta m_\nu} - \left(\partial_\mu \frac{\delta G}{\delta h^\nu} \right) \frac{\delta F}{\delta m_\nu} \right] \right). \tag{5.35}$$

The bracket is identical to (5.32), but for the swapped functional derivatives in the final line. The action (5.19) is unchanged, as are the n equation (5.7) and the σ equation (5.8). The h^μ gains an extra term, and may be written directly as the Maxwell-like equation

$$\partial_\nu (h^\mu u^\nu - u^\mu h^\nu) = \partial_\nu \mathcal{F}^{\mu\nu} = 0$$

without yet imposing a condition on h^μ. Finally, the equation for m^μ ends up with a couple fewer terms than before, yielding

$$\partial_\nu T^{\mu\nu} + \big(h^\mu - (h^\sigma u_\sigma)\, u^\mu\big)\partial_\nu h^\nu = 0 \tag{5.36}$$

where $T^{\mu\nu}$ is the (unchanged) stress-energy tensor (5.11).

However, unlike the prior bracket (5.32), the bracket (5.35) fails to satisfy the Jacobi identity unless the condition $\partial_\nu h^\nu = 0$ holds, as is shown in the Appendix. On the plus side, the momentum equation (5.36) is now reduced to its desired conservation form; on the minus side, the bracket is defined on the constrained class of functionals. The original bracket (5.32) always yields a momentum equation that is not only in conservation form, but also independent of α; however, it will yield differing magnetic equations depending on α, and only those corresponding to $\partial_\nu h^\nu = 0$ produce a Maxwell-like equation.

I regard the first bracket (5.32) to be superior, for then relativistic magnetohydrodynamics may be regarded as a specific example of a broader class of (mostly non-physical) dynamical systems, some of which may be of theoretical interest. For instance, in the non-relativistic case the broader class have been argued to be superior for computational algorithms (see, e.g., Ref. [10]), and the relativistic versions may be as well. Moreover, they may correspond to exotic theories, such as those including magnetic monopoles.

5.3.2 Bivector potential

The divergence-free condition can be made manifest by introducing an antisymmetric bivector potential $A^{\nu\mu}$ such that

$$h^\mu = \partial_\nu A^{\nu\mu} \,. \tag{5.37}$$

Then, assuming $F[h] = \bar{F}[A]$, i.e. functionals of the bivector potential obtain their dependence through h, I obtain

$$\delta F = \int d^4x \, \frac{\delta F}{\delta h^\mu} \delta h^\mu = \int d^4x \, \frac{\delta \bar{F}}{\delta A^{\nu\mu}} \delta A^{\nu\mu} = \delta \bar{F} \, . \tag{5.38}$$

Upon relating δh^μ to $\delta A^{\nu\mu}$ via (5.37), inserting $\delta h^\mu = \partial_\nu \delta A^{\nu\mu}$ into the second equation of (5.38), and assuming $\delta A^{\nu\mu}$ is arbitrary, I obtain the functional chain rule relation

$$\frac{\delta \bar{F}}{\delta A^{\mu\nu}} = \partial_\nu \frac{\delta F}{\delta h^\mu} \tag{5.39}$$

Inserting (5.39) into (5.35) gives the compact expression

$$\{F, G\}_A := 2 \int d^4x \, (\partial_\alpha A^{\alpha\nu}) \left(\frac{\delta F}{\delta m_\mu} \, \frac{\delta G}{\delta A^{\nu\mu}} - \frac{\delta G}{\delta m_\mu} \, \frac{\delta F}{\delta A^{\nu\mu}} \right) \tag{5.40}$$

I will use this form in Sec. 5.4.1, when discussing Casimir invariants.

5.3.3 3-Form bracket

In nonrelativistic MHD I observed in Sec. 2.4.1 that the magnetic equation may be written $\partial \mathbf{B}/\partial t + \pounds_{\mathbf{v}} \mathbf{B} = 0$, where $\pounds_{\mathbf{v}} \mathbf{B}$ is the Lie derivative of the vector density $\mathbf{B}$ dual to a 2-form. Thus one can write $B^i = \epsilon^{ijk} \omega_{jk}$ and $\omega_{jk} = B^i \epsilon_{ijk}/2$, where $i, j, k = 1, 2, 3$, and in terms of the 2-form the equation becomes $\partial \omega/\partial t + \pounds_{\mathbf{v}} \omega = 0$, with $\pounds_{\mathbf{v}}$ now being the appropriate expression for the Lie derivative of a 2-form in three dimensions (e.g., Ref. [41]). In n-dimensions, an $(n-1)$-form has n independent components. This suggests I can introduce the dual 3-form for relativistic MHD as follows:

$$\omega_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\delta} \, h^\delta \qquad \text{and} \qquad h^\delta = \frac{1}{6} \, \epsilon^{\alpha\beta\gamma\delta} \omega_{\alpha\beta\gamma} \, , \tag{5.41}$$

where it is seen that h^μ is a vector density because it is the contraction of the tensorial three-form with $\epsilon_{\alpha\beta\gamma\delta}$ a relative tensor of unit weight. From the above it follows that the 3-form equation of motion is given by $\partial\omega/\partial t + \pounds_u\omega = 0$. If I denote by F^μ_m the 4-vector given by $\delta F/\delta m_\mu$, then the magnetic portion of the Poisson bracket in terms of the 3-form can be compactly written as follows:

$$\{F,G\}_\omega = \int d^4x \left(\frac{\delta F}{\delta\omega_{\alpha\beta\gamma}} \left(\pounds_{G_m}\omega\right)_{\alpha\beta\gamma} - \frac{\delta G}{\delta\omega_{\alpha\beta\gamma}} \left(\pounds_{F_m}\omega\right)_{\alpha\beta\gamma} \right) . \tag{5.42}$$

Although similar forms in terms of Lie derivatives exist for all terms of all brackets, I am concentrating on the magnetic terms which written out are

$$\left(\pounds_{G_m}\omega\right)_{\alpha\beta\gamma} = G^\mu_m \partial_\mu \omega_{\alpha\beta\gamma} + \omega_{\mu\beta\gamma}\partial_\alpha G^\mu_m + \omega_{\alpha\mu\gamma}\partial_\beta G^\mu_m + \omega_{\alpha\beta\mu}\partial_\gamma G^\mu_m \,.$$

The transformation from the bracket $\{F,G\}_h$ of (5.34) to that of (5.42) follows from a chain rule calculation similar to that described in a different section. Thus, it satisfies the Jacobi identity because $\{F,G\}_h$ does, as shown directly in Appendix 1.3.

Relativistic MHD has a natural 3-form dual to b^μ, viz. $F_{\lambda\sigma}u_\nu + F_{\sigma\nu}u_\lambda + F_{\nu\lambda}u_\sigma$, which follows from the definition $b^\mu = \sqrt{4\pi}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}u_\nu/2$ with $u_\mu b^\mu = 0$ and $F_{\mu\nu}u^\nu = 0$. The 3-form dual to h^μ can similarly be represented as $\omega_{\lambda\sigma\nu} = \sqrt{4\pi}\left(F_{\lambda\sigma}w_\nu + F_{\sigma\nu}w_\lambda + F_{\nu\lambda}w_\sigma\right)/6$, where $w_\mu \equiv (h^2u^\mu - \alpha h_\mu)/(b_\lambda b^\lambda)$ is designed so that $h^\mu w_\mu = 0$ and $w_\mu u^\mu = 1$ and evidently $\omega_{\lambda\sigma\nu}h^\mu = 0$. Observe w_μ can be written in various ways using (5.23), (5.20), and other expressions.

The Jacobi identity for the bracket with (5.42) does not require closure of the 3-form. However, if the 3-form ω is exact then it can be written as the exterior derivative of a 2-form $A^*_{\mu\nu}$ as follows:

$$\omega_{\alpha\beta\gamma} = \partial_\alpha A^*_{\beta\gamma} + \partial_\beta A^*_{\gamma\alpha} + \partial_\gamma A^*_{\alpha\beta}$$

and one can rewrite the bracket in terms of $A_{\mu\nu}$. Instead of writing this out, I observe the bivector potential is given by

$$A^{\nu\mu} \equiv \frac{1}{2}\,\epsilon^{\nu\mu\sigma\tau} A^*_{\sigma\tau}$$

and so the closed 3-form bracket is essentially given by (5.40).

When the 3-form $\omega_{\alpha\beta\gamma}$ is exact I have for any 3-surface, Ω, in the four-dimensional Minkowski space-time, $\mathcal{D}$, Stokes theorem

$$\int_\Omega \omega = \int_\Omega \mathrm{d}A^* = \int_{\partial\Omega} A^*\,, \tag{5.43}$$

where in (5.43), $\int_\Omega \omega$ contains the notion of 'flux' in this setting. If Ω contains a time-like direction, one can write this as a conservation law, but such $3+1$ splittings will not be considered here. Instead, I refer to Ref. [22].

5.3.4 Background gravity

Now consider the full formalism generalized to curved spacetimes. In this context, the equations (5.7) - (5.10) are now divergences using the covariant derivative:

$$(nu^\mu)_{;\,\mu} = 0 \tag{5.44}$$

$$(\sigma u^\mu)_{;\,\mu} = 0 \tag{5.45}$$

$$\mathcal{F}^{\mu\nu}{}_{;\,\nu} = 0 \tag{5.46}$$

$$T^{\mu\nu}{}_{;\nu} = 0\,, \tag{5.47}$$

where I use ';' to denote covariant derivative.

Three modifications to the previous action principle are required: (1) because all integrations have tensorial integrands, the integrations must take place over a proper volume $\sqrt{-g}\; d^4x$; (2) h^μ should be treated as a contravariant vector, and m_μ as a covariant one, befitting their definitions (note that treating them any other way would introduce extra factors of $g^{\mu\nu}$ into the bracket); (3) functional derivatives should be defined in a way that makes them tensorial. Specifically, for a field variable v, one implicitly defines the functional derivative via

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(v+\epsilon\delta v) = \int d^4x\, \frac{\delta F}{\delta v}\delta v\, \sqrt{-g}\,.$$

The action is now

$$S = \frac{1}{2}\int d^4x\, \left(\frac{g^{\lambda\sigma}m_\lambda m_\sigma}{\mu} - \frac{\left(h^\lambda m_\lambda\right)^2}{\mu(\mu + g_{\lambda\sigma}h^\lambda h^\sigma)} + p - \rho\right)\sqrt{-g}$$

and its functional derivatives are

$$\frac{\delta S}{\delta n} = -\frac{\partial\rho}{\partial n}\,; \qquad \frac{\delta S}{\delta\sigma} = -\frac{\partial\rho}{\partial\sigma}\,; \qquad \frac{\delta S}{\delta m_\mu} = u^\mu\,; \qquad \frac{\delta S}{\delta h^\mu} = g_{\mu\nu}b^\nu\,.$$

Finally, the bracket becomes

$$\{F,G\} = \int d^4x\sqrt{-g}\Bigg(n\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta n} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta n}\right) + \sigma\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta \sigma}\right)$$
$$+ m_\nu\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta m_\nu} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta m_\nu}\right) + h^\nu\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\nu} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta h^\nu}\right)$$
$$+ h^\mu\left[\left(\partial_\mu\frac{\delta F}{\delta m_\nu}\right)\frac{\delta G}{\delta h^\nu} - \left(\partial_\mu\frac{\delta G}{\delta m_\nu}\right)\frac{\delta F}{\delta h^\nu}\right]\Bigg)\,. \tag{5.48}$$

The ∂_μ operators inside the bracket are still just partial derivatives, but the presence of the metric will tend to convert them into covariant derivatives; see e.g. Gravitation[25] Ch. 21. After an integration by parts, the variation $\{F,S\} = 0$ of the test function $F = \int d^4x\, n(x)\, \delta^4(x-x_0)\, \sqrt{-g}$ gives

$$\partial_\mu\left(nu^\mu\sqrt{-g}\right) = \sqrt{-g}\big(\partial_\mu\left(nu^\mu\right) + nu^\nu\Gamma^\mu{}_{\nu\mu}\big) = \sqrt{-g}\,(nu^\mu)_{;\mu} = 0\,,$$

with a similar result obtaining for the σ variation. The h^μ variation once again requires special attention, as it gives

$$\partial_\nu\left(h^\mu u^\nu\sqrt{-g}\right) - h^\nu\left(\partial_\nu u^\mu\right)\sqrt{-g} = \sqrt{-g}\left(h^\mu u^\nu{}_\nu + h^\mu{}_\nu u^\nu - h^\nu u^\mu{}_\nu + h^\mu u^\lambda\Gamma^\nu{}_{\lambda\nu}\right) = 0\,.$$

This time I choose α so that $h^\mu{}_{;\mu} = \partial_\mu h^\mu + h^\nu\Gamma^\mu{}_{\nu\mu} = 0$. Similar considerations apply to this choice as in the special relativistic case. Subtracting this expression and combining like terms then gives, with $\mathcal{F}^{\mu\nu} = h^\mu u^\nu - h^\nu u^\mu$,

$$\partial_\nu\mathcal{F}^{\mu\nu} + \mathcal{F}^{\mu\lambda}\Gamma^\nu{}_{\lambda\nu} + \mathcal{F}^{\nu\lambda}\Gamma^\mu{}_{\nu\lambda} = \mathcal{F}^{\mu\nu}{}_{;\nu} = 0\,.$$

Note that the third term is zero by the antisymmetry of $\mathcal{F}^{\mu\nu}$ and the symmetry of the covariant indices of $\Gamma^\mu{}_{\nu\lambda}$.

Finally, one obtains the momentum equation (5.47) by varying the test function $F = \int d^4x\, g^{\mu\nu} m_\nu \delta^4(x - x_0)\sqrt{-g}$. This derivation is lengthy, and will only be summarized here: (1) the partial derivative terms appear, and combine, exactly as in the special-relativistic case; (2) the $T^{\mu\lambda}\Gamma^{\nu}{}_{\lambda\nu}$ terms come from taking the partial derivatives of $\sqrt{-g}$; (3) the $T^{\nu\lambda}\Gamma^{\mu}{}_{\nu\lambda}$ terms come from derivatives of extra factors of the metric $g^{\mu\nu}$, some of which come from its inclusion in the test function, others of which come from $\delta S/\delta h^\mu = g_{\mu\nu} b^\nu$.

I conclude with two important notes. First, while I constructed the above formalism to handle curved spacetimes, it also applies to flat spacetimes with arbitrary coordinate systems, such as cylindrical, spherical, or toroidal coordinates. The nonrelativistic version may be generalized the same way (altering volumes d^3x to proper volumes $\sqrt{g}d^3x$), thus solving the problem of MHD coordinate changes in a pleasingly general way. Second, I emphasize that this formulation requires a predetermined spacetime, as including a dynamic $g^{\mu\nu}$ breaks the Jacobi identity for the bracket (5.48). I hope that such a dynamic spacetime can be incorporated into future work.

5.4 Degeneracy, symmetry and gauge in Hamiltonian MHD

Now I consider various issues pertaining to degeneracy. In the first section I obtain Casimir invariants, showing that the action S is not unique. Then I further explore the noninvertiblily of the transformations from (u^μ, b^μ) to (m^μ, h^μ). Here we will see that there is a consequence of a one parameter symmetry, providing an analog of Goldstone's theorem. Finally, in the last section I discuss the how the condition divergence-free condition on h^μ can be constructed for any problem in terms of Φ.

5.4.1 Casimirs and degeneracy

As noted in a previous section, the covariant Poisson bracket possesses degeneracy and associated Casimirs. A functional C is a Casimir if it satisfies

$$\{F, C\} = 0 \qquad \forall\, F\,. \tag{5.49}$$

Equation (5.49) should not be confused with the variational principle of (5.15), $\{F, S\} = 0$ for all functionals F, for the former is an aspect of the bracket alone, and provides no equations of motion. Because of the definition of C, the action S is not unique and can be replaced by $S + C$ for any Casimir C.

Turning to the task of finding Casimirs, I use (5.49) to provide functional equations for the Casimirs. Although difficult to solve in general, some explicit solutions can be found, facilitated by knowledge of Casimirs for nonrelativistic MHD[26, 2]. First, it is easy to obtain a family of what are called the entropy Casimirs,

$$C_s = \int d^4x\, n f(\sigma/n)$$

where f is an arbitrary function. In the nonrelativistic case this is a generalization of the total entropy, for if $f = \sigma/n$ and σ is the entropy per unit volume then $\int d^3x\, n f(\sigma/n) = \int d^3x\, \sigma$ is the total integrated entropy.

Next I seek a Casimir that is a relativistic version of the cross helicity

$\int d^3x\,\mathbf{v}\cdot\mathbf{B}$. For nonrelativistic MHD invariance of cross helicity requires a barotropic equation of state and $\nabla\cdot\mathbf{B}=0$, so I make analogous assumptions here. I assume ρ has no dependence on σ and the analog of $\nabla\cdot\mathbf{B}=0$ is made manifest by introducing the the antisymmetric 'potential' defined by

$$h^\mu = \partial_\gamma A^{\gamma\mu}\,. \tag{5.50}$$

ensuring that $\partial_\mu h^\mu = 0$. Using (5.50), the functional chain rule implies

$$\frac{\delta F}{\delta A^{\gamma\mu}} = -\partial_\gamma \frac{\delta F}{\delta h^\mu} \tag{5.51}$$

as shown earlier in Sec. 5.3.2. With (5.51) the terms of the bracket of (5.32) involving h^μ take on a simplified form. For example, I have

$$h^\nu \frac{\delta F}{\delta m_\mu}\partial_\mu \frac{\delta F}{\delta \beta^\nu} = -h^\nu \frac{\delta F}{\delta m_\mu}\frac{\delta F}{\delta A^{\mu\nu}}\,,$$

and similarly for the remaining terms. When written this way it is easy to show that the following generalization of the cross helicity is a Casimir:

$$C_{ch} = \int d^4x\,\frac{m_\mu}{n}\,\partial_\gamma A^{\gamma\mu} = \int d^4x\,\frac{m_\mu h^\mu}{n}\,, \tag{5.52}$$

This Casimir only exists for the case of divergence-free h^μ. Observe that on the constraint $u_\lambda u^\lambda = 1$, the integrand of (5.52) can be written as $m_\mu\,\partial_\gamma A^{\gamma\mu}/n = m_\mu h^\mu/n = \alpha(p+\rho)/n$, which follows from (5.20). Since α does not exist in the original (u^μ, b^μ) theory, this Casimir is a quantity tied to the Covariant bracket theory in terms of (m^μ, h^μ).

One also expects the existence of a magnetic helicity Casimir, but the nature of linking in four dimensions makes the situation complicated. Relativistic generalizations of magnetic helicity have been found in Refs. [39, 14], but I have yet to demonstrate that a quantity like either of these is in fact a Casimir. I also anticipate the existence of additional Casimirs that are generalizations of the nonrelativistic ones found in Refs. [37, 36], but a full discussion of Casimirs will await a future publication.

5.4.2 Gauge degeneracy

In an earlier section I gave an example that shows Eqs. (5.23) are not invertible. This lack of invertibility, which arises from the gauge freedom associated with α, can be understood in greater generality.

Because the degeneracy is not associated with the thermodynamic variables ρ and σ, I remove them by introducing the following scaled variables:

$$h = (\sqrt{p+\rho})\,\bar{h}\,, \quad m = (p+\rho)\,\bar{m}\,, \quad b = (\sqrt{p+\rho})\,\bar{b}\,, \quad u = \bar{u}\,, \quad \alpha = (\sqrt{p+\rho})\,\bar{\alpha}\,,$$

In terms of these variables (5.23) becomes

$$\bar{\Phi} = \bar{M}^{-1}\cdot\bar{\Psi} \tag{5.53}$$

with

$$\bar{M}^{-1} = \frac{1}{\bar{\mu}}\begin{bmatrix} 1 & -\bar{\alpha} \\ -\bar{\alpha} & \bar{\mu}+\bar{\alpha}^2 \end{bmatrix}, \qquad \bar{M} = \begin{bmatrix} \bar{\mu}+\bar{\alpha}^2 & \bar{\alpha} \\ \bar{\alpha} & 1 \end{bmatrix}, \tag{5.54}$$

and $\bar{\Phi} = (\bar{u}, \bar{b})$, $\bar{\Psi} = (\bar{m}, \bar{h})$. The quantity $\bar{\mu} \equiv 1 - \bar{h}^2$ is a normalized μ, and the quantity $\bar{\alpha}$ satisfies $\bar{\alpha} = \bar{m}_\nu \bar{h}^\nu = \bar{u}_\nu \bar{h}^\nu$. Varying (5.53) gives

$$\delta\bar{\Phi} = \bar{M}^{-1}\cdot\delta\bar{\Psi} + \frac{\partial \bar{M}^{-1}}{\partial\bar{\alpha}}\cdot\bar{\Psi}\,\delta\bar{\alpha} + \frac{\partial \bar{M}^{-1}}{\partial\bar{\mu}}\cdot\bar{\Psi}\,\delta\bar{\mu}\,.$$

Degeneracy follows if I can find a nonzero $\delta\bar{\Psi}$ giving $\delta\bar{\Phi} = 0$. Such would be given by

$$\begin{aligned}
\delta\bar{\Psi} &= -\bar{M} \cdot \frac{\partial \bar{M}^{-1}}{\partial \bar{\alpha}} \cdot \bar{\Psi}\,\delta\bar{\alpha} - \bar{M} \cdot \frac{\partial \bar{M}^{-1}}{\partial \bar{\mu}} \cdot \bar{\Psi}\,\delta\bar{\mu} \\
&= -\bar{M} \cdot \frac{\partial \bar{M}^{-1}}{\partial \bar{\alpha}} \cdot M \cdot \bar{\Phi}\,\delta\bar{\alpha} - \bar{M} \cdot \frac{\partial \bar{M}^{-1}}{\partial \bar{\mu}} \cdot M \cdot \bar{\Phi}\,\delta\bar{\mu} \\
&= \frac{\partial \bar{M}}{\partial \bar{\alpha}} \cdot \bar{\Phi}\,\delta\bar{\alpha} + \frac{\partial \bar{M}}{\partial \bar{\mu}} \cdot \bar{\Phi}\,\delta\bar{\mu} \\
&= \delta\bar{\alpha} \begin{bmatrix} 2\bar{\alpha} & 1 \\ 1 & 0 \end{bmatrix} \cdot \bar{\Phi} + \delta\bar{\mu} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \bar{\Phi}\,.
\end{aligned} \tag{5.55}$$

Thus from (5.55), $\delta\bar{m}^\nu = (2\bar{\alpha}\bar{u}^\nu + \bar{b}^\nu)\delta\bar{\alpha} + \bar{u}^\nu\delta\bar{\mu}$ and $\delta\bar{h}^\nu = \bar{u}^\nu\,\delta\bar{\alpha}$. Using $\delta\bar{\mu} = -2\bar{h}^\nu\delta\bar{h}_\nu = -2\bar{h}^\nu\bar{u}_\nu\,\delta\bar{\alpha} = -2\bar{\alpha}\,\delta\bar{\alpha}$, the two conditions imposed by (5.55) are

$$\delta\bar{h}^\nu = \bar{u}^\nu\,\delta\bar{\alpha} \qquad \text{and} \qquad \delta\bar{m}^\nu = \bar{b}^\nu\,\delta\bar{\alpha}\,, \tag{5.56}$$

reiterating my earlier point that α can vary while leaving u^μ and b^μ unchanged.

In terms of the scaled variables the action becomes

$$S[n,\sigma,\bar{m},\bar{h}] = \frac{1}{2}\int d^4x \left(\frac{p+\rho}{\bar{\mu}} \left(\bar{m}_\lambda\bar{m}^\lambda - \left(\bar{h}_\lambda\bar{m}^\lambda\right)^2 \right) + p - \rho \right)\,. \tag{5.57}$$

Now if I consider variation of the integrand of (5.57) with variations given by (5.56), and restrict to the constraint $u_\mu u^\mu = 1$ as given by the scaled version of (5.27), then the action is easily seen to be invariant. Using the scaled action in the form of (5.26), the integrand becomes upon variation $(p+\rho)(\bar{u}_\lambda\delta\bar{m}^\lambda + \bar{b}_\lambda\delta\bar{h}^\lambda) + \bar{h}_\lambda\delta\bar{h}^\lambda - \bar{\alpha}\delta\bar{\alpha}$, which vanishes upon insertion of (5.56), with the first two terms vanishing individually because $\bar{u}_\lambda b^\lambda = 0$. Thus, degeneracy appears as one transitions from (5.18) to (5.19). I add that in scaled variables $\mathcal{F} \sim \bar{u}^\mu \bar{b}^\nu - \bar{b}^\mu \bar{u}^\nu \sim \bar{u}^\mu \bar{h}^\nu - \bar{h}^\mu \bar{u}^\nu$; thus, at fixed $\bar{u}^\mu$, $\delta\mathcal{F} \sim \bar{u}^\mu \delta\bar{h}^\nu - \delta\bar{h}^\mu \bar{u}^\nu = 0$.

As a consequence of this added degeneracy, the system now possesses an additional symmetry, for one can add to α any solution α^* of the continuity equation $(\alpha^* u^\mu)_{,\mu} = 0$ while leaving the dynamics unchanged. I hope to explore the consequences of this new symmetry in future work.

This degeneracy is related to an the adaptation of Goldstone's theorem[35, 34, 33, 11] described in Ref. [29], where it was proven in the context of degenerate Poisson brackets with Casimir invariants that nonrelativistic Alfvén waves are associated with degeneracy can be thought of as an analog of Goldstone modes. A similar interpretation arises here in this covariant relativistic MHD setting, but discussion is beyond the scope of the present work.

5.4.3 Setting the gauge

Given a relativistic MHD problem posed in terms of (u^μ, b^μ), I must determine the associated problem in terms of (m^μ, h^μ), and this requires the determination of α, which amounts to setting the gauge so that $\partial_\mu h^\mu = 0$. Since this idea sits at the crux of the formalism, I will explain it is some detail.

Posing a relativistic MHD problem requires one specify (u^μ, b^μ) as well as n and σ on a space-like 3-volume, $\Omega \subset \mathcal{D}$, where is $\mathcal{D}$ is the four-dimensional space-time. In addition, a physical problem will have the initial conditions satisfy $u_\lambda u^\lambda = 1$ and $u_\lambda b^\lambda = 0$. Using $u^\alpha \partial_\alpha = \partial/\partial\tau$ where τ is the proper time, one can choose $\tau = 0$ to correspond to the state specified on Ω and then propagate values off of Ω by using the equations of motion to determine $\partial b^\mu/\partial\tau$, $\partial u^\mu/\partial\tau$, $\partial n/\partial\tau$, and $\partial\sigma/\partial\tau$ at $\tau = 0$. This is, in essence, the standard scenario for a Cauchy problem and many references for both MHD and relativistic fluids (e.g., Refs. [16, 3]) describe this in detail. One can imagine an exotic flow in which there exist spacetime points not connected to Ω by any flow lines; however, a modest boundedness condition excludes such cases.

The present situation is complicated by the fact that given b^μ on Ω at $\tau = 0$ I must also have that $\partial_\mu h^\mu = 0$ for all time, in order for the (m^μ, h^μ) dynamics to coincide with the physical (u^μ, b^μ) dynamics. Fortunately, $\partial_\mu h^\mu = 0$ is maintained in time if it is initially true on Ω. To see this I act on (5.33) with ∂_μ and obtain $\partial_\nu(u^\nu\partial_\mu h^\mu) = u^\nu\partial_\nu(\partial_\mu h^\mu) + (\partial_\nu u^\nu)(\partial_\mu h^\mu) = 0$ or

$$\frac{\partial(\partial_\mu h^\mu)}{\partial\tau} + (\partial_\nu u^\nu)(\partial_\mu h^\mu) = 0\,, \tag{5.58}$$

an equation that is analogous to (2.29) for nonrelativistic MHD. From (5.58), one concludes that if $\partial_\mu h^\mu = 0$ on Ω at $\tau = 0$, then $\partial_\mu h^\mu$ remains zero for all time. Thus, one can solve the (m_μ, h_μ) equations and uniquely obtain the (u_μ, b_μ) via (5.23) – provided one can 'set the gauge', i.e., find an α such that $\partial_\mu h^\mu = 0$ on Ω at $\tau = 0$ consistent with the $(u^\mu, b^\mu, n, \sigma)$ of the posed problem.

I will first consider a special example of setting the gauge, corresponding to the case described at the end of a prior section. We are given the MHD problem with initial conditions $\mathbf{v}(0, \mathbf{x}) \equiv 0$, i.e., $u^\mu(0, \mathbf{x}) = (1, \mathbf{0})$ and $b^\mu(0, \mathbf{x}) = (0, \mathbf{B}(0, \mathbf{x}))/\sqrt{4\pi}$ on the space-like 3-volume Ω with coordinates $\mathbf{x}$, and we wish to obtain an $h^\mu(0,\mathbf{x}) = (\alpha, \mathbf{B}/\sqrt{4\pi})$ and $m_\mu(0,\mathbf{x}) = (p+\rho+B^2/4\pi, \alpha\mathbf{B}/\sqrt{4\pi})$ such that $\partial_\mu h^\mu(0,\mathbf{x}) = 0$. Denoting $\partial_0\alpha = \alpha_t$, etc., gives the condition

$$0 = \partial_\mu h^\mu(0,\mathbf{x}) = \frac{1}{\sqrt{4\pi}}\left(\gamma_t \mathbf{v}\cdot\mathbf{B} + \gamma\mathbf{v}_t\cdot\mathbf{B} + \gamma\mathbf{v}\cdot\mathbf{B}_t + \alpha_t\sqrt{4\pi}\right) + \nabla\cdot\mathbf{h}\,, \quad (5.59)$$

where $\mathbf{h}$ is the spatial part of h^μ. Evaluating (5.59) on the initial condition gives

$$0 = \mathbf{v}_t\cdot\mathbf{B}(0,\mathbf{x}) + \alpha_t(0,\mathbf{x})\sqrt{4\pi} + \nabla\cdot\mathbf{B}(0,\mathbf{x})\,,$$

whence, with $\nabla\cdot\mathbf{B}(0,\mathbf{x}) = 0$, I conclude that

$$0 = \mathbf{v}_t\cdot\mathbf{B}(0,\mathbf{x}) + \alpha_t(0,\mathbf{x})\sqrt{4\pi} = -\frac{1}{\rho}\nabla p\cdot\mathbf{B}(0,\mathbf{x}) + \alpha_t(0,\mathbf{x})\sqrt{4\pi} \qquad (5.60)$$

where in (5.60) the MHD equations have been used to make the time derivatives consistent with the initial conditions on Ω, i.e., $\alpha_t(0,\mathbf{x}) = (\sqrt{4\pi}\rho)^{-1}\nabla p\cdot\mathbf{B}(0,\mathbf{x})$ will assure $\partial_\mu h^\mu = 0$ for all time. Observe, $\alpha(0,\mathbf{x})$ has not been specified – I am free to chose it as we please, but in doing so I will obtain different initial conditions $m^\mu(0,\mathbf{x})$ and $h^\mu(0,\mathbf{x})$ and these can be chosen for convenience. One only needs to know that there exists an $\alpha_t(0,\mathbf{x})$ that makes $\partial_\mu h^\mu(0,\mathbf{x}) = 0$, for then it will remain so for all time. Finally, if I solve the equations for m^μ and h^μ and obtain their values at any later time, insert them into (5.23), then values of u^μ and b^μ thus obtained are solutions of the relativistic MHD equations.

Now let us consider the general case, beginning with the expression

$$\partial_\mu h^\mu = \partial_\mu b^\mu + \partial_\mu(\alpha u^\mu) = \partial_\mu b^\mu + \partial_\mu(\alpha n u^\mu/n) = \partial_\mu b^\mu + n\,\frac{\partial}{\partial\tau}\left(\frac{\alpha}{n}\right) \qquad (5.61)$$

where the last equality follows from (5.7). Upon contracting $\partial_\nu(b^\mu u^\nu - u^\mu b^\nu) = 0$ with u_μ I obtain

$$\partial_\nu b^\nu = u_\nu \frac{\partial b^\nu}{\partial \tau} = -b^\nu \frac{\partial u_\nu}{\partial \tau}\,. \tag{5.62}$$

Consequently, (5.61) and (5.62) imply

$$\frac{\partial \alpha}{\partial \tau} - \frac{\alpha}{n}\frac{\partial n}{\partial \tau} = b^\nu \frac{\partial u_\mu}{\partial \tau}\,. \tag{5.63}$$

As stated above, a requisite condition for solving the Cauchy problem is that $\partial n/\partial\tau$ and $\partial u^\mu/\partial\tau$ be given in terms of all variables and their spatial derivatives. Thus on Ω, (5.63) provides a constraint involving α and $\partial\alpha/\partial\tau$ and any consistent choice for these quantities is sufficient to set the gauge. Different choices for α will give different initial conditions for the (m^μ, h^μ) dynamics, in agreement with (5.20), but the corresponding (u^μ, b^μ) at any time will be solutions to the relativistic MHD equations. Also, if the initial conditions match, then the (m^μ, h^μ) satisfy initially (5.27) and the counterpart that arises from $u_\mu b^\mu = 0$, which from (5.23) with (5.20) is

$$0 = \alpha\, m_\mu m^\mu + \alpha h_\mu h^\mu(\mu + \alpha^2) - m_\mu h^\mu(\mu + 2\alpha^2)$$

which is automatically satisfied upon insertion of $m_\mu m^\mu$ from (5.27).

I close this discussion by considering a point that may cause confusion. Given (m^μ, h^μ) on Ω I can certainly calculate $\nabla\cdot\mathbf{h}$, and $\partial h^0/\partial\tau$ will be determined by the equations of motion for (m^μ, h^μ). Thus, one may wonder how we are free to chose α and $\partial\alpha/\partial\tau$ to make $\partial_\mu h^\mu = 0$. The answer lies in the fact that the (m^μ, h^μ) system has a solution space that includes solutions that are not relativistic MHD solutions, and the procedure for picking the gauge selects out those that do indeed correspond – for these the two ways of determining $\partial_\mu h^\mu$ are equivalent.

Chapter 6

More on relativistic fluids and plasmas

This chapter covers further issues arising in relativistic fluids and plasmas. It begins by articulating the difference between Lagrangian and Eulerian coordinates, the origin of the noncanonical bracket of the previous chapter. In doing so it presents an action in Lagrangian coordinates, along the lines of the nonrelativistic ones presented in Chapter 3. I then follow with a derivation of the noncanonical bracket of relativistic MHD. Next follows an exposition of (one version of) the relativistic Navier-Stokes equation, laying out a method of putting such an equation into metriplectic form.

6.1 Lagrangian MHD action

The following two sections present my way of altering the approach of Kawazura et al. [14] to be more closely related to the work I do in my own MHD paper [8]. One big change in this approach is that I use a four-dimensional label space, with proper time treated (until you're done with the variation) as a separate variable. One might object, since now I have two time variables, proper time and one of the components of the label space. However, using a three-dimensional label space (like KYF) implies a choice of reference frame for removing a time component. It may also involve assuming a Minkowskian label space, which is unnecessary. The other change in this approach is that I derive the transformation behavior of the magnetic projection vector, rather than defining it as KYF do a couple of equations before their (24).

I will quickly argue that you need proper time as an independent variable, even when the label space is four-dimensional. Imagine you have the label space, and want to find a 4-velocity. To do so, you would need a limiting process which takes the same fluid element at decreasingly small time intervals. However, in the 4-label space, those two events you're comparing have different labels, so you need to figure out which events correspond to which fluid elements. However, the way to figure out which events in label space correspond to the same fluid elements is to use their 4-velocity. Since you are unable to determine 4-velocities from the label space alone (you need its tangent bundle, as is typical in variational problems), you can't find proper time from it alone, either. So I'm alright in treating it as independent.

I derive the transformation behavior of the magnetic projection vector by assuming this form for the conservation of magnetic flux:

$$\epsilon_{\mu\nu\lambda\sigma} b^\mu dq^\nu dq^\lambda dq^\sigma = \epsilon_{\mu\nu\lambda\sigma} b_0^\mu da^\nu da^\lambda da^\sigma \tag{6.1}$$

See, for example, the discussion Chap. 5 of b^μ and h^μ as duals of three-forms in relativity. From this you can show, after some manipulation,

$$b^\mu = \frac{b_0^\nu}{\mathcal{J}} \frac{\partial q^\mu}{\partial a^\nu} \tag{6.2}$$

However, just as in the Hamiltonian case, it appears that I need h^μ to get the correct Maxwell's equation. For I have

$$\begin{aligned}(b^\mu \dot{q}^\nu - b^\nu \dot{q}^\mu)_{,\nu} =& \frac{\partial}{\partial q^\nu}\left(\frac{b_0^\lambda}{\mathcal{J}}\frac{\partial q^\mu}{\partial a^\lambda}\dot{q}^\nu - \frac{b_0^\lambda}{\mathcal{J}}\frac{\partial q^\nu}{\partial a^\lambda}\dot{q}^\mu\right)\\ =& \frac{A^\sigma_{\ \nu}}{\mathcal{J}}\frac{\partial}{\partial a^\sigma}\left(\frac{b_0^\lambda}{\mathcal{J}}\frac{\partial q^\mu}{\partial a^\lambda}\dot{q}^\nu - \frac{b_0^\lambda}{\mathcal{J}}\frac{\partial q^\nu}{\partial a^\lambda}\dot{q}^\mu\right)\end{aligned}$$

Partially expanded, this becomes

$$\begin{aligned}=&\left(\frac{b_0^\lambda}{\mathcal{J}}\frac{\partial q^\mu}{\partial a^\lambda}\right)\left(\frac{A^\sigma_{\ \nu}}{\mathcal{J}}\frac{\partial \dot{q}^\nu}{\partial a^\sigma}\right) - \left(\frac{b_0^\lambda}{\mathcal{J}}\frac{\partial q^\mu}{\partial a^\lambda}\right)\left(\frac{A^\sigma_{\ \nu}}{\mathcal{J}^2}\dot{q}^\nu\frac{\partial \mathcal{J}}{\partial a^\sigma}\right) + \frac{b_0^\lambda}{\mathcal{J}}\frac{A^\sigma_{\ \nu}\dot{q}^\nu}{\mathcal{J}}\frac{\partial^2 q^\mu}{\partial a^\lambda \partial a^\sigma} \\ &- \frac{b_0^\lambda}{\mathcal{J}}\frac{\partial q^\nu}{\partial a^\lambda}\frac{A^\sigma_{\ \nu}}{\mathcal{J}}\frac{\partial \dot{q}^\mu}{\partial a^\sigma} - \dot{q}^\mu\frac{A^\sigma_{\ \nu}}{\mathcal{J}}\frac{\partial}{\partial a^\sigma}\left(\frac{b_0^\lambda}{\mathcal{J}}\frac{\partial q^\nu}{\partial a^\lambda}\right)\end{aligned} \tag{6.3}$$

In the second term of (6.3), observe that

$$\left(\frac{A^\sigma_{\ \nu}}{\mathcal{J}}\dot{q}^\nu\frac{\partial \mathcal{J}}{\partial a^\sigma}\right) = \frac{\partial \mathcal{J}}{\partial s} = \mathcal{J}\frac{\partial \dot{q}^\nu}{\partial q^\nu} = A^\sigma_{\ \nu}\frac{\partial \dot{q}^\nu}{\partial a^\sigma}$$

Thus the first and the second terms cancel. Similarly, in the third term of (6.3), note that

$$\frac{b_0^\lambda}{\mathcal{J}}\frac{A^\sigma_{\ \nu}\dot{q}^\nu}{\mathcal{J}}\frac{\partial^2 q^\mu}{\partial a^\lambda \partial a^\sigma} = \frac{b_0^\lambda}{\mathcal{J}}\frac{\partial^2 q^\mu}{\partial a^\lambda \partial s} = \frac{b_0^\lambda}{\mathcal{J}}\frac{\partial \dot{q}^\mu}{\partial a^\lambda}$$

The fourth term of (6.3) is the opposite of this, due to the identity

$$\frac{\partial q^\nu}{\partial a^\lambda}\frac{A^\sigma_{\ \nu}}{\mathcal{J}} = \delta^\sigma_\lambda$$

So all terms except the fifth cancel; however, the fifth remains, and is proportional to $b^\mu_{\ ,\mu}$. In other words, I instead have the advective equation

$$(b^\mu u^\nu)_{,\nu} - b^\nu u^\mu_{\ ,\nu} = 0 \tag{6.4}$$

Which, as noted in my MHD paper, represents the Lie dragging of a 3-form, retrospectively justifying my flux conservation equation (6.1). Once again, this advective equation can be converted into Maxwell's equation by using a new 4-vector h^μ with vanishing 4-divergence. The derivation previously undertaken proceeds identically, except now the fifth term of (6.3) vanishes as well.

Now I have to use the action with h^μ and double-check that it gives the correct equation of motion... the result seems to be slightly simpler than what is in KYF, and avoids mixing 3-vectors and 4-vectors in the same expressions. To start, equations (6.1) and (6.2) are the same when expressed in terms of h^μ. In Lagrangian coordinates I write the dual of the EM field tensor as

$$\mathcal{F}^{\mu\nu} = b^\mu \dot{q}^\nu - \dot{q}^\mu b^\nu = \frac{b_0^\lambda}{\mathcal{J}} \left(\frac{\partial q^\mu}{\partial a^\lambda} \dot{q}^\nu - \dot{q}^\mu \frac{\partial q^\nu}{\partial a^\lambda} \right)$$

So the EM portion of the action is

$$\begin{aligned} S_{EM} &= \int -\frac{1}{16\pi} \mathcal{F}^{\mu\nu} \mathcal{F}_{\mu\nu} \, \mathrm{d}^4 q = \int \frac{h_0^\lambda h_0^\sigma}{4\mathcal{J}} \left(\frac{\partial q^\alpha}{\partial a^\lambda} \dot{q}^\beta - \dot{q}^\alpha \frac{\partial q^\beta}{\partial a^\lambda} \right) \left(\frac{\partial q^\alpha}{\partial a^\lambda} \dot{q}^\beta - \dot{q}^\alpha \frac{\partial q^\beta}{\partial a^\lambda} \right) \, \mathrm{d}^4 a \\ &= \int -\frac{h_0^\lambda h_0^\sigma}{2\mathcal{J}} \frac{\partial q^\alpha}{\partial a^\lambda} \frac{\partial q_\alpha}{\partial a^\sigma} \dot{q}_\beta \dot{q}^\beta + \frac{h_0^\lambda h_0^\sigma}{2\mathcal{J}} \frac{\partial q^\alpha}{\partial a^\lambda} \frac{\partial q_\beta}{\partial a^\sigma} \dot{q}_\alpha \dot{q}^\beta \, \mathrm{d}^4 a \end{aligned} \tag{6.5}$$

There are the following new terms in the equation of motion:

$$\begin{aligned} &\frac{\partial}{\partial s} \left(\frac{\partial L_{EM}}{\partial \dot{q}_\mu} \right) + \frac{\partial}{\partial a^\nu} \left(\frac{\partial L_{EM}}{\partial q_{\mu,\nu}} \right) - \frac{\partial L_{EM}}{\partial q_\mu} \\ =&\frac{\partial}{\partial s} \left(\frac{\partial L_{EM}}{\partial \dot{q}_\mu} \right) + \frac{\partial}{\partial a^\nu} \left(\frac{\partial L_{EM}}{\partial q_{\mu,\nu}} \right) - \frac{A_\nu^\mu}{\mathcal{J}} \frac{\partial L_{EM}}{\partial a^\nu} \end{aligned} \tag{6.6}$$

The portions that come from the first term in (6.5) are:

$$-\frac{\partial}{\partial s} \left(\frac{h_0^\lambda h_0^\sigma}{\mathcal{J}} \frac{\partial q^\alpha}{\partial a^\lambda} \frac{\partial q_\alpha}{\partial a^\sigma} \dot{q}^\mu \right) - \frac{\partial}{\partial a^\nu} \left(\frac{h_0^\nu h_0^\sigma}{\mathcal{J}} \dot{q}^\beta \frac{\partial q^\mu}{\partial a^\sigma} \dot{q}_\beta \right) + \frac{A_\nu^\mu}{\mathcal{J}} \frac{\partial}{\partial a^\nu} \left(-\frac{h_0^\lambda h_0^\sigma}{2\mathcal{J}} \frac{\partial q^\alpha}{\partial a^\lambda} \frac{\partial q_\alpha}{\partial a^\sigma} \dot{q}_\beta \dot{q}^\beta \right)$$

After you Eulerianize, which involves dividing by a factor of $\mathcal{J}$, these terms are, keeping their order,

$$\left(-h_\lambda h^\lambda u^\mu u^\nu - h^\mu h^\nu + \frac{1}{2} g^{\mu\nu} h_\lambda h^\lambda\right)_{,\nu} \tag{6.7}$$

as desired.

The second term of (6.5) gives, upon variation,

$$\frac{\partial}{\partial s}\left(\frac{h_0^\lambda h_0^\sigma}{\mathcal{J}} \frac{\partial q^\mu}{\partial a^\lambda} \frac{\partial q_\beta}{\partial a^\sigma} \dot{q}^\beta\right) + \frac{\partial}{\partial a^\nu}\left(\frac{h_0^\nu h_0^\sigma}{\mathcal{J}} \frac{\partial q_\beta}{\partial a^\sigma} \dot{q}^\mu \dot{q}^\beta\right) - \frac{A^\mu_\nu}{\mathcal{J}} \frac{\partial}{\partial a_\nu}\left(\frac{h_0^\lambda h_0^\sigma}{2\mathcal{J}} \frac{\partial q^\alpha}{\partial a^\lambda} \frac{\partial q_\beta}{\partial a^\sigma} \dot{q}_\alpha \dot{q}^\beta\right)$$

When you Eulerianize this, it becomes (keeping the order of the terms)

$$\left((h_\lambda u^\lambda) h^\mu u^\nu + (h_\lambda u^\lambda) u^\mu h^\nu - \frac{1}{2} g^{\mu\nu} (h_\lambda u^\lambda)^2\right)_{,\nu}$$

Combined with (6.7) and set to zero, I have the correct equation: the stress tensor in terms of h^μ matches what I have in (5.14).

I've noticed that my way of doing the fluid action doesn't show up in any of the papers, so I'll write it out too. I have energy density $\rho = n(m+\epsilon)$, and I have conservation of particle number $n\mathrm{d}^4q = n\mathrm{d}^4a$, from which I deduce $n = n_0/\mathcal{J}$. Unlike in the parametrization-independent version, ϵ has no dependency on $R = \sqrt{\dot{q}_\mu \dot{q}^\mu}$. This seems like a disadvantage, but it makes the 4-velocity expressible in terms of the 4-momentum, enabling the Legendre transform that gives a canonical bracket later. I have a pressure defined from an internal energy ϵ via

$$p = n^2 \frac{\partial \epsilon}{\partial n} = \frac{n_0^2}{\mathcal{J}^2} \left.\frac{\partial \epsilon}{\partial n}\right|_{n=n_0/\mathcal{J}} = \frac{p_0}{\mathcal{J}}$$

Note that p_0, unlike n_0, still depends on q. My fluid action is

$$\begin{aligned} S_{fl} &= \int \frac{1}{2}\left(n(m+\epsilon)+p\right)\dot{q}_\mu\dot{q}^\mu + \frac{1}{2}\left(p - n(m+\epsilon)\right)\mathrm{d}^4q \\ &= \int \frac{1}{2}\left(n_0(m+\epsilon)+p_0\right)\dot{q}_\mu\dot{q}^\mu + \frac{1}{2}\left(p_0 - n_0(m+\epsilon)\right)\mathrm{d}^4a \end{aligned}$$

So the variation is

$$\frac{\partial}{\partial s}\left(n_0(m+\epsilon)\dot{q}^\mu + p_0\dot{q}^\mu\right) - \frac{A^\mu_{\ \nu}}{\mathcal{J}}\frac{\partial p_0}{\partial a_\nu} = 0$$

Using the usual tricks, this Eulerianizes to

$$\left((\rho+p)u^\mu u^\nu - p\,g^{\mu\nu}\right)_{,\nu} = 0$$

So my total Lagrangian is

$$\begin{aligned} L =&\quad L_{fl} + L_{EM} \\ =&\ \frac{1}{2}\left[\left(n_0(m+\epsilon)+p_0 - \frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\alpha}{\partial a^\sigma}\right)\dot{q}_\beta\dot{q}^\beta + \Big(p_0 - n_0(m+\epsilon)\Big)\right. \\ &\left. + \frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\beta}{\partial a^\sigma}\dot{q}_\alpha\dot{q}^\beta\right] \end{aligned}$$

It has a canonical momentum

$$\pi_\nu = \frac{\partial L}{\partial \dot{q}^\nu} = \left(n_0(m+\epsilon)+p_0 - \frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\alpha}{\partial a^\sigma}\right)\dot{q}_\nu + \frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\nu}{\partial a^\sigma}\dot{q}_\alpha$$

Define

$$\mu = \left(n_0(m+\epsilon)+p_0 - \frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\alpha}{\partial a^\sigma}\right) \qquad\qquad \alpha = \frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\alpha}{\partial a^\lambda}\dot{q}_\alpha$$

Now μ is off from my definition (27) by a factor of $\mathcal{J}$, but I'll press on anyway.

I can now write the momentum and 4-velocity as

$$\pi_\nu = \mu\dot{q}_\nu + \alpha h_0^\sigma\frac{\partial q_\nu}{\partial a^\sigma} \qquad\qquad \dot{q}_\nu = \frac{1}{\mu}\left(\pi_\nu - \alpha h_0^\sigma\frac{\partial q_\nu}{\partial a^\sigma}\right)$$

Contracting the second term with h^μ, I have

$$\alpha = \frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\nu}{\partial a^\lambda}\dot{q}_\nu = \frac{1}{\mu}\left(\frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\nu}{\partial a^\lambda}\pi_\nu - \alpha\frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q_\nu}{\partial a^\sigma}\frac{\partial q^\nu}{\partial a^\lambda}\right)$$

I can solve this equation for α:

$$\alpha = \frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\nu}{\partial a^\lambda}\pi_\nu \Big/ \left(\mu + \frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q_\nu}{\partial a^\sigma}\frac{\partial q^\nu}{\partial a^\lambda}\right) = \frac{1}{\mu_{fl}}\frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\nu}{\partial a^\lambda}\pi_\nu \tag{6.8}$$

where I write the fluid enthalphy as

$$\mu_{fl} = \mu + \frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q_\nu}{\partial a^\sigma}\frac{\partial q^\nu}{\partial a^\lambda} = n_0(m+\epsilon) + p_0$$

So, in a somewhat more self-contained fashion, I can write

$$\dot{q}_\nu = \frac{\pi_\nu}{\mu} - (h_\lambda \pi^\lambda)\frac{\mathcal{J}h_\nu}{\mu\mu_{fl}}$$

(Yes, I got tired of writing all the conversion factors, just pretend they're still there.) Since I have $\dot{q}^\mu$ and α in terms of momentum, I can convert the entire Lagrangian:

$$L = \frac{1}{2}\left[\frac{\pi_\lambda\pi^\lambda}{\mu} - \mathcal{J}\frac{(h_\lambda\pi^\lambda)^2}{\mu\mu_{fl}} + (p_0 - n_0(m+\epsilon))\right]$$

Except for a factor of $\mathcal{J}$, this matches my expression (30).

Now use a Legendre transformation to attain the Hamiltonian:

$$H = \int \mathcal{H}\,\mathrm{d}^4a = \int \frac{1}{2}\left[\frac{\pi_\lambda\pi^\lambda}{\mu} - \mathcal{J}\frac{(h_\lambda\pi^\lambda)^2}{\mu\mu_{fl}} - (p_0 - n_0(m+\epsilon))\right]\mathrm{d}^4a \tag{6.9}$$

The equations of motion will require the use of a Poisson bracket

$$\{f,g\} = \int\left(\frac{\delta f}{\delta q^\mu}\frac{\delta g}{\delta \pi_\mu} - \frac{\delta g}{\delta q^\mu}\frac{\delta f}{\delta \pi_\mu}\right)\mathrm{d}^4a$$

In particular, I will need to set g to H:

$$\begin{aligned}\{f,H\} &= \int \left(\frac{\delta f}{\delta q^\mu}\frac{\delta H}{\delta \pi_\mu} - \frac{\delta H}{\delta q^\mu}\frac{\delta f}{\delta \pi_\mu} \right) \mathrm{d}^4 a \\ &= \int \left(\frac{\delta f}{\delta q^\mu}\frac{\partial \mathcal{H}}{\partial \pi_\mu} - \frac{\partial \mathcal{H}}{\partial q^\mu}\frac{\delta f}{\delta \pi_\mu} \right) \mathrm{d}^4 a \\ &= \int \left(\frac{\delta f}{\delta q^\mu}\frac{\partial \mathcal{H}}{\partial \pi_\mu} - \frac{\partial \mathcal{H}}{\partial q^\nu_{,\lambda}}\frac{\partial q^\nu_{,\lambda}}{\partial q^\mu}\frac{\delta f}{\delta \pi_\mu} \right) \mathrm{d}^4 a \\ &= \int \left(\frac{\delta f}{\delta q^\mu}\frac{\partial \mathcal{H}}{\partial \pi_\mu} + \frac{\partial}{\partial a^\nu}\frac{\partial \mathcal{H}}{\partial q^\mu_{,\nu}}\frac{\delta f}{\delta \pi_\mu} \right) \mathrm{d}^4 a\end{aligned}$$

The π_μ functional derivative is easy:

$$\frac{\delta H}{\delta \pi_\mu} = \frac{\partial \mathcal{H}}{\partial \pi_\mu} = \frac{\pi^\mu}{\mu} - \mathcal{J}\frac{(h_\lambda \pi^\lambda) h^\mu}{\mu \mu_{fl}} = \dot{q}^\mu \tag{6.10}$$

Further functional derivatives require the identity

$$1 = \dot{q}_\nu \dot{q}^\nu = \frac{1}{\mu^2}\left(\pi_\nu \pi^\nu - \frac{2\mathcal{J}(h_\lambda \pi^\lambda)^2}{\mu_{fl}} + \frac{\mathcal{J}^2 (h_\lambda \pi^\lambda)^2 h_\nu h^\nu}{\mu_{fl}^2} \right)$$

which can only be applied after taking variations. This expression can be manipulated to give another useful identity,

$$\left(-\frac{\pi_\lambda \pi^\lambda}{2\mu^2} + \mathcal{J}\frac{(h_\lambda \pi^\lambda)^2}{2\mu^2 \mu_{fl}} + \mathcal{J}\frac{(h_\lambda \pi^\lambda)^2}{2\mu \mu_{fl}^2} \right) = -\frac{1}{2} \tag{6.11}$$

I need to evaluate the quantity

$$\frac{\partial}{\partial a^\nu}\frac{\partial \mathcal{H}}{\partial q^\mu_{,\nu}}$$

I will start by gathering together all the non-magnetic terms, i.e. those that have no dependence on h^μ, and gain their dependence on $\mathcal{J}$. To simplify things, I first calculate the following, using $\partial \mathcal{J}/\partial q^\mu_{,\nu} = A^\nu_\mu$,

$$\left(\frac{\partial \mu}{\partial q^\mu_{,\nu}} \right)_{fl} = \left(\frac{\partial \mu_{fl}}{\partial q^\mu_{,\nu}} \right)_{fl}$$

$$= \left(-\frac{n_0^2}{\mathcal{J}^2}\frac{\partial \epsilon}{\partial n}A^{\nu}_{\ \mu} - \frac{n_0 s_0}{\mathcal{J}^2}\frac{\partial \epsilon}{\partial s}A^{\nu}_{\ \mu} - \frac{n_0}{\mathcal{J}^2}\frac{\partial p}{\partial n}A^{\nu}_{\ \mu} - \frac{s_0}{\mathcal{J}^2}\frac{\partial p}{\partial s}A^{\nu}_{\ \mu} \right)$$

Next up I have the quantity

$$\begin{aligned}
\left(\frac{\partial \mathcal{H}}{\partial q^{\mu}_{,\nu}}\right)_{fl} &= \left(-\frac{\pi_\lambda \pi^\lambda}{2\mu^2} + \mathcal{J}\frac{(h_\lambda \pi^\lambda)^2}{2\mu^2 \mu_{fl}} + \mathcal{J}\frac{(h_\lambda \pi^\lambda)^2}{2\mu \mu_{fl}^2} \right) \left(\frac{\partial \mu}{\partial q^{\mu}_{,\nu}}\right)_{fl} \\
&\quad + \frac{1}{2}\left(\frac{n_0}{\mathcal{J}^2}\frac{\partial p}{\partial n} + \frac{s_0}{\mathcal{J}^2}\frac{\partial p}{\partial s} - \frac{n_0^2}{\mathcal{J}^2}\frac{\partial \epsilon}{\partial n} - \frac{n_0 s_0}{\mathcal{J}^2}\frac{\partial \epsilon}{\partial s} \right) A^{\nu}_{\ \mu} \\
&= -\frac{1}{2}\left(-\frac{n_0^2}{\mathcal{J}^2}\frac{\partial \epsilon}{\partial n} - \frac{n_0 s_0}{\mathcal{J}^2}\frac{\partial \epsilon}{\partial s} - \frac{n_0}{\mathcal{J}^2}\frac{\partial p}{\partial n} - \frac{s_0}{\mathcal{J}^2}\frac{\partial p}{\partial s} \right) A^{\nu}_{\ \mu} \\
&\quad + \frac{1}{2}\left(\frac{n_0}{\mathcal{J}^2}\frac{\partial p}{\partial n} + \frac{s_0}{\mathcal{J}^2}\frac{\partial p}{\partial s} - \frac{n_0^2}{\mathcal{J}^2}\frac{\partial \epsilon}{\partial n} - \frac{n_0 s_0}{\mathcal{J}^2}\frac{\partial \epsilon}{\partial s} \right) A^{\nu}_{\ \mu} \\
&= \left(\frac{n_0}{\mathcal{J}^2}\frac{\partial p}{\partial n} + \frac{s_0}{\mathcal{J}^2}\frac{\partial p}{\partial s} \right) A^{\nu}_{\ \mu}
\end{aligned}$$

where in the first step I used the identity (6.11). Thus, using $\partial A^{\nu}_{\ \mu}/\partial a^{\nu} = 0$, and an integration by parts (remembering that this expression is inside an integral), I finally acquire

$$\begin{aligned}
\frac{\partial}{\partial a^{\nu}}\left(\frac{\partial \mathcal{H}}{\partial q^{\mu}_{,\nu}}\right)_{fl} &= \frac{\partial}{\partial a^{\nu}}\left(\left(\frac{n_0}{\mathcal{J}^2}\frac{\partial p}{\partial n} + \frac{s_0}{\mathcal{J}^2}\frac{\partial p}{\partial s} \right) A^{\nu}_{\ \mu} \right) \\
&= A^{\nu}_{\ \mu}\frac{\partial}{\partial a^{\nu}}\left(\frac{n_0}{\mathcal{J}^2}\frac{\partial p}{\partial n} + \frac{s_0}{\mathcal{J}^2}\frac{\partial p}{\partial s} \right) \\
&= -\left(\frac{n_0}{\mathcal{J}^2}\frac{\partial p}{\partial n} + \frac{s_0}{\mathcal{J}^2}\frac{\partial p}{\partial s} \right)\frac{\partial \mathcal{J}}{\partial q^{\mu}} \\
&= \frac{\partial p}{\partial n}\frac{\partial n}{\partial q^{\mu}} + \frac{\partial p}{\partial s}\frac{\partial s}{\partial q^{\mu}} = \frac{\partial p}{\partial q^{\mu}}
\end{aligned} \tag{6.12}$$

Finally I have to calculate the magnetic parts, starting with the divergence of the enthalpy,

$$\left(\frac{\partial \mu}{\partial q^{\mu}_{,\nu}}\right)_{EM} = -2\frac{h_0^{\nu}h_0^{\sigma}}{\mathcal{J}}\frac{\partial q_{\mu}}{\partial a^{\sigma}} + \frac{h_0^{\lambda}h_0^{\sigma}}{\mathcal{J}^2}\frac{\partial q^{\alpha}}{\partial a^{\lambda}}\frac{\partial q_{\alpha}}{\partial a^{\sigma}}A^{\nu}_{\ \mu}$$

The full thing, now. Note that the fluid enthalpy μ_{fl} has no magnetic dependence, so the calculation proceeds differently than in the fluid part.

$$
\begin{aligned}
\left(\frac{\partial\mathcal{H}}{\partial q^{\mu}_{,\nu}}\right)_{EM} =& \frac{1}{2}\left[-\frac{\pi_\lambda\pi^\lambda}{\mu^2}+\frac{1}{\mathcal{J}\mu^2\mu_{fl}}\left(h_0^\sigma\frac{\partial q^\lambda}{\partial a^\sigma}\pi_\lambda\right)^2\right]\left(\frac{\partial\mu}{\partial q^{\mu}_{,\nu}}\right)_{EM} \\
& -\frac{1}{\mathcal{J}\mu\mu_{fl}}\left(h_0^\sigma\frac{\partial q^\lambda}{\partial a^\sigma}\pi_\lambda\right)h_0^\nu\pi_\mu+\frac{1}{2\mathcal{J}^2\mu\mu_{fl}}\left(h_0^\sigma\frac{\partial q^\lambda}{\partial a^\sigma}\pi_\lambda\right)^2 A^\nu_{\ \mu} \\
=& \frac{1}{2}\left(-1-\frac{1}{\mathcal{J}\mu\mu_{fl}^2}\left(h_0^\sigma\frac{\partial q^\lambda}{\partial a^\sigma}\pi_\lambda\right)^2\right)\left(-2\frac{h_0^\nu h_0^\sigma}{\mathcal{J}}\frac{\partial q_\mu}{\partial a^\sigma}+\frac{h_0^\lambda h_0^\sigma}{\mathcal{J}^2}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\alpha}{\partial a^\sigma}A^\nu_{\ \mu}\right) \\
& -\frac{1}{\mathcal{J}\mu\mu_{fl}}\left(h_0^\sigma\frac{\partial q^\lambda}{\partial a^\sigma}\pi_\lambda\right)h_0^\nu\pi_\mu+\frac{1}{2\mathcal{J}^2\mu\mu_{fl}}\left(h_0^\sigma\frac{\partial q^\lambda}{\partial a^\sigma}\pi_\lambda\right)^2 A^\nu_{\ \mu} \\
=& \frac{1}{2}\left(1+\mathcal{J}\frac{\alpha^2}{\mu}\right)\left(2\frac{h_0^\nu h_0^\sigma}{\mathcal{J}}\frac{\partial q_\mu}{\partial a^\sigma}-\frac{h_0^\lambda h_0^\sigma}{\mathcal{J}^2}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\alpha}{\partial a^\sigma}A^\nu_{\ \mu}\right) \\
& -\frac{\alpha}{\mu}h_0^\nu\pi_\mu+\frac{1}{2}\frac{\mu_{fl}}{\mu}\alpha^2 A^\nu_{\ \mu}
\end{aligned}
$$

In the first step I used identity (6.11), and in the second I used (6.8) to eliminate most factors of μ_{fl}. Speaking of which, I will look at only the terms containing $A^\nu_{\ \mu}$, and expand the final factor of μ_{fl}:

$$
\begin{aligned}
& -\frac{h_0^\lambda h_0^\sigma}{2\mathcal{J}^2}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\alpha}{\partial a^\sigma}A^\nu_{\ \mu}-\frac{\alpha^2}{2\mu}\frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\alpha}{\partial a^\sigma}A^\nu_{\ \mu} \\
& +\frac{\alpha^2}{2\mu}\left(\mu+\frac{h_0^\lambda h_0^\sigma}{\mathcal{J}}\frac{\partial q^\alpha}{\partial a^\lambda}\frac{\partial q_\alpha}{\partial a^\sigma}\right)A^\nu_{\ \mu} \\
=& \left(-\frac{h_\lambda h^\lambda}{2}+\frac{\left(h_\lambda\dot{q}^\lambda\right)^2}{2}\right)A^\nu_{\ \mu} \qquad (6.13)
\end{aligned}
$$

where h^λ, $\dot{q}^\lambda$ are implicitly written in terms of label space quantities and momenta. The remaining terms are

$$
\begin{aligned}
&\frac{h_0^\nu h_0^\sigma}{\mathcal{J}}\frac{\partial q_\mu}{\partial a^\sigma} + \frac{\alpha^2}{\mu}h_0^\nu h_0^\sigma \frac{\partial q_\mu}{\partial a^\sigma} - \frac{\alpha}{\mu}h_0^\nu \pi_\mu \\
&= \frac{h_0^\nu h_0^\sigma}{\mathcal{J}}\frac{\partial q_\mu}{\partial a^\sigma} + \frac{\alpha^2}{\mu}h_0^\nu h_0^\sigma \frac{\partial q_\mu}{\partial a^\sigma} - \alpha h_0^\nu \left(\dot{q}_\mu + \frac{\mathcal{J}\alpha}{\mu}\frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q_\mu}{\partial a^\lambda}\right) \\
&= \left(\frac{h_0^\lambda h_0^\sigma}{\mathcal{J}^2}\frac{\partial q^\tau}{\partial a^\lambda}\frac{\partial q_\mu}{\partial a^\sigma} - \alpha\frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\tau}{\partial a^\lambda}\dot{q}_\mu\right)A^\nu_{\ \tau} \\
&= \left(h_\mu h^\sigma - (h_\lambda \dot{q}^\lambda)h^\sigma \dot{q}_\mu\right)A^\nu_{\ \sigma}
\end{aligned} \tag{6.14}
$$

Now I can finally use Hamilton's equations. The general form of an equation of motion for f, if f is written in terms of phase-space variables, is

$$
\frac{\partial f}{\partial s} = \dot{q}^\nu \frac{\partial f}{\partial q^\nu} = \{f, H\}
$$

Thanks to the momentum variation (6.10), the n, σ, and h^μ equations are all fairly simple to derive. I use the test function $f(z) = \int z(a, \partial q/\partial a, \pi)\delta(q_0 - a)\mathrm{d}^4 a$ to pick out the equation at a specific location q. First up, the continuity equation:

$$
\begin{aligned}
\dot{q}^\mu \frac{\partial n}{\partial q^\mu} = \{f(n_0/\mathcal{J}), H\} &= \int \dot{q}^\mu \frac{\partial}{\partial q^\mu}\frac{n_0}{\mathcal{J}}\delta(q_0 - a)\mathrm{d}^4 a \\
&= \int -\frac{n_0}{\mathcal{J}^2}\delta(q_0 - a)\mathrm{d}^4 a = -n\frac{\partial \dot{q}^\mu}{\partial q^\mu}
\end{aligned}
$$

where I note that the $\partial/\partial q^\mu$ inside the integral is converted to a, then converted right back a couple of steps later. The same argument produces the σ equation via $\sigma = s_0 n_0/\mathcal{J}$, or the advective s equation via $s = s_0$. The magnetic equation goes

$$\begin{aligned}
\dot{q}^\nu \frac{\partial h^\mu}{\partial q^\nu} =& \left\{ f\left(\frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\mu}{\partial a^\lambda}\right), H\right\} = \int \dot{q}^\nu \frac{\partial}{\partial q^\nu}\left(\frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\mu}{\partial a^\lambda}\right)\delta(q_0 - a)\mathrm{d}^4 a \\
=& - \int \left[\frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\mu}{\partial a^\lambda}\frac{\partial \dot{q}^\nu}{\partial q^\nu} + \frac{\partial}{\partial a^\lambda}\left(\dot{q}^\mu \frac{h_0^\lambda}{\mathcal{J}}\right)\right]\delta(q_0 - a)\mathrm{d}^4 a \\
=& - \int \left[\frac{h_0^\lambda}{\mathcal{J}}\frac{\partial q^\mu}{\partial a^\lambda}\frac{\partial \dot{q}^\nu}{\partial q^\nu} + \frac{h_0^\nu}{\mathcal{J}}\frac{\partial q^\lambda}{\partial a^\nu}\frac{\partial \dot{q}^\mu}{\partial q^\lambda}\right]\delta(q_0 - a)\mathrm{d}^4 a \\
=& - h^\mu \frac{\partial \dot{q}^\nu}{\partial q^\nu} + h^\nu \frac{\partial \dot{q}^\mu}{\partial q^\nu}
\end{aligned}$$

which is the magnetic advection equation (6.4) written in terms of Lagrangian coordinates and h^μ. To write the momentum equation I must use the previously derived eqs. (6.12), (6.13), and (6.14). I get

$$\begin{aligned}
\dot{q}^\nu \frac{\partial \pi^\mu}{\partial q^\nu} =& \dot{q}^\nu \frac{\partial}{\partial q^\nu}\left(\mathcal{J}(\rho + p - h_\lambda h^\lambda)\dot{q}^\mu + \mathcal{J}(h_\lambda \dot{q}^\lambda)h^\mu\right) \\
=& \mathcal{J}\frac{\partial}{\partial q^\nu}\left((\rho + p - h_\lambda h^\lambda)\dot{q}^\mu \dot{q}^\nu + (h_\lambda \dot{q}^\lambda)h^\mu \dot{q}^\nu\right) \\
=& \{f(\pi^\mu), H\} = \int \left(\frac{\partial}{\partial a^\nu}\frac{\partial \mathcal{H}}{\partial q_{\mu,\nu}}\right)\delta(q_0 - a)\mathrm{d}^4 a \\
=& \mathcal{J}\frac{\partial}{\partial q^\nu}\left(p g^{\mu\nu} + \frac{1}{2}\left(-h_\lambda h^\lambda + (h_\lambda \dot{q}^\lambda)^2\right) g^{\mu\nu} + h^\mu h^\nu - (h_\lambda \dot{q}^\lambda)h^\nu \dot{q}^\mu\right)
\end{aligned}$$

which is correct; see, for example, my MHD paper's eqs. (18)-(20).

6.2 Relativistic Euler-Lagrange Map

Next up is converting the MHD canonical bracket,

$$\{f, g\} = \int \left(\frac{\delta f}{\delta q^i}\frac{\delta g}{\delta \pi_i} - \frac{\delta f}{\delta q^i}\frac{\delta g}{\delta \pi_i}\right) d^4 a,$$

into the Eulerian noncanonical one from my paper. To do so I need to express variations of arbitrary functions over spacetime, whether in Eulerian or

Lagrangian coordinates. The total variation will be the same when expressed either way:

$$\delta f = \int \frac{\delta f}{\delta n}\delta n + \frac{\delta f}{\delta \sigma}\delta\sigma + \frac{\delta f}{\delta m^\mu}\delta m^\mu + \frac{\delta f}{\delta h^\mu}\delta h^\mu d^4x \tag{6.15}$$

$$= \int \frac{\delta f}{\delta q^\mu}\delta q^\mu + \frac{\delta f}{\delta \pi^\mu}\delta \pi^\mu d^4a \tag{6.16}$$

To Eulerianize expressions I use delta-function expressions like the following:

$$n(x) = \int n(q)\delta(x-q)d^4q = \int \frac{n_0(a)}{\mathcal{J}}\delta(x-q)d^4q = \int n_0(a)\delta(x-q(a))d^4a$$

The other Eulerian quantities produce similar expressions:

$$\sigma(x) = \int \sigma_0 \delta(x-q)d^4a$$
$$m^\mu(x) = \int \pi_0^\mu \delta(x-q)d^4a$$
$$h^\mu(x) = \int h_0^\nu \frac{\partial q^\mu}{\partial a^\nu}\delta(x-q)d^4a$$

Variations in the Eulerian quantities induce variations in the phase-space variables q^μ and π^μ in this manner:

$$\delta n = \int n_0 \delta_\mu^{'}(x-q)\delta q^\mu d^4a$$
$$\delta \sigma = \int \sigma_0 \delta_\mu^{'}(x-q)\delta q^\mu d^4a$$
$$\delta m^\mu = \int \delta(x-q)\delta\pi_0^\mu + \pi_0^\mu \delta_\nu^{'}(x-q)\delta q^\nu d^4a$$
$$\delta h^\mu = \int h_0^\nu \frac{\partial q^\mu}{\partial a^\nu}\delta_\lambda^{'}(x-q)\delta q^\mu - h_0^\nu \delta_\lambda^{'}(x-q)\frac{\partial q^\lambda}{\partial a^\nu}\delta q^\mu d^4a$$
$$= \int h_0^\nu \frac{\partial q^\mu}{\partial a^\nu}\delta_\lambda^{'}(x-q)\delta q^\mu - h_0^\nu \delta_\lambda^{'}(x-q)\frac{\partial q^\lambda}{\partial a^\nu}\delta q^\mu d^4a$$

Inserting these expressions into the Eulerian side of (6.15) shows that the total variation is

$$\delta f = \int\int \left(\frac{\delta f}{\delta n}n_0 + \frac{\delta f}{\delta \sigma}\sigma_0 + \frac{\delta f}{\delta m^\nu}\pi_0^\nu + \frac{\delta f}{\delta h_\nu}h_0^\lambda \frac{\partial q^\nu}{\partial a^\lambda}\right)\delta'_\mu(x-q)\delta q^\mu$$
$$-\frac{\delta f}{\delta h^\mu}h_0^\nu \frac{\partial q^\lambda}{\partial a^\nu}\delta'_\lambda(x-q)\delta q^\mu + \frac{\delta f}{\delta m^\nu}\delta\pi_0^\nu \delta(x-q)d^4x\, d^4a$$

Comparison with the Lagrangian side of (6.15) shows that the Lagrangian functional derivatives are

$$\frac{\delta f}{\delta \pi^\mu} = \int \frac{\delta f}{\delta m^\mu}\delta(x-q)d^4x = \left.\frac{\delta f}{\delta m^\mu}\right|_{x=q(a)}$$
$$\frac{\delta f}{\delta q^\mu} = \int \left(\frac{\delta f}{\delta n}n_0 + \frac{\delta f}{\delta \sigma}\sigma_0 + \frac{\delta f}{\delta m^\nu}\pi_0^\nu + \frac{\delta f}{\delta h_\nu}h_0^\lambda \frac{\partial q^\nu}{\partial a^\lambda}\right)\delta'_\mu(x-q)$$
$$-\frac{\delta f}{\delta h^\mu}h_0^\nu \frac{\partial q^\lambda}{\partial a^\nu}\delta'_\lambda(x-q)d^4x$$
$$= -\int \left[n_0 \frac{\partial}{\partial x^\mu}\frac{\delta f}{\delta n} + \sigma_0 \frac{\partial}{\partial x^\mu}\frac{\delta f}{\delta \sigma} + \pi_0^\nu \frac{\partial}{\partial x^\mu}\frac{\delta f}{\delta m^\nu}\right.$$
$$\left. + h_0^\lambda \frac{\partial q^\nu}{\partial a^\lambda}\frac{\partial}{\partial x^\mu}\frac{\delta f}{\delta h^\nu} - \frac{\partial}{\partial x^\nu}\left(\frac{\delta f}{\delta h^\mu}h_0^\lambda \frac{\partial q^\nu}{\partial a^\lambda}\right)\right]\delta(x-q)d^4x$$

With these expressions in hand I can convert the bracket:

$$\{f,g\} = \int \frac{\delta f}{\delta q^\mu}\frac{\delta g}{\delta \pi_\mu} - \frac{\delta g}{\delta q^\mu}\frac{\delta f}{\delta \pi_\mu}d^4a$$
$$= -\int\int \left[n_0\left(\frac{\delta g}{\delta m_\mu}\frac{\partial}{\partial x^\mu}\frac{\delta f}{\delta n} - \frac{\delta f}{\delta m_\mu}\frac{\partial}{\partial x^\mu}\frac{\delta g}{\delta n}\right)\right.$$
$$+ \sigma_0\left(\frac{\delta g}{\delta m_\mu}\frac{\partial}{\partial x^\mu}\frac{\delta f}{\delta \sigma} - \frac{\delta f}{\delta m_\mu}\frac{\partial}{\partial x^\mu}\frac{\delta g}{\delta \sigma}\right)$$
$$+ \pi_0^\nu\left(\frac{\delta g}{\delta m_\mu}\frac{\partial}{\partial x^\mu}\frac{\delta f}{\delta m^\nu} - \frac{\delta f}{\delta m_\mu}\frac{\partial}{\partial x^\mu}\frac{\delta g}{\delta m^\nu}\right)$$
$$+ h_0^\nu\left(\frac{\delta g}{\delta m_\mu}\frac{\partial}{\partial x^\mu}\frac{\delta f}{\delta h^\nu} - \frac{\delta f}{\delta m_\mu}\frac{\partial}{\partial x^\mu}\frac{\delta g}{\delta h^\nu}\right.$$
$$\left.\left. + \frac{\delta f}{\delta h^\mu}\frac{\partial}{\partial x^\nu}\frac{\delta g}{\delta m_\mu} - \frac{\delta f}{\delta h^\mu}\frac{\partial}{\partial x^\nu}\frac{\delta g}{\delta m_\mu}\right)\right]\delta(x-q)d^4x d^4a$$

Performing the label-space integration, I acquire the bracket from my other paper:

$$
\begin{aligned}
\{f, g\} = -\int \Bigg[& n \left(\frac{\delta g}{\delta m_\mu} \frac{\partial}{\partial x^\mu} \frac{\delta f}{\delta n} - \frac{\delta f}{\delta m_\mu} \frac{\partial}{\partial x^\mu} \frac{\delta g}{\delta n} \right) \\
& + \sigma \left(\frac{\delta g}{\delta m_\mu} \frac{\partial}{\partial x^\mu} \frac{\delta f}{\delta \sigma} - \frac{\delta f}{\delta m_\mu} \frac{\partial}{\partial x^\mu} \frac{\delta g}{\delta \sigma} \right) \\
& + m^\nu \left(\frac{\delta g}{\delta m_\mu} \frac{\partial}{\partial x^\mu} \frac{\delta f}{\delta m^\nu} - \frac{\delta f}{\delta m_\mu} \frac{\partial}{\partial x^\mu} \frac{\delta g}{\delta m^\nu} \right) \\
& + h^\nu \left(\frac{\delta g}{\delta m_\mu} \frac{\partial}{\partial x^\mu} \frac{\delta f}{\delta h^\nu} - \frac{\delta f}{\delta m_\mu} \frac{\partial}{\partial x^\mu} \frac{\delta g}{\delta h^\nu} \right. \\
& + \left. \frac{\delta f}{\delta h^\mu} \frac{\partial}{\partial x^\nu} \frac{\delta g}{\delta m_\mu} - \frac{\delta f}{\delta h^\mu} \frac{\partial}{\partial x^\nu} \frac{\delta g}{\delta m_\mu} \right) \Bigg] d^4 x
\end{aligned}
$$

The action also converts into that paper's action, as it should.

Next up, something interesting I discovered when investigating the relativistic equivalent of flux conservation. Instead of (6.1), I try

$$F^{\mu\nu} dq_\mu dq_\nu = F_0^{\mu\nu} da_\mu da_\nu \ ,$$

which gives the transformation rule

$$F^{\mu\nu} = \frac{F^{\lambda\sigma}}{\mathcal{J}} \frac{\partial q^\mu}{\partial a^\lambda} \frac{\partial q^\nu}{\partial a^\sigma} \tag{6.17}$$

I can thus convert a standard Eulerian action to one using some fluid's Lagrangian coordinates:

$$L_{EM} = \frac{1}{16\pi} \int F^{\mu\nu} F_{\mu\nu} \, d^4 x = \frac{1}{16\pi} \int \frac{F_0^{\alpha\beta} F_0^{\gamma\delta}}{\mathcal{J}} \frac{\partial q^\nu}{\partial a^\alpha} \frac{\partial q^\lambda}{\partial a^\beta} \frac{\partial q_\nu}{\partial a^\gamma} \frac{\partial q_\lambda}{\partial a^\delta} \, d^4 a$$

Using the variational principle (6.6) gives the new terms

$$\frac{1}{16\pi}\frac{\partial}{\partial a^\nu}\left(\frac{F_0^{\nu\beta}F_0^{\gamma\delta}}{\mathcal{J}}\frac{\partial q^\lambda}{\partial a^\beta}\frac{\partial q_\mu}{\partial a^\gamma}\frac{\partial q_\lambda}{\partial a^\delta}+\frac{F_0^{\alpha\nu}F_0^{\gamma\delta}}{\mathcal{J}}\frac{\partial q^\sigma}{\partial a^\alpha}\frac{\partial q_\sigma}{\partial a^\gamma}\frac{\partial q_\mu}{\partial a^\delta}\right.$$

$$\left.+\frac{F_0^{\alpha\beta}F_0^{\nu\delta}}{\mathcal{J}}\frac{\partial q_\mu}{\partial a^\alpha}\frac{\partial q^\lambda}{\partial a^\beta}\frac{\partial q_\lambda}{\partial a^\delta}+\frac{F_0^{\alpha\beta}F_0^{\gamma\nu}}{\mathcal{J}}\frac{\partial q^\sigma}{\partial a^\alpha}\frac{\partial q_\sigma}{\partial a^\gamma}\frac{\partial q_\mu}{\partial a^\beta}-\frac{F_0^{\alpha\beta}F_0^{\gamma\delta}}{\mathcal{J}^2}A^\nu{}_\mu\frac{\partial q_\nu}{\partial a^\alpha}\frac{\partial q^\lambda}{\partial a^\beta}\frac{\partial q_\nu}{\partial a^\gamma}\frac{\partial q_\lambda}{\partial a^\delta}\right)$$

$$=-\frac{A^\nu{}_\mu}{4\pi}\frac{\partial}{\partial a^\nu}\left(F_{\mu\lambda}F^{\lambda\nu}+\frac{1}{4}\delta^\nu_\mu F^{\lambda\sigma}F_{\lambda\sigma}\right)$$

This is the divergence of a general expression for the electromagnetic stress-energy tensor $T^{\mu\nu}_{EM}$. No condition such as quasineutrality or vanishing proper electric fields had to be imposed. This suggests that a Lagrangian theory of multiple charged fluids might be possible, using an advected field tensor as in (6.17) to get the total Lorentz force.

6.3 Overview of relativistic Navier-Stokes

This section aims to develop the relativistic equivalent of the Navier-Stokes equation for viscous fluids. The field variables are now number density n (the mass density must now include internal energy, via mass-energy equivalence); the specific entropy s or entropy density $\sigma = ns$; and the fluid four-velocity u^μ or four-momentum $m^\mu = (\rho + p)u^\mu$. The continuity equation, Navier-Stokes equation, and entropy production are now expressed as four-divergences of tensorial quantities:

$$(nu^\mu)_{,\mu} = 0 \qquad T^{\mu\nu}{}_{,\nu} = 0 \qquad \sigma^\mu{}_{,\mu} \geq 0$$

The relativistic stress-energy tensor $T^{\mu\nu}$ and entropy vector σ^μ are as yet undetermined, but they will depend on relativized stress and heat, in the form of tensors $\sigma^{\mu\nu}$ and q^{μ}. The tensor $\sigma^{\mu\nu}$ should represent a pure stress, which means that in the local rest frame of the fluid it only contributes to the stress portion of $T^{\mu\nu}$. Such a restriction can be enforced by setting $\sigma^{\mu\nu}u_{\nu} = 0$. In turn, one achieves that restriction by replacing all the Kronecker delta functions in (2.40) with projection operators:

$$\sigma^{\mu\nu} = \Lambda^{\mu\nu\lambda\sigma}\frac{\partial u_{\lambda}}{\partial x^{\sigma}}$$

$$\Lambda^{\mu\nu\lambda\sigma} = \eta\left(P^{\mu\lambda}P^{\nu\sigma} + P^{\mu\sigma}P^{\nu\lambda} - \frac{2}{3}P^{\mu\nu}P^{\lambda\sigma}\right) + \zeta P^{\mu\nu}P^{\lambda\sigma}$$

$$P^{\mu\nu} = u^{\mu}u^{\nu} - g^{\mu\nu}$$

Now to generalize the heat vector. The salient question is whether this heat vector, newly converted into a four-vector, contributes to the stress-energy tensor. To support this view, imagine that you have two identical boxes filled with an ideal gas, separated by a conductive barrier. Start with one box incredibly hot, and the other cool; then, the masses of the gaseous molecules in the first box will be larger by a relativistic gamma factor, causing the center of mass of the entire system to be slightly displaced in the direction of the hot box. After a long time, the boxes will have reached thermal equilibrium; now, with comparable gamma factors, the center of mass will be exactly between the two boxes. Because the center of mass moved from the hot box's side to the middle, there must have been a momentum directed from the hot box to the cold box, in the same direction as the heat flow.

Internal energy will (as in the nondissipative case) be incorporated into the mass density, via the expression $\rho = n(m + \epsilon)$, so the heat four-vector should not contribute to the energy portion of the stress-energy tensor. This can be achieved by setting $u^\mu q_\mu = 0$ and writing the tensor (including all its other portions) as

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu - p g^{\mu\nu} + \sigma^{\mu\nu} + (q^\mu u^\nu + q^\nu u^\mu)$$

Now I will derive an expression for the heat four-vector. For the sake of the following argument, disregard the viscous stress tensor, as it will not affect my conclusion. After an application of the continuity equation, the momentum equation reads

$$\begin{aligned} 0 = T^{\mu\nu}{}_{,\nu} = n\left((m+\epsilon)\, u^\mu\right)_{,\nu} u^\nu + p_{,\nu} u^\mu u^\nu + p u^\mu{}_{,\nu} u^\nu + p u^\mu u^\nu{}_{,\nu} \\ - p_\nu g^{\mu\nu} + q^\mu{}_{,\nu} u^\nu + q^\mu u^\nu{}_{,\nu} + q^\nu{}_{,\nu} u^\mu + q^\nu u^\mu{}_\nu \end{aligned}$$

Contract this equation with u_μ to derive the energy conservation equation. Three terms will vanish via the identity $u_\mu u^\mu{}_{,\nu} = 0$; one term will vanish via $u_\mu q^\mu = 0$; two of the remaining pressure terms will cancel each other; finally, a use of the Leibniz rule will transfer a derivative from a q^μ to a u^μ. In all,

$$u^\mu \epsilon_{,\mu} + p u^\mu{}_{,\mu} + q^\mu{}_{,\mu} - q^\mu u_{\mu,\nu} u^\nu = 0 \tag{6.18}$$

The expression $u_{\mu,\nu} u^\nu$ occuring in the last term is the acceleration four-vector a_μ of the fluid, obeying $a_\mu u^\mu = 0$. Using the continuity equation and thermodynamic definitions, I can also write down an entropy equation:

$$\begin{aligned} (nsu^\mu)_{,\mu} = n u^\mu s_{,\mu} = n u^\mu \left[\left(\frac{\partial s}{\partial n}\right)_\epsilon n_{,\mu} + \left(\frac{\partial s}{\partial \epsilon}\right)_n \epsilon_{,\mu} \right] \\ = n u^\mu \left[\frac{p}{n^2 T} n_\mu + \frac{1}{nT} \epsilon_\mu \right] \end{aligned}$$

Substitute in (6.18) to get

$$(nsu^\mu)_{,\mu} = \frac{1}{T}\left[pu^\mu{}_{,\mu} - pu^\mu{}_{,\mu} - q^\mu{}_{,\mu} + q^\mu a_\mu\right]$$
$$= \frac{1}{T}\left[-q^\mu{}_{,\mu} + q^\mu a_\mu\right] \tag{6.19}$$

Incidentally, note that the RHS would be zero if the heat-related terms were omitted from the stress-energy tensor, and entropy would only be advected, not generated.

The entropy four-vector σ^μ should include not only entropy transported by the fluid (in the form nsu^μ), but also entropy generated by heating, represented schematically by the equation $dS = dQ/T$. So the quantity will be defined as

$$\sigma^\mu = nsu^\mu + \frac{q^\mu}{T}$$

Neglecting entropy generation due to stress, and substituting in (6.19), its four-divergence is

$$\sigma^\mu{}_{,\mu} = \frac{1}{T}\left(-q^\mu{}_{,\mu} + q^\mu a_\mu + q^\mu{}_{,\mu} - \frac{T_{,\mu}}{T}q^\mu\right)$$
$$= \frac{-q^\mu}{T^2}\left(T_{,\mu} - Ta_\mu\right)$$

This expression must be positive. In the fluid's local rest frame, q^0 is zero, so q^i must be parallel to $T_{,i} - Ta_i$. There are thus a total of three conditions on q^μ, including $q^\mu u_\mu = 0$, so q^μ is determined up to its magnitude. Said magnitude is then found by taking the nonrelativistic limit, in which the acceleration term is of order v^2/c^2. So one finds the unique expression for q^μ,

$$q^\mu = -\kappa P^{\mu\nu}\left(T_{,\nu} - Ta_\nu\right)$$

Note that $P^{\mu\nu}$ changes the sign of the inner product, so even though $q^\mu q_\mu$ is negative, $\sigma^\mu{}_{,\mu}$ is positive.

6.4 Metriplectic form of relativistic Navier-Stokes

Now for the brackets! The generator $H + \lambda S$ is composed of

$$H = \int \frac{1}{2(\rho + p)} m_\mu m^\mu + \frac{1}{2}(p - \rho)\, d^4x \qquad\qquad S = \int n\, s\, d^4x$$

and the equations of motion will be generated by

$$\{f, H + \lambda S\} = 0$$

for all functions f of the field variables. The antisymmetric portion of the bracket is given by

$$\begin{aligned}[f,g] = -\int n &\left(\left(\frac{\delta f}{\delta m_\mu}\partial_\mu\right)\frac{\delta g}{\delta n} - \left(\frac{\delta g}{\delta m_\mu}\partial_\mu\right)\frac{\delta f}{\delta n}\right) \\ + m_\mu &\left(\left(\frac{\delta f}{\delta m_\nu}\partial_\nu\right)\frac{\delta g}{\delta m_\mu} - \left(\frac{\delta g}{\delta m_\nu}\partial_\nu\right)\frac{\delta f}{\delta m_\mu}\right) \\ + \sigma &\left(\left(\frac{\delta f}{\delta m_\mu}\partial_\mu\right)\frac{\delta g}{\delta \sigma} - \left(\frac{\delta g}{\delta m_\mu}\partial_\mu\right)\frac{\delta f}{\delta \sigma}\right) d^4x\end{aligned}$$

Finally, the symmetric portion is given by

$$\begin{aligned}(f,g) = \frac{1}{\lambda}\int T\Lambda_{\mu\nu\lambda\sigma} &\left[\partial^\mu \frac{\delta f}{\delta m_\nu} - \frac{1}{T}\frac{\partial u^\mu}{\partial x_\nu}\frac{\delta f}{\delta \sigma}\right]\left[\partial^\lambda \frac{\delta g}{\delta m_\sigma} - \frac{1}{T}\frac{\partial u^\lambda}{\partial x_\sigma}\frac{\delta g}{\delta \sigma}\right] \\ +\kappa T^2 &\left[P^{\mu\nu}\left(\partial_\nu + a_\nu\right)\left(\frac{1}{T}\frac{\delta f}{\delta \sigma}\right) - u_\nu \partial_\mu \frac{\delta f}{\delta m_\nu} - u^\nu \partial_\nu \frac{\delta f}{\delta m^\mu}\right] \\ &\left[P_{\nu\lambda}\left(\partial^\lambda + a^\lambda\right)\left(\frac{1}{T}\frac{\delta g}{\delta \sigma}\right) - u_\sigma \partial^\lambda \frac{\delta g}{\delta m_\sigma} - u^\sigma \partial_\sigma \frac{\delta g}{\delta m_\lambda}\right] d^4x\end{aligned}$$

These brackets have all the properties listed before for the nonrelativistic brackets, except possibly for the positivity of the symmetric part, which I still need to check. At some point I hope to extend this formalism to relativistic MHD with dissipation.

Chapter 7

Relativistic classical spin

I begin this chapter by defining the dipole tensor as an antisymmetric rank-two tensor (following Frenkel [9]), in analogy with the electromagnetic field tensor (throughout I use the $(+---)$ signature, with time components in the zero position and c=1). In the rest frame of the dipole, this tensor has the magnetic dipole moment in the space-space part, and the electric dipole moment in the time-space part:

$$M^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & p_x & p_y & p_z \\ -p_x & 0 & -m_z & m_y \\ -p_y & m_z & 0 & -m_x \\ -p_z & -m_y & m_x & 0 \end{array} \right\}$$

Note, however, that the signs of the magnetic part differs from that of the field tensor, due to the sign convention used for the magnetic moment.

This dipole tensor can have three interpretations, depending on the context: **m** can represent magnetization per unit volume, and **p** the polarization per unit volume; the two can represent the total magnetic and electric dipole moments of a localized charge distribution; finally, they can represent the intrinsic moments of a particle such as an electron, in which case one would expect **p** to be zero in the particle's rest frame. Even prior to developing the apparatus (the Hamiltonian and bracket) necessary to incorporate the dipole moments into the dynamics of some system, one can find several expressions which are simpler in tensorial form than when written in terms of 4-vectors.

A localized charge distribution with changing dipole moments produces the following retarded potentials, with the part due to the electric dipole moment placed on the first line, followed by that due to the magnetic dipole moment:

$$\phi = \frac{\dot{\mathbf{p}} \cdot \mathbf{r}}{r^2} \qquad \mathbf{A} = \frac{\dot{\mathbf{p}}}{r}$$
$$\phi = 0 \qquad \mathbf{A} = \frac{\dot{\mathbf{m}} \times \mathbf{r}}{r^2}$$

Using the dipole tensor for the case of a localized charge distribution, these four equations combine into the single equation

$$A^\mu = \frac{\dot{M}^{\mu\nu} r_\nu}{(r^\lambda u_\lambda)^2} \tag{7.1}$$

where u^μ is the four-velocity of the dipole, and r^μ is a null vector from the retarded position of the dipole to the field point. In order for this expression to be covariant, the dot refers to differentiation with respect to the dipole's proper time. Looking at static fields instead, the classical equations for the potentials of an ideal stationary electric and magnetic dipole are, respectively,

$$\phi = \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \qquad \mathbf{A} = \frac{\mathbf{m} \times \mathbf{r}}{c\, r^3}$$

The two special cases for the dipole potentials then reduce to the unified equation

$$A^\mu = \frac{M^{\mu\nu} r_\nu}{(r^\lambda u_\lambda)^3} - \frac{M^{\mu\nu} u_\nu}{(r^\lambda u_\lambda)^2}$$

where r^μ is the position 4-vector to the retarded position of the dipole, and u^μ is the velocity of the dipole at the retarded time. This expression looks a bit peculiar, but for a particle or charge distribution whose dipole moments are constant and which has zero total charge, it results from a simpler expression for the Hertz vectors of the charge distribution, as shown below.

One can also combine the two equations for bound charge and current,

$$\rho = -\nabla \cdot \mathbf{p} \qquad \mathbf{J} = \frac{\partial \mathbf{p}}{\partial t} + \nabla \times \mathbf{m}$$

into the single equation

$$J^\mu = -M^{\mu\nu}{}_{,\nu}$$

This time one should interpret $m^{\mu\nu}$ as polarization and magnetization per unit volume. The polarization and magnetization also produce a bound surface charge and current:

$$\sigma = \mathbf{P} \cdot \mathbf{n} \qquad \mathbf{K} = \mathbf{M} \times \mathbf{n}$$

where $\mathbf{n}$ is the normal vector to a surface at rest. These two equations become

$$K^\mu = M^{\mu\nu} n_\nu$$

where n^μ is the spacelike 4-vector whose components in the frame where the surface is at rest are $(0, \mathbf{n})$, and K^μ is the surface 4-current.

In a situation where there are no free charge or current, it is convenient to define the Hertz vectors $\mathbf{\Pi}_e$ and $\mathbf{\Pi}_m$, which are related to the potentials as follows:

$$\phi = -\nabla \cdot \mathbf{\Pi}_e \qquad \mathbf{A} = \frac{1}{c}\frac{\partial \mathbf{\Pi}_e}{\partial t} + \nabla \times \mathbf{\Pi}_m$$

Substituting these relations into the inhomogeneous electromagnetic wave equations whose only source terms are bound charges/currents, and grouping terms, one obtains the two equations

$$\nabla \cdot \left(\nabla^2 \mathbf{\Pi}_e - \frac{1}{c^2}\frac{\partial \mathbf{\Pi}_e}{\partial t^2} + 4\pi \mathbf{P} \right) = 0$$

$$\frac{1}{c}\frac{\partial}{\partial t}\left(\nabla^2 \mathbf{\Pi}_e - \frac{1}{c^2}\frac{\partial \mathbf{\Pi}_e}{\partial t^2} + 4\pi \mathbf{P} \right) + \nabla \times \left(\nabla^2 \mathbf{\Pi}_m - \frac{1}{c^2}\frac{\partial \mathbf{\Pi}_m}{\partial t^2} + 4\pi \mathbf{M} \right) = 0$$

These are clearly solved by the two new inhomogeneous wave equations

$$\nabla^2 \mathbf{\Pi}_e - \frac{1}{c^2}\frac{\partial^2 \mathbf{\Pi}_e}{\partial t^2} = -4\pi \mathbf{P}$$

$$\nabla^2 \mathbf{\Pi}_m - \frac{1}{c^2}\frac{\partial^2 \mathbf{\Pi}_m}{\partial t^2} = -4\pi \mathbf{M}$$

These solutions are not unique, as choosing them amounts to a choice of "gauge" for the Hertz vectors. To convert these expressions to covariant form, I define the Hertz tensor

$$\Pi^{\mu\nu} = \left\{ \begin{array}{cccc} 0 & \Pi_{ex} & \Pi_{ey} & \Pi_{ez} \\ -\Pi_{ex} & 0 & -\Pi_{mz} & \Pi_{my} \\ -\Pi_{ey} & \Pi_{mz} & 0 & -\Pi_{mx} \\ -\Pi_{ez} & -\Pi_{my} & \Pi_{mx} & 0 \end{array} \right\}$$

The relation of the potentials to the Hertz vectors becomes

$$A^\mu = -\Pi^{\mu\nu}{}_{,\nu}$$

and the inhomogenous wave equations become

$$\Box \Pi^{\mu\nu} = -4\pi M^{\mu\nu} \qquad \text{where} \quad \Box \equiv \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}$$

For a constant dipole moment, the solution of the wave equations is

$$\Pi^{\mu\nu} = -\frac{M^{\mu\nu}}{(r^\lambda u_\lambda)}$$

From this one can obtain the earlier potentials (7.1).

Contracting $M^{\mu\nu}$ with itself and with its dual tensor, one finds the two invariants

$$\mathbf{p} \cdot \mathbf{m} \qquad m^2 - c^2 p^2$$

These quantities are significant due to the fundamental particles' intrinsic magnetic moments. For instance, the second quantity would be nonzero for any electrons. Were this a relativistic electron in a hot plasma, the covariant transformation of the tensor $M^{\mu\nu}$ could lead to large, perpendicular electric and magnetic dipole moments in another frame, much the same way a static but nonzero electric field in one frame can produce powerful, perpendicular magnetic and electric fields in a frame moving at high velocity relative to the first frame, provided that motion is perpendicular to the original field direction. Such large dipole moments, induced by relativistic motion, would introduce an extra force on the electrons proportional to the gradient of the fields, as can be seen in the equations of motion to follow.

To describe the dynamics of a system that includes spin, I need a Hamiltonian and a bracket. I will focus in particular on the Hamiltonian physics of an electron, for which spin is an intrinsic quantity. For the nonrelativistic spin system, the Hamiltonian and bracket are given by

$$H = H_0 - \mathbf{p} \cdot \mathbf{E} - \mathbf{m} \cdot \mathbf{B}$$

$$\{f, g\} = \{f, g\}_0 + \mathbf{p} \cdot \left(\frac{\partial f}{\partial \mathbf{p}} \times \frac{\partial g}{\partial \mathbf{p}} \right) + \mathbf{m} \cdot \left(\frac{\partial f}{\partial \mathbf{m}} \times \frac{\partial g}{\partial \mathbf{m}} \right) \tag{7.2}$$

where H_0 and $\{,\}_0$ refer to the parts of the Hamiltonian and bracket independent of spin – see for example Marklund and Morrison [21]. These definitions give the proper equations of motion, as is easily checked:

$$\dot{\mathbf{p}} = \{\mathbf{p}, H\} = \mathbf{p} \times \mathbf{E}$$

$$\dot{\mathbf{m}} = \{\mathbf{m}, H\} = \mathbf{m} \times \mathbf{B}$$

The new term in the Hamiltonian, when paired with the field portion of the non-spin bracket, also gives the gradient forces $\nabla(\mathbf{p} \cdot \mathbf{E})$ and $\nabla(\mathbf{m} \cdot \mathbf{B})$.

In order to generalize these equations to the relativistic case, I need a new, more general Hamiltonian and bracket. Here is one simple choice of Hamiltonian:

$$H = H_0 + \frac{1}{2} F^{\alpha\beta} M_{\alpha\beta}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor. The spin part gives $-\mathbf{p} \cdot \mathbf{E} - \mathbf{m} \cdot \mathbf{B}$ in the electron's rest frame. For the electron, specifically, one could also use

$$H = H_0 + \frac{1}{2} M_{\alpha\beta} \left(F^{\alpha\beta} + U^\alpha F^{\beta\gamma} U_\gamma - U^\beta F^{\alpha\gamma} U_\gamma \right).$$

Here the spin part of the Hamiltonian just gives $-\mathbf{m}\cdot\mathbf{B}$ in the particle's rest frame.

Now I need to construct a bracket. I start with the simplest case, a particle whose 4-position and 4-velocity are treated as given functions of its proper time, so that one only needs the spin bracket. Noting that the nonrelativistic bracket (7.2) can be written in index notation as follows,

$$\{f,g\} = \{f,g\}_0 + \epsilon^{ijk} p_i \frac{\partial f}{\partial p^j}\frac{\partial g}{\partial p^k} + \epsilon^{ijk} m_i \frac{\partial f}{\partial m^j}\frac{\partial g}{\partial m^k},$$

I first devise an analogous bracket for the relativistic case. It turns out that a simpler expression is found when using the dual tensors $N^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta}M^{\alpha\beta}/2$ and $G^{\mu\nu} \equiv \epsilon^{\mu\nu\alpha\beta}F^{\alpha\beta}/2$, with corresponding Hamiltonian $H = G^{\alpha\beta}N_{\alpha\beta}/2$. Also, in the relativistic case I have to insert the gyromagnetic ratio g explicitly, whereas before it appeared implicitly in the ratio of magnetic dipole moment to intrinsic spin. After accounting for the symmetries of the dipole tensor, I can devise the following relativistic spin bracket:

$$\{f,g\}_M = \frac{g}{16}\epsilon^{\alpha\beta\gamma\delta}N_{\alpha\beta}U^\mu\left(\frac{\partial f}{\partial N^{\gamma\mu}} - \frac{\partial f}{\partial N^{\mu\gamma}}\right)U^\nu\left(\frac{\partial g}{\partial N^{\delta\nu}} - \frac{\partial g}{\partial N^{\nu\delta}}\right), \qquad (7.3)$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the four-dimensional Levi-Civita tensor, equal to one for indices of even permutation, negative one for indices of odd permutation, and zero otherwise. Using such a bracket gives the following equation of motion:

$$\frac{dN^{\mu\nu}}{d\tau} = -\frac{g}{2}\,\epsilon^{\mu\nu\alpha\beta}N_{\alpha\delta}U^\delta G_{\beta\gamma}U^\gamma \qquad (7.4)$$

From this one can derive the well-known BMT equation [5] for the spin 4-vector S^μ,

$$\frac{dS^\mu}{d\tau} = \frac{g}{2}\left[F^{\mu\alpha}S_\alpha + U^\mu(S_\alpha F^{\alpha\beta}U_\beta)\right] - U^\mu\left(S_\alpha\frac{dU^\alpha}{d\tau}\right), \tag{7.5}$$

as will presently be shown. To start with, the spin 4-vector and the dipole tensor are related through the equations

$$S_\mu = N_{\mu\alpha}U^\alpha \qquad N_{\mu\nu} = U_\mu S_\nu - S_\mu U_\nu$$

I now prove the equivalence, starting with

$$\frac{dS^\mu}{d\tau} = \frac{N^{\mu\alpha}}{d\tau}U_\alpha + N^{\mu\alpha}\frac{dU_\alpha}{d\tau}$$

The first term becomes

$$\begin{aligned}
-\frac{N^{\mu\alpha}}{d\tau}U_\alpha &= \frac{g}{2}\,\epsilon^{\mu\nu\alpha\beta}N_{\alpha\delta}U^\delta G_{\beta\gamma}U^\gamma U_\nu \\
&= \frac{g}{2}\,\epsilon^{\mu\nu\alpha\beta}S_\alpha G_{\beta\gamma}U^\gamma U_\nu \\
&= \frac{g}{4}\,\epsilon^{\mu\nu\alpha\beta}S_\alpha\epsilon_{\beta\gamma\delta\lambda}F^{\delta\lambda}U^\gamma U_\nu \\
&= \frac{g}{4}\,\epsilon^{\beta\mu\nu\alpha}\epsilon_{\beta\gamma\delta\lambda}S_\alpha F^{\delta\lambda}U^\gamma U_\nu
\end{aligned}$$

$$\begin{aligned}
= \frac{g}{4}\,(&\delta^\mu{}_\gamma\delta^\nu{}_\delta\delta^\alpha{}_\lambda + \delta^\mu{}_\delta\delta^\nu{}_\lambda\delta^\alpha{}_\gamma + \delta^\mu{}_\lambda\delta^\nu{}_\gamma\delta^\alpha{}_\delta - \delta^\mu{}_\lambda\delta^\nu{}_\delta\delta^\alpha{}_\gamma \\
&-\delta^\mu{}_\delta\delta^\nu{}_\gamma\delta^\alpha{}_\lambda - \delta^\mu{}_\gamma\delta^\nu{}_\lambda\delta^\alpha{}_\delta)S_\alpha F^{\delta\lambda}U^\gamma U_\nu
\end{aligned}$$

$$\begin{aligned}
= \frac{g}{4}\,(&S_\alpha F^{\nu\alpha}U^\mu U_\nu + S_\alpha F^{\mu\nu}U^\alpha U_\nu + S_\alpha F^{\alpha\mu}U^\nu U_\nu \\
&-S_\alpha F^{\mu\alpha}U^\nu U_\nu - S_\alpha F^{\nu\mu}U^\alpha U_\nu - S_\alpha F^{\alpha\nu}U^\mu U_\nu)
\end{aligned}$$

$$= \frac{g}{4} \left(S_\alpha F^{\nu\alpha} U^\mu U_\nu + S_\alpha F^{\alpha\mu} - S_\alpha F^{\mu\alpha} - S_\alpha F^{\alpha\nu} U^\mu U_\nu \right)$$

$$= -\frac{g}{2} \left(F^{\mu\alpha} S_\alpha + U^\mu (S_\alpha F^{\alpha\nu} U_\nu) \right)$$

where I have used, in order: the definition of S_μ; the definition of the dual tensor $G_{\mu\nu}$ in terms of $F_{\mu\nu}$; the cylic property of the Levi-Civita tensor; an identity of the Levi-Civita tensor; the index-substitution property of the delta tensors; the fact that $S^\alpha U_\alpha = 0$ and $U^\alpha U_\alpha = 1$; and, finally, the antisymmetry of the field tensor. I now have the first two terms of the BMT equation. For the second, thankfully simpler term, one finds:

$$N^{\mu\alpha} \frac{dU_\alpha}{d\tau} = \left(U^\mu S^\alpha - S^\mu U^\alpha \right) \frac{dU_\alpha}{d\tau}$$

$$= U^\mu \left(S^\alpha \frac{dU_\alpha}{d\tau} \right)$$

(Since all forces on an electron are rest-mass-preserving forces, $U^\alpha F_\alpha = 0$, and one term is eliminated.) Sticking all three terms together, I have the BMT equation (7.5). This can in turn be used to derive the Thomas equation of motion[**?**] for the spin 3-vector. If one were to write the equation of motion (7.4) in terms of the original dipole tensor instead of its dual, one would have

$$\frac{dM^{\mu\nu}}{d\tau} = M^{\mu\beta} \left(\eta_{\nu\alpha} + U_\nu U_\alpha \right) F^{\alpha\beta} - M^{\nu\beta} \left(\eta_{\mu\alpha} + U_\mu U_\alpha \right) F^{\alpha\beta}$$

in addition to an altered bracket.

While it is more common among physicists to present brackets in terms of derivatives, one can also define them in terms of basis tensors. To find a more general bracket, one them assumes the analyticity of the functions placed in it, and uses the linearity and Lorentz properties of a bracket:

$$\{f, \alpha g + \beta h\} = \alpha\{f, g\} + \beta\{f, g\}$$
$$\{f, gh\} = \{f, g\}h + \{f, h\}g$$

where f, g, and h are functions, and α and β are scalars (here, real numbers). For instance, the canonical Poisson bracket can be written

$$\begin{aligned}\{X_\mu, X_\nu\} &= 0\\ \{P_\mu, P_\nu\} &= 0\\ \{X^\mu, P_\nu\} &= \delta^\mu_\nu\end{aligned}$$

In order to have all covariant indices, one could also alter the last of these to

$$\{X_\mu, P_\nu\} = \eta_{\mu\nu}.$$

where $\eta_{\mu\nu}$ is the flat-space metric. More useful for my purposes is the transformed bracket which uses 4-velocity in place of 4-momentum, as the latter contains the electromagnetic 4-potential implicitly. Doing so gives the brackets

$$\begin{aligned}\{X_\mu, X_\nu\} &= 0\\ \{X_\mu, U_\nu\} &= \eta_{\mu\nu}\\ \{U_\mu, U_\nu\} &= F_{\mu\nu}\end{aligned}$$

One chooses a Hamiltonian H and bracket with the aim of producing the correct equations of motion

$$\dot{f} = \{f, H\}$$

for the various dynamical variables, i.e. (7.4) plus the standard equations of the electron in its rest frame (dot denotes proper time derivative):

$$\begin{aligned}
\dot{X}_i &= U_i \\
\dot{t} &= 1 \\
\dot{U}_i &= F_{i\alpha} U^\alpha + \nabla_i (\mathbf{m} \cdot \mathbf{B}) \\
\dot{U}_0 &= \mathbf{u} \cdot \mathbf{E} - \frac{\partial}{\partial t} (\mathbf{m} \cdot \mathbf{B})
\end{aligned}$$

One can choose a Hamiltonian freely, but the bracket must satisfy two properties in addition to its built-in linearity, namely antisymmetry and the Jacobi identity. The first is easy, but the second requires checking the following six different basis equations, assuming that the position and velocity already form a valid bracket:

$$\begin{aligned}
\{\{M_{\alpha\beta}, M_{\gamma\delta}\}, M_{\epsilon\zeta}\} + cyclic &= 0 \\
\{\{M_{\alpha\beta}, M_{\gamma\delta}\}, U_\mu\} + cyclic &= 0 \\
\{\{M_{\alpha\beta}, U_\mu\}, U_\nu\} + cyclic &= 0 \\
\{\{M_{\alpha\beta}, M_{\gamma\delta}\}, X_\mu\} + cyclic &= 0 \\
\{\{M_{\alpha\beta}, X_\mu\}, X_\nu\} + cyclic &= 0 \\
\{\{M_{\alpha\beta}, U_\mu\}, X_\nu\} + cyclic &= 0
\end{aligned} \qquad (7.6)$$

There is a general way to construct a representation of SO(m,n) in terms of antisymmetric matrices, which is to define the bracket

$$\{M_{\alpha\beta}, M_{\gamma\delta}\} = M_{\alpha\delta}\eta_{\beta\gamma} + M_{\beta\gamma}\eta_{\alpha\delta} - M_{\alpha\gamma}\eta_{\beta\delta} - M_{\beta\delta}\eta_{\alpha\gamma}$$

where $\eta_{\mu\nu}$ is a symmetric matrix with signature (m,n), in my case the diagonal flat-space metric. This bracket, unfortunately, does not give the correct equations of motion when reasonable Hamiltonians are used. The most promising alteration to date has been a modified version of the above bracket:

$$\begin{aligned}\{M_{\alpha\beta}, M_{\gamma\delta}\} &= M_{\alpha\delta}\,(\eta_{\beta\gamma} + U_\beta U_\gamma) + M_{\beta\gamma}(\eta_{\alpha\delta} + U_\alpha U_\delta) \\ &\quad - M_{\alpha\gamma}(\eta_{\beta\delta} + U_\beta U_\delta) - M_{\beta\delta}(\eta_{\alpha\gamma} + U_\alpha U_\gamma)\end{aligned}$$

This bracket was taken from Yee and Bander[**?**]. When supplied with the basic Hamiltonian

$$H = \frac{1}{2}U_\alpha U^\alpha + \frac{1}{2}M_{\alpha\beta}F^{\alpha\beta}$$

this gives the correct equations of motion, but it has the irreconciliable flaw of failling to satisfy the cross-term Jacobi identities (7.6). Note that the bracket (7.3), which is equivalent to the Yee and Bander bracket, is only valid for a system where the 4-velocity and 4-position are specified beforehand as functions of a particle's proper time.

Thankfully, there is a way to repair it. Hanson, Regge and Teitelboim [12] use a Dirac construction to create a bracket that preserves $M_{\mu\nu}U^\nu$ as a

Casimir invariant. For an uncharged particle, the bracket is as follows:

$$
\begin{aligned}
[X^\mu, X^\nu] &= M^{\mu\nu} \\
[X^\mu, U^\nu] &= \eta^{\mu\nu} \\
[X^\mu, M^{\nu\lambda}] &= U^\lambda M^{\mu\nu} - U^\nu M^{\mu\lambda} \\
[U^\mu, U^\nu] &= 0 \\
[U^\mu, M^{\nu\lambda}] &= 0 \\
[M^{\mu\nu}, M^{\lambda\sigma}] &= \left(M^{\mu\sigma}(\eta^{\nu\lambda} - U^\nu U^\lambda\right) + M^{\nu\lambda}\left(\eta^{\mu\sigma} - U^\mu U^\sigma\right) \\
&\quad - M^{\mu\lambda}\left(\eta^{\nu\sigma} - U^\nu U^\sigma\right) - M^{\nu\sigma}\left(\eta^{\mu\lambda} - U^\mu U^\lambda\right)
\end{aligned}
$$

Because this is a Dirac bracket of the previous one, I do not need to check the Jacobi identity.

From here, it is easy to check that $M_{\mu\nu}U^\nu$ is a Casimir, on the subspace where it is already zero:

$$
\begin{aligned}
[U_\mu, M_{\nu\lambda}U^\lambda] &= [U_\mu, M_{\nu\lambda}]U^\lambda + [U_\mu, U^\lambda]M_{\nu\lambda} = 0 \\
[X_\mu, M_{\nu\lambda}U^\lambda] &= M_{\nu\lambda}[X_\mu, U^\lambda] + [X_\mu, M_{\nu\lambda}]U^\lambda \\
&= M_{\nu\mu} + U_\lambda M_{\mu\nu}U^\lambda - U_\nu M_{\mu\lambda}U^\lambda \\
&= -U_\nu M_{\mu\lambda}U^\lambda \\
[M_{\mu\nu}, M_{\lambda\sigma}U^\sigma] &= [M_{\mu\nu}, M_{\lambda\sigma}]U^\sigma \\
&= M_{\mu\sigma}U^\sigma\left(\eta_{\nu\lambda} - U_\nu U_\lambda\right) - M_{\nu\sigma}U^\sigma\left(\eta_{\mu\lambda} - U_\mu U_\lambda\right)
\end{aligned}
$$

Sadly, reasonable choices for the Hamiltonian do not give the correct equations of motion. For example, a reasonable guess would be to add a coupling term:

$$H = \frac{1}{2}U_\mu U^\mu + M_{\mu\nu}F^{\mu\nu}$$

In this case $dU^\mu/d\tau$ and $dM^{\mu\nu}/d\tau$ have the correct expressions, but now $dX^\mu/d\tau$ is

$$\frac{dX^\mu}{d\tau} = U^\mu + 2M^{\mu\beta}U^\alpha F_{\alpha\beta} + M_{\alpha\beta}F^{\alpha\beta}{}_{,\gamma}M^{\gamma\mu}$$

Either this is incorrect, or U^μ is not the 4-velocity, implying that, while $M^{\mu\nu}$ is purely magnetic, it is purely magnetic in a frame of reference slightly different than that defined by the 4-velocity. This may be an interesting idea to investigate in the future.

Given the appearance of a flat-space metric in the various brackets, and the inability of the authors to produce a fully general particle bracket in the special relativistic case, it appears likely that a full Hamiltonian treatment requires moving to the general relativistic case. Some interesting inroads on this problem have already been made by Marsden et al. [22]. It has already been shown by Papapetrou [38] that spinning particles do not follow geodesics, and an adaptation of Marsden and Morrison's Hamiltonian approach should show this deviation. In addition, one could naturally extend a Hamiltonian theory by switching to a kinetic theory of matter, with distribution functions in place of individual particles, with ready applications in plasma physics. Some work in the nonrelativistic Spin Maxwell-Vlasov equations has already been done by Marklund and Morrison [21]. I remain confident that usefulness of the dipole tensor is far from exhausted.

Chapter 8

Physical Consequences of the Jacobi Identity

I will attempt to show that the homogeneous Maxwell equations (equivalent to the vanishing of magnetic monopoles) can be derived from only a few basic characteristics of the electromagnetic force, given the validity of special relativity. The first such characteristic is that the 4-force on a test particle is linear in its 4-velocity. Such a 4-force K^μ can be written as a linear combination of the components of the 4-velocity U^μ, or

$$K^\mu = M^{\mu\nu} U_\nu$$

where $M^{\mu\nu}$ is some matrix independent of the particle's 4-velocity but varying with its position in space and time. The tensorial character of $M^{\mu\nu}$ is assured by the quotient rule of tensor algebra. Like any rank two tensor, $M^{\mu\nu}$ can be decomposed into the sum of its symmetric part $S^{\mu\nu}$ and antisymmetric part $A^{\mu\nu}$, so that the force can be written

$$K^\mu = S^{\mu\nu} U_\nu + A^{\mu\nu} U_\nu$$

To be a valid 4-force, K^μ must obey the relation $K^\mu U_\mu = 0$, a geometric fact resulting from the constancy of $U^\mu U_\mu$. Written in full,

$$0 = S^{\mu\nu} U_\mu U_\nu + A^{\mu\nu} U_\mu U_\nu$$

The second term vanishes from the antisymmetry of $A^{\mu\nu}$, and by choosing various values of U^μ one can show that $S^{\mu\nu}$ is identically zero. So even the most general force linear in 4-velocity must have the form

$$K^\mu = A^{\mu\nu} U_\nu \tag{8.1}$$

That is, it is characterized by the six components (or two 3-vectors) of an antisymmetric tensor $A^{\mu\nu}$. I next assume that the system is Hamiltonian, which one can reasonably expect of any physical system without dissipation. So there exist a Hamiltonian function H and a bracket $[f, g]$, from which one can derive the equation of motion

$$K^\mu = \frac{dU^\mu}{d\tau} = [U^\mu, H]$$

Using once again the relation $K^\mu U_\mu = 0$, along with the Leibniz rule for brackets, I find

$$0 = U_\mu K^\mu = U_\mu \left[U^\mu, H\right] = \left[\frac{1}{2} U^\mu U_\mu, H\right]$$

So $U_\mu U^\mu/2$ commutes with any valid Hamiltonian H. (It is a peculiarity of relativistic Hamiltonian physics that the geometrical constraint $U_\mu U^\mu = const.$ must be applied after derivations using the bracket, with the paired oddity that the Hamiltonian is just some number, most often zero. So this commutation is not trivial.) An easy way to assure such commutation is to take H to be a function of $U_\mu U^\mu/2$, the simplest of which is that very number, so I will set

$$H = \frac{1}{2} U^\mu U_\mu$$

The force law now reads

$$\frac{dU^\mu}{dt} = A^{\mu\nu} U_\nu = [U^\mu, H] = [U^\mu, U^\nu] \, U_\nu$$

thus defining a portion of the bracket: $[U^\mu, U^\nu] = A^{\mu\nu}$. I fill out the remainder of the bracket by giving it canonical form, i.e. $[X^\mu, U^\nu] = \eta^{\mu\nu}$ and $[X^\mu, X^\nu] = 0$, where X^μ is the particle's 4-position and $\eta^{\mu\nu}$ is the inverse of the flat-space metric $\eta_{\mu\nu}$. Written out in full, the bracket is

$$[f, g] = \eta^{\mu\nu} \left(\frac{\partial f}{\partial X^\mu} \frac{\partial g}{\partial U^\nu} - \frac{\partial g}{\partial X^\mu} \frac{\partial f}{\partial U^\nu} \right) + A^{\mu\nu} \frac{\partial f}{\partial U^\mu} \frac{\partial g}{\partial U^\nu}$$

To be a valid bracket, it must obey the Jacobi identity $[[f, g], h] + [[g, h], f] + [[h, f], g] = 0$, and to check this it suffices to verify the identity by substituting all combinations of U^μ and X^μ for f, g and h. The only nontrivial such identity is

$$\begin{aligned} &[[U_\mu, U_\nu] \, , U_\lambda] + [[U_\nu, U_\lambda] \, , U_\mu] + [[U_\lambda, U_\mu] \, , U_\nu] \\ &= [A_{\mu\nu}, U_\lambda] + [A_{\nu\lambda}, U_\mu] + [A_{\lambda\mu}, U_\nu] \\ &= A_{\mu\nu,\lambda} + A_{\nu\lambda,\mu} + A_{\lambda\mu,\nu} = 0 \end{aligned}$$

These are, barring a factor of q/mc that can be factored out, the four homogeneous Maxwell equations. Note how few assumptions were required to obtain this equation: it all comes from the linearity of the force law and the nature of Hamiltonian systems.

However, I have so far been looking at only a single particle. So far there is no reason to believe that different particles with identical 4-velocities will experience similar forces: there may be no relationship between the tensor $A^{\mu\nu}$ governing one particle and the comparable tensor governing another. It does happen that, in electromagnetism, the tensors $A^{\mu\nu}$ are proportional, with the factor of proportionality being q/m, but this must be regarded as an added assumption if the identity (8.2) is to be regarded as expressing the homogeneous Maxwell equations.

The more general case merits some investigation. Suppose, then, that there are various particles labelled by i, each of which experiences a force linear in its 4-velocity. For each particle there will then be a force of the form (8.1), with an antisymmetric tensor $A^{(i)}_{\mu\nu}$. At a given point, the vector space of antisymmetric tensors at that point has six dimensions, so the various $A^{(i)}_{\mu\nu}$ can be expressed as a linear combinations of at most six basis elements $F^{(j)}_{\mu\nu}$, to wit:

$$\begin{aligned} A^{(i)}_{\mu\nu} &= q^{(i)}_{(1)} F^{(1)}_{\mu\nu} + q^{(i)}_{(2)} F^{(2)}_{\mu\nu} + ... \\ &= \sum_j q^{(i)}_{(j)} F^{(j)}_{\mu\nu} \end{aligned}$$

This can be taken to represent $A^{(i)}_{\mu\nu}$ at any point, provided one acknowledges that the individual $F^{(j)}_{\mu\nu}$ will in general change in different ways as one moves from point to point. This is an additional assumption, but a reasonable one, since it expresses a form of translation invariance of supposed laws of physics. The $q^{(i)}_{(j)}$ are then to be thought of as charges of different kinds. The Jacobi identity for a given particle can still be derived as before, and becomes

$$A^{(i)}_{\mu\nu,\lambda} + A^{(i)}_{\nu\lambda,\mu} + A^{(i)}_{\lambda\mu,\nu}$$
$$= \left(\sum_j q^{(i)}_{(j)} F^{(j)}_{\mu\nu}\right)_{,\lambda} + \left(\sum_j q^{(i)}_{(j)} F^{(j)}_{\nu\lambda}\right)_{,\mu} + \left(\sum_j q^{(i)}_{(j)} F^{(j)}_{\lambda\mu}\right)_{,\nu}$$
$$= \sum_j q^{(i)}_{(j)} \left(F^{(j)}_{\mu\nu,\lambda} + F^{(j)}_{\nu\lambda,\mu} + F^{(j)}_{\lambda\mu,\nu}\right) = 0$$

When summing this identity over the various particles labelled by i, gathering like terms give the further identity

$$\sum_{i,j} q^{(i)}_{(j)} \left(F^{(j)}_{\mu\nu,\lambda} + F^{(j)}_{\nu\lambda,\mu} + F^{(j)}_{\lambda\mu,\nu}\right) = 0$$

Suppose that there are many varieties of particle, but no particular relation among the various $q_{(j)}$ parametrizing any one such particle. Then I expect that, by choosing particles appropriately, I can make the sums $\sum_i q^{(i)}_{(j)}$ disappear for all but one index j, whereupon the remaining charges factor out, and

$$F^{(j)}_{\mu\nu,\lambda} + F^{(j)}_{\nu\lambda,\mu} + F^{(j)}_{\lambda\mu,\nu} = 0$$

for the remaining label j. This can then be done for the remaining labels, so that if I have n charge species that vary independently, I will also get n different replicas of the homogeneous Maxwell equations. This is particularly easy to see if I have "pure" charges of each species, because then no summation is required in (8.2). To avoid this identical behavior among the $F^{(j)}_{\mu\nu}$ there must be at least one relationship $f(q_{(1)}, q_{(2)}, ..., q_{(n)})$ among the various charges of a given particle.

As a specific example of this extended formalism, let's look at a classical theory of magnetic charge. Then each particle has an electric charge q_e and a magnetic charge q_m, and the non-relativistic Lorentz force becomes

$$\mathbf{F} = q_e(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}) + q_m(\mathbf{B} - \frac{1}{c}\mathbf{v} \times \mathbf{E})$$

Switching to covariant notation and unitless quantities, this becomes

$$K^\mu = q_e F^{\mu\nu} U_\nu + q_m G^{\mu\nu} U_\nu$$

where the tensor $G^{\mu\nu}$ is the dual of the field tensor $F^{\mu\nu}$, that is to say $G^{\mu\nu} = (1/2)\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. The Maxwell-like equation is now satisfied by the combined tensor $q_e F^{\mu\nu} + q_m G^{\mu\nu}$. As explained earlier, if the electric and magnetic charge can vary independently (for instance, if you have pure charges of both types), then by summing various such identities I find that both $F^{\mu\nu}$ and $G^{\mu\nu}$ obey the homogeneous Maxwell equations, which is equivalent to the vanishing of both electric and magnetic charges. To avoid this trivial result, there must be some relationship $f(q_e, q_m)$ between the two charges. The simplest such relationship is a linear equation $\alpha q_e + \beta q_m = 0$, with both α and β non-zero. However, then the magnetic charge q_m can be eliminated from the force equation:

$$K^\mu = q_e \left(F^{\mu\nu} - \frac{\alpha}{\beta} G^{\mu\nu} \right) U_\nu = q_e' H^{\mu\nu} U_\nu$$

with

$$q_e' = q_e \sqrt{1 + \frac{\alpha^2}{\beta^2}}$$

and

$$H^{\mu\nu} = \frac{1}{\sqrt{1 + \frac{\alpha^2}{\beta^2}}} \left(F^{\mu\nu} - \frac{\alpha}{\beta} G^{\mu\nu} \right) \tag{8.2}$$

Here I've chosen the constant of proportionality to anticipate the inhomogeneous Maxwell equations $F^{\mu\nu}{}_{,\nu} = q_e U^\mu$ and $G^{\mu\nu}{}_{,\nu} = q_m U^\mu$, which now combine into $H^{\mu\nu}{}_{,\nu} = q_e' U^\mu$. Meanwhile $H^{\mu\nu}$, like any $A^{\mu\nu}$ from (8.1), also obeys the inhomogeneous Maxwell equations as a result of the Jacobi identity. So, with some minor tweaking, a proportional magnetic charge is shown to be equivalent to no magnetic charge at all. This result is shown via different means in Jackson.

No matter what, the Jacobi identity will produce an equation resembling the homogeneous Maxwell equations. However, to interpret this equation the usual way I must add the final assumption that there is only one field tensor $F^{\mu\nu}$ to which all of the tensors $A^{\mu\nu}$ are proportional, possibly following a reduction such as what led to (8.2). This assumption is not necessary, but it does happen to be a familiar quality of electromagnetism.

The fact that the field tensor obeys the homogeneous Maxwell equations, or the equivalent fact that it can be expressed as the curl of a 4-vector potential, is usually taken to be axiomatic. Here, it has been shown to be derivable from four assumptions about electromagnetism: (i) the electromagnetic 4-force is linear in 4-velocity; (ii) two particles with identical 4-velocities will experience proportional such 4-forces; (iii) a particle in an electromagnetic field forms a Hamiltonian system; (iv) the Hamiltonian function has the simple form $H = (1/2)U_\mu U^\mu$. The first two assumptions can be taken to characterize the Lorentz force, as opposed to other hypothetical forces (e.g. quadratic ones or gradient ones), and the third is a reasonable assumption about any nondissipative physical system. The fourth assumption, then, is the weakest. It can, in fact, be generalized somewhat while still yielding half of Maxwell's equations, but the weakened versions are not very illuminating, and the alternative Hamiltonians are more cumbersome. Worse, when one attempts to repeat the above argument for quadratic or higher-order forces, that simple choice of Hamiltonian no longer works. I will show this, and then show how a broader class of forces can be acquired from more general Hamiltonians.

A general quadratic force would have the form

$$\frac{dU^\mu}{d\tau} = M^{\mu\nu\lambda} U_\nu U_\lambda \tag{8.3}$$

for arbitrary $M^{\mu\nu\lambda}$; this time the quotient rule does not apply, and its tensorial nature would have to be established separately. A rank-three contravariant tensor can be decomposed into $M^{\mu\nu\lambda} = S^{\mu\nu\lambda} + A^{\mu\nu\lambda} + R^{\mu\nu\lambda}$, where $S^{\mu\nu\lambda}$ is symmetric in each of its indices, $A^{\mu\nu\lambda}$ is antisymmetric in each of its indices, and $R^{\mu\nu\lambda}$ has the symmetry

$$R^{\mu\nu\lambda} + R^{\nu\lambda\mu} + R^{\lambda\mu\nu} = 0$$

as may be verified by direct computation. The condition $U_\mu K^\mu = 0$ becomes

$$0 = S^{\mu\nu\lambda} U_\mu U_\nu U_\lambda + A^{\mu\nu\lambda} U_\mu U_\nu U_\lambda + R^{\mu\nu\lambda} U_\mu U_\nu U_\lambda \tag{8.4}$$

The middle term drops due to antisymmetry, and in the third term

$$R^{\mu\nu\lambda} U_\mu U_\nu U_\lambda = R^{\nu\lambda\mu} U_\mu U_\nu U_\lambda = R^{\lambda\mu\mu} U_\mu U_\nu U_\lambda$$

since each index is a dummy index. So the cyclic symmetry of $R^{\mu\nu\lambda}$ shows each such term to be zero. In (8.4), then, only the term involving $S^{\mu\nu\lambda}$ contributes, and by choosing various U^μ one can show that every element of $S^{\mu\nu\lambda}$ is zero. Since the antisymmetric $A^{\mu\nu\lambda}$ does not contribute to the force (8.3), a reduction occurs (as in the linear case):

$$\frac{dU^\mu}{d\tau} = R^{\mu\nu\lambda} U_\nu U_\lambda \tag{8.5}$$

Due to the form of this force law, I can also choose $R^{\mu\nu\lambda}$ to be symmetric in the last two indices, leaving it with a total of twenty independent components. The next step would be to choose a Hamiltonian and bracket. However, unlike in the linear case, something undesirable happens if I choose the simplest Hamiltonian $H = (1/2)U_\mu U^\mu$. This Hamiltonian again suggests filling out the brackets via $[X^\mu, X^\nu] = 0$, $[X^\mu, U^\nu] = \eta^{\mu\nu}$, and

$$\begin{aligned} \frac{dU^\mu}{d\tau} &= R^{\mu\nu\lambda} U_\nu U_\lambda \\ &= \left[U^\mu, \frac{1}{2} U_\nu U^\nu \right] \\ &= [U^\mu, U^\nu] \, U_\nu \end{aligned}$$

which suggests $[U^\mu, U^\nu] = R^{\mu\nu\sigma} U_\sigma$. The Jacobi identity then renders the force trivial:

$$\begin{aligned} \left[[U^\mu, U^\nu], X^\lambda \right] + \left[\left[U^\nu, X^\lambda \right], U^\mu \right] + \left[\left[X^\lambda, U^\mu \right], U^\nu \right] &= \\ \left[R^{\mu\nu\sigma} U_\sigma, X^\lambda \right] + \left[-\eta^{\nu\lambda}, U^\mu \right] + \left[\eta^{\lambda\mu}, U^\nu \right] &= \\ R^{\mu\nu\lambda} &= 0 \end{aligned}$$

So, generally speaking, the assumption $H = (1/2)U_\mu U^\mu$ is too strong, and rules out all but the linear case. To solve this problem, I will try a reverse process. I do not assume a particular force law; instead, my first assumption will be that I am looking at a Hamiltonian system consisting of a single particle with a Hamiltonian commuting with $U_\mu U^\mu$, and see what I get from there. This quantity involves an inner product, which means that I have implicitly introduced a metric, so my first try will be to add this metric explicitly:

$$H = \frac{1}{2} g_{\mu\nu} U^\mu U^\nu$$

Moving away from special relativity, I allow this metric to be a function of position, in contrast to the flat-space metric $\eta_{\mu\nu}$. To still be a valid metric, I require that $g_{\mu\nu}$ have signature $(+---)$, which then grants the existence of an inverse $g^{\mu\nu}$. To complete the system, I still need to fill out the bracket $[f, g]$. I will assume analyticity of all functions used in this construction, which means that repeated applications of the Leibniz rule can reduce the general bracket $[f, g]$ to expressions only involving brackets of the phase space basis elements X^μ, U^μ. For X^μ and U^μ to have their usual interpretation, I set $[X^\mu, X^\nu] = 0$ and use the relation

$$U^\mu = \frac{dX^\mu}{d\tau} = [X^\mu, H] = [X^\mu, U^\nu] g_{\nu\lambda} U^\lambda$$

From this I infer the next piece of the bracket, $[X^\mu, U^\nu] = g^{\mu\nu}$. Because $[X^\mu, X^\nu] = 0$, $g^{\mu\nu}$ commutes with X^μ, so I could also write $[X^\mu, U_\nu] = \delta^\mu_\nu$. With nonconstant $g^{\mu\nu}$, I have to start worrying about the Jacobi identity.

Identities involving terms such as $[[X^\mu, X^\nu], X^\lambda]$ and $[[U^\mu, X^\nu], X^\lambda]$ are already satisfied, but a nontrivial one does appear:

$$\begin{aligned}
& \left[\left[U^\mu, U^\nu\right], X^\lambda\right] + \left[\left[U^\nu, X^\lambda\right], U^\mu\right] + \left[\left[X^\lambda, U^\mu\right], U^\nu\right] \\
= & \left[\left[U^\mu, U^\nu\right], X^\lambda\right] + \left[-g^{\nu\lambda}, U^\mu\right] + \left[g^{\lambda\mu}, U^\nu\right] \\
= & \left[\left[U^\mu, U^\nu\right], X^\lambda\right] - g^{\nu\lambda}{}_{,\alpha} g^{\alpha\mu} + g^{\mu\lambda}{}_{,\alpha} g^{\alpha\nu} = 0
\end{aligned}$$

The Jacobi identity will be satisfied if I choose

$$\begin{aligned}
\left[U^\mu, U^\nu\right] &= g^{\mu\sigma}{}_{,\alpha} g^{\alpha\nu} U_\sigma - g^{\nu\sigma}{}_{,\alpha} g^{\alpha\mu} U_\sigma \\
&= g^{\alpha\mu} g^{\nu\beta} g_{\beta\sigma,\alpha} U^\sigma - g^{\alpha\nu} g^{\mu\beta} g_{\beta\sigma,\alpha} U^\sigma
\end{aligned}$$

where one can acquire the expression $g^{\mu\nu}{}_{,\lambda} = -g^{\mu\alpha} g^{\nu\beta} g_{\alpha\beta,\lambda}$ by differentiating $\delta^\mu{}_\nu = g^{\mu\alpha} g_{\alpha\nu}$. The final Jacobi identity involving terms such as $[[U^\mu, U^\nu], U^\lambda]$ is satisfied for symmetric $g^{\mu\nu}$. Now that I have the full bracket, I can get out an equation of motion:

$$\begin{aligned}
\frac{dU^\mu}{d\tau} &= \left[U^\mu, \frac{1}{2} g_{\nu\lambda} U^\nu U^\lambda\right] \\
&= \frac{1}{2}\left[U^\mu, g_{\nu\lambda}\right] U^\nu U^\lambda + \frac{1}{2} g_{\nu\lambda}\left[U^\mu, U^\nu\right] U^\lambda + \frac{1}{2} g_{\nu\lambda}\left[U^\mu, U^\lambda\right] U^\nu \\
&= \frac{1}{2}\left(-g^{\alpha\mu} g_{\nu\lambda,\alpha} U^\nu U^\lambda + (g^{\alpha\mu} g_{\lambda\sigma,\alpha} - g^{\mu\beta} g_{\beta\sigma,\lambda}) U^\sigma U^\lambda + (g^{\alpha\mu} g_{\nu\sigma,\alpha} - g^{\mu\beta} g_{\beta\sigma,\nu}) U^\sigma U^\nu\right) \\
&= -\frac{1}{2} g^{\mu\alpha}\left(g_{\alpha\sigma,\lambda} + g_{\alpha\lambda,\sigma} - g_{\sigma\lambda,\alpha}\right) U^\sigma U^\lambda \\
&= -\Gamma^\mu{}_{\sigma\lambda} U^\sigma U^\lambda
\end{aligned}$$

using the standard definition of the connection coefficients $\Gamma^\mu{}_{\sigma\lambda}$. This time, the Jacobi identity has yielded the geodesic law of motion from general relativity. This is perhaps unsurprising, if you realize that the Hamiltonian system thus constructed can be put into canonical form with the substitution $P_\mu = g_{\mu\nu}U^\nu$, because the canonical Hamiltonian obeys an extremization principle equivalent to the extremization of proper time, which is a more usual basis for deriving the geodesic law of motion. I could also add an antisymmetric, velocity-independent part $A^{\mu\nu}$ to the $[U^\mu, U^\nu]$ bracket and get back the Lorentz force; generally speaking, the various approaches espoused in these notes can be combined to yield multiple types of forces on a single particle.

For another alternative Hamiltonian, I will introduce the rest mass m that has thus far been implicit. Suppose some kind of (special relativistic) interaction yields a nonconstant rest mass; for instance, the particle under consideration could have intrinsic magnetic and/or electric dipole moments $\mathbf{m}$ and $\mathbf{p}$, which can be expressed as the time-space and space-space 3-vectors composing an antisymmetric dipole tensor $M^{\mu\nu}$. The interaction energy $-\mathbf{p} \cdot \mathbf{E} - \mathbf{m} \cdot \mathbf{B}$ can be expressed as $(1/2)F_{\alpha\beta}M^{\alpha\beta}$, which yields a variable rest mass

$$m = m_0 + \frac{1}{2}F_{\alpha\beta}M^{\alpha\beta}$$

The Hamiltonian describing a particle of variable rest mass is

$$H = \frac{1}{2}mU_\mu U^\mu - \frac{1}{2}m$$

Here the second term must be added to make the Hamiltonian commute with $U_\mu U^\mu$. As far as the Jacobi identity is concerned, this problem is identical to the general relativistic case just considered, but with "metric" $g_{\mu\nu} = m\eta_{\mu\nu}$.

Reiterating the previous arguments, I get the bracket

$$
\begin{aligned}
[X^\mu, X^\nu] &= 0 \\
[X^\mu, U^\nu] &= \frac{1}{m}\eta^{\mu\nu} \\
[U^\mu, U^\nu] &= \frac{1}{m^2}\left(\eta^{\mu\alpha} m_{,\alpha} U^\nu - \eta^{\nu\alpha} m_{,\alpha} U^\mu\right)
\end{aligned}
$$

This yields the quadratic equation of motion

$$\frac{dU^\mu}{d\tau} = [U^\mu, H] = \frac{1}{m}\left(\eta^{\mu\alpha} m_{,\alpha} U_\nu U^\nu - m_{,\nu} U^\nu U^\mu\right)$$

If the adjustment to rest mass comes from intrinsic dipole moments, then with $M^{\mu\nu}$ regarded as an independent variable, the equation of motion is

$$m\frac{dU_\mu}{d\tau} = \frac{1}{2}F_{\alpha\beta,\mu}M^{\alpha\beta} - \frac{1}{2}F_{\alpha\beta,\nu}U^\nu M^{\alpha\beta}U_\mu$$

which gives, in the particle's rest frame, the standard force $\mathbf{p}\cdot\nabla\mathbf{E} + \mathbf{m}\cdot\nabla\mathbf{B}$, subject to the constraint that $U_\mu U^\mu$ is constant. In the schema of (8.5), I would have

$$R_{\mu\nu\lambda} = \frac{1}{2}M^{\alpha\beta}\left(F_{\alpha\beta,\mu}\eta_{\nu\lambda} - F_{\alpha\beta,\nu}\eta_{\mu\lambda}\right)$$

The Jacobi identity has been shown to produce remarkable results: (i) the identity, plus a linear force, implies the homogeneous Maxwell's equations (or their equivalent); (ii) the Jacobi identity, plus a nonconstant metric, implies the geodesic law of motion; (iii) the identity, plus a nonconstant mass, produces the relativistic gradient force. All three results are usually found by much different arguments, but bringing the Hamiltonian nature of particle motion to the fore has allowed all three to arise from one oft-neglected identity. I hope that others will find this as remarkable as I do.

Chapter 9

Conclusion

I will conclude by noting the paths for future research that I (and hopefully others) will take, using the preceding research as a starting point.

The discoveries of Chapter 3 provide an important missing link in the study of Hamiltonian generalized MHD models. While Hamiltonian descriptions of Hall and Extended MHD had already been discovered [1], the brackets had to be devised by hand, and the Jacobi identities checked explicitly. Deriving these brackets from a more fundamental one, as I do in Chapter 3, puts these theories on more solid theoretical ground, while at the same time strengthening one's physical intuition about these generalized models. More importantly, to my tastes, it provides a starting point for developing theories of relativistic Hall and Extended MHD. While some such descriptions exist [15, 4], they are based on the simplest method of relativizing a system, by simply replacing 3-vector quantities with 4-vector ones. Because the relativistic version will have an additional equation versus the nonrelativistic one (four, rather than three), there is an inherent ambiguity to this procedure. Such ambiguity turned out to have physical consequences in the theory of Hamiltonian relativistic MHD developed in Chapter 5, for it allowed me to use h^μ instead of b^μ. Use of a relativistic Euler-Lagrange map (as in Chapter 6) may allow me to resolve this ambiguity.

As for regular relativistic MHD, while I have developed some of the implications of my theory, a few more remain. So far, in Chapter 5, I have developed the Hamiltonian theory, found a few alternative brackets, investigated the new gauge freedom, and found a couple of Casimir invariants. For example, a (3+1) split is essential to computational relativistic MHD, and a strength of my formalism is that it can be used to generate infinitely many different such splits, corresponding to different foliations of spacetime into spacelike submanifolds. The full procedure for this split has yet to be developed, and can be a subject of future research. In addition, knowledge of the Casimirs allows one to perform a more general stability analysis on RMHD systems, for it turns out that general critical points correspond to extrema of $H + \lambda_i C_i$, not just H, a point sometimes not realized by those who do energy-stability analysis outside the context of Hamiltonian physics.

A few more topics may be developed in connection with the Lagrangian MHD description of Chapter 6. The chapter (part of a paper still in draft form) already achieves a derivation of Hamiltonian Eulerian MHD from a canonical Lagrangian description. For example, in the nonrelativistic case it turns out that some invariants, most importantly the magnetic helicity, are Noether invariants corresponding to a relabelling symmetry [14]. Kawazura et al have investigated this symmetry using a 3-vectorial magnetic field, while I hope to do so using my 4-vectorial h^μ. Furthermore, my theory has a new degree of freedom in the α that appears in h^μ, which is only determined up to a solution of the continuity equation. If I can find an infinitesimal transformation corresponding to that symmetry, there should be an associated Noether invariant.

As of yet I have made little progress on metriplectic physics: I have put the relativistic Navier-Stokes into metriplectic form, but unfortunately the "the" in that claim is a little suspect, as there are multiple competing versions of Navier-Stokes in the relativity literature. I do not know whether those other versions can be put in metriplectic form. More significantly, I have periodic hopes of deriving the metriplectic formalism from a more fundamental theory, whereas at present it springs full-formed from the mind of Dr. Morrison. The research presented in Chapter 7 has foundered, leading me to the suspicion that there *is* no theory of classical, relativistic Hamiltonian spin: i.e., it is essentially quantum-mechanical. But perhaps I may yet discover some new angle to come at the problem. Finally, while the research in Chapter 8 is complete in its own right, I sometimes wonder whether I can find an equivalent for infinite-dimensional systems.

The research presented in this dissertation has a great deal of promise; perhaps, more promise than success, so far. I hope to be able to fulfill some its promise in the next couple of years, and I hope even more that others will see some of this promise, and attempt to add new discoveries of their own.

Appendix

Appendix 1

Useful Mathematics

1.1 Functional differentiation

In infinite-dimensional systems, important quantities will depend on the basic variables via integral expressions. For instance, in kinetic theory a fluid is described by a distribution function $f(x, v, t)$ over a six-dimensional tangent space $d^6z = d^3x\, d^3v$, and its energy or Hamiltonian is given by

$$H[f] = \int \frac{f}{2} v^2 \, d^6 z$$

One cannot perform partial derivatives directly on this quantity, because the corresponding degrees of freedom have been integrated out. Instead one obtains the first variation,

$$\begin{aligned}
\delta H[f; \delta f] &= \lim_{\epsilon \to 0} \frac{H[f + \delta f] - H[f]}{\epsilon} \\
&= \frac{d}{d\epsilon} H[f + \epsilon \delta f] \Big|_{\epsilon = 0} \\
&= \int \delta f \frac{\delta H}{\delta f} d^6 z = \left\langle \frac{\delta H}{\delta f}, \delta f \right\rangle
\end{aligned}$$

which defines the functional derivative $\delta H / \delta f$. This derivative is uniquely defined, provided the product denoted by $<, >$ is non-degenerate, i.e. has the positivity property of inner products.

Frequently coordinate changes will force one to alter functional derivatives, leading to specific kind of chain rule calculation. The main insight which enables this calculation is that the variation δF is identical, whether written in terms of the new variables or the old variables. For example, in a system involving magnetism, one can use the field $\mathbf{B}$ or the vector potential $\mathbf{A}$, related via $\mathbf{B} = \nabla \times \mathbf{A}$. The variation is the same in terms of either variable, so

$$\delta F = \int \frac{\delta F}{\delta \mathbf{B}} \cdot \delta \mathbf{B} \, d^3x = \int \frac{\delta F}{\delta \mathbf{A}} \cdot \delta \mathbf{A} \, d^3x$$

Furthermore, I have $\delta \mathbf{B} = \nabla \times \delta \mathbf{A}$, so the middle term becomes

$$\int \frac{\delta F}{\delta \mathbf{B}} \cdot (\nabla \times \delta \mathbf{A}) \, d^3x = \int \delta \mathbf{A} \cdot \left(\nabla \times \frac{\delta F}{\delta \mathbf{B}} \right) \, d^3x$$

Thus

$$\frac{\delta F}{\delta \mathbf{A}} = \nabla \times \frac{\delta F}{\delta \mathbf{B}}$$

Another common situation is having to perform a chain rule calculation where the quantity being differentiated depends on multiple field variables. For example, a quantity in fluid physics might depend on the scalar field ρ (density) and the vector field $\mathbf{m}$ (momentum). However, the pair ρ and $\mathbf{v} = \mathbf{m}/\rho$ (velocity) might prove more useful. Because the two must be distinguished, the density will be called ρ in the first pair of variables and $\bar{\rho}$ in the second. The chain rule proceeds as follows (there are no integrations by part, so the integration can be safely omitted):

$$\begin{aligned} \frac{\delta F}{\delta \rho} \delta \rho + \frac{\delta F}{\delta \mathbf{m}} \cdot \delta \mathbf{m} &= \frac{\delta F}{\delta \bar{\rho}} \delta \bar{\rho} + \frac{\delta F}{\delta \mathbf{v}} \cdot \delta \mathbf{v} \\ \frac{\delta F}{\delta \rho} \delta \rho + \frac{\delta F}{\delta \mathbf{m}} \cdot \delta \mathbf{m} &= \frac{\delta F}{\delta \bar{\rho}} \delta \bar{\rho} + \frac{\delta F}{\delta \mathbf{v}} \cdot \frac{\delta \mathbf{m}}{\bar{\rho}} - \frac{1}{\bar{\rho}^2} \frac{\delta F}{\delta \mathbf{v}} \delta \bar{\rho} \end{aligned}$$

Equating coefficients yields

$$\frac{\delta F}{\delta \mathbf{m}} = \frac{1}{\bar{\rho}} \frac{\delta F}{\delta \mathbf{v}} \qquad\qquad \frac{\delta F}{\delta \rho} = \frac{\delta F}{\delta \bar{\rho}} - \frac{1}{\bar{\rho}^2} \frac{\delta F}{\delta \mathbf{v}}$$

This example highlights a quirk of functional (and partial) differentiation: when a variable is unchanged, its derivative will nonetheless change. The same thing happens with the chain rule for partial derivatives. More sophisticated uses of the chain rule can be found in Secs. 2.3.2 and 6.1, and Chap. 3.

1.2 Useful expressions involving the Jacobian determinant

The Euler-Lagrange map of Sections 2.3.2 and 6.1 relies on a dynamic coordinate change $q(a,t)$, where a is the coordinate on a three- or four-dimensional label space, and q typically denotes the position of a fluid element starting at a. The coordinate change defines a Jacobian matrix $\frac{\partial q^i}{\partial a^j}$, whose determinant, the Jacobian scalar, is:

$$\mathcal{J} = \left| \frac{\partial q^i}{\partial a^j} \right| = \frac{1}{6} \epsilon_{ijk} \epsilon^{lmn} \frac{\partial q^i}{\partial a^l} \frac{\partial q^j}{\partial a^m} \frac{\partial q^k}{\partial a^n}$$

The cofactor matrix ${A^i}_j$, for invertible transformations (as all in this dissertation must be), is proportional to the inverse transformation:

$$ {A^i}_l \frac{\partial q^l}{\partial a^j} = \mathcal{J} {\delta^i}_j \tag{1.1}$$

If I write ${q^j}_{,i} = \partial q^j / \partial a^i$ then

$$\frac{\partial \mathcal{J}}{\partial {q^j}_{,i}} = \frac{1}{2} \epsilon_{ijk} \epsilon^{lmn} \frac{\partial q^j}{\partial a^m} \frac{\partial q^k}{\partial a^n} = {A^i}_j$$

Another useful identity is

$$\frac{\partial A^i{}_j}{\partial a^i} = \frac{1}{2}\epsilon_{jkl}\epsilon^{imn}\left(\frac{\partial^2 q^k}{\partial a^m \partial a^i}\frac{\partial q^l}{\partial a^n} + \frac{\partial q^k}{\partial a^m}\frac{\partial^2 q^l}{\partial a^n \partial a^i}\right) = 0$$

which is zero due to contracting a symmetric object with an antisymmetric object. Keeping in mind that time derivatives and label derivatives commute,

$$\dot{\mathcal{J}} = \frac{1}{6}\epsilon_{ijk}\epsilon^{lmn}\frac{\partial \dot{q}^i}{\partial a^l}\frac{\partial q^j}{\partial a^m}\frac{\partial q^k}{\partial a^n} + \dots$$

The first term is

$$\frac{1}{3}A^l{}_i\frac{\partial \dot{q}^i}{\partial a^l} = \frac{1}{3}A^l{}_i\frac{\partial \dot{q}^i}{\partial q^j}\frac{\partial q^j}{\partial a^l} = \frac{1}{3}\mathcal{J}\delta^j{}_i\frac{\partial \dot{q}^i}{\partial q^j} = \frac{1}{3}\mathcal{J}\,\nabla\cdot\dot{\mathbf{q}}$$

So, altogether,

$$\dot{\mathcal{J}} = \mathcal{J}\,\nabla\cdot\dot{\mathbf{q}} \tag{1.2}$$

By taking a partial derivative of (1.1) with regards to $q^m_{\ ,n}$, one finds

$$\frac{\partial A^i{}_l}{\partial q^m{}_{,n}}\frac{\partial q^l}{\partial a^j} + A^i{}_l\,\delta^l{}_m\,\delta^n_j \;=\; \delta^i{}_j\,A^n{}_m$$

which, after multiplying by $A^j{}_k$ and manipulating, reduces to

$$\frac{\partial A^i{}_j}{\partial q^m{}_{,n}} = \frac{1}{\mathcal{J}}\left(A^i{}_jA^n{}_m - A^i{}_mA^n{}_j\right)$$

By taking a time derivative of (1.1), substituting in (1.2), and performing similar manipulations, one acquires

$$\dot{A}^i{}_j = \delta^i{}_j\left(\nabla\cdot\dot{\mathbf{q}}\right) - \frac{1}{\mathcal{J}}\left(A^i{}_k\frac{\partial \dot{q}^k}{\partial a^l}A^l{}_j\right)$$

Jacobian determinants in relativity have analogous properties, with a few substitutions: all indices now range over four values, one uses the 4D Levi-Civita symbol $\epsilon^{\mu\nu\lambda\sigma}$ instead of the 3D ϵ^{ijk}, and the time derivative (for instance, in (1.2)) is replaced by a proper time derivative.

1.3 Jacobi identity for the MHD brackets

Because the Covariant Poisson brackets of (5.32) and (5.35) are direct generalizations of the Lie-Poisson form given in Refs. [30, 31, 26] for non-relativistic MHD the Jacobi identity follows from general Lie algebraic and functional derivative properties (see e.g., Refs. [26] [23] [24, 28]). However, since these may not be known to some readers I include a direct proof in this appendix.

The Jacobi identity is

$$\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0 \tag{1.3}$$

for the two brackets (5.32) and (5.35).

When expanding the expression (1.3), many terms will contain second functional derivatives, for instance

$$nh^{\lambda} \frac{\delta G}{\delta m_{\nu}} \left(\partial_{\nu} \frac{\delta^2 F}{\delta h^{\lambda} \delta m_{\mu}} \right) \partial_{\mu} \frac{\delta H}{\delta n}$$

Thankfully, by a theorem in Ref. [26], all such terms cancel for any antisymmetric bracket. Thus I only have to worry about those terms containing only first functional derivatives. Starting with the bracket (5.32), the needed terms are thus

$$\begin{aligned}
\frac{\delta\{F,G\}}{\delta n} =& \frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta n} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta n} + \ldots \\
\frac{\delta\{F,G\}}{\delta \sigma} =& \frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta \sigma} + \ldots \\
\frac{\delta\{F,G\}}{\delta m_\mu} =& \frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta m_\mu} - \frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta m_\mu} + \ldots \\
\frac{\delta\{F,G\}}{\delta h^\mu} =& \frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta h^\mu} - \frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta h^\mu} + \\
& \partial_\mu\frac{\delta F}{\delta m_\nu}\frac{\delta G}{\delta h^\nu} - \partial_\mu\frac{\delta G}{\delta m_\nu}\frac{\delta F}{\delta h^\nu} + \ldots
\end{aligned} \tag{1.4}$$

with similar expressions for the other two permutations of F, G, and H. Beginning with this expression, it is to be understood that, in the absence of parentheses, the gradient operators act only on the term immediately to their right; when they are followed by an expression in parentheses, they act as normal. This convention will remove many superfluous symbols. The ellipses at the end of each line indicate the terms that may be disregarded thanks to the aforementioned theorem. Upon inserting the expressions (1.4) into the Jacobi identity (1.3), all pertinent terms will be linear in the field variables. Each of these four sets of terms (one for each field variable) must vanish separately.

The terms linear in n are:

$$\int d^4x\, n \left[\left(\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta m_\mu} - \frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta m_\mu}\right)\partial_\mu\frac{\delta H}{\delta n} - \frac{\delta H}{\delta m_\mu}\partial_\mu\left(\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta n} - \frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta n}\right) + \underset{F,G,H}{\circlearrowleft}\right] \tag{1.5}$$

where the circle symbol indicates permutation in F, G, and H. Inside the square braces, the collected second derivative terms are

$$-\frac{\delta H}{\delta m_\mu}\frac{\delta F}{\delta m_\nu}\partial^2_{\mu\nu}\frac{\delta G}{\delta n}+\frac{\delta H}{\delta m_\mu}\frac{\delta G}{\delta m_\nu}\partial^2_{\mu\nu}\frac{\delta F}{\delta n}-\frac{\delta F}{\delta m_\mu}\frac{\delta G}{\delta m_\nu}\partial^2_{\mu\nu}\frac{\delta H}{\delta n}$$
$$+\frac{\delta F}{\delta m_\mu}\frac{\delta H}{\delta m_\nu}\partial^2_{\mu\nu}\frac{\delta G}{\delta n}-\frac{\delta G}{\delta m_\mu}\frac{\delta H}{\delta m_\nu}\partial^2_{\mu\nu}\frac{\delta F}{\delta n}+\frac{\delta G}{\delta m_\mu}\frac{\delta F}{\delta m_\nu}\partial^2_{\mu\nu}\frac{\delta H}{\delta n}$$

which vanish due to the fact that second (partial) derivatives commute. The remaining terms linear in n, keeping the same order they have in the Jacobi identity, follow:

$$\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta n}^{②}-\frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta n}^{⑥}-\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta n}^{③}+\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta n}^{①}+$$
$$\frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta n}^{⑤}-\frac{\delta H}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta n}^{①}-\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta H}{\delta n}^{②}+\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta n}^{④}+$$
$$\frac{\delta H}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta n}^{③}-\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta n}^{④}-\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta n}^{⑤}+\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta H}{\delta n}^{⑥}$$

They vanish in pairs, as labeled by the circled numbers.

So all the terms linear in n have vanished from the Jacobi identity. However, the terms linear in σ are identical, but with functional derivatives $\delta/\delta n$ replaced by $\delta/\delta\sigma$. So the σ terms vanish by an identical calculation. Moreover, the m_λ terms do as well: the $\delta/\delta n$ are replaced with $\delta/\delta m_\lambda$, contracted with the remaining m_λ term outside the square brackets of its version of (1.5), and the calculation proceeds as before.

The only terms remaining to be checked are those linear in h^λ; unfortunately, there are quite a few:

$$
\begin{aligned}
\int d^4x\, h^\lambda \Bigg[& \left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta m_\nu} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta m_\nu}\right)\partial_\nu\frac{\delta H}{\delta h^\lambda}^{\textcircled{1}} \\
& -\frac{\delta H}{\delta m_\nu}\partial_\nu\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\lambda} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\partial F}{\partial h^\lambda}\right)^{\textcircled{1}} - \frac{\delta H}{\delta m_\nu}\partial_\nu\left(\partial_\lambda\frac{\delta F}{\delta m_\mu}\frac{\delta G}{\delta h^\mu} - \partial_\lambda\frac{\delta G}{\delta m_\mu}\frac{\delta F}{\delta h^\mu}\right) \\
& +\partial_\lambda\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta m_\nu} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta m_\nu}\right)\frac{\delta H}{\delta h^\nu} \\
& -\partial_\lambda\frac{\delta H}{\delta m_\nu}\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\nu} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta h^\nu} + \partial_\nu\frac{\delta F}{\delta m_\mu}\frac{\delta G}{\delta h^\mu} - \partial_\nu\frac{\delta G}{\delta m_\mu}\frac{\delta F}{\delta h^\mu}\right) + \underset{F,G,H}{\circlearrowright}\Bigg]
\end{aligned}
$$

The terms labelled by a circled "one" produce a calculation identical to that already performed, and thus cancel. From the remaining terms, I first gather all the second derivative ones inside the square braces:

$$
\begin{aligned}
& -\frac{\delta H}{\delta m_\nu}\frac{\delta G}{\delta h^\mu}\partial^2_{\lambda\nu}\frac{\delta F}{\delta m_\mu}^{\textcircled{5}} + \frac{\delta H}{\delta m_\nu}\frac{\delta F}{\delta h^\mu}\partial^2_{\lambda\nu}\frac{\delta G}{\delta m_\mu}^{\textcircled{2}} + \frac{\delta F}{\delta m_\mu}\frac{\delta H}{\delta h^\nu}\partial^2_{\lambda\mu}\frac{\delta G}{\delta m_\nu}^{\textcircled{1}} - \frac{\delta G}{\delta m_\mu}\frac{\delta H}{\delta h^\nu}\partial^2_{\lambda\mu}\frac{\delta F}{\delta m_\nu}^{\textcircled{4}} \\
& -\frac{\delta F}{\delta m_\nu}\frac{\delta H}{\delta h^\mu}\partial^2_{\lambda\nu}\frac{\delta G}{\delta m_\mu}^{\textcircled{1}} + \frac{\delta F}{\delta m_\nu}\frac{\delta G}{\delta h^\mu}\partial^2_{\lambda\nu}\frac{\delta H}{\delta m_\mu}^{\textcircled{6}} + \frac{\delta G}{\delta m_\mu}\frac{\delta F}{\delta h^\nu}\partial^2_{\lambda\mu}\frac{\delta H}{\delta m_\nu}^{\textcircled{3}} - \frac{\delta H}{\delta m_\mu}\frac{\delta F}{\delta h^\nu}\partial^2_{\lambda\mu}\frac{\delta G}{\delta m_\nu}^{\textcircled{2}} \\
& -\frac{\delta G}{\delta m_\nu}\frac{\delta F}{\delta h^\mu}\partial^2_{\lambda\nu}\frac{\delta H}{\delta m_\mu}^{\textcircled{3}} + \frac{\delta G}{\delta m_\nu}\frac{\delta H}{\delta h^\mu}\partial^2_{\lambda\nu}\frac{\delta F}{\delta m_\mu}^{\textcircled{4}} + \frac{\delta H}{\delta m_\mu}\frac{\delta G}{\delta h^\nu}\partial^2_{\lambda\mu}\frac{\delta F}{\delta m_\nu}^{\textcircled{5}} - \frac{\delta F}{\delta m_\mu}\frac{\delta G}{\delta h^\nu}\partial^2_{\lambda\mu}\frac{\delta H}{\delta m_\nu}^{\textcircled{6}}
\end{aligned}
$$

They cancel in pairs. Finally, the remaining terms, in the same order and bearing the same indices as in the Jacobi identity, are:

$$
\begin{aligned}
&-\frac{\delta H}{\delta m_\nu}\partial_\lambda\frac{\delta F}{\delta m_\mu}\partial_\nu\frac{\delta G}{\delta h^\mu}^{\textcircled{3}} + \frac{\delta H}{\delta m_\nu}\partial_\lambda\frac{\delta G}{\delta m_\mu}\partial_\nu\frac{\delta F}{\delta h^\mu}^{\textcircled{9}} + \frac{\delta H}{\delta h^\nu}\partial_\lambda\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta m_\nu}^{\textcircled{4}} - \frac{\delta H}{\delta h^\nu}\partial_\lambda\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta m_\nu}^{\textcircled{12}} \\
&-\frac{\delta F}{\delta m_\mu}\partial_\lambda\frac{\delta H}{\delta m_\nu}\partial_\mu\frac{\delta G}{\delta h^\nu}^{\textcircled{1}} + \frac{\delta G}{\delta m_\mu}\partial_\lambda\frac{\delta H}{\delta m_\nu}\partial_\mu\frac{\delta F}{\delta h^\nu}^{\textcircled{5}} - \frac{\delta G}{\delta h^\mu}\partial_\lambda\frac{\delta H}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta m_\mu}^{\textcircled{7}} + \frac{\delta F}{\delta h^\mu}\partial_\lambda\frac{\delta H}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta m_\mu}^{\textcircled{2}} \\
&-\frac{\delta F}{\delta m_\nu}\partial_\lambda\frac{\delta G}{\delta m_\mu}\partial_\nu\frac{\delta H}{\delta h^\mu}^{\textcircled{10}} + \frac{\delta F}{\delta m_\nu}\partial_\lambda\frac{\delta H}{\delta m_\mu}\partial_\nu\frac{\delta G}{\delta h^\mu}^{\textcircled{1}} + \frac{\delta F}{\delta h^\nu}\partial_\lambda\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta m_\nu}^{\textcircled{11}} - \frac{\delta F}{\delta h^\nu}\partial_\lambda\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta m_\nu}^{\textcircled{2}} \\
&-\frac{\delta G}{\delta m_\mu}\partial_\lambda\frac{\delta F}{\delta m_\nu}\partial_\mu\frac{\delta H}{\delta h^\nu}^{\textcircled{6}} + \frac{\delta H}{\delta m_\mu}\partial_\lambda\frac{\delta F}{\delta m_\nu}\partial_\mu\frac{\delta G}{\delta h^\nu}^{\textcircled{3}} - \frac{\delta H}{\delta h^\mu}\partial_\lambda\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta m_\mu}^{\textcircled{4}} + \frac{\delta G}{\delta h^\mu}\partial_\lambda\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta H}{\delta m_\mu}^{\textcircled{8}} \\
&-\frac{\delta G}{\delta m_\nu}\partial_\lambda\frac{\delta H}{\delta m_\mu}\partial_\nu\frac{\delta F}{\delta h^\mu}^{\textcircled{5}} + \frac{\delta G}{\delta m_\nu}\partial_\lambda\frac{\delta F}{\delta m_\mu}\partial_\nu\frac{\delta H}{\delta h^\mu}^{\textcircled{6}} + \frac{\delta G}{\delta h^\nu}\partial_\lambda\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta m_\nu}^{\textcircled{7}} - \frac{\delta G}{\delta h^\nu}\partial_\lambda\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta m_\nu}^{\textcircled{8}} \\
&-\frac{\delta H}{\delta m_\mu}\partial_\lambda\frac{\delta G}{\delta m_\nu}\partial_\mu\frac{\delta F}{\delta h^\nu}^{\textcircled{9}} + \frac{\delta F}{\delta m_\mu}\partial_\lambda\frac{\delta G}{\delta m_\nu}\partial_\mu\frac{\delta H}{\delta h^\nu}^{\textcircled{10}} - \frac{\delta F}{\delta h^\mu}\partial_\lambda\frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta H}{\delta m_\mu}^{\textcircled{11}} + \frac{\delta H}{\delta h^\mu}\partial_\lambda\frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta m_\mu}^{\textcircled{12}}
\end{aligned}
$$

They also cancel in pairs, establishing the Jacobi identity. This derivation is also valid in curved spacetimes, for the functional derivative cancels out a factor of $\sqrt{-g}$, and there is no integration by parts to catch another such factor.

Next I will perform a similar calculation for the alternative bracket (5.35). While the same kinds of terms appear as above, there is no longer a complete cancellation. Most of the functional derivatives (1.4) are unchanged, the only differing one being

$$\frac{\delta\{F,G\}}{\delta h^\mu} = \frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\partial G}{\partial h^\mu} - \frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\partial F}{\partial h^\mu} + \partial_\mu\frac{\delta F}{\delta h^\nu}\frac{\delta G}{\delta m_\nu} - \partial_\mu\frac{\delta G}{\delta h^\nu}\frac{\delta F}{\delta m_\nu} + \ldots$$

with the ellipsis again indicating terms with second functional derivatives, all of which can be disregarded.

The terms of the Jacobi identity once more appear in four sets, each linear in one of the field variables. The n, σ, and m^λ terms involve no derivatives with respect to h^λ, and are thus unchanged: they cancel as before. Only the h^λ terms differ. They read:

$$\int d^4x\, h^\lambda \Bigg[\left(\frac{\delta F}{\delta m_\nu}\partial_\nu\frac{\delta G}{\delta m_\mu} - \frac{\delta G}{\delta m_\nu}\partial_\nu\frac{\delta F}{\delta m_\mu}\right)\partial_\mu\frac{\delta H}{\delta h^\lambda}^{①}$$
$$-\frac{\delta H}{\delta m_\nu}\partial_\nu\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\lambda} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta h^\lambda}\right)^{①} - \frac{\delta H}{\delta m_\nu}\partial_\nu\left(\partial_\lambda\frac{\delta F}{\delta h^\mu}\frac{\delta G}{\delta m_\mu} - \partial_\lambda\frac{\delta G}{\delta h^\mu}\frac{\delta F}{\delta m_\mu}\right)$$
$$+\partial_\lambda\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\nu} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta h^\nu} + \partial_\nu\frac{\delta F}{\delta h^\mu}\frac{\delta G}{\delta m_\mu} - \partial_\nu\frac{\delta G}{\delta h^\mu}\frac{\delta F}{\delta m_\mu}\right)\frac{\delta H}{\delta m_\nu}$$
$$-\partial_\lambda\frac{\delta H}{\delta h^\nu}\left(\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta m_\nu} - \frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta m_\nu}\right) + \underset{F,G,H}{\circlearrowright}\Bigg]$$

The terms labelled with a circled "one" cancel as in the previous bracket. The collected second derivative terms are

$$-\frac{\delta H}{\delta m_\nu}\frac{\delta G}{\delta m_\mu}\partial^2_{\nu\lambda}\frac{\delta F}{\delta h^\mu}^{②} + \frac{\delta H}{\delta m_\nu}\frac{\delta F}{\delta m_\mu}\partial^2_{\nu\lambda}\frac{\delta G}{\delta h^\mu}^{①} + \frac{\delta F}{\delta m_\mu}\frac{\delta H}{\delta m_\nu}\partial^2_{\lambda\mu}\frac{\delta G}{\delta h^\nu}$$
$$-\frac{\delta G}{\delta m_\mu}\frac{\delta H}{\delta m_\nu}\partial^2_{\lambda\mu}\frac{\delta F}{\delta h^\nu} + \frac{\delta H}{\delta m_\nu}\frac{\delta G}{\delta m_\mu}\partial^2_{\lambda\nu}\frac{\delta F}{\delta h^\mu}^{②} - \frac{\delta H}{\delta m_\nu}\frac{\delta F}{\delta m_\mu}\partial^2_{\lambda\nu}\frac{\delta G}{\delta h^\mu}^{①} + \underset{F,G,H}{\circlearrowright}$$
$$=\frac{\delta F}{\delta m_\mu}\frac{\delta H}{\delta m_\nu}\partial^2_{\lambda\mu}\frac{\delta G}{\delta h^\nu} - \frac{\delta G}{\delta m_\mu}\frac{\delta H}{\delta m_\nu}\partial^2_{\lambda\mu}\frac{\delta F}{\delta h^\nu} + \frac{\delta G}{\delta m_\mu}\frac{\delta F}{\delta m_\nu}\partial^2_{\lambda\mu}\frac{\delta H}{\delta h^\nu}$$
$$-\frac{\delta H}{\delta m_\mu}\frac{\delta F}{\delta m_\nu}\partial^2_{\lambda\mu}\frac{\delta G}{\delta h^\nu} + \frac{\delta H}{\delta m_\mu}\frac{\delta G}{\delta m_\nu}\partial^2_{\lambda\mu}\frac{\delta F}{\delta h^\nu} - \frac{\delta F}{\delta m_\mu}\frac{\delta G}{\delta m_\nu}\partial^2_{\lambda\mu}\frac{\delta H}{\delta h^\nu}$$

Six terms do not cancel. The other terms (i.e. those that are not second derivatives) are

$$
\begin{aligned}
&-\frac{\delta H}{\delta m_\nu}\partial_\lambda\frac{\delta F}{\delta h^\mu}\partial_\nu\frac{\delta G}{\delta m_\mu}\textcircled{2} + \frac{\delta H}{\delta m_\nu}\partial_\lambda\frac{\delta G}{\delta h^\mu}\partial_\nu\frac{\delta F}{\delta m_\mu}\textcircled{5} + \frac{\delta H}{\delta m_\nu}\partial_\lambda\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\nu} - \frac{\delta H}{\delta m_\nu}\partial_\lambda\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta h^\nu}\\
&+\frac{\delta H}{\delta m_\nu}\partial_\lambda\frac{\delta G}{\delta m_\mu}\partial_\nu\frac{\delta F}{\delta h^\mu} - \frac{\delta H}{\delta m_\nu}\partial_\lambda\frac{\delta F}{\delta m_\mu}\partial_\nu\frac{\delta G}{\delta h^\mu} - \frac{\delta F}{\delta m_\mu}\partial_\lambda\frac{\delta H}{\delta h^\nu}\partial_\mu\frac{\delta G}{\delta m_\nu}\textcircled{1} + \frac{\delta G}{\delta m_\mu}\partial_\lambda\frac{\delta H}{\delta h^\nu}\partial_\mu\frac{\delta F}{\delta m_\nu}\textcircled{3}\\
&-\frac{\delta F}{\delta m_\nu}\partial_\lambda\frac{\delta G}{\delta h^\mu}\partial_\nu\frac{\delta H}{\delta m_\mu}\textcircled{6} + \frac{\delta F}{\delta m_\nu}\partial_\lambda\frac{\delta H}{\delta h^\mu}\partial_\nu\frac{\delta G}{\delta m_\mu}\textcircled{1} + \frac{\delta F}{\delta m_\nu}\partial_\lambda\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta h^\nu} - \frac{\delta F}{\delta m_\nu}\partial_\lambda\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\nu}\\
&+\frac{\delta F}{\delta m_\nu}\partial_\lambda\frac{\delta H}{\delta m_\mu}\partial_\nu\frac{\delta G}{\delta h^\mu} - \frac{\delta F}{\delta m_\nu}\partial_\lambda\frac{\delta G}{\delta m_\mu}\partial_\nu\frac{\delta H}{\delta h^\mu} - \frac{\delta G}{\delta m_\mu}\partial_\lambda\frac{\delta F}{\delta h^\nu}\partial_\mu\frac{\delta H}{\delta m_\nu}\textcircled{4} + \frac{\delta H}{\delta m_\mu}\partial_\lambda\frac{\delta F}{\delta h^\nu}\partial_\mu\frac{\delta G}{\delta m_\nu}\textcircled{2}\\
&-\frac{\delta G}{\delta m_\nu}\partial_\lambda\frac{\delta H}{\delta h^\mu}\partial_\nu\frac{\delta F}{\delta m_\mu}\textcircled{3} + \frac{\delta G}{\delta m_\nu}\partial_\lambda\frac{\delta F}{\delta h^\mu}\partial_\nu\frac{\delta H}{\delta m_\mu}\textcircled{4} + \frac{\delta G}{\delta m_\nu}\partial_\lambda\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta h^\nu} - \frac{\delta G}{\delta m_\nu}\partial_\lambda\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta h^\nu}\\
&+\frac{\delta G}{\delta m_\nu}\partial_\lambda\frac{\delta F}{\delta m_\mu}\partial_\nu\frac{\delta H}{\delta h^\mu} - \frac{\delta G}{\delta m_\nu}\partial_\lambda\frac{\delta H}{\delta m_\mu}\partial_\nu\frac{\delta F}{\delta h^\mu} - \frac{\delta H}{\delta m_\mu}\partial_\lambda\frac{\delta G}{\delta h^\nu}\partial_\mu\frac{\delta F}{\delta m_\nu}\textcircled{5} + \frac{\delta F}{\delta m_\mu}\partial_\lambda\frac{\delta G}{\delta h^\nu}\partial_\mu\frac{\delta H}{\delta m_\nu}\textcircled{6}
\end{aligned}
$$

This time twelve terms do not cancel. All told, eighteen terms remain, which collect in groups of three. Each group reduces to a gradient with a ∂_λ pulled outside the expression. The whole Jacobi identity simplifies to

$$
\begin{aligned}
&\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\}\\
&= \int d^4x\, h^\lambda \partial_\lambda \left(\frac{\delta F}{\delta m_\nu}\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta h^\nu} - \frac{\delta G}{\delta m_\nu}\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta H}{\delta h^\nu} + \frac{\delta G}{\delta m_\nu}\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta h^\nu} - \right.\\
&\qquad \left. \frac{\delta H}{\delta m_\nu}\frac{\delta G}{\delta m_\mu}\partial_\mu\frac{\delta F}{\delta h^\nu} + \frac{\delta H}{\delta m_\nu}\frac{\delta F}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\nu} - \frac{\delta F}{\delta m_\nu}\frac{\delta H}{\delta m_\mu}\partial_\mu\frac{\delta G}{\delta h^\nu}\right)
\end{aligned}
$$

An integration by parts shows that the Jacobi identity is satisfied if ${h^\nu}_{,\nu} = 0$. In a curved spacetime, the above expression is the same, except that d^4x becomes $\sqrt{-g}d^4x$. The integration by parts catches this extra factor, yielding $(h^\nu\sqrt{-g})_{,\nu} = {h^\nu}_{;\nu} = 0$ as a requirement for the Jacobi identity.

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