REGULARITY CRITERION FOR 3D NAVIER-STOKES EQUATIONS IN TERMS OF THE DIRECTION OF THE VELOCITY*

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Abstract. In this short note we give a link between the regularity of the solution u to the 3D Navier-Stokes equation and the behavior of the direction of the velocity $u/|u|$. It is shown that the control of $\operatorname{div}(u/|u|)$ in a suitable $L^p_t(L^q_x)$ norm is enough to ensure global regularity. The result is reminiscent of the criterion in terms of the direction of the vorticity, introduced first by Constantin and Fefferman. However, in this case the condition is not on the vorticity but on the velocity itself. The proof, based on very standard methods, relies on a straightforward relation between the divergence of the direction of the velocity and the growth of energy along streamlines.

Keywords: Navier-Stokes, fluid mechanics, regularity, PRodi-Serrin criteria

MSC 2000: 35Q30, 76D05, 76D03

1. Introduction

This short paper deals with a new formulation of the well-known criteria for regularity of solutions to the incompressible Navier-Stokes equation in dimension 3, namely,

$$(1)\qquad \begin{aligned} \partial_t u + \operatorname{div}(u\otimes u) + \nabla P - \Delta u = 0, \quad t\in\,]0,\infty[, \;\; x\in\mathbb{R}^3,\\ \operatorname{div} u = 0. \end{aligned}$$

The unknown is the velocity field $u(t,x)\in\mathbb{R}^3$. The pressure P is a non local operator of u which can be seen as a Lagrange multiplier associated to the constraint of incompressibility $\operatorname{div} u = 0$. The existence of weak solutions was proved long ago by Leray [10] and Hopf [7]. They showed that for any initial value with finite energy $u^0 \in$

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$L^2(\mathbb{R}^3)$ there exists a function $u \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \times L^2(0,\infty; \dot{H}^1(\mathbb{R}^3))$ verifying (1) in the sense of distributions, and verifying in addition the energy inequality

$$(2) \qquad \|u(t,\cdot)\|^2_{L^2(\mathbb{R}^3)} + 2\int_0^t \|\nabla u(s,\cdot)\|^2_{L^2(\mathbb{R}^3)}\,\mathrm{d}s \leqslant \|u_0\|^2_{L^2(\mathbb{R}^3)}, \quad t \geqslant 0.$$

Such a solution is now called the Leray-Hopf weak solution to (1).

In [12], Serrin showed that a Leray-Hopf solution of (1) lying in $L^p(0,\infty; L^q(\mathbb{R}^3))$ with $p, q \geqslant 1$ such that $2/p + 3/q < 1$ is smooth in the spatial directions. This result was later extended in [13] and [5] to the case of equality for $p < \infty$. Notice that the case of $L^\infty(0,\infty; L^3(\mathbb{R}^3))$ was proven only very recently by Iskauriaza, Serëgin and Shverak [8].

Another class of regularity criteria which involves the gradient of u was introduced by Beirão da Veiga [2]. More precisely, he showed that any Leray-Hopf solution u such that ∇u lies in $L^p(L^q)$ with $2/p + 3/q = 2$, $3/2 < q < \infty$, is smooth. Beale-Kato-Majda [1] dealt with the vorticity $\omega = \operatorname{rot} u$ and proved regularity under the condition $\omega \in L^1(L^\infty)$. This condition was later improved to $L^1(BMO)$ by Kozono and Taniuchi [9].

In [4], Constantin and Fefferman introduced a criterion involving the direction of the vorticity $\omega/|\omega|$. They showed that under a Lipschitz-like regularity assumption on $\omega/|\omega|$, the solution is smooth (see [14] for extension of this result).

Our result is of the same spirit but involves the direction of the velocity itself instead of the vorticity.

Theorem 1. *Let u be a Leray-Hopf solution to the Navier-Stokes equation with an initial value $u_0 \in L^2(\mathbb{R}^3)$. If $\operatorname{div}(u/|u|) \in L^p(0,\infty; L^q(\mathbb{R}^3))$ with*

$$\frac{2}{p} + \frac{3}{q} \leqslant \frac{1}{2}, \quad q \geqslant 6, \quad p \geqslant 4,$$

then u is smooth on $(0,\infty) \times \mathbb{R}^3$.

The result shows that it is enough to control the rate of change of the direction of the velocity to get full regularity of the solution. The main point of this paper is the following straightforward equality coming from the incompressibility of the flow:

$$(3) \qquad |u| \operatorname{div}(u/|u|) = -\frac{u}{|u|} \cdot \nabla |u|.$$

This equality shows that, due to the incompressibility, the growth of $|u|$ along the stream lines is linked to the divergence of the direction of u. It means that to allow some increase of kinetic energy $|u|^2$ along the streamlines, those streamlines need to be bent, producing some divergence on the direction of the velocity.

This remark is the main point of this short note. The proof of the theorem then follows in a very standard way. It uses the fact that the right-hand side term in (3) corresponds, up to the multiplication by a power of $|u|$, to the flux of energy $u\cdot\nabla|u|^2$. Besides, it is also interesting to notice that this term depends only on the symmetric part of the gradient of u. Indeed, it can be rewritten as

$$|u|\,\mathrm{div}(u/|u|) = -\frac{u}{|u|}\cdot\nabla|u| = -\frac{u}{2|u|^2}\cdot\nabla|u|^2 = -\frac{u^T}{|u|^2}\cdot\nabla u\cdot u = -\frac{u^T}{|u|^2}\cdot D(u)\cdot u.$$

It has been already known that if one component of the velocity is bounded in a suitable space, then the solution is smooth (see Penel and Pokorný [11], He [6], Zhou [14], Chae and Choe [3]). Our result states that if the direction of the velocity does not change too drastically, the conclusion is still true.

2. Proof of Theorem 1

Let us first state a technical lemma.

Lemma 2. *For every r, $2\leqslant r<6$, there exists a constant C such that for every $\beta>0$ and every function f lying in $L^2(\mathbb{R}^3)$ and such that ∇f lies in $L^2(\mathbb{R}^3)$, we have*

$$\beta\|f\|^2_{L^r(\mathbb{R}^3)} \leqslant \frac{1}{4}\|\nabla f\|^2_{L^2(\mathbb{R}^3)} + C\beta^{1/\theta}\|f\|^2_{L^2(\mathbb{R}^3)},$$

for $\theta = 3/r - 1/2$.

Proof. The Sobolev inequality gives

$$\|f\|_{L^6(\mathbb{R}^3)} \leqslant C\|\nabla f\|_{L^2(\mathbb{R}^3)}.$$

Interpolation gives

$$\beta\|f\|^2_{L^r(\mathbb{R}^3)} \leqslant (\beta^{1/\theta}\|f\|^2_{L^2(\mathbb{R}^3)})^{\theta}(\|f\|^2_{L^6(\mathbb{R}^3)})^{1-\theta},$$

where

$$\frac{\theta}{2} + \frac{1-\theta}{6} = \frac{1}{r},$$

that is $\theta = 3/r - 1/2$. We complete the proof using the Minkowski inequality

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}$$

with

$$\theta = \frac{1}{p}, \quad 1-\theta = \frac{1}{q}$$

and

$$a = \frac{(\beta^{1/\theta}\|f\|_{L^2}^2)^\theta}{\varepsilon}, \quad b = \varepsilon(\|\nabla f\|_{L^2}^2)^{1-\theta}$$

for ε small enough. $\square$

We consider now u, a Leray-Hopf solution to the Navier-Stokes equation. Since u lies in $L^\infty(0,\infty;L^2(\mathbb{R}^3)) \cap L^2(0,\infty;\dot{H}^1(\mathbb{R}^3))$, for almost every $t_0 > 0$, $u(t_0,\cdot)$ lies in $H^1(\mathbb{R}^3)$. It is classical that there exists a maximal $T > t_0$ such that u is smooth on $(t_0, T) \times \mathbb{R}^3$. Our goal is to show that

$$\lim_{t\to T} \|u\|_{L^3(t_0,t;L^9(\mathbb{R}^3))} < \infty.$$

Thanks to Serrin's criterion, this implies that $T = \infty$. Note that $u(t_0,\cdot) \in L^2(\mathbb{R}^3) \cap L^6(\mathbb{R}^3)$, so it lies in $L^3(\mathbb{R}^3)$. We consider u on $(t_0,T)\times\mathbb{R}^3$. Multiplying (1) by $u|u|$ and integrating in x we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{|u|^3}{3}\,\mathrm{d}x + \int_{\mathbb{R}^3} |u|(|\nabla u|^2 + |\nabla|u||^2)\,\mathrm{d}x - \int_{\mathbb{R}^3} Pu\cdot\nabla|u|\,\mathrm{d}x = 0.$$

Noting that

$$-\Delta P = \sum_{ij} \partial_i\partial_j(u_i u_j),$$

we have, for every $4/3 < r < \infty$,

$$\|P\|_{L^{3r/4}(\mathbb{R}^3)} \leqslant C_r \|u\|_{L^{3r/2}(\mathbb{R}^3)}^2.$$

Since $\operatorname{div}(u/|u|) \in L^p(L^q)$ for $2/p+3/q \leqslant 1/2$, $q \geqslant 6$, and $u \in L^a(L^b)$ for $2/a+3/b = 3/2$, $2 \leqslant b \leqslant 6$, there exist $\overline{p} > 1$ and $2 < \overline{q} < 6$ such that $|u|\operatorname{div}(u/|u|) \in L^{\overline{p}}(L^{\overline{q}})$ with

$$\frac{1}{\overline{p}} = \frac{1}{p} + \frac{1}{a}, \quad \frac{1}{\overline{q}} = \frac{1}{q} + \frac{1}{b}.$$

Note that $2 \leqslant \overline{q} < 6$ and

$$\frac{2}{\overline{p}} + \frac{3}{\overline{q}} \leqslant 2. \tag{4}$$

So, using (3), we have for every fixed time t,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3} \frac{|u|^3}{3}\,\mathrm{d}x + \int_{\mathbb{R}^3} |u||\nabla|u||^2\,\mathrm{d}x \leqslant \int_{\mathbb{R}^3} |P||u|\left|\frac{u}{|u|}\cdot\nabla|u|\right|\mathrm{d}x$$
$$\leqslant C\|u\|_{L^{3r/2}(\mathbb{R}^3)}^3 \||u|\operatorname{div}(u/|u|)\|_{L^{\overline{q}}(\mathbb{R}^3)}$$

with

$$\frac{2}{r}+\frac{1}{\overline{q}}=1.$$

Using Lemma 2 with

$$f=|u|^{3/2}, \quad \nabla f=\frac{3}{2}|u|^{1/2}\nabla|u|,$$

we find that

$$\begin{aligned}C\|u\|^3_{L^{3r/2}(\mathbb{R}^3)}&\||u|\operatorname{div}(u/|u|)\|_{L^{\overline{q}}(\mathbb{R}^3)}\\ &=C\|f\|^2_{L^r(\mathbb{R}^3)}\||u|\operatorname{div}(u/|u|)\|_{L^{\overline{q}}(\mathbb{R}^3)}\\ &\leqslant\frac{9}{16}\||u|^{1/2}\nabla|u|\|^2_{L^2(\mathbb{R}^3)}+C\||u|\operatorname{div}(u/|u|)\|^{1/\theta}_{L^{\overline{q}}(\mathbb{R}^3)}\|u\|^3_{L^3(\mathbb{R}^3)},\end{aligned}$$

where

$$\theta=\frac{3}{r}-\frac{1}{2}=\frac{1}{2}\Big(2-\frac{3}{\overline{q}}\Big).$$

By virtue of (4), this gives $1/\theta\leqslant\overline{p}$, hence $\||u|\operatorname{div}(u/|u|)\|^{1/\theta}_{L^{\overline{q}}(\mathbb{R}^3)}$ lies in $L^1(0,T)$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}\frac{|u|^3}{3}\,\mathrm{d}x+\frac{7}{16}\int_{\mathbb{R}^3}|u||\nabla|u||^2\,\mathrm{d}x\leqslant C\||u|\operatorname{div}(u/|u|)\|^{1/\theta}_{L^{\overline{q}}(\mathbb{R}^3)}\int_{\mathbb{R}^3}\frac{|u|^3}{3}\,\mathrm{d}x.$$

The Gronwall lemma gives that, whenever T is finite,

$$\lim_{t\to T}\int_{\mathbb{R}^3}|u|^3\,\mathrm{d}x<\infty,$$

and so

$$\int_{t_0}^T\int_{\mathbb{R}^3}|u||\nabla|u||^2\,\mathrm{d}x\,\mathrm{d}t=\frac{4}{9}\int_{t_0}^T\int_{\mathbb{R}^3}|\nabla|u|^{3/2}|^2\,\mathrm{d}x\,\mathrm{d}t$$

is finite too. The Sobolev imbedding gives that $u\in L^3(t_0,T;L^9(\mathbb{R}^3))$. The Serrin's criteria, then, contradict the fact that T is finite. This shows that u is smooth on $(t_0,\infty)\times\mathbb{R}^3$ for almost every $t_0>0$. The assertion of Theorem 1 follows.

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