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Roots of Polynomials
and
Their Connections

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Roots of Polynomials
and
Their Connections

by

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Dedication

I have been offered many bits of wisdom and advice throughout my lifetime. Some I have heeded and much I have ignored. Perhaps the best advice I ever embraced was offered to me by my high school English teacher, Claire Miller. I will forever be grateful for her wisdom and humility. It is because of her I became a math teacher. She was that teacher who made a difference in my life. I am a life-long learner. The love of learning was passed on to me by Ms. Miller, and I cherish that gift! I intend to pass it along to my children and as many students as possible.

Acknowledgements

I would like to acknowledge my most ardent supporters. I would not be here if it were not for the consistent, unbounded love and encouragement offered to me by my parents. I am thankful for my mother who will always believe in me and to my father who remains my greatest encourager even though he is gone. How lucky I am to have been given so many incredible opportunities. In addition, I would like to acknowledge my children. This undertaking has required them to make some sacrifices. I appreciate their helpfulness and encouragement which made it possible for me to attain such a meaningful accomplishment.

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Abstract

Roots of Polynomials and Their Connections

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In the study of mathematics, one of the most useful, relevant topics explored in secondary mathematics remains the zeros of polynomials. This paper will present various ways to explore this topic while preserving the fundamental concept as a whole. In addition, this paper will reveal some distinct relationships between roots and their behavior within the different branches of mathematics.

The purpose of this paper is to show how this topic can be inserted at key points in the developmental curriculum to preserve the autonomy of this vital mathematical concept, allowing students to experience the behavior and value of this topic in a variety of contexts.

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Chapter 1: Introduction

The study of functions undoubtedly dominates mathematics curricula beginning in secondary courses. Euler's publication of 1748, *Introductio in analysin infinitorum*, was one of the first important books to be based on the concept of functions, rather than curves. Within the study of functions, students very quickly encounter the task of finding the roots of those functions. Perhaps because of their usefulness in understanding the behavior of certain classes of functions, in utilizing the values as an aid for graphical representation, or in bridging the gap between algebra and calculus, the idea of roots makes an early appearance and remains a recurrent theme throughout the study of mathematics. At the most basic level, students are asked to find x-intercepts for linear functions. As students progress to second degree functions, they find themselves factoring, completing the square, and using the quadratic formula. Subsequent study of more complicated functions tends to involve a deeper exploration of domain, range, and function behavior somewhat abandoning the study and importance of functional roots. Once again in calculus, the idea of roots emerges as one of significance with the introduction to derivatives. It is at this point students begin to grapple with the connectedness of this idea they have been studying for several years. As students initially encounter polynomial roots involving only integral coefficient and constant values, an understanding of how that subset of functions is related provides an example of one of those surprising connections in the field of mathematics. This limited subset produces a framework on which students can practice basic concepts in order to erect their fundamental mathematical foundation. Once established, students can readily move to difficult solutions involving irrational and imaginary components. Herein is found the opportunity to ensure that the essential connection is made by examining the existence and significance of the roots of families of functions.

It is only after serious study and research that one truly begins to appreciate the relationship, power, and usefulness of such a seemingly simple phenomenon. Researching the roots of polynomials offers multiple opportunities to more successfully weave a consistent thread through a general study in mathematics, while exploring their behavior offers a seemingly endless occurrence of connectedness and surprising relationships between the areas of algebra, geometry, and calculus.

Chapter 2: Cubic Polynomials

As a means of providing students a fundamental platform on which to build, the introduction to polynomial roots typically involves those with integral coefficients. The very process used to generate this particular subset of polynomials reveals an intriguing overlap between algebra, geometry and trigonometry. Buddenhagen, Ford, and May introduce this technique using a polynomial of the following form [**1**]:

$$y = (x+a)x(x-b). \qquad (1.1)$$

The right side can be multiplied and simplified to form the expression,
$x^3 + (a-b)x^2 - abx$, whose derivative is :

$$y' = 3x^2 + 2(a-b)x - ab. \qquad (1.2)$$

Restricting coefficients to integral values requires the derivative be restricted to those with perfect square discriminants. And thus Buddenhagen, Ford, and May [**1**] then introduce the Diophantine equation:

$$a^2 + b^2 + ab = c^2. \qquad (1.3)$$

This family of cubic polynomials identified in (1.1) uniquely presents the roots as integral roots with a middle root of zero. Shifting the root of zero to the far left produces two positive integral roots and the following slightly different version of the Diophantine equation:

$$a^2 + b^2 - ab = c^2. \qquad (1.4)$$

The Diophantine equation holds the key to establishing the sought after connection. As an example, consider the familiar *box problem* where *a* and *b* are the dimensions of the rectangular base of the box with height *x* [**1**, p.245]:

> *An open box is constructed from a rectangular piece of metal by cutting four equal squares from the corners and bending up the resulting tabs. Find the dimensions which maximize the volume of the box.*

Students derive the following equation when calculating the volume:

$$y = x(a - 2x)(b - 2x). \qquad (1.5)$$

The simplified expression, $4x^3 - 2ax^2 - 2bx^2 + abx$, yields a derivative of, $12x^2 - 4(a+b)x + ab$. Restricting its discriminant of $(-4)^2(a+b)^2 - 4(12)ab$ to result in a perfect square, yields the following equation:

$$a^2 + 2ab + b^2 - 3ab = c^2. \qquad (1.6)$$

The left side conveniently simplifies to, $a^2 + b^2 - ab$ which is equivalent to the left side of the modified version of the Diophantine equation. Similar results occur in another familiar novice problem when using the law of cosines to find the length of the third side of a triangle. The law of cosines states the square of the third side, *c*, is equal to $a^2 + b^2 - 2ab\cos(C)$, where *C* is the included angle opposite the unknown side. If *g* takes on the value of $-2\cos\lambda$, one obtains the following equation:

$$a^2 + b^2 + gab = c^2. \qquad (1.7)$$

The final triumph occurs when one notices the two Diophantine equations result when the included angle measures 60° and 120° [**1**]. A third observation comes when the included angle measures 90°. Because its cosine is zero, the product of *g* and *ab* results in zero. The third term disappears; and what remains is the familiar Pythagorean Theorem, the theorem most likely familiar to all intermediate mathematics students. This result offers to students a small glimpse of the inescapable overlap existing between the different disciplines of mathematics as well the intricacy of truths described by the use of numbers.

Table 1. Law of Cosine Results.

γ	$\cos\lambda$	$-2\cos\lambda$	Substitution	Equation
60°	$\frac{1}{2}$	-1	$c^2 = a^2 + b^2 - ab$	(1.4)
120°	$-\frac{1}{2}$	1	$c^2 = a^2 + b^2 + ab$	(1.3)
90°	0	0	$c^2 = a^2 + b^2$	Pythagorean

Chapter 3: Roots and Their Relationships

Kalman [**3**] describes another surprising mathematical connection, the marvelous relationship that exists between the roots of a polynomial with those of its derivative. Rolle's Theorem, to which beginning calculus students are exposed, describes the fundamental characteristic of a continuous, differentiable function which will always yield a minimum or maximum value between two of its adjacent roots. Of course, the derivative at this extrema is zero; and is, therefore, a root of the derivative. When considering the graph of a function in the complex plane, Kalman demonstrates the unique relationship which exists between the roots of the following third degree polynomial:

$$p(z) = (z^2 + 1)(z - 1). \qquad (1.8)$$

Solving the polynomial equation $p(z) = 0$ produces an integral root of 1 and two complex roots, i and $-i$. The product $z^3 - z^2 + z - 1$ has $3z^2 - 2z + 1$as its derivative. Using the quadratic formula as a method of calculating the roots of the derivative produces a pair of complex conjugate roots, $\frac{1}{3} \pm \frac{i\sqrt{2}}{3}$. Graphing these points on the complex plane adds further evidence to the accuracy of Rolle's Theorem. In fact Kalman uses this graph to illustrate Lucas' Theorem which asserts the following:

> *All of the roots of the derivative must lie in the convex hull of the roots of the original polynomial.*

The graph of the roots indeed form vertices of a triangle, and the roots of the derivative lie in the hull. In addition, the placement of the roots appears strategically placed. Kalman continues to explain their orientation. If one constructs an inscribed ellipse, which is tangent to the midpoint of the sides of the triangle whose vertices are formed by the roots of the polynomial, the foci of the ellipse are the exact points plotted from the derivative of that same polynomial as seen in Figure 1:

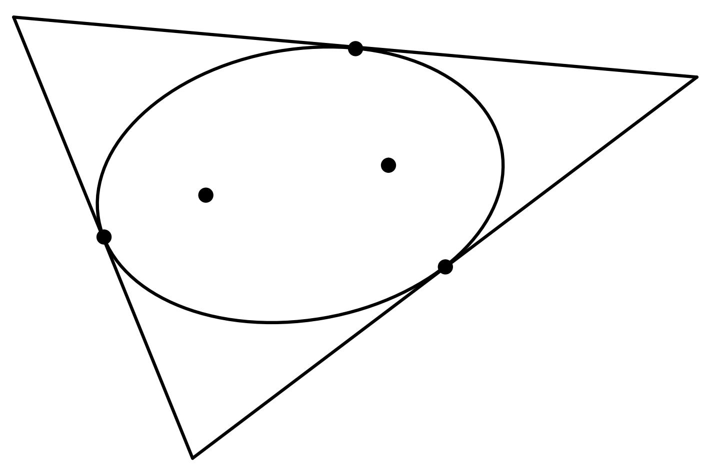

Figure 1. Inscribed Ellipse with Foci [**3**].

According to Kalman, the general principle corresponding to Figure 1 is known as Marden's Theorem [**3**,p.1].

Again, the interrelationships between algebra, geometry, trigonometry and calculus along with the characteristics of basic number systems and their behaviors provide those epiphany moments which may awaken the curiosity of the next great mathematician.

In addition to the opportunity of experiencing mathematical connections between the disciplines, exploring and examining the behaviors of numbers themselves aid in producing students who appropriately appreciate a broad mathematical understanding. Further examination of polynomial roots and the numerical relationships therein provides many of these types of opportunities. Buddenhagen, Ford, and May [**1**] uncover one such example, as further exploration is done on the Diophantine equation in an attempt to generate coefficients of polynomials having only integral values. To achieve this end, a general theorem is used where all rational solutions of the Diophantine equation (1.3) can be found by the following procedure:

$$(a,b,c) = k(u^2 - v^2, 2uv + gv^2, u^2 + guv + v^2). \qquad (1.9)$$

This equation is valid with the restrictions, -2 < g < 2, g and k rational, and u and v both relatively prime. The final form used to give all solutions replaces g with d – 2, and reparameterizes by replacing u with $i + j$ and v with j to obtain:

$$(a,b,c) = k(i^2 + 2ij, 2ij + dj^2, i^2 + dij + dj^2). \qquad (1.10)$$

As an example, simplifying the discriminant of the derivative of the box problem resulted in the modified Diophantine, $a^2 + b^2 - ab$. Because the value of g is 1, the value of d is also 1. Using only the a and b portions of equation (1.9) and a value of 1 for d, all rational dimensions for the box problem can be calculated using, $k(i^2 + 2ij, 2ij + j^2)$. The last criteria for the box problem restricted to integer values dictates the maximum volume be limited to a rational value for the height of the box, x. In order to find this suitable x, the maximum must be calculated from the root of the first derivative, $12x^2 + (-4a - 4b)x + ab$. Using values for *a*, *b*, and *c* of 12, -4(*a* + *b*), and *ab* respectively in the quadratic formula, the extrema values will be determined by $\frac{a + b \pm \sqrt{a^2 - ab + b^2}}{6}$ which simplifies to $\frac{a + b \pm c}{6}$. Testing values identifies $\frac{a + b - c}{6}$ as the maximum. Substitution into equation (1.10) yields $\frac{k(i^2 + 2ij + 2ij + j^2 - i^2 - ij - j^2}{6}$ which then simplifies to $\frac{kij}{2}$. The resulting dimensions of the box are $(ki^2 + kij) \times (kij + kj^2) \times (kij/2)$. As students uncover these relationships they can experiment with different values. For example if one allows *i*, *j*, and *k* to each take on the value of 1, 2, and 3 respectively, the dimensions of the box would be, $9 \times 18 \times 3$. The original rectangle would therefore have values of 15 and 24 for *a* and *b* respectively. Substituting these into the original volume function produces the following cubic polynomial:

$$y = 4x^3 - 78x^2 + 360x. \qquad (1.11)$$

The resulting derivative of $12x^2 - 78x + 360$ has roots at 10 and 3 with a maximum at 3. The value when *x* is 3 indeed agrees with the height dimension previously identified.

Although this certainly is not an exhaustive summary of the relationships of cubic polynomials and their roots, it does provide some working examples of ways to incorporate these connections in a way that provides a thread of continuity between the mathematical disciplines.

Chapter 4: Determining Roots

Redmond [**4**] offers yet another rather intriguing numerical result from examining the process of determining the possible rational roots for polynomials. As students progress from cubic polynomials to those with higher degrees, the tools previously mastered and used no longer prove efficient. Initially, students are introduced to the rational root theorem which allows students to limit their choices for rational roots $\frac{p}{q}$ to those whose numerator is expressed by some factor, *p*, of the constant term and whose denominator is expressed by some factor, *q*, of the lead coefficient. Using Descartes' rule of signs trims the number of possibilities; however, the task of analyzing each possibility can still prove quite an arduous task. One theorem allows students to give special attention to any function *f* for which the lead coefficient, constant term, and $f(1)$ all take on odd values [**4**]. Polynomial functions exhibiting this characteristic yield no rational roots, and recognizing this fact allows the immediate dismissal of the normal considerations regardless of how many possibilities exist for values $\frac{p}{q}$. One can use proof by contradiction to verify this result. Assume $\frac{p}{q}$ is a rational root. As both *p* and *q* are given as relatively prime and were derived from odd numbers, they must themselves be odd. Substituting a root into the function would result in the following equation:

$$f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_1\left(\frac{p}{q}\right) + a_0 = 0. \qquad (1.12)$$

Multiplying both sides by q^n allows for the elimination of the denominators on the right side of the equation and provides a situation where $q^n f\left(\frac{p}{q}\right)$ equals 0. From the initial constraint stating $f(1)$ is odd, subtracting the previously mentioned zero value would leave $f(1)$ unchanged and, furthermore, produce a result congruent to $1(\bmod 2)$. Substituting this value, $f(1)-q^n f\left(\frac{p}{q}\right)$, produces the following[**4**] :

$$f(1)-q^n f\left(\frac{p}{q}\right)=a_n(1-p^n)+a_{n-1}(1-p^{n-1}q)+\ldots+a_1(1-pq^{n-q})+a_0(1-q^n). \qquad (1.13)$$

Because the product of two odd numbers yields an odd product and an odd number raised to any power also yields an odd result, each difference in parentheses on the right must therefore be an even number. The entire equation would, therefore, result in an even number that is congruent to $0(\mathrm{mod}2)$. Herein is found the contradiction providing the proof for the theorem.

Carlyle's Method offers a fascinating example of a geometric approach to determining the roots of second degree polynomials. Hornsby [**2**] describes Carlyle's Method as one which considers the points of intersection of the circle, whose endpoints are $(0,1)$ and (a,b), with the *x*-axis. The number of real solutions is equal to the number of intersections that occur. The center of this particular circle is $\left(\frac{-b+0}{2},\frac{c+1}{2}\right)$. Using the distance formula, the radius is represented by $\sqrt{\left(\frac{-b}{2}-0\right)^2+\left(\frac{c+1}{2}-\frac{2}{2}\right)^2}$. The equation for the circle is, therefore, the following:

$$\left(x+\frac{b}{2}\right)^2+\left(y-\frac{c+1}{2}\right)^2=\left(\sqrt{\left(\frac{-b}{2}\right)^2+\left(\frac{c-1}{2}\right)^2}\right)^2. \quad (1.14)$$

The left side simplifies to $x^2+bx+\frac{b^2}{4}+y^2-(c+1)y+\frac{(c+1)^2}{4}$ and the right side to $\left(\frac{c-1}{2}\right)^2+\frac{b^2}{4}$. The identical terms cancel and the equation for this circle simplifies to [**2**]:

$$x^2+y^2+bx-(1+c)y+c=0. \quad (1.15)$$

The *x*-intercepts can now be determined, and they represent the roots of the given equation.

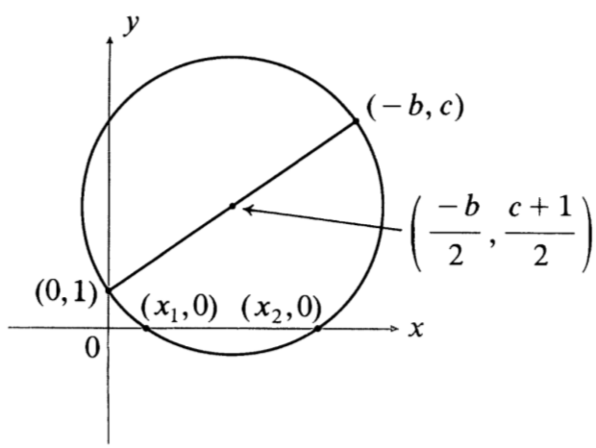

Figure 2. Carlyle's Method [**2**].

For example, consider the quadratic, $x^2 + x - 12$. According to Carlyle's Method, one could construct a circle with endpoints $(0,1)$ and $(-1,-12)$. Using the midpoint formula, the center for this particular circle is $\left(-\frac{1}{2}, -\frac{11}{2}\right)$. The following equation represents the equation for this circle:

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{11}{2}\right)^2 = \frac{170}{4}. \qquad (1.16)$$

Simplification of the equation produces, $x^2 + x + y^2 + 11y - 12$. The intersection of this circle and the *x*-axis occurs when $x^2 + x - 12$ equals 0, and any points of intersection are, in fact, the roots of the original quadratic. Although this method may not be superior to determining the roots of a quadratic, it does provide an opportunity to unite algebraic concepts with fundamental geometric principles.

Chapter 5: Conclusion

The topic of polynomial roots provides a foundational example of a profound mathematical concept which permeates mathematical disciplines. The ideas presented offer a small sample of the ways roots can be integrated and explored. When students are presented with opportunities to explore connections from one course to the next, they are provided with a deeper understanding of the completeness of mathematics. Trigonometry, algebra, geometry, and calculus are not separate, disjointed entities. As these topics are typically taken from one year to the next, opportunities exist to allow students to form connections from what they already know to new things they can discover. The power of mathematics does not stem from the highest level of learning, rather it is demonstrated by how even the simplest concepts remain constant throughout all levels.

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Vita

Cathy Jo Wardlaw (West) was born in the west Texas town of El Paso and raised in the Houston area metropolis. After graduating high school in 1978, she attended Southwest Texas State University and in 1984 earned her Bachelor of Science in computer science with a minor in mathematics. After a brief career as a software engineer in Austin, she entered the teaching field as a high school mathematics, computer science and Spanish teacher. After completing the necessary coursework, she earned her teaching certificate in 1986 and continued her teaching career for the next 25 years. In that time, she was privileged to teach virtually all levels of math and computer science at the elementary and secondary levels. She enrolled at the University of Texas at Austin and earned her Master of Arts degree in mathematics education. She will be employed at Taylor High School and will be teaching the AP Calculus and Statistics courses as well as other high school mathematics classes.

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