The Dissertation Committee for Diego Ribeiro Moreira
certifies that this is the approved version of the following dissertation:

Least Supersolution Approach to Regularizing Elliptic Free Boundary Problems

Committee:

Luis Caffarelli, Supervisor

Panagiotis Souganidis

Irene Gamba

Mikhail Vishik

Clint Dawson

Least Supersolution Approach to Regularizing Elliptic Free Boundary Problems

by

Diego Ribeiro Moreira, B.S., M.S.

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Dedicated to my wife, Vanessa Araujo, and my parents, Neuman Ribeiro and Almir Moreira.

"The profound study of nature is the most fertile source of mathematical discoveries." –Joseph Fourier

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Least Supersolution Approach to Regularizing Elliptic Free Boundary Problems

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Diego Ribeiro Moreira, Ph.D.
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Supervisor: Luis Caffarelli

In this dissertation, we study a free boundary problem obtained as a limit as $\varepsilon \to 0$ to the following regularizing family of semilinear equations $\Delta u = \beta_\varepsilon(u)F(\nabla u)$, where β_ε approximates the Dirac delta in the origin and F is a Lipschitz function bounded away from 0 and infinity. The least supersolution approach is used to construct solutions satisfying geometric properties of the level surfaces that are uniform. This allows to prove that the free boundary of the limit has the "right" weak geometry, in the measure theoretical sense. By the construction of some barriers with curvature, the classification of global profiles for the blow-up analysis is carried out and the limit function is proven to be a viscosity and pointwise solution (a.e) to a free boundary problem. Finally, the free boundary is proven to be a $C^{1,\alpha}$ surface around $\mathcal{H}^{n-1}$ a.e. point.

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Chapter 1

Introduction

Regularizing methods in free boundary problems are models for a wide spectrum of problems in nature. They are of particular interest in the theory of flame propagation to describe laminar flames as an asymptotic limit for high energy activation. These methods go back to Zeldovich and Frank-Kamenetski, [29], in 1938. However, the rigorous mathematical study was postponed until recently with the pionerring works of Berestycki-Caffarelli-Nirenberg [3] and by Caffarelli-Vazquez [11].

In the last decade, some attention has been given to the study of the limit as $\varepsilon \to 0$ of solutions to the elliptic equation

$$\Delta u = \beta_\varepsilon(u) \tag{1.0.1}$$

where $\beta_\varepsilon(s) = 1/\varepsilon \beta(s/\varepsilon)$ and β is a Lipschitz continuous function, with $\beta > 0$ in $(0,1)$, $\text{supp}\,(\beta) = [0,1]$ and $\int \beta = M > 0$. It is known from the series of important papers of Luis Caffarelli, Claudia Lederman and Noemi Wolanski, ([19],[18],[17]) that under certain geometric conditions about the limit function u_0 and its free boundary, it is a viscosity solution of the following free boundary problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \{u > 0\} \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 2M & \text{on } \Omega \cap \partial \{u > 0\}\,, \end{cases} \tag{1.0.2}$$

and the free boundary is locally a $C^{1,\alpha}$ surface. These assumptions are necessary if one intends to obtain further regularity results since there are limits which do not satisfy the free boundary condition in the classical sense in any portion of the free boundary ([19], remark 5.1).

Recently, in [15], Luis Caffarelli, David Jerison and Carlos Kenig proved some new monotonicity results so that it applies to inhomogeneous equations in which the right-hand side of the equation does not need vanish on the free boundary. The new versions of the monotonicity Theorem led to some existence and regularity results to the Prandtl-Batchelor equation. In connection with these results, a uniform Lipschitz estimate for solutions to a family of semilinear equations was proven. These regularizing approximations generalize the type of elliptic equations in (1.0.1) and they are the object of study of this dissertation. More concretelly, we study the limit free boundary problem and its regularity theory as $\varepsilon \to 0$ of the following family of semilinear equations

$$\Delta u = \beta_\varepsilon(u) F(\nabla u) \tag{1.0.3}$$

Here, F is a Lipchitz continuous function bounded away from 0 and infinity.

The strategy used here is the following: We use the least supersolution approach to construct solutions u_ε, which are more "stable" from the geometric viewpoint. This is done for equations more general than (1.0.3) and also allows to obtain a limit function with some geometric properties and its free boundary having some "weak" geometry. We then move to study the limit problem. The key part here is the classification of global profiles (2-plane functions) of the blow-up analysis. We remark however that, the typical integration by parts method developed in [19] and extensively used in similar problems does not seem to work for this case. Here, the classification depends upon a delicate construction of barriers with some uniform control on the curvature of their free boundaries as well as the asymptotic behavior of their slopes. Finally, the limit of the least supersolutions is proven to be a viscosity and pointwise ($\mathcal{H}^{N-1}$) a.e. solution to

$$\left\{ \begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega \setminus \{u > 0\} \\ H_\nu(u_\nu^+) - H_\nu(u_\nu^-) & = & M & \text{on } \Omega \cap \partial \{u > 0\}, \end{array} \right. \tag{1.0.4}$$

with $H_\nu(t) = \int_0^t \dfrac{s}{F(s\nu)} ds.$

and the free boundary $\Omega \cap \partial \{u > 0\}$ to be a $C^{1,\alpha}$ surface around $\mathcal{H}^{n-1} a.e.$ point.

In this case, the free boundary condition

$$H_\nu(u_\nu^+) - H_\nu(u_\nu^-) = M \text{ on } F(u)$$

also depends on the normal direction to the free boundary. This type of free boundary conditions appear as a limit of homogenization problems in periodic media. For homogenization free boundary problems, we refer to [20], [21].

To finish this introduction, we give a quick description on how the chapters are organized. In chapter 2, we establish the basic results as existence of the least supersolutions of the regularizing problems, as well as their regularity theory. We also investigate the geometric properties of the level surfaces of the ε least supersolutions. The key point is this section is to prove the uniform estimates for their geometry. In chapter 3, we discuss the properties of the limit function and we also establish all the measure theoretical properties of its free boundary, what we called the "weak geometry". In chapter 4, we introduce the assumptions on the equation (1.0.3) and also discuss the blow-up convergence results later on used in the "blow-up" analysis. In chapter 5, we prove all the qualitative ingredients as No interior Contact Lemma, Cubic decayment in the inteiror that will play a key role in the classifications of global profiles. Chapter 6 contains the construction of the barriers with uniformed curved free boundaries that will be used to force the free boundary condition to follow the pattern dictated by the 1-dimensional profiles. Chapter 7 we present the heart of the whole argument, namely, the classification of global solutions. It also contains a geometric and heuristic discussion about the free boundary condition. Chapter 8 is in charge of the blow-up analysis *per se* , the limit of the least supersolutions is proven to be a solution in the viscosity sense to the free boundary problem and finally the (partial) regularity of the free boundary is obtained. The first Appendix (Appendix A), brings the results of the Alt-Caffarelli Theory in [8] related with harmonicity, linear growth and nondegeneracy. Appendix B contains some results in [6],[17] about linear behaviour of Harmonic functions at boundary regular points that are used in some of the chapters.

Chapter 2

The Least Supersolutions

2.1 Existence, Continuity, Regularity Theory

In this section we will consider the following ε- regularized equations

$$(E_\varepsilon) \qquad \Delta u = F_\varepsilon(u, \nabla u) \text{ in } \Omega$$

where $\Omega \subset \mathbb{R}^N$ is a Lipschitz domain and $\{F_\varepsilon\}_{\varepsilon>0}$ is under the following structural conditions:

$$F_\varepsilon \in C(\mathbb{R} \times \mathbb{R}^N), \tag{2.1.1}$$

$$0 \leq F_\varepsilon(z,p) \leq \frac{A}{\varepsilon}\chi_{\{0<z<\varepsilon\}} \text{ in } \mathbb{R} \times \mathbb{R}^N, \quad A > 0 \tag{2.1.2}$$

Since our goal is the study of the free boundary of the limit configuration as $\varepsilon \to 0$, we will be interested to investigate geometric properties of some level sets of u_ε. For this reason, we should choose in some sense, more "stable" solutions u_ε to deal with. This was the approach done in [24], where solutions were chosen to be the minimizers of the corresponding functional associated to the ε - perturbed equations. In this case, due to lack of variational characterization for solutions of (E_ε), we will consider the least viscosity supersolution of the equation above. This will be accomplished by Perron's method.

Let φ be in $C(\partial\Omega)$ and let us define,

$$\mathcal{S}_\varphi^\varepsilon = \mathcal{S}_\varepsilon := \left\{ w \in C(\overline{\Omega}), w \text{ viscosity supersolution of } (E_\varepsilon); w \geq \varphi \text{ on } \partial\Omega \right\}$$

Clearly, $\mathcal{S}_\varepsilon \neq \emptyset$ since $h_\varphi \in \mathcal{S}_\varepsilon$, where h_φ is the harmonic in Ω such that $h = \varphi$ in $\partial\Omega$. Besides, there is also a natural barrier from below for the functions in set $\mathcal{S}_\varepsilon$. Indeed, if for each $\varepsilon > 0$, we define

$$L_\varepsilon := \sup_{(z,p)\in(0,\varepsilon)\times\mathbb{R}^N} F_\varepsilon(z,p) < +\infty$$

and let Ψ_ε be the unique solution to

$$\begin{cases} \Delta\Psi &= L_\varepsilon & \text{in } \Omega \\ \Psi &= \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.1.3}$$

by maximum principle, we have

$$\mathcal{S}_\varepsilon = \left\{ w \in C(\overline{\Omega}), w \text{ viscosity supersolution of } (E_\varepsilon);\ w \geq \Psi_\varepsilon \text{ in } \overline{\Omega} \right\}$$

We define the function

$$u_\varepsilon(x) := \inf_{w\in\mathcal{S}_\varepsilon} w(x) \tag{2.1.4}$$

It will be called the least supersolution of the equation (E_ε). From the discussion above, there exists natural barriers for u_ε, namely, $\Psi_\varepsilon \leq u_\varepsilon \leq h_\varphi$ in $\overline{\Omega}$.

Remark 2.1.1. It worths to notice that, in general, comparison principle for supersolutions and subsolutions of equation (E_ε) is not available. This way, uniqueness of solutions is not expected to hold.

Remark 2.1.2. We recall some definitions that are going to be used in next Theorem. If $u : \Omega \to \mathbb{R}$ is locally bounded, we define

$$u^*(x) = \inf \{v(x) \mid v \in USC(\Omega) \text{ and } v \geq u \text{ in } \Omega\}$$

$$u_*(x) = \sup \{v(x) \mid v \in LSC(\Omega) \text{ and } v \leq u \text{ in } \Omega\}$$

Clearly, $u^* \in USC(\Omega), u_* \in LSC(\Omega)$ and $u_* \leq u \leq u^*$. Besides, we have

$$u^*(x) = \lim_{r \searrow 0} \sup \{u(y) \mid y \in \Omega \cap B_r(x)\}$$

$$u_*(x) = \lim_{r \searrow 0} \inf \{u(y) \mid y \in \Omega \cap B_r(x)\}$$

The functions u^*, u_* are called upper semicontinuous envelope and lower semicontinuous envelope of u respectively.

Theorem 2.1.3. For each $\varepsilon > 0$, the least supersolution of equation (E_ε), u_ε, belongs to $C(\overline{\Omega}) \cap C^{1,\alpha}_{loc}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ for any $0 < \alpha < 1$ and any $1 \leq p < \infty$ and it is a viscosity solution of (E_ε). Besides, u_ε is a strong solution of (E_ε) and assume the boundary values φ continuously on the boundary, i.e,

$$\begin{cases} \Delta u_\varepsilon = F_\varepsilon(u_\varepsilon, \nabla u_\varepsilon) & \text{a.e. in} \quad \Omega \\ u_\varepsilon = \varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1.5}$$

In particular, $u_\varepsilon \in \mathcal{S}_\varepsilon$.

Proof. Let us observe first that $u_\varepsilon = (u_\varepsilon)^*$. It follows from Perron's method developed by Ishii in [14] that u_ε is a a viscosity subsolution and $(u_\varepsilon)_*$ is a viscosity supersolution of (E_ε). Since $\Delta u_\varepsilon \geq 0$ in the viscosity sense and u_ε is upper semicontinuous, from the uniqueness of the subharmonic upper semicontinuous representative ([22], Theorem 9.3), we conclude

$$u_\varepsilon(x) = \lim_{r \to 0} \fint_{B_r(x)} u_\varepsilon(y) dy \tag{2.1.6}$$

Moreover, for any $w \in \mathcal{S}_\varepsilon$, $\Delta w \leq L_\varepsilon$ in the viscosity sense. In Particular,

$$\Delta(w - \Psi_\varepsilon) \leq 0 \text{ in } \mathcal{D}'(\Omega)$$

which implies, by the average characterization of superharmonicity,

$$\Delta(u_\varepsilon - \Psi_\varepsilon) \leq 0 \text{ in } \mathcal{D}'(\Omega)$$

Again, from superharmonicity theory, there exists a unique superharmonic and lower semicontinuous representative ω_ε such that $\omega_\varepsilon = u_\varepsilon - \Psi_\varepsilon$ a.e in Ω and it is given by

$$\omega_\varepsilon(x) = \lim_{r\to 0} \fint_{B_r(x)} [u_\varepsilon(y) - \Psi_\varepsilon(y)]dy = u_\varepsilon(x) - \Psi_\varepsilon(x)$$

Where we use (2.1.6) in the second inequality. In particular, u_ε is lower semicontinuous, and so, $u_\varepsilon = (u_\varepsilon)_*$ is a continuous viscosity solution of (E_ε). From the structural conditions of F_ε and the regularity theory developed in [27], there is a universal $0 < \gamma < 1$ such that, $u_\varepsilon \in C^{1,\gamma}_{loc}(\Omega)$. It also follows from [28] that u_ε is twice differentiable almost everywhere in Ω, with equation (E_ε) then holding almost everywhere. To finish the proof, observe that if we define, $f_\varepsilon(x) = F_\varepsilon(u_\varepsilon(x), \nabla u_\varepsilon(x))$, then $f_\varepsilon \in C(\Omega) \cap L^\infty(\Omega)$ and $\Delta u_\varepsilon = f_\varepsilon$ in the viscosity sense. From $W^{2,p}$ estimates in ([9], Theorem 7.1), $u_\varepsilon \in W^{2,p}_{loc}(\Omega)$ for any $1 \le p < \infty$ and thus $u_\varepsilon \in C^{1,\alpha}_{loc}(\Omega)$ for any $0 < \alpha < 1$. To finish, let $x_0 \in \partial\Omega$, and $x_n \to x_0$. Since, $\Psi_\varepsilon(x_n) \le u_\varepsilon(x_n) \le h_\varphi(x_n)$, letting $n \to \infty$, we conclude $u(x_0) = \varphi(x_0)$.

□

Remark 2.1.4. It follows from the proof of the Theorem (2.1.3), that under the continuity assumption of F_ε and structural condition (2.1.2), any continuous viscosity solution of (E_ε) belongs to $C^{1,\alpha}_{loc}(\Omega) \cap W^{2,p}_{loc}(\Omega)$ for any $0 < \alpha < 1$ and $1 \le p < \infty$ and satisfies the equation almost everywhere in Ω and in the distributional sense.

Remark 2.1.5. The twice differentiability of u_ε in the Theorem above could also be justified by the fact that any function in $W^{2,p}_{loc}(\Omega)$ with $n < 2p$ is twice diferrentiable almost everywhere. This fact is a consequence of the Calderon-Zygmund theory. A direct proof can be found in ([16], Appendix C)

To finish this section, we state a result about local uniform Lipschitz regularity. This result is due to Luis Caffarelli.

Theorem 2.1.6 ([7], Corollary 2). Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a family of continuous viscosity solutions to (E_ε) such that $||v_\varepsilon||_{L^\infty(\Omega)} \leq \mathcal{A}$. Then, if $\Omega' \subset\subset \Omega$ there exists a universal contant $C = C(\Omega', \mathcal{A})$ such that

$$||\nabla v_\varepsilon||_{L^\infty(\Omega')} \leq C$$

In particular, the family $\{v_\varepsilon\}_{\varepsilon>0}$ is locally uniformly Lipschitz continuous.

2.2 Geometric Properties

In this section, we prove important geometric properties of the least supersolutions. We will be focused in two properties: Linear growth away from certain level sets and strong nondegeneracy. In general, those properties are not expected to hold for general solutions of the equation (E_ε). Those properties rely heavily on the special kind of solutions considered, the least supersolutions to (E_ε). These features will be crucial for the study of the regularity of the free boundary of the limit function later on. As we will see, these geometric facts will imply a rather restrictive geometry of the free boundary.

Some notation is now introduced.

$$B_\alpha^\star = B_{\delta_\varepsilon}(x_\varepsilon) \text{ where } u_\varepsilon(x_\varepsilon) = \alpha \text{ and } \delta_\varepsilon = \frac{1}{2} dist(x_\varepsilon, \partial\Omega)$$

$$\Omega_\alpha = \{x \in \Omega;\ 0 \le u_\varepsilon(x) \le \alpha\} \text{ and } d_\alpha(x) = dist(x, \Omega_\alpha)$$

$$\Omega_\alpha^+ = \{x \in \Omega;\ u_\varepsilon(x) > \alpha\}$$

$$\Omega^{'} \subset\subset \Omega \text{ and } \Delta = dist(\Omega^{'}, \mathbb{R}^N \setminus \Omega)$$

Theorem 2.2.1 (Linear growth away from level set ε). There exists a universal constant $C_3 > 0$ such that if $x_0 \in B_\varepsilon^\star \cap \Omega_\varepsilon^+$

$$u_\varepsilon(x_0) \ge C_3 d_\varepsilon(x_0)$$

Proof. Let us prove by contradiction. If this is not the case, for $\varepsilon > 0$ small enough, there exists $y_\varepsilon \in B_\varepsilon^\star \cap \Omega_\varepsilon^+$ such that $u_\varepsilon(y_\varepsilon) << d_\varepsilon(y_\varepsilon) = d_\varepsilon$. The idea now, is to construct an admissible supersolution (in $\mathcal{S}_\varepsilon$) strictly below u_ε in some point providing a contradiction. Since, $y_\varepsilon \in B_\varepsilon^\star \cap \Omega_\varepsilon^+$, we have $B_{d_\varepsilon}(y_\varepsilon) \subset \Omega_\varepsilon^+$ and thus

$$\Delta u_\varepsilon = 0 \text{ in } B_{d_\varepsilon}(y_\varepsilon)$$

By Harnack inequality, there exists a universal constant $C > 0$ such that

$$u_\varepsilon \le C u_\varepsilon(y_\varepsilon) \text{ in } B_{d_\varepsilon/2}(y_\varepsilon)$$

Now, consider the following function:

$$\begin{cases} \Delta v_\varepsilon &= 0 \quad \text{in } \mathcal{R} = B_{d_\varepsilon/2}(y_\varepsilon) \setminus \overline{B_{d_\varepsilon/4}(y_\varepsilon)} \\ v_\varepsilon &= 0 \quad \text{on } \partial B_{d_\varepsilon/4}(y_\varepsilon) \\ v_\varepsilon &= 1 \quad \text{on } \partial B_{d_\varepsilon/2}(y_\varepsilon) \end{cases} \tag{2.2.1}$$

and define,

$$\overline{w_\varepsilon} = \begin{cases} 0, & \text{in } \overline{B_{d_\varepsilon/4}(y_\varepsilon)} \\ \min\{u_\varepsilon, d_\varepsilon v_\varepsilon\} & \text{in } \mathcal{R} = B_{d_\varepsilon/2}(y_\varepsilon) \setminus \overline{B_{d_\varepsilon/4}(y_\varepsilon)} \\ u_\varepsilon & \text{in } \Omega \setminus \overline{B_{d_\varepsilon/2}(y_\varepsilon)} \end{cases} \tag{2.2.2}$$

Since $C > 0$ is a universal constant (that appears in the Harnack inequality) and $u_\varepsilon(x_0) << d_\varepsilon$, we can assume for ε small enough that, $Cu_\varepsilon(x_0) < d_\varepsilon$, and thus, $\overline{w_\varepsilon}$ is continuous along $\partial B_{d_\varepsilon/4}(y_\varepsilon)$. It is easy to check that, $\overline{w_\varepsilon}$ is a supersolution ([9], Proposition 2.8, for example) and so $\overline{w_\varepsilon} \in \mathcal{S}_\varepsilon$, providing a contradiction since $\overline{w_\varepsilon}(x_0) = 0 < u_\varepsilon(x_0)$. This finishes the proof of the Theorem.

□

In what follows, we will assume that the family $\{u_\varepsilon\}_{\varepsilon>0}$ of least supersolutions of the equation (E_ε) is uniformly bounded, i .e,

$$||u_\varepsilon||_{L^\infty(\Omega)} \leq \mathcal{A} \tag{2.2.3}$$

Corollary 2.2.2. There exists a universal constant $C = C(\Omega', \mathcal{A})$ such that

$$x \in \Omega' \cap \Omega_\varepsilon^+, \quad d_\varepsilon(x) \leq \frac{\Delta}{4} \Longrightarrow \quad C_3 d_\varepsilon(x) \leq u_\varepsilon(x) \leq C d_\varepsilon(x) + \varepsilon$$

Proof. The first inequality follows from the Theorem (2.2.1) just by observing that if $d_\varepsilon(x) < \frac{\Delta}{3}$, then $x \in B_\varepsilon^\star$. Indeed, let $x_\varepsilon \in \partial\Omega_\varepsilon^+$ with $d_\varepsilon(x) = |x - x_\varepsilon|$, then

$$2|x - x_\varepsilon| = 2d_\varepsilon(x) < dist(x, \partial\Omega) - d_\varepsilon(x) \leq dist(x_\varepsilon, \partial\Omega) = 2\delta_\varepsilon(x)$$

The other inequality follows from uniform Lipschitz continuity, Theorem (2.1.6). □

We turn our attention to a strong nondegeneracy result for the least supersolutions. Below, we state and prove the Strong Nondegeneracy Lemma. Although our second order elliptic operator is the Laplacian the Lemma will be proven in a general way for elliptic operators in divergence form.

Lemma 2.2.3. (Strong Nondegeneracy Lemma) Assume $A \in C^\alpha(\overline{B_R(\xi)} \cap \Omega)$ is a matrix with $0 < \alpha < 1$ and that $v \geq 0$ is a Lipschitz solution of:

$$Lv = \operatorname{div}(A(x)\nabla v) = 0 \text{ in } \Omega \cap B_R(\xi) \text{ such that:} \tag{2.2.4}$$

1. $v \equiv \delta$ on $\partial\Omega \cap B_R(\xi), 0 \in \partial\Omega$
2. $v(x_0) \geq C\delta > 0,\ C >> 1$ with $x_0 \in B_{R/2}(\xi)$
3. $v(x) \geq D \cdot dist(x, \partial\Omega)$ in $\{v \geq C\delta\} \cap B_{R/2}(\xi)$

Then, there exists a universal constant $M = M(C, D, Lip(v))$ such that:

$$\sup_{B_r(x_0)} v \geq Mr \quad \text{for} \quad 0 < r \leq \frac{R}{4}$$

Proof. Let $B_\rho(x_0)$ be the largest ball contained in $\{v > \delta\}$. Consider $y_0 \in \partial B_\rho(x_0)$ such that $\rho = |x_0 - y_0| = \text{dist}(x_0, \partial\Omega)$ with $v(y_0) = \delta$. From the assumptions, $B_\rho(x_0) \subset \Omega \cap B_R(\xi)$. By (3),

$$v(x_0) \geq D \cdot \rho$$

Now, by Lipschitz continuity, taking $h = \frac{D}{CLip(v)}$ we have:

$$v \leq \left(\frac{1}{C} + \frac{Lip(v)h}{D}\right) v(x_0) \leq \frac{v(x_0)}{2} \text{ in } B_{\rho h}(y_0)$$

Therefore,

$$v \leq \frac{v(x_0)}{2} \text{ in } A = B_{\rho h}(y_0) \cap \partial B_\rho(x_0)$$

Notice that $\mathcal{H}^{N-1}(A)$ depends only on the fixed constant h. This way, it is independent of the particular points x_0, y_0 considered above. Since v is a solution of the equation (2.2.4), $v \in C^{1,\alpha}(\Omega \cap B_1(0))$ and

$$v(x_0) = \int_{\partial B_\rho(x_0)} v(y) K(x_0, y) d\mathcal{H}^{n-1}(y)$$

where

$$K(x_0, y) = \left\langle A(y) \nabla_y G(x_0, y), \overrightarrow{N}(y) \right\rangle$$

here, $G(x, y)$ is the Green function of $B_\rho(x_0)$ and $\overrightarrow{N}$ is the inward unit normal vector to the ball $B_\rho(x_0)$. Since constants are solutions of $Lu = 0$, we also have:

$$\int\limits_{\partial B_\rho(x_0)} K(x_0, y) d\mathcal{H}^{N-1}(y) = 1$$

As a consequence of Hopf's lemma, there exists a constant $\mu > 0$ such that:

$$K(x_0, \cdot) \geq \frac{\mu}{\rho^{N-1}} \text{ on } \partial B_\rho(x_0)$$

this way,

$$\int\limits_{\partial B_\rho(x_0)} \chi_{B_{h\rho}(y_0)}(y) K(x_0, y) d\mathcal{H}^{N-1}(y) \geq C(h, \mu)$$

Estimating $v(x_0)$

$$v(x_0) \leq \int\limits_{\partial B_\rho(x_0)} \chi_{B_{h\rho}(y_0)}(y) K(x_0, y) \frac{v(x_0)}{2} d\mathcal{H}^{N-1}(y) +$$
$$+ \left(\sup_{\partial B_\rho(x_0)} v \right) \int\limits_{\partial B_\rho(x_0)} \left(1 - \chi_{B_{h\rho}(y_0)}(y)\right) K(x_0, y) d\mathcal{H}^{N-1}(y)$$

which implies

$$\left(1 - \frac{C'}{2}\right) v(x_0) \leq (1 - C') \sup_{\partial B_\rho(x_0)} v$$

For some $C(h, \mu) \leq C' < 1$. This way, we obtain

$$\sup_{\partial B_\rho(x_0)} v \geq \left[1 + \overline{C}(\mu, h)\right] v(x_0)$$

In particular, we can find $x_1 \in \partial B_\rho(x_0)$ such that,

$$v(x_1) \geq (1 + \overline{C}) v(x_0)$$

and

$$v(x_1) - v(x_0) \geq \overline{M} |x_1 - x_0| \text{ where } \overline{M} = \overline{C} D$$

The idea is to construct a polygonal along which v grows linearly, starting from x_0. If We can iterate this process, we produce a sequence $\{x_n\}_{n\geq 0}$ such that:

A1) $v(x_n) - v(x_0) \geq \overline{M}|x_n - x_0|$ and $v(x_n) - v(x_{n-1}) \geq \overline{M}|x_n - x_{n-1}|$

A2) $v(x_n) \geq (1 + \overline{C})^n v(x_0)$

A3) $|x_n - x_{n-1}| = dist(x_{n-1}, \partial\Omega)$

Since $v(x_n) \to +\infty$ as $n \to +\infty$ (by (A2)) there exist a last x_n in the ball B_r, i.e, $\exists n_0 \in \mathbb{N}$ such that $x_{n_0} \in B_r(x_0)$ and $x_{n_0+1} \notin B_r(x_0)$. Now let us observe that:

$$\sup_{B_r(x_0)} v \geq v(x_{n_0}) + \overline{M}|x_{n_0} - x_0| \geq \overline{M}|x_{n_0} - x_0|$$

This way, the result will be proven if we can show there exists a universal constant $M > 0$:

$$|x_{n_0} - x_0| \geq Mr \text{ where } M = M(C, D, Lip(v))$$

This is indeed the case, since by A1)

$$|x_{n_0} - x_0| \geq \frac{1}{Lip(v)}(v(x_{n_0}) - v(x_0)) \geq \sum_{k=1}^{n_0} \overline{M}|x_k - x_{k-1}| \geq \overline{M}|x_{n_0} - x_{n_0-1}|$$

Using A3), we conclude

$$dist(x_{n_0}, \partial\Omega) \leq 2|x_{n_0} - x_{n_0-1}| \leq \frac{2Lip(v)}{\overline{M}}|x_{n_0} - x_0|$$

We finish the proof by observing

$$r \leq |x_{n_0+1} - x_0| \leq |x_{n_0+1} - x_{n_0}| + |x_{n_0} - x_0| \leq$$
$$\leq dist(x_{n_0}, \partial\Omega) + |x_{n_0} - x_0| \leq (1 + \frac{2Lip(v)}{\overline{M}})|x_{n_0} - x_0|$$

□

Now, we can come back to the least supersolutions and apply the Theorem above. This will provide us the following

Theorem 2.2.4 (Strong Nondegeneracy). Given $C_4 >> 1$ there exists $C = C(\Omega^{'}, C_3, C_4, \mathcal{A})$ such that

$$\sup_{B_\rho(x_0)} u_\varepsilon \geq C\rho \qquad \text{for } \rho \leq \frac{\Delta}{12}$$

for

$$x_0 \in \Omega^{'} \cap \{u_\varepsilon \geq C_4\varepsilon\}\,, \quad d_\varepsilon(x_0) \leq \frac{\Delta}{6}$$

Proof. Let x_0 be in the conditions above. If $d_\varepsilon = d_\varepsilon(x_0) = |x_0 - x_\varepsilon|$ and $\delta_\varepsilon = dist(x_\varepsilon, \partial\Omega)$, we have:

$$\frac{2\Delta}{3} = \Delta - \frac{\Delta}{3} < dist(x_0, \partial\Omega) - d_\varepsilon(x_0) \leq \delta_\varepsilon$$

This way,

$$x_0 \in \frac{1}{2}\overline{B_{\frac{\Delta}{3}}(x_\varepsilon)} \subset B_{\frac{\Delta}{3}}(x_\varepsilon) \subset \frac{1}{2}B_{\delta_\varepsilon}(x_\varepsilon) = B_\varepsilon^\star \tag{2.2.5}$$

Besides,

$$\bigcup B_{\frac{\Delta}{3}}(x_\varepsilon) \subset \mathcal{N}_{\frac{\Delta}{2}}(\Omega') \subset\subset \Omega \tag{2.2.6}$$

The inclusion (2.2.5) and Theorem (2.2.1) say that the previous Lemma can be used with $B_{\frac{\Delta}{3}}(x_\varepsilon)$ in place of $B_R(\xi)$ and the last inclusion (2.2.6) says that we can take uniform Lipschitz for all the appearing balls there. This concludes the Theorem.

□

Chapter 3

The Limit Function and Its Weak Geometry

This section will be devoted to establish the first results about the limit function and the weak geometry of its free boundary. Before, we introduce the following notation for a continuous function $v : \Omega \to \mathbb{R}$

$$\Omega^+(v) = \{x \in \Omega \mid v(x) > 0\} \quad ; \quad \Omega^-(v) = (\Omega \setminus \Omega^+(v))^\circ$$

$$F(v) = \partial \{x \in \Omega \mid v(x) > 0\} \cap \Omega = \partial\Omega^+ \cap \Omega$$

The set $F(v)$ is called the free boundary of v. Again, in what follows, we assume $\Omega' \subset\subset \Omega$.

Theorem 3.0.5 (Properties of the limit of the least supersolutions). Let $\{u_\varepsilon\}_{\varepsilon>0}$ the family of least supersolutions of (E_ε). Assume,

$$||u_\varepsilon||_{L^\infty(\Omega)} \leq \mathcal{A}$$

Then for every sequence $\varepsilon_k \to 0$ there exists a subsequence $\varepsilon_k' \to 0$ such that

a) $u_{\varepsilon_k'} \to u_0 \in C^{0,1}_{loc}(\Omega)$ uniformly on compact subsets of Ω.

b) (Regularity) $u_0 \in C^{0,1}_{loc}(\Omega)$, $\Delta u_0 \geq 0$ in $\mathcal{D}'(\Omega)$ and

$$\Delta u_0 = 0 \text{ in } \Omega^+(u_0) \text{ and in } \Omega^-(u_0)$$

c) (Linear growth away of the free boundary) Let $C_3 > 0$ be the constant given by Theorem (2.2.1), then

$$u_0^+(x_0) \geq C_3 \text{dist}(x_0, \{u_0 \leq 0\})$$

$$\text{if} \quad x_0 \in \Omega', \text{dist}(x_0, \{u_0 \leq 0\}) \leq \frac{\Delta}{4}$$

d) (Strong nondegeneracy) There exists a constant $C = C(\Omega', \mathcal{A})$ such that:

$$\sup_{B_\rho(x_0)} u_0 \geq C\rho \quad \text{for} \quad \rho \leq \frac{\Delta}{12}$$

provided

$$x_0 \in \Omega' \cap (\Omega_0 \cup F(u_0)) \text{ with } \text{dist}(x_0, \{u_0 \leq 0\}) \leq \frac{\Delta}{6}$$

e) (Nondegeneracy) There exists a constant $\underline{C} = \underline{C}(\Omega', \mathcal{A})$ and $\overline{C} = \overline{C}(\Omega', \mathcal{A})$ such that:

$$\underline{C} \leq \frac{1}{\rho} \fint_{\partial B_\rho(x_0)} u_0^+(y) d\mathcal{H}^{N-1}(y) \leq \overline{C} \quad \text{for } \rho \leq \frac{\Delta}{12}$$

whenever

$$x_0 \in \Omega' \cap F(u_0) \text{ with } \mathrm{dist}(x_0, \{u_0 \le 0\}) \le \frac{\Delta}{6}$$

Proof. $a)$ follows imediately from Theorem (2.1.6) and Ascoli-Arzela Theorem. In $b)$, the subharmonicity of u_0 is a straightforward consequence of the average characterization of subharmonic functions and the uniform convergence. To prove $\Delta u_0 = 0$ in $\Omega^+(u_0)$, let B be a ball $B \subset\subset \Omega^+(u_0)$. There exists $c > 0$ such that $u_0 \ge c$ in $\overline{B}$. From the uniform convergence, if ε_k' is taken small enough, $u_{\varepsilon_k'} \ge \frac{c}{2} \ge \varepsilon_k'$ in $\overline{B}$. So, $\Delta u_{\varepsilon_k'} = 0$ in B and thus, $\Delta u_0 = 0$ in B. Analogously, we conclude, $\Delta u_0 = 0$ in $\{u_0 < 0\}$. This implies, $\Delta u_0 \le 0$ in $\Omega^-(u_0)$. From the global subharmonicity of u_0 we conclude $b)$. To prove $c)$, we can assume $x_0 \in \Omega^+(u_0) \cap \Omega'$. For $\varepsilon_k' > 0$ small enough, we have $x_0 \in \Omega^+_{\varepsilon_k'}$ and $d_{\varepsilon_k'}(x_0) \le \frac{\Delta}{4}$. By Corollary (2.2.2), there exists $y_k \in \Omega_{\varepsilon_k'}, d_{\varepsilon_k'}(x_0) = |x_0 - y_k|$ such that:

$$u_{\varepsilon_k'}(x_0) \ge C_3 d_{\varepsilon_k'}(x_0) = C_3|x_0 - y_k|$$

Since $\{y_k\}_{k\ge 1}$ is bounded, we can assume, $y_k \to y_0,\ y_0 \in \Omega \setminus \Omega^+(u_0)$ and thus,

$$u_0(x_0) \ge C_3|x_0 - y_0| \ge C_3 \mathrm{dist}(x_0, \{u_0 \le 0\})$$

For $d)$, let us study two cases: (i) $x_0 \in \Omega^+(u_0)$ and (ii) $x_0 \in F(u_0)$. In case (i) again, for $\varepsilon_k' > 0$ small enough, we have $x_0 \in \Omega^+_{C_4\varepsilon_k'}$ and $d_{\varepsilon_k'}(x_0) \le \frac{\Delta}{6}$. By Theorem (2.2.4),

$$\sup_{B_\rho(x_0)} u_{\varepsilon'_k} \geq C\rho$$

Since $u_{\varepsilon'_k} \to u_0$ uniformly in compact subsets, passing the limit in the previous expression, the result is proven. Now, to prove (ii) let us observe that we can find $x_1 \in \Omega^+(u_0) \cap B_{\rho/4}(x_0)$. Setting, $K = \mathcal{N}_{\frac{\Delta}{8}}(\Omega')$ and applying (i) to K, we conclude

$$\sup_{B_\rho(x_0)} u_0 \geq \sup_{B_{\rho/4}(x_1)} u_0 \geq \frac{C\rho}{4}$$

$e)$ follows from $b)$ and $d)$. Indeed, if we define $K = \mathcal{N}_{\frac{\Delta}{2}}(\Omega')$, then $\overline{B_\rho(x_0)} \subset K$ and by Lipschitz continuity,

$$u_0^+ \leq \left(\frac{Lip(u_0 \mid K)}{12}\right) \rho \text{ in } B_\rho(x_0)$$

yielding,

$$\frac{1}{\rho} \fint_{\partial B_\rho(x_0)} u_0^+ d\mathcal{H}^{N-1} \leq \overline{C}$$

To prove the other inequality, let us consider x_0 in the conditions described in $e)$. This way, by $d)$, there exists $x_1 \in \overline{B_{\rho/2}(x_0)}$ such that $u_0(x_1) \geq \frac{C\rho}{4}$. By, Lipschitz continuity, if $\tau \leq \frac{1}{3}$, since $B_{\rho\tau}(x_1) \subset\subset B_\rho(x_0) \subset K$

$$u_0 \geq \left(\frac{C}{4} - Lip(u_0 \mid K)\tau\right) \rho \text{ in } B_{\rho\tau}(x_1)$$

Taking τ small enough, $u_0 \geq \frac{C\rho}{8} > 0$ in $B_{\rho\tau}(x_1)$ and thus,

$$\fint_{B_\rho(x_0)} u_0^+ dx \geq \tau^{N-1} \fint_{B_{\rho\tau}(x_1)} u_0^+ dx \geq \frac{\tau^N C}{8}\rho$$

By now, we have proven $e)$ for the volume average, i.e, there exist a constant $C_1 = C_1(\Omega', \mathcal{A}) > 0$ such that, whenever, x_0 is in the conditions of (5), we have:

$$\frac{1}{\rho}\fint_{B_\rho(x_0)} u_0^+ dx \geq C_1 \tag{3.0.1}$$

From the fact that $u_0^+ \geq 0$ is locally Lipschitz constinuous and harmonic in $\left\{u_0^+ > 0\right\}$, the same conclusion holds for the area average as in the statement of $e)$. Indeed, suppose by contradiction, that this is not the case. Then, we can find a sequence $\{x_n\}_{n\geq 1} \subset F(u_0) \cap \Omega'$ with $\text{dist}(x_n, \{u_0 \leq 0\}) \leq \frac{\Delta}{6}$, such that

$$\fint_{\partial B_{\rho_n}(x_n)} u_0^+ d\mathcal{H}^{N-1} \leq \frac{1}{n}\rho_n \quad \text{with } \rho_n \to 0. \tag{3.0.2}$$

Considering the rescaling functions, $v_n(x) := \frac{1}{\rho_n} u_0^+(x_n + \rho_n x)$, it follows from remark (3.0.6), there exists a subsequence, that we still denote by v_n, such that $v_n \to V$ uniformly in compact sets of $\mathbb{R}^N$, $V \geq 0$, V Lipschitz continuous an harmonic in $\{V > 0\}$. Now, rewriting, (3.0.2), in terms of v_n we find,

$$\fint_{\partial B_1(0)} v_n d\mathcal{H}^{N-1} \leq \frac{1}{n}\rho_n$$

Since, u_0^+ is globally subharmonic, we have

$$0 \leq \fint_{B_1(0)} u_0^+ dx \leq \fint_{\partial B_1(0)} u_0^+ d\mathcal{H}^{N-1} = 0$$

which implies that $u_0^+ \equiv 0$ in $B_1(0)$. On the other hand, we have proven that

$$\fint_{B_1(0)} v_n dx \geq C_1 > 0$$

Letting $n \to \infty$, we get $\fint_{B_1(0)} u_0^+ dx \geq C_1 > 0$, a contradiction. This finishes the proof of the Theorem.

□

Remark 3.0.6. Let us observe that the Lipschitz constant is invariant under the rescaling $v_n(x) := \frac{1}{\rho_n} u_0^+(x_n + \rho_n x)$. Moreover, $v_n(0) = 0$ for all $n \geq 1$. So, we can obtain a function V described as in the proof of theorem (3.0.5) by Ascoli-Arzela Theorem. Since $v_n's$ are harmonic whenever positive, by the uniform convegernce, the same holds for V.

Now, we establish some properties of the free boundary of u_0, $F(u_0)$. Before, we need the following definition

Definition 3.0.7. Let $v : \Omega \to \mathbb{R}$ be a continuous function. A unit vector $\nu \in \mathbb{R}^N$ is said to be the inward unit normal in the measure theoretic sense to the free boundary $F(v)$ at a point $x_0 \in F(v)$ if

$$\lim_{\rho\to 0}\frac{1}{\rho^N}\int\limits_{B_\rho(x_0)} \mid \chi_{\{v>0\}} - \chi_{H_\nu^+(x_0)} \mid dx = 0 \tag{3.0.3}$$

where $H_\nu^+(x_0) = \left\{x \in \mathbb{R}^N \mid \langle x - x_0, \nu\rangle > 0\right\}$. If A is a set of locally finite perimeter, then for every point in the reduced boundary, $\partial_{red}A$, the inward unit normal is defined. The details can be found in ([12], section 5.7).

Theorem 3.0.8. (Properties of the free boundary $F(u_0)$)

Let u_0 be the functions given by Theorem (3.0.5). Then,

a) $\mathcal{H}^{N-1}(\Omega' \cap \partial\{u_0 > 0\}) < \infty$

b) There exist borelian functions $q_{u_0}^+$ and $q_{u_0}^-$ defined on $F(u_0)$ such that

$$\Delta u_0^+ = q_{u_0}^+ \mathcal{H}^{N-1} \lfloor \partial\{u_0 > 0\}$$

$$\Delta u_0^- = q_{u_0}^- \mathcal{H}^{N-1} \lfloor \partial\{u_0 > 0\}$$

c) There exists universal constants $\underline{C} > 0$, $\overline{C} > 0$ and $\rho_0 > 0$ depending on $\Omega', \mathcal{A}$ such that

$$\underline{C}\rho^{N-1} \le \mathcal{H}^{N-1}(B_\rho(x_0) \cap \partial\{u_0 > 0\}) \le \overline{C}\rho^{N-1}$$

for every $x_0 \in \Omega' \cap \partial\{u_0 > 0\}, 0 < \rho < \rho_0$

d) $0 < \underline{C} \leq q_{u_0}^{+} \leq \overline{C}$ and $0 \leq q_{u_0}^{-} \leq \overline{C}$ in $\Omega' \{u_0 > 0\}$. In addition, $q_{u_0}^{-} = 0$ in $\partial \{u_0 > 0\} \setminus \{u_0 < 0\}$.

e) u_0 has the following asymptotic development at $\mathcal{H}^{N-1}$-almost every point x_0 in $F(u_0)_{red}$

$$u_0(x) = q_{u_0}^{+}(x_0) \langle x - x_0, \nu \rangle^{+} - q_{u_0}^{-}(x_0) \langle x - x_0, \nu \rangle^{-} + o(|x - x_0|)$$

f) There exists a constant $\tau = \tau(\Omega', \mathcal{A}) > 0$ such that

$$\mathcal{H}^{N-1}(F(u_0)_{red} \cap B_\rho(x_0)) \geq \tau^{\star} \rho^{N-1}$$

for any $x_0 \in F(u_0) \cap \Omega'$. In Particular, we have

$$\mathcal{H}^{N-1}(F(u_0) \setminus F(u_0)_{red}) = 0 \tag{3.0.4}$$

Proof. In order to prove the Theorem, we will make use of some results from the Alt-Caffarelli Theory. These results are stated in the Appendix. It follows from Theorem (3.0.5) that all the assumptions of the Alt-Caffarelli Theory (section 4 in [8]) (see Appendix) are satisfied. This way, it follows that $a), b), c), d)$ and $e)$ holds.

We observe however, that because of the lack of variational characterization for solutions u_ε (and therefore for u_0), we are unable to obtain a positive uniform density from below of the positive phase, like Lemma (3.7) in [8]. Then, the $\mathcal{H}^{N-1}$ measure totality in (3.0.4) of the reduced free boundary, $F(u_0)_{red}$, does not follow from Alt-Caffarelli theory in [8]. Instead, a sublte construction like one developed in [5] is necessary. This way, let us concentrate in proving f). By rescaling, i.e, considering the function

$$(u_0)_\rho(x) = \frac{1}{\rho} u_0(\rho(x - x_0))$$

it is enough to prove the case where $\rho = 1$ and $x_0 = 0$. For $0 < \sigma < 1/4$, let us define the following auxiliary function v_σ

$$\begin{cases} \Delta v_\sigma &= -\frac{1}{|B_\sigma(0)|}\chi_{B_\sigma(0)} \quad \text{in } B_1(0) \\ v_\sigma &= 0 \quad \text{on } \partial B_1(0), \end{cases} \tag{3.0.5}$$

In fact, if $G(x, y)$ denotes the Green function of the unit ball, we have

$$v_\sigma(x) = -\!\!\!\!\!\!\int_{B_\sigma(0)} G(x, y)dy$$

By maximum principle, $v_\sigma \geq 0$. It follows from Litmann-Weinberger-Stampacchia Theorem, ([26], Theorem 7.1), that $v_\sigma \leq \overline{C}\sigma^{2-N}$ outside $B_{2\sigma}(0)$ ($\overline{C} > 0$ universal constant) and $\partial_\nu v_\sigma \sim C > 0$ (here, C is also a universal constant) along $\partial B_1(0)$, where ν is the unit outwards normal vector to $\partial B_1(0)$. Now, by Harnack inequality, ([13], Theorems 8.17 and 8.18), for any $q > \frac{N}{2}$,

$$\sup_{B_{2\sigma}(0)} v_\sigma \leq C^\star \left\{ \inf_{B_{2\sigma}(0)} v_\sigma + \sigma^{2-\frac{2N}{q}} || \frac{1}{|B_\sigma(0)|}\chi_{B_\sigma(0)} ||_{L^q(B_{2\sigma}(0))} \right\}$$

where $C^\star = C^\star(N,q)$. Since $\inf\limits_{B_\sigma(0)} v_\sigma \le \overline{C}\sigma^{2-N}$, we finally obtain that

$$v_\sigma \le C\sigma^{2-N} \text{ in } B_1(0), \text{ where } C = C(N,\sigma) \tag{3.0.6}$$

Since, $u_{\varepsilon_k}, v_\sigma \in C^{1,\alpha}(\overline{B_1(0)})$ for any $0 < \alpha < 1$, we can apply the second Green's formula, obtaining

$$\int\limits_{\Omega^+(u_0)\cap B_1(0)} (v_\sigma \Delta u_{\varepsilon_k} - u_{\varepsilon_k}\Delta v_\sigma)dx = \tag{3.0.7}$$

$$= \int\limits_{B_1(0)\cap F(u_0)_{red}} (v_\sigma \partial_\nu u_{\varepsilon_k} - u_{\varepsilon_k}\partial_\nu v_\sigma)d\mathcal{H}^{N-1} - \int\limits_{\partial B_1(0)\cap\Omega^+(u_0)} u_{\varepsilon_k}\partial_\nu v_\sigma \; d\mathcal{H}^{N-1}$$

From the uniform Lipschitz continuity of u_{ε_k} in $\overline{B_1(0)}$ and (3.0.6),

$$\Big| \int\limits_{B_1(0)\cap F(u_0)_{red}} v_\sigma \partial_\nu u_{\varepsilon_k} d\mathcal{H}^{N-1} \Big| \le C\sigma^{2-N}\mathcal{H}^{N-1}(F(u_0)_{red}\cap B_1(0))$$

Moreover, as $\varepsilon_k \to 0$,

$$\int\limits_{B_1(0)\cap F(u_0)_{red}} u_{\varepsilon_k}\partial_\nu v_\sigma d\mathcal{H}^{N-1} \to 0$$

$$\int_{\partial B_1(0)\cap\Omega^+(u_0)} u_{\varepsilon_k}\partial_\nu v_\sigma \; d\mathcal{H}^{N-1} \to \int_{\partial B_1(0)} u_0^+\partial_\nu v_\sigma \; d\mathcal{H}^{N-1}$$

$$-\int_{\Omega^+(u_0)\cap B_1(0)} u_{\varepsilon_k}\Delta v_\sigma dx = \frac{1}{|B_\sigma(0)|}\int_{\Omega^+(u_0)\cap B_\sigma(0)} u_{\varepsilon_k} dx \; \to \fint_{B_\sigma(0)} u_0^+ dx$$

Since $v_\sigma \Delta u_{\varepsilon_k} \geq 0$, from (3.0.7), we deduce

$$\fint_{B_\sigma(0)} u_0^+ dx + \int_{\partial B_1(0)} u_0^+\partial_\nu v_\sigma \; d\mathcal{H}^{N-1} \leq C\sigma^{2-N}\mathcal{H}^{N-1}(F(u_0)_{red}\cap B_1(0)) \quad (3.0.8)$$

By Theorem (3.0.5)-e),

$$\int_{\partial B_1(0)\cap\Omega^+(u_0)} u_0^+\partial_\nu v_\sigma \; d\mathcal{H}^{N-1} \geq C_1 > 0$$

In Particular, again by nondegeneracy (3.0.1), the relation (3.0.8) implies

$$C^\star\sigma \leq \fint_{B_\sigma(0)} u_0^+ dx \leq C\sigma^{2-N}\mathcal{H}^{N-1}(F(u_0)_{red}\cap B_1(0))$$

To conclude, since there exist $\underline{C}, \overline{C}$ depending on Ω' and $\mathcal{A}$, such that

$$\underline{C} \leq \frac{1}{\sigma}\fint_{B_\sigma(0)} u_0^+ dx \leq \overline{C}$$

We can then choose, $\sigma = \underline{C}/8\overline{C}$, a universal constant. The last conclusion follows from the density Theorem for lower dimensional Hausdorff Measure ([12], Theorem 1, pp. 72), just by observing that $\mathcal{H}^{N-1}\lfloor F(u_0)$ is a Radon Measure. $\square$

Chapter 4

Regularizing Problems

4.1 The Special Form

In the previous sections, we have described the "weak" geometry of the free boundary $F(u_0)$ for the limit of the least supersolutions u_ε to the equation (E_ε). In order to study in more depth the limit free boundary problem, we will restrict ourselves to deal with the special case where equation (E_ε) assumes the following form

$$(SE_\varepsilon) \qquad \Delta u = \beta_\varepsilon(u)F(\nabla u) \quad \text{in } \Omega$$

where F satisfies

$F-1)$ $F \in C^{0,1}(\mathbb{R}^N)$;

$F-2)$ $0 < F_{min} \leq F(p) \leq F_{max} < \infty \quad \forall p \in \mathbb{R}^N$;

and β satisfies the conditions in specified in [11], i.e,

$\beta-1)$ $\beta \in C^{0,1}(\mathbb{R})$;

$\beta-2)$ $\beta > 0$ in $(0,1)$ and support of β is $[0,1]$;

$\beta - 3)$ β is increasing in $[0, 1/2)$ and decreasing in $(1/2, 1]$;

$\beta - 4)$ $\displaystyle\int_0^1 \beta(s)ds := M > 0$;

and additionally,

$\beta - 5)$ $\beta(t) \geq B_0 t^+$ for all $t \leq 3/4$, where $B_0 > 0$.

Observe that from the condition $\beta - 2$, we conclude that there exists $\tau_0 > 0$ such that

$$\beta(t) \geq \frac{\tau_0}{F_{min}} \text{ for } t \in [1/4, 3/4] \tag{4.1.1}$$

and we define the following universal constant

$$A_0 := \frac{\tau_0}{3N} > 0 \tag{4.1.2}$$

As we pointed out in the introduction, the semilinear equations (SE_ε) have connections with the Prandtl-Batchelor free boundary problems as they were pointed out by Luis Caffarelli, David Jerison and Carlos Kenig in [15].

Remark 4.1.1. From the assumption $F - 1$, we can improve the regularity obtained in Theorem (2.1.3). Indeed, it follows from ([9], Theorem 8.1) or ([23], Theorem 5.20) that if v_ε is a continuous viscosity solution to (SE_ε), then v_ε is actually a classical solution of (SE_ε).

4.2 Blow-up Convergence Results

The presence of the gradient in the equations (SE_ε) does not affect rescaling properties (see remark (4.2.4), below). This way, the convergence of blow-up's and their compatibility condition proven in [19] and [17] are preserved. Since the proofs are a small variant of the original ones, they will be ommited.

Proposition 4.2.1 (Blow-up convergence - [19], Lemma 3.2)*.* Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a family of viscosity solutions to (SE_ε). Assume for a subsequence $\varepsilon_j \to 0$, $v_{\varepsilon_j} \to v$ uniformly in compact subsets of Ω. Let $x_0, x_n \in \Omega \cap \partial\{v > 0\}$ be such that $x_n \to x_0$ as $k \to \infty$. Let $\lambda_n \to 0, v_{\lambda_n}(x) = (1/\lambda_n)v(x_n + \lambda_n x)$ and $(v_{\varepsilon_j})_{\lambda_n}(x) = (1/\lambda_n)v_{\varepsilon_j}(x_n + \lambda_n x)$. Suppose, that $v_{\lambda_n} \to V$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^N$. Then, there exists $j(n) \to \infty$ such that for every $j_n \geq j(n)$ there holds that $\varepsilon_{j_n}/\lambda_n \to 0$, and

i) $(v_{\varepsilon_{j_n}})_{\lambda_n} \to V$ uniformly in compact subsets of $\mathbb{R}^N$;

ii) $\nabla(v_{\varepsilon_{j_n}})_{\lambda_n} \to \nabla V$ in $L^2_{loc}(\mathbb{R}^N)$;

iii) $\nabla v_{\lambda_n} \to \nabla V$ in $L^2_{loc}(\mathbb{R}^N)$.

Proposition 4.2.2 (Blow-up compatibility condition - [17], Lemma 3.1)*.* Let $\{v_\varepsilon\}_{\varepsilon>0}$ be a family of viscosity solutions to (SE_ε). Assume for a subsequence $\varepsilon_j \to 0$, $v_{\varepsilon_j} \to v$ uniformly in compact subsets of Ω. Let $x_0 \in F(v)$ and, for $\lambda > 0$, let $v_\lambda(x) = \frac{1}{\lambda}v(x_0 + \lambda x)$. Let $\lambda_n \to 0$ and $\widetilde{\lambda}_n \to 0$ be such that

$$v_{\lambda_n} \to V = \alpha x_1^+ - \gamma x_1^- + o(|x|),$$

$$v_{\widetilde{\lambda_n}} \to \widetilde{V} = \widetilde{\alpha} x_1^+ - \widetilde{\gamma} x_1^- + o(|x|),$$

uniformly in compact sets of $\mathbb{R}^N$, with $\alpha, \widetilde{\alpha}, \gamma, \widetilde{\gamma} \geq 0$. Then $\alpha\gamma = \widetilde{\alpha}\widetilde{\gamma}$.

Definition 4.2.3. A continuous family $\{v_\varepsilon\}_{\varepsilon>0}$ of viscosity solutions to (SE_ε) is said to be a family of least viscosity supersolutions to (SE_ε) in Ω if for every open set $V \subset\subset \Omega$, we have for every $\varepsilon > 0$

$$v_\varepsilon \mid V = \omega_\varepsilon^V$$

where

$$w_\varepsilon^V(x) := \inf_{w \in \mathcal{S}_\varepsilon(V)} w(x)$$

$$\mathcal{S}_\varepsilon(V) := \left\{ w \in C^0(\overline{V}), w \text{ viscosity supersolution of } (SE_\varepsilon); w \geq v_\varepsilon \text{ on } \partial V \right\}$$

Clearly, proceeding by Perron's method, as in Theorem (2.1.3), ω_ε^V is a continuous viscosity solution of (SE_ε) in V. It follows directly from the Theory developed in Theorem (2.1.3) that $\{u_\varepsilon\}_{\varepsilon>0}$ is a family of least viscosity supersolutions of (SE_ε).

Remark 4.2.4. (Transformations that preserves (SE_ε))

i) (Rescaling) Assume that v is a solution to (SE_ε) in Ω. If $x_0 \in \Omega$ and $\lambda > 0$, let $T^\lambda_{x_0}(x) := x_0 + \lambda x$ we define the open set $\Omega^\lambda_{x_0} = (T^\lambda_{x_0})^{-1}(\Omega) = \left\{x \in \mathbb{R}^N \mid x_0 + \lambda x \in \Omega\right\}$ and the function $(v_{x_0})_\lambda(x) := \frac{1}{\lambda} v(T^\lambda_{x_0}(x)) = \frac{1}{\lambda} v(x_0 + \lambda x)$. It is imediate that , $(v_{x_0})_\lambda$ is a solution in $\Omega^\lambda_{x_0}$ to

$$\Delta u = \beta_{\frac{\varepsilon}{\lambda}}(u) F(\nabla u)$$

Conversely, if w is a solution to $(SE_{\frac{\varepsilon}{\lambda}})$ in $\Omega^\lambda_{x_0}$, we define in Ω the function $(w_{x_0})^\lambda(y) := \lambda w((T^\lambda_{x_0})^{-1}(y)) = \lambda w(\frac{y - x_0}{\lambda})$. Again, it is clear that $(w_{x_0})^\lambda$ is a solution to (SE_ε).

This way, the correspondences $v \mapsto (v_{x_0})_\lambda$ and $w \mapsto (w_{x_0})^\lambda$ establish a bijection among solutions of (SE_ε) and $(SE)_{\frac{\varepsilon}{\lambda}}$. Since those maps preserve order, i.e, $v^1 \leq v^2 \Longrightarrow (v^1_{x_0})_\lambda \leq (v^2_{x_0})_\lambda$ and $w^1 \leq w^2 \Longrightarrow (w^1_{x_0})^\lambda \leq (w^2_{x_0})^\lambda$, we conclude: $\{v_\varepsilon\}_{\varepsilon > 0}$ is a family of least viscosity supersolution to (SE_ε) in Ω if and only if $\{((v_\varepsilon)_{x_0})_\lambda\}_{\varepsilon > 0}$ is a family of least viscosity solutions to $(SE)_{\frac{\varepsilon}{\lambda}}$ in $\Omega^\lambda_{x_0}$.

ii) (Invariance under translations) Since the equation (SE_ε) does not dependend on x, the equation is translation invariant, i.e, translations of solutions (subsolutions, supersolutions) $u, \quad v = u(\cdot + h), h \in \mathbb{R}^N$ are still solutions (subsolutions, supersolutions) respectively.

Chapter 5

Qualitative Results

5.1 Scalings of β

In this section, we will prove some results that will be used in a decisive way to obtain the classifications of global profiles later on. We will start with some definitons.

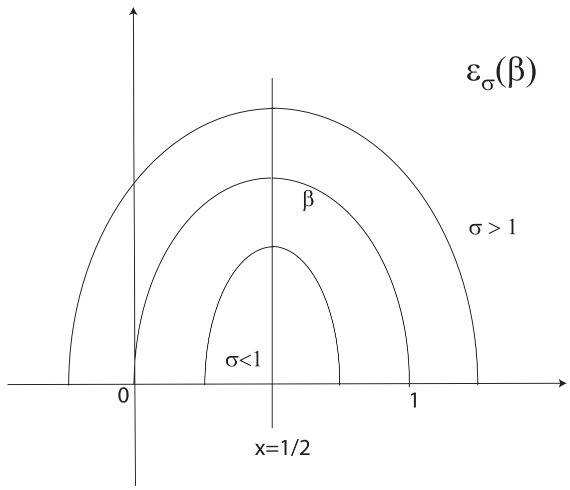
Definition 5.1.1. We set for $\sigma > 0$ the scaled function

$$\mathcal{E}_\sigma(\beta)(x) := \sigma\beta(\frac{x}{\sigma} - \frac{1}{2\sigma} + \frac{1}{2}) \tag{5.1.1}$$

Geometrically, the graph of $\mathcal{E}_\sigma(\beta)$ corresponds to a $\sigma-$ rescaling of the graph of β with respect to $x = \frac{1}{2}$. So, supp $(\mathcal{E}_\sigma(\beta)) = [\kappa_\sigma^-, \kappa_\sigma^+]$, where, $\kappa_\sigma^- := \frac{1}{2} - \frac{\sigma}{2}$ and $\kappa_\sigma^+ := \frac{1}{2} + \frac{\sigma}{2}$. Also, for any $\sigma > 0$, $\mathcal{E}_\sigma(\beta) \in C^{0,1}(\mathbb{R})$ with $Lip(\mathcal{E}_\sigma(\beta)) = Lip(\beta)$. By $\beta - 3)$, it is easy to verify that

$$0 < \sigma < 1 \Longrightarrow \mathcal{E}_\sigma(\beta)(t) < \beta(t) \quad \text{ for } t \in \text{supp}(\mathcal{E}_\sigma(\beta))$$

$$\sigma > 1 \Longrightarrow \mathcal{E}_\sigma(\beta)(t) > \beta(t) \quad \text{ for } t \in [0,1] = \text{supp}(\beta)$$



Scalings of β

Moreover, from the relation

$$\sigma_1, \sigma_2 > 0 \Longrightarrow \mathcal{E}_{\frac{\sigma_2}{\sigma_1}}(\mathcal{E}_{\sigma_1}(\beta)) = \mathcal{E}_{\sigma_2}(\beta)$$

It follows that

$$0 < \sigma_1 < \sigma_2 \Longrightarrow \mathcal{E}_{\sigma_1}(\beta)(t) < \mathcal{E}_{\sigma_2}(\beta)(t) \quad \text{for } t \in \text{supp}(\mathcal{E}_{\sigma_1}(\beta)) \tag{5.1.2}$$

We set,

$$M_\sigma := \int_{\kappa_\sigma^-}^{\kappa_\sigma^+} \mathcal{E}_\sigma(\beta)(t)dt = \sigma^2 M \tag{5.1.3}$$

As usual, we use the same notation for the $\varepsilon-$rescaling, i.e,

$$(\mathcal{E}_\sigma(\beta))_\varepsilon(t) = \frac{1}{\varepsilon}\mathcal{E}_\sigma(\beta)(\frac{t}{\varepsilon}).$$

Let us also define, for $|\mu| < F_{min}/2$ and $|\delta| < \frac{1}{2}$,

$$F_{\delta,\mu}(p) = (1+\delta)(F(p)+\mu) > \frac{\delta F_{min}}{4} > 0 \tag{5.1.4}$$

and finally, let $e_1 = (1, 0, ..., 0)$ be the first canonical vector in $\mathbb{R}^N$,

$$H_{\delta,\mu}(t) = \int_0^t \frac{s}{(F_{\delta,\mu}(se_1))} ds = \int_0^t \frac{s}{(1+\delta)(F(se_1)+\mu)} ds \tag{5.1.5}$$

We also denote,

$$H(t) = H_{0,0}(t) = \int_0^t \frac{s}{F(se_1)} ds \tag{5.1.6}$$

In the next Lemma, we show that the monotonicity relation in (5.1.2) still holds if we perturb $\mathcal{E}_\sigma(\beta)$ by a scaling factor close enough to 1.

Lemma 5.1.2. Assume $0 < \sigma_1 < \sigma_2$. If θ is close enough to 1, then for every $\varepsilon > 0$ we have the following inequalities

$$(\mathcal{E}_{\sigma_2}(\beta))_\varepsilon(t) \geq (\mathcal{E}_{\sigma_1}(\beta))_\varepsilon(\theta t) \quad \text{for all } t \in \mathbb{R} \tag{5.1.7}$$

$$(\mathcal{E}_{\sigma_2}(\beta))_\varepsilon(\theta t) \geq (\mathcal{E}_{\sigma_1}(\beta))_\varepsilon(t) \quad \text{for all } t \in \mathbb{R} \tag{5.1.8}$$

Proof. Clearly, by rescaling, it is enough to prove the Lemma for $\varepsilon = 1$. So, let us define the following functions in $\mathbb{R}$,

$$G_\theta(t) = \mathcal{E}_{\sigma_2}(\beta)(\theta t) - \mathcal{E}_{\sigma_1}(\beta)(t)$$

$$J_\theta(t) = \mathcal{E}_{\sigma_2}(\beta)(t) - \mathcal{E}_{\sigma_1}(\beta)(\theta t)$$

We will prove that $G_\theta, J_\theta \geq 0 \quad \forall t \in \mathbb{R}$.

Indeed, let $K \subset \mathbb{R}$ be a compact interval such that supp $\mathcal{E}_{\sigma_1}(\beta) \subset K \subset$ supp $\mathcal{E}_{\sigma_2}(\beta)$, where the inclusions are proper.

Setting $G(t) = \mathcal{E}_{\sigma_2}(\beta)(t) - \mathcal{E}_{\sigma_1}(\beta)$, since $\mathcal{E}_\sigma(\beta)(t)$ is Lipschitz continuous for $\sigma > 0$, we have $G_\theta \to G$ and $J_\theta \to G$ locally uniformly in compact subsets of $\mathbb{R}$. By, (5.1.2), $G > 0$ in K. In particular, by the uniform convergence, $G_\theta, J_\theta > 0$ in K for θ close enough to 1. By the other hand, clearly, $G_\theta(t) \geq 0$ for $t \notin K$. If $g_\theta(t) = \mathcal{E}_{\sigma_1}(\beta)(\theta t)$ then for θ close enough to 1, supp $g_\theta = \dfrac{1}{\theta}(\text{supp } \mathcal{E}_{\sigma_1}(\beta)) \subset K$ (proper inclusion) and thus, $J_\theta(t) \geq 0$ for $t \notin K$. This finishes the Lemma.

□

5.2 No Interior Contact Lemma

Now, we prove a Lemma that says essentially that if a "almost" strict subsoluton to (SE_ε) is below a supersolution to (SE_ε), then they cannot touch inside the domain. This Lemma will be used later on with the help of some barriers to prevent the slopes of the blow-up limits to have a "too closed" aperture.

Lemma 5.2.1 (No interior contact). Let $u_1, u_2 \in C^2(B_1) \cap C^0(\overline{B}_1)$ and $\sigma > 1$ such that

$$\Delta u_1 \leq \beta_\varepsilon(u_1) F(\nabla u_1) \text{ in } B_1$$

$$\Delta u_2 \geq (\mathcal{E}_\sigma(\beta))_\varepsilon(u_2) F(\nabla u_2) \text{ in } B_1$$

$$u_1 \geq u_2 \text{ in } B_1$$

Then, u_2 cannot touch u_1 in an interior point.

Proof. Let us prove the renormalized case $\varepsilon = 1$. The general case will follow analogously. So, let us assume, by contradiction, that u_2 touches u_1 by below at $x_0 \in B_1$. This way $\Delta u_2(x_0) \leq \Delta u_1(x_0)$. Moreover, since $\nabla u_1(x_0) = \nabla u_2(x_0)$ and $\mathcal{E}_\sigma(\beta) \geq \beta \;\; (\sigma > 1)$, we have the opposite inequality and thus

$$\Delta u_2(x_0) = \Delta u_1(x_0)$$

If we choose $1 < \overline{\sigma} < \sigma$, then $\beta \leq \mathcal{E}_{\overline{\sigma}} < \mathcal{E}_\sigma$ in supp $\mathcal{E}_{\overline{\sigma}} = [a, b]$, where $a < 0$ and $b > 1$. Thus, $c = u_1(x_0) = u_2(x_0) \notin [a, b]$. Let us suppose $c > b$. Consider $r = dist(x_0, \left\{u_1 \leq \frac{1+b}{2}\right\})$ and consider the convex set $A = \overline{B}_r(x_0) \cap \overline{B}_1$. Since u_1 is harmonic in A°, u_2 is subharmonic in A° and A° is connected, the strong maximum principle implies $u_1 \equiv u_2$ in A. In particular, $\nabla u_1 \equiv \nabla u_2$ and $\Delta u_1 \equiv \Delta u_2$ in A°. If x_1 is such that $r = |x_1 - x_0|$, then $u_1(x_1) = \frac{1+b}{2}$. This way, the segment $(x_1, x_0) \subset A^\circ$. In particular, by the mean value Theorem, we can find x_2 in the open segment, for which $\frac{1+b}{2} < u_1(x_2) = \frac{1+b}{2} + \frac{b-1}{8} = \overline{b} < b$.

This way, since $x_2 \in A^\circ$, we have $u_1(x_2) = \bar{b} = u_2(x_2), \nabla u_1(x_2) = p = \nabla u_2(x_2)$ and $\Delta u_1(x_2) = \Delta u_2(x_2)$. Thus,

$$\beta(\bar{b})F(p) = \Delta u_1(x_2) = \Delta u_2(x_2) = \mathcal{E}_\sigma(\beta)(\bar{b})F(p)$$

which implies, since $F > 0$, $\beta(\bar{b}) = \mathcal{E}_\sigma(\bar{b})$, a contraditcion since $\bar{b} \in (a, b)$. If $c \leq a$ we proceed similarly. So, u_2 never touches u_1 and the Lemma is proven.

□

5.3 Radially Symmetric Supersolution and Cubic Interior Decay

In the next Proposition, we contruct a radially symmetric supersolution to (SE_ε) where its value in a inner disk is much smaller that its value on the boundary. This will be used to prove that the least supersolution u_ε have some type of expontential decay inside the domain.

Proposition 5.3.1 (Radially symmetric supersolution). Given $\eta > 0$, there exists radially symmetric functions $\Theta_\varepsilon \in C^1(\mathbb{R}^N) \cap W^{2,\infty}_{loc}(\mathbb{R}^N)$ and universal constants $\kappa_2 > 0$ and $0 < \kappa_1 < 1$ such that

i) $\Theta_\varepsilon \equiv \dfrac{\varepsilon}{4}$ in $B_{\kappa_1 \eta}$

ii) $\Theta_\varepsilon \geq \kappa_2 \eta$ in $\mathbb{R}^N \setminus B_\eta$

iii) Θ_ε is a a viscosity supersolution to (SE_ε) for ε small enough.

Proof. We will work assuming first that $\varepsilon = 1$. After that, we will rescale the construction to obtain Θ_ε. Let $L \geq \dfrac{10}{\sqrt{2A_0}}$, we define,

$$\overline{\Theta}(r) = \begin{cases} 1/4, & \text{for } 0 \leq r \leq L \\ G(r) = A_0(r-L)^2 + 1/4 & \text{for } L \leq r \leq L + 1/\sqrt{2A_0} \\ \Gamma(r) & \text{for } r \geq L + 1/\sqrt{2A_0} \end{cases} \tag{5.3.1}$$

where Γ solves

$$\Gamma_{rr} + \frac{N-1}{r}\Gamma_r = 0 \text{ for } r \geq L + 1/\sqrt{2A_0} \tag{5.3.2}$$

$$\Gamma(L + 1/\sqrt{2A_0}) = 3/4, \quad \Gamma_r(L + 1/\sqrt{2A_0}) = \sqrt{2A_0}$$

Let us assume $N \geq 3$. Then

$$\Gamma(r) = 3/4 + \frac{\sqrt{2A_0}}{N-2}(L + 1/\sqrt{2A_0}) - \frac{\sqrt{2A_0}}{N-2}(L + 1/\sqrt{2A_0})^{N-1} r^{2-N} =$$

$$= K_L - f(r), \text{ respectively.}$$

This way,

$$\kappa^{2-N} < \frac{1}{2}\left(\frac{10}{11}\right)^{N-1} \Rightarrow \kappa^{2-N} < \left(\frac{10}{11}\right)^{N-1} \frac{1}{2L}\left(L + 1/\sqrt{2A_0}\right) \Longrightarrow$$

$$\Longrightarrow \frac{\sqrt{2A_0}}{N-2}\left(\frac{11}{10}\right)^{N-1} \kappa^{2-N} L \leq \frac{1}{2}\frac{\sqrt{2A_0}}{N-2}\left(L + 1/\sqrt{2A_0}\right)$$

Translating the inequality above in terms of K_L and $f(r)$, and recalling that $L > \dfrac{10}{\sqrt{2A_0}}$

$$f(\kappa_3 L) \leq \frac{1}{2} K_L \qquad \text{for } \kappa_3^{2-N} = \frac{1}{4}\left(\frac{10}{11}\right)^{N-1} < 1$$

In particular, since Γ is increasing, $\Gamma(r) > \frac{1}{2}K_L \geq \kappa_4 L$ for $r \geq \kappa_3 L$, where $\kappa_4 = \sqrt{2A_0}/2(N-2)L$. Finally, let us observe, that for $r \in (L, L + 1/\sqrt{2A_0})$, $1/4 \leq \Theta \leq 3/4$, so

$$\Theta_{rr} + \frac{N-1}{r}\Theta_r = G_{rr} + \frac{N-1}{r}G_r \leq 2A_0 N \leq \tau_0 \leq \beta(\Theta(r))F(\Theta_r(r)\frac{x}{|x|})$$

Thus, setting $\Theta(x) := \overline{\Theta}(|x|)$, by construction, $\Theta \in C^1(\mathbb{R}^N) \cap W^{2,\infty}_{loc}(\mathbb{R}^N)$ is a L^∞_{loc}-strong solution to the equation (i.e, it belongs to $W^{2,\infty}_{loc}(\mathbb{R}^N)$ and solves the equation a.e.)

$$\Delta u = \beta(u)F(\nabla u)$$

if $\varepsilon < \varepsilon_0 := \dfrac{\eta\sqrt{2A_0}}{10\kappa_3}$, we can find $L > \frac{10}{\sqrt{2A_0}}$ such that $\varepsilon = \frac{\eta}{\kappa_3 L}$ and set

$$\Theta_\varepsilon(x) := \varepsilon\Theta(\frac{x}{\varepsilon})$$

We see that $\Theta_\varepsilon \in C^1(\mathbb{R}^N) \cap W^{2,\infty}_{loc}(\mathbb{R}^N)$ and $i)$ and $ii)$ are satisfied with $k_1 = 1/\kappa_3$ and $\kappa_2 = \kappa_4/\kappa_3$. The fact the Θ_ε are viscosity solutions of (SE_ε) follows from Theorem (2.1) in [25] or more generally by the results in [16].

The case $N = 2$, where $\Gamma(r) = 3/4 + \sqrt{2A_0}(L + 1/\sqrt{2A_0})log(\frac{r}{L + \sqrt{2A_0}})$, is proven similarly. $\square$

We will prove a interesting geometric property of family of least supersolution to (SE_ε). Essentially, it says that if they are small in a certain domain, as soon as we get a little bit inside the domain, they become much smaller. In some sense, this decay is exponentially fast in ε as further inside we go into the domain. For our purposes, it is enough to show that the decay is cubic. This is the content of next Proposition, where we use the notation

$$Q_r = \left\{(x_1, x^{'}) \in \mathbb{R}^N; |x_1| \leq r, |x^{'}| \leq r\right\}$$

Proposition 5.3.2 (Cubic decay inside). Suppose $\{v_\varepsilon\}_{\varepsilon>0}$ is a family of least supersolutions of (SE_ε) and that for some $\eta > 0$ (small), $||v_\varepsilon^+||_{L^\infty(Q_1)} < \kappa_2\eta$. Then, there exist a constant $C_\eta > 0$ depending on η such that

$$v_\varepsilon^+(x) \leq C_\eta\varepsilon^3 \text{ for all } x \in Q_{1-2\eta} \text{ and } \varepsilon \text{ small enough.}$$

Proof. Indeed, if $x_0 \in Q_{1-\eta}$, $B_\eta(x_0) \subset Q_1$. We can now place the radially symmetric barrier constructed in the previous Proposition (5.3.1) in this ball, and since v_ε is the least supersolution of (SE_ε), we conclude, $v_\varepsilon(x_0) \leq \frac{\varepsilon}{4}$. This way,

$$v_\varepsilon(x) \leq \frac{\varepsilon}{4} \text{ for all } x \in Q_{1-\eta}$$

Let us denote by G_x the positive Green's function of the ball $B_\eta(x)$. If $x_1 \in Q_{1-2\eta}$, $\overline{B}_\eta(x_1) \subset Q_{1-\eta}$. Using the Green's representation formula

$$v_\varepsilon(x_1) = \fint_{\partial B_\eta(x_1)} v_\varepsilon d\mathcal{H}^{N-1} - \int_{B_\eta(x_1)} G_{x_1}(y)\Delta v_\varepsilon(y)dy$$

We have by property $\beta - 5)$,

$$\frac{F_{min}B_0}{\varepsilon^2}\left(\inf_{B_{\frac{\eta}{2}}(x_1)} G_{x_1}\right)\int_{B_{\eta/2}(x_1)} v_\varepsilon^+(y)dy \leq F_{min}\int_{B_{\eta/2}(x_1)} G_{x_1}(y)\beta_\varepsilon(v_\varepsilon(y)) \leq \frac{\varepsilon}{2} \tag{5.3.3}$$

Since v_ε^+ is subharmonic,

$$v_\varepsilon^+(x_1) \leq \fint_{B_{\eta/2}(x_1)} v_\varepsilon^+(y)dy \tag{5.3.4}$$

Recalling that $\inf_{B_{\eta/2}(x_1)} G_{x_1} = A_\eta$, where A_η is a universal constant depending on η and combining (5.3.3) and (5.3.4), we have

$$v_\varepsilon^+(x_1) \leq \frac{\varepsilon^3}{2F_{min}B_0A_\eta|B_{\eta/2}(x_1)|} = C_\eta\varepsilon^3$$

□

5.4 1-Dimensional Profiles

Finally, to end this section, we study the 1-dimensional profiles of our family of regularizing equations. This profiles will be modified in the next section, to create barriers with uniformly curved free boundaries.

Lemma 5.4.1 (1-dimensional profiles). Assume that $P \in C^2(\mathbb{R})$ is the unique solution of

$$u_{ss} = \mathcal{E}_\sigma(\beta)(u) F_{\delta,\mu}(u_s e_1) = (1+\delta)(\mathcal{E}_\sigma(\beta)(u))(F(u_s e_1) + \mu) \tag{5.4.1}$$

$$u(0) = \kappa_\sigma^+ \text{ and } u_s(0) = \alpha > 0$$

Then,

a) If $\gamma \geq 0$ and $H_{\delta,\mu}(\alpha) - H_{\delta,\mu}(\gamma) > M_\sigma$, there exist $\overline{\gamma} > \gamma$ and $\overline{s} < 0$ depending on $\alpha, \gamma, \delta, \sigma, \mu$ such that

$$P(s) = \begin{cases} \kappa_\sigma^+ + \alpha s, & s \geq 0 \\ \overline{\gamma}(s - \overline{s}) + \kappa_\sigma^-, & s \leq \overline{s}, \end{cases} \tag{5.4.2}$$

b) If $\gamma \geq 0$ with $H_{\delta,\mu}(\alpha) - H_{\delta,\mu}(\gamma) < M_\sigma$ we have two cases:

b.1) If $H_{\delta,\mu}(\alpha) > M_\sigma$, there exist $\overline{\gamma} < \gamma$ and $\overline{s} < 0$ depending on $\alpha, \gamma, \delta, \sigma, \mu$ such that

$$P(s) = \begin{cases} \kappa_\sigma^+ + \alpha s, & s \geq 0 \\ \overline{\gamma}(s - \overline{s}) + \kappa_\sigma, & s \leq \overline{s}, \end{cases} \tag{5.4.3}$$

or

b.2) If $H_{\delta,\mu}(\alpha) < M_\sigma$, there exist $\overline{\gamma} > 0$ and $\overline{s} < 0$ depending on $\alpha, \gamma, \delta, \sigma, \mu$ such that

$$P(s) = \begin{cases} \kappa_\sigma^+ + \alpha s, & s \geq 0 \\ \kappa_\sigma^+ - \overline{\gamma}(s - \overline{s}) & s \leq \overline{s}, \end{cases} \tag{5.4.4}$$

Moreover, in this case, there exists $\overline{\kappa}_\sigma$ such that $\kappa_\sigma^- < \overline{\kappa}_\sigma < P(s) < \kappa_\sigma^+$ for $\overline{s} < s < 0$. Furthermore, setting $P_\varepsilon(s) = \varepsilon P(\frac{s}{\varepsilon})$, it solves

$$(\mathcal{E}^\varepsilon_{\alpha,\delta,\mu,\sigma}) \qquad u_{ss} = (\mathcal{E}_\sigma(\beta))_\varepsilon(u) F_{\delta,\mu}(u_s e_1)$$

$$u(0) = \varepsilon\kappa_\sigma^+ \text{ and } u_s(0) = \alpha > 0$$

Proof. We start by observing that $H_{\delta,\mu}$ is a bijection from $[0, +\infty)$ over itself. This follows since $H_{\delta,\mu}(s) \geq \dfrac{s^2}{3F_{max}}$, and $(H_{\delta,\mu})_s > 0$ for $s > 0$. Multiplying the equation (5.4.1) by P_s we find,

$$(H_{\delta,\mu}(P_s))_s = B^\sigma(P)_s$$

where $B^\sigma(\zeta) = \int_{\kappa_\sigma^-}^{\zeta} \mathcal{E}_\sigma(\beta)(t)dt$. Integrating this equation, we obtain, in cases a) and $b.1$), for some $\overline{\gamma} > 0$,

$$H_{\delta,\mu}(P_s(s)) - B^\sigma(P(s)) = H_{\delta,\mu}(\alpha) - M_\sigma = H_{\delta,\mu}(\overline{\gamma}) > 0 \qquad (5.4.5)$$

This way, since from the expression above, $P_s \geq 0$

$$0 < \overline{\gamma} \leq P_s(s) \leq \alpha, \qquad \text{for } t \in \mathbb{R}$$

In case a), we have $H_{\delta,\mu}(\overline{\gamma}) > H_{\delta,\mu}(\gamma) \geq 0$ and so $\overline{\gamma} > \gamma$. In case b), $H_{\delta,\mu}(\overline{\gamma}) < H_{\delta,\mu}(\gamma)$, and thus $\overline{\gamma} < \gamma$. From the inequality (5.4.5) above, the conclusion of $a)$ and $b.1)$ is straightforward. To prove $b.2)$, we observe again, the relation (5.4.5), i.e,

$$H_{\delta,\mu}(P_s(s)) - B^{\sigma}(P(s)) = H_{\delta,\mu}(\alpha) - M_{\sigma} < 0 \tag{5.4.6}$$

Since $\overline{P}_{ss} \geq 0$, $\overline{P}_s$ is nondecreasing, thus, $P_s > 0$ in $s \geq 0$. This way, we conclude, $P(s) = \kappa_{\sigma}^{+} + \alpha s$ for $s \geq 0$. Observing that, $H_{\delta,\mu} \geq 0$, we see that relation (5.4.6), implies, in particular, $P > 0$ in $\mathbb{R}$. Actually, $\inf_{\mathbb{R}} P = \overline{\kappa}_{\sigma} > \kappa_{\sigma}^{-}$, otherwise, we could take a minimizing sequence s_n, (5.4.6) would provide

$$0 \leq H_{\delta,\mu}(P_s(s_n)) = B^{\sigma}(P(s_n)) + H(\alpha) - M$$

letting $n \to \infty$, we would obtain a contradiction. Our assetion will follow, if we can show that $lim_{s \to -\infty} P(s) = +\infty$. For this purpose, it is enough to show that there exists $b < 0$ such that $P_s(b) < 0$, since by convexity we have

$$P(s) \geq P(b) + P_s(b)(s - b) \quad \forall s \in \mathbb{R}$$

So, let us suppose by contradiction, that $P_s \geq 0$ for all $s \in \mathbb{R}$. This way, P is nondecreasing and thus $lim_{s \to -\infty} P(s) = \overline{\kappa}_{\sigma} > \kappa_{\sigma}^{-}$ and $\lim_{s \to -\infty} P_s(s) = 0$. Applying limit as $s \to -\infty$ in (5.4.6), we find

$$0 < B^{\sigma}(\overline{\kappa}_{\sigma}) = M_{\sigma} - H_{\delta,\mu}(\alpha) = \rho < M_{\sigma}$$

Since B^σ is invertible in $(\kappa_\sigma^-, \kappa_\sigma^+)$, we conclude that $P(s) \to \overline{\kappa}_\sigma \in (\kappa_\sigma^-, \kappa_\sigma^+)$ as $s \to -\infty$. In particular, for $\eta > 0$ small enough, $P(s) \in A_\eta = [\overline{\kappa}_\sigma - \eta, \overline{\kappa}_\sigma + \eta] \subset (\kappa_\sigma^-, \kappa_\sigma^+)$ for $s \leq c, c < 0$. This way, if $\tau = inf_{A_\eta} \mathcal{E}_\sigma(\beta)$, we have

$$P_{ss}(s) = (\mathcal{E}_\sigma(\beta))(P(s))F_{\delta,\mu}(P_s(s)e_1) \geq \tau \frac{F_{min}}{4} > 0 \text{ for all } s \leq c$$

But, this implies $P_s(c) = \infty$, clearly a contradiction since, $P \in C^2(\mathbb{R})$. This finishes $b.2$). □

Chapter 6

Barriers with Curved Free Boundaries

In this section, we will construct some barriers with uniformly curved free boundaries. The are essentially obtained by a uniform bending of the 1-dimensional profiles given by Lemma (5.4.1). The key tool used to acomplish this is a sequence of Kelvin transforms with respect to large spheres, i.e, spheres having centers and radii approaching infinity. These barriers will be the fundamental ingridient to classify global profiles (2-plane functions) in the next section.

Remark 6.0.2. For later reference, we will recall some facts about Inversion and Kelvin transform that will be used in the sequel. For $L > 0$, we denote

$$\mathbb{S}_L = \left\{ x \in \mathbb{R}^N ; |x + Le_1| = L \right\}$$

$$\mathbb{S}^\star_L = \left\{ x \in \mathbb{R}^N ; |x - Le_1| = L \right\}$$

The Kelvin transforms of a continuous function u with respect to $\mathbb{S}_L$ and $\mathbb{S}^\star_L$ are given, respectively, by K_L and T_L below

$$K_L[u](x) = (\rho_L(x))^{N-2}\, u(I_L(x)) \tag{6.0.1}$$

$$T_L[u](x) = (\varrho_L(x))^{N-2}\, u(J_L(x)) \tag{6.0.2}$$

where I_L, J_L are the inversions with respect to $\mathbb{S}_L$ and $\mathbb{S}_L^\star$, respectively, given by

$$I_L(x) = -Le_1 + \frac{L^2}{|x + Le_1|^2}(x + Le_1)$$

$$J_L(x) = Le_1 + \frac{L^2}{|x - Le_1|^2}(x - Le_1)$$

$$\rho_L(x) = \frac{L}{|x + Le_1|} \qquad \text{and} \qquad \varrho_L(x) = \frac{L}{|x - Le_1|}$$

it follows also that,

$$\Delta K_L[u](x) = (\rho_L(x))^{N+2}\, \Delta u(I_L(x)) \tag{6.0.3}$$

$$\Delta T_L[u](x) = (\varrho_L(x))^{N+2}\, \Delta u(J_L(x)) \tag{6.0.4}$$

Furthermore, if $\mathcal{R}_1$ is the orthogonal reflection with respect the hyperplane $\{x_1 = 0\}$, then for any $L_0 > 0$, and $L > L_0$ we have

$$\rho_L \to 1 \text{ in } C^1_{loc}(\mathbb{R}^N \setminus \{le_1; l \leq -10L_0\}) \tag{6.0.5}$$

$$\varrho_L \to 1 \text{ in } C^1_{loc}(\mathbb{R}^N \setminus \{le_1; l \geq 10L_0\}) \tag{6.0.6}$$

$$I_L \to \mathcal{R}_1 \text{ in } C^1_{loc}(\mathbb{R}^N \setminus \{le_1; l \leq -10L_0\}) \tag{6.0.7}$$

$$J_L \to \mathcal{R}_1 \text{ in } C^1_{loc}(\mathbb{R}^N \setminus \{le_1; l \geq 10L_0\}) \tag{6.0.8}$$

For more details about Inversions and Kelvin transforms, check ([1]) and ([2]).

In what follows, we use the cylinder for $L_0 > 0$,

$$Q_{L_0} = \left\{ x = (x_1, x') \in \mathbb{R}^N | \quad |x|_\infty = \max \left\{ |x_1|, |x^{'}| \right\} \leq 4L_0 \right\}$$

Proposition 6.0.3 (Above condition barrier). Suppose, $\overline{\sigma} > \sigma > 1$, $\delta, \mu > 0$ and $\overline{\alpha} > 0$, $\gamma \geq 0$ are such that $H_{\delta,\mu}(\overline{\alpha}) - H_{\delta,\mu}(\gamma) > M_{\overline{\sigma}}$. There exists $\vartheta_\varepsilon \in C^2(Q_{L_0})$ such that

a) $\Delta \vartheta_\varepsilon(x) \geq (\mathcal{E}_\sigma(\beta))_\varepsilon(\vartheta_\varepsilon(x)) F(\nabla \vartheta_\varepsilon(x))$ for $x \in Q_{L_0}$;

b) one has

$$\vartheta_\varepsilon < 0 \text{ in } Q_{L_0} \cap \mathbb{B}^C$$

$$\vartheta_\varepsilon > 0 \text{ in } Q_{L_0} \cap \mathbb{B}^\circ$$

$$\vartheta_\varepsilon = 0 \text{ on } Q_{L_0} \cap \mathbb{S}$$

$$\mathbb{S} \cap \partial Q_{L_0} \subset \{x_1 = d\}, \quad d = d(\text{radius of } \mathbb{S}, L_0) > 0$$

where $\mathbb{S} = \partial\mathbb{B}$, and $\mathbb{B}$ is a closed ball completely contained in the half space $\{x_1 \geq 0\}$, centered in the positive semi-axis generated by e_1 and tangent to the hyperplane $\{x_1 = 0\}$;

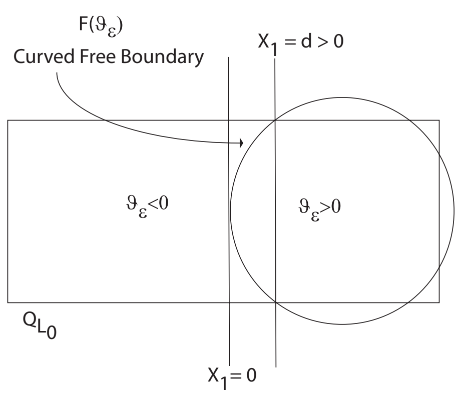
c) There exists $\widetilde{\alpha} > \overline{\alpha} > \widetilde{\gamma} > \gamma$ such that for $\mathcal{W}(x) = \widetilde{\alpha}x_1^+ - \widetilde{\gamma}x_1^-$, we have

$$\mathcal{W}(x) \geq \vartheta_\varepsilon(x) \text{ in } Q_{L_0} \text{ and } \mathcal{W}(0) = \vartheta_\varepsilon(0) \tag{6.0.9}$$

$$\mathcal{W}(x - de_1) \geq \vartheta_\varepsilon(x) \text{ for } x \in Q_{L_0} \cap \left\{ x = (x_1, x^{'}) \in \mathbb{R}^N; |x^{'}| = L_0 \right\} \tag{6.0.10}$$

$$Q_\varepsilon \leq \vartheta_\varepsilon \text{ along span}\,\{e_1\} \tag{6.0.11}$$

where $Q_\varepsilon(x) := \overline{Q}_\varepsilon(x_1)$ and $\overline{Q}_\varepsilon(s) := P_\varepsilon(s + a_\varepsilon)$ is the solution to $\mathcal{E}^\varepsilon_{\overline{\alpha},\delta,\mu,\sigma}$ and a_ε is chosen such that $\overline{Q}_\varepsilon(0) = 0$. Moreover, $\widetilde{\alpha}$ can be taken as close as we wish from α.



Curved Free Boundary of ϑ_ε

Proof. As suggested in c), let us define

$$Q_\varepsilon(x) := \overline{Q}_\varepsilon(x_1)$$

and recall that $\mathcal{R}_1$ denotes the reflection with respect the hyperplane $\{x_1 = 0\}$. Taking $L > 20L_0$, we set

$$\vartheta_\varepsilon^L(x) := (K_L[Q_\varepsilon] \circ \mathcal{R}_1)(x) = K_L[Q_\varepsilon](\mathcal{R}_1(x)) = (\overline{\rho}_L(x))^{N-2} Q_\varepsilon(\overline{I}_L(x)) \quad (6.0.12)$$

where

$$\overline{I}_L = I_L \circ \mathcal{R}_1, \quad \overline{\rho}_L = \rho \circ \mathcal{R}_1$$

By remark (6.0.2),

$$\Delta\vartheta_\varepsilon^L(x) = (\Delta K_L[Q_\varepsilon] \circ \mathcal{R}_1)(x) = \Delta K_L[Q_\varepsilon](\mathcal{R}_1(x)) = \quad (6.0.13)$$

$$= (\overline{\rho}_L(x))^{N+2} \Delta Q_\varepsilon(\overline{I}_L(x))$$

But

$$\Delta Q_\varepsilon(\overline{I}_L(x)) = (\mathcal{E}_{\overline{\sigma}}(\beta))_\varepsilon (Q_\varepsilon(\overline{I}_L(x)) F_{\delta,\mu}(\nabla Q_\varepsilon(\overline{I}_L(x)) = \quad (6.0.14)$$

$$= (\mathcal{E}_{\overline{\sigma}}(\beta))_\varepsilon((1/\rho_L(x))^{N-2}\vartheta_\varepsilon^L(x))F_{\delta,\mu}(\nabla\vartheta_\varepsilon^L(x) + A_L^\varepsilon(x))$$

where

$$A_L^\varepsilon(x) = \nabla Q_\varepsilon(\overline{I}_L(x)) - \nabla\vartheta_\varepsilon^L(x) =$$

$$= \nabla Q_\varepsilon(\overline{I}_L(x)) - \nabla[(\overline{\rho}_L(x))^{N-2}]Q_\varepsilon(\overline{I}_L(x)) - (\overline{\rho}_L(x))^{N-2}\nabla[Q_\varepsilon(\overline{I}_L(x))]$$

This way,

$$|\nabla Q_\varepsilon(\overline{I}_L(x)) - (\overline{\rho}_L(x))^{N-2}\nabla[Q_\varepsilon(\overline{I}_L(x))]| =$$

$$= \sup_{|v|=1}\left\langle \nabla Q_\varepsilon(\overline{I}_L(x)) - (\overline{\rho}_L(x))^{N-2}\nabla[Q_\varepsilon(\overline{I}_L(x))], v\right\rangle \leq$$

$$\leq |1 - (\overline{\rho}_L(x))^{N-2}| \cdot |\nabla Q_\varepsilon(\overline{I}_L(x))| +$$

$$+ |\overline{\rho}_L(x)|^{N-2}|\nabla Q_\varepsilon(\overline{I}_L(x))| \cdot ||Id_{\mathbb{R}^N} - D\overline{I}_L(x)||_{\mathcal{L}(\mathbb{R}^N)}$$

This way, since $Q_\varepsilon(\overline{I}_L(x))$ and $\nabla Q_\varepsilon(\overline{I}_L(x))$ are uniformly bounded in Q_{L_0} (recall Q_ε are translations of rescalings of P given in Lemma (5.4.1)) by (6.0.5) and (6.0.7)

$A_L^\varepsilon \to 0$ uniformly in Q_{L_0} as $L \to \infty$ uniformly in ε

Since F is Lipschitz continuous, we have for $x \in Q_{L_0}$ and L large enough

$$F(\nabla\vartheta_\varepsilon^L(x)+A_L^\varepsilon)+\mu \geq F(\nabla\vartheta_\varepsilon^L(x)+A_L^\varepsilon)+Lip(F)|A_L^\varepsilon| \geq F(\nabla\vartheta_\varepsilon^L(x)) \quad (6.0.15)$$

$$(1+\delta)(\overline{\rho}_L(x))^{N-2} \geq 1+\frac{\delta}{2} \quad (6.0.16)$$

Also, by Lemma (5.1.2), since $\overline{\sigma} > \sigma > 1$

$$(\mathcal{E}_{\overline{\sigma}}(\beta))_\varepsilon((1/\rho_L(x))^{N-2}\vartheta_\varepsilon^L(x)) \geq (\mathcal{E}_\sigma(\beta))_\varepsilon(\vartheta_\varepsilon^L(x))$$

Combining the estimates above, we conclude that choosing L large enough, for $x \in Q_{L_0}$ uniformly in ε

$$\Delta\vartheta_\varepsilon^L(x) \geq (1+\frac{\delta}{2})(\mathcal{E}_\sigma(\beta))_\varepsilon(\vartheta_\varepsilon^L(x))F(\nabla\vartheta_\varepsilon^L(x)) \geq \quad (6.0.17)$$

$$\geq (\mathcal{E}_\sigma(\beta))_\varepsilon(\vartheta_\varepsilon^L(x))F(\nabla\vartheta_\varepsilon^L(x))$$

It follows from the proof of Lemma (5.4.1)a) that there exists $\overline{\gamma} > \gamma$ such that

$$\overline{\gamma} < (\overline{Q}_\varepsilon)_s < \overline{\alpha} \text{ with } \overline{Q}_\varepsilon(0) = 0$$

We can easily check that the following properties below hold

1. $\overline{Q}_\varepsilon(s) \le \overline{\gamma} s$ for $s \in (-\infty, 0]$ and $\overline{Q}_\varepsilon(s) \le \overline{\alpha} s$ for $s \in [0, \infty)$;

2. $x \in Q_{L_0} \Rightarrow (\overline{I}_L(x))_1 \le -L + \frac{L^2}{L - x_1} =: \tau_L(x_1)$ with

$$\tau_L \ge 0 \text{ in } \{x_1 \ge 0\} \text{ and } \tau_L \le 0 \text{ in } \{x_1 \le 0\};$$

3. If $\tau > 0$ is a small number, for L large enough, we have

$$1 - \tau \le \overline{\rho}_L \le 1 + \tau \text{ in } Q_{L_0}$$

$$1 - \tau \le \frac{d}{dx_1}\tau_L \le 1 + \tau \text{ in } [-4L_0, 4L_0]$$

Since,

$$\frac{d}{dx_1}\tau_L(x_1) = \frac{L^2}{(L - x_1)^2} \to 1 \text{ uniformly in } [-4L_0, 4L_0]$$

From these, it is easy to observe the following estimates

For $x \in \{x_1 \le 0\} \cap Q_{L_0}$,

$$\vartheta_\varepsilon^L(x) = (\overline{\rho}_L(x))^{N-2} Q_\varepsilon(\overline{I}_L(x)) \le (1 - \tau)^{N-2}\overline{Q}_\varepsilon((\overline{I}_L(x))_1) \le$$

$$\leq (1-\tau)^{N-2}\overline{Q}_{\varepsilon}(\tau_L(x_1)) \leq$$

$$\leq (1-\tau)^{N-1}\overline{\gamma}x_1 = -\widetilde{\gamma}x_1^-$$

Similarly, for $x \in \{x_1 \geq 0\} \cap Q_{L_0}$

$$\vartheta_{\varepsilon}^L(x) = (\overline{\rho}_L(x))^{N-2}\overline{Q}_{\varepsilon}((\overline{I}_L(x))_1) \leq (1+\tau)^{N-2}\overline{Q}_{\varepsilon}(\tau_L(x_1)) \leq$$

$$(1+\tau)^{N-1}\overline{\alpha}x_1 = \widetilde{\alpha}x_1^+$$

We can use also, similar ideas, to obtain estimates along the boundary. In this case, the estimates will be 1-dimensional. Indeed, if $x \in Q_{L_0} \cap \left\{x = (x_1, x^{'}) \in \mathbb{R}^N; |x^{'}| = L_0\right\}$ then

$$\vartheta_{\varepsilon}(x) = (\widetilde{\rho}_L(x_1))^{N-2}Q_{\varepsilon}(\overline{I}_L(x)) = (\widetilde{\rho}_L(x_1))^{N-2}\overline{Q}_{\varepsilon}(\varphi_L(x_1))$$

where

$$\widetilde{\rho}_L(x_1) = \frac{L}{\sqrt{(L-x_1)^2 + L_0^2}}$$

and

$$\varphi_L(x_1) := (\widetilde{I}_L(x))_1 = -L + \frac{L^2}{(L-x_1)^2 + L_0^2}(L-x_1)$$

Now, let us observe that φ_L has the following properties,

$$\frac{d\varphi_L}{dx_1}(x_1) = \frac{L^2((L-x_1)^2 - L_0^2)}{[(L-x_1)^2 + L_0^2]^2} \to 1 \text{ uniformly in } [-4L_0, 4L_0]$$

$$\varphi_L(x_1) = 0 \Longleftrightarrow x_1 = d := \frac{L - \sqrt{L^2 - 4L_0^2}}{2} > 0$$

$$\varphi_L \geq 0 \text{ in } [-4L_0, d], \qquad \varphi_L \leq \text{ in } [d, 4L_0]$$

Also,

$$\left\{\vartheta_\varepsilon^L(x) = 0\right\} \cap Q_{L_0} \cap \left\{x \in Q_{L_0}; |x^{'}| = L_0\right\} \Longleftrightarrow x_1 = d$$

(using that $g(x) = \sqrt{x}$ is Lipschitz away from the origin, we can easily estimate $d < \frac{L_0}{10}$). If $\tau > 0$ is a small enough, again for L large enough,

$$1 - \tau \leq \frac{d\varphi_L}{dx_1}(x_1) \leq 1 + \tau \text{ for } x_1 \in [-4L_0, 4L_0];$$

and

$$1 - \tau \leq \widetilde{\rho}_L(x_1) \leq 1 + \tau \text{ for } x_1 \in [-4L_0, 4L_0]$$

This way, we have for $x \in Q_{L_0} \cap \left\{x = (x_1, x^{'}) \in \mathbb{R}^N; |x^{'}| = L_0\right\} \cap \{x_1 \geq d\}$

$$\vartheta_\varepsilon^L(x) \leq (1+\tau)^{N-2} \overline{Q}_\varepsilon(\varphi_L(x_1)) \leq (1+\tau)^{N-2} \overline{\alpha} \varphi_L(x_1) \leq$$

$$\leq (1+\tau)^{N-1}\overline{\alpha}(x_1-d)=\widetilde{\alpha}(x_1-d)^+$$

Similarly,

$$x\in Q_{L_0}\cap\left\{x=(x_1,x^{'})\in\mathbb{R}^N;|x^{'}|=L_0\right\}\cap\{x_1\leq d\}\Rightarrow$$

$$\Rightarrow \vartheta_\varepsilon^L(x)\leq -(1-\tau)^{N-1}\overline{\gamma}(d-x_1)=-\widetilde{\gamma}(x_1-d)^-$$

The fact that $Q_\varepsilon\leq\vartheta_\varepsilon^L$ along *span* $\{e_1\}$ is straightforward. If we choose now $\overline{L}$ large enough in such way that all the estimates above holds, we define for every ε

$$\vartheta_\varepsilon:=\vartheta_\varepsilon^{\overline{L}}$$

Thus, *a*) and *c*) are proven. *b*) follows from the geometric properties of inversions. □

Proposition 6.0.4 (Below condition barrier - I). $0<\overline{\sigma}<\sigma<1$, $\delta,\mu<0$ and $\overline{\alpha},\gamma>0$ be such that

$$0<H_{\delta,\mu}(\overline{\alpha})-M_{\overline{\sigma}}<H_{\delta,\mu}(\gamma)$$

Let $0<\alpha^\star<\overline{\alpha}$ be close to $\overline{\alpha}$. There exists a function $\chi_\varepsilon\in C^2(Q_{L_0})$ such that for every $\varepsilon>0$

a) $\Delta\chi_\varepsilon(x)\leq\beta_\varepsilon(\chi_\varepsilon(x))F(\nabla\chi_\varepsilon(x))$ for x in Q_{L_0};

b) one has

$$\chi_\varepsilon > 0 \text{ in } Q_{L_0} \cap {\mathbb{B}_\star}^C$$

$$\chi_\varepsilon < 0 \text{ in } Q_{L_0} \cap \mathbb{B}_\star^\circ$$

$$\chi_\varepsilon = 0 \text{ on } Q_{L_0} \cap \mathbb{S}_\star$$

$$\mathbb{S}_\star \cap \partial Q_{L_0} \subset \{x_1 = d_\star\}, \quad d_\star = d_\star(\text{radius of } \mathbb{S}_\star, L_0) < 0$$

where $\mathbb{S}_\star = \partial\mathbb{B}_\star$, and $\mathbb{B}_\star$ is a closed ball completely contained in the half space $\{x_1 \leq 0\}$, centered in the negative semi-axis generated by e_1 and tangent to the hyperplane $\{x_1 = 0\}$;

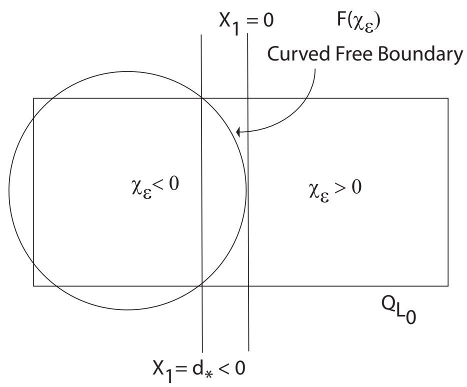
c) There exist $0 < \widetilde{\alpha} < \alpha^\star$ and $0 < \widetilde{\gamma} < \gamma$ and constants $C, D > 0$ not depending on ε such that if $\mathcal{W}^\star(x) = \widetilde{\alpha}x_1^+ - \widetilde{\gamma}x_1^-$, then

$$\mathcal{W}_\varepsilon^\star(x) := \mathcal{W}^\star(x - \varepsilon De_1) + C\varepsilon \leq \chi_\varepsilon(x) \text{ for all } x \in Q_{L_0} \tag{6.0.18}$$

$$\mathcal{W}^\star(x+(d_\star-\varepsilon D)e_1) \leq \chi_\varepsilon(x) \text{ for } x \in Q_{L_0}\cap\Big\{x = (x_1, x') \in \mathbb{R}^N; |x'| = L_0\Big\} \tag{6.0.19}$$

$$Q_\varepsilon \geq \chi_\varepsilon \text{ along } \quad \text{span}\,\{e_1\} \tag{6.0.20}$$

where $Q_\varepsilon(x) := \overline{Q}_\varepsilon(x_1)$, and $\overline{Q}_\varepsilon(s) = P_\varepsilon(s + a_\varepsilon)$, P_ε is the solution $(\mathcal{E}^{\varepsilon}_{\widetilde{\alpha},\delta,\mu,\sigma})$ where a_ε is chosen such that $\overline{Q}_\varepsilon(0) = 0$. Moreover, $\widetilde{\alpha}$ can be taken as close as we wish from $\alpha^\star$.



Curved Free Boundary of χ_ε

Proof. The proof is very similar to the proof of Proposition(6.0.3). As suggested in c), if we define

$$Q_\varepsilon(x) := \overline{Q}_\varepsilon(x_1)$$

and for $\overline{J}_L = J_L \circ \mathcal{R}_1$ and $\overline{\varrho}_L = \varrho \circ \mathcal{R}_1$ we set

$$\chi^L_\varepsilon(x) := (T_L[Q_\varepsilon] \circ \mathcal{R}_1)(x) = T_L[Q_\varepsilon](\mathcal{R}_1(x)) = (\overline{\varrho}_L(x))^{N-2} Q_\varepsilon(\overline{J}_L(x)) \quad (6.0.21)$$

with

$$\Delta\chi^L_\varepsilon(x) = (\Delta T_L[Q_\varepsilon] \circ \mathcal{R}_1)(x) = \Delta T_L[Q_\varepsilon](\mathcal{R}_1(x)) = \quad (6.0.22)$$

$$= (\overline{\varrho}_L(x))^{N+2}\Delta Q_\varepsilon(\overline{J}_L(x)) =$$

$$= (\overline{\varrho}_L(x))^{N+2}(\mathcal{E}_{\overline{\sigma}}(\beta))_\varepsilon((1/\varrho_L(x))^{N-2}\chi_\varepsilon^L(x))F_{\delta,\mu}(\nabla\chi_\varepsilon^L(x) + \overline{A}_L^\varepsilon(x))$$

where,

$$\overline{A}_L^\varepsilon(x) = \nabla Q_\varepsilon(\overline{J}_L(x)) - \nabla\chi_\varepsilon^L(x)$$

Proceeding as in the proof of Proposition (6.0.3), we obtain

$$\overline{A}_L^\varepsilon \to 0 \text{ uniformly in } Q_{L_0} \text{ as } L \to \infty \text{ uniformly in } \varepsilon$$

Since $\delta, \mu < 0$, for $x \in Q_{L_0}$ and $L > 20L_0$ large enough

$$F(\nabla\chi_\varepsilon^L(x)+\overline{A}_L^\varepsilon)+\mu \leq F(\nabla\chi_\varepsilon^L(x)+A_L^\varepsilon)-Lip(F)|A_L^\varepsilon| \leq F(\nabla\chi_\varepsilon^L(x)) \quad (6.0.23)$$

$$(1+\delta)(\overline{\varrho}_L(x))^{N-2} \leq 1+\frac{\delta}{2} < 1 \quad (6.0.24)$$

by Lemma (5.1.2), if $\overline{\sigma} < \sigma < 1$,

$$(\mathcal{E}_{\overline{\sigma}}(\beta))_\varepsilon((1/\varrho_L(x))^{N-2}\chi_\varepsilon^L(x)) \leq (\mathcal{E}_\sigma(\beta))_\varepsilon(\chi_\varepsilon^L(x))$$

and thus for L large enough and for $x \in Q_{L_0}$,

$$\Delta\chi_\varepsilon^L(x) \leq (1+\frac{\delta}{2})(\mathcal{E}_\sigma(\beta))_\varepsilon(\chi_\varepsilon^L(x))F(\nabla\chi_\varepsilon^L(x)) \leq$$

$$(\mathcal{E}_\sigma(\beta))_\varepsilon(\chi_\varepsilon^L(x))F(\nabla\chi_\varepsilon^L(x)) \leq \beta_\varepsilon(\chi_\varepsilon^L(x))F(\nabla\chi_\varepsilon^L(x))$$

Analogously to the proof of Proposition (6.0.3), by Lemma (5.4.1)b.1), there exists $0 < \overline{\gamma} < \gamma$ such that

$$\overline{\gamma} \leq (\overline{Q}_\varepsilon)_s \leq \overline{\alpha}$$

It is easy to check properties below

1) There exists a constant $\overline{D}$ such that

$$\begin{cases} \overline{Q}_\varepsilon(s) \geq \alpha^\star s, & \text{for } s \geq \overline{D}\varepsilon \\ \overline{Q}_\varepsilon(s) = \overline{\gamma}s, & \text{for } s \leq 0 \\ \overline{Q}_\varepsilon(s) \geq \overline{\gamma}s, & \text{for every } s \end{cases} \tag{6.0.25}$$

2) $x \in Q_{L_0} \Rightarrow (J_L(x))_1 \geq L - \frac{L^2}{L+x_1} := \tau_L^\star(x_1)$, with

$$\tau_L^\star \geq 0 \text{ in } \{x_1 \geq 0\} \quad \textit{and} \quad \tau_L^\star \leq 0 \text{ in } \{x_1 \leq 0\}$$

3) If $\tau > 0$ is a small number, for L large enough we have

$$1 - \tau \leq \overline{\varrho}_L \leq 1 + \tau \text{ in } Q_{L_0}$$
$$1 - \tau \leq \frac{d}{dx_1}\tau_L^\star \leq 1 + \tau \text{ in } [-4L_0, 4L_0]$$

From (3), there exists D such that $x_1 \geq D\varepsilon \Rightarrow \tau_L^\star(x_1) \geq \overline{D}\varepsilon$, and thus,

$$\left\{x_1 \geq \overline{D}\varepsilon\right\} \cap Q_{L_0} \Rightarrow \chi_\varepsilon^L(x) = (\varrho_L(x))^{N-2}\overline{Q}_\varepsilon((\overline{J}_L(x))_1) \geq$$

$$\geq (1-\tau)^{N-2}\overline{Q}_\varepsilon(\tau_L^\star(x_1)) \geq (1-\tau)^{N-1}\alpha^\star x_1$$

Proceeding similarly, we find

$$x_1 \leq 0 \Rightarrow \chi_\varepsilon^L(x) \geq -(1+\tau)^{N-1}\overline{\gamma}x_1^- = -\widetilde{\gamma}x_1^-$$

$$0 \leq x_1 \leq \overline{D}\varepsilon \Rightarrow \chi_\varepsilon^L(x) \geq (1-\tau)^{N-1}\overline{\gamma}x_1 = \gamma^\star x_1^+$$

Setting $C = \gamma^\star D$, it follows that

$$\mathcal{W}_\varepsilon^\star(x) = \mathcal{W}^\star(x - D\varepsilon) + C\varepsilon \leq \chi_\varepsilon(x) \text{ for all } x \in Q_{L_0}$$

Following the ideas above and proceeding as in the proof of Proposition (6.0.3), we finish this proof.

□

Proposition 6.0.5 (Below condition barrier - II). Let $0 < \sigma < \overline{\sigma} < 1$ and $\delta, \mu < 0$ with $\overline{\alpha} > 0$ such that

$$H_{\delta,\mu}(\overline{\alpha}) < M_{\overline{\sigma}}$$

Then, there exist a function $\chi_\varepsilon \in C^2(Q_{L_0})$ and constants $C, D > 0$ (independent of ε) satisfying for every ε

a) $\Delta\chi_\varepsilon(x) \le \beta_\varepsilon(\chi_\varepsilon(x))F(\nabla\chi_\varepsilon(x))$ for x in Q_{L_0};

b) $\chi_\varepsilon \ge C\varepsilon$ in Q_{L_0} and $\chi_\varepsilon \le Q_\varepsilon$ for $\{x_1 \ge 0\}$; where $Q_\varepsilon(x) := P_\varepsilon(x_1)$, P_ε solution to $(\mathcal{E}^{\varepsilon}_{\overline{\alpha},\delta,\mu,\sigma})$;

c) There exists $0 < \widetilde{\alpha} < \overline{\alpha}$ and a constant $C > 0$ independent of ε such that

$$\chi_\varepsilon \ge \widetilde{\alpha}x_1^+ + D\varepsilon \text{ for } x \in Q_{L_0} \cap \{x_1 \ge 0\}$$

Moreover, $\widetilde{\alpha}$ can be taken as close as we wish from $\overline{\alpha}$.

d) There exists a negative number $d_\star$ independent of ε such that on

$$\chi_\varepsilon(x) \to g(x_1) \text{ uniformly on } \left\{x = (x_1, x^{'}) \in Q_{L_0}; |x^{'}| = L_0\right\}$$

and

$$g(x_1) \ge \widetilde{\alpha}(x_1 - d_\star) \qquad \text{for } x_1 \ge d_\star$$

Proof. Defining $\chi_\varepsilon^L(x) = (T_L[Q_\varepsilon] \circ \mathcal{R}_1)(x)$ as in Proposition (6.0.4), where $Q_\varepsilon(x)$ is specified above, then, for L large enough,

$$\Delta\chi_\varepsilon^L(x) \le \beta_\varepsilon(\chi_\varepsilon(x))F(\nabla\chi_\varepsilon(x)) \text{ for } x \text{ in } Q_{L_0}$$

But now, by Lemma (5.4.1)$b.2$), we have for $C = (1-\tau)^{N-2}\overline{\kappa}_{\overline{\sigma}}$

$$\chi_\varepsilon^L(x) = (\varrho_L(x))^{N-2}\overline{Q}_\varepsilon((\overline{J}_L(x))_1) \geq C\varepsilon \quad \forall x \in Q_{L_0}$$

and also, for $D = (1-\tau)^{N-2}\kappa_{\overline{\sigma}}^+$ and $x \in Q_{L_0} \cap \{x_1 \geq 0\}$

$$\chi_\varepsilon^L(x) \geq (1-\tau)^{N-2}\overline{Q}_\varepsilon(\tau_L^\star(x_1)) \geq (1-\tau)^{N-1}\alpha x_1^+ + D\varepsilon = \widetilde{\alpha}x_1^+ + D\varepsilon$$

Now, as before, we fix a universal L for which the estimates above hold uniformly in ε. From, Lemma (5.4.1)b.2), we conclude that for some $\overline{\gamma} > 0$

$$Q_\varepsilon \to P^\star(x) := \overline{\alpha}x_1^+ + \overline{\gamma}x_1^- \text{ uniformly in } \mathbb{R}^N$$

Since, Kelvin Transforms preserve uniform convergence, we have

$$\chi_\varepsilon \to T_L[P^\star] \circ \mathcal{R}_1 \text{ uniformly in } Q_{L_0} \text{ as } \varepsilon \to 0$$

In particular, for $x \in Q_{L_0} \cap \left\{x = (x_1, x^{'}) \in \mathbb{R}^N; |x^{'}| = L_0\right\}$

$$\chi_\varepsilon \to g(x_1) := (\widetilde{\varrho}_L(x_1))^{N-2}P^\star(\varphi_L^\star(x_1)) \text{ uniformly as } \varepsilon \to 0 \tag{6.0.26}$$

where,

$$\widetilde{\varrho}_L(x_1) = \frac{L}{\sqrt{(L+x_1)^2 + L_0^2}}$$

and

$$\varphi_L^\star(x_1) = L - \frac{L^2}{(L+x_1)^2 + L_0^2}(L + x_1)$$

Clearly,

$$x_1 \in [-4L_0, 4L_0] \text{ with } g(x_1) = 0 \Longleftrightarrow x_1 = d_\star := \frac{1}{2}(-L + \sqrt{L^2 - 4L_0^2}) < 0$$

L can be taken large enough such that if τ is a small number

$$\frac{d}{dx_1}\varphi_L^\star(x_1) \geq 1 - \tau \qquad \forall x \in Q_{L_0}$$

and thus,

$$\varphi_L^\star(x_1) \geq (1-\tau)(x_1 - d_\star) \qquad \forall x \in Q_{L_0}$$

So,

$$x_1 \geq d_\star \Rightarrow g(x_1) \geq (1-\tau)^{N-1}\overline{\alpha}(x_1 - d_\star) = \widetilde{\alpha}(x_1 - d_\star)$$

From the convergence, (6.0.26), d) follows, finishing the proof. □

Chapter 7

Classification of Global Profiles

7.1 Heuristic Considerations

In this section, we discuss in a heurestic and geometric way, the classification of global profiles for the equations (SE_ε). Essentially, the free boundary condition is dictated by the behaviour of 1-dimensional profiles of the ODE

$$u_{ss} = \beta_\varepsilon(u)F(u_s e_1)$$

Let us assume that we have a family of functions which are the least supersolutions $\{u_\varepsilon\}_{\varepsilon>0}$ to (SE_ε) and

$$u_\varepsilon \to \alpha x_1^+ - \gamma x_1^- \text{ locally uniformly in } \mathbb{R}^N \text{ as } \varepsilon \to 0$$

Then, we claim that

$$H(\alpha) - H(\gamma) = M$$

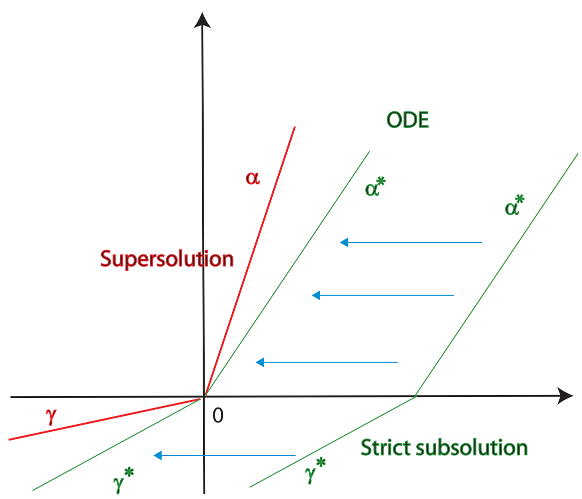
where

$$H(t) = \int_0^t \frac{s}{F(se_1)} ds$$

Indeed, the 2-plane function $P(x) = \alpha x_1^+ - \gamma x_1^-$ can be tought as the (least) supersolution of our free boundary problem since it is approximated by least supersolutions u_ε. If $H(\alpha) - H(\gamma) > M$, then we can find $\alpha^* < \alpha, \;\; \gamma^* > \gamma$ and $\sigma > 1$ such that $H(\alpha^*) - H(\gamma^*) > M_\sigma > M$. In particular, the 2-plane function $P_0(x) = \alpha^* x_1^+ - \gamma^* x_1^-$ is "more open than P" and it can be thought as "almost strict subsolution" of our limit free boundary problem since it can be approximated by solutions to ODE's of the form

$$u_{ss}(s) = \mathcal{E}_\sigma(\beta_\varepsilon(u))F(u_s e_1) > \beta_\varepsilon(u)F(u_s e_1)$$

But, if we "bring" P_0 from the right, starting at "infinity", the conditions on the slopes force "the strict subsolution P_0 " to touch from below "the supersolution P " which is a contradiction.



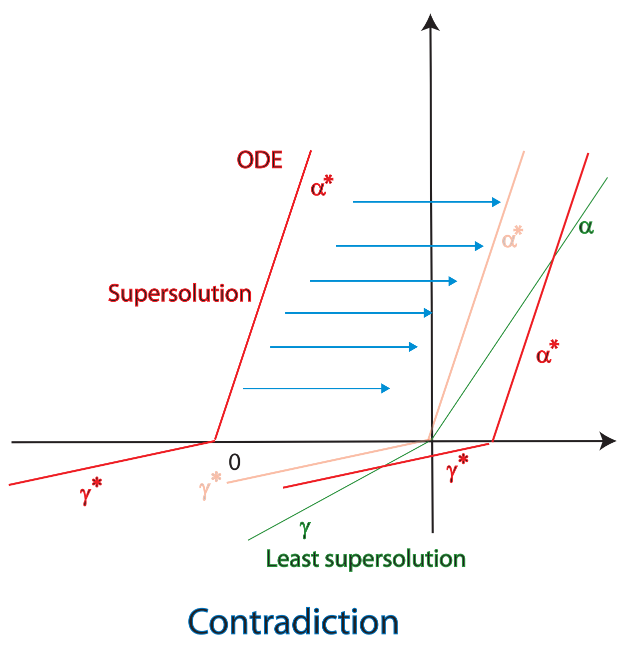
Contradiction

Touching by Below

Conversely, if $H(\alpha) - H(\gamma) < M$, then we can find $\alpha^* > \alpha$, $\gamma^* < \gamma$ and $\sigma < 1$ such that $H(\alpha^*) - H(\gamma^*) < M_\sigma < M$. Now however, $P_0(x) = \alpha^* x_1^+ - \gamma^* x_1^-$ is "more closed than P" and it can be tought as a "supersolution" since it can be approximated by solutions to ODE's of the form

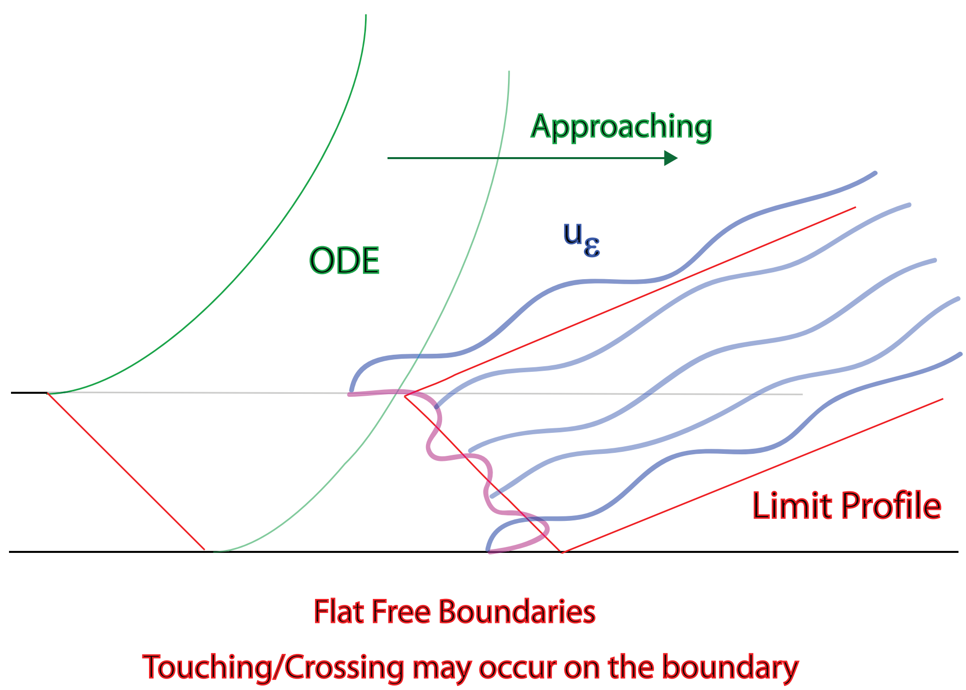
$$u_{ss}(s) = \mathcal{E}_\sigma(\beta_\varepsilon(u))F(u_s e_1) < \beta_\varepsilon(u)F(u_s e_1)$$

Now, if we "bring" P_0 from the left, starting at "infinity", once more, the conditions on the slopes force "the supersolution P_0" to touch from above "the supersolution P". However, if we translate a little bit further, P_0 will cross P inside, which is a contradiction, since P is supposed to be the least supersolution of the free boundary problem.



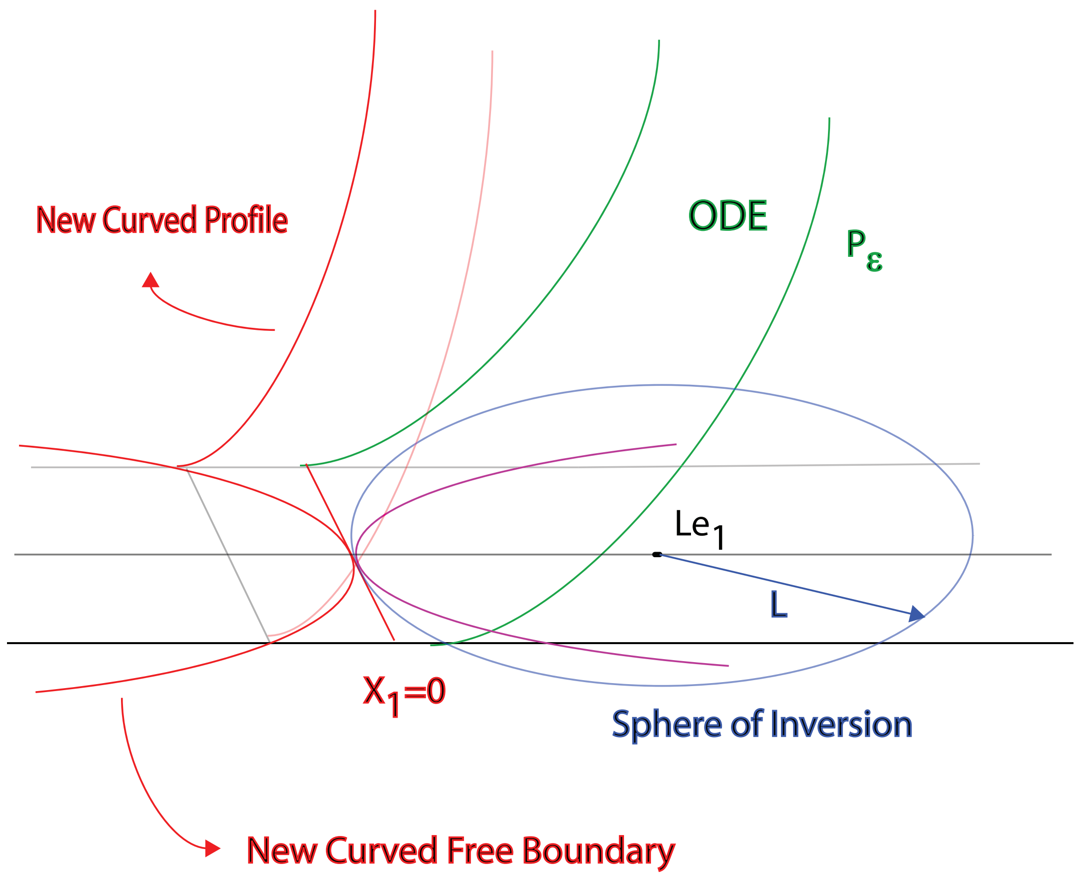
Crossing Inside

It is important to notice that, we need to garantee somehow that both, "the touching point" and "the crossing point" described heuristically above happens in the ε level (i.e, for the equation equation (SE_ε)) and also in the inteiror of the domain. This is a key point in the argument. Profiles of the ODE have flat free boundaries in higher dimensions, and so there is no guarantee that the touching will not happen on the boundary, as the picture below schematically indicates

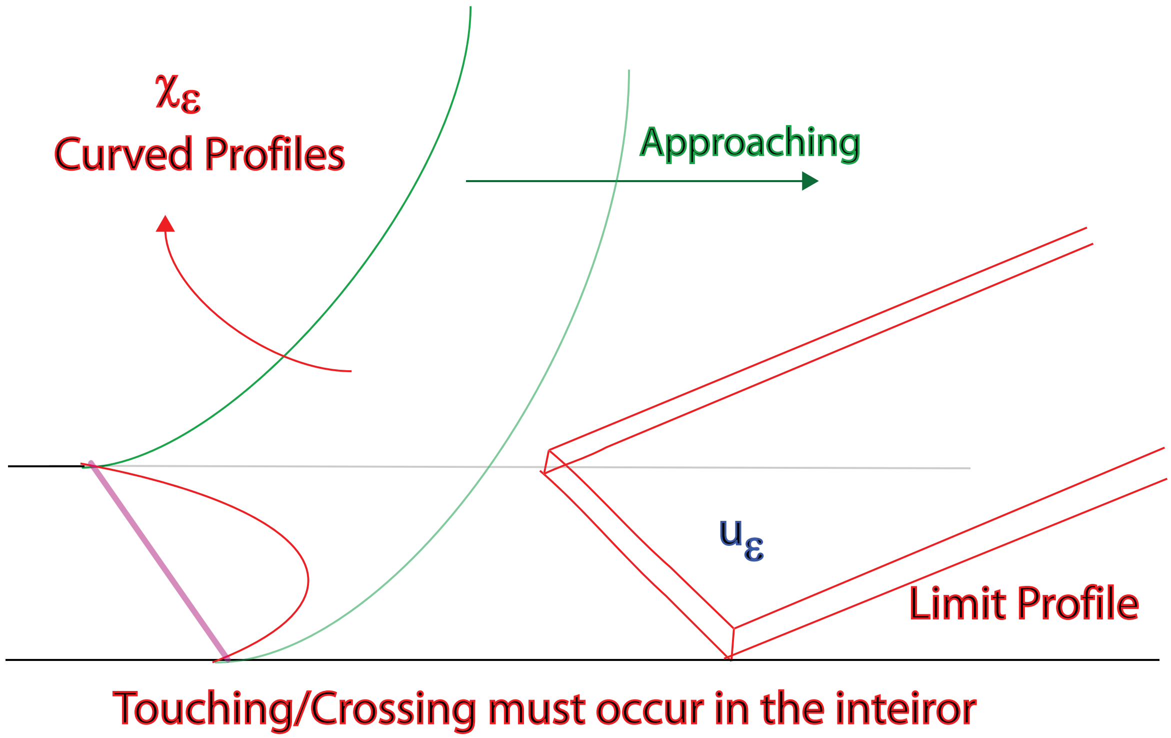


Touching on the Boundary

The idea to avoid this type of situation is to obtain a uniformly bending of the ε free boundaries to make sure that the contact/crossing happens in the interior. This uniform bending will be accomplished using a sequence of Kelvin transforms with respect to large spheres, i.e, spheres having radii and centers approaching infinity. This way, a uniform choice for the pole and radius of inversion can be done in such way that (super,sub) solutions to (SE_ε) do not change "geometry" to much, preserving the heuristic idea described above. The pictures below indicate schematically, how the inversion of the free boundary is done, avoiding the touching on the boundary.



Bending Uniformly The Free Boundaries



Interior Touching

Finally, we should mention that in the one-phase case, to reproduce the argument above and rule out the case $H(\alpha) < M$, we need the cubic decay in ε of the least supersolution in subdomains proven before to show that an undesirable "early touching" does not happen in the process of approximation from the left. The rigorous argument for all the above is the purpose of the next section.

7.2 Classification of 2-planes Global Profiles

The purpose of this section is to classify the global profiles (2-plane functions) that will appear in the blow-up analysis of our free boundary problem in the next section. The precise statement of the result is the following

Theorem 7.2.1 (Classification of Global Profiles)*.* Let v_{ε_j} be a family of least viscosity solutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v(x)$ uniformly on compact subsets of $\mathbb{R}^N$. Then we have,

$$v(x) = \alpha x_1^+ - \gamma x_1^- \text{ with } \alpha > 0, \gamma \geq 0 \Longrightarrow H(\alpha) - H(\gamma) = M$$

$$v(x) = \alpha x_1^+ + \gamma x_1^+ \text{ with } \alpha > 0, \gamma \geq 0 \Longrightarrow H(\alpha) \leq M$$

The proof of this result will be divided in several Propositions, analyzing different scenarios.

Proposition 7.2.2*.* Let v_{ε_j} be viscosity solutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v = \alpha x_1^+ - \gamma x_1^-$ uniformly on compact subsets of $\mathbb{R}^N$, with $\alpha > 0, \gamma \geq 0$ and $\varepsilon_j \to 0$. Then,

$$H(\alpha) - H(\gamma) \leq M \tag{7.2.1}$$

Proof. Let us suppose by contradiction that,

$$H_{0,0}(\alpha) - H_{0,0}(\gamma) = H(\alpha) - H(\gamma) > M$$

This way, we can find $0 < \overline{\alpha} < \alpha$ and $\overline{\sigma} > 1$ such that $H_{0,0}(\overline{\alpha}) - H_{0,0}(\gamma) > M_\sigma = \overline{\sigma}^2 M > M$. So, by continuity, there exist $\delta > 0, \mu > 0$ such that

$$H_{\delta,\mu}(\overline{\alpha}) - H_{\delta,\mu}(\gamma) > M_{\overline{\sigma}} \tag{7.2.2}$$

Thus, we are in conditions to use the above condition barriers constructed in Proposition (6.0.3) with $\alpha > \widetilde{\alpha}$. In what follows, we will freely use them as well as the notation employed there. Let $\eta > 0$ small be given. By assumption, we can find $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that

$$\varepsilon < \varepsilon_0 \Longrightarrow ||v_\varepsilon - v||_{L^\infty(Q_{L_0})} < \eta$$

Setting $c_1 = \dfrac{1}{\widetilde{\gamma}}$ and $c_2 = c_1 + \dfrac{1}{\overline{\gamma}}$, we may assume that η is so small that

$$(c_1 + c_2)\,\eta < \frac{d}{4} < \frac{L_0}{4} \tag{7.2.3}$$

Setting $Q_0 = \frac{1}{2} Q_{L_0}$ and $Q_{00} = \frac{1}{4} Q_{L_0}$, we have that for every $\xi \in \mathbb{R}^N$ with $|\xi| \leq L_0$, the functions $(\vartheta_\varepsilon)_\xi : Q_0 \to \mathbb{R}$ given by $(\vartheta_\varepsilon)_\xi(x) = \vartheta_\varepsilon(x + \xi)$ are well defined. In particular, we can define $\vartheta_\varepsilon^\star : Q_0 \to \mathbb{R}$, given by

$$\vartheta_\varepsilon^\star(x) = \vartheta_\varepsilon(x - c_1 \eta e_1)$$

It is easy to see that

$$\vartheta_\varepsilon^\star(x) \leq \mathcal{W}(x - c_1\eta e_1) < v(x) - \eta < v_\varepsilon(x) \quad \text{for } x \in Q_{00}$$

If $|T| \leq \frac{L_0}{4}$ then we can define $(\vartheta_\varepsilon^\star)_T : Q_{00} \to \mathbb{R}$ by

$$(\vartheta_\varepsilon^\star)_T(x) := \vartheta_\varepsilon^\star(x + Te_1)$$

So, let us consider the set of translations

$$\Gamma_\varepsilon = \left\{ 0 < T \leq \frac{L_0}{4}; \quad (\vartheta_\varepsilon^\star)_T \leq v_\varepsilon \quad \text{in } Q_{00} \right\} \quad \text{and} \quad \mathcal{T}_\varepsilon = \sup \Gamma_\varepsilon$$

Let us recall that $Q_\varepsilon(x) = \overline{Q}_\varepsilon(x_1) \geq \overline{\gamma} x_1$ for $x_1 \geq 0$ and (6.0.11). In particular, considering $x = le_1$, with $|l| \leq \frac{L_0}{4}$ and $l \geq -\dfrac{\eta}{\overline{\gamma}}$

$$\vartheta_\varepsilon((l + \frac{\eta}{\overline{\gamma}})e_1) \geq \overline{\gamma} l + \eta$$

but,

$$\vartheta_\varepsilon((l + \frac{\eta}{\overline{\gamma}})e_1) = \vartheta_\varepsilon((l - c_1\eta + c_2\eta)e_1)) = (\vartheta_\varepsilon^\star)_{c_2\eta}(le_1)$$

Taking now, $l = 0$, we find

$$(\vartheta_\varepsilon^\star)_{c_2\eta}(0) \geq \eta > v_\varepsilon(0)$$

In other words, if we translate $\vartheta_\varepsilon^\star$ by $c_2\eta$, we have gone to far in terms of touching v_ε by below. This implies that

$$\mathcal{T}_\varepsilon \leq c_2\eta \tag{7.2.4}$$

Moreover, there exists $x_n \in Q_{00}$ such that for all $n \geq 1$

$$(\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon+1/n}(x_n^\varepsilon) > v_\varepsilon(x_n)$$

Passing to a subsequence if necessary, we can assume $x_n^\varepsilon \to x_0^\varepsilon$ as $n \to \infty$ where $x_0^\varepsilon \in Q_{00}$. Thus we have,

$$(\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x_0^\varepsilon) = v_\varepsilon(x_0).$$

$$(\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon} \leq v_\varepsilon \text{ in } Q_{00}$$

Now, since for $x \in Q_{00}$, by (6.0.9),

$$v_\varepsilon(x) - (\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x) \geq v(x) - \eta - (\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x) \geq v(x) - \eta - \mathcal{W}(x - c_1\eta e_1 + \mathcal{T}_\varepsilon e_1)$$

We have

$$x \in \partial Q_{00} \cap \{x_1 = \pm L_0\} \Rightarrow v_\varepsilon - (\vartheta_\varepsilon^\star)_{\mathcal{T}_\varepsilon}(x) \geq$$

$$\min\{(\alpha-\widetilde{\alpha})L_0 + A(\varepsilon,\eta), (\widetilde{\gamma}-\gamma)L_0 + B(\varepsilon,\eta)\} \geq c_3 > 0$$

if η are chosen small enough, since

$$A(\varepsilon,\eta) = (\widetilde{\alpha}c_1\eta - \widetilde{\alpha}\mathcal{T}_\varepsilon - \eta) \to 0 \text{ as } \eta \to 0$$

$$B(\varepsilon,\eta) = (\widetilde{\gamma}c_1\eta - \widetilde{\gamma}\mathcal{T}_\varepsilon - \eta) \to 0 \text{ as } \eta \to 0$$

Now, once

$$\rho > 0 \Longrightarrow v(x) - \mathcal{W}(x-\rho e_1) \geq \min\{\alpha, \widetilde{\gamma}\}\rho \quad \forall x \in \mathbb{R}^N$$

we can estimate, using (6.0.10),

$$x \in \partial Q_{00} \cap \left\{x = (x_1, x^{'}) \in \mathbb{R}^N; |x^{'}| = L_0\right\} \Rightarrow$$

$$\Rightarrow v_\varepsilon(x) - (\vartheta^\star_\varepsilon)_{\mathcal{T}_\varepsilon}(x) \geq v(x) - \eta - \vartheta_\varepsilon(x - c_1\eta e_1 + \mathcal{T}_\varepsilon e_1) \geq$$

$$\geq v(x) - \eta - \mathcal{W}(x - c_1\eta e_1 + \mathcal{T}_\varepsilon e_1 - de_1) \geq$$

$$\geq \min\{\alpha, \widetilde{\gamma}\}(c_1\eta - \mathcal{T}_\varepsilon + d) - \eta \geq \frac{\min\{\alpha, \widetilde{\gamma}\}}{4} d,$$

for η small enough, since $c_1\eta - \mathcal{T}_\varepsilon \to 0$ as $\eta \to 0$. In particular, we conclude that if $\eta > 0$ is chosen small enough, on the boundary of Q_{00}, $(\vartheta^\star_\varepsilon)_{\mathcal{T}_\varepsilon}$ is striclty below v_ε for ε small enough. This forces, the contact point $x_0^\varepsilon \in int(Q_{00})$. Now, from the translation invariance, Remark (4.2.4), $\overline{\vartheta}_\varepsilon = (\vartheta^\star_\varepsilon)_{\mathcal{T}_\varepsilon}$ satisfies for some $\sigma > 1$,

$$\Delta\overline{\vartheta}_\varepsilon(x) \geq (\mathcal{E}_\sigma(\beta))_\varepsilon(\overline{\vartheta}_\varepsilon(x))F(\nabla\overline{\vartheta}_\varepsilon(x)) \text{ in } Q_{00}$$

Since v_ε are solutions to (SE_ε), this contradicts Lemma (5.2.1). This way,

$$H(\alpha) - H(\gamma) \leq M$$

and the Theorem is proven.

□

Using the same ideas of Theorem (7.2.2), we can state next corollary. The proof will follow *mutatis-mutandis.*

Corollary 7.2.3. Let v_{ε_j} be viscosity solutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v = \alpha(x - x_0)_1^+ + \gamma(x - x_0)_1^+$ uniformly on compact subsets of $\mathbb{R}^N$, with $\alpha > 0, \gamma \geq 0$ and $\varepsilon_j \to 0$. Then,

$$H(\alpha) \leq M$$

Now, we study the situation where the limit is a strict 2 -phase case. The idea is very similar to the proposition (7.2.2) but "approaching the curved barriers from the other side".

Proposition 7.2.4. Let v_{ε_j} be a family of least viscosity solutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v = \alpha(x-x_0)_1^+ - \gamma(x-x_0)_1^-$ uniformly on compact subsets of $\mathbb{R}^N$, with $\alpha > 0, \gamma > 0$ and $\varepsilon_j \to 0$. Then,

$$H(\alpha) - H(\gamma) \geq M \tag{7.2.5}$$

Proof. Let us suppose by contradiction that,

$$H_{0,0}(\alpha) - H_{0,0}(\gamma) = H(\alpha) - H(\gamma) < M$$

This way, we can find $0 < \overline{\sigma} < 1$ such that

$$H_{0,0}(\alpha) - M_{\overline{\sigma}} < H_{0,0}(\gamma)$$

Since $\gamma > 0$, we can find $\overline{\alpha} > \alpha$ such that

$$0 < H_{0,0}(\overline{\alpha}) - M_{\overline{\sigma}} < H_{0,0}(\gamma)$$

By continuity, there exist $\delta, \mu < 0$ such that

$$0 < H_{\delta,\mu}(\overline{\alpha}) - M_{\overline{\sigma}} < H_{\delta,\mu}(\gamma)$$

Now, let $\alpha < \alpha^{\star} < \overline{\alpha}$. This way, we are in conditions to use the below condition barriers χ_{ε} constructed in Proposition (6.0.4). Let $\eta > 0$ small be given. We can find $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that

$$\varepsilon < \varepsilon_0 \Rightarrow ||v_{\varepsilon} - v||_{L^{\infty}(Q_{L_0})} < \eta$$

Let us set $c_1 := \max\left\{\frac{1}{\alpha}, \frac{1}{\gamma}\right\}$, $c_2 = c_1 + \frac{1}{\overline{\gamma}}$ and assume η is so small that

$$(c_1 + c_2)\,\eta < \frac{d_{\star}}{4} < \frac{L_0}{4}$$

Setting $Q_0 = \frac{1}{2}Q_{L_0}$ and $Q_{00} = \frac{1}{4}Q_{L_0}$, we see that if $\xi \in \mathbb{R}^N$ with $|\xi| \leq L_0$, the functions $(\chi_{\varepsilon})_{\xi} : Q_0 \to \mathbb{R}$ given by $(\chi_{\varepsilon})_{\xi}(x) = \chi_{\varepsilon}(x + \xi)$ are well defined. In particular, taking $\xi = c_1\eta e_1$, we define

$$\chi_{\varepsilon}^{\star}(x) = \chi_{\varepsilon}(x + c_1\eta e_1)$$

It is easy to check that

$$\chi_{\varepsilon}^{\star}(x) \geq \mathcal{W}^{\star}(x + c_1\eta e_1) > v(x) + \eta > v_{\varepsilon}(x) \text{ for } x \in Q_{00} \tag{7.2.6}$$

For $|T| \leq \dfrac{L_0}{4}$, we can define $(\chi_\varepsilon^\star)_T : Q_{00} \to \mathbb{R}$ by

$$(\chi_\varepsilon^\star)_T(x) := \chi_\varepsilon^\star(x - Te_1)$$

and consider the set of translations

$$\Gamma_\varepsilon = \left\{ 0 \leq T \leq \frac{L_0}{4}; (\chi_\varepsilon^\star)_T \geq v_\varepsilon \text{ in } Q_{00} \right\} \text{ and } \mathcal{T}_\varepsilon = \sup \Gamma_\varepsilon$$

Let us recall that $Q_\varepsilon(x) = \overline{Q}_\varepsilon(x_1) \leq \overline{\gamma} x_1$ for $x_1 \leq 0$ and (6.0.20). In particular, considering $x = le_1$, with $|l| \leq \frac{L_0}{4}$ and $l \leq \dfrac{\eta}{\overline{\gamma}}$

$$\chi_\varepsilon((l - \frac{\eta}{\overline{\gamma}})e_1) \leq \overline{\gamma} l - \eta$$

but

$$\chi_\varepsilon((l - \frac{\eta}{\overline{\gamma}})e_1) = \chi_\varepsilon((l + \eta c_1 - \eta c_2)e_1) = (\chi_\varepsilon^\star)_{c_2\eta}(le_1)$$

Taking now, $l = 0$, we find

$$(\chi_\varepsilon^\star)_{c_2\eta}(0) \leq -\eta < v_\varepsilon(0)$$

This means that if we translate $\chi_\varepsilon^\star$ by $c_2\eta$ we have gone too far in terms of touching v_ε by above. In particular,

$$\mathcal{T}_\varepsilon \leq c_2\eta$$

Setting

$$Z_\varepsilon^\tau(x) := (\chi_\varepsilon^\star)_{\mathcal{T}_\varepsilon+\tau}(x), \quad \tau > 0$$

then, we can find $x_\varepsilon^\tau \in Q_{00}$ such that

$$Z_\varepsilon^\tau(x_\varepsilon^\tau) < v_\varepsilon(x_\varepsilon^\tau)$$

Let us observe that for $x \in Q_{00}$, by (6.0.18),

$$Z_\varepsilon^\tau(x) - v_\varepsilon(x) \geq \mathcal{W}_\varepsilon^\star(x + (c_1\eta - \mathcal{T}_\varepsilon - \tau)e_1) - v(x) - \eta \geq$$

$$\geq \mathcal{W}^\star((x + (c_1\eta - \mathcal{T}_\varepsilon - \tau)e_1) - \varepsilon De_1) + C\varepsilon - v(x) - \eta$$

In particular,

$$x \in Q_{00} \cap \{x_1 = \pm L_0\} \Rightarrow$$

$$\Rightarrow Z_\varepsilon^\tau(x) - v_\varepsilon(x) \geq$$

$$\geq \min\left\{(\widetilde{\alpha} - \alpha)L_0 + \overline{A}(\eta, \varepsilon, \tau), (\gamma - \widetilde{\gamma})L_0 + \overline{B}(\eta, \varepsilon, \tau)\right\} \geq c_4 > 0$$

if η and τ are chosen small enough, since

$$\overline{A}(\eta,\varepsilon,\tau)=\widetilde{\alpha}(c_1\eta-\mathfrak{T}_\varepsilon-\tau-\varepsilon D)+C\varepsilon-\eta\to 0 \text{ as } \varepsilon,\eta,\tau\to 0$$

$$\overline{B}(\eta,\varepsilon,\tau)=\widetilde{\gamma}(c_1\eta-\mathfrak{T}_\varepsilon-\tau-\varepsilon D)+C\varepsilon-\eta\to 0 \text{ as } \varepsilon,\eta,\tau\to 0$$

Furthermore, by (6.0.19), if $x\in Q_{00}\cap\left\{x=(x_1,x^{'})\in\mathbb{R}^N;|x^{'}|=L_0\right\}$

$$Z_\varepsilon^\tau(x)-v_\varepsilon(x)\geq \mathcal{W}^\star(x+(c_1\eta-\mathfrak{T}_\varepsilon-\tau)e_1+(d_\star-\varepsilon D)e_1)-v(x)-\eta\geq$$

$$\geq\min\left\{\overline{\alpha},\gamma\right\}(c_1\eta-\mathfrak{T}_\varepsilon-\tau+d_\star-\varepsilon D)>c_5>0$$

for ε,η,τ small enough, since $c_1\eta-\mathfrak{T}_\varepsilon-\tau-\varepsilon D\to 0$ as $\varepsilon,\eta,\tau\to 0$. This way, by the translation invariance of (SE_ε), Remark (4.2.4), Z_ε^τ is a supersolution of (SE_ε) in Q_{00}. Finally, for η,τ,ε small enough, we have

$$Z_\varepsilon^\tau\geq v_\varepsilon \text{ in } Q_{00}$$

$$Z_\varepsilon^\tau(x_\varepsilon^\tau)<v_\varepsilon(x_\varepsilon^\tau) \text{ with } x_\varepsilon^\tau\in \text{ int}(Q_{00})$$

which contradicts the fact that v_ε is the least supersolution of (SE_ε). This way, $H(\alpha)-H(\gamma)\geq M$ and the Theorem is proven.

□

Finally, we treat the case where the case where the profile is of one-phase. The idea is the same as the previous Theorem, taking into account, the cubic decay of the least supersolutions to prevent an "early" touching.

Proposition 7.2.5. Let v_{ε_j} be a family of least viscosity solutions to $(SE)_{\varepsilon_j}$ in a domain $\Omega_j \subset \mathbb{R}^N$ such that $\Omega_j \subset \Omega_{j+1}$ and $\cup_{j=1}^{\infty}\Omega_j = \mathbb{R}^N$. Suppose v_{ε_j} converge to $v = \alpha(x - x_0)_1^+$ uniformly on compact subsets of $\mathbb{R}^N$, with $\alpha > 0$ and $\varepsilon_j \to 0$. Then,

$$H(\alpha) \geq M \tag{7.2.7}$$

Proof. The proof is similar to the proof of Theorem (7.2.4). Once more, let us assume by contradiction that $H(\alpha) < M$. As before, we can find $\overline{\alpha} > \alpha$, $\delta, \mu < 0$ and $\overline{\sigma} < 1$ such that

$$H_{\delta,\mu}(\overline{\alpha}) < M_{\overline{\sigma}}$$

Let us now choose, $\alpha < \widetilde{\alpha} < \overline{\alpha}$. We are now in conditions to use the barriers constructed in Proposition (6.0.5). By assumption, there exists $\varepsilon_0 = \varepsilon_0(\eta)$ such that

$$\varepsilon \leq \varepsilon_0 \Rightarrow ||v_\varepsilon - v||_{L^\infty(Q_{L_0})} < \kappa_2 \eta$$

Setting $Q_0 = \frac{1}{2}Q_{L_0}$ and $Q_{00} = \frac{1}{4}Q_{L_0}$, we see that if $\xi \in \mathbb{R}^N$ with $|\xi| \leq L_0$, the functions $(\chi_\varepsilon)_\xi : Q_0 \to \mathbb{R}$ given by $(\chi_\varepsilon)_\xi(x) = \chi_\varepsilon(x + \xi)$ are well defined.

Let us set

$$Q^{\eta}_{L_0} = \left\{ x = (x_1, x') \in \mathbb{R}^N | \ \ |x|_{\infty} = \max\left\{ |x_1|, |x^{'}| \right\} \leq 4L_0 - 2\eta \right\}$$

Let us define $c_1 := 2\kappa_2/\widetilde{\alpha} + 3$ and $c_2 := c_1 + 2/\overline{\alpha}$, and consider η so small that,

$$(c_1 + c_2)\eta < \frac{d_\star}{4} < \frac{L_0}{4}$$

Observe, that by the cubic decay in the interior, Lemma (5.3.2), there exists a constant C_η such that

$$x \in Q_{00} \cap \{x_1 \leq -2\eta\} \Rightarrow v^+_\varepsilon \leq C_\eta \varepsilon^3$$

Now, taking the barrier constructed in Proposition (6.0.5), by b), $\chi_\varepsilon \geq C\varepsilon$ in Q_{L_0}. Let us define

$$\chi^\star_\varepsilon(x) = \chi_\varepsilon(x + c_1\eta e_1)$$

Then, if for $x_1 \geq -c_1\eta$ we have

$$\chi^\star_\varepsilon(x) \geq \widetilde{\alpha}(x_1 + c_1\eta e_1) + D\varepsilon \geq \widetilde{\alpha}x_1 + 2\kappa_2\eta + 3\widetilde{\alpha}\eta + D\varepsilon$$

In particular, $x_1 \geq -3\eta \Rightarrow \chi^\star_\varepsilon(x) > 2\kappa_2\eta$. This way, there exists $\varepsilon_1 = \varepsilon_1(\eta) < \varepsilon_0$, we have $\chi_\varepsilon - v_\varepsilon > 0$ in Q_0 since

$$\chi_\varepsilon^\star - v_\varepsilon \geq C\varepsilon - C_\eta \varepsilon^3 > 0 \quad in \quad Q_0 \cap \{x_1 \leq -2\eta\}$$

$$\chi_\varepsilon^\star - v_\varepsilon \geq \kappa_2 \eta \quad in \quad Q_0 \cap \{x_1 \geq -3\eta\}$$

For $|T| \leq \dfrac{L_0}{4}$, we can define $(\chi_\varepsilon^\star)_T : Q_{00} \to \mathbb{R}$ by

$$(\chi_\varepsilon^\star)_T(x) := \chi_\varepsilon^\star(x - Te_1)$$

and consider, as before, the set of translations

$$\Gamma_\varepsilon = \left\{ 0 \leq T \leq \frac{L_0}{4}; (\chi_\varepsilon^\star)_T \geq v_\varepsilon \text{ in } Q_{00} \right\} \text{ and } \mathcal{T}_\varepsilon = \sup \Gamma_\varepsilon$$

Now, let us recall that

$$\chi_\varepsilon(le_1) \leq Q_\varepsilon(le_1) = P_\varepsilon(l) = \overline{\alpha} l + \varepsilon \kappa_{\overline{\sigma}}^+ \quad \text{for} \quad l \geq 0$$

In particular, if $l \geq (c_2 - c_1)\eta$ and $l \leq \frac{L_0}{4}$, then

$$(\chi_\varepsilon^\star)_{c_2\eta}(le_1) = \chi_\varepsilon(le_1 + (c_1 - c_2)\eta e_1) \leq \overline{\alpha} l + \overline{\alpha}(c_1 - c_2)\eta + \varepsilon \kappa_{\overline{\sigma}}^+$$

Taking $l = 2\eta/\overline{\alpha}$, for ε small enough,

$$(\chi_\varepsilon^\star)_{c_2\eta}\left(\frac{2\eta}{\overline{\alpha}}e_1\right) = \chi_\varepsilon(0) = \varepsilon\kappa_{\overline{\sigma}}^{+} < \frac{2\alpha\eta}{\overline{\alpha}} = v\left(\frac{2\eta}{\overline{\alpha}}e_1\right)$$

In other words, if we translate $\chi_\varepsilon^\star$ by $c_2\eta$ we have gone to far in terms of touching v_ε by above. This implies, that

$$0 \leq \mathcal{T}_\varepsilon \leq c_2\eta$$

Let us we define for $0 < \tau < L_0/10$ a small number,

$$Z_\varepsilon^\tau(x) := (\chi_\varepsilon^\star)_{\mathcal{T}_\varepsilon+\tau}(x), \quad \tau > 0$$

Now, we estimate

$$x \in \partial Q_{00} \cap \{x_1 = \ L_0\} \Rightarrow Z_\varepsilon^\tau(x) - v_\varepsilon(x) \geq Z_\varepsilon^\tau(x) - v(x) - \eta \geq$$

$$\geq \chi_\varepsilon(x + c_1\eta e_1 - \mathcal{T}_\varepsilon e_1 - \tau e_1) - v(x) - \eta \geq$$

$$\geq (\widetilde{\alpha} - \alpha)L_0 + \overline{A}(\varepsilon,\eta,\tau) \geq$$

$$\geq \frac{1}{4}(\widetilde{\alpha} - \alpha)L_0$$

since $\overline{A}(\varepsilon,\eta,\tau) = \widetilde{\alpha}(c_1\eta - c_2\eta - \tau) + D\varepsilon - \eta \to 0$ as $\varepsilon,\eta,\tau \to 0$.

Clearly, for η,τ,ε small enough,

$$x \in \partial Q_{00} \cap \{x_1 = -L_0\} \Rightarrow Z_\varepsilon^\tau - v_\varepsilon > C\varepsilon - C_\eta\varepsilon^3 > 0$$

Finally, let us see that

$$x \in \partial Q_{00} \cap \left\{ x = (x_1, x^{'}) \in Q_{L_0}; |x^{'}| = L_0 \right\} \Rightarrow Z_\varepsilon^\tau(x) > v_\varepsilon(x)$$

Indeed, choosing η, τ small enough, $d_\star - c_1\eta + \mathcal{T}_\varepsilon + \tau < \frac{3}{4}d_\star$. We can assume, passing to a subsquence, if necessary that, $\mathcal{T}_\varepsilon \to \mathcal{T}$ as $\varepsilon \to 0$. By the convergence given in Proposition (6.0.5)d),

$$Z_\varepsilon^\tau \to G \qquad \text{uniformly in} \qquad \partial Q_{00} \cap \left\{ x = (x_1, x^{'}) \in Q_{L_0}; |x^{'}| = L_0 \right\}$$

where,

$$G(x_1) = g(x_1 + c_1\eta - \mathcal{T} - \tau)$$

Additionally, if $x_1 \geq d_\star - c_1\eta + \mathcal{T} + \tau$, then

$$G(x_1) \geq \widetilde{\alpha}(x_1 - d_\star + c_1\eta - \mathcal{T} - \tau)$$

So, for ε small enough and $x \in \partial Q_{00} \cap \{x = (x_1, x^{'}) \in Q_{L_0}; |x^{'}| = L_0\} \cap \{x_1 \geq \frac{3}{4}d_\star\}$,

$$Z_\varepsilon^\tau \geq G - \eta$$

This way,

$$x_1 \geq d_\star/2 \Rightarrow Z_\varepsilon^\tau > \mathfrak{L}(x_1) = \widetilde{\alpha}(x_1 - d_\star) + \overline{B}(\eta, \mathfrak{T}, \tau)$$

where $\overline{B}(\eta, \mathfrak{T}, \tau) = \widetilde{\alpha}(c_1\eta - \mathfrak{T} - \tau) - \eta \to 0$ as $\eta, \tau \to 0$.

Since, $\mathfrak{L}(d_\star/2) > -\widetilde{\alpha}/4 > \kappa_2\eta$ for η, τ small enough, and $\frac{d}{dx_1}\mathfrak{L}(x_1) = \widetilde{\alpha} > \alpha$, we conclude that

$$x_1 \geq d_\star/2 \Rightarrow Z_\varepsilon^\tau \geq \mathfrak{L} > v + \eta > v_\varepsilon$$

and clearly,

$$x_1 \leq d_\star/2 \Rightarrow Z_\varepsilon^\tau - v_\varepsilon > C\varepsilon - C_\eta\varepsilon^3 > 0$$

This way, by the translation invariance of (SE_ε), Remark (4.2.4), Z_ε^τ is a supersolution of (SE_ε) in Q_{00}. Finally, for η, τ, ε small enough,

$$Z_\varepsilon^\tau \geq v_\varepsilon \text{ in } Q_{00}$$

$$Z_\varepsilon^\tau(x_\varepsilon^\tau) < v_\varepsilon(x_\varepsilon^\tau) \text{ with } x_\varepsilon^\tau \in \text{ int}(Q_{00})$$

which contradicts the fact that v_ε is the least supersolution of (SE_ε). This way, $H(\alpha) \geq M$ and the Theorem is proven.

$\square$

Chapter 8

Limit Free Boundary Problem and Regularity of the Free Boundary

8.1 Limit Free Boundary Problem

In this section, we prove that the limit of the least viscosity, u_0 provided by Theorem (3.0.5) is a solution in the Caffarelli's viscosity sense as well as in the pointwise sense ($\mathcal{H}^{N-1}$ a.e.) to the free boundary problem

$$(FBP) \qquad \Delta u = 0 \text{ in } \Omega \setminus \partial\{u > 0\}$$

$$H_\nu(u_\nu^+) - H_\nu(u_\nu^-) = M \text{ on } \Omega \cap \partial\{u > 0\}$$

where $u^+ = \max(u, 0), u^- = max(-u, 0)$, ν is the inward unit normal to the free boundary $F(u) = \Omega \cap \partial\{u > 0\}$ and

$$H_\nu(t) = \int_0^t \frac{s}{F(s\nu)} ds$$

This notion of weak solution was introduced by Luis A. Caffarelli in the classical papers [6, 5]. Now, we provide these definitions to our problem.

Definition 8.1.1. Let Ω be a domain in $\mathbb{R}^N$ and $u \in C^0(\Omega)$. Then, u is called a viscosity supersolution to (FBP) if

i) $\Delta u \leq 0$ in $\Omega^+ = \Omega \cap \{u > 0\}$

ii) $\Delta u \leq 0$ in $\Omega^- = (\Omega \setminus \Omega^+)^\circ$

iii) Along $F(u)$, u satisfies

$$H_\nu(u_\nu^+) - H_\nu(u_\nu^-) \leq M$$

in the following sense:

If $x_0 \in F(u)$ is a regular point from the nonnegative side(i.e, there exists $B_r(y) \subset \Omega^+$ with $x_0 \in \partial B_r(x_0)$) and

$$u^+(x) \geq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \text{ in } B_r(x_0), \qquad (\alpha > 0)$$

and

$$u^-(x) \geq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \text{ in } B_r(x_0)^C, \qquad (\beta \geq 0)$$

with equality along every nontangential domain in both cases, then

$$H_\nu(\alpha) - H_\nu(\beta) \leq M$$

Analogously, we have

Definition 8.1.2. Let Ω be a domain in $\mathbb{R}^N$ and $u \in C^0(\Omega)$. Then, u is called a viscosity subsolution to (FBP) if

i) $\Delta u \geq 0$ in $\Omega^+ = \Omega \cap \{u > 0\}$

ii) $\Delta u \geq 0$ in $\Omega^- = (\Omega \setminus \Omega^+)^\circ$

iii) Along $F(u)$, u satisfies

$$H_\nu(u_\nu^+) - H_\nu(u_\nu^-) \geq M$$

in the following sense:

If $x_0 \in F(u)$ is a regular point from the nonpositive side (i.e, there exists $B_r(y) \subset \Omega^-$ with $x_0 \in \partial B_r(x_0)$) and

$$u^-(x) \geq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \text{ in } B_r(x_0), \qquad (\beta \geq 0)$$

and

$$u^+(x) \geq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \text{ in } B_r(x_0)^C, \qquad (\alpha \geq 0)$$

with equality along every nontangential domain in both cases, then

$$H_\nu(\alpha) - H_\nu(\beta) \geq M$$

Remark 8.1.3. There are equivalent definitions for supersolutions and subsolutions to (FBP) above. We mention an equivalent one for supersolutions that will be used in the next results. For this and further details, see [10], chapter 2.

Equivalentely, $u \in C^0(\Omega)$ is a supersolution of (FBP) if conditions $i), ii)$ of defintion (8.1.1) are satisfied and if x_0 is regular point from the nonnegative side with tangent ball B

$$u^+(x) \geq \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|) \text{ in } B, \qquad (\alpha \geq 0)$$

then,

$$u^-(x) \geq \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \text{ in } B^C, \qquad (\beta \geq 0)$$

For any β such that

$$H_\nu(\alpha) - H_\nu(\beta) > M$$

Now, we move towards the proof of the major results in this section. In what follows, u_0 will always denote the limit of the least supersolutions given by Theorem(3.0.5).

Proposition 8.1.4. u_0 is a viscosity subsolution to (FBP)

Proof. Clearly, conditions $i), ii)$ of definiton (8.1.1) are satisfied. Now, let us suppose that $x_0 \in F(u_0)$ is a regular point from the nonpositive side with tangent ball B. We can assume without lost of generality that $x_0 = 0$ and $\nu = e_1$. This way, by linear behavior at regular boundary points, Lemma (11.7) in [10] (see Appendix), there exist $\alpha \geq 0$ and $\beta > 0$

$$u_0^+(x) = \alpha x_1^+ + o(|x|) \text{ in } B^C$$

and

$$u_0^-(x) = \beta x_1^- + o(|x|) \text{ in } B.$$

Since u_0^+ is nondegenerate, by Theorem (3.0.5)e) or more specifically, since (3.0.1) holds, we conclude that $\alpha > 0$ and thus, B is tangent to $F(u_0)$. This way, u_0 admits full asymptotic development, i.e,

$$u_0(x) = \alpha x_1^+ - \beta x_1^- + o(|x|)$$

Taking now any sequence $\lambda_n \to 0$ and using the blow-up sequence $(u_{\varepsilon_k^{'}})_{\lambda_n}$ given in proposition (4.2.1), we conclude that there exists a subsequence that we still denote by $\varepsilon_k^{'}$ such that $(u_{\varepsilon_k^{'}})_{\lambda_k} \to \alpha x_1^+ - \beta x_1^-$ uniformly in compact subsets of $\mathbb{R}^N$. Since by remark (4.2.4) the equation (SE_ε) and the least supersolution property are preserved under blow-up process, by Theorem (7.2.1), we conclude that

$$H(\alpha) - H(\beta) = M$$

where $H = H_{e_1}$.

□

Proposition 8.1.5. u_0 is a supersolution to (FBP).

Proof. As we already observed, u_0 satisfies conditions $i), ii)$ of definition (8.1.1). We will show that the condition in the remark (8.1.3) holds. This way, let us assume that $B = B_r(y)$ be a touching ball from the nonnegative side at x_0 and let us assume that for some $\alpha \geq 0$,

$$u_0^+(x) \geq \alpha \langle x - x_0 \rangle^+ + o(|x - x_0|) \quad \text{in } B \tag{8.1.1}$$

where ν given by the inward unit radial direction of the ball at x_0. If $H_\nu(\alpha) \leq M$ there is nothing to prove. Otherwise, if

$$H_\nu(\alpha) > M, \tag{8.1.2}$$

let $\gamma \geq 0$ such that $H_\nu(\alpha) - H_\nu(\gamma) > M$ (we can find such γ, since H_ν is a bijection from $[0, +\infty]$ into itself). We will show that

$$u_0^-(x) \geq \gamma \langle x - x_0, \nu \rangle^- + o(|x - x_0|) \text{ in } B^C \tag{8.1.3}$$

As usual, we assume without loss of generality that $\nu = e_1$ and $x_0 = 0$. We will prove the following

Claim: There exist $\overline{\alpha}, \overline{\gamma} > 0$ such that

$$u_0(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^- + o(|x|).$$

Indeed, by the Lemma (4.1) in [17] (see Appendix),

$$u_0^-(x) = \overline{\gamma} x_1^- + o(|x|) \quad \text{ in } \{x_1 < 0\} \tag{8.1.4}$$

for some $\overline{\gamma} \geq 0$. Let us consider the blow-up sequence, i.e, for $\lambda > 0$,

$$(u_0)_\lambda(x) = \frac{1}{\lambda} u_0(\lambda x).$$

Since u_0 is locally Lipschitz continuous and $u_0(0) = 0$ then, for every sequence, $\lambda_n \to 0$, there exists a subsequence, that we still denote by λ_n, such that $(u_0)_{\lambda_n} \to U_0$ uniformly in compact sets of $\mathbb{R}^N$, where U_0 is Lipschitz in $\mathbb{R}^N$. By (8.1.1) and (8.1.4) we know that

$$U_0^- = \overline{\gamma} x_1^- \text{ in } \mathbb{R}^N$$

and

$$U_0 > 0 \quad \text{and harmonic in} \, \{x_1 > 0\}$$

We have to analyze two cases:

Case I: $\overline{\gamma} > 0$.

In this case, $U_0 < 0$ in $\{x_1 < 0\}$. Therefore $U_0 = 0$ on the hyperplane $\{x_1 = 0\}$ and since it is Lipschitz continuous, we have

$$U_0^+(x) = \overline{\alpha} x_1^+ \quad \text{in } \mathbb{R}^N$$

for some $\overline{\alpha} > 0$. This way, we conclude that

$$U_0(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^-, \quad \overline{\alpha}, \overline{\gamma} > 0. \tag{8.1.5}$$

Case II: $\overline{\gamma} = 0$.

In this case, $U_0 \geq 0$ in $\mathbb{R}^N$. Since $U_0 > 0$ and harmonic in harmonic in $\{x_1 > 0\}$, then by Lemma A1 in [6] (see Appendix), there exist $\overline{\alpha} > 0$ such that

$$U_0(x) = \overline{\alpha} x_1^+ + o(|x|) \text{ in } \{x_1 > 0\}. \tag{8.1.6}$$

Since $\overline{\alpha} > \alpha$, then $(H = H_{e_1})$

$$H(\overline{\alpha}) \geq H(\alpha) > M \tag{8.1.7}$$

Let us consider, for $\lambda > 0$, the blow-up sequence

$$(U_0)_\lambda(x) = \frac{1}{\lambda} U_0(\lambda x)$$

Since U_0 is Lipschitz continuous and $U_0(0) = 0$, there exists a subsequence $\overline{\lambda}_n \to 0$, such that $(U_0)_{\overline{\lambda_n}} \to U_{00}$ uniformly on compact sets of $\mathbb{R}^N$, where $U_{00} \in Lip(\mathbb{R}^N)$. By (8.1.6),

$$U_{00}(x) = \overline{\alpha} x_1^+ \text{ in } \{x_1 > 0\}\,.$$

Let us observe that, $U_{00} \geq 0$ in $\mathbb{R}^N$, it is harmonic in its positivity set $\{U_{00} > 0\}$ and $u_{00} = 0$ on the hyperplane $\{x_1 = 0\}$, again by Lemma A1 in [6], we have

$$u_{00}(x) = \widetilde{\alpha} x_1^- + o(|x|) \text{ in } \{x_1 < 0\}\,,$$

for some $\widetilde{\alpha} \geq 0$. Finally, if we consider once more for $\lambda > 0$ the blow-up sequence

$$(u_{00})_\lambda = \frac{1}{\lambda} u_{00}(\lambda x).$$

As before, there is still a subsequence $\widetilde{\lambda}_n \to 0$ and $U_{000} \in Lip(\mathbb{R}^N)$, such that $(U_{00})_{\widetilde{\lambda_n}} \to U_{000}$ uniformly on compact subsets of $\mathbb{R}^N$. From the computations above, we conclude

$$U_{000}(x) = \overline{\alpha} x_1^+ + \widetilde{\alpha} x_1^-, \qquad \overline{\alpha} > 0,\ \widetilde{\alpha} \geq 0.$$

Applying proposition (4.2.1) and recalling that least supersolutions are preserved under blow-ups, we can see that there exists a sequence $\delta_n \to 0$ and least supersolutions u_{δ_n} to (SE_{δ_n}) such that

$$u_{\delta_n} \to U_0 \tag{8.1.8}$$

uniformly on compact sets of $\mathbb{R}^N$. Applying the same poposition twice, we see that there exist a sequence $\widetilde{\delta_n} \to 0$ and solutions $u_{\widetilde{\delta_n}}$ to $(SE_{\widetilde{\delta}_n})$ such that $u_{\widetilde{\delta_n}} \to U_{000}$ uniformly on compact sets of $\mathbb{R}^N$. By Theorem (7.2.1) and by (8.1.2)

$$H(\overline{\alpha}) \leq M < H(\alpha)$$

which contradicts (8.1.7). Then, case II does not occur and (8.1.5) holds, proving the claim. This way, by (8.1.8), we can apply again Theorem (7.2.1) to U_0 to conclude

$$H_{e_1}(\overline{\alpha}) - H_{e_1}(\overline{\gamma}) = M \tag{8.1.9}$$

By Proposition (4.2.2), the blow-up compatibility condition, there exists a $\delta > 0$ independent of the sequence λ_n such that

$$\overline{\alpha\gamma} = \delta \tag{8.1.10}$$

So, $\overline{\alpha}$ and $\overline{\gamma}$ are determinded in a unique way and therefore, U_0 does not depend on the sequence λ_n. In particular,

$$(u_0)_\lambda \to U_0$$

uniformly in compact subsets of $\mathbb{R}^N$(as $\lambda \to 0$). Thus,

$$u_0(x) = \overline{\alpha} x_1^+ - \overline{\gamma} x_1^- + o(|x|)$$

In particular,

$$u_0^-(x) = \overline{\gamma} x_1^- + o(|x|) \qquad \text{in } B^C \tag{8.1.11}$$

By (8.1.9), we obtain since $\overline{\alpha} \geq \alpha$

$$H_{e_1}(\overline{\gamma}) = H_{e_1}(\overline{\alpha}) - M \geq H_{e_1}(\alpha) - M > H_{e_1}(\gamma) \tag{8.1.12}$$

from which we conclude $\overline{\gamma} > \gamma$ and therefore by (8.1.11),

$$u_0^-(x) > \gamma x_1^- + o(|x|) \qquad \text{in } B^C$$

This finishes the proof. □

8.2 Flatness and Regularity of the Free Boundary

We establish now, the pointwise result.

Theorem 8.2.1. For $\mathcal{H}^{n-1}$ a.e. $x_0 \in F(u_0)$, u_0 has the following asymptotic development

$$u_0(x) = \alpha \langle x - x_0, \nu \rangle^+ - \gamma \langle x - x_0, \nu \rangle^- + o(|x - x_0|)$$

where

$$H_\nu(\alpha) - H_\nu(\gamma) = M$$

In particular, around such points, the free boundary $F(u_0)$ is flat in the sense of Theorem $2'$ in [6].

Proof. Indeed, by Theorem (3.0.8), for $\mathcal{H}^{N-1}$ a.e. in $F(u_0)$ we have,

$$u_0(x) = q_{u_0}^+(x_0) \langle x - x_0, \nu \rangle^+ - q_{u_0}^-(x_0) \langle x - x_0, \nu \rangle^- + o(|x - x_0|)$$

Considering now, the blow-up sequence, $(u_0)_\lambda(x) = \frac{1}{\lambda} u(x_0 + \lambda x), \lambda > 0$, we have

$$(u_0)_\lambda \to q_{u_0}^+(x_0) \langle x - x_0, \nu \rangle^+ - q_{u_0}^-(x_0) \langle x - x_0, \nu \rangle^-$$

Since least supersolutions are preserved under blow-up process, as in the previous Theorem, by proposition (4.2.1) and Theorem (7.2.1), we conclude that

$$H_\nu(q^+_{u_0}(x_0)) - H_\nu(q^-_{u_0}(x_0)) = M$$

The flatness follows now by the arguments in [5]. This finishes the proof.

□

At last, we prove our last Theorem concerning about the regularity of the free boundary $F(u_0)$.

Theorem 8.2.2 (Free boundary regularity). Let u_0 be the limit of the least supersolutions given by Theorem (3.0.5). Then, the free boundary $F(u_0) = \partial\{u_0 > 0\} \cap \Omega$ is a $C^{1,\gamma}$ surface in a neighborhood of $\mathcal{H}^{N-1}$ a.e. point $x_0 \in F(u_0)_{red}$. In particular, $F(u_0)$ is a $C^{1,\gamma}$ surface in a neighborhood of $\mathcal{H}^{N-1}$ a.e. point in $F(u_0)$.

Proof. We already know that u_0 is a viscosity solution of (FBP). In this case, u_0 satisfies

$$u^+_\nu = G(u^-_\nu, \nu) \text{ on } F(u)$$

in the viscosity sense, where

$$G(z,\nu) = H_\nu^{-1}(M + H_\nu(z)) \tag{8.2.1}$$

Let us observe that G depends on ν in a Lipschitz continuous fashion. Indeed, there is a constant $C > 0$ such that $G(z,\nu) \geq C$. To see that, since $t^2/2F_{max} \leq H_\nu(t) \leq t^2/2F_{min}$, for $t \geq 0$, we obtain

$$\frac{G(z,\nu)^2}{2F_{min}} \geq H_\nu(G(z,\nu)) \geq M + H_\nu(z) \geq M$$

Furthermore,

$$|H_\nu(x) - H_\nu(y)| \geq \frac{\sigma}{F_{min}}|x-y| \text{ for } x, y \in [\sigma, +\infty).$$

This way, for $\nu_1, \nu_2 \in \mathbb{S}^{N-1}$, by (8.2.1)

$$|H_{\nu_1}(G(z,\nu_1) - H_{\nu_2}(G(z,\nu_2)| = |H_{\nu_1}(z) - H_{\nu_2}(z)|$$

therefore for $|z| \leq C_0$, there exists $\overline{C_0} = \overline{C_0}(C_0, Lip(F))$ such that

$$\frac{C}{F_{min}}|G(\nu_1,z) - G(\nu_2,z)| \leq |H_{\nu_1}(G(z,\nu_1) - H_{\nu_2}(G(z,\nu_2)| =$$

$$= |H_{\nu_1}(z) - H_{\nu_2}(z)| \leq \overline{C_0}|\nu_1 - \nu_2|$$

Moreover, by Theorem (3.0.5) u_0 is locally Lipschitz continuous and it has linear growth away from its free boundary $F(u_0)$. Also, since $F(u_0)_{red}$ has full $\mathcal{H}^{N-1}$ measure in $F(u_0)$, u_0 is for $\mathcal{H}^{N-1}$ a.e. point on $F(u_0)$ a 2-plane solution. In particular, for any such point x_0, a suitable dilation

$$(u_0)_\tau = \frac{u(\tau(x - x_0))}{\tau}, \quad \tau \text{ small enough}$$

falls under conditions of Theorem 3 in [6], concluding the proof. $\square$

Appendices

Appendix A

Some Results From Alt-Caffarelli Theory

In this Appendix A, we state some important results contained in the so called Alt-Caffarelli Theory developed in [8]. These results were used to study the properties of the free boundary $F(u_0)$ of the limit of the least supersolutions.

The idea of these results below is essentially the following: The linear growth away from its free boundary for a Harmonic function u is equivalent to a density property of its free boundary, $F(u) := \partial \{u > 0\} \cap \Omega$. This implies that $F(u)$ in an $(N-1)$ dimensional object (in the geometric measure theoretical sense), and as a consequence, a representation Theorem is available. This Theorem says that Δu is absolutely continuous measure with respect to $\mathcal{H}^{N-1}\lfloor\{u > 0\}$. In addition, near almost all points x_0 in the reduced free boundary u behaves like the positive part of a linear function with slope given by the density of the Laplacian with respect to the $\mathcal{H}^{N-1}$ at x_0.

Assumption: $(\star)$ We will assume that $u \in C(\Omega)$ is nonnegative and harmonic in $\Omega \cap \{u > 0\}$.

Although the next Lemma is not need anywhere in this paper, we state it for completeness.

Lemma A.0.3 (Remark 4.2, [8]). Under the above assumption, $u \in H^1_{loc}(\Omega)$ and $\mu := \Delta u$ is a positive Radon measure with support on its free boundary $F(u) = \partial \{u > 0\} \cap \Omega$.

The following Theorem plays a crucial role in this theory.

Theorem A.0.4 (Theorem 4.3, [8]). Assume u is as in $(\star)$. The following statements are equivalent

a) For $D \subset\subset \Omega$ there are constants $0 < c \leq C < \infty$, such that for balls $B_\rho \subset D$ with center on $F(u)$,

$$c \leq \frac{1}{\rho} \fint_{\partial B_\rho} u d\mathcal{H}^{N-1} \leq C$$

a) For $D \subset\subset \Omega$ there are constants $0 < c \leq C < \infty$, such that for balls $B_\rho \subset D$ with center on $F(u)$,

$$c\rho^{N-1} \leq \int_{B_\rho} d\mu \leq C\rho^{N-1}$$

Now, we state the representation Theorem.

Theorem A.0.5 (Representation Theorem - Theorem 4.5, [8]). Suppose u satisfies $(\star)$ and conditions $a)$ or $b)$ of the last Theorem. Then,

1) $\mathcal{H}^{N-1}(D \cap \partial \{u > 0\}) < \infty$ for every $D \subset\subset \Omega$,

2) There is a Borel function q_u, such that

$$\Delta u = q_u \mathcal{H}^{N-1} \lfloor \partial \{u > 0\},$$

that is, for every $\zeta \in C_0^\infty(\Omega)$, we have

$$-\int_\Omega \langle \nabla u, \nabla \zeta \rangle \, dx = \int_{F(u)} \zeta q_u d\mathcal{H}^{N-1}$$

3) For $D \subset\subset \Omega$ there are constants $0 < c \leq C < \infty$ depending on N, Ω, D, and the constants in A.0.4(a), such that for balls $B_\rho(x) \subset D$ with $x \in \partial \{u > 0\}$,

$$c < q_u(x) \leq C, \qquad c\rho^{N-1} \leq \mathcal{H}^{N-1}(B_\rho(x) \cap \partial \{u > 0\}) \leq C\rho^{N-1}$$

We now recall some concepts: For any set E and $x_0 \in E$, we define the (topological) tangent cone of E at x_0 by

$$Tan(E, x_0) = \left\{ v \mid \; v = \lim_{m \to \infty} r_m v_m, \; r_m > 0, x_0 + v_m \in E, \; v_m \to 0 \text{ as } m \to \infty \right\}$$

Also, if μ is a Radon Measure in $\mathbb{R}^N$, we define its upper-density at $x_0 \in \mathbb{R}^N$, by

$$\Theta^{*N-1}(\mu, x_0) := \limsup_{\rho \to 0} \frac{\mu(B_\rho(x_0))}{\alpha(N-1)\rho^{N-1}}$$

where,

$$\alpha(s) := \frac{\pi^{s/2}}{\Gamma(s/2+1)} \qquad \text{and} \qquad \Gamma(s) = \int_0^\infty e^{-x} x^{s-1} dx$$

Now, we can state the Theorem that identifies the density q_u. In this Theorem, we will need the concept of inner normal vector in the measure theoretical sense and reduced boundary introduced in Definition (3.0.7).

Theorem A.0.6 (Identification of the density q_u, Theorem 4.8, Remark 4.9 [8]). Suppose u satisifies $(\star)$ and the properties of Theorem (A.0.4). Let $x_0 \in \partial_{red}\{u>0\}$ with

$$\Theta^{*N-1}(\mathcal{H}^{N-1}\lfloor\{u>0\}, x_0) \leq 1$$

Then, $Tan(\partial\{u>0\}, x_0) = \{x / \langle x, \nu(x_0)\rangle = 0\}$. In addition, if

$$\fint_{B_\rho(x_0)\cap F(u)} |q_u - q_u(x_0)| d\mathcal{H}^{N-1} = o(\rho^{N-1}) \quad as \quad \rho \to 0$$

then,

$$u(x) = q_u(x_0)\langle x - x_0, \nu\rangle^+ + o(|x - x_0|) \quad \text{as} \quad x \to 0$$

Furthermore, this conclusion holds for $\mathcal{H}^{N-1} a.e.$ $x_0 \in \partial_{red}\{u>0\}$.

Appendix B

Linear Behaviour at Regular Boundary Points

In this Appendix B, we state the fundamental results proven by Luis Caffarelli about the asymptotic behavior of Harmonic functions at regular boundary points.

Theorem B.0.7 (Linear behaviour at regular boundary points - Lemma 11.17, [10]; Lemma A1, [6]). Let $u \in C^{0,1}_{loc}(\overline{\Omega})$ be a positive harmonic function in a domain Ω. Assume that $x_0 \in \partial\Omega$ and that u vanishes in $B_1(x_0)$. Then the following hold,

a) If x_0 is regular from the *right* , with touching ball B, then near x_0, u has the following asymptotic development,

$$u(x) = \alpha \langle x - x_0, \nu \rangle^+ + o(|x - x_0|)$$

with $\alpha > 0$, where ν is the unit normal to ∂B at x_0, inward to Ω.

b) If x_0 is regular from the *left*, near x_0,

$$u(x) = \beta \langle x - x_0, \nu \rangle^- + o(|x - x_0|)$$

with $\beta \geq 0$. Moreover, if $\beta > 0$, then B is tangent to $\partial\Omega$ at x_0.

A small modification in the proof of the Theorem above, provides the following

Proposition B.0.8 (Lemma 4.1, [17]). Let U be a Lipschitz function in some ball B centerd at the origin. Assume that U is nonnegative and subharmonic in B, $U(0) = 0$. Assume, in addition that $U \equiv 0$ in some ball $B_\rho(y) \subset \{x_1 = 0\}$, $B_\rho(y) \subset\subset B$, $0 \in \partial B_\rho(y)$. Then, near the origin, U has the asymptotic development

$$U(x) = \alpha x_1^+ + o(|x|) \qquad \text{in} \qquad \{x_1 > 0\}\,,$$

with $\alpha \geq 0$.

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Vita

Diego Ribeiro Moreira

Born: August 18, 1977

Citzenship: Brazilian

Education: M.S., Univesidade Federal do Ceara, Brazil - 2001

B.S., Universidade Federal do Ceara, Brazil, 2000.

Permanent address: Rua Dr. Gilberto Sutdart No. 955 Apt. 101
60190-750
Fortaleza, Ceara, Brasil

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