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Volume-Dependent Field Theories

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Volume-Dependent Field Theories

by

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Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

December 2023

Dedication

To my father who opened my senses to Nature.

To my mother who, beneath a fog of evaporated memories, harbors a soul filled with Love.

Acknowledgments

Thank you, Dan, for teaching me how to fish.

Thank you to all of my friends both in and out of the department. Your energy, thoughtfulness, kindness, and love have made me who I am.

Abstract

Volume-Dependent Field Theories

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The University of Texas at Austin, 2023

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We develop the axiom system in Kontsevich and Segal (2021) to define *volume-dependent field theories (VFTs)*, a class of non-topological quantum field theories whose dependence on the background metric factors through the associated density. We construct a well-defined Lorentzian limit of a Wick-rotated VFT defined on smooth, possibly degenerate Lorentzian bordisms with incoming and outgoing boundary both nonempty. When the VFT is reflection positive, this extends the main theorem in Kontsevich and Segal (2021).

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Chapter 1: Introduction

The axiomatization of topological field theories (Atiyah (1988), following Segal (1988)) has allowed the development of a fruitful mathematical enterprise that has in turn shed light back onto physics. It is natural to wonder what a general mathematical framework for non-topological field theories would add to our understanding of the world.

A non-topological quantum field theory depends on a background Lorentzian metric of space-time. To study such theories, Kontsevich and Segal (2021) proposed *Wick-rotating* to the larger space of *allowable complex metrics*. These metrics form a convex open subset of the space of non-degenerate complex symmetric 2-tensors on space-time. The space of allowable metrics contains the space of Riemannian metrics and its *Shilov boundary*, the holomorphic generalization of the set of extremal points for a complex domain, contains the space of Lorentzian metrics. Wick-rotated quantum field theory can then be defined as a functor out of a bordism category of *manifold germs* equipped with a background allowable complex metric to a suitable category of topological vector spaces such that the vector spaces and linear maps assigned by the functor depend *holomorphically* on the infinite dimensional complex manifold of allowable metrics. One can then attempt to recover the Lorentzian theory by evaluating on manifolds whose allowable metric approaches the Shilov boundary.

In two dimensions, an allowable metric is equivalent to a pair of complex structures together with a complex density. A topological theory has no dependence on the metric and a conformal theory is one that depends only on the pair of complex structures. In this thesis, we study field theories which depend solely on the density.

Given any manifold M, there is a holomorphic map

$$\sqrt{\det} : \mathrm{Met}_{\mathbf{C}}(M) \to \mathrm{Dens}_{\mathbf{C}}(M) \tag{1.1}$$

from the complex manifold of allowable complex metrics on M to the complex manifold of *allowable densities* on M which extends to a map on Shilov boundaries

$$\mathrm{Met}_{\mathrm{Lor}}(M) \to \mathrm{Dens}_{i\mathbf{R}}(M) \tag{1.2}$$

from possibly degenerate Lorentzian metrics to purely imaginary densities. We define a *volume-dependent field theory (VFT)* to be a field theory whose dependence on the allowable metric factors through (1.1). Equivalently, it is a field theory whose background fields are allowable densities and whose dependence on the allowable density is holomorphic.

To construct the Lorentzian limit of a Wick-rotated field theory, one would like to evaluate the functor on a sequence of bordisms whose allowable metric approaches the boundary and take a limit. In general, there is no clear reason why this limit should be independent of the limiting sequence of metrics.

However, in one dimension the situation is nicer. In this case, (1.1) is a biholomorphic equivalence – allowable metrics and densities determine the same data on 1-manifolds. In particular, a Riemannian metric is equivalent to a real and positive density and Moser's theorem Moser (1965) becomes relevant:

Theorem 1.1 (Moser)**.** *Let X be a compact, connected n-manifold and let ω_0, ω_1 be real and positive densities of the same total volume. Then there exists a diffeomorphism $\varphi \in \mathrm{Diff}(X)$ such that*

$$\varphi^*(\omega_1) = \omega_0 \tag{1.3}$$

Together with diffeomorphism invariance of the functor, this implies that 1-dimensional Euclidean field theories depend only on the total volume. Put differently: given any density ω_t in a path of positive densities with constant volume, Moser's argument shows that there exists an infinitesimal diffeomorphism which can be identified with a real vector field ξ_t whose action by Lie derivative on ω_t satisfies

$$\mathcal{L}_{\xi_t}\omega_t = \dot{\omega}_t. \tag{1.4}$$

Diffeomorphism invariance then shows that the value of the functor does not change as the density is varied along the path ω_t.

When the metric is allowable, Moser's argument can be complexified to produce a *complex* vector field ξ_t satisfying (1.4). The same argument together with holomorphicity of the functor then shows that the field theory on a connected bordism depends only on the *total complex volume* valued in the right half plane $\mathbf{C}_{>0}$ of complex numbers with positive real part. This dependence implies that the functor obeys the semigroup law on $\mathbf{C}_{>0}$ which allows one to show that the Lorentzian limit is independent of the sequence of metrics used to construct it.

If the field theory is *reflection positive*, then one can see this explicitly. An allowable germ of a 0-manifold fixed under co-orientation reversal and complex conjugation is assigned a *Hermitian* object in the target category which defines a Hilbert space $\mathcal{H}$. Composition of bordisms respects the semigroup law on $\mathbf{C}_{>0}$ and holomorphicity, together with the assumption of reflection positivity, implies that an interval of length $s \in \mathbf{C}_{>0}$ is assigned the trace class operator

$$e^{-sH} : \mathcal{H} \to \mathcal{H} \tag{1.5}$$

where H is a self-adjoint, unbounded operator on $\mathcal{H}$ with discrete spectrum bounded below. The Lorentzian limit then assigns to a bordism of imaginary length it the unitary operator

$$e^{-itH} : \mathcal{H} \to \mathcal{H}. \tag{1.6}$$

To apply this argument in higher dimensions, Kontsevich and Segal (2021) restrict to *real analytic and globally hyperbolic* bordisms. A bordism $X : \Sigma_0 \rightsquigarrow \Sigma_1$ is globally hyperbolic if every maximally extended time-like geodesic travels from Σ_0 to Σ_1. This implies that $X \cong \Sigma_0 \times I$ is diffeomorphic to a cylinder and there is a global time function $\tau : X \to i\mathbf{R}$ whose level surfaces foliate X by Riemannian manifolds. The Lorentzian metric on X can be expressed as $g = d\tau^2 + h_\tau$, where h_τ is a Riemannian metric on the fiber above $\tau \in i\mathbf{R}$. Real analyticity of X, g and τ implies that g, τ extend holomorphically to a holomorphic thickening $X_{\mathbf{C}}$ of X. Specifically, τ extends to a holomorphic bundle

$$\tau_{\mathbf{C}} : X_{\mathbf{C}} \to U \tag{1.7}$$

where $U \subset \mathbf{C}$ is a neighborhood containing the purely imaginary image of τ. There is a holomorphic trivialization of $\tau_{\mathbf{C}}$ that extends the diffeomorphism $X \cong \Sigma_0 \times I$. This implies that above every curve $\gamma : I \to U$ in the base satisfying $\gamma'(t) > 0$, there is a unique totally real allowable bordism $X_\gamma \subset X_{\mathbf{C}}$ in the preimage $\tau_{\mathbf{C}}^{-1}(\gamma)$.

Adapting the one-dimensional argument above, Kontsevich and Segal (2021) show that the functor applied to X_γ depends only on the total length of γ in $\mathbf{C}_{>0}$ measured using the allowable metric obtained by restricting the complex symmetric 2-tensor $d\tau^2$ to $\gamma \subset U$. This means that the functor obeys a semigroup law on $\mathbf{C}_{>0}$ which allows one to construct a well-defined Lorentzian limit:

Theorem 1.2 (Kontsevich and Segal (2021) Theorem 5.2)**.** *A reflection positive quantum field theory induces a functor*

$$\mathscr{L} : \mathbf{Bord}^{\omega}_{n,n-1}(\mathrm{Met}_{g.h.}) \to \mathbf{Hilb} \tag{1.8}$$

out of the category of real analytic bordisms equipped with a real-analytic globally hyperbolic metric to the category of Hilbert spaces and bounded maps which takes bordisms to unitary operators.

The main goal of this thesis is to extend this theorem when the theory is volume-dependent. In this case, Moser's argument can be applied in all dimensions to show:

Theorem 3.5. *A VFT of general dimension depends only on the total complex volume of a given bordism.*

This implies the key fact that *on cylinders, a VFT obeys the semigroup law on* $\mathbf{C}_{>0}$. This makes possible the construction of a well-defined Lorentzian limit. Furthermore, Moser's argument applies in the smooth setting so one can discard the real analyticity assumption and enlarge the domain to smooth Lorentzian manifolds. In fact, if one allows limits to degenerate Lorentzian metrics in the Shilov boundary, one can go beyond cylinders and construct Lorentzian limits of general bordisms so long as their incoming and outgoing boundaries are both nonempty. However, the operators assigned to such bordisms may no longer be bounded. Instead, they will be morphisms in a certain category of rigged Hilbert spaces which we denote $\mathcal{NP}^h$. We summarize this in the following.

Theorem 3.24. *A reflection positive volume-dependent field theory induces a functor*

$$\mathscr{L} : \mathbf{Bord}_{n,n-1}^{in \wedge out}(\mathrm{Met}_{\mathrm{Lor}}) \to \mathcal{NP}^h \tag{1.9}$$

out of the category of smooth, possibly degenerate Lorentzian bordisms with nonempty incoming and nonempty outgoing boundary. It sends cylindrical bordisms to unitary morphisms and general bordisms to unbounded morphisms of rigged Hilbert spaces.

We now give an overview of the thesis. Chapter 2 develops the definition of field theory debuted in Kontsevich and Segal (2021). Given a sheaf $\mathscr{F}$ on manifolds, we define a bordism category $\mathbf{Bord}_{n,n-1}(\mathscr{F})$ whose objects are *cylindrical germs* equipped with a co-orientation and a germ of a section of $\mathscr{F}$, and whose morphisms are thickened germs of bordisms also equipped with a section of $\mathscr{F}$. Disjoint union defines a symmetric monoidal product. For the purposes of defining holomorphicity, we restrict ourselves to sheaves of smooth sections of fiber bundles whose fibers are finite dimensional complex manifolds. Allowable metrics and densities are examples of such sheaves.

In Section 2.2, we define the codomain category $\mathcal{NP}$ of *nuclear pairs* whose objects are continuous, injective maps with dense image

$$\check{E} \hookrightarrow \hat{E} \tag{1.10}$$

from a *nuclear dual Fréchet space* $\check{E}$ to a *nuclear Fréchet space* $\hat{E}$. The symmetric monoidal structure is given by the topological tensor products on the nuclear spaces $\check{E}$ and $\hat{E}$. In Kontsevich and Segal (2021), nuclear pairs arise as the direct and inverse limits of systems of nuclear maps between Fréchet spaces assigned by the field theory to direct and inverse systems of cylindrical bordisms. In this thesis, we take the symmetric monoidal category of nuclear pairs as fundamental and impose a condition we term *coherence* on the functor to ensure that the nuclear pair that arises from the direct and inverse limits associated to a cylindrical germ is isomorphic to the nuclear pair the functor assigns to it.

In Section 2.3 we define an $\mathscr{F}$*-field theory* to be a coherent symmetric monoidal functor

$$Z : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathcal{NP} \tag{1.11}$$

that depends *holomorphically* on the sections of $\mathscr{F}$. We also require that the functor sends bordisms to morphisms of nuclear pairs consisting of nuclear maps, which we term *nuclearity*. Coherence and nuclearity give way to an action of germs of diffeomorphisms which we elucidate in Theorem 2.16.

When the sheaf is equipped with an *antiholomorphic involution*, there is a twisted involution $\tau_{\mathscr{F}}$ on $\mathbf{Bord}_{n,n-1}(\mathscr{F})$ consisting of conjugating sections composed with reversing co-orientations of the boundary. Likewise, there is a twisted involution $\tau_{\mathcal{NP}}$ on $\mathcal{NP}$ of complex conjugation and taking strong duals of nuclear spaces. We say Z is *reflection positive* if it is $(\tau_{\mathscr{F}}, \tau_{\mathcal{NP}})$-equivariant and if the images of fixed points land in the subcategory $\mathcal{NP}^h$ of *Hermitian nuclear pairs* consisting of $\tau_{\mathcal{NP}}$-fixed points which satisfy a positivity condition. This positivity condition gives Hermitian nuclear pairs the structure of a rigged Hilbert space.

In Chapter 3, we define a VFT to be a $\mathrm{Met}_{\mathbf{C}}$-field theory

$$Z : \mathbf{Bord}_{n,n-1}(\mathrm{Met}_{\mathbf{C}}) \to \mathcal{NP} \tag{1.12}$$

whose dependence on the allowable complex metric factors through (1.1). We use Moser's argument to show total volume dependence and use this to prove that every VFT is naturally isomorphic to one that factors through a nuclear functor

$$V : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathcal{NP} \tag{1.13}$$

out of the category of bordisms whose connected components are labeled by an element of the semigroup $\mathbf{C}_{>0}$ representing the total volume.

In Section 3.2 we construct a Lorentzian limit for general VFTs

$$L : \mathbf{Bord}^{in \wedge out}_{n,n-1}(i\mathbf{R}) \to \mathcal{NP} \tag{1.14}$$

and define the short-distance topological limit as the restriction of this functor to bordisms with 0 volume which in particular is a topological field theory partially defined on a subcategory of the full bordism category.

In Section 3.3, we study reflection positive VFTs. In this case the functor (1.13) can be refined to a functor

$$V^{Hilb} : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathbf{Hilb} \tag{1.15}$$

where the codomain is now the category of separable Hilbert spaces and bounded maps. The value of V^{Hilb} on a closed $n-1$ manifold Σ is a Hilbert space $\mathcal{H}_\Sigma$ and evaluating on a cylindrical bordism gives a family of trace-class operators $V^{Hilb}(\Sigma \times I, s) \in \mathrm{End}(\mathcal{H}_\Sigma)$ depending holomorphically on the total volume $s \in \mathbf{C}_{>0}$. Reflection positivity and the semigroup law imply there exists an unbounded operator $H_\Sigma \in \mathrm{End}(\mathcal{H}_\Sigma)$ with discrete spectrum bounded below such that

$$V^{Hilb}(\Sigma \times s) = \exp(-sH_\Sigma) \tag{1.16}$$

for all $s \in \mathbf{C}_{>0}$. We then prove Theorem 3.24, where we recover that the Lorentzian limit (1.14) of a reflection positive VFT assigns unitary operators to cylindrical bordisms. Furthermore, to general bordisms it assigns an unbounded operator of rigged

Hilbert spaces satisfying a sub-exponential growth condition specified by the eigenvalues of the Hamiltonians H_Σ.

Finally, in Section 3.4 we define the *long-distance topological limit* of a reflection positive VFT. This is a topological field theory obtained from taking the limit as the real part of the total volume goes to positive infinity.

Chapter 2: Geometric Axioms for QFT

In this chapter we define quantum field theory with background fields valued in a sheaf of complex manifolds following Kontsevich and Segal (2021).

2.1 The Bordism Category of Germs

Let $\mathbf{Man}_n$ be the site of n-manifolds without boundary and embeddings between them and let $\mathscr{F} : \mathbf{Man}_n^{op} \to \mathbf{Set}$ be a sheaf. In this section we define a symmetric monoidal category $\mathbf{Bord}_{n,n-1}(\mathscr{F})$ of bordisms between germs of manifolds with background fields valued in $\mathscr{F}$.

2.1.1 Preliminaries on Germs

Let $k \leqslant n$ and let Y be a compact k-manifold possibly with boundary. We will use $[Y]$ to denote the diffeomorphism type of Y as a manifold with boundary. Let $\mathcal{G}_{[Y]}$ be the category whose objects $Y \subset N$ consist of n-manifolds $N \in \mathbf{Man}_n$ containing a smoothly embedded manifold of diffeomorphism type $[Y]$ and whose morphisms are embeddings

$$\begin{array}{ccc} N & \hookrightarrow & N' \\ \cup & & \cup \\ Y & \xrightarrow{\sim} & Y' \end{array} \tag{2.1}$$

restricting to a diffeomorphism of manifolds with boundary. Each object $Y \subset N$ determines a subcategory $\mathfrak{U}_Y^N \subset \mathcal{G}_{[Y]}$ of open neighborhoods $Y \subset U \subset N$ directed by inclusion. Set

$$\mathring{U}_Y^N := \varprojlim_{U \in \mathfrak{U}_Y^N} U \tag{2.2}$$

which is an object in the pro-completion $\mathrm{Pro}(\mathcal{G}_{[Y]})$. We call $\mathring{U}_Y^N$ a *manifold germ* of Y.

The set of morphisms between germs in the pro-completion is by definition

$$\mathrm{Pro}(\mathcal{G}_{[Y]})(\mathring{U}_Y^N, \mathring{U}_{Y'}^{N'}) = \varprojlim_{V \in \mathfrak{U}_{Y'}^{N'}} \varinjlim_{U \in \mathfrak{U}_Y^N} \mathcal{G}_{[Y]}(U, V) \tag{2.3}$$

We set $\mathring{\mathcal{G}}_{[Y]}$ to be the full subcategory of $\mathrm{Pro}(\mathcal{G}_{[Y]})$ generated by all manifold germs $\mathring{U}_Y^N$.

Lemma 2.1. *For all $V \in \mathfrak{U}_{Y'}^{N'}$, the canonical map*

$$\mathring{\mathcal{G}}_{[Y]}(\mathring{U}_Y^N, \mathring{U}_{Y'}^{N'}) \to \varinjlim_{U \in \mathfrak{U}_Y^N} \mathcal{G}_{[Y]}(U, V) \tag{2.4}$$

is a bijection of sets.

Proof. Let $V \overset{j}{\hookrightarrow} V'$ be a morphism in $\mathfrak{U}_{Y'}^{N'}$. There is a map of sets

$$j_* : \varinjlim_{U \in \mathfrak{U}_Y^N} \mathcal{G}_{[Y]}(U, V) \to \varinjlim_{U \in \mathfrak{U}_Y^N} \mathcal{G}_{[Y]}(U, V') \tag{2.5}$$

induced from post-composing an embedding $f : U \hookrightarrow V$ with j. If $\mathring{f}, \mathring{f}$ are elements of $\varinjlim_{U \in \mathfrak{U}_Y^N} \mathcal{G}_{[Y]}(U, V)$ satisfying $j_* \mathring{f} = j_* \mathring{f}'$, then there exists $U \in \mathfrak{U}_Y^N$ and embeddings $f, f' : U \hookrightarrow V$ such that $j \circ f = j \circ f'$. Injectivity of j implies $\mathring{f} = \mathring{f}'$ which implies injectivity of j_*. If $\mathring{g}$ is a germ of an embedding $g : U \hookrightarrow V'$, let $\tilde{U} := g^{-1}(V) \subset U$. Then $g|_{\tilde{U}} : \tilde{U} \hookrightarrow V$ has a germ whose image under j_* is $\mathring{g}$ which implies surjectivity, and therefore bijectivity, of j_*. As each map in the inverse system defining $\mathring{\mathcal{G}}_{[Y]}(\mathring{U}_Y^N, \mathring{U}_{Y'}^{N'})$ is a bijection, this proves the claim. ∎

Given $U \in \mathfrak{U}_Y^N, V \in \mathfrak{U}_{Y'}^{N'}$ there is a composition

$$\mathcal{G}_{[Y]}(U, V) \to \varinjlim_{\tilde{U} \in \mathfrak{U}_Y^N} \mathcal{G}_{[Y]}(\tilde{U}, V) \xrightarrow{\sim} \mathring{\mathcal{G}}_{[Y]}(\mathring{U}_Y^N, \mathring{U}_{Y'}^{N'}) \tag{2.6}$$

of the canonical map to the colimit and the inverse of (2.4). We will refer to the image of $f : U \hookrightarrow V$ under this composition as the *germ* of f.

Proposition 2.2. *Let $\mathring{f} \in \mathring{\mathcal{G}}_{[Y]}(\mathring{U}^N_Y, \mathring{U}^{N'}_{Y'})$. There exists $U \in \mathfrak{U}^N_Y, V \in \mathfrak{U}^{N'}_{Y'}$ and a diffeomorphism $f : U \xrightarrow{\sim} V$ in $\mathcal{G}_{[Y]}(U, V)$ whose germ is $\mathring{f}$.*

Proof. By definition of the colimit and the previous lemma, there exists $U \in \mathfrak{U}^N_Y, \tilde{V} \in \mathfrak{U}^{N'}_{Y'}$, and $f : U \hookrightarrow \tilde{V} \in \mathcal{G}_{[Y]}(U, \tilde{V})$ whose germ if $\mathring{f}$. Then f is a diffeomorphism onto its image $V := \mathrm{Im}(f)$ whose germ is $\mathring{f}$. ∎

Composition of morphisms in $\mathring{\mathcal{G}}_{[Y]}$ can be expressed as follows. Let $\mathring{f} \in \mathring{\mathcal{G}}_{[Y]}(\mathring{U}^N_Y, \mathring{U}^{N'}_{Y'})$ and $\mathring{g} \in \mathring{\mathcal{G}}_{[Y]}(\mathring{U}^{N'}_{Y'}, \mathring{U}^{N''}_{Y''})$. By Proposition 2.2, we can choose $U \in \mathfrak{U}^N_Y, V \in \mathfrak{U}^{N'}_{Y'}, W \in \mathfrak{U}^{N''}_{Y''}$ and diffeomorphisms $f : U \xrightarrow{\sim} V$ and $g : V \xrightarrow{\sim} W$ whose germs are $\mathring{f}, \mathring{g}$. The composition $\mathring{g} \circ \mathring{f}$ is the germ of $g \circ f$.

Corollary 2.3. *$\mathring{\mathcal{G}}_{[Y]}$ is a groupoid.*

Proof. Let $\mathring{f} \in \mathring{\mathcal{G}}_{[Y]}(\mathring{U}^N_Y, \mathring{U}^{N'}_{Y'})$. Proposition 2.2 implies there exists a diffeomorphism $f : U \xrightarrow{\sim} V$ whose germ is $\mathring{f}$. The inverse $\mathring{f}^{-1}$ is the germ of $f^{-1} : V \xrightarrow{\sim} U$. ∎

Remark 2.1. We call an element of (2.3) an *isomorphism of manifold germs.*

Notation. Denote $\mathrm{Diff}(\mathring{U}^N_Y) := \mathring{\mathcal{G}}_{[Y]}(\mathring{U}^N_Y, \mathring{U}^N_Y)$.

Remark 2.2. The sheaf $\mathscr{F} : \mathbf{Man}^{op}_n \to \mathbf{Set}$ induces a pre-sheaf $\mathscr{F} : \mathring{\mathcal{G}}^{op}_{[Y]} \to \mathbf{Set}$ which assigns to a manifold germ $\mathring{U}^N_Y$ the set

$$\mathscr{F}(\mathring{U}^N_Y) = \underset{\overrightarrow{U \in \mathfrak{U}^N_Y}}{\mathrm{colim}}\, \mathscr{F}(U) \tag{2.7}$$

When $Y := \Sigma^{n-1}$ is a closed $n-1$-manifold, we set $\mathcal{G}'_{[\Sigma]} \subset \mathcal{G}_{[\Sigma]}$ to be the full sub-category of co-orientable inclusions $\Sigma \subset N$ and $\mathring{\mathcal{G}}'_{[\Sigma]} \subset \mathring{\mathcal{G}}_{[\Sigma]}$ the corresponding sub-groupoid of co-orientable manifold germs. There is a pre-sheaf $\mathscr{P}^c_{[\Sigma]} : (\mathcal{G}'_{[\Sigma]})^{op} \to \mathbf{Set}$ that assigns to a co-orientable inclusion $\Sigma \subset N$ the set of co-orientations which induces a pre-sheaf $\mathring{\mathscr{P}}^c_{[\Sigma]} : (\mathring{\mathcal{G}}'_{[\Sigma]})^{op} \to \mathbf{Set}$ of co-orientations on manifold germs.

Definition 2.1. Let $\mathring{\mathcal{G}}^c_{[\Sigma]}$ be the groupoid of pairs $(\mathring{U}^N_\Sigma, \mathring{c})$ of a manifold germ $\mathring{U}^N_\Sigma \in \mathring{\mathcal{G}}'_{[\Sigma]}$ and a co-orientation $\mathring{c} \in \mathring{\mathscr{P}}^c_{[\Sigma]}(\mathring{U}^N_\Sigma)$ with morphisms $(\mathring{U}^N_\Sigma, \mathring{c}) \rightsquigarrow (\mathring{U}^{N'}_{\Sigma'}, \mathring{c}')$ consisting of co-orientation preserving isomorphisms $\mathring{f} \in \mathring{\mathcal{G}}_{[\Sigma]}(\mathring{U}^N_\Sigma, \mathring{U}^{N'}_{\Sigma'})$ satisfying $\mathring{f}^*\mathring{c}' = \mathring{c}$.

Remark 2.3. When Y is a k-manifold with corners for all $k \leqslant n$, there will be a corresponding presheaf $\mathscr{P}^c_{[Y]} : \mathring{\mathcal{G}}^{op}_{[Y]} \to \mathbf{Set}$ of co-orientation data needed to glue bordisms between manifolds of higher codimension.

Definition 2.2. Let $C_\Sigma := \Sigma \times \mathbf{R}$ be the cylinder and let $\Sigma \subset C_\Sigma$ denote the inclusion of the 0-slice $\Sigma \times \{0\} \subset \Sigma \times \mathbf{R}$. We will call $\mathring{\Sigma} := \mathring{U}^{C_\Sigma}_\Sigma \in \mathring{\mathcal{G}}'_{[\Sigma]}$ the *cylindrical germ of* Σ. Let $\mathring{c}_+ \in \mathring{\mathscr{P}}^c_{[\Sigma]}(\mathring{\Sigma})$ be the positive co-orientation. We will call $\mathring{\Sigma}^+ := (\mathring{\Sigma}, \mathring{c}_+) \in \mathring{\mathcal{G}}^c_{[\Sigma]}$ the *positively co-oriented cylindrical germ of* Σ.

Proposition 2.4. $\mathring{\Sigma}^+$ *is an initial object in* $\mathring{\mathcal{G}}^c_{[\Sigma]}$.

Proof. Let $(\mathring{U}^N_\Sigma, \mathring{c}) \in \mathring{\mathcal{G}}^c_{[\Sigma]}$. Choose a Riemannian metric on N. The co-orientation $\mathring{c}$ determines a co-orientation and a trivialization of the normal bundle on $\Sigma \subset N$. For $\varepsilon > 0$ small, the exponential map is a co-orientation preserving diffeomorphism

$$\exp_\varepsilon : U_\varepsilon \xrightarrow{\sim} V \tag{2.8}$$

from $U_\varepsilon := \Sigma \times (-\varepsilon, \varepsilon)$ in $\mathfrak{U}^{C_\Sigma}_\Sigma$ onto a tubular neighborhood $V \in \mathfrak{U}^N_\Sigma$ which restricts to the identity on $\Sigma \subset C_\Sigma$. The germ of $\exp_\varepsilon \in \mathcal{G}_{[\Sigma]}(U_\varepsilon, V)$ is an isomorphism from $(\mathring{U}^{C_\Sigma}_\Sigma, \mathring{c}_+)$ to $(\mathring{U}^N_\Sigma, \mathring{c})$. ∎

We now suppose $Y := X$ is a compact n-manifold with boundary. Let $X \subset N$ be an object in $\mathcal{G}_{[X]}$ and let $p : \partial X \to \{0, 1\}$ be a partition of the boundary. Set $\partial X_i := p^{-1}(i)$. The inclusion $X \subset N$ restricts to inclusions of the boundary components $\partial X_i \subset N$ which determine manifold germs $\mathring{U}^N_{\partial X_i}$ and we let $\mathring{c}_i \in \mathring{\mathscr{P}}^c_{[\partial X_i]}(\mathring{U}^N_{\partial X_i})$ be the incoming and outgoing co-orientations for $i = 0$ and $i = 1$.

Let $X' \subset N'$ be another object in $\mathcal{G}_{[X]}$ and $p' : \partial X' \to \{0, 1\}$ a partition. An isomorphism $\mathring{f} \in \mathring{\mathcal{G}}_{[X]}(\mathring{U}^N_X, \mathring{U}^{N'}_{X'})$ restricts to a diffeomorphism $f|_{\partial X} : \partial X \xrightarrow{\sim} \partial X'$

and we say $\mathring{f}$ *respects partitions* if $(f|_{\partial X})^* p' = p$. Such an isomorphism $\mathring{f}$ induces isomorphisms $\mathring{f}_i \in \mathring{\mathcal{G}}_{[\partial X_i]}(\mathring{U}^N_{\partial X_i}, \mathring{U}^{N'}_{\partial X'_i})$.

Definition 2.3. The *groupoid of bordisms of type* $[X]$ is the groupoid $\mathring{\mathcal{G}}^*_{[X]}$ with objects consisting of tuples

$$\mathcal{X} := (\mathring{U}^N_X, p, \mathring{\theta}_0, \mathring{\theta}_1) \tag{2.9}$$

where

- $\mathring{U}^N_X \in \mathring{\mathcal{G}}_{[X]}$ is a manifold germ
- $p : \partial X \to \{0, 1\}$ is a partition of the boundary
- $\mathring{\theta}_i : \mathring{\Sigma}^+_i \xrightarrow{\sim} (\mathring{U}^N_{\partial X_i}, \mathring{c}_i)$ are co-orientation preserving isomorphisms in $\mathring{\mathcal{G}}^c_{[\Sigma_i]}$

and isomorphisms

$$(\mathring{U}^N_X, p, \mathring{\theta}_0, \mathring{\theta}_1) \xrightarrow{\sim} (\mathring{U}^{N'}_{X'}, p', \mathring{\theta}'_0, \mathring{\theta}'_1) \tag{2.10}$$

for each partition-respecting isomorphism $\mathring{f} \in \mathring{\mathcal{G}}_{[X]}(\mathring{U}^N_X, \mathring{U}^{N'}_{X'})$ satisfying $\mathring{f}_i \circ \mathring{\theta}_i = \mathring{\theta}'_i$. We say $\mathcal{X}$ is a *bordism from* $\mathring{\Sigma}^+_0$ *to* $\mathring{\Sigma}^+_1$ which we denote $\mathcal{X} : \mathring{\Sigma}^+_0 \rightsquigarrow \mathring{\Sigma}^+_1$ and we say $\mathring{f}$ is a *diffeomorphism of bordisms*.

2.1.2 $\mathscr{F}$-Bordisms

Let $\mathcal{X} := (\mathring{U}^N_X, p, \mathring{\theta}_0, \mathring{\theta}_1)$ be a bordism from $\mathring{\Sigma}^+_0$ to $\mathring{\Sigma}^+_1$. For $i = 0, 1$, there are maps

$$r^{\mathcal{X}}_i : \mathscr{F}(\mathring{U}^N_X) \to \mathscr{F}(\mathring{U}^N_{\partial X_i}) \xrightarrow{\mathring{\theta}^*_i} \mathscr{F}(\mathring{\Sigma}_i) \tag{2.11}$$

which restrict a germ of a background field on X to the boundary components ∂X_i and pull it back to $\mathring{\Sigma}_i$ along $\mathring{\theta}_i$.

Definition 2.4. Let X be a compact n-manifold with boundary. The *groupoid of $\mathscr{F}$-bordisms of type* $[X]$ is the groupoid $\mathring{\mathcal{G}}^{\mathscr{F}}_{[X]}$ with objects consisting of pairs

$$(\mathcal{X}, \mathring{\sigma}_X) \tag{2.12}$$

where

- $\mathcal{X} := (\mathring{U}^N_X, p, \mathring{\theta}_0, \mathring{\theta}_1)$ is a bordism of type $[X]$
- $\mathring{\sigma}_X \in \mathscr{F}(\mathring{U}^N_X)$ is a germ of a section

and isomorphisms

$$(\mathcal{X}, \mathring{\sigma}_X) \xrightarrow{\sim} (\mathcal{X}', \mathring{\sigma}_{X'}) \tag{2.13}$$

for each $\mathring{f} \in \mathring{\mathcal{G}}^*_{[X]}(\mathcal{X}, \mathcal{X}')$ satisfying $\mathring{f}^*\mathring{\sigma}_{X'} = \mathring{\sigma}_X$. We say $(\mathcal{X}, \mathring{\sigma}_X)$ is an *$\mathscr{F}$-bordism from* $(\mathring{\Sigma}^+_0, \mathring{\sigma}_0)$ *to* $(\mathring{\Sigma}^+_1, \mathring{\sigma}_1)$ where $\mathring{\sigma}_i := r^{\mathcal{X}}_i \mathring{\sigma}_X$. We say $\mathring{f}$ is a *diffeomorphism of $\mathscr{F}$-bordisms.*

Remark 2.4. If $\mathscr{F} : \mathbf{Man}^{op}_n \to \mathbf{Set}$ is the trivial sheaf that assigns to every n-manifold the singleton set $\{*\}$ then $\mathring{\mathcal{G}}^{\mathscr{F}}_{[X]} = \mathring{\mathcal{G}}^*_{[X]}$.

Definition 2.5 (Composition of $\mathscr{F}$-bordisms)**.** Let $\mathring{\sigma}_i \in \mathscr{F}(\mathring{\Sigma}_i)$ for $i = 0, 1, 2$ and let

$$\begin{aligned} (\mathcal{X}, \mathring{\sigma}_X) &: (\mathring{\Sigma}^+_0, \mathring{\sigma}_0) \rightsquigarrow (\mathring{\Sigma}^+_1, \mathring{\sigma}_1) \\ (\mathcal{Y}, \mathring{\sigma}_Y) &: (\mathring{\Sigma}^+_1, \mathring{\sigma}_1) \rightsquigarrow (\mathring{\Sigma}^+_2, \mathring{\sigma}_2) \end{aligned} \tag{2.14}$$

be $\mathscr{F}$-bordisms where

$$\begin{aligned} \mathcal{X} &:= (\mathring{U}^N_X, p, \mathring{\theta}_0, \mathring{\theta}_1) : \mathring{\Sigma}^+_0 \rightsquigarrow \mathring{\Sigma}^+_1 \\ \mathcal{Y} &:= (\mathring{U}^M_Y, q, \mathring{\eta}_1, \mathring{\eta}_2) : \mathring{\Sigma}^+_1 \rightsquigarrow \mathring{\Sigma}^+_2 \end{aligned} \tag{2.15}$$

are bordisms. We assume there are sections $\sigma_N \in \mathscr{F}(N)$ and $\sigma_M \in \mathscr{F}(M)$ whose restrictions to $\mathring{U}^N_X$ and $\mathring{U}^M_Y$ are $\mathring{\sigma}_X$ and $\mathring{\sigma}_Y$; if not, we replace N, M by open neighborhoods containing X, Y that admit such sections.

Let

$$\begin{aligned} \theta_0 &: U_0 \hookrightarrow N \qquad & \eta_1 &: U_1 \hookrightarrow M \\ \theta_1 &: U_1 \hookrightarrow N \qquad & \eta_2 &: U_2 \hookrightarrow M \end{aligned} \tag{2.16}$$

be embeddings of neighborhoods $\Sigma_i \subset U_i \subset C_{\Sigma_i}$ restricting to $\mathring{\theta}_0, \mathring{\theta}_1, \mathring{\eta}_1, \mathring{\eta}_2$. Set

$$\begin{aligned} \tilde{X} &:= X \cup \operatorname{Im}(\theta_1) \\ \tilde{Y} &:= \operatorname{Im}(\eta_1) \cup Y \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} V &:= \operatorname{Im}(\theta_0) \cup X \cup \operatorname{Im}(\theta_1) \\ W &:= \operatorname{Im}(\eta_1) \cup Y \cup \operatorname{Im}(\eta_2). \end{aligned} \tag{2.18}$$

Let

$$L := V \cup_{U_1} W \tag{2.19}$$

be the smooth manifold obtained from gluing V, W along U_1 with θ_1, η_1. The sections σ_N, σ_M restrict to sections σ_V, σ_W on V, W which agree on the overlap inside L. By the sheaf property, they glue uniquely to a section $\sigma_L \in \mathscr{F}(L)$. The smooth manifold

$$Y \circ X := \tilde{X} \cup_{U_1} \tilde{Y} \tag{2.20}$$

is a submanifold of L with boundary isomorphic to $p^{-1}(0) \sqcup q^{-1}(1)$. The section σ_L restricts to a germ $\mathring{\sigma}_{Y \circ X} \in \mathscr{F}(\mathring{U}^L_{Y \circ X})$. Set $r : \partial(Y \circ X) \to \{0, 1\}$ to be the partition that sends $p^{-1}(0)$ to 0 and $q^{-1}(1)$ to 1. Regard $\mathring{\theta}_0, \mathring{\eta}_2$ as germs of embeddings into L. We say the bordism

$$\mathcal{Y} \circ \mathcal{X} := (\mathring{U}^L_{Y \circ X}, r, \mathring{\theta}_0, \mathring{\eta}_2) : \mathring{\Sigma}^+_0 \rightsquigarrow \mathring{\Sigma}^+_2 \tag{2.21}$$

is *a composition of* $\mathcal{X}$ *and* $\mathcal{Y}$ and the $\mathscr{F}$-bordism

$$(\mathcal{Y} \circ \mathcal{X}, \mathring{\sigma}_{Y \circ X}) : (\mathring{\Sigma}^+_0, \mathring{\sigma}_0) \rightsquigarrow (\mathring{\Sigma}^+_2, \mathring{\sigma}_2) \tag{2.22}$$

is *a composition of* $(\mathcal{X}, \mathring{\sigma}_X)$ *and* $(\mathcal{Y}, \mathring{\sigma}_Y)$.

Proposition 2.5. *Composition is well-defined and unique on diffeomorphism classes of $\mathscr{F}$-bordisms.*

Proof. Let $(\mathcal{X}, \mathring{\sigma}_X), (\mathcal{Y}, \mathring{\sigma}_Y)$ be fixed $\mathscr{F}$-bordisms with notation as above. We first show that the compositions obtained from different sets of representatives (2.16) are diffeomorphic. Let θ_i, η_j be as above and let

$$\begin{aligned} \theta_0' &: U_0' \hookrightarrow N & \eta_1' &: U_1' \hookrightarrow M \\ \theta_1' &: U_1' \hookrightarrow N & \eta_2' &: U_2' \hookrightarrow M \end{aligned} \tag{2.23}$$

be another set of embeddings restricting to $\mathring{\theta}_0, \mathring{\theta}_1, \mathring{\eta}_1, \mathring{\eta}_2$. Let $\tilde{X}, \tilde{Y}, V, W, L$ be as before, and set

$$\begin{aligned} \tilde{X}' &:= X \cup \operatorname{Im}(\theta_1') \\ \tilde{Y}' &:= \operatorname{Im}(\eta_1') \cup Y \end{aligned} \tag{2.24}$$

$$\begin{aligned} V' &:= \operatorname{Im}(\theta_0') \cup X \cup \operatorname{Im}(\theta_1') \\ W' &:= \operatorname{Im}(\eta_1') \cup Y \cup \operatorname{Im}(\eta_2'). \end{aligned} \tag{2.25}$$

$$L' := V' \cup_{U_1'} W' \tag{2.26}$$

There is a diffeomorphism

$$\begin{aligned} (Y \circ X)' &:= \tilde{X}' \cup_{U_1'} \tilde{Y}' \\ &\cong \tilde{X} \cup_{U_1} \tilde{Y} \\ &=: Y \circ X \end{aligned} \tag{2.27}$$

that commutes with the embeddings of X and Y. As θ_0, θ_0' and η_2, η_2' define the same germs, they restrict to the same embeddings

$$\theta_0'' : \tilde{U}_0 \hookrightarrow N \quad \text{and} \quad \eta_2'' : \tilde{U}_2 \hookrightarrow M \tag{2.28}$$

on small enough open neighborhoods $\Sigma_0 \subset \tilde{U}_0 \subset U_0 \cap U_0'$ and $\Sigma_2 \subset \tilde{U}_2 \subset U_2 \cap U_2'$. Set

$$\begin{aligned} \tilde{V} &:= \operatorname{Im}(\theta_0'') \cup X \cup \operatorname{Im}(\theta_1') \\ \tilde{W} &:= \operatorname{Im}(\eta_1) \cup Y \cup \operatorname{Im}(\eta_2'') \end{aligned} \quad \text{and} \quad \begin{aligned} \tilde{V}' &:= \operatorname{Im}(\theta_0'') \cup X \cup \operatorname{Im}(\theta_1') \\ \tilde{W}' &:= \operatorname{Im}(\eta_1') \cup Y \cup \operatorname{Im}(\eta_2'') \end{aligned} \tag{2.29}$$

The diffeomorphism (2.27) extends to a diffeomorphism

$$\tilde{V}' \cup_{U_1'} \tilde{W}' \cong \tilde{V} \cup_{U_1} \tilde{W} \tag{2.30}$$

which commutes with the embeddings (2.28) and defines a diffeomorphism of bordisms

$$\mathcal{Y} \circ \mathcal{X} \cong (Y \circ \mathcal{X})' \tag{2.31}$$

where

$$\begin{aligned} \mathcal{Y} \circ \mathcal{X} &:= (\mathring{U}^{L}_{Y \circ X}, r, \mathring{\theta}_0, \mathring{\eta}_2) \\ (\mathcal{Y} \circ \mathcal{X})' &:= (\mathring{U}^{L'}_{(Y \circ X)'}, r, \mathring{\theta}_0, \mathring{\eta}_2) \end{aligned} \tag{2.32}$$

The section $\sigma_{L'}$ pulls back to σ_L along (2.30) which gives a diffeomorphism

$$(\mathcal{Y} \circ \mathcal{X}, \mathring{\sigma}_{Y \circ X}) \cong ((\mathcal{Y} \circ \mathcal{X})', \mathring{\sigma}_{(Y \circ X)'}) \tag{2.33}$$

of $\mathscr{F}$-bordisms.

The proof that if

$$(\mathcal{X}, \mathring{\sigma}_X) \cong (\mathcal{X}', \mathring{\sigma}_{X'}) \quad \text{and} \quad (\mathcal{Y}, \mathring{\sigma}_Y) \cong (\mathcal{Y}', \mathring{\sigma}_{Y'}) \tag{2.34}$$

are diffeomorphic $\mathscr{F}$-bordisms, then

$$(\mathcal{Y} \circ \mathcal{X}, \mathring{\sigma}_{Y \circ X}) \cong (\mathcal{Y}' \circ \mathcal{X}', \mathring{\sigma}_{Y' \circ X'}) \tag{2.35}$$

is similar. ■

Definition 2.6. We define the symmetric monoidal bordism semicategory $\mathbf{Bord}^s_{n,n-1}(\mathscr{F})$ of germs with background fields valued in $\mathscr{F}$ as follows. An object is a pair $(\mathring{\Sigma}^+, \mathring{\sigma})$ consisting of a positively co-oriented cylindrical germ of a closed $n-1$-manifold Σ and a germ of a section $\mathring{\sigma} \in \mathscr{F}(\mathring{\Sigma})$. Morphisms are $\mathscr{F}$-bordisms up to diffeomorphism. Composition of morphisms is composition of bordisms up to diffeomorphism. The symmetric monoidal product is given by disjoint union with the empty manifold germ acting as the unit.

Notation. Denote $\mathrm{Diff}(\mathring{\Sigma}^+) := \mathring{\mathcal{G}}^c_{[\Sigma]}(\mathring{\Sigma}^+, \mathring{\Sigma}^+)$.

Definition 2.7. The bordism category $\mathbf{Bord}_{n,n-1}(\mathscr{F})$ is the category containing the bordism semicategory with the addition of isomorphisms $\mathring{f} : (\mathring{\Sigma}^+, \mathring{\sigma}) \xrightarrow{\sim} (\mathring{\Sigma}^+, \mathring{\sigma}')$ for each $\mathring{f} \in \mathrm{Diff}(\mathring{\Sigma}^+)$ satisfying $\mathring{f}^*\mathring{\sigma}' = \mathring{\sigma}$.

Proposition 2.6. *A morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ induces a symmetric monoidal functor*

$$\varphi_* : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathbf{Bord}_{n,n-1}(\mathscr{G}) \tag{2.36}$$

Proof. The sheaf morphism φ extends to a morphism of presheaves on $\mathrm{Pro}(\mathbf{Man}_n)$. We use this to define φ_* which acts on objects and morphisms by

$$\begin{aligned} (\mathring{\Sigma}^+, \mathring{\sigma}) &\mapsto (\mathring{\Sigma}^+, \varphi(\mathring{\sigma})) \\ (\mathcal{X}, \mathring{\sigma}_X) &\mapsto (\mathcal{X}, \varphi(\mathring{\sigma}_X)) \\ \mathring{f} &\mapsto \mathring{f} \end{aligned} \tag{2.37}$$

where the action on diffeomorphisms is justified by functoriality of sheaf morphisms.

∎

2.2 The Category of Nuclear Pairs

Let $\mathcal{NF}$ and $\mathcal{NDF}$ denote the symmetric monoidal categories of nuclear Fréchet and nuclear dual Fréchet spaces. For a review of these categories and nuclear spaces in general we refer the reader to Appendix A which we will peruse throughout this section.

Definition 2.8. A *nuclear pair* is a continuous dense injection $\check{E} \overset{\iota_E}{\hookrightarrow} \hat{E}$ with $\check{E} \in \mathcal{NDF}, \hat{E} \in \mathcal{NF}$. A morphism of nuclear pairs is a commutative diagram

$$\begin{array}{ccc} \check{E} & \overset{\iota_E}{\hookrightarrow} & \hat{E} \\ \check{f}\downarrow & & \downarrow\hat{f} \\ \check{F} & \overset{\iota_F}{\hookrightarrow} & \hat{F} \end{array} \tag{2.38}$$

The category of such pairs will be denoted $\mathcal{NP}$.

By Proposition A.38, the map $\check{E} \overset{\iota_E}{\hookrightarrow} \hat{E}$ is nuclear.

Definition 2.9. We will say a morphism of nuclear pairs is *nuclear* if there exists a factorization

$$\begin{array}{ccc} \check{E} & \xhookrightarrow{\iota_E} & \hat{E} \\ {\scriptstyle \check{f}}\downarrow & \swarrow & \downarrow{\scriptstyle \hat{f}} \\ \check{F} & \xhookrightarrow{\iota_F} & \hat{F} \end{array} \tag{2.39}$$

through a continuous map $\hat{E} \to \check{F}$.

Lemma 2.7. *The maps $\check{f}$ and $\hat{f}$ in (2.39) are nuclear.*

Proof. Both maps are a composition of a nuclear map with a continuous map. Apply Proposition A.23. ■

Lemma 2.8. $\mathcal{NP}$ *is a symmetric monoidal category.*

Proof. The monoidal product is given by

$$[\check{E} \xhookrightarrow{\iota_E} \hat{E}] \otimes [\check{F} \xhookrightarrow{\iota_F} \hat{F}] := [\check{E}\widehat{\otimes}\check{F} \xhookrightarrow{\iota_E \widehat{\otimes} \iota_F} \hat{E}\widehat{\otimes}\hat{F}]. \tag{2.40}$$

where $\iota_E\widehat{\otimes}\iota_F$ is the map from Corollary A.18 which by Proposition A.19 is injective with dense image. We omit the verification of the axioms for a symmetric monoidal category, which are implied by Proposition A.30. ■

Lemma 2.9. *The transpose*

$$\hat{E}^* \xrightarrow{\iota_E^*} \check{E}^* \tag{2.41}$$

is a nuclear pair and there is a canonical isomorphism

$$\begin{array}{ccc} \check{E} & \xhookrightarrow{\quad\iota_E\quad} & \hat{E} \\ \wr\Vert\downarrow & & \wr\Vert\downarrow \\ (\check{E}^*)^* & \xhookrightarrow{(\iota_E^*)^*} & (\hat{E}^*)^* \end{array} \tag{2.42}$$

of nuclear pairs.

Proof. By Proposition A.6, the transpose $\hat{E}^* \xrightarrow{\iota_E^*} \check{E}^*$ is continuous. By Corollary A.32 and Proposition A.33, $\check{E}$ and $\hat{E}$ are reflexive spaces which implies the isomorphism (2.42). Together with Proposition A.5, we see that ι_E is injective and dense if and only if ι_E^* is injective and dense. Finally, Proposition A.27 implies that $\hat{E}^* \overset{\iota_E^*}{\hookrightarrow} \check{E}^*$ is nuclear. ■

Let $\delta_{\mathcal{NP}} : \mathcal{NP} \to \mathcal{NP}^{op}$ be the functor that sends $\check{E} \overset{\iota_E}{\hookrightarrow} \hat{E}$ to $\hat{E}^* \overset{\iota_E^*}{\hookrightarrow} \check{E}^*$ and a morphism (2.38) to the morphism

$$\begin{array}{ccc} \hat{F}^* & \overset{\iota_F^*}{\hookrightarrow} & \check{F}^* \\ \hat{f}^*\downarrow & & \downarrow\check{f}^* \\ \hat{E}^* & \overset{\iota_E^*}{\hookrightarrow} & \check{E}^* \end{array} \tag{2.43}$$

The isomorphism (2.42) is functorial – thus $\delta_{\mathcal{NP}}^2$ is naturally isomorphic to the identity functor which makes $\delta_{\mathcal{NP}}$ a twisted involution. For an overview of involutions on categories we refer the reader to Appendix B of Freed and Hopkins (2021).

There is another functor $\alpha_{\mathcal{NP}} : \mathcal{NP} \to \mathcal{NP}$ which sends a nuclear pair $\check{E} \overset{\iota}{\hookrightarrow} \hat{E}$ to its complex conjugate $\overline{\check{E}} \overset{\bar{\iota}}{\hookrightarrow} \overline{\hat{E}}$ and a morphism (2.38) to the morphism

$$\begin{array}{ccc} \overline{\check{E}} & \overset{\overline{\iota_E}}{\hookrightarrow} & \overline{\hat{E}} \\ \overline{\check{f}}\downarrow & & \downarrow\overline{\hat{f}} \\ \overline{\check{F}} & \overset{\overline{\iota_F}}{\hookrightarrow} & \overline{\hat{F}} \end{array} \tag{2.44}$$

The canonical natural isomorphism of $\alpha_{\mathcal{NP}}^2$ with the identity functor makes $\alpha_{\mathcal{NP}}$ an involution on $\mathcal{NP}$.

The involutions $\delta_{\mathcal{NP}}$ and $\alpha_{\mathcal{NP}}$ commute which implies the composition

$$\tau_{\mathcal{NP}} := \delta_{\mathcal{NP}} \circ \alpha_{\mathcal{NP}} \tag{2.45}$$

is a twisted involution on $\mathcal{NP}$. A fixed point of $\tau_{\mathcal{NP}}$ is a nuclear pair $\check{E} \overset{\iota}{\hookrightarrow} \hat{E}$ equipped with an isomorphism

$$\begin{array}{ccc} \check{E} & \overset{\iota}{\hookrightarrow} & \hat{E} \\ \wr\!\!\parallel\big\downarrow\check{\theta} & & \wr\!\!\parallel\big\downarrow\hat{\theta} \\ \overline{\hat{E}}^* & \overset{\bar{\iota}^*}{\hookrightarrow} & \overline{\check{E}}^* \end{array} \tag{2.46}$$

which we will denote θ.

Given a fixed point (2.46), Proposition A.35 implies the composition

$$\hat{\theta} \circ \iota : \check{E} \hookrightarrow \overline{\check{E}}^{*} \tag{2.47}$$

is equivalent to a continuous map

$$\check{E} \widehat{\otimes} \overline{\check{E}} \to \mathbf{C} \tag{2.48}$$

and injectivity of ι implies that the corresponding sesquilinear form

$$\langle , \rangle_{\iota,\theta} : \check{E} \times \overline{\check{E}} \to \mathbf{C} \tag{2.49}$$

is non-degenerate.

Definition 2.10. A $\tau_{\mathcal{NP}}$-fixed point $(\check{E} \overset{\iota}{\hookrightarrow} \hat{E}, \theta)$ will be called *Hermitian* if $\langle , \rangle_{\iota,\theta}$ is a positive definite inner product. We will call a nuclear pair $\check{E} \overset{\iota}{\hookrightarrow} \hat{E}$ *Hermitian* if there exists an isomorphism θ that makes (ι, θ) a Hermitian fixed point.

The domain $\check{E}$ of a Hermitian nuclear pair is a pre-Hilbert space. Let E^{Hilb} be its Hilbert space completion.

Lemma 2.10. *A Hermitian nuclear pair $\check{E} \overset{\iota}{\hookrightarrow} \hat{E}$ admits a factorization*

$$\check{E} \hookrightarrow E^{Hilb} \hookrightarrow \hat{E} \tag{2.50}$$

into a composition of nuclear inclusions with dense image.

Proof. The first map is the injection of $\check{E}$ into its Hilbert space completion and is thus dense. It is continuous because the sesquilinear form (2.49) is jointly continuous. It is therefore a continuous map from a nuclear space to a Banach space and Proposition A.28 implies that it is nuclear. By [Treves Proposition 47.4], its conjugate transpose is a nuclear map and it is equipped with an isomorphism

$$\begin{array}{ccc} \overline{E^{Hilb}}^{*} & \hookrightarrow & \overline{\check{E}}^{*} \\ \wr\!\!\| \big\downarrow \theta^{Hilb} & & \wr\!\!\| \big\downarrow \hat{\theta} \\ E^{Hilb} & \hookrightarrow & \hat{E} \end{array} \tag{2.51}$$

where θ^{Hilb} is the isomorphism induced by the Hilbert space inner product. The bottom inclusion is the second map in (2.50). That the composition is ι is a consequence of the definition of the sesquilinear form (2.49). ■

The proof of Lemma 2.10 implies the isomorphism (2.46) of a Hermitian nuclear pair extends to an isomorphism of Hilbert spaces

$$\begin{array}{ccccc} \check{E} & \hookrightarrow & E^{Hilb} & \overset{\iota}{\hookrightarrow} & \hat{E} \\ \cong\downarrow \check{\theta} & & \cong \downarrow \theta^{Hilb} & & \cong \downarrow \hat{\theta} \\ \overline{\hat{E}}^* & \overset{\bar{\iota}^*}{\hookrightarrow} & \overline{E^{Hilb}}^* & \hookrightarrow & \overline{\check{E}}^* \end{array} \tag{2.52}$$

Let $\mathcal{NP}^h$ denote the full subcategory of $\mathcal{NP}$ whose objects are Hermitian nuclear pairs.

Definition 2.11. A morphism in $\mathcal{NP}^h$

$$\begin{array}{ccccc} \check{E} & \hookrightarrow & E^{Hilb} & \hookrightarrow & \hat{E} \\ \downarrow \check{f} & & & & \downarrow \hat{f} \\ \check{F} & \hookrightarrow & F^{Hilb} & \hookrightarrow & \hat{E} \end{array} \tag{2.53}$$

will be called *bounded (resp. unitary, ...)* if there exists a bounded (resp. unitary, ...) map

$$f^{Hilb} : E^{Hilb} \to F^{Hilb} \tag{2.54}$$

making the diagram (2.53) commute. If there is no such bounded map, we will say it is *unbounded with domain* $\check{E}$ and will say that the unbounded operator f^{Hilb} is *induced* from the morphism (2.53).

Let $\mathcal{NP}^{h,nuc} \subset \mathcal{NP}^h$ be the subcategory with the same objects but with morphisms restricted to those that are nuclear or the identity.

Lemma 2.11. *Every morphism in* $\mathcal{NP}^{h,nuc}$

(2.55)
$$\begin{array}{ccccc} \check{E} & \hookrightarrow & E^{Hilb} & \overset{i}{\hookrightarrow} & \hat{E} \\ \downarrow & & f \swarrow & & \downarrow \\ \check{F} & \underset{j}{\hookrightarrow} & F^{Hilb} & \hookrightarrow & \hat{F} \end{array}$$

is trace-class.

Proof. By Lemma 2.10, the map $f^{Hilb} := j \circ f \circ i$ is a composition of continuous and nuclear maps. Thus Proposition A.23 implies that f^{Hilb} is a trace-class operator. ■

Let **Hilb** denote the symmetric monoidal category of separable Hilbert spaces equipped with the Hilbert-Schmidt tensor product. Let

(2.56)
$$\mathcal{NP}^{h,nuc} \to \mathbf{Hilb}$$

be the functor that sends a Hermitian nuclear pair (2.50) to E^{Hilb} and a nuclear morphism (2.55) to f^{Hilb}. Let $\tau_{\mathbf{Hilb}} : \mathbf{Hilb}^{op} \to \mathbf{Hilb}$ denote the twisted involution that sends a Hilbert space to its conjugate dual and a bounded linear map to its conjugate transpose.

Proposition 2.12. *The functor (2.56) is symmetric monoidal and* $(\tau_{\mathcal{NP}}, \tau_{\mathbf{Hilb}})$*-equivariant.*

Proof. Let $\check{E}_1 \hookrightarrow E_1^{Hilb} \hookrightarrow \hat{E}_1$ and $\check{E}_2 \hookrightarrow E_2^{Hilb} \hookrightarrow \hat{E}_2$ be Hermitian nuclear pairs and let $\langle , \rangle_i$ be the Hilbert space inner product on E_i^{Hilb}. Their monoidal product is the nuclear pair

(2.57)
$$\check{E}_1 \widehat{\otimes} \check{E}_2 \xhookrightarrow{\iota_1 \widehat{\otimes} \iota_2} \hat{E}_1 \widehat{\otimes} \hat{E}_2$$

which is equivalent, under the isomorphism (2.46), to a map

(2.58)
$$(\check{E}_1 \widehat{\otimes} \check{E}_2) \widehat{\otimes} \overline{(\check{E}_1 \widehat{\otimes} \check{E}_2)} \to \mathbf{C}.$$

This defines an inner product $\langle , \rangle_{\iota_1 \widehat{\otimes} \iota_2}$ on $\check{E}_1 \widehat{\otimes} \check{E}_2$ which satisfies

(2.59)
$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle_{\iota_1 \widehat{\otimes} \iota_2} = \langle x_1, y_1 \rangle_{\iota_1} \cdot \langle x_2, y_2 \rangle_{\iota_2}$$

for $x_i, y_i \in \check{E}_i$. Its Hilbert space completion is $E_1^{Hilb} \widehat{\otimes}_{HS} E_2^{Hilb}$, where $\widehat{\otimes}_{HS}$ is the Hilbert-Schmidt tensor product, and thus the functor is symmetric monoidal. The isomorphism (2.52) is functorial and implies equivariance. ■

2.3 Field Theory

In this section we give a functorial definition of field theory with background fields in a sheaf $\mathscr{F}$ of complex manifolds following Kontsevich and Segal (2021).

2.3.1 Sheaves of Complex Manifolds

Let $\mathbf{Fib}_n$ denote the category of smooth fiber bundles

$$\begin{array}{c} E \\ \downarrow \\ M \end{array} \tag{2.60}$$

on n-manifolds $M \in \mathbf{Man}_n$ admitting local trivializations

$$\begin{array}{ccc} X \times U & \xrightarrow{\sim} & E|_U \\ & \searrow \quad \swarrow & \\ & U & \end{array} \tag{2.61}$$

whose restriction to a fiber X is a biholomorphism of finite dimensional complex manifolds. We let morphisms consist of maps of smooth bundles covering embeddings of n-manifolds

$$\begin{array}{ccc} E & \longrightarrow & F \\ \downarrow & & \downarrow \\ M & \hookrightarrow & N \end{array} \tag{2.62}$$

We will say a morphism (2.62) is *holomorphic* if it is fiberwise holomorphic, and *antiholomorphic* if it is fiberwise antiholomorphic. Henceforth we will restrict ourselves to sheaves $\mathscr{F}$ that admit a factorization

$$\mathscr{F} : \mathbf{Man}_n^{op} \xrightarrow{\mathbf{X}^{op}} \mathbf{Fib}_n^{op} \xrightarrow{C^\infty} \mathbf{Set} \tag{2.63}$$

where the functor $C^\infty : \mathbf{Fib}_n^{op} \to \mathbf{Set}$ is the contravariant functor of smooth global sections and where $\mathfrak{X} : \mathbf{Man}_n \to \mathbf{Fib}_n$ is a functor that sends embeddings $M \hookrightarrow N$ of n-manifolds to pullback squares

$$\begin{array}{ccc} \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(N) \\ \downarrow & \lrcorner & \downarrow \\ M & \hookrightarrow & N. \end{array} \tag{2.64}$$

which are in particular holomorphic morphisms in $\mathbf{Fib}_n$. We will refer to such sheaves as *sheaves of complex manifolds*

Remark 2.5. The condition (2.64) means that $\mathfrak{X}(M)$ has a fixed complex manifold as fiber for every M.

Definition 2.12. Let $\mathscr{F}$ be a sheaf of complex manifolds of the form (2.63), S a finite dimensional real (resp. complex) manifold, and $U \in \mathbf{Man}_n$. A map of sets

$$f : S \to \mathscr{F}(U) \tag{2.65}$$

defines a section σ_f of the pullback

$$\begin{array}{ccc} pr_2^*\mathfrak{X}(U) & \longrightarrow & \mathfrak{X}(U) \\ \sigma_f \circlearrowleft \downarrow & & \downarrow \\ S \times U & \xrightarrow{pr_2} & U \end{array} \tag{2.66}$$

We say f is *smooth* (resp. *holomorphic, antiholomorphic*) if σ_f is smooth (resp. holomorphic, antiholomorphic) in S.

Definition 2.13. Let S be a finite dimensional real (resp. complex) manifold and $M \in \mathrm{Pro}(\mathbf{Man}_n)$. We say a map $S \to \mathscr{F}(\mathring{M})$ of sets is *smooth* (resp. *holomorphic, antiholomorphic*) if it admits a factorization through a smooth (resp. holomorphic, antiholomorphic) map

$$S \xrightarrow{f} \mathscr{F}(U) \to \mathscr{F}(M) \tag{2.67}$$

for some morphism $M \hookrightarrow U$ with $U \in \mathbf{Man}_n$.

Definition 2.14. Let $M, N \in \mathrm{Pro}(\mathbf{Man}_n)$. A map of sets

$$\mathscr{F}(M) \to \mathscr{F}(N) \tag{2.68}$$

is *holomorphic* if the composition

$$S \xrightarrow{f} \mathscr{F}(M) \to \mathscr{F}(N) \tag{2.69}$$

is holomorphic for all holomorphic maps f where S is a finite dimensional complex manifold.

Definition 2.15. Let

$$\begin{aligned} \mathscr{F} &: \mathbf{Man}_n^{op} \xrightarrow{\mathbf{X}^{op}} \mathbf{Fib}_n^{op} \xrightarrow{C^\infty} \mathbf{Set} \\ \mathscr{G} &: \mathbf{Man}_n^{op} \xrightarrow{\mathbf{Y}^{op}} \mathbf{Fib}_n^{op} \xrightarrow{C^\infty} \mathbf{Set} \end{aligned} \tag{2.70}$$

be sheaves of complex manifolds. A morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ is *holomorphic* (resp. *antiholomorphic*) if for every manifold $M \in \mathbf{Man}_n$ and holomorphic map $S \to \mathscr{F}(M)$, the composition

$$S \to \mathscr{F}(M) \to \mathscr{G}(M) \tag{2.71}$$

is holomorphic (resp. antiholomorphic).

Definition 2.16. A morphism of sheaves of complex manifolds $\varphi : \mathscr{F} \to \mathscr{G}$ *admits a holomorphic right inverse* if for every $M \in \mathbf{Man}_n$, the map $\varphi(M) : \mathscr{F}(M) \to \mathscr{G}(M)$ has a holomorphic right inverse.

Remark 2.6. A morphism of sheaves that admits a holomorphic right inverse is surjective.

Definition 2.17. Let $\mathfrak{X}, \mathfrak{Y} : \mathbf{Man}_n \to \mathbf{Fib}_n$ be functors satisfying condition (2.64). We will say a natural transformation $\psi : \mathfrak{X} \implies \mathfrak{Y}$ is *holomorphic* (resp. *antiholomorphic*) if it defines maps covering the identity embedding

$$\begin{array}{ccc} \mathfrak{X}(M) & \xrightarrow{\psi(M)} & \mathfrak{Y}(M) \\ & \searrow \quad \swarrow & \\ & M & \end{array} \tag{2.72}$$

that are holomorphic (resp. antiholomorphic). We will say ψ is *submersive* if the maps (2.72) are fiberwise surjective submersions.

Finally, we record the following proposition which follows from applying the definitions.

Proposition 2.13. *A holomorphic (resp. antiholomorphic) natural transformation* $\mathfrak{X} \implies \mathfrak{Y}$ *induces a holomorphic (resp. antiholomorphic) morphism of sheaves of smooth sections* $\mathscr{F} \to \mathscr{G}$.

2.3.2 Nuclearity

Definition 2.18. A symmetric monoidal functor out of the bordism semicategory

$$Z^s : \mathbf{Bord}^s_{n,n-1}(\mathscr{F}) \to \mathcal{NP} \tag{2.73}$$

is *nuclear* if it sends $\mathscr{F}$-bordisms to nuclear morphisms of nuclear pairs (cf. Definition 2.9). We say a symmetric monoidal functor of categories

$$Z : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathcal{NP} \tag{2.74}$$

is *nuclear* if its restriction to $\mathbf{Bord}^s_{n,n-1}(\mathscr{F})$ is nuclear.

2.3.3 Holomorphicity

Let $Z : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathcal{NP}$ be a nuclear functor. For each cylindrical germ $\mathring{\Sigma}$, Z determines families of nuclear dual Fréchet and nuclear Fréchet spaces

$$\begin{aligned} \check{\pi}_{\mathring{\Sigma}} &: \check{\mathscr{E}}_{\mathring{\Sigma}} \to \mathscr{F}(\mathring{\Sigma}) \\ \hat{\pi}_{\mathring{\Sigma}} &: \hat{\mathscr{E}}_{\mathring{\Sigma}} \to \mathscr{F}(\mathring{\Sigma}) \end{aligned} \tag{2.75}$$

related by a map

$$\begin{array}{ccc} \check{\mathscr{E}}_{\mathring{\Sigma}} & \hookrightarrow & \hat{\mathscr{E}}_{\mathring{\Sigma}} \\ & {\scriptstyle \check{\pi}_{\mathring{\Sigma}}} \searrow \quad \swarrow {\scriptstyle \hat{\pi}_{\mathring{\Sigma}}} & \\ & \mathscr{F}(\mathring{\Sigma}) & \end{array} \tag{2.76}$$

whose restriction to a fiber $\check{E}_{\mathring{\sigma}} \hookrightarrow \hat{E}_{\mathring{\sigma}}$ over $\mathring{\sigma} \in \mathscr{F}(\mathring{\Sigma})$ is the nuclear pair assigned to $(\mathring{\Sigma}^+, \mathring{\sigma})$.

For each bordism $\mathcal{X} := (\mathring{U}_X^N, p, \mathring{\theta}_0, \mathring{\theta}_1) : \mathring{\Sigma}_0^+ \rightsquigarrow \mathring{\Sigma}_1^+$, there is a correspondence of sets

$$\begin{array}{ccccc} & & \mathscr{F}(\mathring{U}_X^N) & & \\ & \overset{r_0^{\mathcal{X}}}{\swarrow} & & \overset{r_1^{\mathcal{X}}}{\searrow} & \\ \mathscr{F}(\mathring{\Sigma}_0) & & & & \mathscr{F}(\mathring{\Sigma}_1) \end{array} \tag{2.77}$$

where $r_i^{\mathcal{X}}$ is the map (2.11). Applying Z gives a family of maps

$$\begin{array}{ccc} \underline{\hat{\mathscr{E}}}_{\mathring{\Sigma}_0} & \longrightarrow & \underline{\check{\mathscr{E}}}_{\mathring{\Sigma}_1} \\ & \searrow \quad \swarrow & \\ & \mathscr{F}(\mathring{U}_X^N) & \end{array} \tag{2.78}$$

where $\underline{\hat{\mathscr{E}}}_{\mathring{\Sigma}_0} := (r_0^{\mathcal{X}})^* \hat{\mathscr{E}}_{\mathring{\Sigma}_0}$ and $\underline{\check{\mathscr{E}}}_{\mathring{\Sigma}_1} := (r_1^{\mathcal{X}})^* \check{\mathscr{E}}_{\mathring{\Sigma}_1}$ are obtained by pulling back $\hat{\pi}_{\mathring{\Sigma}_0}$ and $\check{\pi}_{\mathring{\Sigma}_1}$ along $r_i^{\mathcal{X}}$.

Definition 2.19. Z is *holomorphic* if

1. for every cylindrical germ $\mathring{\Sigma}$ and holomorphic map $f : S \to \mathscr{F}(\mathring{\Sigma})$ from a finite dimensional complex manifold S (cf. Definition 2.13), the pullback of (2.76) along f

$$\begin{array}{ccc} f^* \check{\mathscr{E}}_{\mathring{\Sigma}} & \hookrightarrow & f^* \hat{\mathscr{E}}_{\mathring{\Sigma}} \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{2.79}$$

   is a map of holomorphic vector bundles (Definition B.7).

2. for every bordism $\mathcal{X} := (\mathring{U}_X^N, p, \mathring{\theta}_0, \mathring{\theta}_1) : \mathring{\Sigma}_0^+ \rightsquigarrow \mathring{\Sigma}_1^+$ and holomorphic map $f : S \to \mathscr{F}(\mathring{U}_X^N)$, the pullback of (2.78) along f is a holomorphic map of vector bundles

$$\begin{array}{ccc} f_0^* \hat{\mathscr{E}}_{\mathring{\Sigma}_0} & \longrightarrow & f_1^* \check{\mathscr{E}}_{\mathring{\Sigma}_1} \\ & \searrow \quad \swarrow & \\ & S & \end{array} \tag{2.80}$$

   where we have set $f_i := r_i^{\mathcal{X}} \circ f$.

Remark 2.7. The family of maps (2.78) determined by Z is equivalent to a section $Z_{\mathcal{X}}$ of the family of nuclear dual Fréchet spaces

$$\begin{array}{c} \underline{\hat{\mathscr{E}}}^*_{\mathring{\Sigma}_0}\widehat{\otimes}\underline{\check{\mathscr{E}}}_{\mathring{\Sigma}_1} \\ \downarrow\Big)\, Z_{\mathcal{X}} \\ \mathscr{F}(\mathring{U}^N_X) \end{array} \tag{2.81}$$

where we have used Proposition A.35 to identify the fiber $\mathrm{Hom}(\hat{E}_{\mathring{\sigma}_0}, \check{E}_{\mathring{\sigma}_1}) \cong \hat{E}^*_{\mathring{\sigma}_0}\widehat{\otimes}\check{E}_{\mathring{\sigma}_1}$ over $\mathring{\sigma}$ in the preimage

$$\mathscr{F}(\mathring{U}^N_X; \mathring{\sigma}_0, \mathring{\sigma}_1) := (r_0^{\mathcal{X}}, r_1^{\mathcal{X}})^{-1}(\mathring{\sigma}_0, \mathring{\sigma}_1) \tag{2.82}$$

of the map

$$(r_0^{\mathcal{X}}, r_1^{\mathcal{X}}) : \mathscr{F}(\mathring{U}^N_X) \to \mathscr{F}(\mathring{\Sigma}_0) \times \mathscr{F}(\mathring{\Sigma}_1). \tag{2.83}$$

Thus the second condition in Definition 2.19 is equivalent to requiring that the pull-back of (2.81)

$$\begin{array}{c} f^*\left(\underline{\hat{\mathscr{E}}}^*_{\mathring{\Sigma}_0}\widehat{\otimes}\underline{\check{\mathscr{E}}}_{\mathring{\Sigma}_1}\right) \\ \downarrow\Big)\, f^* Z_{\mathcal{X}} \\ S \end{array} \tag{2.84}$$

is a holomorphic section of a holomorphic vector bundle for all holomorphic maps $f : S \to \mathscr{F}(\mathring{U}^N_X)$.

The section (2.81) restricted to $\mathscr{F}(\mathring{U}^N_X; \mathring{\sigma}_0, \mathring{\sigma}_1)$ determines a map

$$Z_{\mathcal{X}} : \mathscr{F}(\mathring{U}^N_X; \mathring{\sigma}_0, \mathring{\sigma}_1) \to \hat{E}^*_{\mathring{\sigma}_0}\widehat{\otimes}\check{E}_{\mathring{\sigma}_1} \tag{2.85}$$

which is holomorphic in the sense that if $f : S \to \mathscr{F}(\mathring{U}^N_X; \mathring{\sigma}_0, \mathring{\sigma}_1)$ is holomorphic then $Z_{\mathcal{X}} \circ f : S \to \hat{E}^*_{\mathring{\sigma}_0}\widehat{\otimes}\check{E}_{\mathring{\sigma}_1}$ is holomorphic.

The group $\mathrm{Diff}(\mathcal{X}) := \mathring{\mathcal{G}}^{\mathscr{F}}_{[X]}(\mathcal{X}, \mathcal{X})$ of self-diffeomorphisms of a bordism $\mathcal{X} := (\mathring{U}^N_X, p, \mathring{\theta}_0, \mathring{\theta}_1)$ acts on $\mathscr{F}(\mathring{U}^N_X)$ by pulling back germs of sections and the maps $r_i^{\mathcal{X}} : \mathscr{F}(\mathring{U}^N_X) \to \mathscr{F}(\mathring{\Sigma}_i)$ are invariant under this action. Thus $\mathrm{Diff}(\mathcal{X})$ restricts to an action on $\mathscr{F}(\mathring{U}^N_X; \mathring{\sigma}_0, \mathring{\sigma}_1)$. Since the field theory is defined on diffeomorphism classes of bordisms, the map (2.85) is invariant under the action of $\mathrm{Diff}(\mathcal{X})$.

2.3.4 Coherence

We will make use of the following lemmas which we simply state; they are consequences of $\mathscr{F}$ being a sheaf of smooth sections of a fiber bundle.

Lemma 2.14. *Let $\mathring{\sigma} \in \mathscr{F}(\mathring{\Sigma})$. Then there exists a section $\sigma \in \mathscr{F}(C_\Sigma)$ whose restriction to $\mathring{\Sigma}$ is $\mathring{\sigma}$.*

Lemma 2.15. *Let $\mathring{f} \in \mathrm{Diff}(\mathring{\Sigma})$. Then there exists a diffeomorphism $f \in \mathrm{Diff}(C_\Sigma)$ whose restriction to $\mathring{\Sigma}$ is $\mathring{f}$.*

Let $(\mathring{\Sigma}^+, \mathring{\sigma})$ be an object in $\mathbf{Bord}^s_{n,n-1}(\mathscr{F})$. By Lemma 2.14, there is a section $\sigma \in \mathscr{F}(C_\Sigma)$ whose germ at $\mathring{\Sigma}$ is $\mathring{\sigma}$.

Denote the slice at time t by $\Sigma_t := \Sigma \times \{t\} \subset C_\Sigma$ and let $\mathring{\Sigma}_t, \mathring{\Sigma}_t^+$ its manifold germ without and with the positive co-orientation. Let $X_{a,b} := \Sigma \times [a, b] \subset C_\Sigma$ for $a < b \in \mathbf{R}$, $\mathring{U}^{C_\Sigma}_{X_{a,b}}$ its manifold germ, $\mathring{\sigma}_{X_{a,b}}$ the restriction of σ to $\mathring{U}^{C_\Sigma}_{X_{a,b}}$, and p be a partition of $\partial X_{a,b}$ that designates Σ_a incoming and Σ_b outgoing. Let

$$\begin{aligned} \theta_t : \Sigma \times \mathbf{R} &\xrightarrow{\sim} \Sigma \times \mathbf{R} \\ (x, s) &\mapsto (x, s + t) \end{aligned} \tag{2.86}$$

be translation by $t \in \mathbf{R}$ whose germ at $\mathring{\Sigma}$ is a co-orientation preserving isomorphism of germs $\mathring{\theta}_t \in \mathring{\mathcal{G}}^c_{[\Sigma]}(\mathring{\Sigma}^+, \mathring{\Sigma}_t^+)$. Then the tuple

$$\mathcal{X}_{a,b} := (\mathring{U}^{C_\Sigma}_{X_{a,b}}, p, \mathring{\theta}_a, \mathring{\theta}_b) \tag{2.87}$$

is a cylindrical bordism from $\mathring{\Sigma}^+$ to itself. If we set $\mathring{\sigma}_t$ to be the pullback by $\mathring{\theta}_t$ of the restriction of σ to $\mathring{\Sigma}_t$ then

$$(\mathcal{X}_{a,b}, \mathring{\sigma}_{X_{a,b}}) \tag{2.88}$$

is an $\mathscr{F}$-bordism from $(\mathring{\Sigma}^+, \mathring{\sigma}_a)$ to $(\mathring{\Sigma}^+, \mathring{\sigma}_b)$.

Let $Z : \mathbf{Bord}^s_{n,n-1}(\mathscr{F}) \to \mathcal{NP}$ be a nuclear symmetric monoidal functor. We will denote by

$$\check{E}_{\mathring{\sigma}} \hookrightarrow \hat{E}_{\mathring{\sigma}} \tag{2.89}$$

the nuclear assigned by Z to the object $(\mathring{\Sigma}^+, \mathring{\sigma})$

Applying Z to the $\mathscr{F}$-bordism (2.88) gives a nuclear morphism of nuclear pairs

$$\begin{array}{ccc} \check{E}_{\mathring{\sigma}_a} & \xhookrightarrow{\iota_a} & \hat{E}_{\mathring{\sigma}_a} \\ {\scriptstyle \check{Z}_{a,b}}\downarrow & \swarrow & \downarrow{\scriptstyle \hat{Z}_{a,b}} \\ \check{E}_{\mathring{\sigma}_b} & \xhookrightarrow{\iota_b} & \hat{E}_{\mathring{\sigma}_b} \end{array} \tag{2.90}$$

where $\check{Z}_{a,b}, \hat{Z}_{a,b}$ are nuclear by Lemma 2.7.

Definition 2.20. The *direct cylindral system of* σ is the direct system $\mathcal{C}_\sigma := \{\check{E}_{\mathring{\sigma}_s} \mid s < 0\}$ whose morphisms consist of the maps

$$\check{Z}_{s,s'} : \check{E}_{\mathring{\sigma}_s} \to \check{E}_{\mathring{\sigma}_{s'}} \tag{2.91}$$

for all $s < s' < 0$. The *inverse cylindrical system of* σ is the inverse system $\mathcal{D}_\sigma := \{\hat{E}_{\mathring{\sigma}_t} \mid t > 0\}$ whose morphisms are

$$\hat{Z}_{t,t'} : \hat{E}_{\mathring{\sigma}_t} \to \hat{E}_{\mathring{\sigma}_{t'}} \tag{2.92}$$

for all $0 < t < t'$.

We will denote the direct and inverse limits of the cylindrical systems associated to σ by

$$\check{E}_{\mathcal{C}_\sigma} := \varinjlim_{\mathcal{C}_\sigma} \check{E}_s \quad \text{and} \quad \hat{E}_{\mathcal{D}_\sigma} := \varprojlim_{\mathcal{D}_\sigma} \hat{E}_t \tag{2.93}$$

The universal properties of direct and inverse limits implies that (2.93) are isomorphic to the direct and inverse limits along any countable final and cofinal subsequence in $\mathcal{C}_\sigma, \mathcal{D}_\sigma$; it follows that we can apply Proposition A.36 and Corollary A.37 to conclude they are in $\mathcal{NDF}$ and $\mathcal{NF}$ respectively. The universal properties also imply the existence of a continuous map

$$\check{E}_{\mathcal{C}_\sigma} \to \hat{E}_{\mathcal{D}_\sigma} \tag{2.94}$$

which is nuclear by Proposition A.38.

The universal properties of the direct and inverse limits further imply the existence of unique maps making the diagrams

$$
\begin{array}{ccc}
\check{E}_{\mathring{\sigma}_s} & \longrightarrow & \check{E}_{\mathcal{C}_\sigma} \\
& {\scriptstyle \check{Z}_{s,0}} \searrow & \Big\downarrow {\scriptstyle \exists!} \\
& & \check{E}_{\mathring{\sigma}}
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\hat{E}_{\mathcal{D}_\sigma} & \longrightarrow & \hat{E}_{\mathring{\sigma}_t} \\
{\scriptstyle \exists!} \Big\uparrow & \nearrow {\scriptstyle \hat{Z}_{0,t}} & \\
\hat{E}_{\mathring{\sigma}} & &
\end{array}
\tag{2.95}
$$

commute for all $s < 0 < t$; here the horizontal maps $\check{E}_{\mathring{\sigma}_s} \to \check{E}_{\mathcal{C}_\sigma}$ and $\hat{E}_{\mathcal{D}_\sigma} \to \hat{E}_{\mathring{\sigma}_t}$ are the canonical maps to and from the direct and inverse limits.

Definition 2.21. We say Z is *coherent* if the above maps are isomorphisms that make the square

$$
\begin{array}{ccc}
\check{E}_{\mathcal{C}_\sigma} & \longrightarrow & \hat{E}_{\mathcal{D}_\sigma} \\
\wr\Big\downarrow & & \wr\Big\uparrow \\
\check{E}_{\mathring{\sigma}} & \hookrightarrow & \hat{E}_{\mathring{\sigma}}
\end{array}
\tag{2.96}
$$

commute; here the top map is (2.94).

Remark 2.8. The above definition is independent of which section we choose to represent the germ $\mathring{\sigma}$, for if $\sigma' \in \mathscr{F}(C_\Sigma)$ with germ $\mathring{\sigma}$ at $\mathring{\Sigma}$, there are final and cofinal subsequences on which $\mathcal{C}_\sigma, \mathcal{C}_{\sigma'}$ and $\mathcal{D}_\sigma, \mathcal{D}_{\sigma'}$ agree which implies canonical isomorphisms

$$
\begin{array}{ccc}
\check{E}_{\mathcal{C}_\sigma} & \longrightarrow & \hat{E}_{\mathcal{D}_\sigma} \\
\wr\Big| & & \wr\Big| \\
\check{E}_{\mathcal{C}_{\sigma'}} & \longrightarrow & \hat{E}_{\mathcal{D}_{\sigma'}}
\end{array}
\tag{2.97}
$$

2.3.5 The Action of Diffeomorphisms

There is an action

$$
\begin{aligned}
\mathrm{Diff}(\mathring{U}_Y^N) \times \mathscr{F}(\mathring{U}_Y^N) &\to \mathscr{F}(\mathring{U}_Y^N) \\
(\mathring{f}, \mathring{\sigma}) &\mapsto (\mathring{f}^{-1})^* \mathring{\sigma}
\end{aligned}
\tag{2.98}
$$

For a fixed $\mathring{\sigma} \in \mathscr{F}(\mathring{U}_Y^N)$ we get an orbit map

$$\begin{aligned} \alpha_{\mathring{\sigma}} : \mathrm{Diff}(\mathring{U}_Y^N) &\to \mathscr{F}(\mathring{U}_Y^N) \\ \mathring{f} &\mapsto (\mathring{f}^{-1})^*\mathring{\sigma} \end{aligned} \tag{2.99}$$

We will now see that a consequence of coherence is that the action (2.98) of $\mathrm{Diff}(\mathring{\Sigma})$ on $\mathscr{F}(\mathring{\Sigma})$ lifts to an action on the family of nuclear pairs (2.76). We first describe the action of the subgroup $\mathrm{Diff}(\mathring{\Sigma}^+)$ of co-orientation preserving germs of diffeomorphisms.

Let $\mathring{f} \in \mathrm{Diff}(\mathring{\Sigma}^+)$; by Lemma 2.15 there exists $f \in \mathrm{Diff}(C_\Sigma)$ whose germ at Σ is $\mathring{f}$. Let $\mathring{\sigma} \in \mathscr{F}(\mathring{\Sigma})$ and let $\sigma \in \mathscr{F}(C_\Sigma)$ whose germ at Σ is $\mathring{\sigma}$. The action of $\mathring{f}$ on $\mathring{\sigma}$ gives the germ $\mathring{\sigma}' := (\mathring{f}^{-1})^*\mathring{\sigma}$. The section $\sigma' := (f^{-1})^*\sigma \in \mathscr{F}(C_\Sigma)$ has germ $\mathring{\sigma}'$ at the 0-slice. As before, we set $\mathring{\sigma}_t, \mathring{\sigma}'_t \in \mathscr{F}(\mathring{\Sigma})$ to be the pullback by $\mathring{\theta}_t$ of the restriction of σ, σ' to $\mathring{\Sigma}_t$.

Let $m, M : \mathbf{R} \to \mathbf{R}$ be the continuous monotone increasing functions defined by

$$\begin{aligned} m(t) &:= \min_{x \in \Sigma_t} pr_2(f(x)) \\ M(t) &:= \max_{x \in \Sigma_t} pr_2(f(x)) \end{aligned} \tag{2.100}$$

where $pr_2 : \Sigma \times \mathbf{R} \to \mathbf{R}$ is the projection onto the second factor. They satisfy $m(0) = M(0) = 0$ and $m(t) \leqslant M(t)$ for all $t \in \mathbf{R}$. In particular, we can find monotone sequences $\{s_i\}, \{s'_i\} \subset \mathbf{R}_{<0}$ and $\{t_i\}, \{t'_i\} \subset \mathbf{R}_{>0}$ converging to 0 such that

$$M(s_i) < s'_i < m(s_{i+1}) \quad \text{and} \quad M(t_{i+1}) < t'_i < m(t_i). \tag{2.101}$$

Set

$$\begin{aligned} X_{s_i,s'_i} &:= f(\Sigma \times (s_i, \infty)) \cap \Sigma \times (-\infty, s'_i) & X_{t_{i+1},t'_i} &:= f(\Sigma \times (t_{i+1}, \infty)) \cap \Sigma \times (-\infty, t'_i) \\ Y_{s'_i,s_{i+1}} &:= \Sigma \times (s'_i, \infty) \cap f(\Sigma \times (-\infty, s_{i+1})) & Y_{t'_i,t_i} &:= \Sigma \times (t'_i, \infty) \cap f(\Sigma \times (-\infty, t_i)) \end{aligned} \tag{2.102}$$

and let $p_X : \partial X_{a',b} \to \{0,1\}$, $p_Y : \partial Y_{a,b'} \to \{0,1\}$ be partitions that designate $\Sigma_{a'}, f(\Sigma_a)$ as incoming and $f(\Sigma_b), \Sigma_{b'}$ as outgoing. If we define

(2.103)
$$\mathcal{X}_{s_i,s_i'} := (\mathring{U}^{C_\Sigma}_{X_{s_i,s_i'}}, p_X, \mathring{\theta}_{s_i}, \mathring{f}^{-1} \circ \mathring{\theta}_{s_i'}) \qquad \mathcal{X}_{t_{i+1},t_i'} := (\mathring{U}^{C_\Sigma}_{X_{t_{i+1},t_i'}}, p_X, \mathring{\theta}_{t_{i+1}}, \mathring{f}^{-1} \circ \mathring{\theta}_{t_i'})$$
$$\mathcal{Y}_{s_i',s_{i+1}} := (\mathring{U}^{C_\Sigma}_{Y_{s_i',s_{i+1}}}, p_Y, \mathring{f}^{-1} \circ \mathring{\theta}_{s_i'}, \mathring{\theta}_{s_{i+1}}) \qquad \mathcal{Y}_{t_i',t_i} := (\mathring{U}^{C_\Sigma}_{Y_{t_i',t_i}}, p_Y, \mathring{f}^{-1} \circ \mathring{\theta}_{t_i'}, \mathring{\theta}_{t_i})$$

and set $\mathring{\sigma}_X, \mathring{\sigma}_Y$ to be the restrictions of σ to $\mathring{U}^{C_\Sigma}_{X_{a',b}}$ and $\mathring{U}^{C_\Sigma}_{Y_{a,b'}}$ then

(2.104)
$$(\mathcal{X}_{s_i,s_i'}, \mathring{\sigma}_X) : (\mathring{\Sigma}^+, \mathring{\sigma}_{s_i}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\sigma}'_{s_i'}) \qquad (\mathcal{X}_{t_{i+1},t_i'}, \mathring{\sigma}_X) : (\mathring{\Sigma}^+, \mathring{\sigma}_{t_{i+1}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\sigma}'_{t_i'})$$
$$(\mathcal{Y}_{s_i',s_{i+1}}, \mathring{\sigma}_Y) : (\mathring{\Sigma}^+, \mathring{\sigma}'_{s_i'}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\sigma}_{s_{i+1}}) \qquad (\mathcal{Y}_{t_i',t_i}, \mathring{\sigma}_Y) : (\mathring{\Sigma}^+, \mathring{\sigma}'_{t_i'}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\sigma}_{t_i})$$

are $\mathscr{F}$-bordisms.

Applying a nuclear symmetric monoidal functor

$$Z^s : \mathbf{Bord}^s_{n,n-1}(\mathscr{F}) \to \mathcal{NP} \tag{2.105}$$

to the above gives the direct system

$$\check{E}_{\mathring{\sigma}_{s_1}} \xrightarrow{Z^s(\mathcal{X}_{s_1,s_1'}, \mathring{\sigma}_X)} \check{E}_{\mathring{\sigma}'_{s_1'}} \xrightarrow{Z^s(\mathcal{Y}_{s_1',s_2}, \mathring{\sigma}_Y)} \check{E}_{\mathring{\sigma}_{s_2}} \to \cdots \tag{2.106}$$

and the inverse system

$$\cdots \hat{E}_{\mathring{\sigma}_{t_2}} \xrightarrow{Z^s(\mathcal{X}_{t_2,t_1'}, \mathring{\sigma}_X)} \hat{E}_{\mathring{\sigma}'_{t_1'}} \xrightarrow{Z^s(\mathcal{Y}_{t_1',t_1}, \mathring{\sigma}_Y)} \hat{E}_{\mathring{\sigma}_{t_1}} \tag{2.107}$$

The unprimed alternating terms give final and cofinal subsequences $\tilde{\mathcal{C}}_\sigma := \{\check{E}_{\sigma_{s_i}}\}$ and $\tilde{\mathcal{D}}_\sigma := \{\hat{E}_{\sigma_{t_i}}\}$ of the direct and inverse systems $\mathcal{C}_\sigma$ and $\mathcal{D}_\sigma$. Likewise, the primed alternating terms give final and cofinal subsequences $\tilde{\mathcal{C}}_{\sigma'}$ and $\tilde{\mathcal{D}}_{\sigma'}$ of $\mathcal{C}_{\sigma'}$ and $\mathcal{D}_{\sigma'}$. Set

(2.108)
$$\check{E}_{\tilde{\mathcal{C}}_\sigma} := \varinjlim_{\tilde{\mathcal{C}}_\sigma} \check{E}_{\sigma_{s_i}} \qquad \hat{E}_{\tilde{\mathcal{D}}_\sigma} := \varprojlim_{\tilde{\mathcal{D}}_\sigma} \hat{E}_{\sigma_{t_i}}$$
$$\check{E}_{\tilde{\mathcal{C}}_{\sigma'}} := \varinjlim_{\tilde{\mathcal{C}}_{\sigma'}} \check{E}_{\sigma'_{s_i'}} \qquad \hat{E}_{\tilde{\mathcal{D}}_{\sigma'}} := \varprojlim_{\tilde{\mathcal{D}}_{\sigma'}} \hat{E}_{\sigma'_{t_i'}}$$

If Z^s is coherent, we have the following sequence of isomorphisms

$$
\begin{array}{ccccccccccc}
\check{E}_{\mathring{\sigma}} & \xrightarrow{\cong} & \check{E}_{\mathcal{C}_\sigma} & \xrightarrow{\cong} & \check{E}_{\tilde{\mathcal{C}}_\sigma} & \xrightarrow{\cong} & \check{E}_{\tilde{\mathcal{C}}_{\sigma'}} & \xrightarrow{\cong} & \check{E}_{\mathcal{C}_{\sigma'}} & \xrightarrow{\cong} & \check{E}_{\mathring{\sigma}'} \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\hat{E}_{\mathring{\sigma}} & \xrightarrow{\cong} & \hat{E}_{\mathcal{D}_\sigma} & \xrightarrow{\cong} & \hat{E}_{\tilde{\mathcal{D}}_\sigma} & \xrightarrow{\cong} & \hat{E}_{\tilde{\mathcal{D}}_{\sigma'}} & \xrightarrow{\cong} & \hat{E}_{\mathcal{D}_{\sigma'}} & \xrightarrow{\cong} & \hat{E}_{\mathring{\sigma}'}
\end{array}
\tag{2.109}
$$

where the left- and right-most isomorphisms are from coherence and the rest are canonical from the universal properties of the direct and inverse limits. The middle isomorphism is from the interwoven systems (2.106) and (2.107).

Remark 2.9. The composition (2.109) is an isomorphism of nuclear pairs that is independent of the choice of f, σ representing $\mathring{f}$, $\mathring{\sigma}$.

We summarize this in the following.

Theorem 2.16. *Let $Z^s : \mathbf{Bord}^s_{n,n-1}(\mathscr{F}) \to \mathcal{NP}$ be a nuclear and coherent symmetric monoidal functor of semicategories. Then Z^s extends canonically to a nuclear and coherent symmetric monoidal functor of categories*

$$Z : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathcal{NP} \tag{2.110}$$

We now describe the action of reversing the co-orientation. Let $\mathring{\delta} \in \mathrm{Diff}(\mathring{\Sigma})$ be the germ at $\Sigma \subset C_\Sigma$ of the co-orientation reversing diffeomorphism

$$
\begin{aligned}
\delta : \Sigma \times \mathbf{R} &\xrightarrow{\sim} \Sigma \times \mathbf{R} \\
(x, t) &\mapsto (x, -t)
\end{aligned}
\tag{2.111}
$$

We use this to define the twisted involution

$$\delta_{\mathscr{F}} : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathbf{Bord}_{n,n-1}(\mathscr{F})^{op} \tag{2.112}$$

acting on objects and morphisms by

$$
\begin{aligned}
(\mathring{\Sigma}^+, \mathring{\sigma}) &\mapsto (\mathring{\Sigma}^+, \mathring{\delta}^* \mathring{\sigma}) \\
(\mathcal{X}, \mathring{\sigma}_X) &\mapsto (\mathcal{X}^*, \mathring{\sigma}_X)
\end{aligned}
\tag{2.113}
$$

where $\mathcal{X} := (\mathring{U}_X^N, p, \mathring{\theta}_0, \mathring{\theta}_1)$ is a bordism and $\mathcal{X}^* := (\mathring{U}_X^N, 1-p, \mathring{\delta}^*\mathring{\theta}_1, \mathring{\delta}^*\mathring{\theta}_0)$ is $\mathcal{X}$ with incoming and outgoing boundaries reversed.

Remark 2.10. Although we defined the $\delta_{\mathscr{F}}$ on $\mathscr{F}$-bordisms, it descends to diffeomorphism classes of bordisms since $\mathring{f}_i \circ \mathring{\delta}^*\mathring{\theta}_i = \mathring{\delta}^*(\mathring{f}_i \circ \mathring{\theta}_i)$ for all diffeomorphisms $\mathring{f}$ of $\mathscr{F}$-bordisms.

Proposition 2.17. *Let $Z : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathcal{NP}$ be a nuclear and coherent symmetric monoidal functor. Then Z is $(\delta_{\mathscr{F}}, \delta_{\mathcal{NP}})$-equivariant.*

We refer the reader to the appendix to Section 3 of Kontsevich and Segal (2021) for a proof.

2.3.6 Definition of Field Theory

Definition 2.22. An *$\mathscr{F}$-field theory* is a symmetric monoidal functor

$$\mathbf{Bord}^s_{n,n-1}(\mathscr{F}) \to \mathcal{NP} \tag{2.114}$$

that is nuclear, holomorphic, and coherent.

Remark 2.11. By Theorem 2.16, an $\mathscr{F}$-field theory (2.114) extends canonically to a symmetric monoidal functor out of the full $\mathscr{F}$-bordism category

$$\mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathcal{NP} \tag{2.115}$$

This extension will also be called an $\mathscr{F}$-field theory.

We record the following proposition for use in the next chapter.

Proposition 2.18. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a holomorphic morphism between sheaves of complex manifolds admitting a holomorphic right inverse and let $Z_{\mathscr{F}} : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathcal{NP}$ be an $\mathscr{F}$-field theory that admits a factorization*

$$\begin{array}{ccc} \mathbf{Bord}_{n,n-1}(\mathscr{F}) & \xrightarrow{Z_{\mathscr{F}}} & \mathcal{NP} \\ \downarrow{\scriptstyle \varphi_*} & \nearrow{\scriptstyle Z_{\mathscr{G}}} & \\ \mathbf{Bord}_{n,n-1}(\mathscr{G}) & & \end{array} \tag{2.116}$$

where φ_* is the functor (2.36) induced from the sheaf morphism. Then $Z_{\mathscr{G}}$ is a $\mathscr{G}$-field theory.

Proof. Let $(\mathfrak{X}, \mathring{\tau}_X)$ be a $\mathscr{G}$-bordism in $\mathbf{Bord}_{n,n-1}(\mathscr{G})$. The sheaf morphism φ is surjective (cf. Remark 2.6) which implies there exists $\mathring{\sigma}_X \in \mathscr{F}(\mathring{U}_X^N)$ satisfying $\varphi(\mathring{\sigma}_X) = \mathring{\tau}_X$. Thus $Z_{\mathscr{G}}(\mathfrak{X}, \mathring{\tau}_X) = Z_{\mathscr{F}}(\mathfrak{X}, \mathring{\sigma}_X)$ is a nuclear morphism of nuclear pairs, which shows that $Z_{\mathscr{G}}$ is nuclear.

Let $\sigma \in \mathscr{G}(C_\Sigma)$ with germ $\mathring{\sigma} \in \mathscr{G}(\mathring{\Sigma})$ and let

$$\psi(\mathring{\Sigma}) : \mathscr{G}(\mathring{\Sigma}) \to \mathscr{F}(\mathring{\Sigma}) \tag{2.117}$$

be a holomorphic right inverse to $\varphi(\mathring{\Sigma}) : \mathscr{F}(\mathring{\Sigma}) \to \mathscr{G}(\mathring{\Sigma})$. Surjectivity of φ implies there exists $\tau \in \mathscr{F}(C_\Sigma)$ with germ $\mathring{\tau} \in \mathscr{F}(\mathring{\Sigma})$ satisfying $\varphi(\tau) = \sigma$ and $\varphi(\mathring{\tau}) = \mathring{\sigma}$. The direct and inverse cylindrical systems obtained from applying Z to the systems of bordisms given by σ, τ are identical, which proves coherence of $Z_{\mathscr{G}}$.

Let

$$\begin{array}{ccc} \check{\mathscr{E}}_{\mathring{\Sigma}} & \hookrightarrow & \hat{\mathscr{E}}_{\mathring{\Sigma}} \\ & \searrow \quad \swarrow & \\ & \mathscr{G}(\mathring{\Sigma}) & \end{array} \tag{2.118}$$

be the family of nuclear pairs assigned to $\mathring{\Sigma}$ by $Z_{\mathscr{G}}$ and let $f : S \to \mathscr{G}(\mathring{\Sigma})$ be a holomorphic map. The pullback of (2.118) along $\varphi(\mathring{\Sigma}) : \mathscr{F}(\mathring{\Sigma}) \to \mathscr{G}(\mathring{\Sigma})$ is the holomorphic bundle of nuclear pairs assigned to $\mathring{\Sigma}$ by $Z_{\mathscr{F}}$, which in turn pulls back along $\psi(\mathring{\Sigma}) \circ f$ to a holomorphic bundle of nuclear pairs on S. We can identify this with the pullback of (2.118) along f. A similar argument shows that the bundle of maps assigned to a bordism $\mathfrak{X} : \mathring{\Sigma}_0 \rightsquigarrow \mathring{\Sigma}_1$ by $Z_{\mathscr{G}}$

$$\begin{array}{ccc} \hat{\mathscr{E}}_{\mathring{\Sigma}_0} & \longrightarrow & \check{\mathscr{E}}_{\mathring{\Sigma}_1} \\ & \searrow \quad \swarrow & \\ & \mathscr{G}(\mathfrak{X}) & \end{array} \tag{2.119}$$

pulls back along a holomorphic map $f : S \to \mathscr{G}(\mathfrak{X})$ to a holomorphic map of holomorphic bundles. ■

2.3.7 Reflection Positivity

We now restrict to sheaves $\mathscr{F} : \mathfrak{Man}_n^{op} \xrightarrow{\mathfrak{X}} \mathbf{Fib}_n^{op} \to \mathbf{Set}$ such that $\mathfrak{X} : \mathbf{Man}_n \to \mathbf{Fib}_n$ is equipped with an anti-holomorphic involution, i.e. an involutive natural isomorphism $\alpha : \mathfrak{X} \implies \mathfrak{X}$ such that $\alpha(U)$ for $U \in \mathbf{Man}_n$ is a bundle map

$$\begin{array}{ccc} \mathfrak{X}(U) & \xrightarrow{\alpha(U)} & \mathfrak{X}(U) \\ & \searrow \quad \swarrow & \\ & U & \end{array} \tag{2.120}$$

which is fiberwise an antiholomorphic involution.

By Proposition 2.13, this induces an antiholomorphic involution of sheaves $\alpha_* : \mathscr{F} \to \mathscr{F}$. This in turn induces an involution (2.36) on $\mathbf{Bord}_{n,n-1}(\mathscr{F})$ which we denote

$$\alpha_{\mathscr{F}} : \mathbf{Bord}_{n,n-1}(\mathscr{F}) \to \mathbf{Bord}_{n,n-1}(\mathscr{F}) \tag{2.121}$$

The involutions $\alpha_{\mathscr{F}}$ and $\delta_{\mathscr{F}}$ commute and we denote their composition by $\tau_{\mathscr{F}} := \alpha_{\mathscr{F}} \circ \delta_{\mathscr{F}}$. If Z is $(\alpha_{\mathscr{F}}, \alpha_{\mathcal{NP}})$-equivariant, then Proposition 2.17 implies that Z is $(\tau_{\mathscr{F}}, \tau_{\mathcal{NP}})$-equivariant. In particular, Z induces a functor

$$Z^\tau : \mathbf{Bord}_{n,n-1}(\mathscr{F})^{\tau_{\mathscr{F}}} \to \mathcal{NP}^{\tau_{\mathcal{NP}}} \tag{2.122}$$

between fixed point categories. We say Z is *Hermitian* if Z^τ sends $\tau_{\mathscr{F}}$ fixed points to Hermitian nuclear pairs (cf. Definition 2.10).

Definition 2.23. An $\mathscr{F}$-field theory is *reflection positive* if Z is $(\alpha_{\mathscr{F}}, \alpha_{\mathcal{NP}})$-equivariant and Hermitian.

Chapter 3: Volume-dependent Field Theories

3.1 Definitions and Properties

We first recall the definition of an allowable complex metric. Let V be a real vector space of dimension n and let $g \in \mathrm{Sym}^2(V^*) \otimes \mathbf{C}$ be a complex symmetric 2-tensor on V inducing a non-degenerate quadratic form on $V \otimes \mathbf{C}$. Let $e_1, ..., e_n$ be a basis of $V \otimes \mathbf{C}$ that is orthonormal with respect to g. Nondegeneracy implies an isomorphism

$$T_g : V \otimes \mathbf{C} \cong V^* \otimes \mathbf{C} \tag{3.1}$$

and we let $e^1, ..., e^n$ be the basis of $V^* \otimes \mathbf{C}$ defined by $e^i := T_g(e_i)$. Set

$$\mathrm{vol}_g := e^1 \wedge \cdots \wedge e^n \tag{3.2}$$

which we view as a complex density in

$$|\Lambda^n(V^*)|_{\mathbf{C}} := (\Lambda^n(V^*) \otimes \mathbf{C}) \otimes \mathfrak{o}(V) \tag{3.3}$$

where $\mathfrak{o}(V)$ is the orientation line of V. Tensoring by the orientation line implies that vol_g is independent of the choice of orthonormal basis. Contraction with vol_g gives a map

$$\begin{aligned} *_g : \Lambda^k V^* \otimes \mathbf{C} &\to \Lambda^{n-k} V^* \otimes \mathbf{C} \otimes \mathfrak{o}(V) \\ \alpha &\mapsto \iota_{(\Lambda^k T_g^{-1})\alpha} \mathrm{vol}_g \end{aligned} \tag{3.4}$$

which restricts to the ordinary Hodge star operator on $\Lambda^\bullet V$ when g is real. Define the quadratic form

$$\begin{aligned} Q_g : \Lambda^\bullet V^* \otimes \mathbf{C} &\to |\Lambda^n V^*|_{\mathbf{C}} \\ \alpha &\mapsto \alpha \wedge *\alpha \end{aligned} \tag{3.5}$$

which is also independent of the choice of orthonormal basis.

Definition 3.1 (allowable metric on a vector space)**.** A nondegenerate complex symmetric 2-tensor $g \in \mathrm{Sym}^2(V^*) \otimes \mathbf{C}$ is *allowable* if the real part of the quadratic form Q_g is positive definite. We denote the set of allowable 2-tensors by $\mathrm{met}_{\mathbf{C}}(V)$.

Definition 3.2 (allowable density on a vector space)**.** A complex density in $|\Lambda^n V^*|_{\mathbf{C}}$ will be called an *allowable density on* V if its real part is positive. We will denote by $\mathrm{dens}_{\mathbf{C}}(V)$ the set of allowable densities.

We now recall an equivalent characterization of allowable metrics whose proof can be found in Kontsevich and Segal (2021).

Proposition 3.1 (Kontsevich and Segal (2021) Theorem 2.2)**.** *A complex quadratic form* $g \in \mathrm{Sym}^2(V^*) \otimes \mathbf{C}$ *is allowable if and only if there exists a basis* $e_1, ..., e_n$ *of* V *in which* g *can be expressed as*

$$g = \sum_{i=1}^{n} \lambda_i e^i \otimes e^i \tag{3.6}$$

where (e^i) *is the coordinate dual basis and* $\lambda_i \in \mathbf{C} \backslash \mathbf{R}_{\leqslant 0}$ *are nonzero complex numbers not on the negative real axis satisfying*

$$\sum_{i=1}^{n} |\arg(\lambda_i)| < \pi. \tag{3.7}$$

The condition (3.7) is open which implies that allowable metrics $\mathrm{met}_{\mathbf{C}}(V)$ form an open subset of $\mathrm{Sym}^2(V^*) \otimes \mathbf{C}$ and therefore a complex manifold. Likewise, the allowable densities $\mathrm{dens}_{\mathbf{C}}(V)$ are an open subset of $|\Lambda^n V^*|_{\mathbf{C}}$ and also form a complex manifold. If g is allowable, applying Q_g to $1 \in \Lambda^0 V^*$ shows that $\mathrm{vol}_g \in \mathrm{dens}_{\mathbf{C}}(V)$.

Proposition 3.2. *The map*

$$\begin{aligned} \sqrt{\det} : \mathrm{met}_{\mathbf{C}}(V) &\to \mathrm{dens}_{\mathbf{C}}(V) \\ g &\mapsto \mathrm{vol}_g \end{aligned} \tag{3.8}$$

is holomorphic, equivariant with respect to the action of $\mathrm{GL}(V)$ *on* $\mathrm{met}_{\mathbf{C}}(V)$ *and* $\mathrm{dens}_{\mathbf{C}}(V)$*, and has a holomorphic right inverse.*

Proof. Let $g \in \mathrm{met}_{\mathbf{C}}(V)$ be fixed. By Proposition 3.1, there exists a basis (e_i) of V such that $g = \sum_i \lambda_i e^i \otimes e^i$ with $\sum_i |\arg(\lambda_i)| < \pi$. Let

$$z^{1/2} : \mathbf{C}\backslash\mathbf{R}_{\leqslant 0} \to \mathbf{C}_{>0} \tag{3.9}$$

be the holomorphic square root defined on the branch away from the negative real axis. Then $(\lambda_i^{1/2} e^i)$ is a basis of $V^* \otimes \mathbf{C}$ which is orthonormal with respect to g. The map (3.8) applied to g then becomes

$$\sum_i \lambda_i e^i \otimes e^i \mapsto \prod_i \lambda_i^{1/2} \cdot |e^1 \wedge \cdots \wedge e^n| \tag{3.10}$$

where $|e^1 \wedge \cdots \wedge e^n|$ is a positive density in $\mathrm{dens}_{\mathbf{R}}(V) := \Lambda^n(V^*) \otimes \mathfrak{o}(V)$.

The map (3.8) can be expressed in terms of any other basis (f_i) of V as

$$\sum_{i,j} g_{ij} f^i \otimes f^j \mapsto \det(g_{ij})^{1/2} \cdot |f^1 \wedge \cdots \wedge f^n| \tag{3.11}$$

with

$$\det(g_{ij})^{1/2} = |\det(A)| \cdot \prod_i \lambda_i^{1/2} \tag{3.12}$$

an element of $\mathbf{C}\backslash\mathbf{R}_{\leqslant 0}$, where $A \in \mathrm{GL}(V)$ is the unique invertible linear operator satisfying $Ae_i = f_i$. Equation (3.11) shows holomorphicity and (3.12) shows equivariance.

Let $h \in \mathrm{met}_{\mathbf{R}}(V) \subset \mathrm{met}_{\mathbf{C}}(V)$ be a metric of Euclidean signature and let $\mathrm{vol}_h \in \mathrm{dens}_{\mathbf{R}}(V)$ be the associated positive real density on V. Any element $\omega \in \mathrm{dens}_{\mathbf{C}}(V)$ can be expressed as

$$\omega = c \cdot \mathrm{vol}_h \tag{3.13}$$

for $c \in \mathbf{C}_{>0}$. Then the map

$$\begin{aligned} \mathrm{dens}_{\mathbf{C}}(V) &\to \mathrm{met}_{\mathbf{C}}(V) \\ c \cdot \mathrm{vol}_h &\mapsto c^{2/n} \cdot h \end{aligned} \tag{3.14}$$

is a holomorphic right inverse to $\sqrt{\det}$. ■

Remark 3.1. The *Shilov boundary* of an open subset $U \subset \mathbb{A}$ of a finite dimensional complex affine space is the smallest compact subset K of the closure $\overline{U}$ such that every holomorphic function defined on a neighborhood of $\overline{U}$ attains its maximum modulus on K when restricted to $\overline{U}$. The set $\mathrm{met}_{\mathrm{Lor}}(V)$ of possibly degenerate metrics of Lorentzian signature is a subset of the Shilov boundary of $\mathrm{met}_{\mathbf{C}}(V)$. Likewise, the set $\mathrm{dens}_{i\mathbf{R}}$ of purely imaginary densities on V can be identified with the Shilov boundary of $\mathrm{dens}_{\mathbf{C}}(V)$. The map (3.8) extends to a $\mathrm{GL}(V)$-equivariant map

$$\sqrt{\det} : \mathrm{met}_{\mathrm{Lor}}(V) \to \mathrm{dens}_{i\mathbf{R}}(V) \tag{3.15}$$

For a more detailed discussion of the Shilov boundary we refer the reader to Kontsevich and Segal (2021).

Let M be a smooth n-manifold, let $\mathcal{B}(M) \to M$ be the principal $\mathrm{GL}_n(\mathbf{R})$ bundle of bases on M, and let V be a real vector space of dimension n. The bundle of allowable metrics on M is defined to be the associated bundle

$$\mathrm{met}_{\mathbf{C}}(M) := \mathcal{B}(M) \times_{\mathrm{GL}_n} \mathrm{met}_{\mathbf{C}}(V) \tag{3.16}$$

Similarly, we define the bundle of allowable densities on M to be the associated bundle

$$\mathrm{dens}_{\mathbf{C}}(M) := \mathcal{B}(M) \times_{\mathrm{GL}_n} \mathrm{dens}_{\mathbf{C}}(V). \tag{3.17}$$

Equivariance of the map (3.8) gives a map between associated bundles

$$\sqrt{\det} : \mathrm{met}_{\mathbf{C}}(M) \to \mathrm{dens}_{\mathbf{C}}(M) \tag{3.18}$$

which we also showed is holomorphic on the fibers.

We denote by

$$\mathrm{Met}_{\mathbf{C}} : \mathbf{Man}_n^{op} \xrightarrow{\mathrm{met}_{\mathbf{C}}} \mathbf{Fib}_n^{op} \xrightarrow{C^\infty} \mathbf{Set} \tag{3.19}$$

$$\mathrm{Dens}_{\mathbf{C}} : \mathbf{Man}_n^{op} \xrightarrow{\mathrm{dens}_{\mathbf{C}}} \mathbf{Fib}_n^{op} \xrightarrow{C^\infty} \mathbf{Set} \tag{3.20}$$

the sheaves of allowable metrics and densities on the site of smooth n-manifolds and embeddings between them which, particular, are sheaves of complex manifolds.

Lemma 3.3. *The morphism of sheaves*

$$\sqrt{\det} : \mathrm{Met}_{\mathbf{C}} \to \mathrm{Dens}_{\mathbf{C}} \tag{3.21}$$

whose value on $M \in \mathbf{Man}_n$ is induced by (3.18), admits a holomorphic right inverse (cf. Definition 2.16).

Proof. Let $M \in \mathbf{Man}_n$ and let $h \in \mathrm{Met}_{\mathbf{R}}(M)$ be a Riemannian metric. If $\omega \in \mathrm{Dens}_{\mathbf{C}}(M)$ then it can be expressed as

$$\omega = f \cdot |\,\mathrm{vol}_h\,| \tag{3.22}$$

where $f \in C^\infty(M; \mathbf{C}_{>0})$ and $|\,\mathrm{vol}_h\,| \in \mathrm{Dens}_{\mathbf{R}}(M)$ is the positive real density on M associated to h. Then Proposition 3.2 implies the map

$$\begin{aligned} \mathrm{Dens}_{\mathbf{C}}(M) &\to \mathrm{Met}_{\mathbf{C}}(M) \\ f \cdot |\,\mathrm{vol}_g\,| &\mapsto f^{2/n} \cdot h \end{aligned} \tag{3.23}$$

is a holomorphic right inverse to $(\sqrt{\det})(M)$. ■

Likewise, we set

$$\mathrm{Met}_{\mathrm{Lor}} : \mathbf{Man}_n^{op} \xrightarrow{\mathrm{met}_{\mathrm{Lor}}} \mathbf{Fib}_n^{op} \xrightarrow{C^\infty} \mathbf{Set} \tag{3.24}$$

$$\mathrm{Dens}_{i\mathbf{R}} : \mathbf{Man}_n^{op} \xrightarrow{\mathrm{dens}_{i\mathbf{R}}} \mathbf{Fib}_n^{op} \xrightarrow{C^\infty} \mathbf{Set} \tag{3.25}$$

to be the sheaves of possibly degenerate Lorentzian metrics and purely imaginary densities. As in Lemma 3.3, the map (3.15) induces a morphism of sheaves

$$\sqrt{\det} : \mathrm{Met}_{\mathrm{Lor}} \to \mathrm{Dens}_{i\mathbf{R}} \tag{3.26}$$

which we record for later use.

Definition 3.3. A *Wick-rotated quantum field theory* is a $\mathrm{Met}_{\mathbf{C}}$-field theory.

Definition 3.4. A *volume-dependent field theory (VFT)* is a $\mathrm{Met}_{\mathbf{C}}$-field theory admitting a factorization

$$\begin{array}{ccc} \mathbf{Bord}_{n,n-1}(\mathrm{Met}_{\mathbf{C}}) & \longrightarrow & \mathcal{NP} \\ {\scriptstyle \sqrt{\det}_*}\big\downarrow & \nearrow_{Z} & \\ \mathbf{Bord}_{n,n-1}(\mathrm{Dens}_{\mathbf{C}}) & & \end{array} \tag{3.27}$$

(cf. Definition 2.22).

Lemma 3.4. *The functor Z in (3.27) is a $\mathrm{Dens}_{\mathbf{C}}$-field theory.*

Proof. This follows from an application of Proposition 2.18. ■

Let $\check{E}_i \hookrightarrow \hat{E}_i$ be the nuclear pair assigned to $(\mathring{\Sigma}_i^+, \mathring{\omega}_i) \in \mathbf{Bord}_{n,n-1}(\mathrm{Dens}_{\mathbf{C}})$ by Z for $i = 0, 1$. Let $\mathfrak{X} := (\mathring{U}_X^N, p, \mathring{\theta}_0, \mathring{\theta}_1)$ be a bordism from $\mathring{\Sigma}_0^+$ to $\mathring{\Sigma}_1^+$ with X connected. Recall there is a correspondence of background fields (2.77). As in (2.82), given $\mathring{\omega}_i \in \mathrm{Dens}_{\mathbf{C}}(\mathring{\Sigma}_i)$ we set

$$\mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N; \mathring{\omega}_0, \mathring{\omega}_1) := (r_0^{\mathfrak{X}}, r_1^{\mathfrak{X}})^{-1}(\mathring{\omega}_0, \mathring{\omega}_1) \tag{3.28}$$

to be the set of allowable densities on $\mathring{U}_X^N$ whose restriction to $\mathring{U}_{\partial X_i}^N$ pulled back along $\mathring{\theta}_i$ to $\mathring{\Sigma}_i$ is $\mathring{\omega}_i$.

The field theory Z determines a holomorphic map

$$Z_{\mathfrak{X}} : \mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N; \mathring{\omega}_0, \mathring{\omega}_1) \to \mathrm{Hom}(\hat{E}_0, \check{E}_1) \tag{3.29}$$

In fact, this map only depends on the total volume of the density.

Theorem 3.5. *There is a factorization*

$$\begin{array}{ccc} \mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N; \mathring{\omega}_0, \mathring{\omega}_1) & \xrightarrow{Z_{\mathfrak{X}}} & \mathrm{Hom}(\hat{E}_0, \check{E}_1) \\ {\scriptstyle \int_X}\big\downarrow & \nearrow_{V_X} & \\ \mathbf{C}_{>0} & & \end{array} \tag{3.30}$$

through a holomorphic map V_X, where $\int_X$ is the map that integrates a density over X.

Proof. Let $\mathring{\nu}_0, \mathring{\nu}_1 \in \mathrm{Dens}_{\mathbf{C}}(\mathring{U}^N_X; \mathring{\omega}_0, \mathring{\omega}_1)$ be densities on $\mathring{U}^N_X$ whose restrictions to X have the same total volume $v := \int_X \mathring{\nu}_0 = \int_X \mathring{\nu}_1$. For $t \in [0,1]$, let

$$\mathring{\nu}_t := (1-t)\mathring{\nu}_0 + t\mathring{\nu}_1 \tag{3.31}$$

be the straight-line path in $\mathrm{Dens}_{\mathbf{C}}(\mathring{U}^N_X; \mathring{\omega}_0, \mathring{\omega}_1)$. It satisfies $\int_X \mathring{\nu}_t = v$ for all t.

Let $\tilde{U} \in \mathfrak{U}^N_X$ be a neighborhood of X in N and let $\tilde{\nu}_i \in \mathrm{Dens}_{\mathbf{C}}(\tilde{U})$ be densities restricting to $\mathring{\nu}_i$ on $\mathring{U}^N_X$. Since $\tilde{\nu}_0, \tilde{\nu}_1$ restrict to the same germ at $\mathring{U}^N_{\partial X_i}$, there exist neighborhoods $V_i \in \mathfrak{U}^N_{\partial X_i}$ for $i = 0, 1$ such that $\tilde{\nu}_0|_{V_i} = \tilde{\nu}_1|_{V_i}$. The density $\tilde{\nu}_t := (1-t)\tilde{\nu}_0 + t\tilde{\nu}_1 \in \mathrm{Dens}_{\mathbf{C}}(\tilde{U})$ has germ $\mathring{\nu}_t$ and the restriction of $\tilde{\nu}_t$ to V_i is constant for all t. Set

$$\begin{aligned} \omega_i &:= \tilde{\nu}_t|_{V_i} \\ U &:= X \cup V_0 \cup V_1 \\ \nu_t &:= \tilde{\nu}_t|_U. \end{aligned} \tag{3.32}$$

Connectedness of X implies U is also connected.

Let $\Omega^n_U(\mathbf{C}_{>0}; \omega_0, \omega_1)$ be the set of n-forms valued in $\mathbf{C}_{>0}$ that restrict to ω_i on V_i. We assume that U is orientable and $\nu_t \in \Omega^n_U(\mathbf{C}_{>0}; \omega_0, \omega_1)$. The unorientable case reduces to the orientable case by using a partition of unity and identifying the density bundle with top dimensional forms on local neighborhoods. The n-form $\dot{\nu}_t := \frac{d}{dt}\nu_t = \nu_1 - \nu_0$ satisfies $\int_U \dot{\nu}_t = 0$. By connectedness of U, we conclude that $\dot{\nu}_t$ is exact, i.e.

$$\dot{\nu}_t = \nu_1 - \nu_0 = d\eta \tag{3.33}$$

for some $\eta \in \Omega^{n-1}_U(\mathbf{C})$ constant on $V_0 \cup V_1$; without loss of generality we assume that η vanishes on this set. Since ν_t is a nowhere vanishing n-form, there exists a smooth complexified vector field $\xi_t \in C^\infty(U; TU \otimes \mathbf{C})$ satisfying $\iota_{\xi_t}\nu_t = \eta$ with $\xi_t|_{V_i} = 0$. Applying the exterior derivative gives

$$\mathcal{L}_{\xi_t}\nu_t = \nu_1 - \nu_0 \tag{3.34}$$

where $\mathcal{L}_{\xi_t}$ denotes the Lie derivative with respect to ξ_t.

Let

$$Z_U : \Omega^n_U(\mathbf{C}_{>0}; \omega_0, \omega_1) \to \mathrm{Hom}(\hat{E}_0, \check{E}_1) \tag{3.35}$$

be the composition of (3.29) with the holomorphic map restricting n-forms on U to $\mathring{U}^N_X$.

Let $\mathrm{Diff}(U; V_0 \cup V_1)$ be the group of diffeomorphisms of U that restrict to the identity on $V_0 \cup V_1$ and let $\mathrm{Lie}(\mathrm{Diff}(U; V_0 \cup V_1))$ denote the Lie algebra of vector fields on U that vanish on $V_0 \cup V_1$. For fixed $t \in [0, 1]$, let

$$\begin{aligned} \alpha_{\nu_t} : \mathrm{Diff}(U; V_0 \cup V_1) &\to \Omega^n_U(\mathbf{C}_{>0}; \omega_0, \omega_1) \\ f &\mapsto (f^{-1})^* \nu_t \end{aligned} \tag{3.36}$$

be the orbit map. The derivative at the identity gives a map

$$\begin{aligned} d\alpha_{\nu_t} : \mathrm{Lie}(\mathrm{Diff}(U; V_0 \cup V_1)) &\to T_{\nu_t}\Omega^n_U(\mathbf{C}_{>0}; \omega_0, \omega_1) \\ \xi &\mapsto \mathcal{L}_\xi \nu_t \end{aligned} \tag{3.37}$$

where

$$T_{\nu_t}\Omega^n_U(\mathbf{C}_{>0}; \omega_0, \omega_1) = \{\alpha \in \Omega^n_U(\mathbf{C}) \mid \alpha|_{V_i} = 0\} \tag{3.38}$$

is the tangent space at ν_t to the space of allowable densities on U that restrict to ω_i on V_i.

By diffeomorphism invariance, the composition $Z_U \circ \alpha_{\nu_t}$ is constant. Thus the composition

$$\mathrm{Lie}(\mathrm{Diff}(U)) \xrightarrow{d\alpha_{\nu_t}} T_{\nu_t}\Omega^n_U(\mathbf{C}_{>0}; \omega_0, \omega_1) \xrightarrow{dZ_U} \mathrm{Hom}(\hat{E}_0, \check{E}_1) \tag{3.39}$$

is 0. Holomorphicity of Z_U implies that the complexification

$$\mathrm{Lie}(\mathrm{Diff}(U)) \otimes \mathbf{C} \xrightarrow{d\alpha_{\nu_t}} T_{\nu_t}\Omega^n_U(\mathbf{C}_{>0}; \omega_0, \omega_1) \xrightarrow{dZ_U} \mathrm{Hom}(\hat{E}_0, \check{E}_1) \tag{3.40}$$

is also 0. The form $\dot{\nu}_t = \mathcal{L}_{\xi_t}\nu_t$ is in the image of the complexified map $d\alpha_{\nu_t}$ which implies $dZ_U(\dot{\nu}_t) = 0$. Thus $Z_U(\nu_t)$ is constant for all $t \in [0, 1]$ and therefore $Z_X(\mathring{\nu}_t)$ is constant for all $t \in [0, 1]$.

To see that V_X is holomorphic, consider the holomorphic map

$$\begin{aligned} m_\nu : \mathbf{C}_{>0} &\to \mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N; \mathring{\omega}_0, \mathring{\omega}_1) \\ \lambda &\mapsto \lambda \mathring{\nu} \end{aligned} \tag{3.41}$$

for some density $\mathring{\nu}$ satisfying $\int_X \mathring{\nu} = 1$. Then the composition

$$\mathbf{C}_{>0} \xrightarrow{m_\nu} \mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N; \mathring{\omega}_0, \mathring{\omega}_1) \xrightarrow{\int_X} \mathbf{C}_{>0} \tag{3.42}$$

is the identity. By holomorphicity of Z, the map

$$V_X : \mathbf{C}_{>0} \xrightarrow{m_\nu} \mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N; \mathring{\omega}_0, \mathring{\omega}_1) \xrightarrow{Z_X} \mathrm{Hom}(\hat{E}_0, \check{E}_1) \tag{3.43}$$

is holomorphic. ■

Definition 3.5. Let Σ be a connected closed $n-1$-manifold and let $\omega \in \mathrm{Dens}_{\mathbf{C}}(C_\Sigma)$ be a density. The *signed volume curve of* ω is the curve $\gamma_\omega : \mathbf{R} \to \mathbf{C}$ given by

$$\gamma_\omega(t) := \begin{cases} -\int_{\Sigma \times [t,0]} \omega & t < 0 \\ 0 & t = 0 \\ \int_{\Sigma \times [0,t]} \omega & t > 0 \end{cases} \tag{3.44}$$

It is smooth and satisfies $\gamma'(t) \in \mathbf{C}_{>0}$ for all $t \in \mathbf{R}$. When $\Sigma := \bigsqcup_i \Sigma^i$ has multiple connected components, we define the signed volume curve $\gamma_\omega : \mathbf{R} \to \mathbf{C}^{\#\pi_0(\Sigma)}$ to be the curve whose ith component is the signed volume curve $\gamma_{\omega_i} : \mathbf{R} \to \mathbf{C}$ of the restriction $\omega|_{\Sigma^i}$ to the ith connected component.

Proposition 3.6. *Let* $Z : \mathbf{Bord}_{n,n-1}(\mathrm{Dens}_{\mathbf{C}}) \to \mathcal{NP}$ *be a volume-dependent field theory, let* $\mathring{\omega}, \mathring{\omega}' \in \mathrm{Dens}_{\mathbf{C}}(\mathring{\Sigma})$, *and let* $\check{E}_{\mathring{\omega}} \hookrightarrow \hat{E}_{\mathring{\omega}}$ *and* $\check{E}_{\mathring{\omega}'} \hookrightarrow \hat{E}_{\mathring{\omega}'}$ *be the nuclear pairs assigned to them by* Z. *Then there exists a canonical isomorphism*

$$\begin{array}{ccc} \check{E}_{\mathring{\omega}} & \hookrightarrow & \hat{E}_{\mathring{\omega}} \\ \wr\!\!\downarrow \check{\alpha} & & \wr\!\!\downarrow \hat{\alpha} \\ \check{E}_{\mathring{\omega}'} & \hookrightarrow & \hat{E}_{\mathring{\omega}'} \end{array} \tag{3.45}$$

of nuclear pairs.

Proof. If Σ is empty, the isomorphism is the identity map on the trivial nuclear pair $\mathbf{C} \hookrightarrow \mathbf{C}$. If not, let $\omega, \omega' \in \mathrm{Dens}_{\mathbf{C}}(C_\Sigma)$ be allowable densities whose germs at $\mathring{\Sigma}$ are $\mathring{\omega}, \mathring{\omega}'$ and let $\gamma_\omega, \gamma_{\omega'}$ be their respective signed volume curves. The real parts $\Re(\gamma_\omega)$, $\Re(\gamma_{\omega'})$ are continuous monotone increasing functions vanishing at 0. Thus there exist monotone real sequences $\{s_i\}_{i=0}^\infty, \{s'_i\}_{i=0}^\infty \subset \mathbf{R}_{<0}$ increasing to 0 and $\{t_i\}_{i=0}^\infty, \{t'_i\}_{i=0}^\infty \subset \mathbf{R}_{>0}$ decreasing to 0 such that

(3.46)

$$\Re(\gamma_\omega(s_i)) < \Re(\gamma_{\omega'}(s'_i)) < \Re(\gamma_\omega(s_{i+1})) \quad , \quad \Re(\gamma_\omega(t_{i+1})) < \Re(\gamma_{\omega'}(t'_i)) < \Re(\gamma_\omega(t_i))$$

for all $i \geqslant 0$. Let

$$\begin{aligned} X_i &:= \Sigma \times [s_i, s_{i+1}] \qquad & Y_i &:= \Sigma \times [t_{i+1}, t_i] \\ X'_i &:= \Sigma \times [\tilde{s}_i, \tilde{s}_{i+1}] \qquad & Y'_i &:= \Sigma \times [t'_{i+1}, t'_i] \end{aligned} \tag{3.47}$$

which are compact manifolds smoothly embedded in C_Σ whose boundaries we co-orient positively. In general for $W := \Sigma \times [a, b] \subset C_\Sigma$, we set $\mathring{\omega}_W, \mathring{\omega}'_W$ to be the restrictions of ω, ω' to $\mathring{U}_W^{C_\Sigma}$ and p to be the partition on ∂W that designates $\Sigma \times \{a\}$ incoming and $\Sigma \times \{b\}$ outgoing. Let $\mathring{\theta}_t$ be the germ of the translation map (2.86) and set $\mathring{\omega}_t := \mathring{\theta}_t^* \omega$ and $\mathring{\omega}'_t := \mathring{\theta}_t^* \omega'$. Define

$$\begin{aligned} \mathcal{X}_i &:= (\mathring{U}_{X_i}^{C_\Sigma}, p, \mathring{\theta}_{s_i}, \mathring{\theta}_{s_{i+1}}) \qquad & \mathcal{Y}_i &:= (\mathring{U}_{Y_i}^{C_\Sigma}, p, \mathring{\theta}_{t_{i+1}}, \mathring{\theta}_{t_i}) \\ \mathcal{X}'_i &:= (\mathring{U}_{X'_i}^{C_\Sigma}, p, \mathring{\theta}_{s'_i}, \mathring{\theta}_{s'_{i+1}}) \qquad & \mathcal{Y}'_i &:= (\mathring{U}_{Y'_i}^{C_\Sigma}, p, \mathring{\theta}_{t'_{i+1}}, \mathring{\theta}'_{t_i}) \end{aligned} \tag{3.48}$$

from which we form the $\mathrm{Dens}_{\mathbf{C}}$-bordisms

$$\begin{aligned} (\mathcal{X}_i, \mathring{\omega}_{X_i}) &: (\mathring{\Sigma}^+, \mathring{\omega}_{s_i}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}_{s_{i+1}}) \qquad & (\mathcal{Y}_i, \mathring{\omega}_{Y_i}) &: (\mathring{\Sigma}^+, \mathring{\omega}_{t_{i+1}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}_{t_i}) \\ (\mathcal{X}'_i, \mathring{\omega}'_{X'_i}) &: (\mathring{\Sigma}^+, \mathring{\omega}'_{s'_i}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}'_{s'_{i+1}}) \qquad & (\mathcal{Y}'_i, \mathring{\omega}'_{Y'_i}) &: (\mathring{\Sigma}^+, \mathring{\omega}'_{t'_{i+1}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}'_{t'_i}) \end{aligned} \tag{3.49}$$

Choose cylindrical $\mathrm{Dens}_{\mathbf{C}}$-bordisms

$$\begin{aligned} \mathcal{A}_i &: (\mathring{\Sigma}^+, \mathring{\omega}_{s_i}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}'_{s'_i}) \qquad & \mathcal{B}_i &: (\mathring{\Sigma}^+, \mathring{\omega}'_{t'_i}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}_{t_i}) \\ \mathcal{A}'_i &: (\mathring{\Sigma}^+, \mathring{\omega}'_{s'_i}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}_{s_{i+1}}) \qquad & \mathcal{B}'_i &: (\mathring{\Sigma}^+, \mathring{\omega}_{t_{i+1}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}'_{t'_i}) \end{aligned} \tag{3.50}$$

with volumes

$$\begin{aligned} &\gamma_{\omega'}(\tilde{s}'_i) - \gamma_\omega(s_i) \qquad & &\gamma_\omega(t_i) - \gamma_{\omega'}(s'_i) \\ &\gamma_\omega(s_{i+1}) - \gamma_{\omega'}(s'_i) \qquad & &\gamma_{\omega'}(t'_i) - \gamma_\omega(t_{i+1}) \end{aligned} \tag{3.51}$$

respectively.

By Theorem 3.5,

$$
\begin{aligned}
Z(\mathcal{X}_i, \mathring{\omega}_{X_i}) &= Z(\mathcal{A}_i') \circ Z(\mathcal{A}_i) & \qquad Z(\mathcal{Y}_i, \mathring{\omega}_{Y_i}) &= Z(\mathcal{B}_i) \circ Z(\mathcal{B}_i') \\
Z(\mathcal{X}_i', \mathring{\omega}_{X_i'}') &= Z(\mathcal{A}_{i+1}) \circ Z(\mathcal{A}_i') & \qquad Z(\mathcal{Y}_i', \mathring{\omega}_{Y_i'}') &= Z(\mathcal{B}_i') \circ Z(\mathcal{B}_{i+1})
\end{aligned} \tag{3.52}
$$

There are direct and inverse systems

$$
\check{E}_{\mathring{\omega}_{s_0}} \xrightarrow{\check{Z}(\mathcal{A}_0)} \check{E}_{\mathring{\omega}'_{s_0'}} \xrightarrow{\check{Z}(\mathcal{A}_0')} \check{E}_{\mathring{\omega}_{s_1}} \to \cdots \to \hat{E}_{\mathring{\omega}_{t_1}} \xrightarrow{\hat{Z}(\mathcal{B}_0')} \hat{E}_{\mathring{\omega}'_{t_0'}} \xrightarrow{\hat{Z}(\mathcal{B}_0)} \hat{E}_{\mathring{\omega}_{t_0}} \tag{3.53}
$$

Let $\mathcal{C}_\omega, \mathcal{D}_\omega$ be the direct and inverse cylindrical systems of ω and similarly let $\mathcal{C}_{\omega'}, \mathcal{D}_{\omega'}$ be the cylindrical systems of ω' (cf. Definition 2.20). Let $\tilde{\mathcal{C}}_\omega, \tilde{\mathcal{D}}_\omega$ be the direct and inverse systems consisting of the unprimed terms in (3.53) and $\tilde{\mathcal{C}}_{\omega'}, \tilde{\mathcal{D}}_{\omega'}$ the direct and inverse systems consisting of the primed terms. Then $\tilde{\mathcal{C}}_\omega \subset \mathcal{C}_\omega$, $\tilde{\mathcal{C}}_{\omega'} \subset \mathcal{C}_{\omega'}$ are cofinal subsystems and $\tilde{\mathcal{D}}_\omega \subset \mathcal{D}_\omega, \tilde{\mathcal{D}}_{\omega'} \subset \mathcal{D}_{\omega'}$ are final subsystems.

There is a sequence of isomorphisms

$$
\begin{array}{ccccccccccc}
\check{E}_{\mathring{\omega}} & \xrightarrow{\cong} & \varinjlim_{\mathcal{C}_\omega} \check{E}_{\mathring{\omega}_s} & \xrightarrow{\cong} & \varinjlim_{\tilde{\mathcal{C}}_\omega} \check{E}_{\mathring{\omega}_{s_{2i}}} & \xrightarrow{\cong} & \varinjlim_{\tilde{\mathcal{C}}_{\omega'}} \check{E}_{\mathring{\omega}'_{s_{2i+1}}} & \xrightarrow{\cong} & \varinjlim_{\mathcal{C}_{\omega'}} \check{E}_{\mathring{\omega}'_s} & \xrightarrow{\cong} & \check{E}_{\mathring{\omega}'} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\hat{E}_{\mathring{\omega}} & \xrightarrow{\cong} & \varprojlim_{\mathcal{D}_\omega} \hat{E}_{\mathring{\omega}_t} & \xrightarrow{\cong} & \varprojlim_{\tilde{\mathcal{D}}_\omega} \hat{E}_{\mathring{\omega}_{t_{2i}}} & \xrightarrow{\cong} & \varprojlim_{\tilde{\mathcal{D}}_{\omega'}} \hat{E}_{\mathring{\omega}'_{t_{2i+1}}} & \xrightarrow{\cong} & \varprojlim_{\mathcal{D}_{\omega'}} \hat{E}_{\mathring{\omega}'_t} & \xrightarrow{\cong} & \hat{E}_{\mathring{\omega}'}
\end{array} \tag{3.54}
$$

where the first and last are implied by coherence, the second and fourth are canonical isomorphisms implied by the universal properties of limits and colimits applied to final and cofinal subsequences, and the middle isomorphism results from applying the universal properties to (3.53). The composition of these isomorphisms gives what was desired. ∎

Remark 3.2. We can interpret this proposition as giving a canonical trivialization of the bundle (2.76) for a VFT.

Proposition 3.7. *Let* $\mathring{f} \in \mathrm{Diff}(\mathring{\Sigma}^+)$ *be a germ of a diffeomorphism satisfying* $\mathring{\omega} = \mathring{f}^*\mathring{\omega}'$. *Then* $Z(\mathring{f})$ *is the canonical isomorphism (3.45).*

Proof. Let $f \in \mathrm{Diff}(C_\Sigma)$ be a diffeomorphism of the cylinder whose germ at $\mathring{\Sigma}$ is $\mathring{f}$ and let $\omega, \omega' \in \mathrm{Dens}_{\mathbf{C}}(C_\Sigma)$ be allowable densities restricting to $\mathring{\omega}, \mathring{\omega}'$ on $\mathring{\Sigma}$. Let $\gamma_\omega, \gamma_{\omega'}$ be the signed volume curves of ω, ω'. As in Subsection 2.3.5, let

$$\begin{aligned} m(t) &:= \min_{x \in \Sigma_t} pr_2(f(x)) \\ M(t) &:= \max_{x \in \Sigma_t} pr_2(f(x)). \end{aligned} \tag{3.55}$$

There exist monotone increasing sequences $\{s_i\}, \{s_i'\} \subset \mathbf{R}_{<0}$, monotone decreasing sequences $\{t_i\}, \{t_i'\} \subset \mathbf{R}_{>0}$ converging to 0 satisfying

$$M(s_i) < s_i' < m(s_{i+1}) \quad \text{and} \quad M(t_{i+1}) < t_i' < m(t_i) \tag{3.56}$$

and $\mathrm{Dens}_{\mathbf{C}}$-bordisms

$$\begin{aligned} (\mathcal{X}_{s_i,s_i'}, \mathring{\omega}_X) : (\mathring{\Sigma}^+, \mathring{\omega}_{s_i}) &\rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}'_{s_i'}) \qquad (\mathcal{X}_{t_{i+1},t_i'}, \mathring{\omega}_X) : (\mathring{\Sigma}^+, \mathring{\omega}_{t_{i+1}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}'_{t_i'}) \\ (\mathcal{Y}_{s_i',s_{i+1}}, \mathring{\omega}_Y) : (\mathring{\Sigma}^+, \mathring{\omega}'_{s_i'}) &\rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}_{s_{i+1}}) \qquad (\mathcal{Y}_{t_i',t_i}, \mathring{\omega}_Y) : (\mathring{\Sigma}^+, \mathring{\omega}'_{t_i'}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}_{t_i}) \end{aligned} \tag{3.57}$$

which have volumes

$$\begin{aligned} \gamma_{\omega'}(s_i') - \gamma_\omega(s_i) &\qquad \gamma_{\omega'}(t_i') - \gamma_\omega(t_{i+1}) \\ \gamma_\omega(s_{i+1}) - \gamma_{\omega'}(s_i') &\qquad \gamma_\omega(t_i) - \gamma_{\omega'}(t_i'). \end{aligned} \tag{3.58}$$

respectively. Applying the VFT Z gives the interwoven system (3.53) which shows that the sequences of isomorphisms (2.109) and (3.54) are the same. ■

Proposition 3.8. *Let* $\mathcal{X} := (\mathring{U}_X^N, p, \mathring{\theta}_0, \mathring{\theta}_1)$ *be a bordism from* $\mathring{\Sigma}_0^+$ *to* $\mathring{\Sigma}_1^+$ *and let*

$$\begin{aligned} (\mathcal{X}, \mathring{\omega}_X) &: (\mathring{\Sigma}_0, \mathring{\omega}_0) \rightsquigarrow (\mathring{\Sigma}_1, \mathring{\omega}_1) \\ (\mathcal{X}, \mathring{\omega}'_X) &: (\mathring{\Sigma}_0, \mathring{\omega}'_0) \rightsquigarrow (\mathring{\Sigma}_1, \mathring{\omega}'_1) \end{aligned} \tag{3.59}$$

be $\mathrm{Dens}_{\mathbf{C}}$*-bordisms satisfying* $\int_X \mathring{\omega}_X = \int_X \mathring{\omega}'_X$*. Then the square*

$$\begin{array}{ccc} \hat{E}_{\mathring{\omega}_0} & \xrightarrow{Z(\mathcal{X}, \mathring{\omega}_X)} & \check{E}_{\mathring{\omega}_1} \\ {\scriptstyle \wr\!\!\parallel}\big\downarrow {\scriptstyle \hat{\alpha}_0} & & {\scriptstyle \wr\!\!\parallel}\big\downarrow {\scriptstyle \check{\alpha}_1} \\ \hat{E}_{\mathring{\omega}'_0} & \xrightarrow[Z(\mathcal{X}, \mathring{\omega}'_X)]{} & \check{E}_{\mathring{\omega}'_1} \end{array} \tag{3.60}$$

commutes, where $\hat{\alpha}_0, \check{\alpha}_1$ *are the canonical isomorphisms implied by Proposition 3.6.*

Proof. If Σ_0, Σ_1 are both empty then the result is implied by Theorem 3.5. Suppose Σ_0 is nonempty and let $\omega_N, \omega'_N \in \mathrm{Dens}_{\mathbf{C}}(N)$ be densities on N whose restrictions to $\mathring{U}^N_X$ are $\mathring{\omega}_X, \mathring{\omega}'_X$. Let $\theta_i : C_{\Sigma_i} \hookrightarrow N$ be an embedding onto a cylindrical neighborhood of $\partial X_i \subset N$ whose germ is $\mathring{\theta}_i$. Without loss of generality we assume that $N = X \cup \mathrm{Im}(\theta_0) \cup \mathrm{Im}(\theta_1)$. Set $\omega_i := \theta_i^* \omega_N$ and $\omega'_i := \theta_i^* \omega'_N$ which are densities on the cylinder C_{Σ_i} and let $\mathring{\omega}_{i,t} := \mathring{\theta}_t^* \omega_i$, $\mathring{\omega}'_{i,t} := \mathring{\theta}_t^* \omega'_i$.

As in the previous proof, there exist sequences $\{t_i\} \subset \mathbf{R}_{>0}$ and $\{s_i\} \subset \mathbf{R}_{<0}$ converging to 0 and $\mathrm{Dens}_{\mathbf{C}}$-bordisms

(3.61)
$$\begin{aligned} \mathcal{B}_{2i} &: (\mathring{\Sigma}^+, \mathring{\omega}'_{0,t_{2i+1}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}_{0,t_{2i}}) & \mathcal{A}_{2i} &: (\mathring{\Sigma}^+, \mathring{\omega}_{1,s_{2i}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}'_{1,s_{2i+1}}) \\ \mathcal{B}_{2i+1} &: (\mathring{\Sigma}^+, \mathring{\omega}_{0,t_{2i+2}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}'_{0,t_{2i+1}}) & \mathcal{A}_{2i+1} &: (\mathring{\Sigma}^+, \mathring{\omega}'_{1,s_{2i+1}}) \rightsquigarrow (\mathring{\Sigma}^+, \mathring{\omega}_{1,s_{2i+2}}) \end{aligned}$$

forming a system of maps

(3.62)
$$\cdots \hat{E}_{\mathring{\omega}_{0,t_2}} \xrightarrow{\hat{Z}(\mathcal{B}_1)} \hat{E}_{\mathring{\omega}'_{0,t_1}} \xrightarrow{\hat{Z}(\mathcal{B}_0)} \hat{E}_{\mathring{\omega}_{0,t_0}} \xrightarrow{\quad Z(\mathcal{X}) \quad} \check{E}_{\mathring{\omega}_{1,s_0}} \xrightarrow{\check{Z}(\mathcal{A}_0)} \check{E}_{\mathring{\omega}'_{1,s_1}} \xrightarrow{\check{Z}(\mathcal{A}_1)} \check{E}_{\mathring{\omega}_{1,s_2}} \cdots$$

such that the even terms are final and cofinal subsequences of $\mathcal{D}_{\omega_0}, \mathcal{C}_{\omega_1}$, and the odd terms are final and cofinal subsequences of $\mathcal{D}_{\omega'_0}$ and $\mathcal{C}_{\omega'_1}$. Here $\mathcal{X} : (\mathring{\Sigma}^+_0, \mathring{\omega}_{0,t_0}) \rightsquigarrow (\mathring{\Sigma}^+_1, \mathring{\omega}_{1,s_0})$ is any $\mathrm{Dens}_{\mathbf{C}}$ bordism of type X with volume $\int_{X_{t_0,s_0}} \mathring{\omega}_X$, where

$$X_{t_0,s_0} = X \backslash (\theta_0([0,t_0]) \cup \theta_1([s_0,0])) \tag{3.63}$$

Taking limits and colimits of the odd and even subsequences and applying coherence and the universal properties gives the result. ∎

Definition 3.6. Let $(S,+)$ be a commutative semigroup. Define $\mathbf{Bord}^s_{n,n-1}(S)$ to be the semicategory with objects consisting of closed $n-1$-manifolds and morphisms consisting of pairs (X,s) of a bordism X with $m := \#\pi_0(X)$ connected components each labeled by an element of S which we represent by the tuple $s := (s_1, ..., s_m) \in S^m$. Composition of morphisms is defined by

$$(X,s) \circ (Y,t) := (X \circ Y, s \oplus t) \tag{3.64}$$

where $X \circ Y$ is composition of ordinary bordisms and $s \oplus t \in S^{\#\pi_0(X \circ Y)}$ is the tuple obtained from composing labels whenever they are part of the same connected component in $X \circ Y$. The symmetric monoidal product is disjoint union. We define $\mathbf{Bord}_{n,n-1}(S)$ to be the category obtained by adding identity morphisms to $\mathbf{Bord}^s_{n,n-1}(S)$.

Remark 3.3. $\mathbf{Bord}_{n,n-1}(S) = \mathbf{Bord}^s_{n,n-1}(S)$ if and only if S is a monoid, i.e. has a unit.

We will now set $S := (\mathbf{C}_{>0}, +)$ to be the semigroup of complex numbers with positive real part.

Definition 3.7. Let $F : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathcal{NP}$ be a nuclear symmetric monoidal functor and let $\check{E}_\Sigma \hookrightarrow \hat{E}_\Sigma$ be the nuclear pair assigned to a closed $n-1$-manifold Σ. We say F is *holomorphic* if for every bordism $X : \Sigma_0 \rightsquigarrow \Sigma_1$, the map

$$\begin{aligned} F_X : (\mathbf{C}_{>0})^{\#\pi_0(X)} &\to \mathrm{Hom}(\hat{E}_{\Sigma_0}, \check{E}_{\Sigma_1}) \\ s &\mapsto Z(X,s) \end{aligned} \tag{3.65}$$

is holomorphic.

Construction 3.1. For each closed $n-1$-manifold Σ, pick a positive $n-1$-dimensional density ν_Σ of total volume 1 on Σ and let $\omega_\Sigma := \nu_\Sigma \wedge dt \in \mathrm{Dens}_{\mathbf{R}}(C_\Sigma)$. We note that its germ $\mathring{\omega}_\Sigma$ at $\mathring{\Sigma}$ is time symmetric, i.e. $\tau_{\mathrm{Dens}_{\mathbf{C}}}(\mathring{\Sigma}^+, \mathring{\omega}_\Sigma) = (\mathring{\Sigma}^+, \mathring{\omega}_\Sigma)$ is a $\tau_{\mathrm{Dens}_{\mathbf{C}}}$-fixed point in $\mathbf{Bord}^s_{n,n-1}(\mathrm{Dens}_{\mathbf{C}})$.

Let $\check{E}_\Sigma \hookrightarrow \hat{E}_\Sigma$ be the nuclear pair assigned to $\mathring{\omega}_\Sigma$ by Z. Define a functor

$$V : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathcal{NP}^{h,nuc} \tag{3.66}$$

that sends an object Σ to the nuclear pair $\check{E}_\Sigma \hookrightarrow \hat{E}_\Sigma$ and a bordism $(X,s) : \Sigma_0 \rightsquigarrow \Sigma_1$ to the nuclear morphism

$$\begin{array}{ccc} \check{E}_{\Sigma_0} & \hookrightarrow & \hat{E}_{\Sigma_0} \\ {\scriptstyle \check{V}(X,s)} \downarrow & \swarrow & \downarrow {\scriptstyle \hat{V}(x,s)} \\ \check{E}_{\Sigma_1} & \hookrightarrow & \hat{E}_{\Sigma_1} \end{array} \tag{3.67}$$

defined by

$$\check{V}(X,s) := \check{Z}(\mathfrak{X}, \mathring{\omega}_X) \quad \text{and} \quad \hat{V}(X,s) := \hat{Z}(\mathfrak{X}, \mathring{\omega}_X) \tag{3.68}$$

where

$$(\mathfrak{X}, \mathring{\omega}_X) : (\mathring{\Sigma}_0, \mathring{\omega}_{\Sigma_0}) \rightsquigarrow (\mathring{\Sigma}_1, \mathring{\omega}_{\Sigma_1}) \tag{3.69}$$

is any $\mathrm{Dens}_{\mathbf{C}}$-bordism of type X whose total volume on each connected component of X is given by $s \in (\mathbf{C}_{>0})^{\#\pi_0(X)}$.

Remark 3.4. V is well defined because by Theorem 3.5 any choice of $\mathrm{Dens}_{\mathbf{C}}$-bordism $(\mathfrak{X}, \mathring{\omega}_X)$ of total volume s gives the same map (3.67).

Remark 3.5. By Theorem 3.5, V is holomorphic (cf. Definition 3.7).

Lemma 3.9. *Let $\mathfrak{X} := (\mathring{U}_X^N, p, \mathring{\theta}_0, \mathring{\theta}_1)$ be a bordism from $\mathring{\Sigma}_0$ to $\mathring{\Sigma}_1$. Then the integration map*

$$\int_X : \mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N) \to \mathbf{C}_{>0}^{\#\pi_0(X)} \tag{3.70}$$

is holomorphic.

Proof. The map (3.70) is a composition

$$\mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N) \to \mathrm{Dens}_{\mathbf{C}}(X) \xrightarrow{\int'_X} \mathbf{C}_{>0}^{\#\pi_0(X)} \tag{3.71}$$

where the first map is the restriction map and the second is the map that integrates a density on each connected component of X. That the restriction map is holomorphic follows from the definitions (cf. Definition 2.14).

There is an extension of $\int'_X$ to a complex linear map

$$\int'_X : C^\infty(\Lambda^n(T^*X) \otimes \mathfrak{o}(X) \otimes \mathbf{C}) \to \mathbf{C}^{\#\pi_0(X)} \tag{3.72}$$

out of the vector space of complex top dimensional forms twisted by the orientation line bundle on X. As $\mathrm{Dens}_{\mathbf{C}}(X)$ is an open subset of this complex vector space, this shows holomorphicity. ∎

Define a functor

$$\int : \mathbf{Bord}_{n,n-1}(\mathrm{Dens}_{\mathbf{C}}) \to \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \tag{3.73}$$

that sends an object $(\mathring{\Sigma}, \mathring{\omega})$ to Σ, isomorphisms to the identity morphism, and a $\mathrm{Dens}_{\mathbf{C}}$-bordism $(\mathcal{X}, \mathring{\omega}_X) : (\mathring{\Sigma}_0, \mathring{\omega}_0) \rightsquigarrow (\mathring{\Sigma}_1, \mathring{\omega}_1)$ to $(X, s) : \Sigma_0 \rightsquigarrow \Sigma_1$ where $s := (\int_{X_j} \mathring{\omega}_X) \in (\mathbf{C}_{>0})^{\#\pi_0(X)}$ is the tuple of total volumes on each connected component X_j of X.

Proposition 3.10. *The composition*

$$Z' : \mathbf{Bord}_{n,n-1}(\mathrm{Dens}_{\mathbf{C}}) \xrightarrow{\int} \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \xrightarrow{V} \mathcal{NP}^{h,nuc} \tag{3.74}$$

is a $\mathrm{Dens}_{\mathbf{C}}$*-field theory.*

Proof. We check the following criteria:

1. *Z' is nuclear*: by construction.

2. *Z' is holomorphic*: The bundle (2.79) is trivial with fiber $V(\Sigma) = \check{E}_\Sigma \hookrightarrow \hat{E}_\Sigma$ and in particular it is holomorphic.

   Let $\mathcal{X} := (\mathring{U}_X^N, p, \mathring{\theta}_0, \mathring{\theta}_1)$ be a bordism from $\mathring{\Sigma}_0$ to $\mathring{\Sigma}_1$. By Lemma 3.9, the composition

$$\mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N) \xrightarrow{\int_X} \mathbf{C}_{>0}^{\#\pi_0(X)} \xrightarrow{V_X} \mathrm{Hom}(\hat{E}_{\Sigma_0}, \check{E}_{\Sigma_1}) \tag{3.75}$$

   is holomorphic. This is equivalent to a holomorphic section of the trivial bundle

$$\begin{array}{c} \underline{\mathrm{Hom}(\hat{E}_{\Sigma_0}, \check{E}_{\Sigma_1})} \\ \downarrow \curvearrowright Z'_{\mathcal{X}} \\ \mathrm{Dens}_{\mathbf{C}}(\mathring{U}_X^N) \end{array} \tag{3.76}$$

   which is (2.81) for the field theory Z' and shows holomorphicity.

3. *Z' is coherent*: Let $(\mathring{\Sigma}, \mathring{\omega})$ be an object in $\mathbf{Bord}_{n,n-1}(\mathrm{Dens}_{\mathbf{C}})$ and let $\omega \in \mathrm{Dens}_{\mathbf{C}}(C_\Sigma)$ be a density with germ $\mathring{\omega}$ at $\mathring{\Sigma}$. Let $\gamma_\omega : \mathbf{R} \to \mathbf{C}^{\#\pi_0(\Sigma)}$ be the signed volume curve of ω (cf. Definition 3.5).

   The direct and inverse cylindrical systems $\mathcal{C}_\omega, \mathcal{D}_\omega$ (cf. Definition 2.20) are given by the systems of maps

$$\begin{gathered} \check{E}_\Sigma \xrightarrow{\check{V}(\gamma_\omega(s')-\gamma_\omega(s))} \check{E}_\Sigma \\ \hat{E}_\Sigma \xrightarrow{\hat{V}(\gamma_\omega(t')-\gamma_\omega(t))} \hat{E}_\Sigma \end{gathered} \tag{3.77}$$

   and the systems $\mathcal{C}_{\omega_\Sigma}, \mathcal{D}_{\omega_\Sigma}$ are given by

$$\begin{gathered} \check{E}_\Sigma \xrightarrow{\check{V}(s'-s)} \check{E}_\Sigma \\ \hat{E}_\Sigma \xrightarrow{\hat{V}(t'-t)} \hat{E}_\Sigma \end{gathered} \tag{3.78}$$

   for all $s < s' < 0 < t < t'$.

   By universal property, there are unique maps making the diagrams

$$\begin{array}{ccc} \check{E}_\Sigma & \longrightarrow & \check{E}_{\mathcal{C}_\omega} \\ & {\scriptstyle \check{V}(-\gamma(s))} \searrow & \downarrow {\scriptstyle \check{\varphi}} \\ & & \check{E}_\Sigma \end{array} \qquad \begin{array}{ccc} \hat{E}_{\mathcal{D}_\omega} & \longrightarrow & \hat{E}_\Sigma \\ {\scriptstyle \hat{\varphi}} \uparrow & \nearrow {\scriptstyle \hat{V}(\gamma(s))} & \\ \hat{E}_\Sigma & & \end{array} \tag{3.79}$$

   commute. Coherence follows from noticing that $\check{\varphi}, \hat{\varphi}$ have a factorization into isomorphisms

$$\begin{array}{ccc} \check{E}_{\mathcal{C}_\omega} & \longrightarrow & \hat{E}_{\mathcal{D}_\omega} \\ \wr\Vert & & \wr\Vert \\ \check{E}_{\mathcal{C}_{\omega_\Sigma}} & \longrightarrow & \hat{E}_{\mathcal{D}_{\omega_\Sigma}} \\ \wr\Vert & & \wr\Vert \\ \check{E}_\Sigma & \hookrightarrow & \hat{E}_\Sigma \end{array} \quad (\check{\varphi}: \check{E}_{\mathcal{C}_\omega} \to \check{E}_\Sigma,\ \hat{\varphi}: \hat{E}_\Sigma \to \hat{E}_{\mathcal{D}_\omega}) \tag{3.80}$$

   where the top isomorphism square is from Proposition 3.6 and the bottom isomorphism square is from coherence of Z.

■

Proposition 3.11. *Z is naturally isomorphic to Z'.*

Proof. For any object $(\mathring{\Sigma}, \mathring{\omega})$ in $\mathbf{Bord}^s_{n,n-1}(\mathrm{Dens}_{\mathbf{C}})$, let $\check{E}_{\mathring{\omega}} \hookrightarrow \hat{E}_{\mathring{\omega}}$ denote the nuclear pair assigned to it by Z. Define $\alpha(\mathring{\Sigma}, \mathring{\omega})$ to be the canonical isomorphism

$$\begin{array}{ccc} \check{E}_{\mathring{\omega}} & \hookrightarrow & \hat{E}_{\mathring{\omega}} \\ \cong\downarrow & & \cong\downarrow \\ \check{E}_{\Sigma} & \hookrightarrow & \hat{E}_{\Sigma} \end{array} \tag{3.81}$$

given by Proposition 3.6. Then Propositions 3.7 and 3.8 show that this defines a natural isomorphism $\alpha : Z \implies Z'$. ■

3.2 The Lorentzian Limit

Let $V : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathcal{NP}$ be the functor from Construction 3.1. Let $X : \Sigma_0 \rightsquigarrow \Sigma_1$ be a bordism of volume $\tau \in (\mathbf{C}_{>0})^{\#\pi_0(X)}$ between closed $n-1$-manifolds Σ_0 and Σ_1. The theory V assigns to (X, τ) a morphism of nuclear pairs

$$\begin{array}{ccc} \check{E}_{\Sigma_0} & \xhookrightarrow{\iota_{\Sigma_0}} & \hat{E}_{\Sigma_0} \\ {\scriptstyle \check{V}_X(\tau)}\downarrow & \swarrow & \downarrow{\scriptstyle \hat{V}_X(\tau)} \\ \check{E}_{\Sigma_1} & \xhookrightarrow{\iota_{\Sigma_1}} & \hat{E}_{\Sigma_1} \end{array} \tag{3.82}$$

Notation. Given bordisms $X : \Sigma_0 \rightsquigarrow \Sigma_1$ and $Y : \Sigma_1 \rightsquigarrow \Sigma_2$, let

$$\oplus_{Y \circ X} : (\mathbf{C}_{>0})^{\#\pi_0(Y)} \times (\mathbf{C}_{>0})^{\#\pi_0(X)} \to (\mathbf{C}_{>0})^{\#\pi_0(Y \circ X)} \tag{3.83}$$

be the function defined by the composition

$$(Y \circ X, s \oplus_{Y \circ X} t) := (Y, s) \circ (X, t) \tag{3.84}$$

of bordisms in $\mathbf{Bord}_{n,n-1}(\mathbf{C}_{\geqslant 0})$ (cf. Definition 3.6). When the context is clear we will omit the subscript and simply write $s \oplus t$.

Theorem 3.12. *Let $X : \Sigma_0 \rightsquigarrow \Sigma_1$ be a bordism and let $\zeta \in \mathbf{R}^{\#\pi_0(X)}$. Then there exist unique maps*

$$\check{L}(X, i\zeta) : \check{E}_{\Sigma_0} \to \check{E}_{\Sigma_1} \qquad \textit{and} \qquad \hat{L}(X, i\zeta) : \hat{E}_{\Sigma_0} \to \hat{E}_{\Sigma_1} \tag{3.85}$$

such that for all $\tau^{(i)} = (\tau_j^{(i)}) \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_i)}$, the diagrams

$$\begin{array}{ccc} \check{E}_{\Sigma_0} & \xrightarrow{\check{V}_{\Sigma_0 \times I}(\tau^{(0)})} & \check{E}_{\Sigma_0} \\ & \searrow^{\check{V}_X(i\zeta \oplus \tau^{(0)})} & \downarrow{\scriptstyle \check{L}(X,i\zeta)} \\ & & \check{E}_{\Sigma_1} \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} \hat{E}_{\Sigma_0} & & \\ {\scriptstyle \hat{L}(X,i\zeta)}\downarrow & \searrow^{\hat{V}_X(i\zeta \oplus \tau^{(1)})} & \\ \hat{E}_{\Sigma_1} & \xrightarrow[\hat{V}_{\Sigma_1 \times I}(\tau^{(1)})]{} & \hat{E}_{\Sigma_1} \end{array} \tag{3.86}$$

commute when

$$\Sigma_0 \neq \varnothing \qquad \textit{and} \qquad \Sigma_1 \neq \varnothing \tag{3.87}$$

respectively.

Proof. We assume that both Σ_0 and Σ_1 are nonempty. For $i = 0, 1$ let

$$\gamma^{(i)} : \mathbf{R} \to \mathbf{C}^{\#\pi_0(\Sigma_i)} \tag{3.88}$$

be curves satisfying $\frac{d}{dt}\gamma^{(i)}(t) \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_i)}$ and $\gamma^{(i)}(0) = 0$.

Let $\mathcal{C}_{\gamma^{(0)}} := \{\check{E}_s := \check{E}_{\Sigma_0} \mid s < 0\}$ be the direct system consisting of maps

$$\check{E}_s \xrightarrow{\check{V}_{\Sigma_0 \times I}(\gamma^{(0)}(s') - \gamma^{(0)}(s))} \check{E}_{s'} \tag{3.89}$$

for all $s < s' < 0$. Similarly, let $\mathcal{D}_{\gamma^{(1)}} := \{\hat{E}_t := \hat{E}_{\Sigma_1} \mid t > 0\}$ be the inverse system consisting of maps

$$\hat{E}_t \xrightarrow{\hat{V}_{\Sigma_1 \times I}(\gamma^{(1)}(t') - \gamma^{(1)}(t))} \hat{E}_{t'} \tag{3.90}$$

for all $0 < t < t'$.

As in the proof of coherence in Proposition 3.10, there are canonical isomorphisms

$$\check{E}_{\Sigma_0} \cong \underset{\mathcal{C}_{\gamma^{(0)}}}{\underrightarrow{\operatorname{colim}}}\, \check{E}_s \tag{3.91}$$

$$\hat{E}_{\Sigma_1} \cong \varprojlim_{\mathcal{D}_{\gamma(1)}} \hat{E}_t. \tag{3.92}$$

The maps

$$\check{E}_s \xrightarrow{\check{V}_X(i\zeta \oplus (-\gamma^{(0)}(s)))} \check{E}_{\Sigma_1} \tag{3.93}$$

form a cocone to $\mathcal{C}_{\gamma(0)}$. By the universal property of the colimit and the identification (3.91), there exists a unique map

$$\check{L}(X, i\zeta) : \check{E}_{\Sigma_0} \to \check{E}_{\Sigma_1} \tag{3.94}$$

making the diagrams

$$\begin{array}{ccc} \check{E}_s & \xrightarrow{\check{V}_{\Sigma_0 \times I}(-\gamma^{(0)}(s))} & \check{E}_{\Sigma_0} \\ & \searrow^{\check{V}_X(i\zeta \oplus (-\gamma^{(0)}(s)))} & \Big\downarrow{\scriptstyle \check{L}(X, i\zeta)} \\ & & \check{E}_{\Sigma_1} \end{array} \tag{3.95}$$

commute for all $s < 0$.

Similarly, the maps

$$\hat{E}_{\Sigma_0} \xrightarrow{\hat{V}_X(i\zeta \oplus \gamma^{(1)}(t))} \hat{E}_t \tag{3.96}$$

form a cone to $\mathcal{D}_{\gamma(1)}$. By the universal property of the limit and the identification (3.92) there exists a unique map

$$\hat{L}(X, i\zeta) : \hat{E}_{\Sigma_0} \to \hat{E}_{\Sigma_1} \tag{3.97}$$

making the diagram

$$\begin{array}{ccc} \hat{E}_{\Sigma_0} & & \\ {\scriptstyle \hat{L}(X, i\zeta)} \Big\downarrow & \searrow^{\hat{V}_X(i\zeta \oplus \gamma^{(1)}(t))} & \\ \hat{E}_{\Sigma_1} & \xrightarrow[\hat{V}_{\Sigma_1 \times I}(\gamma^{(1)}(t))]{} & \hat{E}_t \end{array} \tag{3.98}$$

commute for all $t > 0$.

For all $\tau^{(0)} \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_0)}$, continuity of $\gamma^{(0)}$ implies that there exists $s < 0$ such that $\tau^{(0)} + \gamma^{(0)}(s) \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_0)}$. Precomposing diagram (3.95) by the map

$$\check{V}_{\Sigma_0 \times I}(\tau^{(0)} + \gamma^{(0)}(s)) : \check{E}_{\Sigma_0} \to \check{E}_s \tag{3.99}$$

shows that the left hand diagram in (3.86) commutes. Likewise, for each $\tau^{(1)} \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_1)}$ continuity of $\gamma^{(1)}$ implies that there exists $t > 0$ such that $\tau^{(1)} - \gamma^{(1)}(t) \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_0)}$. Postcomposing diagram (3.98) by

$$\hat{V}_{\Sigma_1 \times I}(\tau^{(1)} - \gamma^{(1)}(t)) : \hat{E}_t \to \hat{E}_{\Sigma_1} \tag{3.100}$$

shows that the right hand diagram in (3.86) commutes. ■

Remark 3.6. The theorem in particular implies that the constructions of $\check{L}(X, i\zeta), \hat{L}(X, i\zeta)$ are independent of the choice of right-moving curve γ.

Corollary 3.13. *Let Σ be a closed $n-1$-manifold. Then $\check{L}(\Sigma \times I, 0) = id_{\check{E}_\Sigma}$ and $\hat{L}(\Sigma \times I, 0) = id_{\hat{E}_\Sigma}$.*

Proof. When $X = \Sigma \times I$ and $\zeta = 0 \in \mathbf{R}^{\#\pi_0(\Sigma)}$, the identity maps make the diagrams (3.86) commute for all $\tau^{(i)} \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma)}$. ■

Lemma 3.14. *Let $X : \Sigma_0 \rightsquigarrow \Sigma_1$ be a bordism and let $\zeta \in \mathbf{R}$. Then the square*

$$\begin{array}{ccc} \check{E}_{\Sigma_0} & \hookrightarrow & \hat{E}_{\Sigma_0} \\ {\scriptstyle \check{L}(X,i\zeta)}\downarrow & & \downarrow{\scriptstyle \hat{L}(X,i\zeta)} \\ \check{E}_{\Sigma_1} & \hookrightarrow & \hat{E}_{\Sigma_1} \end{array} \tag{3.101}$$

commutes.

Proof. The solid arrows in the diagram

$$\begin{array}{ccccccc} \check{E}_s & \xrightarrow{\check{V}_{\Sigma_0 \times I}(\tau^{(0)})} & \check{E}_{\Sigma_0} & \xhookrightarrow{\iota_{\Sigma_0}} & \hat{E}_{\Sigma_0} & & \\ & \searrow^{\check{V}_X(i\zeta \oplus \tau^{(0)})} & \downarrow{\scriptstyle \check{L}(X,i\zeta)} & & \downarrow{\scriptstyle \hat{L}(X,i\zeta)} & \searrow^{\hat{V}_X(i\zeta \oplus \tau^{(1)})} & \\ & & \check{E}_{\Sigma_1} & \xhookrightarrow{\iota_{\Sigma_1}} & \hat{E}_{\Sigma_1} & \xrightarrow[\hat{V}_{\Sigma_1 \times I}(\tau^{(1)})]{} & \hat{E}_{\Sigma_1, t} \end{array} \tag{3.102}$$

commute for all $\tau^{(i)} \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_i)}$. ■

Let $(i\mathbf{R}, +)$ be the group of purely imaginary numbers. Let $\mathbf{Bord}^{in}_{n,n-1}(i\mathbf{R})$, $\mathbf{Bord}^{out}_{n,n-1}(i\mathbf{R})$, $\mathbf{Bord}^{in\wedge out}_{n,n-1}(i\mathbf{R})$ be the subcategories of $\mathbf{Bord}_{n,n-1}(i\mathbf{R})$ (cf. Definition 3.6) whose morphisms consist of diffeomorphism classes of bordisms with non-empty incoming boundary, non-empty outgoing boundary, or both, respectively.

Proposition 3.15. *The maps* $\check{L}(X, i\zeta), \hat{L}(X, i\zeta)$ *can be assembled into symmetric monoidal functors*

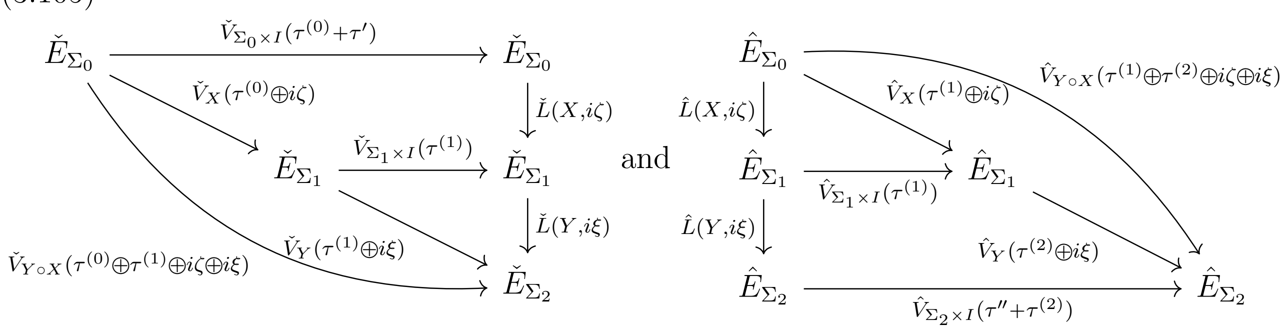
$$\begin{aligned} \check{L} &: \mathbf{Bord}^{in}_{n,n-1}(i\mathbf{R}) \to \mathcal{NDF} \\ \hat{L} &: \mathbf{Bord}^{out}_{n,n-1}(i\mathbf{R}) \to \mathcal{NF} \\ L &: \mathbf{Bord}^{in\wedge out}_{n,n-1}(i\mathbf{R}) \to \mathcal{NP} \end{aligned} \tag{3.103}$$

Proof. For Σ a closed $n-1$-manifold, define

$$\begin{aligned} \check{L}(\Sigma) &:= \check{E}_\Sigma \\ \hat{L}(\Sigma) &:= \hat{E}_\Sigma \\ L(\Sigma) &:= \check{E}_\Sigma \hookrightarrow \hat{E}_\Sigma \end{aligned} \tag{3.104}$$

which all inherit the property from Z that they send disjoint unions to topological tensor products. Let $X : \Sigma_0 \rightsquigarrow \Sigma_1$ and $Y : \Sigma_1 \rightsquigarrow \Sigma_2$ be bordisms and let $\zeta \in \mathbf{R}^{\#\pi_0(X)}$, $\xi \in \mathbf{R}^{\#\pi_0(Y)}$. For all $\tau^{(i)} \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_i)}$, the diagrams

(3.105)



commute, where τ' is any element of $(\mathbf{C}_{>0})^{\#\pi_0(\Sigma_0)}$ satisfying

$$0_X \oplus \tau' = \tau^{(1)} \oplus 0_X \tag{3.106}$$

and τ'' is any element of $(\mathbf{C}_{>0})^{\#\pi_0(\Sigma_2)}$ satisfying

$$\tau'' \oplus 0_Y = 0_Y \oplus \tau^{(1)}. \tag{3.107}$$

By Theorem 3.12, $\check{L}(Y \circ X, i\zeta \oplus i\xi)$ and $\hat{L}(Y \circ X, i\zeta \oplus i\xi)$ are the unique maps making the outer triangles commute and thus we have

$$\begin{aligned} \check{L}(Y, i\xi) \circ \check{L}(X, i\zeta) &= \check{L}(Y \circ X, i\zeta \oplus i\xi) \\ \hat{L}(Y, i\xi) \circ \hat{L}(X, i\zeta) &= \hat{L}(Y \circ X, i\zeta \oplus i\xi) \end{aligned} \tag{3.108}$$

which proves functoriality of $\check{L}$ and $\hat{L}$. Functoriality of L follows from applying Lemma 3.14. ∎

Definition 3.8. We refer to L as the *Lorentzian limit of V*.

Let

$$\begin{aligned} \check{L}_0 &: \mathbf{Bord}^{in}_{n,n-1} \to \mathcal{NDF} \\ \hat{L}_0 &: \mathbf{Bord}^{out}_{n,n-1} \to \mathcal{NF} \\ L_0 &: \mathbf{Bord}^{in \wedge out}_{n,n-1} \to \mathcal{NP} \end{aligned} \tag{3.109}$$

be the symmetric monoidal functors obtained from restricting $\check{L}, \hat{L}, L$ to the subcategories

$$\begin{aligned} \mathbf{Bord}^{in}_{n,n-1} &:= \mathbf{Bord}^{in}_{n,n-1}(0) \subset \mathbf{Bord}^{in}_{n,n-1}(i\mathbf{R}) \\ \mathbf{Bord}^{out}_{n,n-1} &:= \mathbf{Bord}^{out}_{n,n-1}(0) \subset \mathbf{Bord}^{out}_{n,n-1}(i\mathbf{R}) \\ \mathbf{Bord}^{in \wedge out}_{n,n-1} &:= \mathbf{Bord}^{in \wedge out}_{n,n-1}(0) \subset \mathbf{Bord}^{in \wedge out}_{n,n-1}(i\mathbf{R}) \end{aligned} \tag{3.110}$$

The functors (3.109) are topological field theories partially defined on subcategories of the ordinary bordism category.

Definition 3.9. We refer to L_0 as the *short distance topological limit.*

Remark 3.7. There is a family of VFTs controlled by a parameter $\kappa \in \mathbf{R}_{>0}$ called the *coupling constant*

$$V_\kappa : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathcal{NP} \tag{3.111}$$

defined by

$$V_\kappa(\Sigma) := V(\Sigma) \tag{3.112}$$

$$V_\kappa(X,\tau) := V(X,\kappa\tau) \tag{3.113}$$

and the short-distance topological limit is typically studied by taking a limit as κ approaches 0. This corresponds to taking the curves (3.88) to be

$$\gamma^{(i)}(\kappa) = \kappa\tau \tag{3.114}$$

where κ is now a real parameter.

3.3 Reflection Positive VFTs

Let $Z : \mathbf{Bord}_{n,n-1}(\mathrm{Dens}_{\mathbf{C}}) \to \mathcal{NP}$ be a reflection positive VFT and let $V : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathcal{NP}$ be the nuclear symmetric monoidal functor that results when Construction 3.1 is applied to Z. Recall that for Σ a closed $n-1$-manifold, $V(\Sigma) := Z(\mathring{\Sigma}, \mathring{\omega}_\Sigma)$ where $(\mathring{\Sigma}, \mathring{\omega}_\Sigma)$ is a $\tau_{\mathrm{Dens}_{\mathbf{C}}}$ fixed point. Reflection positivity of Z implies that $V(\Sigma)$ is Hermitian and V is in fact a nuclear symmetric monoidal functor

$$V : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathcal{NP}^{h,nuc} \tag{3.115}$$

valued in Hermitian nuclear pairs and nuclear morphisms. Postcomposing (3.115) with (2.56) gives a symmetric monoidal functor

$$V^{Hilb} : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathbf{Hilb}\,. \tag{3.116}$$

We set

$$\mathcal{H}_\Sigma := E_\Sigma^{Hilb} \tag{3.117}$$

to be the Hilbert space assigned to Σ by V^{Hilb}.

Lemma 3.16. *V^{Hilb} is holomorphic, i.e. for all bordisms $X : \Sigma_0 \rightsquigarrow \Sigma_1$ the map*

$$\begin{aligned} (\mathbf{C}_{>0})^{\#\pi_0(X)} &\to \mathrm{Hom}(\mathcal{H}_{\Sigma_0}, \mathcal{H}_{\Sigma_1}) \\ s &\mapsto V^{Hilb}(X,s) \end{aligned} \tag{3.118}$$

is holomorphic.

Proof. This follows from holomorphicity of V (cf. Remark 3.5). ■

Proposition 3.17. *Let* $X : \Sigma_0 \rightsquigarrow \Sigma_1$ *be a bordism of volume* $s \in (\mathbf{C}_{>0})^{\#\pi_0(X)}$*. Then*

$$V^{Hilb}(X^*, \bar{s}) = V^{Hilb}(X, s)^\dagger \tag{3.119}$$

where $X^* : \Sigma_1 \rightsquigarrow \Sigma_0$ *is the bordism* X *with incoming and outgoing boundaries reversed,* $\bar{s} \in (\mathbf{C}_{>0})^{\#\pi_0(X)}$ *is the complex conjugate of* s*, and* $V^{Hilb}(X, s)^\dagger$ *is the Hermitian adjoint of* $V^{Hilb}(X, s) : \mathcal{H}_{\Sigma_0} \to \mathcal{H}_{\Sigma_1}$*.*

Proof. Let $(\mathcal{X}, \mathring{\omega}_X) : (\mathring{\Sigma}_0^+, \mathring{\omega}_{\Sigma_0}) \rightsquigarrow (\mathring{\Sigma}_1^+, \mathring{\omega}_{\Sigma_1})$ be a $\text{Dens}_{\mathbf{C}}$-bordism of volume s. Reflection positivity of Z implies that

$$\begin{aligned} V(X^*, \bar{s}) &= Z(\tau_{\text{Dens}_{\mathbf{C}}}(\mathcal{X}, \mathring{\omega}_X)) \\ &= \tau_{\mathcal{NP}} Z(\mathcal{X}, \mathring{\omega}_X) \\ &= \tau_{\mathcal{NP}} V(X, s) \end{aligned} \tag{3.120}$$

Applying the $(\tau_{\mathcal{NP}}, \tau_{\mathbf{Hilb}})$-equivariant functor (2.56) then gives

$$V^{Hilb}(X^*, \bar{s}) = V^{Hilb}(X, s)^\dagger \tag{3.121}$$

■

For Σ a closed $n-1$-manifold, set

$$V^{Hilb}_\Sigma(s) := V^{Hilb}(\Sigma \times I, s) \tag{3.122}$$

which is a trace-class operator in $\text{End}(\mathcal{H}_\Sigma)$ for all $s \in (\mathbf{C}_{>0})^{\#\pi_0(X)}$.

Proposition 3.18. *Let* Σ *be a connected closed* $n-1$*-manifold. There exists a self-adjoint operator* $H_\Sigma \in \text{End}(\mathcal{H}_\Sigma)$ *with discrete spectrum unbounded from above and bounded from below with finite multiplicity such that*

$$V^{Hilb}_\Sigma(s) = \exp(-sH_\Sigma) \tag{3.123}$$

Proof. For all $s, s' \in \mathbf{C}_{>0}$ we have

$$V_\Sigma^{Hilb}(s)V_\Sigma^{Hilb}(s') = V_\Sigma^{Hilb}(s+s') = V_\Sigma^{Hilb}(s')V_\Sigma^{Hilb}(s) \tag{3.124}$$

and when $s' = \bar{s}$ this becomes

$$V_\Sigma^{Hilb}(s)V_\Sigma^{Hilb}(s)^\dagger = V_\Sigma^{Hilb}(s)^\dagger V_\Sigma^{Hilb}(s) \tag{3.125}$$

by Proposition 3.17. Thus $\{V_\Sigma^{Hilb}(s) \mid s \in \mathbf{C}_{>0}\}$ is a holomorphic family of mutually commuting compact (trace-class) normal operators on $\mathcal{H}_\Sigma$. By the spectral theorem, there exists an isometry

$$\mathcal{H}_\Sigma \cong \ell^2(\mathbf{N}) \tag{3.126}$$

under which V_Σ^{Hilb} becomes a family of diagonal operators

$$V_\Sigma^{Hilb}(s) = \sum_{i \geqslant 0} \mu_i(s)\Pi_i \tag{3.127}$$

where $\mu_i : \mathbf{C}_{>0} \to \mathbf{C}$ is a holomorphic function whose value at s is the ith eigenvalue of $V_\Sigma^{Hilb}(s)$, and $\Pi_i^2 = \Pi_i \in \mathrm{End}(\mathcal{H}_\Sigma)$ is the orthogonal projection onto the ith eigenspace, which has finite dimension. The functions μ_i cannot be identically 0 for otherwise this would imply that $V_\Sigma^{Hilb}(s)$ has a kernel for all s which would imply that the induced inclusion of nuclear spaces $\check{E}_\Sigma \hookrightarrow \hat{E}_\Sigma$ is not dense and this would contradict our assumption that Z is a field theory.

Relation (3.124) implies

$$\mu_i(s+s') = \mu_i(s)\mu_i(s'). \tag{3.128}$$

If $\mu_i(s_0) = 0$ for some $s \in \mathbf{C}_{>0}$, then (3.128) implies $\mu_i(s) = 0$ for all $s \in \mathbf{C}_{>0}$ satisfying $\Re(s) > \Re(s_0)$. Holomorphicity then implies $\mu_i \equiv 0$, which is not allowed. Thus μ_i does not vanish anywhere. The relation (3.128) also implies that the function

$$f_{i,s}(\tau) := \frac{\mu_i(\tau+s) - \mu_i(\tau)}{s\mu_i(\tau)} \tag{3.129}$$

is constant on $\mathbf{C}_{>0}$. In particular the limit

$$\lim_{s \to 0} f_{i,s}(\tau) = \frac{\mu_i'(\tau)}{\mu_i(\tau)} \tag{3.130}$$

is a holomorphic function equal to a constant λ_i which implies $\mu_i'(\tau) = \lambda_i \mu_i(\tau)$ for all $\tau \in \mathbf{C}_{>0}$. Solving the differential equation gives $\mu_i(\tau) = c_i e^{-\lambda_i \tau}$ and (3.128) implies $c_i \in \{0, 1\}$. Since μ_i is not identically zero, we have $c_i = 1$.

By Proposition 3.17, $\mu_i(s) \in \mathbf{R}$ when $s \in \mathbf{R}_{>0}$ which implies $\lambda_i \in \mathbf{R}$. The operator V_Σ is trace-class which implies that $\{\lambda_i\}$ is unbounded above and bounded below with no accumulation points. Set

$$H_\Sigma := \sum_{i \geqslant 0} \lambda_i \Pi_i \tag{3.131}$$

which is unbounded above and bounded below. ■

When $\Sigma := \bigsqcup_{i=1}^k \Sigma^i$ has k connected components, we set

$$\mathcal{H}_\Sigma := \bigotimes_{i=1}^{k} \mathcal{H}_{\Sigma^i} \tag{3.132}$$

$$\hat{H}_{\Sigma^i} := 1 \otimes \cdots \otimes H_{\Sigma^i} \otimes \cdots \otimes 1 \tag{3.133}$$

$$H_\Sigma := \sum_{i=1}^{k} \hat{H}_{\Sigma^i} \tag{3.134}$$

and

$$\begin{aligned} \exp(-sH_\Sigma) &:= \exp(-\sum_{i=1}^{k} s_i \hat{H}_{\Sigma_i}) \\ &= \bigotimes_{i=1}^{k} \exp(-s_i H_{\Sigma^i}) \end{aligned} \tag{3.135}$$

for $s = (s_1, ..., s_k) \in (\mathbf{C}_{>0})^k$. We note that H_Σ is a self-adjoint operator with discrete spectrum unbounded from above and bounded from below on $\mathcal{H}_\Sigma$. That V^{Hilb} is symmetric monoidal implies the following.

Corollary 3.19. *Proposition 3.18 holds for all closed $n-1$-manifolds Σ.*

Let

$$X : \Sigma_0 \rightsquigarrow \Sigma_1 \tag{3.136}$$

be a bordism with $\Sigma_0 = \bigsqcup_i \Sigma_0^i$ and $\Sigma_1 = \bigsqcup_j \Sigma_1^j$.

Proposition 3.20. *There is an equality of maps*

$$V^{Hilb}(X, s) \circ \hat{H}_{\Sigma_0^i} = \hat{H}_{\Sigma_1^j} \circ V^{Hilb}(X, s) \tag{3.137}$$

for all connected components Σ_0^i, Σ_1^j.

Proof. Theorem 3.5 implies that for all $\tau \in \mathbf{C}_{>0}$

$$V^{Hilb}(X, s) \circ \exp(-\tau \hat{H}_{\Sigma_0^i}) = \exp(-\tau \hat{H}_{\Sigma_1^j}) \circ V^{Hilb}(X, s) \tag{3.138}$$

where each side is a holomorphic family of bounded operators from $\mathcal{H}_{\Sigma_0}$ to $\mathcal{H}_{\Sigma_1}$ in the parameter $\tau \in \mathbf{C}_{>0}$. Taking the derivative with respect to τ on both sides followed by the limit $\tau \to 0$ gives the result. ∎

When Σ is a closed connected $n-1$-manifold, there is an orthogonal decomposition

$$\mathcal{H}_\Sigma \cong \widehat{\bigoplus}_{\lambda \in \mathrm{Spec}(H)} \mathcal{H}_\Sigma^\lambda \tag{3.139}$$

where $\mathcal{H}_\Sigma^\lambda$ is the finite dimensional Hilbert space corresponding to the λ-eigenspace of H_Σ and $\widehat{\oplus}$ denotes the Hilbert space completion of the direct sum.

If $\Sigma := \bigsqcup_i \Sigma^i$ has multiple connected components, there is again an orthogonal decomposition of the form (3.139). There is an isomorphism

$$\mathcal{H}_\Sigma^\lambda \cong \bigotimes_i \mathcal{H}_{\Sigma^i}^\lambda \tag{3.140}$$

of the λ-eigenspace of H_Σ, where λ is an eigenvalue in

$$\mathrm{Spec}(H_\Sigma) = \bigcap_i \mathrm{Spec}(H_{\Sigma^i}). \tag{3.141}$$

For X a bordism as in (3.136), we set

$$\Lambda_{\Sigma_0,\Sigma_1} := \mathrm{Spec}(H_{\Sigma_0}) \cap \mathrm{Spec}(H_{\Sigma_1}) \tag{3.142}$$

to be the set of eigenvalues shared by the incoming and outgoing Hamiltonians. For $k = 0, 1$, let

$$\pi_{k,\lambda} : \mathcal{H}_{\Sigma_k} \to \mathcal{H}^\lambda_{\Sigma_k} \tag{3.143}$$

denote the orthogonal projection and

$$\iota_{k,\lambda} : \mathcal{H}^\lambda_{\Sigma_k} \hookrightarrow \mathcal{H}_{\Sigma_k} \tag{3.144}$$

the inclusion.

Corollary 3.21. *There is a decomposition*

$$V^{Hilb}(X, s) = \sum_{\lambda \in \Lambda_{\Sigma_0,\Sigma_1}} \iota_{1,\lambda} \circ V^{Hilb}_\lambda(X, s) \circ \pi_{0,\lambda} \tag{3.145}$$

with $V^{Hilb}_\lambda(X, s) \in \mathrm{Hom}(\mathcal{H}^\lambda_{\Sigma_0}, \mathcal{H}^\lambda_{\Sigma_1})$*, such that*

$$\sum_{\lambda \in \Lambda_{\Sigma_0,\Sigma_1}} \|V^{Hilb}_\lambda(X, s)\| < \infty \tag{3.146}$$

where $\|\cdot\|$ *denotes the operator norm.*

Proof. The decomposition follows immediately from Proposition 3.20. $V^{Hilb}(X, s)$ being trace-class implies (3.146). ■

Let $\gamma : \mathbf{R} \to \mathbf{C}$ be any curve satisfying $\gamma'(t) \in \mathbf{C}_{>0}$ and $\gamma(0) = 0$. For Σ a closed $n-1$-manifold, let

$$\mathcal{C}^{Hilb}_\gamma := \{\mathcal{H}_s := \mathcal{H}_\Sigma \mid s < 0\} \tag{3.147}$$

be the direct system consisting of maps

$$\mathcal{H}_s \xrightarrow{V_\Sigma^{Hilb}(\gamma(s')-\gamma(s))} \mathcal{H}_{s'} \tag{3.148}$$

for all $s < s' < 0$ and

$$\mathcal{D}_\gamma^{Hilb} := \{\mathcal{H}_t := \mathcal{H}_\Sigma \mid t > 0\} \tag{3.149}$$

the inverse system consisting of maps

$$\mathcal{H}_t \xrightarrow{V_\Sigma^{Hilb}(\gamma(t')-\gamma(t))} \mathcal{H}_{t'} \tag{3.150}$$

for all $0 < t < t'$.

Let $\check{E}_\Sigma \hookrightarrow \mathcal{H}_\Sigma \hookrightarrow \hat{E}_\Sigma$ be the Hermitian nuclear pair assigned to Σ by V. By coherence and Proposition 3.6, $\check{E}_\Sigma$ and $\hat{E}_\Sigma$ are canonically isomorphic to the direct and inverse limits of $\mathcal{C}_\gamma^{Hilb}, \mathcal{D}_\gamma^{Hilb}$. Under the isometry (3.139), these can be expressed as

$$\begin{aligned}\check{E}_\Sigma \cong \underset{\mathcal{C}_\gamma^{Hilb}}{\underrightarrow{\operatorname{colim}}}\, \mathcal{H}_s &= \{(v_\lambda) \in \prod_{\lambda \in \operatorname{Spec}(H_\Sigma)} \mathcal{H}_\Sigma^\lambda \mid \exists s < 0 \ , \ (e^{-\gamma(s)\lambda} v_\lambda) \in \mathcal{H}_\Sigma\} \\ &= \{(v_\lambda) \in \prod_{\lambda \in \operatorname{Spec}(H_\Sigma)} \mathcal{H}_\Sigma^\lambda \mid \exists \tau \in \mathbf{C}_{>0} \ , \ (e^{\tau\lambda} v_\lambda) \in \mathcal{H}_\Sigma\}\end{aligned} \tag{3.151}$$

$$\begin{aligned}\hat{E}_\Sigma \cong \underset{\mathcal{D}_\gamma^{Hilb}}{\underleftarrow{\lim}}\, \mathcal{H}_t &= \{(v_\lambda) \in \prod_{\lambda \in \operatorname{Spec}(H_\Sigma)} \mathcal{H}_\Sigma^\lambda \mid \forall t > 0 \ , \ (e^{-\gamma(t)\lambda} v_\lambda) \in \mathcal{H}_\Sigma\} \\ &= \{(v_\lambda) \in \prod_{\lambda \in \operatorname{Spec}(H_\Sigma)} \mathcal{H}_\Sigma^\lambda \mid \forall \tau \in \mathbf{C}_{>0} \ , \ (e^{-\tau\lambda} v_\lambda) \in \mathcal{H}_\Sigma\}\end{aligned} \tag{3.152}$$

The Fréchet topology on $\hat{E}_\Sigma$ can be prescribed by a family of seminorms

$$\mathfrak{p}_i : (v_\lambda) \mapsto \|(e^{-\tau_i\lambda} v_\lambda)\|_{\mathcal{H}_\Sigma} \tag{3.153}$$

where $\{\tau_i\} \subset \mathbf{C}_{>0}$ is any sequence of complex numbers whose real parts are decreasing to 0. The isomorphism

$$\overline{\check{E}} \cong \hat{E}^* \tag{3.154}$$

of (2.46) can be expressed via the nondegenerate pairing

$$\begin{aligned} \overline{\check{E}_\Sigma} \times \hat{E}_\Sigma &\to \mathbf{C} \\ (v_\lambda), (w_\lambda) &\mapsto \langle (e^{\bar{\tau}\lambda} v_\lambda), (e^{-\tau\lambda} w_\lambda) \rangle \end{aligned} \tag{3.155}$$

where $\tau \in \mathbf{C}_{>0}$ is such that $(e^{\bar{\tau}\lambda} v_\lambda), (e^{-\tau\lambda} w_\lambda) \in \mathcal{H}_\Sigma$ and $\langle , \rangle$ is the sesquilinear inner product on $\mathcal{H}_\Sigma$.

Lemma 3.22. *Let Σ_0, Σ_1 be nonempty closed $n-1$-manifolds and let $f_\lambda : \mathcal{H}^\lambda_{\Sigma_0} \to \mathcal{H}^\lambda_{\Sigma_1}$ be a collection of linear maps between finite dimensional Hilbert spaces for each $\lambda \in \Lambda_{\Sigma_0,\Sigma_1}$. Then the not necessarily bounded operator*

$$f^{Hilb} := \sum_{\lambda \in \Lambda_{\Sigma_0,\Sigma_1}} \iota_{1,\lambda} \circ f_\lambda \circ \pi_{0,\lambda} \tag{3.156}$$

from $\mathcal{H}_{\Sigma_0}$ to $\mathcal{H}_{\Sigma_1}$ is induced from a morphism of nuclear pairs

$$\begin{array}{ccccc} \check{E}_{\Sigma_0} & \hookrightarrow & \mathcal{H}_{\Sigma_0} & \hookrightarrow & \hat{E}_{\Sigma_1} \\ \downarrow{\scriptstyle \check{f}} & & & & \downarrow{\scriptstyle \hat{f}} \\ \check{E}_{\Sigma_1} & \hookrightarrow & \mathcal{H}_{\Sigma_1} & \hookrightarrow & \hat{E}_{\Sigma_1} \end{array} \tag{3.157}$$

if and only if for all $t > 0$ there exists $C > 0$ such that

$$\|f_\lambda\| < Ce^{t\lambda} \tag{3.158}$$

for every $\lambda \in \Lambda_{\Sigma_0,\Sigma_1}$.

Proof. Assume the identifications (3.151) and (3.152). In order for $\check{f}$ to have image in $\check{E}_{\Sigma_1}$ we need that for all $v = (v_\lambda)$ in the closed subspace $\widehat{\bigoplus}_{\lambda \in \Lambda_{\Sigma_0,\Sigma_1}} \mathcal{H}^\lambda_{\Sigma_0} \subset \mathcal{H}_{\Sigma_0}$ and $s \in \mathbf{C}_{>0}$, there exists $s' \in \mathbf{C}_{>0}$ such that

$$e^{s'\lambda} \sum_{\lambda \in \Lambda_{\Sigma_0,\Sigma_1}} f_\lambda(e^{-s\lambda} v_\lambda) \in \mathcal{H}_{\Sigma_1} \tag{3.159}$$

which is true if and only if (3.158) is satisfied. Conversely, if (3.158) holds, the maps

$$\begin{aligned} \check{f} : \check{E}_{\Sigma_0} &\to \check{E}_{\Sigma_1} \\ (x_\lambda) &\mapsto (f_\lambda(x_\lambda)) \end{aligned} \tag{3.160}$$

and

$$\begin{aligned} \hat{f} : \hat{E}_{\Sigma_0} &\to \hat{E}_{\Sigma_1} \\ (x_\lambda) &\mapsto (f_\lambda(x_\lambda)) \end{aligned} \tag{3.161}$$

are well-defined and fit into the commutative diagram (3.157). Let $s > 0$ and let

$$\begin{aligned} \mathfrak{p}_s : \hat{E}_{\Sigma_1} &\to \mathbf{R} \\ y &\mapsto \|e^{-sH_{\Sigma_1}} y\|_{\mathcal{H}_{\Sigma_1}} \end{aligned} \tag{3.162}$$

be a seminorm on $\hat{E}_{\Sigma_1}$ and let

$$\begin{aligned} \mathfrak{q}_t : \hat{E}_{\Sigma_0} &\to \mathbf{R} \\ x &\mapsto \|e^{-tH_{\Sigma_0}} x\|_{\mathcal{H}_{\Sigma_0}} \end{aligned} \tag{3.163}$$

be a seminorm on $\hat{E}_{\Sigma_0}$ with $0 < t < s$. By assumption, there exists $C > 0$ with

$$\|f_\lambda\| < Ce^{(s-t)\lambda} \tag{3.164}$$

for all $\lambda \in \Lambda_{\Sigma_0,\Sigma_1}$. If $\mathfrak{q}_t(x) < \delta$, then

$$\begin{aligned} \mathfrak{p}_s(f_\lambda x_\lambda) &= \|(e^{-s\lambda} f_\lambda x_\lambda)\|_{\mathcal{H}_{\Sigma_1}} \\ &\leqslant C\|e^{-t\lambda} x_\lambda\| \\ &< C\delta \end{aligned} \tag{3.165}$$

which shows continuity of (3.161).

The adjoint maps

$$f_\lambda^\dagger : \mathcal{H}_{\Sigma_1}^\lambda \to \mathcal{H}_{\Sigma_0}^\lambda \tag{3.166}$$

satisfy the same bound (3.158) and thus fit together into a continuous map

$$\check{f}^\dagger : \hat{E}_{\Sigma_1} \to \hat{E}_{\Sigma_0} \tag{3.167}$$

which can be identified with the conjugate transpose $\overline{\check{f}^*} : \overline{\check{E}_{\Sigma_1}^*} \to \overline{\check{E}_{\Sigma_0}^*}$ via the isomorphism (3.154). Proposition (A.12) then implies (3.160) is continuous. ■

When the VFT is reflection positive, the Lorentzian limit of V is a functor

$$L : \mathbf{Bord}_{n,n-1}^{in\wedge out}(i\mathbf{R}) \to \mathcal{NP}^h \tag{3.168}$$

taking values in Hermitian nuclear pairs. If X is a bordism having both incoming and outgoing boundary, $L(X, i\zeta)$ is the morphism

$$\begin{array}{ccccc} \check{E}_{\Sigma_0} & \hookrightarrow & \mathcal{H}_{\Sigma_0} & \hookrightarrow & \hat{E}_{\Sigma_0} \\ {\scriptstyle \check{L}(X,i\zeta)}\downarrow & & & & \downarrow{\scriptstyle \hat{L}(X,i\zeta)} \\ \check{E}_{\Sigma_1} & \overset{j}{\hookrightarrow} & \mathcal{H}_{\Sigma_1} & \hookrightarrow & \hat{E}_{\Sigma_1} \end{array} \tag{3.169}$$

If $X = \Sigma \times I$ is a cylinder, $L(X, i\zeta)$ is a unitary morphism (cf. Definition 2.11) with

$$L^{Hilb}(\Sigma \times I, i\zeta) = e^{-i\zeta H_\Sigma}. \tag{3.170}$$

However for a general bordism $X : \Sigma_0 \rightsquigarrow \Sigma_1$ equipped with imaginary volume $i\zeta \in (i\mathbf{R})^{\#\pi_0(X)}$, there is no bounded map of Hilbert spaces that fits into the diagram. Nevertheless, we can define an unbounded operator

$$L^{Hilb}(X, i\zeta) := j \circ \check{L}(X, i\zeta) \tag{3.171}$$

whose domain of definition is $\check{E}_{\Sigma_0}$ which we identify with its dense image in $\mathcal{H}_{\Sigma_0}$.

Proposition 3.23. *The unbounded operator $L^{Hilb}(X, i\zeta)$ has a decomposition*

$$L^{Hilb}(X, i\zeta) = \sum_{\lambda \in \Lambda_{\Sigma_0,\Sigma_1}} \iota_{1,\lambda} \circ L_\lambda^{Hilb}(X, i\zeta) \circ \pi_{0,\lambda} \tag{3.172}$$

with $L_\lambda^{Hilb}(X, i\zeta) \in \mathrm{Hom}(\mathcal{H}_{\Sigma_0}^\lambda, \mathcal{H}_{\Sigma_1}^\lambda)$, such that for all $t > 0$, there exists $C > 0$ with

$$\|L_\lambda^{Hilb}(X, i\zeta)\| < Ce^{t\lambda} \tag{3.173}$$

for all $\lambda \in \Lambda_{\Sigma_0,\Sigma_1}$.

Proof. The map $L^{Hilb}(X, i\zeta)$ is the unique map making the diagram

$$\begin{array}{ccc} \mathcal{H}_{\Sigma_0} & \xrightarrow{\exp(-\tau H_{\Sigma_0})} & \check{E}_{\Sigma_0} \\ & {\scriptstyle V^{Hilb}(X,\tau\oplus i\zeta)}\searrow & \downarrow{\scriptstyle L^{Hilb}(X,i\zeta)} \\ & & \mathcal{H}_{\Sigma_1} \end{array} \tag{3.174}$$

commute for all $\tau \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_0)}$. If $L^{Hilb}(X, i\zeta)$ did not satisfy (3.172), there would be some $v_\lambda \in \mathcal{H}^\lambda_{\Sigma_0}$ such that $\pi_{1,\mu}(L^{Hilb}(X, i\zeta)v_\lambda) \neq 0$ for $\mu \neq \lambda$ in $\mathrm{Spec}(H_{\Sigma_1})$. Since $\mathcal{H}^\lambda_{\Sigma_0}$ is an eigenspace of $\exp(-\tau H_{\Sigma_0})$, this would imply that $\pi_{1,\mu}(V^{Hilb}(X, \tau \oplus i\zeta)v_\lambda) \neq 0$ which contradicts Corollary 3.21. The growth condition (3.173) is implied by Lemma 3.22. ■

Thus (3.168) sends cylindrical bordisms to unitary morphisms and general bordisms to unbounded morphisms satisfying the sub-exponential growth condition (3.173).

By Proposition 2.6, the sheaf morphism (3.26) induces a symmetric monoidal functor

$$\sqrt{\det}_* : \mathbf{Bord}^{in \wedge out}_{n,n-1}(\mathrm{Met}_{\mathrm{Lor}}) \to \mathbf{Bord}^{in \wedge out}_{n,n-1}(\mathrm{Dens}_{i\mathbf{R}}). \tag{3.175}$$

The integration functor (3.73) extends to a symmetric monoidal functor

$$\int : \mathbf{Bord}^{in \wedge out}_{n,n-1}(\mathrm{Dens}_{i\mathbf{R}}) \to \mathbf{Bord}^{in \wedge out}_{n,n-1}(i\mathbf{R}) \tag{3.176}$$

which sends a bordism equipped with a purely imaginary density with nonempty incoming and outgoing boundary to the same bordism labeled by its purely imaginary total volume. We set

$$\mathscr{L} : \mathbf{Bord}^{in \wedge out}_{n,n-1}(\mathrm{Met}_{\mathrm{Lor}}) \xrightarrow{\sqrt{\det}_*} \mathbf{Bord}^{in \wedge out}_{n,n-1}(\mathrm{Dens}_{i\mathbf{R}}) \xrightarrow{\int} \mathbf{Bord}^{in \wedge out}_{n,n-1}(i\mathbf{R}) \xrightarrow{L} \mathcal{NP}^h \tag{3.177}$$

to be the composition. We summarize the above in the following:

Theorem 3.24. *A reflection positive volume-dependent field theory induces a symmetric monoidal functor*

$$\mathscr{L} : \mathbf{Bord}^{in \wedge out}_{n,n-1}(\mathrm{Met}_{\mathrm{Lor}}) \to \mathcal{NP}^h \tag{3.178}$$

out of the category of bordisms equipped with smooth, possibly degenerate Lorentzian metrics with nonempty incoming and outgoing boundary. It sends cylindrical bordisms to unitary morphisms and general bordisms to unbounded morphisms satisfying the sub-exponential growth condition (3.173).

Definition 3.10. We call (3.178) the *Lorentzian limit of* Z.

Remark 3.8. Theorem 5.2 of Kontsevich and Segal (2021) constructs the Lorentzian limit of a reflection positive $\text{Met}_{\mathbf{C}}$-field theory on the subcategory of real-analytic and globally hyperbolic bordisms valued in Hilbert spaces and unitary operators. Theorem 3.24 extends the domain of this functor to smooth, possibly degenerate Lorentzian bordisms with nonempty incoming/outgoing boundary when the reflection positive field theory is volume-dependent, provided we also enlarge the codomain category to allow unbounded morphisms of Hermitian nuclear pairs.

When the field theory is reflection positive, the short distance limit (cf. Definition 3.9) is a functor

$$L_0 : \mathbf{Bord}_{n,n-1}^{in \wedge out} \to \mathcal{NP}^h. \tag{3.179}$$

We record a lemma which can be checked by unraveling definitions.

Lemma 3.25. *Let* $X : \Sigma_0 \rightsquigarrow \Sigma_1$ *be a bordism with volume* $s \in (\mathbf{C}_{>0})^{\#\pi_0(X)}$ *and let* $s_i \in (\mathbf{C}_{>0})^{\#\pi_0(\Sigma_i)}$ *for* $i = 0, 1$ *such that* $s_0 \oplus 0_X \oplus s_1 = s$. *Then*

$$V^{Hilb}(X, s) = \begin{cases} \exp(-s_1 H_{\Sigma_1}) \circ \hat{L}_0(X) & \Sigma_0 = \varnothing, \Sigma_1 \neq \varnothing \\ \check{L}_0(X) \circ \exp(-s_0 H_{\Sigma_0}) & \Sigma_0 \neq \varnothing, \Sigma_1 = \varnothing \\ \exp(-s_1 H_{\Sigma_1}) \circ L_0^{Hilb}(X) \circ \exp(-s_0 H_{\Sigma_0}) & \Sigma_0, \Sigma_1 \neq \varnothing \end{cases} \tag{3.180}$$

where in our notation we regard $\check{E}_{\Sigma_i} \subset \mathcal{H}_{\Sigma_i} \subset \hat{E}_{\Sigma_i}$ *as subspace inclusions and* $L_0^{Hilb}(X) : \mathcal{H}_{\Sigma_0} \to \mathcal{H}_{\Sigma_1}$ *is the unbounded operator induced by* $L_0(X)$.

3.4 The Long-Distance Topological Limit

In this section we define the long-distance topological limit of a reflection positive VFT. Let Z be a reflection positive volume-dependent theory with induced functor

$$V^{Hilb} : \mathbf{Bord}_{n,n-1}(\mathbf{C}_{>0}) \to \mathbf{Hilb}. \tag{3.181}$$

Recall that by Proposition 3.18 V^{Hilb} evaluates on a cylindrical bordism $\Sigma \times I$ of volume $s \in \mathbf{C}_{>0}$ to the operator

$$V_\Sigma^{Hilb}(s) := \exp(-sH_\Sigma) \tag{3.182}$$

where H_Σ are self-adjoint unbounded operators on $\mathcal{H}_\Sigma$ with discrete spectrum bounded below with finite multiplicity. In this section we assume that all H_Σ have non-negative spectrum with lowest eigenvalue 0.

Using isomorphisms (3.151) and (3.152), we identify $\ker(H_\Sigma)$ as a finite dimensional subspace in the sequence of inclusions

$$\ker(H_\Sigma) \subset \check{E}_\Sigma \subset \mathcal{H}_\Sigma \subset \hat{E}_\Sigma. \tag{3.183}$$

Let $\pi_\Sigma \in \mathrm{End}(\mathcal{H}_\Sigma)$ be the orthogonal projection onto $\ker(H_\Sigma)$ and let $\iota_\Sigma : \ker(\Sigma) \hookrightarrow \mathcal{H}_\Sigma$ be the inclusion. If $\Sigma = \varnothing$ is the empty manifold we set $\pi_\Sigma = \iota_\Sigma = id$ to be the identity map on $\mathbf{C}$.

As $\Re(s) \to +\infty$, the operator $V_\Sigma^{Hilb}(s)$ converges to π_Σ uniformly. In light of Lemma 3.25, we make the following definition.

Definition 3.11. The *long-distance topological limit* of Z is the functor

$$L_\infty : \mathbf{Bord}_{n,n-1} \to \mathbf{Vect} \tag{3.184}$$

which sends Σ to $\ker(H_\Sigma)$ and a bordism $X : \Sigma_0 \rightsquigarrow \Sigma_1$ with at least one nonempty boundary to

$$L_\infty(X) := \pi_{\Sigma_1} \circ \tilde{L}_0(X) \circ \iota_{\Sigma_0} \tag{3.185}$$

where

$$\tilde{L}_0(X) := \begin{cases} \hat{L}_0(X) & \Sigma_1 \neq \varnothing \\ \check{L}_0(X) & \Sigma_0 \neq \varnothing \end{cases} \tag{3.186}$$

Remark 3.9. If both $\Sigma_0 = \Sigma_1 = \varnothing$, we can express X as the composition of bordisms with at least one nonempty boundary. If neither Σ_0 nor Σ_1 are empty, $\pi_{\Sigma_1} \circ \check{L}_0(X) \circ \iota_{\Sigma_0} = \pi_{\Sigma_1} \circ \hat{L}_0(X) \circ \iota_{\Sigma_0}$. L_∞ is symmetric monoidal because $\ker(H_{\Sigma \sqcup \Sigma'}) = \ker(H_\Sigma) \otimes \ker(H_{\Sigma'})$ and thus defines a topological field theory.

Appendix A: Nuclear Spaces

A.1 Topological Vector Spaces

Definition A.1. A *filter* on a set X is a family of subsets $\mathscr{F} \subset 2^X$ satisfying:

1. $\varnothing \notin \mathscr{F}$
2. $U, V \in \mathscr{F} \implies U \cap V \in \mathscr{F}$
3. $A \supset U$ and $U \in \mathscr{F} \implies A \in \mathscr{F}$

A topology on X determines a filter $\mathscr{F}_x$ for every $x \in X$ consisting of the neighborhoods of x. We recall that a neighborhood U of x contains an open subneighborhood x and we can regard U as a neighborhood of y for every $y \in V$. Thus for each $x \in X$, $\mathscr{F}_x$ satisfies

1. $x \in \bigcap_{U \in \mathscr{F}_x} U$
2. For every $U \in \mathscr{F}_x$, there exists $V \in \mathscr{F}_x$ such that for all $y \in V$, $U \in \mathscr{F}_y$.

Conversely, a collection of filters $\mathscr{F}_x$ for each $x \in X$ satisfying the above two properties defines a topology on X.

Definition A.2. Let X be a topological space. A filter $\mathscr{F}$ *converges* to $x \in X$ if every neighborhood of x belongs to $\mathscr{F}$.

Let E be a topological vector space.

Definition A.3. A filter $\mathscr{F} \subset 2^E$ is *Cauchy* if to every neighborhood U of $0 \in E$ there is a subset $M \in \mathscr{F}$ such that $M - M \subset U$.

Definition A.4. E is *complete* if every Cauchy filter on E converges to a point $x \in E$.

Before defining completions, we record some definitions while E is a general topological vector space.

Definition A.5. A subset $B \subset E$ is *bounded* if for every neighborhood U of $0 \in E$ there exists $t > 0$ such that $B \subset tU$.

Definition A.6. A subset $D \subset E$ is a *disk* if it is convex and balanced.

Definition A.7. A subset $T \subset E$ is a *barrel* if it is a closed and absorbing disk.

Definition A.8. E is *barrelled* if every barrel is a neighborhood of $0 \in E$.

We now assume E is Hausdorff. Let $\mathscr{C}$ be the set of all Cauchy filters on E. Let $\mathscr{F}, \mathscr{G} \in \mathscr{C}$ be two Cauchy filters. We will write $\mathscr{F} \sim \mathscr{G}$ if for all neighborhoods U of $0 \in E$, there is an element A of $\mathscr{F}$ and an element B of $\mathscr{G}$ such that $A - B \subset U$.

Lemma A.1. *The relation $\sim$ is an equivalence relation.*

Proof. Treves (2006) Theorem 5.2 (1) ∎

Definition A.9. The *completion* of E is the quotient $\tilde{E} := \mathscr{C} / \sim$

Proposition A.2. *$\tilde{E}$ is a complete Hausdorff topological vector space and there exists a bicontinuous, dense injection $E \hookrightarrow \tilde{E}$ satisfying the following universal property: for every continuous linear map $E \to F$ with F Hausdorff, there exists a unique continuous extension*

$$\begin{array}{ccc} E & \longrightarrow & F \\ \big\downarrow & \nearrow_{\exists!} & \\ \tilde{E} & & \end{array} \tag{A.1}$$

Proof. Treves (2006) Theorem 5.2 ∎

A.2 Locally Convex Hausdorff Spaces

Let E be a locally convex Hausdorff vector space.

Proposition A.3. *The completion $\tilde{E}$ is a locally convex Hausdorff space.*

Let $E' := \mathrm{Hom}(E, \mathbf{C})$ denote the dual vector space of continuous linear functionals with no choice of topology.

Definition A.10. The *polar* of a subset $A \subset E$ is the subset $A^0 \subset E'$ defined by

$$A^0 := \{x' \in E' \mid \sup_{x \in A} \langle x', x \rangle \leqslant 1\} \tag{A.2}$$

Definition A.11. A subset $S \subset E'$ is *equicontinuous* if for all $\varepsilon > 0$, there is a neighborhood U of $0 \in E$ such that for all $f \in E'$ and $x \in U$, $|f(x)| < \varepsilon$.

Proposition A.4. *A subset $S \subset E'$ is equicontinuous if and only if it is contained in the polar of a neighborhood of $0 \in E$.*

Proof. Treves (2006) Proposition 32.7 ■

Denote by E'_σ and E^* the dual of E endowed with the weak and strong topologies respectively, i.e. the topologies of uniform convergence on finite and bounded sets respectively.

Definition A.12. Given a continuous map $u : E \to F$ between locally convex Hausdorff vector spaces, the *transpose of* u is the map

$$u^* : F' \to E' \tag{A.3}$$

which precomposes a linear functional on F by u.

Proposition A.5. *Let $u : E \to F$ be a continuous map between locally convex Hausdorff spaces and let $u^* : F' \to E'$ be its transpose. Then $\mathrm{Im}(u)$ is dense in F if and only if u^* is injective.*

Proof. This is a consequence of the Hahn Banach theorem – see Treves (2006) Corollary 5 of Theorem 18.2. ■

Proposition A.6. *The transpose is a continuous linear map*

$$u^* : F^* \to E^* \tag{A.4}$$

between strong duals.

Proof. Treves (2006) Corollary 19.5 ■

Proposition A.7. *There is an isomorphism of vector space* $E \simeq (E'_\sigma)'$

Proof. Treves (2006) Proposition 35.1 ■

Proposition A.8. *Let* E, F *be locally convex Hausdorff spaces. Then there is an isomorphism of vector spaces*

$$E \otimes F \simeq B(E'_\sigma, F'_\sigma) \tag{A.5}$$

Proof. Treves (2006) Proposition 42.4 ■

Definition A.13. Let $A \subset E$ and $A^0 \subset E'$ its polar. The *bipolar* A^{00} of A is the polar of A^0 regarded as a subset $E \simeq (E'_\sigma)'$

Proposition A.9. *Let* $A \subset E$ *be a subset. The bipolar* A^{00} *is the closed convex balanced hull of* A*.*

Proof. Treves (2006) Proposition 35.3 ■

Definition A.14. E is *Montel* if E is barrelled and if every closed bounded subset is compact.

Proposition A.10. *Suppose* E *is Montel. Then every closed, bounded subset of* E^* *is compact. Moreover, the strong and weak topologies coincide on bounded subsets of* E^**.*

Proof. Treves (2006) Proposition 34.6 ■

Proposition A.11. *Let E be a Montel space. Then every weakly convergent sequence in E' is strongly convergent.*

Proof. Treves (2006) Corollary 34.6 ■

Definition A.15. A locally convex Hausdorff vector space E is *reflexive* if $(E^*)^*$.

Proposition A.12. *A linear map $u : E \to F$ between reflexive spaces is continuous if and only if u^* is continuous.*

Proof. This follows from Proposition A.6. ■

Proposition A.13. *Let E be a Montel space. Then E is reflexive and E^* is also Montel.*

Proof. Treves (2006) Proposition 36.10 ■

A.3 Topological Tensor Products

Let E, F be locally convex Hausdorff spaces.

Definition A.16. The π-topology on $E \otimes F$ is the finest locally convex topology such that the map

$$\begin{aligned} E \times F &\to E \otimes F \\ (x, y) &\mapsto x \otimes y \end{aligned} \tag{A.6}$$

is continuous. We will call the tensor product endowed with this topology the *projective tensor product* and denote it by $E \otimes_\pi F$.

Let $B(E, F)$ denote the vector space of continuous bilinear forms on $E \times F$ and $\mathscr{B}(E, F)$ the vector space of separately continuous bilinear forms on $E \times F$. By Proposition A.8 we can regard $E \otimes F$ as a vector subspace of $\mathscr{B}(E'_\sigma, F'_\sigma)$.

Definition A.17. The ε-topology on $E \otimes F$ is the subspace topology when $\mathscr{B}(E'_\sigma, F'_\sigma)$ is equipped with the topology of uniform convergence on products of equicontinuous sets. This topological tensor product is called the *injective tensor product* and will be denoted by $E \otimes_\varepsilon F$.

Notation. We will denote the completions of these two topological tensor products by $E \widehat{\otimes}_\pi F$ and $E \widehat{\otimes}_\varepsilon F$.

We now examine some properties of the projective tensor product. The first is a universal property.

Proposition A.14. *Let E, F, G be locally convex Hausdorff spaces and $E \times F \to G$ a continuous bilinear map. Then there exists a unique continuous extension*

$$\begin{array}{ccc} E \times F & \longrightarrow & G \\ \downarrow & \overset{\exists !}{\nearrow} & \\ E \otimes_\pi F & & \end{array} \tag{A.7}$$

Proof. Treves (2006) Proposition 43.4 ∎

Corollary A.15. *The map in (A.7) extends uniquely to a map $E \widehat{\otimes}_\pi F \to G$.*

Proof. Apply Proposition A.2. ∎

We note that by definition of the injective tensor product, the map

$$\begin{aligned} E \times F &\to E \otimes_\varepsilon F \\ (x, y) &\mapsto x \otimes y \end{aligned} \tag{A.8}$$

is continuous.

Corollary A.16. *The map (A.8) extends to a continuous map*

$$E \widehat{\otimes}_\pi F \to E \widehat{\otimes}_\varepsilon F. \tag{A.9}$$

Thus the projective topology is finer than the injective topology.

Lemma A.17. *Let* $E_1 \xrightarrow{u} E_2, F_1 \xrightarrow{v} F_2$ *be continuous maps between locally convex Hausdorff spaces. Then* $u \otimes v : E_1 \otimes_\tau F_1 \to E_2 \otimes_\tau F_2$ *is continuous for* $\tau = \pi, \varepsilon$.

Proof. Treves (2006) Proposition 43.6 ■

Corollary A.18. *The previous maps extend uniquely to maps on completions*

$$u \widehat{\otimes}_\pi v : E_1 \widehat{\otimes}_\pi F_1 \to E_2 \widehat{\otimes}_\pi F_2 \tag{A.10}$$

$$u \widehat{\otimes}_\varepsilon v : E_1 \widehat{\otimes}_\varepsilon F_1 \to E_2 \widehat{\otimes}_\varepsilon F_2 \tag{A.11}$$

Proof. Apply Proposition A.2. ■

Proposition A.19. *Let* u, v *be as above. If* u, v *are injective, then* $u \widehat{\otimes}_\varepsilon v$ *is injective. Likewise, if* u, v *have dense images, then* $u \widehat{\otimes}_\pi v$ *has dense image.*

Proof. Treves (2006) Exercise 39.3 and Proposition 43.9 ■

We will also need the following description of elements of the projective tensor product of Fréchet spaces.

Proposition A.20. *Let* E, F *be Fréchet spaces. Then every* $\theta \in E \widehat{\otimes}_\pi F$ *can be expressed as an absolutely convergent series*

$$\theta = \sum_n \lambda_n x_n \otimes y_n \tag{A.12}$$

where $\{\lambda_n\} \in \ell^1$ *and* $\{x_n\}, \{y_n\}$ *are sequences converging to 0 in* E, F.

Proof. Treves (2006) Theorem 45.1 ■

A.4 Nuclear Maps

Let E, F be Banach spaces. There is a continuous map $E^* \times F \to \mathrm{Hom}(E, F)$, where the codomain is equipped with the topology induced by the operator norm. By Proposition A.14, there is a unique continuous map $E^* \widehat{\otimes}_\pi F \to \mathrm{Hom}(E, F)$ and we denote its image by $L^1(E, F) \subset \mathrm{Hom}(E, F)$.

Definition A.18. Let E, F be Banach spaces. A continuous map $u : E \to F$ is *nuclear* if $u \in L^1(E, F)$.

We now describe nuclear maps when E, F are general locally convex Hausdorff spaces. Let $D \subset E$ be a disk and let $E_D := \mathrm{span}(D) \subset E$ be the subspace spanned by D. There is a seminorm $\mathfrak{p}_D : E_D \to \mathbf{R}$ given by

$$\mathfrak{p}_D(x) := \inf_{\{\rho \in \mathbf{R}_{>0} \mid x \in \rho U\}} \rho. \tag{A.13}$$

Note that if B is a bounded disk, $\mathfrak{p}_B$ is a norm on E_B.

Definition A.19. We say a subset $B \subset E$ is a *Banach disk* if B is a bounded disk and $(E_B, \mathfrak{p}_B)$ is a Banach space.

Lemma A.21. *Let $B \subset E$ be a complete, bounded disk. Then B is a Banach disk.*

Proof. Treves (2006) Lemma 36.1 ∎

Corollary A.22. *Let $B \subset E$ be a compact disk. Then B is a Banach disk.*

Proof. Treves (2006) Corollary 36.1 ∎

Definition A.20. A subset $U \subset E$ is a *dual Banach disk* if it is a closed disk that is a neighborhood of $0 \in E$.

A dual Banach disk U is absorbing which implies $\mathfrak{p}_U$ is a seminorm on all of E. We remark that U is the closed unit semiball of $\mathfrak{p}_U$.

Construction A.1. Let $\mathfrak{p} : E \to \mathbf{R}$ be a seminorm and V its closed unit semiball. There is a norm induced by $\mathfrak{p}$ on the quotient $E/\ker(\mathfrak{p})$. We will use $E_{\mathfrak{p}}$ and E^V interchangeably to denote the Banach space completion $\overline{E/\ker(\mathfrak{p})}$.

Applying the construction to $\mathfrak{p}_U$ yields the Banach space E^U.

Let $\mathfrak{U}$ be the collection of all dual Banach disks on E and $\mathfrak{B}$ the collection of all Banach disks on F. Let $U \in \mathfrak{U}$, $B \in \mathfrak{B}$ and suppose $u \in L^1(E^U, F_B)$. There are continuous maps $E \xrightarrow{q_U} E^U$ and $F_B \xrightarrow{i_B} F$ induced by the vector space quotient $E \to E/\ker(\mathfrak{p}_U)$ and inclusion $F_B \hookrightarrow F$ respectively. Thus there is a map

$$\begin{aligned} L^1(E^U, F_B) &\to \operatorname{Hom}(E, F) \\ u &\mapsto i_B \circ u \circ q_U \end{aligned} \tag{A.14}$$

and we denote its image by $L^1_{U,B}(E, F)$. Set

$$L^1(E, F) := \bigcup_{U \in \mathfrak{U}, B \in \mathfrak{B}} L^1_{U,B}(E, F) \tag{A.15}$$

Definition A.21. A map $u : E \to F$ is *nuclear* if $u \in L^1(E, F)$.

Proposition A.23. *Let $f : G \to E \xrightarrow{u} F \to H$ be a composition of continuous maps between locally convex Hausdorff spaces and suppose u is nuclear. Then f is nuclear.*

Proof. Treves (2006) Proposition 47.1 ∎

Proposition A.24. *Let $u : E \to F$ be continuous. The following are equivalent:*

1. *u is nuclear*

2. *u is a composition*

$$u : E \to V \xrightarrow{v} W \to F$$

with V, W Banach spaces and v nuclear

3. u can be expressed as a map

$$u : x \mapsto \sum_k \lambda_k \langle x'_k, x \rangle y_k \tag{A.16}$$

where $\{x'_k\} \subset E'$ *is an equicontinuous sequence,* $\{y_k\} \subset B$ *is a sequence contained in a Banach disk* $B \subset F$*,* $\{\lambda_k\} \subset \mathbf{C}$ *is absolutely summable.*

Proof. Treves (2006) Proposition 47.2 ■

Corollary A.25. $L^1(E, F) \subset \mathrm{Hom}(E, F)$ *is a subspace.*

We have the following refinement of Proposition A.24 for maps between Fréchet spaces.

Proposition A.26. *Let* $u : E \to F$ *be a continuous map between Fréchet spaces. Then u is nuclear if and only if u has a representation (A.16) with* $\{x'_k\}, \{y_k\}$ *bounded sequences in* E^*, F*.*

Proof. Treves (2006) Proposition 47.2 Corollary 2 ■

We now record some properties of the transpose of a nuclear map. Recall that by Proposition A.6 the transpose of a continuous map is continuous.

Proposition A.27. *Let* $E \xrightarrow{u} F$ *be a nuclear map between two locally convex Hausdorff spaces. Then* $u^* : F^* \to E^*$ *is nuclear. (Here,* E^* *denotes the dual vector space equipped with the strong topology.)*

Proof. Treves (2006) Proposition 47.4 ■

A.5 Nuclear Spaces

Let E be a locally convex Hausdorff space. Suppose $\mathfrak{p}, \mathfrak{q}$ are continuous seminorms on E with $\mathfrak{q} \geqslant \mathfrak{p}$. Then the topology defined by $\mathfrak{q}$ on E is finer than the topology defined by $\mathfrak{p}$ and $\ker(\mathfrak{q}) \subset \ker(\mathfrak{p})$. Thus there is a continuous map $E/\ker(\mathfrak{q}) \to E/\ker(\mathfrak{p})$ which extends to a continuous map of completions $E_{\mathfrak{q}} \to E_{\mathfrak{p}}$.

Definition A.22. A locally convex Hausdorff vector space E is *nuclear* if for every seminorm $\mathfrak{p} : E \to \mathbf{R}$ there exists a seminorm $\mathfrak{q} \geqslant \mathfrak{p}$ such that

$$E_{\mathfrak{q}} \to E_{\mathfrak{p}} \tag{A.17}$$

is a nuclear map of Banach spaces.

Proposition A.28. *Let E be a locally convex Hausdorff space. The following are equivalent:*

1. *E is nuclear*
2. *The map (A.9) induces an isomorphism*

   $$E \widehat{\otimes}_{\pi} F \simeq E \widehat{\otimes}_{\varepsilon} F \tag{A.18}$$

   for every F locally convex Hausdorff.
3. *Every continuous map $E \to B$ to a Banach space B is nuclear.*

Proof. Treves (2006) Theorem 50.1 ∎

Notation. In light of the isomorphism (A.18), we will denote $E \widehat{\otimes} F := E \widehat{\otimes}_{\pi} F \simeq E \widehat{\otimes}_{\varepsilon} F$ when either E or F is nuclear.

Proposition A.29. *The following are true:*

1. *E is nuclear if and only if $\tilde{E}$ is nuclear.*
2. *A linear subspace of a nuclear space is nuclear.*
3. *Let $F \subset E$ be a closed subspace. Then E/F is nuclear.*
4. *Arbitrary colimits of nuclear spaces are nuclear.*
5. *Countable limits of nuclear spaces are nuclear.*
6. *If E, F are nuclear, then $E \widehat{\otimes} F$ is nuclear.*

Proof. Treves (2006) Proposition 50.1 ∎

A.6 The categories $\mathcal{NF}$ and $\mathcal{NDF}$

Definition A.23. A locally convex Hausdorff space is *dual Fréchet* if it is the strong dual of a Fréchet space.

We will denote by $\mathcal{F}$, $\mathcal{NF}$, $\mathcal{NDF}$ the categories of Fréchet, nuclear Fréchet, and nuclear dual Fréchet spaces respectively.

Proposition A.30. *$\mathcal{NF}$ and $\mathcal{NDF}$ are symmetric monoidal categories*

Proof. Costello (2011) Appendix 2 ■

Proposition A.31. *Suppose $E \in \mathcal{NF}$ or $E \in \mathcal{NDF}$. Then E is Montel.*

Corollary A.32. *Let $E \in \mathcal{NF}$. Then E is reflexive.*

Proposition A.33. *Let $E \in \mathcal{F}$. Then $E \in \mathcal{NF}$ if and only if $E^* \in \mathcal{NDF}$.*

Proposition A.34. *Taking strong duals gives an equivalence of symmetric monoidal categories*

$$\mathcal{NF}^{op} \simeq \mathcal{NDF} \tag{A.19}$$

Proposition A.35. *Let $E, F \in \mathcal{NF}$. Then we have the following isomorphisms*

$$E\widehat{\otimes}F \simeq \mathrm{Hom}(E^*, F) \tag{A.20}$$

$$E^*\widehat{\otimes}F \simeq \mathrm{Hom}(E, F) \tag{A.21}$$

$$E^*\widehat{\otimes}F^* \simeq (E\widehat{\otimes}F)^* \simeq B(E, F) \tag{A.22}$$

Proposition A.36. *The inverse limit of a countable inverse system of Fréchet spaces with nuclear maps is nuclear Fréchet.*

Proof. Let $(E_i)_{i\in I}$ be the inverse system of nuclear maps between Fréchet spaces, i.e. I is a countable directed poset where if $i \geqslant j$ there exists a nuclear map $f_{ij} : E_i \to E_j$ of Fréchet spaces. The inverse limit $\hat{E} := \varprojlim_{i\in I} E_i$ is the vector space

$$\hat{E} = \{(x_i) \in \prod_{i\in I} E_i \mid x_j = f_{ij}(x_i) \text{ for all } i \geqslant j\} \tag{A.23}$$

equipped with the topology defined by the seminorms given by composing

$$\hat{\mathfrak{p}} : \hat{E} \overset{\iota}{\hookrightarrow} \prod_{i\in I} E_i \xrightarrow{\pi_k} E_k \xrightarrow{\mathfrak{p}} \mathbf{R} \tag{A.24}$$

where ι is the inclusion, π_k is the canonical projection onto the kth factor and $\mathfrak{p}$ is a seminorm defining the topology on E_k. Set $\sigma_k := \pi_k \circ \iota$ and $F_k := \overline{\sigma_k(\hat{E})} \subset E_k$ be the closure of the image of σ_k. Since Fréchet spaces are closed under taking countable products and closed subspaces, $\hat{E}$ and F_k are both Fréchet. Observe that for any $j \geqslant k$, $\sigma_k = f_{j,k} \circ \sigma_j : \hat{E} \to F_k$ is a composition of a nuclear map with a continuous map and is therefore nuclear by Proposition A.23.

Let V be a Banach space and let $u : \hat{E} \to V$ be a continuous map. Then for all $\varepsilon > 0$, there exists $i \in I$ and $\mathfrak{p} : E_i \to \mathbf{R}$ such that $u(U_{\hat{p}}) \subset B_\varepsilon$, where $U_{\hat{\mathfrak{p}}}$ is the unit semiball of $\hat{\mathfrak{p}}$ and B_ε is the ball of radius ε in V. Thus u admits a factorization

$$u = u_i \circ \sigma_i \tag{A.25}$$

with $u_i : F_i \to V$ continuous. Proposition A.23 implies u is nuclear and therefore Proposition A.28 implies $\hat{E}$ is nuclear. ∎

Corollary A.37. *The direct limit of a direct system of dual Fréchet spaces with nuclear maps is nuclear dual Fréchet.*

Proof. Dualize and apply Proposition A.36. ∎

Proposition A.38. *Let $u : \check{E} \to \hat{F}$ be a continuous map between $\check{E} \in \mathcal{NDF}$ and $\hat{F} \in \mathcal{NF}$. Then u is nuclear.*

Proof. Under the isomorphism (A.20) of Proposition A.35, we can view $u \in \check{E}^* \widehat{\otimes} \check{F}$, where $\check{E}^* \in \mathcal{NF}$ by Corollary A.32 and Proposition A.33. By Proposition A.20, u can be expressed as an absolutely convergent series

$$u = \sum_n \lambda_n x'_n \otimes y_n \tag{A.26}$$

with $\lambda_n \in \ell^1$ and $\{x'_n\}, \{y_n\}$ sequences converging to 0 in E', F. In particular, $\{x'_n\}, \{y_n\}$ are bounded. Applying isomorphism (A.20) and Proposition A.26 shows that u is nuclear. ∎

Appendix B: Bundles of Topological Vector Spaces

For thorough treatments of infinite dimensional differential geometry we refer the reader to Schmeding (2022) and Kriegl and Michor (1997).

We assume all topological vector spaces are locally convex and Hausdorff.

Definition B.1. A continuous map $T : E \to F$ between real topological vector spaces is C^1-*differentiable* on an open subset $U \subset E$ if the limit

$$dT|_x(v) := \lim_{t\to 0} \frac{T(x+tv) - T(x)}{t} \tag{B.1}$$

exists for all $x \in U, v \in E$ and defines a continuous map

$$dT : U \times E \to F \tag{B.2}$$

Remark B.1. If T is C^1-differentiable on $U \subset E$, then its differential $dT : U \times E \to F$ is linear in the second factor.

Definition B.2. We inductively define smooth maps as follows. Let $T : E \to F$ be a continuous map between real topological vector spaces, $U \subset E$ be an open subset, and set $d^1T := dT$. We say T is C^{k+1}-*differentiable on* U if it is C^k-differentiable on U and the map

$$d^{k+1}T : U \times E^{k+1} \to F \tag{B.3}$$

given by

$$d^{k+1}(x, v_1, .., v_{k+1}) = \lim_{t\to 0} \frac{d^kT(x + tv_{k+1}, v_1, ..., v_k) - d^kT(x, v_1, ..., v_k)}{t} \tag{B.4}$$

exists and is continuous. We say T is C^∞-*differentiable on* U or *smooth on* U if it is C^k-differentiable on U for all k.

Definition B.3. Let $T : E \to F$ be smooth on an open subset $U \subset E$, where E, F are complex topological vector spaces. We say T is *holomorphic on* U if $dT : U \times E \to F$ is complex linear in the second factor.

Definition B.4. Let S be a finite dimensional manifold of dimension d, E and F real topological vector spaces, and suppose

$$f : S \times E \to F \tag{B.5}$$

is a continuous map. We say f is *smooth* if for every smooth embedding $\psi : \mathbf{R}^d \hookrightarrow S$, the composition

$$\mathbf{R}^d \times E \overset{\psi}{\hookrightarrow} S \times E \xrightarrow{f} F \tag{B.6}$$

is a smooth map of real topological vector spaces.

Definition B.5. Let S be a finite dimensional complex manifold of dimension d. Suppose E, F are complex topological vector spaces and let

$$g : S \times E \to F \tag{B.7}$$

be a continuous map. We say g is *holomorphic* if for every holomorphic embedding $\psi : \mathbf{C}^d \hookrightarrow S$, the composition

$$\mathbf{C}^d \times E \overset{\psi}{\hookrightarrow} S \times E \xrightarrow{g} F \tag{B.8}$$

is a holomorphic map of complex topological vector spaces.

Definition B.6. Let

$$\pi : \mathscr{E} := \bigsqcup_{s \in S} E_s \to S \tag{B.9}$$

be a set of real topological vector spaces parametrized by a finite dimensional manifold S. Suppose there exists an open cover $S = \bigcup_{\alpha \in A} U_\alpha$, topological vector spaces E_α for $\alpha \in A$, and maps

$$\begin{array}{ccc} U_\alpha \times E_\alpha & \xrightarrow{\varphi_\alpha} & \bigsqcup_{s \in U_\alpha} E_s \\ & {\scriptstyle pr_1} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & U_\alpha & \end{array} \tag{B.10}$$

such that $\varphi_\alpha|_{\{s\}\times E}$ is an isomorphism of topological vector spaces. Set $U_{\alpha\beta} := U_\alpha \cap U_\beta$. We say $\pi : \mathscr{E} \to S$ is a *smooth vector bundle* if the compositions

$$g_{\alpha\beta} : U_{\alpha\beta} \times E_\alpha \xrightarrow{\varphi_\beta^{-1}\circ\varphi_\alpha} U_{\alpha\beta} \times E_\beta \xrightarrow{pr_2} E_\beta \tag{B.11}$$

are smooth maps.

Definition B.7. Let $\pi : \mathscr{E} \to S$ be a smooth vector bundle over a finite dimensional complex manifold S, whose fibers are complex vector spaces. We say π is *holomorphic* if the maps (B.11) are holomorphic.

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