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# Multiplicative and Dynamical Analysis on Idèles and Idèle Class Groups

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# Multiplicative and Dynamical Analysis on Idèles and Idèle Class Groups

**by**

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Dedicated to Mathematics, which as brought immeasurable joy to my life.

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# Multiplicative and Dynamical Analysis on Idèles and Idèle Class Groups

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We prove an extension of a result due to Allcock and Vaaler from 2009. In the main theorem we show that an idèle group associated to $\overline{\mathbb{Q}}$ is naturally dense in a banach algebra normed by the Weil height. We establish bounds for the dynamics of generic idèlic points of a field modulo the diagonally-embedded multiplicative groups of the associated fields.

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# Chapter 1

# Introduction

In [1] the authors establish the curio that non-zero algebraic numbers modulo roots of unity are an archimedean $\mathbb{Q}$-vector space normed by the (absolute, logarithmic) Weil height and hence that its completion is a real Banach space. In [19] Vaaler successfully uses this result to establish norm (height) bounds on multiplicative dependencies between algebraic numbers. In the present work we realize an extension of the construction from [1], including an extension to the positive characteristic case to better understand its applications to problems in number theory. The main theorem in the paper is Theorem 7.1, which we reproduce here in a more self-contained format.

We begin by letting $k$ be a global field and $Y = Y_k$ the set of all places of $\overline{k}$, i.e. equivalence classes of absolute values on a separable closure of $\mathbb{Q}$ or $\mathbb{F}_p(t)$. In [1], the authors show that

$$Y \cong \varprojlim_{K} W_v(K)$$

is an inverse limit of *sets* $W_v(K)$ of all places of finite, Galois extensions $K/k$ extending those of $k$ in the case one has a set of absolute values for finite extensions which satisfy the fundamental equality and product formula for non-zero elements of the field

$$\sum_{w|v}[L_w : k_v] = [L : k] \qquad\qquad \prod_{v} |\alpha|_v = 1.$$

In the present work we extend this to an inverse limit of coset spaces using the absolute Galois group of $k$.

The measure on $Y_{\mathbb{Q}}$ is defined on the sets $Y(\mathbb{Q}, v)$ and will be seen to be induced by the normalized Haar measure on the absolute Galois group using the structural description 2.5. The authors of [1] do this by using the Riesz representation theorem rather than the isomorphism from lemma 2.5 directly, but the underlying constructions are equivalent. The norm on $\mathcal{G}_{\mathbb{Q}}$ from [1] is naturally $2h$, twice the absolute, logarithmic Weil height. We note–for the sake of consistency–that the set $Y = \{\text{places of } \overline{\mathbb{Q}}\}$ will turn out to be equal to the set of places of $\overline{k}$ for any $k \subseteq \overline{\mathbb{Q}}$ and similarly for $Y$ associated to any finite extension $E/\mathbb{F}_p(t)$, since both will amount to all places of a separable closure of the bottom fields. We will mimic the notation of the authors of [1] and utilize the notation $\mathcal{G}_k = \overline{k}^{\times}/\mathrm{Tor}(\overline{k}^{\times})$. Although the authors of [1] only treat the case of $\mathcal{G}_{\mathbb{Q}}$, the construction can be generalized to the case $\mathcal{G}_{\mathbb{F}_p(t)}$ and other $\mathcal{G}_k$ when $k$ is a global field

In § 3.1 we construct an idèle group $I$ associated to $\overline{k}$ and show it is well-defined. We further define $\mathcal{N} \subseteq I$ to be the maximal, compact subgroup of $I$, for which a typical element is represented by some $K$-idèle, $(x_v)$, such that $|x_v|_v = 1$ for each place $v$ of $K$.

We exploit several factors in the sequel: an embedding of $\mathcal{G}_k$ into $I/\mathcal{N}$ wherein elements are found as simple functions with compact support, the description of idèle groups as described in appendix C, and the set $I$ as constructed in section 3.1. We will see in the sequel that the constructed embedding of $\mathcal{G}_{\mathbb{Q}}$ into $L^1(Y_{\mathbb{Q}})$ from [1] is also possible for $\mathcal{G}_k$ for other global fields, $k$, and is a straightforward construction to carry out for $I/\mathcal{N}$.

This point-of-view is furthered by the following two conclusions that occur when we use $I/\mathcal{N}$ in place of $\mathcal{G}_k$: (1) the completion of $I/\mathcal{N}$ will be found to have no codimension restrictions present in [1] and (2) the integral of an idèle function is substantially the classical idèle volume–through the formula

$$|\mathbf{x}| = |\mathbf{x_0}| = \prod_v |x_v|_v = \exp\left(\int_Y f_{\mathbf{x}}\, d\lambda(y)\right)$$

holds for a certain function $f_{\mathbf{x}}$ associated to an idèle $\mathbf{x}$ of a global field. This substantially explains the presence of the integral 0 condition found in [1] for the set $\mathcal{G}_{\mathbb{Q}}$ and its completion which the authors denote by $\mathcal{X}$.

We will, in the sequel, have the basic structures from [1], and a primary interest in the case of number fields, but provide proofs in the general context. That is to say: $Y$ will be the set of places of any $k$ such that $\overline{k} = \overline{\mathbb{Q}}$ or $\overline{k} = \overline{\mathbb{F}_p(t)}$ and the space $I/\mathcal{N}$ will be defined analogously for both cases.

**Theorem 1:** With the notation as above, we have the following structural descriptions:

(i) There is an inclusion of groups $\mathcal{G}_k \subseteq I/\mathcal{N}$, where $\overline{k}^\times$ is the group of non-zero algebraic elements of $\overline{k}$, and the set of torsion points, $\mathrm{Tor}(\overline{k}^\times)$, is the set

$$\mathrm{Tor}(\overline{k}^\times) = \{\zeta \in \overline{k} : \exists n \in \mathbb{N},\, \zeta^n = 1\}$$

of roots of unity.

(ii) $I/\mathcal{N}$ is a $\mathbb{Q}$-vector space with the operation "+" being ordinary idèlic multiplication and the scalar action of $\mathbb{Q}$ being done by exponentiation. Furthermore, $\mathcal{G}_k$ is a $\mathbb{Q}$-sub vector space of $I/\mathcal{N}$.

(iii) $I/\mathcal{N}$ carries an archimedean norm given by twice the (absolute, logarithmic) Weil height of an idèle,

$$2h(\mathbf{x}) = \sum_v |\log |x_v|_v|$$

where $x_v$ is the $v$ component of an element of $I/\mathcal{N}$, represented by some $\widetilde{\mathbf{x}} \in I_K$, for some finite, Galois extension $K/k$. The absolute value $|\cdot|_v$ is a normalized choice of $v$-adic absolute value (normalized so that the product formula holds for non-zero algebraic numbers), and the unadorned $|\cdot|$ is the classical absolute value of real numbers, and the sum is over all places $v$ of the field $K$. This norm restricts to the subset $\overline{k}^\times/\mathrm{Tor}(\overline{k}^\times)$ so that it is a *normed* $\mathbb{Q}$-subvector space.

(iv) The completion of $I/\mathcal{N}$ with respect to this Weil height norm is isometrically isomorphic to $L^1(Y, \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the Borel $\sigma$-algebra of the set $Y$ of places of $\overline{k}$, and $\lambda$ is a Galois-invariant measure on $Y$, that is for all $\tau \in G_k = \mathrm{Gal}\,(\overline{k}^\times/k)$

$$\int_Y f(y)\, d\lambda(y) = \int_Y f(y)\, d\lambda(\tau y)$$

(v) The completion of $\mathcal{G}_k$ is the codimension-1 subspace of $L^1(Y)$ with mean value (i.e. integral) zero. That is to say:

$$\mathcal{G}_k \subseteq \{f \in L^1(Y) : \int f\, d\lambda = 0\}$$

and for all $g \in L^1(Y)$ such that $\int_Y g\, d\lambda = 0$ and for all $\epsilon > 0$, there is $f \in \mathcal{G}_k$ such that $\|f - g\|_{L^1(Y)} < \epsilon$.

(vi) $L^1(Y)$ is naturally a Banach algebra. Furthermore the convolution/multiplication is inherited from the convolution on the space $L^1(G_k)$ with $G_k$ being given the normalized Haar measure and Borel $\sigma$-algebra of open sets as a topological group.

We use this theorem to reformulate the classical problem of algebraic independence (or lack thereof) of the logarithms of algebraic numbers. This question in transcendental number theory has its origins in the Lindemann-Weierstrass theorem, which states that $n$ complex numbers $z_1, \ldots, z_n$ which are $\mathbb{Q}$-linearly independent give rise to $n$ algebraically independent numbers, $e^{z_1}, \ldots, e^{z_n}$ and a conjecture of Schanuel (described in [9, pp. 30-31]) itself an important, unproven result asserting that, among the $2n$ complex numbers, $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$, there are at least $n$ algebraically independent such numbers–which would settle many old questions concerning transcendence of $e + \pi$, $e^{\pi^2}$, and $e^e$ as well as algebraic independence of $e$ and $\pi$. Baker's 1968 theorem in [2] is one of the strongest partial results on Schanuel's conjecture to date, proving that linear independence over $\mathbb{Q}$ of logarithms of algebraic numbers, $\alpha_1, \ldots, \alpha_n$ implies linear independence of the same logarithms over $\overline{\mathbb{Q}}$, a result still, prima facie, far-removed from algebraic independence.

The algebra structure described in chapter 7 gives a setting in which multiplication of logarithms occurs in a more natural (i.e. not ad hoc) fashion. We mention this idea again in chapter 10. We will find the object from Theorem 1 enables the use of abstract harmonic and functional analysis through (vi) to potentially approach these old problems in a way which also emphasizes both the algebra structure and the *completeness* of $L^1(Y)$, and we note the result cannot be formulated in the context of $\mathcal{G}_k$ given in [1] because the convolution operation in $L^1(Y)$ does not preserve the mean-0 condition for $\mathcal{G}_k$.

Along the way we establish a stucture theorem for $L^1(Y)$ based on a decomposition of $Y$ as a disjoint union of homogeneous $G_k$ spaces which is the content of §2 and §7 which is summarized as

**Theorem 2:** The set, $Y$, of places of the separable closure, $\overline{k}$, is canonically, topologically a disjoint union of homogeneous spaces of the form $G_k/D_v$ with $v$ a place of $k$ and $D_v$ a choice of decomposition group for $v$, i.e.

$$Y = \coprod_v G_k/D_v$$

with the union being taken over all places $v$ of $k$. Furthermore, the Banach algebra structure alluded to in Theorem 1 on $L^1(Y)$ has as a dense sub-algebra a direct image of

$$\mathcal{A} = \bigoplus_v L^1(G_k)$$

with each component endowed with its canonical Banach algebra structure as $L^1$ of a locally compact group. We can further describe this image quite explicitly. With

$$\widetilde{f}\,(xD_v) = \int_{D_v} f(xh)\,dm_{D_v}(h)$$

the projection which endows $L^1(Y)$ with a Banach algebra structure is given by projecting each direct sum of component of $\mathcal{A}$ along the map

$$\begin{cases} T_v : L^1(G_k) \to L^1(G/D_v) \\ f \mapsto \widetilde{f} \end{cases}$$

so that

$$L^1(Y) = \overline{\bigoplus_v T_v\,(L^1(G_k))}.$$

We also consider sub $\sigma$-algebras of the Borel $\sigma$-algebra, $\mathcal{B}$, with which we initially equip $Y$, and establish the following"measure-theoretic fundamental theorem of Galois theory" (FTGT) listed in the present work as theorem 8.15

**Theorem 3:** If $K/k$ is an extension of global fields and $\sigma(K)$ is the $\sigma$-algebra generated by functions $f_{\mathbf{x}} \in I_K/\mathcal{N} \cap I_K \subseteq I/\mathcal{N}$, then there is a canonical anti-equivalence–i.e. inclusion reversing bijection–between closed subgroups of $G_k$ and such $\sigma$-algebras associated to subextensions $k \subseteq K \subseteq \overline{k}$.

With the framework of §8.1 we are also able to prove a relationship between an idèle and it's relative norm for relatively finite extensions in the text as proposition 8.16 which we formulate here as

**Theorem 4:** Let $k \subseteq K \subseteq L$ be a tower of Galois extensions of the global field $k$ with $[L : k] < \infty$. Let $\mathbf{x} = (x_w) \in I_L$ and denote the relative norm of idèle groups by

$$\mathfrak{N}_K^L = \begin{cases} I_L \to I_K \\ \mathbf{x} \mapsto \prod_v \left( \prod_{w|v} N_{K_v}^{L_w}(x_w) \right) \end{cases}$$

i.e.

$$\mathfrak{N}_K^L(\mathbf{x}) = \left( \prod_{w|v} N_{K_v}^{L_w}(x_w) \right)_v .$$

Then the "nearest" element–i.e. the Radon-Nikodym derivative in $L^1(Y, \sigma(K), \lambda)$–of $f_{\mathbf{x}} \in L^1(Y, \sigma(L), \lambda)$ is given by a sum of partial averages

$$df_{\mathbf{x}}/d\mu_K = \sum_v \sum_{w|v} \frac{1}{r_v} \log |\mathfrak{N}_K^L(x_v)| \mathbf{1}_{Y(K,v_w)}$$

where $r_v = \dfrac{[L:K]}{[L_w:K_v]}$. In particular, if $\mathbf{x}$ is a unit at all primes of $K$ which split in $L$, $df_{\mathbf{x}}/d\mu_K = f_{\mathfrak{N}_K^L(\mathbf{x})}$.

For the reader unfamiliar with the basic theory of heights, idèles, or inverse and direct limits, we refer to the appendices. We also provide references in the main text when results depend on development of theory presented in the appendices. In the sequel we provide proofs of the basic results we will need and recast some of the objects from [1] in a new way in order to better understand the structure of this Banach space. We develop the cases of zero and positive characteristic separately for clarity, though much of the basic theory can be placed on equal footing at its inception. In chapter 7 we will discover that the Banach space is naturally a Banach algebra, graded by information on places of the base field. We end the discussion of the main results in chapter 9 to talk about distributions of orbits in idèle class groups.

Throughout this dissertation, unless explicitly stated or clear from context, all base fields are assumed to be global fields.

# Chapter 2

# The space $Y$

For this chapter we begin to make use of the absolute Galois group, $G_k$, of a global field, $k$. This will be the first instance where we treat characteristics 0 and $p > 0$ separately, but we note this is for clarity more than distinction–the theories parallel one another. For economy's sake, we will devote the positive characteristic section to noting how one adapts the case of number fields, because a full redevelopment would ultimately be an exercise in copying, pasting, and added a few sentences, which the author believes will confuse rather than elucidate.

## 2.1 Characteristic 0

Throughout this section the global field, $k$, is assumed to have characteristic 0, so that all its algebraic extensions are separable.

**Definition 2.1:** Given a place $v$ of $k$, we define, for a finite, extension $L/k$, the set $W_{v,k}(L)$ to be the (finite) set of all places of $L$ lying over the place $v$. When clear from context, we will suppress the base field, $k$, in the notation.

**Proposition 2.2:** The sets $W_{v,k}(L)$ are canonically homeomorphic to $G_k/\text{stab}_{G_k}(w)$ for any $w|v$ a place of $L$, where here stab is the stabilizer subgroup of $G_k$ which fixes $v$.

*Proof.* It is easily seen that an extension of number fields, $L/k$, satisfies the fundamental equality and product formula for non-zero elements

$$\sum_{w|v}[L_w : k_v] = [L : k] \qquad\qquad \prod_v |\alpha|_v = 1$$

with the sum being taken over extensions, $w$ of $L$, of the place $v$ of $k$ and the product over all places $v$ of $k$. We note that the action of the absolute Galois group of $k$, $G_k$, is continuous and transitive on places above a fixed place of the base field by classical theory (see for example [13, Proposition 9.1 Chapter 2 §9]). That is to say: for finite, Galois extensions $K/k$ we have that $G_k$ acts continuously and transitively on $W_v(K)$. Hence $W_{v,k}(L)$ is what is called a *homogeneous space* for $G_k$, i.e. topologically the quotient of $G_k$ by a closed subgroup. If we let $w \in W_v(K)$ and denote by $D_{w,K}$ the stabilizer, $\mathrm{stab}_{G_k}(w)$ then the map

$$\begin{cases} G_k \to G_k/D_{w,K}G_K \\ \sigma \mapsto \sigma D_{w,K}G_K \end{cases}$$

onto the coset space induces the map

$$\begin{cases} G_k/D_{v,K}G_K \to W_v(K) \\ \sigma D_{v,K}G_K \mapsto \sigma(w) \end{cases}$$

between the quotient space and the set of places $W_v(K)$ which is manifestly a bijection by the orbit-stabilizer theorem and the $G_k$ action on $W_v(K)$'s transitivity. As each of the given sets is a discrete set, the map between the two is automatically continuous, hence a homeomorphism.

□

**Remark 2.3:** It is well-known that a decomposition group, $D_{v,L/k} = \mathrm{stab}_{G_k/G_L}(w)$, of any place $w|v$ of $L$ over the place $v$ of $k$, is unique up to conjugation (eg. [13, Proposition 9.1, Chapter II-9 ]) and so the quotient space $(G_k/G_L)/D_{v,L}$ is independent of the choice of $w|v$ as a topological space, hence our notation $D_{v,L}$ instead of $D_{w,L}$, and we may dispense with the notation $W_{v,k}(L)$ in favor of the noticeably superior $(G_k/G_L)/D_{v,L}$. If we further denote a choice of stabilizer for a given $y \in Y(k,v)$ by $D_v$, then it is clear that $D_{v,L} = D_v/(G_L \cap D_v)$, using the second isomorphism theorem to identify $D_v/(G_L \cap D_v)$ with $D_v G_L/G_L$, and we can then simplify notation further to $G_k/D_v G_L$.

**Lemma 2.4** (Y structure I)**:** The set $Y$, of places of $\overline{k}$, can be written canonically as a disjoint union–indexed by places, $v$, of $k$–of profinite sets

$$Y_v = Y(k,v) = \varprojlim_{\deg_k(L)<\infty} G_k/D_v G_L.$$

*Proof.* We begin by showing $Y_v$ has the claimed form. We note that restriction of a place of $\overline{k}$ to $k$ implies that each place $y$ of $\overline{k}$ lies above some place $v$ of $k$, so that each place in $Y$ is indeed in some $Y_v$ treated as an inverse system over the $W_v(K)$ sets. If $y \neq y'$ are elements of $Y$, then there is a finite extension $L/k$ on which they differ, so that by the definition of the inverse limit, $y \in Y_v \iff y\Big|_L \in W_v(L)$, since the latter is a typical element of the inverse system associated to all *finite* sets of places over $v$ in all finite, Galois extensions of $k$.

Now, for each place, $v$, of $k$, the homeomorphisms from 2.2 form a system of homeomorphisms, $f_{v,K} : G_k/D_{w,K} \to W_v(K)$. If we can show these homeomorphisms are compatible with the transition maps, this will imply that the two inverse systems are homeomorphic and hence their limits are homeomorphic by the definition of an inverse limit. To see this, we first note that the transition maps are given, for $k \subseteq K \subseteq L$ with both $L, K$ finite and

Galois over $k$ by choosing $w'|w|v$ for places $w'$ of $L$ and $w$ of $L$ and are explicitly realized as the maps

$$g_{K,L} = \begin{cases} G_k/D_{w,K}G_K \mapsto G_k/D_{w',L}G_L \\ \sigma D_{w',L} \mapsto \sigma D_{w,K} \end{cases}$$

and

$$h_{K,L} = \begin{cases} W_v(L) \to W_v(K) \\ w' \mapsto w \end{cases}$$

The first is well defined as $D_{w',L} \subseteq D_{w,K}$ by Galois theory, the second is just the restriction map.

By definition of the $f_{v,L}$ we have that $f_{v,K} \circ g_{K,L} = h_{K,L} \circ f_{v,L}$, verifying the compatibility with the transition functions and establishing the topological isomorphism, hence $Y_v$ has the indicated form. It remains to show disjointness. If $y \in Y_v \cap Y_w$ then $y$ induces both $v$-adic and $w$-adic absolute values on $k$, but since places are defined to be equivalence classes of absolute values, the only way this can be true is if $v = w$, completing the proof. □

In fact an even nicer result is true

**Lemma 2.5** (Y structure II)**:** If $k$ is a number field with absolute Galois group, $G_k = \text{Gal}\left(\overline{k}/k\right)$, then denoting a choice of decomposition group of a place $w \in Y_v$ by $D_v$ we have

$$Y_v \cong G_k/D_v.$$

Furthermore the different choices of $D_v$ differ only by conjugation by an element of $G_k$.

*Proof.* The action of $G_k$ on each $G_k/D_vG_L$ continuous and transitive. By definition of an inverse limit, $G_k$ is continuous and transitive on $Y_v$. Hence

$$Y_v \cong \varprojlim_{\deg_k(L)<\infty} G_k/D_vG_L = G_k\Big/\left(D_v \varprojlim_{\deg_k(L)<\infty} G_L\right).$$

Now since all the $G_L \subseteq G_k$, we know by definition of an inverse limit that

$$\varprojlim_{\deg_k(L)<\infty} G_L = \bigcap_{\deg_k(L)<\infty} G_L$$

with the equality $\bigcap_{\deg_k(L)<\infty} G_L = \{e\}$ due to the fact the group $G_k$ is Hausdorff and the collection of subgroups $\{G_L\}$ forms a fundamental system of neighborhoods for the identity. We conclude $Y_v \cong G_k/D_v\{e\} = G_k/D_v$. (see for example [17, Chapter 1]) □

Now, we wish to describe explicitly the structure of the stabilizer of a place $w \in Y_v$.

**Lemma 2.6** (Structure of $D_v$)**:** The stabilizer of a place $w$ of $\overline{k}/k$ is canonically isomorphic to the absolute Galois group, $G_{k_v} = \mathrm{Gal}\left(\overline{k}_v/k_v\right)$, with $\overline{k}_v$ the algebraic closure of $k_v$, i.e. $k_v \otimes \overline{k}$.

*Proof.* Choose a place $v$ of $k$. Then choosing a $w \in Y_v$ is the same as making a choice of embedding, $\iota_w : \overline{k} \to \overline{k}_v$, which gives us a function $\varphi : G_{\overline{k}_v} \to G_k$ defined by taking $s \in G_{k_v}$ and defining its image $\varphi(s)$ to be defined as $\iota_w^{-1} \circ s \circ \iota_w$. That this map is well-defined follows from [20, Corollary 2 to Proposition 3, Chapter III-2 ] and [20, Corollary 4 to Theorem 4, Chapter III-4 ], and continuity is implicit in the construction, but we are able to be explicit as well. Choose $F/k$ finite, and examining $G_F = \mathrm{Gal}\left(\overline{k}/F\right) \leq G_k$, a basic open set in the Krull topology and set $F' = \iota_w(F)k_v \subseteq \overline{k}_v$. It is clear that $\iota_w(F)$ generates $F'/k_v$ and if $\varphi(s) \in G_F$ then–by continuity and density–$s$ fixes $k_v$ and cleary $\iota_w(s)$ fixes $\iota_w(F)$, so it fixes $F'$. On the other hand, anything in $G_{F'}$ is sent to something in $G_F$ by $\varphi$, so any $s \in G_{F'}$ is also contained in $\varphi^{-1}(G_F)$, proving continuity. The image of this function is $D_v$, the decomposition group of $w$, which is another name for the stabilizer of $w$ relative to the group action of $G_k$. The kernel is, of course, Gal $\left(\overline{k}_v/\overline{k}_v\iota(\overline{k})\right)$, which will be trivial by Krasner's lemma, so this is an injection, and $D_v \cong G_{\overline{k}_v}$ as topological groups. (The homeomorphism part comes from the fact that this is a continuous bijection with a compact domain and Hausdorff codomain, which is an exercise in undergraduate topology). $\square$

From here we want to take one more step in developing the essential structure of $Y$ by giving it a $\sigma$-algebra and measure to provide a straightforward generalization of [1, Theorem 4 ].

**Lemma 2.7:** Let $G$ be a compact group with normalized Haar measure $dm$, and let $\{X_n\}_{n\in\mathbb{N}}$ be homogeneous $G$-spaces, i.e. topologically isomorphic to $G/H_n$ for some closed subgroups, $H_n \leq G$. Then there exists a $G$-invariant measure on $X = \coprod X_n$.

*Proof.* We first define for $f \in C(X_n)$ the functional $I_n$ as follows: first, select any point $xH_n \in X_n(\cong G/H_n)$ and define

$$I_n = \int_G f(g(xH_n))\, dm(g).$$

This is well-defined because $dm$ is a Haar measure on $G$ and the action of $G$ is transitive on $X_n$. Next we define for $F \in C_c(X)$ the functional

$$I(F) = \sum_n I_n(F \cdot 1_{X_n}).$$

The sum is finite because $F$ has compact support, the latter condition being equivalent to having support contained in only finitely many of the $X_n$. By the Riesz-representation theorem see [16, Theorem 3.14 ], there is a $\sigma$-algebra, $\Sigma$, containing the Borel $\sigma$-algebra of $X$, along with a unique radon measure, $\lambda$ on $X$ induced by the linear functional $I$. Furthermore, the definition of $I$ demonstrates that $\lambda$ is invariant under $G$, that is to say, for $F \in C_c(X)$ and $\tau \in G$, we have that

$$I(\tau^*(F)) = \int_X F(\tau x)\, d\lambda(x) = \sum_n \int_G F(\sigma\tau(x)) \cdot 1_{X_n}(\sigma\tau x)\, dm(\sigma)$$

$$= \sum_n \int_G F(\sigma x) \cdot 1_{X_n}(\sigma x)\, dm(\sigma) = \int_X F(x)\, d\lambda(x) = I(F).$$

□

**Corollary 2.8:** Let $K/k$ be a finite extension of a number field . Then with the measure $\lambda$ on $Y(k,v)$ induced by the construction in lemma 2.7, we have that for each $w|v$ (i.e. $w$ is a place of $K$ lying over the place $v$ of $k$)

$$\lambda(Y(K,w)) = \frac{[K_w : k_v]}{[K : k]}.$$

*Proof.* Let $K/k$ be a finite, *Galois* extension of number fields. By the invariance of the measure under the Galois action and the transitivity of that action, we see that each $Y(K,w)$ has the *same* measure, since we need only select a $\sigma \in G_k$ such that $\sigma(w) = w'$ for any two $w, w'|v$ and then $\sigma(Y(K,w)) = Y(K,w')$, whence $\lambda(Y(K,w)) = \lambda(Y(K,w'))$. From classical number theory we have that any prime, $v$, of $k$ and any Galois extension $K/k$ thereof gives rise to the identity $[K : k] = efr$ where $e = e(w|v)$ is the ramification index, $f = f(w|v)$ is the inertial degree, and $r = r_K(v)$ is the number of primes above $v$ in $K$. Since $[K_w : k_v] = ef$, we see that there are $r = \frac{[K:k]}{[K_w:k_v]}$ such $Y(K,w)$ , so that each piece, being of equal measure, must have measure $\frac{[K_w:k_v]}{[K:k]}$. The case of more general extensions simply requires us to group together primes in a Galois extension above an intermediate extension and use the multiplicativity of inertial degrees and ramification indices. □

**Remark 2.9:** For the interested reader, the removal of the hypothesis that $K/k$ be Galois is also treated in [19].

## 2.2 Characteristic $p$

Because positive characteristic field extensions are not always separable, not all finite extensions have a Galois closure. This difference complicates our discussion from the characteristic-0 case where we used Galois theory extensively in the construction of the measure and the structure of the space $Y$. However, we will find this difference is ultimately superficial; In fact, the former problem is a consequence of the latter: Consider the case $\mathbb{F}_p(t)[x]/(x^p - t)$: the degree of the completions not to add up to the total degree of the extension because there is only one $p^{th}$ root of 1 in $\mathbb{F}_p$. In this section we include details of how we adapt the proof in the universally separable case (i.e. the characteristic-0 case) for the sections of those proofs where we invoke the separability hypothesis to prove the results.

We note, first, that the transitive Galois action is still a fact, so that the homeomorphism type of $Y_v$–defined as in the previous section–is $G/D_v$. This is because the purely inseparable extensions $E/F$ have the place extension from $F$ to $E$ is unique, so that for general finite, inseparable extensions, we may write $E \supseteq E_{\text{sep}} \supseteq F$ and use the Galois action on the separable extension to classify the places of $E$.

Next, we note lemma 2.6 refers to the exact structure of $D_v$ in terms of the Galois group of the algebraic closure of $k_v$. It should be amended to use $\text{Gal}\left((\overline{k}_v)_{\text{sep}}/k_v\right)$ instead of $\text{Gal}\left(\overline{k}_v/k\right)$–which is nonsense since Galois groups are only defined for separable extensions. Because the fundamental result which relates the action of $G$ to the topology on $Y_v$ is the stabilizer result, because the places of $\overline{k}_v$ are exactly those of the separable closure, the rest of the proof goes off unchanged after the initial statement is corrected.

Corollary 2.8 seems to depend on the fundamental equality in a way which simply cannot be recovered for inseparable extensions. We note, however, that despite the result and proof only holding for separable extensions verbatim, we may recover the *utility* of corollary 2.8 so that we recover what we were after when we wrote it: the measure of the most basic sets. We claim that we may once again reduce to the case of a purely inseparable extension, $K/k$. In this case $Y(L,v) = Y(K,v)$ since the place $v$ is the same on $L$ as $K$. It is true that the actual measure for inseparable extensions is not simply $[K_w : k_v]/[K : k]$, however we note that determining the measure is as simple as writing $k \subseteq K_{\text{sep}} \subseteq K$ and measuring in $K_{\text{sep}}$. Then we can use the corollary for the maximal separable-over-$k$ subfield to determine the basic measures, and then lift to an arbitrary extension with impunity, resolving the measurement difficulty initially introduced by the inseparability.

# Chapter 3

# The space $I$

In this chapter we establish basic facts about the space $I$ and it's distinguished subgroup $\mathcal{N}$ and state our generalization of the function $h$ defined on $\overline{k}^{\times}$ to be a function attached to the idèle groups of our favorite global field, $k$–denoted by $I_k$–which restricts to the usual height on the subgroup $k^{\times} \subseteq I_k$ of $k$-idèles and prove the essential lemmata about this function.

## 3.1 $I$ and $\mathcal{N}$

The central object of interest is the space $I$ of idèles of $\overline{k}$.

**Definition 3.1:** With $I_k$ as the group of idèles associated to $k$, equipped with it's usual restricted product topology relative to the non-archimedean local units, $\mathcal{O}_v^{\times}$, and $\overline{k}$ an algebraic closure of $k$ we define

$$I = \varinjlim_{[L:k]<\infty} I_L$$

as their direct limit.

**Lemma 3.2:** $I$ is well-defined.

*Proof.* Given a tower of finite, Galois extensions of $k$, $L \supseteq K \supseteq k$, with $[L : k] < \infty$, we can define the usual maps $f_K^L : I_K \to I_L$ which are the canonical inclusion maps of idèle groups when the corresponding fields are included, which are represented by a product of diagonal maps, $f_K^L = \prod_v \Delta_v$ with

$$\begin{cases} \Delta_v : K_v \to L_w \\ K_v \to L_w \cong K_v^{[L_w:K_v]} \end{cases}$$

and these will be the connecting homomorphisms for the well-defined inductive system, which hence has an inductive (i.e. direct) limit. □

**Remark 3.3:** All element of $I$ may be thought of as elements of some $I_K$ and two elements $\mathbf{x}, \mathbf{y} \in I$ are *equivalent* if we have an idèle $(x_v) \in I_K$ its image in $I_L$ under the canonical inclusion $f_K^L$ is just $(x_w)$ where for each $w|v_0$, a fixed, but arbitrary place of $K$, we have $x_v$ repeated $[L_w : K_{v_0}]$ times at the $L_w$ local factor of $I_L$. This makes $I$ into what amounts to the *union* of all idèle groups for finite, Galois extensions of the base field, $k$, with a typical element represented by some $(x_v) \in I_K$ for some $K/k$ finite and Galois.

We next turn our attention to the an important family of subgroups $\mathcal{N}_K = \prod \mathcal{O}_v^\times \subseteq I_K$, with $\mathcal{O}_v^\times = \{x_v \in k_v^\times : |x_v|_v = 1\}$ and where the product is taken over *all* places, $v$, of $K$.

**Definition 3.4:** We define

$$\mathcal{N} = \varinjlim_{\deg_k(K)<\infty} \mathcal{N}_K.$$

**Lemma 3.5:** $\mathcal{N}$ is well-defined.

*Proof.* Clearly the transition functions $f_K^L$ takes $\mathcal{N}_L \to \mathcal{N}_K$ and the consistency for the $I_E$ descends to consistency on the $\mathcal{N}_E$. □

**Remark 3.6:** The definition of $\mathcal{N}_K$ is analogous to $\mathrm{Tor}(k^\times)$ in [1], as the latter is exactly the maximal, compact subgroup of $k\times$ just as $\mathcal{N}$ will be shown to be for $I$.

**Remark 3.7:** In theory we should denote the group $\mathcal{N}$ by $\mathcal{N}_{\overline{k}}$, but as with $I$ we omit the subscript when there is no reason to expect confusion.

Note the canonical embedding of $k \hookrightarrow I_k$ which extends by definition to the direct limit (see B.3 for details) to an embedding $\overline{k}^\times$ into $I$. This induces an embedding of $\overline{k}^\times/\mathrm{Tor}(\overline{k}) = \overline{k}^\times/\mathcal{N} \cap \overline{k}^\times$ into $I/\mathcal{N}$.

**Lemma 3.8:** $\mathcal{N}$ is the maximal, compact subgroup of $I$ in the sense that if $H$ is a compact subgroup of $I$ then $H \leq \mathcal{N}$.

*Proof.* Let $K$ be a global field, and let $Q_K$ be the maximal, compact subgroup of $I_K$. By definition of $\mathcal{N}$ as a direct limit, we have

$$\mathcal{N} \cap I_K = \prod_v \mathcal{O}_v^\times$$

for the idèle groups of which they are subsets, when $v$ ranges over all the places, $v$, of $K$. To show $Q_K \subseteq \mathcal{N}_K$ it is sufficient to show that all local factors of a compact subgroup are contained in some $\mathcal{O}_v^\times$, as clearly $\mathcal{N}_K = \prod_v \mathcal{O}_v^\times \subseteq Q_K$. We use the homomorphism

$$\prod_v |\cdot|_v : I_K \to \mathbb{R}_+^\times$$

and note the image of $Q_K$ is a compact subgroup of the codomain, hence is the trivial subgroup, so $Q_K \subseteq \prod_v \mathcal{O}_v^\times$. Since $I$ has the direct limit topology, a compact subgroup $Q \subseteq I$ is compact iff $Q \cap I_K$ is compact for each $I_K$, hence $Q \subseteq \mathcal{N}$, proving the assertion. □

We follow this with

**Proposition 3.9:** The set $I/\mathcal{N}$ is naturally a $\mathbb{Q}$ vector space with the scalar action given by $\frac{r}{s} \cdot \mathbf{x} = \mathbf{x}^{r/s} = (x_v^{r/s})_v$.

*Proof.* Let $r/s \in \mathbb{Q}$ be a reduced fraction and let $\mathbf{x} \in I$ and let $\mathbf{x}$ be represented by $\mathbf{x}' \in I_K$ for some finite, Galois $K/k$. Let $\{x_{v_i}\}$ be the (finite) set of coordinates of $\mathbf{x}$ for which $|x_v|_v \neq 1$. Then select $K = k(\zeta_s)$ to be the field $k$ adjoined a primitive $s^{th}$ root of 1. Then the choices of $x_{v_i}^{r/s}$ have at most $s$ representatives, all of which differ by an element of $\mathcal{N}$, and similarly any representative of $x_w$ for $w \notin \{v_i\}$ are already in $\mathcal{N}$ and $r/s^{th}$ powers of these are also in $\mathcal{N}$ by definition. Hence $I/\mathcal{N}$ is a $\mathbb{Q}$-vector space. □

## 3.2 The Generalized Height

**Definition 3.10:** Let $\mathbf{x} = (x_v)$ be a $k$-idèle with local factor $x_v$ at the place $v$ of $k$. Then the height of $\mathbf{x}$, is defined by

$$h(\mathbf{x}) = \frac{1}{2} \sum_v |\log |x_v|_v|.$$

By definition of an idèle all but finitely many $x_v$ have $|x_v|_v = 1$, so this sum has only finitely many non-zero terms and has a well-defined value for each $\mathbf{x} \in I_k$. By D.2 this agrees with the definition of "height" for the diagonally embedded subgroup, $k \subseteq I_k$, and by the normalization choices for the product formula, this definition is independent of choice of field. That is to say for $k \subseteq K \subseteq L$ with $[L : k] < \infty$, that under the canonical embedding

$I_K \overset{f_K^L}{\hookrightarrow} I_L$ described in appendix C, we have that

$$h(\mathbf{x}) = h(f_K^L(\mathbf{x})).$$

**Remark 3.11:** We note that the ideal map, which sends a non-zero $a \in \mathcal{O}_k$ to an idèle with entries equal to 1 for infinite places and a local uniformizing parameter to the power $v(a)$ which is the (additive) valuation of $a$ at the place $v$, for finite $v$ gives a definition of "height" for ideals which is independent of choice of uniformizing parameter and satisfies the basic inequality

$$h((\alpha)) \leq h(\alpha)$$

where $(\alpha)$ is the (fractional) ideal generated by $\alpha$. This is materially just the fact that

$$h((\alpha)) = \frac{1}{[k:\mathbb{Q}]} \log \left| N_{\mathbb{Q}}^k(\alpha) \right| = \sum_{v \nmid \infty} \log |\alpha|_w.$$

and here $|\cdot|_w$ is normalized to satisfy the product formula.

From here we proceed yet one step further to finalize our height function to work on "any" idèle in a consistent context. We begin by considering the injective system of topological groups

$$\{I_K : \deg_k(K) < \infty\}$$

with transition functions $\{f_K^L : I_K \to I_L\}$ the inclusions of $I_K$ into $I_L$, as described in § C, where $K \subseteq L$ is an inclusion of global fields. By standard field theory this is a well-defined injective system, i.e. if $k \subseteq K \subseteq L \subseteq F$ with $[F : k] < \infty$ then $f_L^F \circ f_K^L = f_K^F$, and so has a well-defined injective limit,

$$I_{\overline{k}} = \varinjlim_{\deg_k(K) < \infty} I_K.$$

When there is no confusion about the base field, or when that information is not relevant to the proof, we simply refer to this topological group as $I$.

**Lemma 3.12:** The function $h$ is well-defined on $I$.

*Proof.* By definition of an injective limit, each $\mathbf{x} \in I$ has a representative in some $I_K$ so that $h(\mathbf{x})$ is well-defined there, and because $h$ is compatible with the transition functions–recall our use of the $|\cdot|$ absolute values in forming the height–$f_K^L$, $h$ extends to a well-defined map from $I$ to $[0, \infty)$. For general extensions, we recall from § 2.2 that the places of $K/k$ are totally determined by $K_{\text{sep}}$. Then for $k \subseteq K \subseteq L$ and $[L : k] < \infty$, and $L = L_{\text{sep}}$ we have

$$\frac{1}{2}\sum_{v}\sum_{v' \in Y(K,v)} |\log |x_{v'}|_{v'}|$$

by definition of the $\|\cdot\|$ and $|\cdot|$ absolute values we have this is then

$$= \frac{1}{2}\sum_{v}\sum_{v' \in Y(K,v)} \left|\log \|x_{v'}\|_{v'}^{[K_{v'}:k_v]/[K:k]}\right|$$

since the fundamental identity

$$\sum_{w|w_0} [F_w : E_{w_0}] = [F : E]$$

holds for all finite, separable extensions, $F/E$. We have that this is then

$$= \frac{1}{2}\sum_{v}\sum_{v'' \in Y(L,v)} \left|\log \|x_{v''}\|_{v''}^{[L_{v''}:k_v]/[L:k]}\right|$$

and finally we use again the definitions of the two absolute values, $\|\cdot\|$ and $|\cdot|$ to get that this is

$$= \frac{1}{2}\sum_{v}\sum_{v'' \in Y(L,v)} |\log |x_{v''}|_{v''}|$$

hence the choice of field is immaterial to the definition of the height. $\square$

**Lemma 3.13:** The function $h$ is a pseudo-norm on the space $I$ as a $\mathbb{Z}$-module.

*Proof.* A pseudo-norm, $p$, is defined on a $\mathbb{Z}$-module $A$ by the three properties

(i) $p(a) \geq 0$ for all $a \in A$

(ii) $p(n \cdot a) = |n|p(a)$ for all $n \in \mathbb{Z}, a \in A$

(iii) $p(a+b) \leq p(a) + p(b)$ for all $a, b \in A$

Clearly properties (i) and (ii) hold from the definition of $h$ and the functional equation for the logarithm function. For property (iii) we let $\mathbf{x}, \mathbf{y} \in I$. Let $K/k$ be a finite extension such that $\mathbf{x}, \mathbf{y}$ have representatives in $I_K$. Then by definition of the vector addition in $I$ as idèlic multiplication we have

$$h(\mathbf{xy}) = \sum_v |\log |x_v y_v|_v|$$
$$= \sum_v |\log |x_v|_v + \log |y_v|_v|$$

and the result follows from the ordinary triangle inequality on real numbers. □

Recall the subgroup $\mathcal{N}$ from section 3.1. We describe it in terms of the norm properties of $h$ in

**Lemma 3.14:** Let $I$ be equipped with the pseudo-norm $h$, and let

$$A = \{\mathbf{x} \in I : h(\mathbf{x}) = 0\}.$$

Then $A = \mathcal{N}$.

*Proof.* We observe if $\mathbf{x} \notin \mathcal{N} \cap I_K$ for some $K$ then $\{\mathbf{x}^n : n \in \mathbb{Z}\}$ is unbounded in the height pseudo-norm, and so cannot be a set height pseudo-norm 0 elements, and so $A \subseteq \mathcal{N}$. By the definition of $\mathcal{N}$ and the height on $I$, we have that $\mathcal{N} \subseteq A$, completing the proof. □

**Corollary 3.15:** The set $(I/\mathcal{N}, h)$ is a normed $\mathbb{Q}$ vector space.

**Definition 3.16:** We define the sets $A_{C,k} = \{\mathbf{x} \in I_k/\mathcal{N}_k : h(\mathbf{x}) \leq \log C\}$, which are the basic closed balls around the identity of $I_k/\mathcal{N}_k$ since they have the subspace topology from $I/\mathcal{N}$.

**Proposition 3.17:** Considered as a subspace of $I/\mathcal{N}$, an idèle group modulo its maximal compact subgroup is locally compact in the height topology iff it is locally compact in its usual topology induced by the natural projection $\pi_k : I_k \to I_k/\mathcal{N}_k$.

*Proof.* Let $\mathbf{x} = (x_v) \in I_k$. If $h(\mathbf{x}) \leq \log C$ then we use the definition of the direct limit topology as the coarsest one for which the inclusions are continuous to see that $I_k$ in its usual topology is a sub pseudo metric space of $\varinjlim I_k$, and so any compact neighborhood of a point in $I_k/\mathcal{N}_k$ is compact in the image in $I/\mathcal{N}$. □

**Corollary 3.18:** For all global fields, $k$, $I_k$ is locally compact in the height topology.

*Proof.* Let $k$ be a global field. Then $k$ has finitely many archimedean places, $[k : \mathbb{Q}]$ if $k$ has prime field $\mathbb{Q}$ and none at all if $k$ has prime field $\mathbb{F}_p$. Then the idèle group, $I_k$, has the structure of a restricted direct product of locally compact groups, relative to compact subgroups as indicated in appendix C, hence is locally compact. Now let $\mathbf{x} \in I_K/\mathcal{N}_K$. Because $I_K/\mathcal{N}_K$

has the quotient topology, we select a lift $\widetilde{\mathbf{x}} \in I_K$ and a compact neighborhood, $C$ of $\widetilde{\mathbf{x}}$. Then the image of $C$ under the canonical projection, $\pi$ to $I_K/\mathcal{N}_K$ is a compact by continuity, and since $C$ is a neighborhood it contains an open set, $U \subseteq C$, and since $\pi$ is an open mapping, $\pi(U) \subseteq \pi(C)$ is also open, hence $\pi(C)$ is a compact neighborhood of $\mathbf{x} \in I_K/\mathcal{N}_K$. From here the result follows from proposition 3.17. □

## 3.3 The Embedding

As we have classified the pseudo-norm kernel for $h$ and wish to establish an embedding $I/\mathcal{N} \to L^1(Y)$, we will henceforth only use $2h$ for norm computations because–although any equivalent norm will give density-style results–it is with the norm $2h$ that $I/\mathcal{N}$ is *isometrically* embedded into $L^1(Y)$ by definition of the idèle height and the embedding in

**Proposition 3.19:** There is an isometric embedding $(I/\mathcal{N}, 2h) \to (L^1(Y), \|\cdot\|_1)$.

*Proof.* Define the map to be

$$\begin{cases} I/\mathcal{N} \to L^1(Y) \\ \mathbf{x} = (x_v)_v \mapsto \sum_v \log |x_v|_v \cdot \mathbf{1}_{Y_v} \end{cases}.$$

Then as all elements of $I$ are mapped to locally constant functions supported on finitely many $Y_v$, the map is well-defined. To see the isometry property, we simply compute. Let $\mathbf{x} = (x_v)_v \in I$. Then by definition

$$2h(\mathbf{x}) = \sum_v \big| \log |x_v|_v \big|$$

Because the functions $\log |x_v|_v \cdot \mathbf{1}_{Y_v}$ have disjoint support, $|f_{\mathbf{x}}| = \sum_v \big| \log |x_v|_v \big|$ hence

$$\|f_{\mathbf{x}}\|_1 = \int_Y |f_{\mathbf{x}}(y)| \, d\lambda(y) = \sum_v \int_{Y_v} |f_{\mathbf{x}(y)}| \, d\lambda(y)$$

$$= \sum_v \big|\log|x_v|_v\big|\lambda(Y_v) = \sum_v \big|\log|x_v|_v\big|.$$

□

To prove density, we begin with an important

**Lemma 3.20:** Let $k$ be a global field. Then $\mathcal{G}_k = \overline{k}^\times/\mathrm{Tor}(\overline{k}^\times)$ is dense in the codimension-1 space of $L^1(Y_{\overline{k}})$ given by

$$\mathcal{X} = \left\{ f \in L^1(Y_{\overline{k}}) : \int f = 0 \right\}.$$

*Proof.* Let $k$ be a global field. By the results in appendix D we already know that an isometric embedding exists into $L^1(Y_{\overline{k}})$, so we need only show $\mathcal{G}_k$ is dense in

$$\mathcal{X} = \left\{ f \in L^1(Y) : \int f = 0 \right\}.$$

To this end we let $n \geq 2$ by an integer and select any finite collection, $S = \{v_1, \ldots, v_n\}$ of places of the global field, $k$, containing all the infinite places. It is possible to ensure *all* infinite places are in $S$ because because $k$ is a *finite* extension of $\mathbb{F}$. Select $n-1$ multiplicatively independent $S$ units, $\xi_j$. By Dirichlet's unit theorem, the matrix

$$M_S = ([k_v : \mathbb{F}_v] \log|\xi_j|_{v_i})_i$$

has that $\dim(\ker M_S) = 1$ for any $S$ satisfying our hypotheses, i.e. the matrix has maximal rank. Since $S$ is arbitrary, the only linear forms that annihilate all elements of $\mathcal{G}_k = \overline{k}^\times/\mathrm{Tor}(\overline{k}^\times)$ are those which are multiples of the form which restricts to $\mathbf{y} \mapsto \mathbf{y} \cdot (1, 1, \ldots, 1)$ on *every* finite-dimensional subspace, i.e. the integration functional; the density follows immediately. □

**Remark 3.21:** We note that, by restriction to the case $k = \mathbb{Q}$, this induces the same embedding of $\mathcal{G}_k$ into $L^1(Y)$ constructed in [1, §5 ] when we treat $\mathcal{G}_Q \subseteq I_{\overline{\mathbb{Q}}}/\mathcal{N}_{\overline{\mathbb{Q}}}$.

**Corollary 3.22:** $I/\mathcal{N}$ also naturally contains $K_w^\times/U_w$, by restricting our embedding to the $w$ coordinate of $I_K/\mathcal{N}_K$.

**Lemma 3.23:** Given a global field, $k$ and a finite, extension, $K/k$, and an idèle $\mathbf{x} \in I_K$ and with associated function $f_{\mathbf{x}} \in L^1(Y)$, we have

$$\log\left(\prod_v |x_v|_v\right) = \int_Y f_{\mathbf{x}}(y)\, d\lambda(y).$$

*Proof.* By definition, the image of an idèle, $\mathbf{x} = (x_v)_v \in I_K$ in $L^1(Y)$ is the compositum of the canonical maps

$$I_K \hookrightarrow I \twoheadrightarrow I/\mathcal{N} \hookrightarrow L^1(Y)$$

i.e. we first think of injecting $\mathbf{x}$ into the direct limit, $I$, which is the union of all $I_K$ as a set, then we project out along $\mathcal{N}$, and use the embedding map into $L^1(Y)$ which is defined by

$$\mathbf{x} = (x_v) \mapsto \sum_v \log|x_v|_v \cdot \mathbf{1}_{Y(K,v)}$$

But then by the functional equation for the logarithm, this is just

$$\log\left(\prod_v |x_v|_v\right).$$

Since finitely many of the factors are non-zero this is a valid application of the functional equation, and the result follows. □

**Corollary 3.24:** If we define the volume of an element $\mathbf{x} \in I$ in the natural way, and denote the image of $\mathbf{x}$ in $L^1(Y)$ (after projecting along $\mathcal{N}$) by $f_{\mathbf{x}}$ then

$$|\mathbf{x}| = \exp\left(\int_Y f_{\mathbf{x}}\right).$$

*Proof.* We first note that the "natural" definition of the volume of an element of $I$ is to take a representative $(x_v) = \mathbf{x}_0 \in I_K$ for some finite, Galois extension $K/k$ of our base field and compute *its* idèle volume

$$|\mathbf{x}_0| = \prod_v |x_v|$$

where, as usual $v$ ranges over all the places $v$ of $K$. Then we simply *declare* $|\mathbf{x}| = |\mathbf{x}_0|$, and the definition of the $|\cdot|$ absolute values gives us a consistent definition independent of choice of $K$. Again, since $I$ is a direct limit of $I_K$ of this type, this verifies well-definition on $I$. But then the result follows from the lemma by applying the exponential map to both sides. □

**Remark 3.25:** This lemma allows us a *continuous* analog of product for idèles–as well as other elements of $L^1(Y)$. Just as summation of discrete things became integration in a continuous setting, the expansion of $I/\mathcal{N}$ to $L^1(Y)$ has enabled us to have a continuous analog of a product, which agrees with the old one in the cases we care about.

We close this section with

**Theorem 3.26:** The normed $\mathbb{Q}$ vector space $(I/\mathcal{N}, 2h)$ is isometrically isomorphic to a dense subset of the Banach space $L^1(Y)$.

*Proof.* We already know that $I/\mathcal{N}$ embeds into $L^1(Y)$ isometrically by Proposition 3.19, and that $\mathcal{G}_k \subseteq I/\mathcal{N}$ is dense in

$$\mathcal{X} = \left\{ f \in L^1(Y) : \int f = 0 \right\}.$$

by lemma 3.20. Then by corollary 3.24 we see that–so long as there is an idèle of volume not equal to 1–there is an element of $I/\mathcal{N}$ which does not project into the proper subspace. However there are plenty of such idèles, so that density follows immediately by the Hahn-Banach Theorem as described in appendix E. □

# Chapter 4

# Functional Analysis Lemmata

We wish to exploit the work we did earlier in understanding the structure of the space $Y$ so that we will have more structure on $L^1(Y)$ than just its natural Banach space structure. We will see in the sequel that we have the structure of a Banach algebra similar to $L^1(G)$ for a locally compact group, $G$. To begin, we formulate some basic lemmata.

**Lemma 4.1:** Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. If $(A, \mu_A), (B, \mu_B)$ are compact, Hausdorff spaces with Radon measures, $\mu_A, \mu_B$ such that $C_c(A, \mathbb{F})$ and $C_c(B, \mathbb{F})$ are algebras over $\mathbb{F}$ with the $L^1$ norms $\|\cdot\|_A$ and $\|\cdot\|_B$, then $C_c(A \sqcup B, \mathbb{F})$ has a canonical $\mathbb{F}$-algebra structure given by a direct sum of algebras in the form

$$C_c\left(A \sqcup B, \mathbb{F}\right) \cong C_c(A, \mathbb{F}) \oplus C_c(B, \mathbb{F}).$$

*Proof.* First we supply the map and show it is a linear bijection. For simplicity we suppress the value field in the notation. Recall that for a set, $S \subseteq X$, the *indicator function* for $X$ is defined by

$$1_S(x) = \begin{cases} 1 & x \in X \\ 0 & \text{o/w} \end{cases}$$

For $f \in C_c(A \sqcup B)$ set $f_A = f \cdot 1_A$ and $f_B = f \cdot 1_B$. Then we claim the map given by

$$f \mapsto (f_A, f_B)$$

is a linear isomorphism of topolgical vector spaces.

This map is a bijection because it has a well-defined inverse which is computed by sending a pair $(f, g) \in C_c(A) \oplus C_c(B)$ to the function $f + g$ which is equal to $f$ on $A$ and $g$ on $B$, each thought of as (obviously disjoint) subsets of $A \sqcup B$. It is manifestly linear over $\mathbb{F}$ from the definitions. All that remains is to declare the norm on $C_c(A \sqcup B)$ and check continuity. In the spirit of the map we let a function $f \in C_c(A \sqcup B)$ decompose as $f_A + f_B$ and declare the norm, $\|\cdot\|_1$, on $C_c(A \sqcup B)$ to be $\|f\|_1 = \|f_A\|_A + \|f_B\|_B$. Then since the map is linear we need only check continuity at the origin.

Let $\epsilon > 0$ be given and let $\{f_n : A \to \mathbb{F}\}_n, \{g_n : B \to \mathbb{F}\}_n$ be two sequences of $\mathbb{F}$-valued functions converging to 0 in the norms $\|\cdot\|_A$ and $\|\cdot\|_B$, respectively. Then let $N = N\left(\frac{\epsilon}{2}\right)$ be such that all $\|f_n\|_A \leq \frac{\epsilon}{2}$ when $n > N$ and $M = M\left(\frac{\epsilon}{2}\right)$ be such that $\|g_n\|_B < \frac{\epsilon}{2}$ whenever $n > M$. Then for $n > \max\{N, M\}$ we have that $\|f_n + g_n\|_1 \leq \|f_n\|_A + \|g_n\|_B < \epsilon$. So then if the components of a sequence of functions $f : A \sqcup B \to \mathbb{F}$ are going to 0 in the norms on $A$ and $B$, then so too is the function in the norm on $C_c(A \sqcup B)$, proving continuity and establishing the linear isomorphism between the two vector spaces.

We then *declare* the algebra structure on $C_c(A \sqcup B)$ to be the one supplied by the algebraic direct sum. Because the normed vector space structures are already compatible, this construction is as natural as can be hoped for, and is, moreover, compatible with the structure already present on $C_c(A)$ and $C_c(B)$. We need now only show that the multiplication $\star$ on $C_c(A \sqcup B)$ satisfies $\|f \star g\| \leq \|f\| \cdot \|g\|$ in order to complete the proof. By definition of our convolution we can write $f = f \cdot 1_A + f \cdot 1_B = f_A + f_B$ and similarly $g = g_A + g_B$. Then by definition of our algebra structure as the direct sum structure, we have that $f \star g = f_A \star_A g_A + f_B \star_B g_B$ where here the products come from the structures in $C_c(A)$ and $C_c(B)$. But then they still have disjoint support and the norm is an $L^1$ integration norm hence the norm of the sum is the sum of the norms

$$\|f \star g\|_1 = \|f_A \star_A g_A\|_1 + \|f_B \star_B g_B\|_1$$

Then by definition of $\|\cdot\|_1$ this is

$$= \|f_A \star_A g_A\|_A + \|f_B \star_B g_B\|_B.$$

By the Banach inequality on the spaces $C_c(A)$, $C_c(B)$ we have this is

$$\leq \|f_A\|_A \|g_A\|_A + \|f_B\|_B \|g_B\|_B.$$

Finally, putting in the terms we're missing, we get that this is

$$\leq (\|f_A\|_A + \|f_B\|_B)(\|g_A\|_A + \|g_B\|_B) = \|f\| \cdot \|g\|.$$

□

**Lemma 4.2:** With the assumptions and notation of lemma 2.7, there is an $\mathbb{F}$-algebra structure on $C_c(X)$.

*Proof.* By lemma 4.1 and induction, if we know that the $C(X_n)$ have $\mathbb{F}$-algebra structures, then we have a natural algebra structure on the disjoint union of any finite number of them. However we also know that the direct sum of *all* of the $C_c(X_n)$ breaks up as

$$C_c\left(\bigsqcup_n X_n\right) \cong \bigoplus_n C_c(X_n) = \bigcup_m \bigoplus_{n=1}^m C_c(X_n)$$

because the definition of the direct sum demans that only finitely many coordinates be non-zero. So by definition of $C_c(X)$ and predicated on the existence of algebra structures on the $C(X_n) = C_c(X_n)$ we can also induce an algebra structure on $C_c(X)$. However, the existence of an $\mathbb{F}$-algebra structure on the $C_c(X_n)$ is the content of [6, Theorem 3.1]. $\square$

# Chapter 5

# The Group Action

We now notice that there is a representation of the absolute Galois group, $G_k$, (of a separable closure of our base field) induced by this machinery, by the map $f_{\mathbf{x}} \mapsto f_{\sigma(\mathbf{x})}$ on the dense subspace $I/\mathcal{N}$. To see this is actually a continuous action of $G_k$ on $L^1(Y)$ we note that it is sufficient to show it is a continuous action on $I/\mathcal{N}$ and extend by continuity to all of $L^1(Y)$.

**Lemma 5.1:** For any finite, separable extension, $K/k$ of a global field $k$, the action of $\mathrm{Gal}(K/k)$ on $I_K$ given by $(\sigma, \mathbf{x}) \mapsto \sigma(\mathbf{x})$ is continuous.

*Proof.* We consider basic open sets of $I_k$, which are sets of the form

$$V = \prod_{v \in S} U_v \times \prod_{v \notin S} \mathcal{O}_v^\times$$

where $|S| < \infty$, $S$ contains all the archimedean places, and the $U_v \subseteq \mathbb{F}_v^\times$ are open subsets. But then we know that $\mathrm{Gal}(K/k)$ is a continuous action on each of the factors, and that, labeling the action map $h : \mathrm{Gal}(K/k) \times I_K \to I_K$, that $h^{-1}(V)$ is the union of sets of the form

$$V_\sigma = \prod_{v \in S} \{\sigma\} \times \sigma^{-1}(U_v) \times \prod_{v \notin S} \{\sigma\} \times \mathcal{O}_v^\times$$

all of which are open in $\text{Gal}(K/k) \times I_K$, hence $V$ is itself open, proving continuity.

□

**Lemma 5.2:** For separable extensions $K/k$ of global fields, the action of $G_K$ on $I$ is continuous.

*Proof.* $I$ has the final topology, which means (by definition) that it is sufficient to show that the action is continuous on each $I_K$, when the degree of $K/k$ is finite, but this is exactly the content of lemma 5.1. □

**Corollary 5.3:** If $k$ is a global field, then there is a continuous action of $G_k$ on $I/\mathcal{N}$ which extends to a continuous action on $L^1(Y)$.

*Proof.* It is clear that $\text{Gal}(\overline{k}/k)$ maps $\mathcal{N}$ bijectively onto $\mathcal{N}$, so the action on $I$ descends to an action on $I/\mathcal{N}$ which has the quotient topology, so by definition the action is continuous if it is continuous before taking quotients. By the previous lemma, this is the case. Then as $I/\mathcal{N}$ is dense in $L^1(Y)$ it follows from basic topology that the desired extension exists and is unique by density. □

**Corollary 5.4:** There is a continuous representation of the absolute Galois group of any completion (with respect to a place) of a global on the assciated Banach space, $L^1(Y)$.

*Proof.* From the construction in section 2, we know that the stabilizer of a given place of $Y$ is isomorphic to the absolute Galois group of the desired completion of the base field at that place. Since we have already proven continuity for $G_{\overline{k}}$ subgroups inherit this property. □

**Lemma 5.5:** For any $\mathbf{x} \in I$ and any $\sigma \in G_k$, we have that $h(\mathbf{x}) = h(\sigma\mathbf{x})$.

*Proof.* The embedding $\mathbf{x} \mapsto f_{\mathbf{x}}$ assigns to $\sigma\mathbf{x}$ the function $f_{\sigma\mathbf{x}}(y)$. Say $\mathbf{x}$ has a representative in $I_K$, we use this one to compute the norm

$$||f_{\sigma\mathbf{x}}|| = \sum_v \lambda(Y(K, v))| \log |\sigma x_v|_v|.$$

We may now break the sum into sub-sums over primes above a given prime and note that the Galois action merely permutes the summands, giving us the same result for $||f_{\mathbf{x}}||$ as $||f_{\sigma\mathbf{x}}||$. □

**Proposition 5.6:** The action from corollary 5.4 is an isometry, i.e. it preserves the norm in $L^1(Y)$.

*Proof.* Write $f \in L^1(Y) = \lim_n \mathbf{x}_n$ with $\mathbf{x}_n \in I$, by density this is possible. Then by lemma 5.5 we have that the sequence $||\mathbf{x}_n|| = ||\sigma\mathbf{x}_n||$. Since the action of $G_k$ on $L^1(Y)$ is continuous, $\sigma\mathbf{x}_n \stackrel{n\to\infty}{\longrightarrow} \sigma f$, and so $||f|| = ||\sigma f||$ as desired. □

**Lemma 5.7:** Let $\alpha \in \overline{k}$ for some global field, $k$. Then there is a minimal (with respect to inclusion of fields) extension $L_\alpha/k$ such that all fields $k(\alpha^n)$ contain $L_\alpha$ and which contains $k(\alpha^n)$ for some $n \in \mathbb{N}$.

*Proof.* First we consider the set $\{k(\alpha^n) : n \in \mathbb{N}\}$ partially ordered by inclusion. Denote respectively the greatest common divisor and the least common multiple of integers $m, n \in \mathbb{Z}$ by $(m, n)$ and $[m, n]$. Then it is clear from figure 5.1 that there is a unique minimal element for this poset, and this is the desired $L_\alpha$. We will denote by $n(\alpha)$ the smallest natural number $n$ such that $L_\alpha = k(\alpha^n)$.

$$
\begin{array}{ccc}
 & k(\alpha^{(m,n)}) & \\
 \diagup & & \diagdown \\
k(\alpha^n) & & k(\alpha^m) \\
 \diagdown & & \diagup \\
 & k(\alpha^{[m,n]}) &
\end{array}
$$

Figure 5.1: The lattice of fields

□

We define a relation $\sim$ on $\overline{k}^{\times}$ where $\alpha \sim \beta$ whenever there are integers $m, n \in \mathbb{Z}$ such that $\alpha^n = \beta^m$. It is clear that this is the equivalence relation $\alpha, \beta$ are in the same linear subspace of $\mathcal{G}_k$. In the notation of lemma 5.7, we consider the (obviously finite) Galois orbit $G_k \cdot \alpha^{n(\alpha)} = \{g \cdot \alpha^{n(\alpha)} : g \in G_k\}$. Then we have the following:

**Lemma 5.8:** The action of the absolute Galois group on $\mathcal{G}_k/\sim$ can be uniquely determined by it's action on the set $\{\alpha^{n(\alpha)} : \alpha \in \overline{k}^{\times}\}$. Moreover if we denote the image of $\alpha$ in $\mathcal{G}_k$ by $\overline{\alpha}$, then the dimension of the vector space $V_{\overline{\alpha}} = \text{span}_{\mathbb{Q}}\{g \cdot \overline{\alpha} : g \in G_k\}$ is finite.

*Proof.* The second claim follows from finiteness of the Galois orbit of any algebraic number, in particular of $\alpha^{n(\alpha)}$. Because we have introduced the equivalence relation $\sim$ on $\mathcal{G}_k$, all $\mathbb{Q}$ sub-vector spaces of $\mathcal{G}_k$ are identified with a unique vector $\alpha^{n(\alpha)}$ defined in lemma 5.7. But then it is clear that the (linear) action of the Galois group is uniquely determined by its action on these sets for the (finite dimensional) space $V_\alpha$ defined in the statement of the lemma. □

**Remark 5.9:** It appears, at first glance, that this gives a strategy for forming a basis for $\mathcal{G}$, which would answer a question asked by Allcock and Vaaler in [1], however note that 2, 3, and 6 are all in different subspaces, but 6 depends linearly on 2 and 3 in $\mathcal{G}$, so that despite that the action is determined by the action on the subspaces $V_{\overline{\alpha}}$, it is also *over-determined.* That is to say $V_{\overline{2}} \cap V_{\overline{3}} = V_{\overline{2}} \cap V_{\overline{6}} = \{0\}$, but $\text{span}_{\mathbb{Q}}\{\overline{2}, \overline{3}, \overline{6}\}$ is not three dimensional, so the choices of vectors are generating, but *not* a basis, albeit it is still a much smaller generating set than all of $\mathcal{G}$.

**Remark 5.10:** It is of interest to note that lemma 5.8 indicates that the action of $G_k$ on any $\alpha^{n(\alpha)}$ is synonymous with the action of $\text{Gal}(k_\alpha/k)$ where $k_\alpha$ is the Galois closure of $k(\alpha^{n(\alpha)})$ in $\overline{k}$. This will be of interest in the sequel.

# Chapter 6

# Miscellaneous Lemmata

In this chapter we provide some miscellaneous lemmata which relate to the conjectures in chapter 10 and the results of chapter 9.

We begin by establishing some properties of idèle class groups using the properties of $I/\mathcal{N}$ as a metric space.

**Proposition 6.1:** If $k$ is a global field, then there exists a constant $c(k)$ depending only upon $k$ so that each element of $I_k^1$ is at most distance $c(k)$ from an element from $\mathcal{G}_k$.

*Proof.* Consider the following diagram of inclusions

$$\begin{array}{ccccc}
 & & & & \mathcal{G}_k \\
 & & & & \cap| \\
k^\times & & \text{———————} & & k^\times \\
\cup| & & & & \cap| \\
\mathrm{Tor}\,(k^\times) = k^\times \cap \mathcal{N}_k & \subseteq & \mathcal{N}_k & \subseteq & I_k^1
\end{array}$$

Figure 6.1: Canonical inclusions

Poking around a bit with the basic isomorphism theorems and the inclusions in figure 6.1, we find that

$$I_k^1/k^\times \cong \left(I_k^1/\mathrm{Tor}(k^\times)\right) \Big/ \left(k^\times/\mathrm{Tor}(k^\times)\right) \qquad (*)$$

and also

$$I_k^1/\mathcal{N}_k \subseteq I_k^1/\mathrm{Tor}(k^\times). \qquad (**)$$

We conclude that there is a natural inclusion

$$I_k^1/\mathcal{N}_k k^\times = \left(I_k^1/\mathcal{N}_k\right) \Big/ \mathcal{G}_k \quad \subseteq \quad I_k^1/k^\times. \qquad (***)$$

Lifting the image of the LHS–to which we refer, for the remainder of the proof, as $D_k$–back up to $L^1(Y)$, we have that a fundamental domain for this group has diameter bounded by that of $I_k^1/k^\times$ in the height norm. Since the larger group in $(***)$ is compact, we have by proposition 3.17 and [20, Chapter IV §4, Theorem 6] that it is bounded as a subset of a metric space. Hence the closed subset, $D_k$ of the compact $I_k^1/k^\times$–being compact–is also bounded. By selecting a fundamental domain for $D_k$ which has the induced metric

$$d(\mathbf{x},\mathbf{y}) = \min_{\alpha \in \mathcal{G}_k} h(\mathbf{x}\mathbf{y}^{-1}\alpha^{-1})$$

the result follows. □

**Remark 6.2:** In the proof of proposition 6.1 we have that the bounding constant is actually *equal* to that for height approximations for $I_k^1/k^\times$ because the qualitative difference between $D_k$ and $I_k^1/k^\times$ is only by norm-0 elements and elements in each differ from one another by norm-0 elements only.

**Remark 6.3:** Note that the metric from the proof of proposition 6.1 is of essentially the same quality as the circle metric used in Diophantine approximation where we write

$$\|x\| = \min_{n \in \mathbb{Z}} |x - n|$$

and declare the metric on the circle group, $\mathbb{R}/\mathbb{Z}$ to be $d(x, y) = \|x - y\|$.

**Corollary 6.4:** Let $k$ be a global field. Select some idèle $\mathbf{x}_0 \in I_k$ so that $|x_v|_v = 1$ everywhere except at $v_0$ where $|x_{v_0}|_{v_0} > 1$. Then there is a constant $c'(k) > 0$ so that each element of $\mathbf{y} \in I_k/\mathcal{N}_k$ has a corresponding $\alpha \in k^\times$ and $n \in \mathbb{Z}$ such that

$$h(\mathbf{y}\mathbf{x}_0^{-n}\alpha^{-1}) < c'(k)$$

*Proof.* This follows from proposition 6.1 and the polar decomposition, $I_k \cong R \times I_k^1$, given in appendix C. □

We next mention some applications of the space $I/\mathcal{N}$ and its height norm to classical properties of interest in the study of heights and algebraic numbers. Of particular interest is the application of the theory to studying behavior such as the Northcott and Bogomolov properties, (B) and (N) respectively. The former is a property of a field wherein the number of non-zero points of bounded height is finite and the latter is one wherein the number of non-zero, non-torsion (multiplicative torsion) points is finite. Property (N) is named for D.G. Northcott for his famous theorem in [14]. We begin with

**Definition 6.5:** We say a field, $k/\mathbb{Q}$, has the Bogomolov property, (B), if $\mathcal{G}_k \subseteq I/\mathcal{N}$ is discrete in the metric induced by the height.

**Lemma 6.6:** If a topological group, $G$, has a metrizable topology, then a subgroup, $H \leq G$ is discrete iff the identity $e$ is isolated as a point of $H$.

*Proof.* Let $d$ be a metric on $G$ inducing the topology and let $\epsilon > 0$ be given. Then if $d(e, x) \geq \epsilon$ for every $x \in H \setminus \{e\}$, we have, since $d(x, y) = d(xy^{-1}, e)$ that $d(x, y) < \epsilon \implies x = y$. Conversely if $H$ is discrete in $G$, then $H$ is a discrete group with a metric topology, hence there exists an $\epsilon > 0$ such that $d(x, y) > \epsilon$ for every $x \neq y$, and so setting $y = e$ the result follows. □

**Corollary 6.7:** If $(G, ||\cdot||)$ is a pseudo normed group with norm kernel $N = \{g \in G : ||g|| = 0\}$. Then a subgroup, $H \leq G$, is a pseudo-discrete subgroup (i.e. discrete after passing to $H/H \cap N \leq G/N$ iff there exists $\epsilon > 0$ such that the identity coset of $H/H \cap N$ is an isolated point in $G/N$.

*Proof.* This follows directly from lemma 6.6. □

**Remark 6.8:** It is essential that we are dealing with subgroups here, and not subsets, the set $\{\frac{1}{n} | n \in \mathbb{N}\}$ is discrete, but there is no uniform choice of $\epsilon$; the homogeneity of the group structure is what allows the proof to work.

**Proposition 6.9:** A field $k$ has (B) iff there exists $\epsilon > 0$ such that set $A_{\epsilon,k} \cap k^\times = \mathrm{Tor}(k^\times)$. That is, for some $\epsilon > 0$ the set of points in $k$ of height less than $\epsilon$ is just the torsion points.

*Proof.* This follows immediately from applying corollary 6.7 to $G = I_k$ and $N = \mathcal{N}_k$ and appealing to the classical definition of (B). □

**Corollary 6.10** (Weak Northcott Theorem)**:** The set of all points in a number field, $k$, of bounded height is finite.

*Proof.* Let $K$ be a number field. Then as $K^\times/\mathrm{Tor}(K^\times)$ has the Bogomolov property, it is a discrete (hence closed) subset of $I_K/\mathcal{N}_K$, and the set $A_{C,K}$ is compact–hence so is its image in $I_K/\mathcal{N}_K$–the set of points of $K^\times$ of bounded height is both and hence is finite. $\square$

# Chapter 7

# The Algebra Structure

In this section we work to understand the structure of $L^1(Y)$ as a Banach algebra, deduce some of its basic properties from that of $L^1(G_k)$, and apply our results to obtain a better understanding of idèles and algebraic numbers in this new context.

The author of the present work has found [15] an invaluable resource on the basics of harmonic analysis on homogeneous spaces, $G/H$, in the general (i.e. non-abelian, $H \ntrianglelefteq G$) case, and for the reader unfamiliar with the Haar measure on a (locally) compact group, [5] has an ample introduction.

**Theorem 7.1:** The set $L^1(Y)$ possesses the structure of a Banach algebra.

*Proof.* Let $f_n \to f$ and $g_n \to g$ be two sequences of $C_c(Y)$ functions converging in the $L^1$ norm. Then by continuity of multiplication, we know that

$$f_n \star g_n \overset{n\to\infty}{\longrightarrow} f \star g$$

if $L^1(Y)$ has a Banach algebra stucture with multiplication $\star$, so that it is sufficient to prove the result for $C_c(Y)$ by density of the latter in the former. By the results of chapter 2, $Y$ is of the form $Y = \bigsqcup_v G_k/D_v$, a disjoint union of homogeneous $G_k$ spaces, and so by lemma 4.2 the desired $\mathbb{F}$-algebra structure exists for $C_c(Y)$, and hence for $L^1(Y)$. □

**Remark 7.2:** In the case $k = \mathbb{Q}$ note that given $f \in L^1(Y)$ the function $f_\infty = f \cdot 1_\infty$ is what we would call the "infinite" part of a $\mathbb{Q}$-idèle and the $f - f_\infty$ corresponds to the "finite" or "ideal" part of an idèle as in the classical decomposition of a $k$-idèle given by the classical ideal map $\mathrm{id} : I_k \to J_k$ where $J_k$ is the ideal group of $k$. In this way we can justify calling the sub-algebra $L^1(G_k)$ the "infinite component" of any $\mathbf{x} \in I/\mathcal{N}$–regardless of the base-field–and the rest the "ideal component."

**Remark 7.3:** One may wonder if the structure on the $L^1(G_k/D_v)$ is recognizable in a more natural or canonical form and [6, Corollary 3.1] ensures that it is indeed the intuitive one: functions in the set $L^1(G_k : D_v) = \{f \in L^1(G_k) | f(xh) = f(x)\ \forall x \in G_k, \forall h \in D_v\}$ of right $D_v$-periodic functions from $L^1(G_k)$. This–along with lemma 2.6 which produces an identification of $D_v$ with $\mathrm{Gal}\left(\overline{k}_v/k_v\right)$–provides come helpful context and a more computation-friendly understanding of the functions in $L^1(Y)$.

**Remark 7.4** (How to compute)**:** For the sake of self-containment we include the "how-to" of computing convolutions in $L^1(Y)$ which is developed abstractly in [6]. We wish to compute in the same fashion as we do for integers modulo $n$. The process there is to choose integers $a, b \in \mathbb{Z}$ such that $a \equiv [a] \mod n$, $b \equiv [b] \mod n$ so that we can compute $[a] + [b]$ by just adding the normal integers $a, b$ and reducing mod $n$, i.e. $[a] + [b] \equiv [a + b] \mod n$, using the fact that $\mathbb{Z}$ projects naturally onto $\mathbb{Z}/n\mathbb{Z}$, and that this is independent of choice of lifted integer–this is the essence of canonical projections of algebras, rings, groups, et cetera. The procedure we use for $L^1(Y)$ convolutions is in this vein and is essentially Weil's formula for locally compact groups (proven by Weil for $H \trianglelefteq G$). That is, if $G$ is such a locally compact group and is equipped with a Haar measure, $dx$, and a closed subgroup, $H \leq G$, with Haar measure $dh$, we have the integral equality

$$\int_G f(x)\,dx = \int_{G/H} \left( \int_H f\,dh \right) d\dot{x}.$$

(see [15, pp 88].)

The point is that the canonical projection of $L^1$ of a locally compact group–in our case the absolute Galois group, $G_k$ of our field, $k$–onto the corresponding space $L^1(G/H)$ is given by integrating along $H$. In our case we may also think of this as *averaging* along the subgroup, $H$, since our $H$ is $D_v$ which is compact and so possess a canonical *normalized* Haar measure. Let the map $T_H$ be defined as the "averaging along $H$" map, i.e.

$$T_H(f)(x) = \int_H f(xh)\,dh.$$

It is this map that [6] uses to induce the algebra structure on $L^1\left(G/H\right)$ and he shows that one can define a multiplication $\star$ on $L^1\left(G/H\right)$ such that $T_H$ is the canonical surjection of an algebra onto its quotient algebra. That is

$$T_H : L^1(G) \to L^1\left(G/H\right)$$

is surjective and if $f \star g = h$ as elements of $L^1(G)$ then $T_H(f \star g) = T_H(f) \star T_H(g)$ where the second $\star$ is understood to be in the algebra structure on $L^1\left(G/H\right)$. This means that when we wish to compute, for $f, g \in L^1(\sqcup_n X_n)$, the quantity $f \star g$ we need only break $f, g$ into $X_n$-local components, $f = \sum f_n$, $g = \sum g_n$ then find representatives $\widetilde{f}_n, \widetilde{g}_n \in L^1(G)$ such that $f_v = T_H(\widetilde{f}_n)$ and $g_n = T_H(\widetilde{g}_v)$. From there we simply compute $\widetilde{f}_n \star \widetilde{g}_n$ in the classical way via convolution on groups, and then average along $H$ and add up the results. For our context we summarize this as

1 For each place, $v$, of $k$ find functions $\widetilde{f}_v$, $\widetilde{g}_v$ projecting onto the components $f_v = f \cdot 1_{G/D_v}$, $g_v = g \cdot 1_{G/D_v}$ of $f$ and $g$ (respectively). This is possible by surjectivity of the maps, $T_{D_v}$.

2 Compute $\widetilde{f}_v \star_G \widetilde{g}_v$ and apply $T_{D_v}$ to the result. Because the map $T_H$ is an algebra homomorphism for any compact subgroup, $H$ of a locally compact group, $G$, according to [6]–our case is simply $H = D_v$, $G = G_k$ each time, and repeat the procedure for each $v$ then put the results back into the direct sum component-wise. So we get that this is well-defined immediately, and the only thing that $f_v \star g_v$ could be.

3 Put all the components back together, that is: recall the identification from lemma 4.2 and theorem 7.1 and conclude that $f \star g \leftrightarrow (f_{v_i} \star g_{v_i})_i$.

**Remark 7.5:** We note that the inequality in lemma 4.1 almost surely always loses some since we are always missing the cross terms from $(\|f_A\| + \|f_B\|)(\|g_A\| + \|g_B\|)$, namely $\|f_A\|\|g_B\|$ and $\|f_B\|\|g_A\|$. In the case of $L^1(Y)$ it is an interesting question to ask when equality holds (if ever). There is also a subtlety in the fact that $2h$ is our norm instead of simply $h$, which we illustrate in the following

**Example 7.6:** We compute here $f_5 \star f_7$, i.e. the function associated to the rational integers (hence idèles) 5 and 7 after convolution with the base field $k = \mathbb{Q}$. In the spirit of section 2 we write $f_5 = \log 5 \cdot 1_{G_\mathbb{Q}} - \log 5 \cdot 1_{G_\mathbb{Q}/D_5}$ and $f_7 = \log 7 \cdot 1_{G_\mathbb{Q}} - \log 7 \cdot 1_{G/D_7}$. Here $1_X$ is the characteristic function of the set, $X$, treated as a subset of $Y$. As usual $G_\mathbb{Q}$ is identified with $Y(\mathbb{Q}, \infty)$ and each $G_\mathbb{Q}/D_v$ are the $Y(\mathbb{Q}, v)$. According to our structure theorem, 7.1, and the explicit description of the convolution procedure as described in remark 7.4 we can proceed.

So for this example, we first note that the only common $G_{\mathbb{Q}}/D_v$ on which both $f_5$ and $f_7$ are supported is $v = \infty$ and $G_{\mathbb{Q}}/D_\infty = G_{\mathbb{Q}}$, on $v = 5$, $f_7$ has a representative from $G_{\mathbb{Q}}$ which is the 0 function, since clearly $T_{D_5}(0) = 0$ and step 2 says that I just convolve any lift of $\log 5 \cdot 1_{G_{\mathbb{Q}}/D_5}$ against any lift of 0, in particular, 0, so I get 0. Similarly for $f_5$ on $G_{\mathbb{Q}}/D_7$. On $G_{\mathbb{Q}}$ we needn't lift at all, since we're already all the way up at $G_{\mathbb{Q}}$. Also, $f_5 \equiv \log 5$ on $G_{\mathbb{Q}}$ and similarly for $f_7 \equiv \log 7$, so the convolution of these two constant functions just gives us the function which is identically $\log 5 \log 7 \cdot m_G(G)$, where $m_G$ is the (normalized) Haar measure on $G$. We conclude

$$f_5 \star f_7 = \log 5 \log 7 \cdot 1_{G_{\mathbb{Q}}}.$$

**Remark 7.7:** We note that example 7.6 implies that $\overline{k}^\times/\mathrm{Tor}(\overline{k}^\times)$ is not a sub-*algebra* of $L^1(Y)$ since clearly $f_5 \star f_7$ does not have integral 0–indeed it has integral $\log 5 \log 7$.

The proof of theorem 7.1 has a feature which we now exploit: the base field can be chosen to be any global field. More precisely

**Corollary 7.8:** Let $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F}_p(t)$ and let $k/\mathbb{F}$ be a global field with $v$ an aribtrary place of $v$. Then there is an isometric isomorphism between the Banach algebras $L^1\left(G_k/D_{v,k}\right)$ and $L^1\left(G_{\mathbb{F}}/D_{v,\mathbb{F}}\right)$

*Proof.* We begin by noting that since $k/\mathbb{F}$ is an extension, $I_{\overline{k}} = I_{\overline{\mathbb{F}}}$ because $I_{\mathbb{F}} \subseteq I_k$ and this inclusion is compatible with the transition maps $f_{ij}$ used to define $I_{\overline{k}}$. Indeed, if $\left(\{K/k\}_K, \{f_{ij}\}_{i,j\in I}\right)$ is the directed system used to define $I_{\overline{k}}$ we have that it includes *without any alteration whatsoever* into $\left(\{K'/\mathbb{F}\}_{K'}, \{f'_{ij}\}_{i,j\in I}\right)$. Further as $k\mathbb{F} = k$ and $Lk$ is an extension of both $k$ and $\mathbb{F}$ for any $L/\mathbb{F}$ we see that in order to be consistent, the sub-system must have an identical limit to the original. Indeed this is just the observation that field extensions of $\mathbb{F}$ form a lattice and that field extensions of $k$ are a sub-lattice. $\mathcal{N}_k = \mathcal{N}_{\mathbb{F}}$ also follows from an identical argument. Since the norms on both formations of $I$ are identical, both are isometric and have the same completions–again, this is not only a canonical identification, it is *equality.*

From this we deducethere is an isometric–hence continuous–linear algebra-isomorphism

$$\varphi_k^0 : \bigoplus_v C\left(G_k/D_{v,k}\right) \longrightarrow \bigoplus_v{}' C\left(G_{\mathbb{F}}/D_{v,\mathbb{F}}\right)$$

where, as usual, the $\oplus'$ means to sum over representative restrictions of places $v$ from $k$. By continuity, this uniquely extends to an isometric algebra-isomorphism $\varphi_k : L^1(Y_k) \to L^1(Y_{\mathbb{F}})$. Then given any $K$ a finite extension of either $k$ or $\mathbb{F}$ we know that $L^1\left(G_K/D_{v,K}\right) \subseteq L^1(Y)$ in the canonical, isometric way. (We purposely omit the subscript here to empahsize this inclusion is true with either choice.) By considering the support of $f \in L^1(Y_k) = L^1(Y_{\mathbb{F}})$ locally at $v$, we see $\varphi_k$ induces the desired isometric isomorphism

$$\varphi_{v,k} : L^1\left(G_k/D_{v_i,k}\right) \to L^1\left(G_{\mathbb{F}}/D_{v,\mathbb{F}}\right).$$

□

**Remark 7.9:** By Galois theory, $G_k \leq G_{\mathbb{F}}$. The most striking part of this result is that $D_v \cap G_k = D_{v_i,k}$ implies $G_k/D_{v_i,k}$ is canonically a subspace of $G_{\mathbb{F}}/D_{v,\mathbb{F}}$. And this is not just a simple decomposition either. In the infinite place, for example, this says there is an isometric isomorphism $\varphi_{\infty,k} : L^1\left(G_{\mathbb{F}}\right) \to L^1\left(G_k\right)$ the latter being a *proper subspace* of the former. Even more fantastically: considering $k, \ell$ both extensions of the same $\mathbb{F}$ we have

$\varphi_{\infty,k} \circ \varphi_{\infty,\ell}^{-1}$ gives an isometry between two Banach algebras over base fields that are not even obviously related other than via $\mathbb{F}$. Without the common ground of the idèles, it is not clear how one would establish this isometry.

From here we prove

**Lemma 7.10:** The closed sub-algebra generated by the functions associated to elements of the base idèles, $\{f_{\mathbf{x}}\}_{\mathbf{x}\in I_k}$, are contained in the center of $L^1(Y)$.

*Proof.* By continuity, it is sufficient to show this for the set $\{f_{\mathbf{x}}\}_{\mathbf{x}\in I_k}$ in order to show it for the closure. Let $f_{\mathbf{x}} \in L^1(Y)$ come from an idèle, $\mathbf{x} = (x_v)$ of the base field, $k$, and let $g \in L^1(Y)$ be arbitrary. Then write $f_{\mathbf{x}} = \sum_v f_{x_v}$ and similarly write $g = \sum_v g \cdot 1_{G_k/H_v} = \sum_v g_v$. Then $f_{\mathbf{x}} \star g = \sum_v f_{x_v} \star_v g_v$. Since the $f_v$ are constant on the $G_k/H_v$ we can lift it to a constant function multiplying a lift, $\widetilde{g_v} \in L^1(G)$. But they commute in $L^1(G)$ trivially by definition of convolution on a group. So after applying $T_{D_v}$ to each component we find

$$\begin{aligned} g_v \star f_{x_v} &= T_{D_v}(\widetilde{g_v}) \star T_{D_v}(\widetilde{f_{x_v}}) \\ &= T_{D_v}\left(\widetilde{g_v} \star \widetilde{f_{x_v}}\right) \\ &= T_{D_v}\left(\widetilde{f_{x_v}} \star \widetilde{g_v}\right) \\ &= T_{D_v}(f_{x_v} \star \widetilde{g_v}) = f_{x_v} \star g_v \end{aligned}$$

hence each $f_{x_v} \star_v g_v = g_v \star_v f_{x_v}$, and so $f_{\mathbf{x}} \in Z(L^1(Y))$ as claimed. □

**Remark 7.11:** Lemma 7.10 and example 7.6 allow us to discuss algebraic combinations of logarithms of rational primes using some simple observations

1 When $k = \mathbb{Q}$ we see all $f_p \in Z(L^1(Y))$ for rational primes, $p$.

2 One can extrapolate how to get any monomial in the set $\{\log p_i\}_i$ from the example by using the appropriate number of convolutions to get powers.

3 For addition, one need only add or subtract functions then ask about the representation theory on the algebra $L^1(Y)$ to further investigate the problem.

We illusterate this philosophy with the following

**Remark 7.12:** One might be tempted to say that the content of example 7.6 indicates that functions with disjoint supports have convolution 0. This is not necessarily the case, however, as we only saw 0 interaction when there were pieces completely supported on different primes, one might consider the field $\mathbb{Q}(\sqrt{2})$ with its two infinite places $\infty_1, \infty_2$. The corresponding decomposition of $Y(\mathbb{Q}, \infty)$ for this extension is $Y(\mathbb{Q}, \infty) = Y(\mathbb{Q}(\sqrt{2}), \infty_1) \sqcup Y(\mathbb{Q}(\sqrt{2}), \infty_2)$. It is not difficult to find an idèle supported on just $Y(\mathbb{Q}(\sqrt{2}), \infty_1)$ and another on just $Y(\mathbb{Q}(\sqrt{2}), \infty_2)$ with non-zero convolution, since the action of $G_{\mathbb{Q}}$ permutes the two places $\infty_1$, $\infty_2$.

**Remark 7.13:** It's a bit of a shame that there is no canonical C*-algebra structure on $L^1(Y)$. This is because the place stabilizers, $D_v$, defined as in remark 2.3, are not normal subgroups for any $w \not| \infty$, and in [6, Theorem 3.2] it is established that a canonical involution on the algebra $L^1(G/H)$ is given by the projection we use–$T_H : L^1(G) \to L^1(G/H)$–if and only if $H \trianglelefteq G$. It is further shown that that $T_H$ is a *-homomorphism iff $H \trianglelefteq G$, which is false for all of our $D_v$ except $D_\infty = \{e\}$.

# Chapter 8

# Measure Theorems

## 8.1 Measure Theory for $Y$ I

In this section we develop the theory of the space $L^1(Y)$ by considering different $\sigma$-algebras rather than the one given by the construction in [1]which contains the Borel $\sigma$-algebra—-although this $\sigma$-algebra will still be distinguished for the main theorem in chapter 7. For those unfamiliar with the concepts in measure theory related to concepts like the Radon-Nikodym derivative, any graduate text in analysis such as [21] will have the requisite material and in appendix E we have a statement of the relevant result. For the duration of this section, all fields, base or not are assumed to be separable extensions of $\mathbb{Q}$ or $\mathbb{F}_p(t)$. The reader interested in the most common case may assume that all fields are Galois and finite over $\mathbb{Q}$ or $\mathbb{F}_p(t)$, as the full generality is only done because it is possible. Throughout this section we frequently denote by $\mathcal{P}_K$ the set of all placess of a field $K$, by $N_K^L$ the (relative) norm of field elements for the extension $L/K$, and by $\mathfrak{N}_K^L : I_L \to I_K$ the (relative) norm between the idèle groups $I_L$ and $I_K$, which we recall is defined as $\mathfrak{N}_K^L(\mathbf{x}) = \left( \prod_{w|v} N_{K_v}^{L_w}(x_w) \right)_v$.

We begin with the following

**Definition 8.1** (preliminary definition)**:** If $K$ is a field, we define the $\sigma$-algebra, $\sigma(K)$ to be the $\sigma$-algebra generated by the sets $Y(K, v)$ with $v \in \mathcal{P}_K$.

We make this as a preliminary definition and revise it to definition 8.14 after theorem 8.13 primarily in order to avoid the awkward and unmotivated theorem statements that would result from a proper definition at this point.

A simple, initial observation is that if $K$ is a number field, $\sigma(K)$ characterizes $K$ up to isomorphism: for the places that split completely in $K$ determine $K$ up to isomorphism, and if $K/\mathbb{Q}$ is Galois then $K$ is truly unique, and this is the only case we consider in the sequel. We have the following

**Lemma 8.2:** Let $K$ be a field contained in the separable closure of a global field, and let $S \subseteq \mathcal{P}_K$ be a countable subset of the set of all places of $K$. Then the algebra, $\sigma(K)$, is composed of the sets $\left\{ \bigsqcup_{v \in S} Y(K, v) \right\}_{S \subseteq \mathcal{P}_K}$ and their complements.

*Proof.* This follows from the fact that $Y(K, v) \cap Y(K, w) = \varnothing$ unless $w = v$, hence they form a partition of the set $Y$ with the $Y(K, v)$ as atoms in the $\sigma$-algebra. As such, a measurable set is determined solely based on the elements of $\mathcal{P}_K$ contained in it. The countability condition comes because we are only allowed countable unions in $\sigma$-algebras. □

**Corollary 8.3:** If $k$ is a global field and $K/k$ a finite, Galois extension, and $S \subseteq \mathcal{P}_K$ is *any* subset of the set of places of $K$ then $\sigma(K)$ is the collection of all sets of the form $\left\{ \bigsqcup_{v \in S} Y(K, v) \right\}_{S \subseteq \mathcal{P}_K}$.

*Proof.* For there are only boundedly many places over a given place of $k$, hence $\mathcal{P}_K$ itself is countable. □

**Corollary 8.4:** If the base field, $k$ has only countably many places and $K/k$ has the property that each place $v$ of $k$ has only countably many places above it in $K$, then $\sigma(k) \subseteq \sigma(K)$.

*Proof.* This follows from corollary 8.3, the fact that

$$Y(k,v) = \bigcup_{w|v} Y(K,w)$$

and that the countable union of countable sets is countable. □

**Remark 8.5:** If $K/k$ is an extension of number fields, then $\sigma(k) \subseteq \sigma(K)$ is obviously true becuase there are only finitely many places over a fixed place.

The following proposition is of great comfort and and is one of the "if there is any justice in the world" results.

**Proposition 8.6** (justice)**:** If $\mathcal{P}_K$ is countable and $\mathbf{x} \in I$ has a representative in $I_K$, then $f_{\mathbf{x}}$–as defined in §3.3–is measurable for $\sigma(K)$.

*Proof.* For in this case, $f_{\mathbf{x}}$ is a locally constant function on each $Y(K,v)$, and hence takes on only countably many values (in fact only finitely many!). If $A \subseteq \mathbb{R}$ is a set in the Borel $\sigma$-algebra on $\mathbb{R}$, then $f_{\mathbf{x}}^{-1}(A)$ is just the union of the $Y(K,v)$ for which $f_{\mathbf{x}}(Y(K,v)) \in A$ i.e. is a countable union of measurable sets, and so is itself measurable. Hence $f_{\mathbf{x}}$ is measurable. □

**Corollary 8.7** (more justice)**:** If $\mathcal{P}_K$ is countable, then $\sigma(\{f_{\mathbf{x}}\}_{\mathbf{x}\in I_K}) = \sigma(K)$.

*Proof.* Let $\Sigma = \sigma(\{f_{\mathbf{x}}\}_{\mathbf{x}\in I_K})$. The inclusion $\Sigma \subseteq \sigma(K)$ is clear from proposition 8.6. To see that $\sigma(K) \subseteq \Sigma$ it suffices to show that $Y(K,v) \in \Sigma$ for each $v$ of $K$. For this Select a $v$ and consider the idèle $\mathbf{x}$ which has local component 1 for every $w \neq v$ and at $w$ choose $x_v \in K_v$ such that $|x_v|_v > 1$. Then $\log |x_v|_v \neq 0$ and this is the only coordinate for which this is the case, hence $Y(K, v) = f_{\mathbf{x}}^{-1}(\log |x_v|_v)$. □

**Remark 8.8** (needs more justice)**:** By setting $k = \mathbb{Q}$ we note that $\sigma(\overline{\mathbb{Q}}) \subset \mathcal{B}$, the full Borel $\sigma$-algebra, so there is still some space in-between all of $\mathcal{B}$ and $\sigma(\overline{\mathbb{Q}})$.

**Remark 8.9:** Note that using $I_K$ instead of $K$ allows a great simplification in the proof since we can freely choose finitely many coordinates without affecting the others, whereas the product formula for $K$ would give us much less control, albeit it can be done with the CRT or the approximation theorem and some juggling.

**Lemma 8.10:** If we restrict ourselves to a base field, $k$, with countably many places and the class, $\mathcal{C}$, of extensions $K/k$ with countably many places, then for all $K, L \in \mathcal{C}$ the following are equivalent

(1) $K \subseteq L$

(2) $\sigma(K) \subseteq \sigma(L)$

*Proof.* (1) $\implies$ (2) is just corollary 8.7. To see that (2) $\implies$ (1), we note that since all the $Y(\infty_K, K) \in \sigma(L)$ for every infinite place, $\infty_K$ of $K$ that every element of the absolute Galois group, $G_k$ of the base field, $k$, which fix $L$ also fix $K$, hence the subgroup $G_L \subseteq G_K$ and by the FTGT, $K \subseteq L$. □

**Corollary 8.11:** The finite, Galois extensions of a field with countably many places have the property that field containment is equivalent to $\sigma$-algebra containment. In particular, there is an anti-equivalence between the compact, open subgroups of the absolute Galois group, $G_k$, of $k$ and these $\sigma$-algebras.

*Proof.* The first statement follows from the fact that the number of places above a fixed place is bounded by the degree of a field extension, and the second from the FTGT. □

**Remark 8.12:** Note that this anti-equivalence is the same as in the case of fields. By corollary 8.7 we may consider the collection of the $\sigma(K)$ and define the action of $G_k$ as the action on the sets $Y(k, v)$ by permuting the places according to the Galois action on places. Then point-wise fixing subgroups give the same natural 1-1 correspondence from Galois theory. Also, the proof of lemma 8.10 doesn't require us to necessarily look at the infinite places, both $L, K$ are contained in a common separable closure, so we really could find distinct places of any kind between the two which situates them differently in the separable closure of any completion and derive the same conclusion.

## 8.2 Measure Theory for $Y$ II

In this section we reach the main conclusions of our analysis of the measure theory associated to $Y$. Here we come to a final definition for the $\sigma$-algebra of a global field, $K$

**Theorem 8.13:** If we remove the requirement that the $K/k$ be finite, then we no longer have an anti-equivalence of $\sigma$-algebras, $\sigma(K)$, associated to separable, algebraic extensions $K/k$ and closed subgroups of $G_k$ under definition 8.1. Furthermore this failure is as slight as possible.

*Proof.* We assume by means of contradiction that the anti-equivalence holds with definition 8.1. Firstly we must lift from the finite extension case to that of infinite extensions. We note it is sufficient to show, for any separably algebraic extension, $K/k$, that $\sigma(K)$ is the $\sigma$-algebra generated by the $f_{\mathbf{x}} \in I_K/\mathcal{N} \cap I_K$, for we can detect the difference between two fields by detecting separate places and this immediately produces a place $w$ of $K$ which is unequal to one of $L$ or vice-versa. To this end, we let $L/k$ be any separably algebraic extension of $k$, we note that all sets $Y(L, w)$ can be written as

$$\bigcap_{k=1}^{\infty} \bigcup_{i=1}^{n_k} Y(L_k, w_{k_i})$$

with $k = L_0$ and $[L_k : L_{k-1}] < \infty$ Galois. But then $w_0 \in Y(L, w)$ iff for all $m$ there is a $j$ such that $w_0 \in Y(L_m, w_{m_j})$. The complement of this set is $\{w_0 : \exists m \forall j \text{ such that } w_0 \notin Y(L_m, w_{m_j})\}$, i.e. the set

$$\bigcup_{k=1}^{\infty} \bigcap_{i=1}^{n_k} Y(L_k, w_{k_j})^c.$$

From this point, we note that the sets

$$\left\{ \bigcap_{i=1}^{n_k} Y(L_k, w_{k_j}) \right\}_{k=1}^{\infty}$$

form a $\pi$-system, i.e. a collection of sets closed under finite intersections. By Dynkin's celebrated $\lambda - \pi$ theorem as stated in appendix E, we conclude that the $\sigma$-algebra generated by these sets–which is $\sigma(K)$ because the complements of these sets are a generating set for $\sigma(K)$–is contained in any $\lambda$-system–i.e. a collection of sets including $Y$ which is closed under relative complements and limits of countable chains of sets–containing the original $\pi$-system. In particular, $\sigma(L)$ is contained in the $\sigma$-algebra generated by all $f_{\mathbf{x}} \in I_L/\mathcal{N} \cap I_L$ since we can detect any of the sets $Y(L_k, w_{k_j})$ by selecting elements of $L_k$ as in the finite case. To see the reverse inclusion we let $f_{\mathbf{x}} \in I_L/\mathcal{N} \cap I_L$. Then since $f_{\mathbf{x}}$ is represented on some $k \subseteq L' \subseteq L$ with $[L' : k] < \infty$ as a locally constant function with compact support, we need only find the sets $Y(L', w_0) \in \sigma(L)$ for each fixed, but arbitrary $w_0$ that has a $w \in \mathrm{supp}(f_{\mathbf{x}})$ such that $w|w_0$. Now we instantiate $L' = \mathbb{Q}$ and $L = \overline{\mathbb{Q}}$. Then as all sets $Y(\overline{\mathbb{Q}}, y)$ are singletons, we see that it is impossible to write the uncountable set (with uncountable complement) $Y(\mathbb{Q}, \infty)$ as the countable union of singleton sets or the complement thereof. This quantifies what we mean by the failure of the anti-equivalence holding to be "as slight as possible" as one inclusion actually holds independent of countability conditions.

□

To this end we make the revised

**Definition 8.14** (Final definition)**:** We define the $\sigma$-algebra of a field, K, contained in the separable closure of $\overline{\mathbb{Q}}$ or $\overline{\mathbb{F}_p(t)}$ to be the smallest $\sigma$-algebra for which all of the $f_{\mathbf{x}}$ are measurable.

This leads us to

**Theorem 8.15:** With the assumptions as in theorem 8.13, the canonical inclusion-reversing bijection of $\sigma$-algebras associated to subfields of $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}_p(t)}$ and closed subgroups of the corresponding absolute Galois groups holds.

*Proof.* This was shown in the proof of theorem 8.13. □

At this point we wish to talk about the Radon-Nikodym derivative with respect to a given $\sigma(K)$. For simplicity, in the sequel we will assume we are in the case of corollary 8.4, where $k$ has only countably many primes, and the class of extensions, $K/k$ which are both finite and Galois so that we may use all the results of this section with the more convenient description of $\sigma(K)$ from lemma 8.2. We have the following

**Theorem 8.16:** Let $[L : k] < \infty$ with $k \subseteq K \subseteq L$, and $\mathbf{x} \in I_L$. Denote by $\mu_K$ the restriction to $\sigma(K)$ of the measure, $\lambda$, on $Y$ furnished to us by the Riesz-representation theorem and similarly for $\mu_L$ on $\sigma(L)$. Then we have the following interpretation of the Radon-Nikodym derivative

$$df_{\mathbf{x}}/d\mu_K = \sum_v \sum_{w|v} \frac{1}{r_v} \log |\mathfrak{N}_K^L(x_w)| \cdot \mathbf{1}_{Y(K,v)}.$$

With $r_v = [L : K]/[L_w : K_v]$, $\mathbf{1}_X$ the characteristic function of a set $X$, the sums being on places $v$ of $K$ and $w$ of $L$, and $\mathbf{x} = (x_{v_1}, x_{v_2}, \ldots)$ with each $x_{v_i}$ shorthand for the collection of places $x_w$ with $w|v_i$.

*Proof.* By definition of the Radon-Nikodym derivative, we must find $g$ measurable for $\sigma(K)$ such that

$$\int_A g \, d\mu_K = \int_A f_{\mathbf{x}} \, d\mu_L. \qquad (*)$$

First we use the representation of an idèle as a product of prime idèles, $\mathbf{x} = (x_w)$ to write $f_{\mathbf{x}} = \sum_w f_{x_w}$–a finite sum and use linearity of the derivative to reduce to the case $\mathbf{x} = (1, 1, \ldots, 1, x_w, 1, \ldots)$ so that $f_{\mathbf{x}} = f_{x_w}$.

Then we have the following classification

$$\int_A f_{\mathbf{x}} \, d\mu_L = \begin{cases} \frac{[L_w:k_w]}{[L:k]} \log |x_w|_w & Y(L,w) \subseteq A \\ 0 & \text{o/w} \end{cases}.$$

Based on the classification in 8.3, this is equivalent to

$$\int_A f_{\mathbf{x}}\, d\mu_L = \begin{cases} \frac{[L_w:k_w]}{[L:k]} \log |x_w|_w & Y(K, v_w) \subseteq A \\ 0 & \text{o/w} \end{cases}$$

where $v_w$ is the unique place of $K$ under $w$. This together with $\lambda(Y(K, v_w)) = [K_w : k_w]/[K : k]$ we see that

$$g = \frac{[L_w : K_w]}{[L : K]} \log |x_w|_w \mathbf{1}_{Y(K,v_w)}$$

satisfies the defining relation, $(*)$ on the prime piece $f_{x_w}$.

Write $[L : K] = efr$ with $e = e(L : K)$ the ramification index, $f = f(L|K)$ the intertial degree, and $r = r(L|K)$ the splitting number. By standard local field theory, $[L_w : K_w] = ef$, so by writing $x_w$ as a power of a uniformizer and using the definition and multiplicativity of the norm, we see that this is simply

$$g = \frac{1}{r} \log |\mathfrak{N}_K^L(x_w)| \mathbf{1}_{Y(K,v_w)}.$$

Each prime of $L$ above $v_w$ has the same, uniform value for $r$ because all the extensions in sight are Galois, hence we can extend our result to the case $\mathbf{x} = (1, \ldots, x_{w_1}, \ldots, x_{w_r}, 1, \ldots)$ where the $w_i$ are exactly the primes dividing a fixed $v_w$ of $K$, hence for such $\mathbf{x}$ we conclude that

$$\frac{df_{\mathbf{x}}}{d\mu_K} = \frac{1}{r} \log |\mathfrak{N}_K^L(\mathbf{x})| \mathbf{1}_{Y(K,v_w)}.$$

The remainder of the theorem follows from linearity of the derivative. □

**Corollary 8.17:** If $\mathbf{x}$ has local component a unit for all split primes of $L/K$, then $df_{\mathbf{x}}/d\mu_K = f_{\mathfrak{N}_K^L(\mathbf{x})}$.

# Chapter 9

# Dynamics

Here we establish some bounds on the diameter of product-1 idèle class groups.

**Theorem 9.1:** Let $\mathbf{x} \in I_k^1/\mathcal{N}_k$. Then there is an explicitly computable constant $M$ such that

$$\min_{\alpha \in \mathcal{G}_k} h(\mathbf{x}\alpha^{-1}) \leq M.$$

If $k$ is a number field of degree $n = [k : \mathbb{Q}]$ with discriminant $D_k$ we can simplify the above bound to

$$\min_{\alpha \in \mathcal{G}_k} h(\mathbf{x}\alpha^{-1}) \leq \frac{1}{2}\left(\log |D_k| + M'\right)$$

where $D_k$ is the discriminant of $k$ and $M'$ depends only on the units of $k$.

*Proof.* Let $r = |S_\infty|$ be the number of infinite places of $k$, $\varepsilon_1, \ldots, \varepsilon_{r-1}$ to be multiplicatively independent fundamental units of $k$, and $\Omega_\infty = \prod_{v \in S_\infty} k_v^\times$. Let $\mathbf{a}_i$, $1 \leq i \leq h$ be idèlic representatives for the distinct ideal classes of $k$. We know that $I_k^1/\Omega_\infty k^\times \cong C_k$, the class group of $k$, which is finite. Select representative idèles, $\mathbf{a}_1, \ldots, \mathbf{a}_h$ for each of the $h$ ideal classes.

As in [20, p. 93] we write the logarithm map

$$l : \begin{cases} \Omega_\infty \to \mathbb{R}^{r+1} \\ (x_{\infty_i}) \mapsto (\log |x_{\infty_i}|_{\infty_i}) \end{cases}$$

and set the standard fundamental domain of order $e = |\{\zeta \in k : \exists n \in \mathbb{N},\ \zeta^n = 1\}|$ for the quotient group $I_k^1/k^\times$ (see [20, Chapter V-4, Proposition 8]) to be

$$E_k = \bigcup_{j=1}^{h} \mathbf{a}_j l^{-1}\left(\sum_{i=1}^{r} c_i l(\varepsilon_i)\right) = \bigcup_{j=1}^{h} \mathbf{a}_j E_k^0, \qquad 0 \le c_i < 1.$$

We modify this fundamental domain in order to make it more centrally symmetric and denote it as

$$F_k = \bigcup_{j=1}^{h} \mathbf{a}_j l^{-1}\left(\sum_{i=1}^{r} c_i l(\varepsilon_i)\right) = \bigcup_{j=1}^{h} \mathbf{a}_j F_k^0, \qquad |c_i| \le \frac{1}{2}.$$

Setting $M_1 = \max_{1\le i\le h} h(\mathbf{a}_i)$ and $M_2 = \max_{1\le j\le r-1} h(\varepsilon_j)$, we have each $\mathbf{x} \in I_k^1$ differs from an element of $E$ by an element $\alpha \in k^\times$ by an element of height at most $M = M_1 + \frac{r-1}{2} M_2$, proving the general case.

For the number field case we use the classical Minkowski bound (see, for example [13, Chapter I-6, Exercise 3]) which states that each element of the ideal class group, $C_k$, of $k$ has an (integral) ideal, $\mathfrak{a}$, of norm at most

$$N(\mathfrak{a}) \le \left(\frac{4}{\pi}\right)^n \frac{n!}{n^n}\sqrt{|D_k|}.$$

We now assume, WLOG, that the representative idèles $\mathbf{a}_1, \ldots, \mathbf{a}_h$ associated to ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ representing the $h$ distinct equivalence classes of $C_k$, each satisfying the Minkowski bound. From the definition of the idèle height we see $h(\mathbf{a}_i) = \log(N(\mathfrak{a}_i))$. By considering proposition 6.1 we see that a fundamental domain, $F_k$, for $I_k^1/\mathcal{N}_k k^\times$ has height diameter equal to that for a fundamental domain for $I_k^1/k^\times$ with the heigh pseudo-metric, because all elements of $F_k$ differ from those in $F_k^0$ by norm-0 idèles.

By using the Stirling approximation for the factorial with error estimate–see [4, Chapter 1-9, Remark 1.15]–we have

$$\max_{1\le i\le h} h(\mathbf{a}_h) = \max_{1\le i\le h} \log N(\mathfrak{a}_i)$$
$$\le \frac{1}{2}\log|D_k| + \left(\log\left(\frac{4}{\pi}\right) - 1\right)n + \frac{1}{2}\log n + \frac{1}{2}\log(2\pi) + \frac{1}{12n}$$
$$\le \frac{1}{2}\log|D_k|$$

and this bound on the heights holds for all $n$. We conclude that we have the desired bound by setting $M' = 2(r-1)M_2$. □

**Remark 9.2:** It is classical that $\max\left\{\left(\frac{4}{\pi}\right)^n \frac{n!}{n^n}\right\} < 1$ occurs for small $n$ using the same Stirling approxmation method.

**Remark 9.3:** The number $M' = 2(r-1)M_2$ is the sharp choice for the bounding constant. This comes from the fact that the fundamental domain, $F_k^0$ is an $\ell^\infty$ box in the trace-0 subspace of $\mathbb{R}^r$ isomorphic to $\mathbb{R}^{r-1}$, and the classical inequalities for norms in $\mathbb{R}^n$ that for any $x \in \mathbb{R}^n$ we have $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$. The factor of 2 comes from the fact that our $\ell^1$ norm on $\mathbb{R}^{r-1}$ is given by *twice* the Weil height for the isometric properties of the embedding into $L^1(Y)$.

**Remark 9.4:** Since $\sqrt{|D_k|} \ge \left(\frac{\pi}{4}\right)^{n/2}\frac{n^n}{n!}$ (see [11, p. 120]), the bound in theorem 9.1 is of essentially the best quality we can hope, as given $0 < \epsilon < 1 - \frac{1}{2}\log\left(\frac{4}{\pi}\right)$ we have

$$\frac{1}{2}\log|D_k| - \left(1 - \log\left(\frac{4}{\pi}\right)\right)n \ge \left(1 - \frac{1}{2}\log\left(\frac{4}{\pi}\right) - \epsilon\right)n$$

when $n \gg 0$.

**Corollary 9.5:** If $k$ is a global field with PID integer ring $\mathcal{O}_k$, then with the notation and hypotheses of Theorem 9.1 and $R_k$ the regulator of the field, $k$, we have

$$\min_{\alpha \in \mathcal{G}_k} h(\mathbf{x}\alpha^{-1}) \leq \frac{6^{n-3}(n-1)!(n-2)^{n-2}n^{3n-8}R_k}{\log n}.$$

*Proof.* We first note that the discriminant factor can be ignored because when $\mathcal{O}_k$ is a PID, the number $M_1$ in the proof of the theorem is just 0, since 1 represents the identity ideal class. We may combine the classical estimate on heights of fundamental units in [18] and the trivial observation that $r-1 < n$ to obtain the bound on the number $M_2$ in Theorem 9.1 as

$$M_2 \leq 6^{n-3}(n-1)!(n-2)^{n-2}n^{3n-9}R_K(\log n)^{-1}.$$

□

**Remark 9.6:** Note that the corollary applies regardless of characteristic, despite the estimate involving the discriminant being only applicable in the number field case where we had the Minkowski estimate.

**Remark 9.7:** We can also eliminate the mysterious $M_1$ terms from the theorem if we are willing to use the points in $h_k^{-1}\mathcal{G}_k$–here $h_k$ is the class number of $k$– of points of $I_k^1/\mathcal{N}_k$ which are $h_k^{th}$ roots of those in $\mathcal{G}_k$, that is if $f_\alpha \in \mathcal{G}_k$, we allow functions from $I_k^1/\mathcal{N}_k$ of the form $\frac{1}{h}f_\alpha$. Then because the class group has order $h_k$, any ideal class is represented by some $\frac{1}{h}f_\alpha$ with $\alpha \in \mathcal{G}_k$, and we need only use the estimate for the diameter of $F_k^0$, which doesn't depend on the height of non-identity ideals, and so does not require estimation.

# Chapter 10

# Remarks and Conjectures

**Conjecture 10.1:** For any global field, $k$, there is a constant $c_k$ such that

$$\min_{1\le n\le N}\left\{\min_{\alpha\in\mathcal{G}_k}\|nf_{\mathbf{x}}-f_\alpha\|_{L^1(Y)}\right\}\le c_kN^{-1}$$

for almost every $\mathbf{x}$. That is to say there is some $1\le n\le N$ and $\alpha\in\mathcal{G}_k$ such that each element of $I_k^1/\left(\mathcal{N}_k\right)$

$$2h(\mathbf{x}^n\alpha^{-1})\le c_kN^{-1}$$

and since this differs from $I_k^1/k^\times$ only by norm-0 vectors, the same holds if $\mathbf{x}\in I_k^1$ and $\alpha$ is allowed to be any element of $k^\times$.

**Conjecture 10.2:** Let $k$ be a global field and $p(x)\in\mathbb{Z}[x]$ be a non-trivial polynomial such that $p(0)=0$. Let $\overline{\mathbf{x}}\in\left(I_k^1/\mathcal{N}_k\right)\Big/\mathcal{G}_k$ be a non-torsion point represented by $\mathbf{x}\in I_K^1/\left(\mathcal{N}_k\right)\subseteq L^1(Y)$. Then the sequence $\{p(n)\mathbf{x}\}$ is uniformly distributed modulo $\mathcal{G}_k$.

**Remark 10.3:** The main theorem of [8] leads us to believe both of the preceding conjectures to be true due to the isometry given in theorem 9.1.

Because we are able to prove, using only simple definitions, corollary 6.10 it would be of considerable interest to find a simple, topological property of a given $I_k$ which is equivalent to $k$ possessing the Northcott property–i.e. that the set of points in $k$ of bounded degree and height is finite. Clearly an analytic formulation is simply that $\sum_{h(\alpha)<T} 1 < \infty$ for all $T$, so the discreteness property, (B) as noted in chapter 6, is necessary but not sufficient and local compactness of $I_k$ is a bit too strong. Such a property would lead us to a necessary condition for counting relative densities of points as in the classical theory by considering summatory functions

$$\left(\sum_{\alpha\in S, h(\alpha)\leq x)} 1\right) \Big/ \left(\sum_{h(\alpha)\leq x} 1\right).$$

Examining lemma 5.8 we see that there are finite-sized blocks which can completely determine the entirety of the Galois action on $L^1(Y)$. Though these spaces are not all of $Y$ by simple dimension counting, it leads us to make

**Conjecture 10.4:** The representation of $G_k$ on $L^1(Y)$ given in chapter 5 is semi-simple.

A question raised by chapter 8 is as to the statistical properties of $Y$ when restricting to the conditional expectation relative to the $\sigma$-algebras described in that chapter. Do they have some sort of Martingale-esque properties?

When $(X, \Sigma, \mu)$ is a measure space, the interpretation of the Radon-Nikodym derivative in $L^2(X, \Sigma, \mu)$ is literally a projection onto subspaces, a fact exploited in the work of Fili and Miner in [7] where they considered $L^2(Y)$ projection operators. In the $L^1(Y)$ case where we have shown the Radon-Nikodym derivative of an element relative to a $\sigma$-algebra of a smaller field as described in chapter 8, can we utilize this result to examine how some points in $L^1(Y)$ which are not already given by algebraic numbers or idèles might be distinguished as "more algebraic" by seeing which give projections which are algebraic?

In [1] the authors use a theorem of Krein, Milman, and Rutman [12] described in appendix E to show that the closure of $\mathcal{G}_k$ has a (Schauder) basis (see appendix E) from $\mathcal{G}_k$. The results of chapter 5 seem to highlight radical elements of $\overline{k}$ as relevant for determining the action of $G_k$ on $L^1(Y)$. Is it so that a basis can be taken from these elements? If so, what is an appropriate choice? Is there one which is canonical rather than ad hoc, determined by an intrinsic algebraic property rather than brute force?

From the functional analysis side, the decomposition of $Y$ as in chapter 2 seems to indicate that there might be a natural choice for a basis of $L^1(Y)$ by examining the functions on $L^1(G_k)$ and using the fact that $\bigoplus_v L^1(G_k/D_v)$ is dense in $L^1(Y)$ and each of the $G_k/D_v$ is a compact space.

There are other completions of $I/\mathcal{N}$. One could, for example, complete with respect to a $p$-adic norm, rather than the Weil height, and find an infinite-dimensional representation of $G_k$. This would not longer have the advantage of using the familiar height function as a norm, but might be interesting from an algebraic standpoint.

# Appendices

# Appendix A

# Inverse limits

We begin with inverse limits. Say we have a collection of sets (groups, rings, et cetera), and we would like to produce an object which behaves like an intersection ought to, i.e. something where there's some sort of consistency condition required so that if something in one of the $X_i$ has to do the same stuff if it has a clone in $X_j$ and where we're being *really* strict in that we are *only* considering elements which are all "in" each of the $X_i$. This motivates the following

**Definition A.1:** Let $\{X_i\}_{i \in I}$ be a collection of objects where the indexing set, $I$, has a partial ordering, $\leq$. Further assume we have a collection of morphisms–i.e. group homomorphisms if the $X_i$ are groups, continuous functions if the $X_i$ are topological spaces, et cetera–$f_{ij} : X_j \to X_i$ whenever $j \geq i$ subject to two conditions

(i) $f_{ii} = \mathrm{id}_{X_i}$

(ii) $f_{ij} \circ f_{jk} = f_{ik}$ (whenever this makes sense, i.e. $i \leq j \leq k$)

We call such a collection, $(\{X_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ an *inverse system* or sometimes a *projective system.* We further define the inverse (projective) limit of this system to be the set/group/topological space/ring/et cetera defined by the formula

$$X = \varprojlim_{i \in I} X_i = \{(x_i)_{i \in I} \in \prod_{i \in I} X_i : f_{ij}(x_j) = x_i \, \forall i \leq j\}.$$

If the sets $X_i$ are topological spaces, then the space $X$ is given the inverse limit topology, which is the coarsest topology so that the natural projection maps, $\pi_i : X \to X_i$ which picks out the $i$th component of an element of $X$ is a continuous function. Similarly if the $X_i$ are groups, the group structure on $X$ is given as inherited from the product group, $\prod X_i$.

We observe that the set $X$ is non-empty if all of the $X_i$ are, because directly from its definition we can see that natural projection maps, $\pi_i$, each surject onto $X_i$, so that the cardinality of $X$ satisfies $|X| \geq |X_i|$ for each $i \in I$.

**Proposition A.2:** If all the $X_i$ are compact Hausdorff spaces, then so is $X = \varprojlim_{i \in I} X_i$.

*Proof.* First we observe that, $X$ is Hausdorff. If $(x_i) \neq (y_i)$ then there is some $i_0 \in I$ such that $x_{i_0} \neq y_{i_0}$. Then lifting a basic open set around $x_{i_0}$ and one around $y_{i_0}$ to the inverse limit (recall all the canonical projections are continuous by definition of the inverse limit topology) we get an open set around $(x_i)$ which does not contain $(y_i)$.

We proceed to compactness. By Tychonov's theorem we can see the product $\prod_{i \in I} X_i$ is compact. But then with the demand that all the $f_{ij}$ be continuous (recall the $f_{ij}$ are morphisms in whatever category they sit in, since we are considering $X_i$ as topological spaces, this automatically requires we need the $f_{ij}$ to be continuous maps) the conditions $\{f_{ij}(x_j) = x_i\}_{i \leq j}$ are all closed conditions, so imposing all of them is an intersection of closed sets, hence a closed subset of the product, implying that the inverse limit is indeed compact. □

**Remark A.3:** It's not often mentioned, but I think it merits saying out loud at least once: the inverse limit philosophy is that of an universal statement. I.e. to get a property for an inverse limit you want all of its constituents to share that property. This was true for compactness, non-emptiness, and Hausdorff condition, for some basic examples. We don't use the property that inverse limits are left-exact functors, but the proof of this is in [13, Proposition 2.7 ] and one can easily observe that it involves similar "for all" statements. For sets contained in a common universe, the inverse limit is the intersection, and this truly solidifies the analogy, as any beginning mathematics undergrad should be able to tell you, proving a point is in the intersection of sets is the same as showing it is in *all* of the sets. The dual philosophy will be true for direct (inductive) limits, where the analogy is with the union and the philosophy parallels *existential* statements. This is reflected well in the following.

**Definition A.4:** If $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ is an inverse system with (inverse) limit, $X$, and collections of morphisms, $\{g_i : X_i \to Y\}_{i\in I}$ and $\{g'_i : Z \to X_i$ are given such that whenever $i \leq j$ we have

$$\begin{cases} f_{ij} \circ g_i = g_j \\ g'_i \circ f_{ij} = g'_j \end{cases} .$$

When we have maps

$$\begin{cases} g : X \to Y \\ g' : Z \to X \end{cases}$$

defined by the rules

$$g = \left( \prod_{i \in I} g_i \right) \Big|_X : X \longrightarrow Y \quad , \quad g' = \left( \prod_{i \in I} g'_i \right) : Z \longrightarrow X$$

we call the map $g$ (respectively $g'$) a map *into* (respectively *out of*) a direct limit.

It is readily checked that these are well-defined notions and produce morphisms of the appropriate type, i.e. continuous if the $X_i$ are topological spaces, group theoretic if the $X_i$ are groups, et cetera. It is worth noting that this means that providing a maps into and out of an inverse limit object is simply an exercise in providing a map into or out of each of them in a way consistent with the structural morphisms, $\{f_{ij}\}_{i \leq j}$. Equally important is to see that any map into or out of an inverse limit is of this type, by Definition A.1.

# Appendix B

# Direct limits

Again, we find ourselves in possession of a collection of sets (groups, topological spaces, et cetera), and we would like to create an object$\{X_i\}_{i\in I}$ where, again we're looking for consistency, but this time we want something more like a *union*, so that things should still behave the same if some element $x \in X_i$ has a clone in $X_j$, but we're being looser and not looking at stuff that has a copy in *all* of the $X_i$, rather it only needs to exist in *one* of them. This motives the idea of a direct limit.

**Definition B.1:** Let $\{X_i\}_{i\in I}$ again be a collection of objects, and let there be a partial ordering, $\leq$ on $I$, and morphisms $f_{ij} : X_i \to X_j$ when $i \leq j$ (note the direction of the inequality has changed from the previous section) satisfying the twin conditions

(i) $f_{ii} = \mathrm{id}_{X_i}$

(ii) $f_{jk} \circ f_{ij} = f_{jk}$ (whenever this makes sense, i.e. $i \leq j \leq k$)

then we call this pairing $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ a *direct system* or an *inductive system.* The *direct limit* (*inductive limit*) of this system is

$$X = \varinjlim_{i\in I} X_i = \coprod_{i\in I} X_i / \sim$$

where two points $x \in X_i$ and $y \in X_j$ are equivalent if $i \leq j$ and $f_{ij}(x_i) = x_j$. This set is manifestly non-empty, if *any* of the $X_i$ are.

**Remark B.2:** One should think of the inductive limit as a sort of union, when union isn't directly available, i.e. there is no obvious universe set in which all of the other sets sit, and it is not strictly necessary that there are actual inclusions, simply the maps $f_{ij}$ described in the structure of the inductive system. Again we have an exactness condition as in A.3 (see [13, Proposition 2.6 ] for a statement and proof) but just as existential statements are more relaxed than universal ones, we find this is sufficient for us to drop the compactness requirement that was needed for inverse limits, and indeed the proof of the proposition follows directly from specific "there exists" statements and some basic diagram pushing. We won't have need for this in the present work, but it is meant to illustrate some of the differences between inverse and direct limits so that the reader can get a better feel for them. We conclude with

**Definition B.3:** Let $(\{X_i\}_{i\in I}, \{f_{ij}\}_{i\leq j})$ be a direct system with (direct) limit $X$. Let $\{h_i : X_i \to Y\}_{i\in I}$ and $\{h'_i : Z \to X_i\}_{i\in I}$ be collections of morphisms such that whenever $i \leq j$

$$\begin{cases} h_j \circ f_{ij} = h_i \\ f_{ij} \circ h'_i = h'_j \end{cases} .$$

When we have maps

$$\begin{cases} h : X \to Y \\ h' : Z \to X \end{cases}$$

defined by the rules

$$h = \left( \coprod_{i \in I} h_i \right) \Big/ \sim : X \longrightarrow Y \quad , \quad h' = \left( \coprod_{i \in I} h_i' \right) : Z \longrightarrow X$$

i.e. for $x \in X$, find some $X_i \ni x'$ so that the canonical projection $\pi : \coprod_{i \in} X_i \to X$ has $\pi(x') = x$, then $h(x)$ is defined to be $h_i(x')$, and $h'$ requires no further information. We call respectively $h$ (respectively $h'$) a map out of (respectively *into*) the direct limit.

It is checked from the definitions that $h$ and $h'$ are morphisms, just as in the case of the inverse limit. As was the case with the inverse limit, any map into or out of a direct limit is merely a collection of maps consistent with the structural morphisms, $f_{ij}$ by Definition B.1.

# Appendix C

# Idèles

We use the topology and algebra of idèle groups heavily in our work here, so we recall some basic facts about them. Our treatment mostly follows that of [20, Chapter IV ], expanding or omitting as necessary for our application. The notation for the idèle group of a given field, $k$, will be $I_k$–rather than $k_{\mathbb{A}}^{\times}$, as in [20]–for the sake of simplifying the notation in the sequel where the $I_k$ provides the necessary flexibility of notation for our applications.

Given a global field, $k$, there is an associated ring of adeles, $k_{\mathbb{A}}$, which is canonically the inductive (i.e. direct) limit of the sets

$$k(P) = \prod_{v \in P} k_v \times \prod_{v \notin P} \mathcal{O}_v$$

where $P$ is any finite set containing the (possibly empty) set $P_\infty$ of all archimedean places of $k$, $k_v$ is the local field associated to the place $v$–i.e. the completion of $k$ at $v$, $\mathcal{O}_v = \{x \in k_v : |x|_v \leq 1\}$ is the local integer ring determined by the place $v$, and the limit is taken relative to the canonical injections induced by inclusions of the indexing sets, $P$–i.e. $k(P) \subseteq k(Q) \iff P \subseteq Q$. It is easiest to think of this limit, as was true for the set $I$ in the introduction, as the set-theoretic *union* of vectors $\mathbf{x} = (x_v)_v$ where for each place $v$ of the global field, we have the $v$-local component, which is an element of $k_v$. Our emphasis on the "limit" terminology is to reinforce that this is a topological space and not simply a set. This is sometimes referred to as the "restricted" direct product of the $k_v$ relative to the compact subgroups, $\mathcal{O}_k$. The "restriction" in the direct product means that we only an element to have finitely many coordinates, $x_v$, which are not elements of the integer ring, $\mathcal{O}_v$, though infinitely many components could still be non-zero. It is well-known that $k$ canonically sits inside of $k_{\mathbb{A}}$ diagonally as a lattice–i.e. a discrete, co-compact subset [20, Theorem 2, Chapter IV-2]. The embedding being very direct

$$\begin{cases} k \longrightarrow k_{\mathbb{A}} \\ \alpha \mapsto (\alpha)_v \end{cases}$$

where we just repeat $\alpha$ at each place $v$. Because the group of fractional ideals of $k$, $J(k)$, is a free abelian group on the prime ideals of $k$, it is readily seen that there is a (unique) factorization

$$(\alpha) = \prod_{i=1}^{n} \mathfrak{p}_i^{e_i}, \quad e_i \in \mathbb{Z} \qquad (*)$$

and so $|\alpha|_v = 1$ for all but finitely many places $v$, since there are finitely many archimedean places, and only finitely many non-archimedean places where it is possible $|\alpha|_v > 1$, namely those places associated to primes, $\mathfrak{p}_i$ where $e_i < 0$ in the factorization $(*)$, so that the claimed map actually does land inside of $k_{\mathbb{A}}$.

With the topology and algebraic structure induced by restricted direct product relative to the compact subrings, $\mathcal{O}_v$, we see that $k_{\mathbb{A}}$ is a locally compact topological ring, with operations defined componentwise.

Inside of $k_{\mathbb{A}}$ is the multiplicative group, $I_k = k_{\mathbb{A}}^{\times}$, of idèles: the invertible elements of $k_{\mathbb{A}}$. With the given topology on $k_{\mathbb{A}}$, $I_k$ is *not* a topological group with the subset topology.

It is readily seen that inversion is not a continuous operation in this topology, for example. To remedy this one gives $I_k$ the coarsest topology so that it inversion is also a continuous operation. Most directly this can be done via the injection

$$\begin{cases} I_k \to k_{\mathbb{A}} \times k_{\mathbb{A}} \\ \mathbf{x} \mapsto (\mathbf{x}, \mathbf{x}^{-1}) \end{cases}$$

identifying $I_k$ with its image and giving this image the subset topology.

A more topologically and computationally useful means to achieve this is to see that $I_k$ also has a canonical decomposition as a restricted direct product (of topological groups) and looks itself like the inductive limit of sets

$$k(P)^\times = \prod_{v \in P} k_v^\times \times \prod_{v \notin P} \mathcal{O}_v^\times$$

where again $P$ runs over the finite sets containing the possibly empty set, $P_\infty$, of archimedean primes, and the maps are the same, canonical inclusions as with $k_{\mathbb{A}}$. Here $\mathcal{O}_k^\times$ is the group of $v$-units, and as a *set* is equal to $\{x \in k_v : |x|_v = 1\}$. As was the case for the underlying set object, $I$, from the introduction and for $k_{\mathbb{A}}$ this can be thought of as the *union* of the sets, $k(P)^\times$, so that a typical element is $\mathbf{x} = (x_v)_v$ with (1) all components $x_v$ invertible elements of the local fields $k_v$ and (2) for all but finitely many $v$, we have $x_v \in \mathcal{O}_v^\times$.

It is directly seen that this is *necessary* to ensure that $I_k$ carries the structure of a locally compact topological group. That it is locally compact is immediate from the definitions, because the $\mathcal{O}_v^\times$ are compact groups from classical algebraic number theory. To see that using $\mathcal{O}_v \setminus \{0\}$ is insufficient for the group structure is because inversion on the this set can produce an element which has infinitely many components in $k_v^\times \setminus \mathcal{O}_v$. Take, for example, an adele, $(x_v)$ where at every non-archimedean place, we choose some local uniformizing parameter, $\pi_v = x_v$ and just let $x_v = 1$ at all the archimedean places. It is immediate that $x_v$ is in $\mathcal{O}_v \setminus \{0\}$ for all the non-archimedean places, so that $(x_v)$ is a well-defined adele, however the inverse of this element is $(1, 1, \ldots, 1)_{v|\infty} \times (\pi_v^{-1})_{v\not|\infty}$, so that at every one of the infinitely many non-archimedean places, $(x_v)$ has absolute value greater than 0, hence this object is not an adele at all, which destroys the topological group structure.

For an inclusion of global fields, $k \subseteq K$, there are natural injections $k_\mathbb{A} \to K_\mathbb{A}$ and $I_k \to I_K$ defined by taking $\mathbf{x} = (x_v)$ and mapping it to $(x_v)_w$ where $x_v$ is repeated at all $w|v$ places of $K$, i.e.

$$k_\mathbb{A} \to K_\mathbb{A} \quad , \quad I_k \to I_K$$
$$(x_v) \mapsto \prod_v \left( \prod_{w|v} (x_v)_w \right)$$

Both inclusions are continuous ring or group homomorphisms in their respective topologies.

We know here that if we start off with an idèle, $\mathbf{x}$, given in coordinates by $(x_v)$, the standard product

$$|\mathbf{x}|_\mathbb{A} = \prod_v |x_v|_v$$

called the *volume* or *module* of $\mathbf{x}$, is a well-defined map, trivial on $k^\times$ which establishes $k^\times$ as a discrete subset of $k_\mathbb{A}^\times$ and gives us a decomposition of $I_k$ as a product isomorphic to

$$I_k \cong R \times I_k^1$$

with $I_k^1 = \ker |\cdot|_{\mathbb{A}}$. [20, Theorem 6, Chapter IV-4 ] uses the notation $N$ in place of $R$, and gives that the structure of $R$ is either $\mathbb{R}$ or $\mathbb{Z}$ depending on whether $k$ is of characteristic 0 or not. We choose not to use $N$ because we have plans for that letter as a central object for the main theorems. We think of this decomposition as a sort of "polar coordinates" on $I_k$, justifying our use of the letter $R$ for the "radial" component of the idèle group and $I_k^1$ as the "directional" component.

By [20, Theorem 6, Chapter IV-4 ], we have that $k^\times \subseteq I_k^1$ is cocompact, and so along with discreteness, we have that $k^\times$ is a lattice in $I_k^1$. Since the canonical topological group projection is an open mapping and $I_k = I_k^1 R$ is an internal direct product, we have–by virtue of the ideal map sending $a \in k^\times$ to $\mathrm{id}(a)$ which is an idèle $\mathbf{x} = (x_v)$ with its infinite component, $\varphi_\infty(\mathbf{x}) = (1, 1, \ldots 1)$, everywhere equal to 1 and its finite component, $\varphi_0(\mathbf{x}) = \left(\pi_v^{-v(a)}\right)$, which is 1 in each entry for which the associated additive valuation $v$ is trivial on $a$ and each $\pi_v$ is a uniformizing parameter for $\mathcal{O}_k$. It is classical that $\varphi_0(I_k)$ is canonically isomorphic–to the ideal group of $k$ and that the image of $k^\times$ in this group via the ideal map is canonically isomorphic to the subgroup of principal, fractional ideals.

If we denote by

$$\Omega_\infty = \prod_{v|\infty} k_v^\times \times \prod_{v\nmid\infty} \mathcal{O}_v^\times$$

the open subset of $I_k$ which has only infinite components with non-units, we have the celebrated isomorphism

$$I_k / \Omega_\infty k^\times \cong C_k$$

with $C_k$ the class group of $k$, which is finite, a fact we will make use of in chapter 9 in forming the fundamental domain.

# Appendix D

# Heights

In this section we review some of the most basic properties of standard heights on global fields. A far more complete reference would be [3, Chapter 1-2 ]. We assume, WLOG, that all fields in this section are global fields, which are number fields in the characteristic 0 and finitely generated extensions of $\mathbb{F}_p$ of transcendence degree 1 over $\mathbb{F}_p$ [20, Definition 1, Chapter III-1] (Weil uses the term $\mathbb{A}$-fields).

For the rest of this section, we fix a global field, $k$. Here we consider algebraic elements of an arbitrary, but *fixed*, algebraic closure of $k$, $\overline{k}/k$. Later we will extend our consideration to a new object which behaves like an idèle group for all of $\overline{k}$. This will also give us a definition for the height of fractional ideals of such $k$ and other important objects in number theory as corollaries.

For the purposes of this work, we will consider two sets of absolute values on a finite extension, $K/k$

(i) The auxiliary absolute values, $\|\cdot\|_v$, are the absolute values on $K$, associated to the places, $v$ of $K$, which are extensions of the usual absolute values on the base field, $\mathbb{Q}$ or $\mathbb{F}_p(t)$.

(ii) The main absolute values, $|\cdot|_v$, are defined as $\|\cdot\|^p$ with $p = [K_v : k_v]/[K : k]$ are normalized by local field extension degree divided by the global field extension degree in order to satisfy the product formula

$$\prod_v |\alpha|_v = 1$$

for any non-zero $\alpha \in \overline{k}$. The set of all absolute values of type (ii) are "well-behaved" in the sense of [10] when the extension is separable over the base field; in this case we have the fundamental relation:

$$\sum_{w|v} [L_w : K_v] = [L : K].$$

For inseparable extensions, one writes the extension as a tower of separable and purely inseparable extensions, and proves the product formula in this case through a standard argument.

We recall a real-valued function $f$ has a unique decomposition as $f^+ - f^-$ with $f^+ = \max\{f, 0\}$ and $f^- = \max\{0, -f\}$. By definition the (absolute, logarithmic) Weil height of an algebraic element $\alpha \in \overline{k}^\times$ is defined by first selecting any global field $K$ containing $\alpha$ and then computing the sum

$$h(\alpha) = \sum_v \log^+ |\alpha|_v \qquad (*)$$

where the sum is on all places of $v$ of $K$. Because $K^\times \subseteq I_K$ for any global field, $K$, we know that, for almost all $v$, $|\alpha|_v = 1$ so the sum $(*)$ is actually finite. It is well-known and easily-checked from our normalization of the absolute values that this definition does not depend on the choice of $K$ containing $\alpha$, so this implies naturally that $h$ is a well-defined map from $\overline{k}^\times$ to $[0,\infty)$. The three most central properties of the height are

1. $h(\alpha) = 0 \iff \alpha \in \mathrm{Tor}(\overline{k}^\times)$ (i.e. iff $\alpha$ is a root of unity)

2. $h(\alpha\beta^{-1}) \leq h(\alpha) + h(\beta)$

3. $h(\alpha^n) = |n| h(\alpha)$ for every $n \in \mathbb{Z}$

It is easy to prove (2) and (3) directly from the definition of the height and the logarithm's functional equation. (1) follows immediately from Northcott's theorem in the case of a number field (see [14]) using (2) and (3): the ($\Rightarrow$) direction by an application of Northcott and the ($\Leftarrow$) by noting that $\alpha^n = 1$ means that $h(\alpha)^n \leq |n|h(\alpha)$ implies all powers of a root of unity have height bounded by $nh(1) = 0$. In the function field case (1) follows from [3, Example 1.5.23, p.21 ], since the only polynomials with no poles in projective space are the constants, i.e. elements of $\overline{\mathbb{F}}_p^\times$. With this is makes $h$ into a pseudo-norm on $\overline{k}^\times$ with its group operation. If we project out the torsion subgroup of $\overline{k}^\times$ we produce the quotient group, $\overline{k}^\times/\mathrm{Tor}(\overline{k}^\times)$, which we will henceforth refer to simply as $\mathcal{G}_k$. We claim the map $h$ becomes a well-defined archimedean norm on $\mathcal{G}_k$ treated as a $\mathbb{Z}$-module.

To check well-definition of $h$ as a norm, we note that the identity $\log^+ |\alpha\beta|_v = (\log |\alpha\beta|_v)^+ = (\log |\alpha|_v + \log |\beta|_v)^+$ holds so that if $\zeta$ is a root of unity, $\log^+(\alpha\zeta) = \log^+(\alpha)$, so that $h$ is independent of choice of coset representative. Further, this shows that if $x^s = \alpha^r$ then

$$|s|h(x) = |r|h(\alpha) \iff h(x) = \left|\frac{r}{s}\right| h(\alpha). \qquad (*)$$

This leads us to conclude the following

**Proposition D.1:** Let $k$ be a global field with algebraic closure, $\overline{k}$. Then $\mathcal{G}_k$ naturally has the structure of a normed $\mathbb{Q}$-vector space.

*Proof.* We have already proven $\mathcal{G}_k$ has the structure of a normed $\mathbb{Z}$-module. The proof that $h$ is a well-defined norm establishes, as a corollary that the $\mathbb{Q}$-action is well-defined and compatible with the height norm, which completes the proof of the extension to of scalars. $\square$

We will find it useful to note the equivalent definition of "height" as it avoids the somewhat awkward properties of the $\log^+$ function in favor of the more classical functional equation and properties of the log function.

**Proposition D.2:** Let $k$ be a global field and let $\alpha \in \overline{k}^\times$ be a non-zero algebraic element over $k$. Then the following formula holds

$$h(\alpha) = \frac{1}{2} \sum_v \left|\log |\alpha|_v\right|.$$

*Proof.* By the product formula and the finite number of $v$ such that $|\alpha|_v \neq 1$ we have

$$\sum_v \log^+ |\alpha|_v - \sum_v \log^- |\alpha_v| = \log\left(\prod_v |\alpha|_v\right) = 0.$$

As $|f| = f^+ + f^-$ for all functions, $f$, the result follows. $\square$

# Appendix E

# Functional Analysis

**Theorem E.1** (Radon-Nikodym)**:** Given a measure space $(X, \Sigma)$ with $\sigma$-finite measure $\mu$ absolutely continuous relative to another $\sigma$-finite measure $\nu$ on $(X, \Sigma)$, then there is a non-negative, measurable function, $f : X \to \mathbb{R}$ such that for all $S \in \Sigma$

$$\mu(S) = \int_A f \, d\nu.$$

**Theorem E.2** (Hahn-Banach)**:** Set $F = \mathbb{R}$ or $\mathbb{C}$ and let $V$ be a normed vector space over $F$. Let $K \subseteq V$ be compact and $H \subseteq V$ be closed such that $K, H$ are convex and $K \cap H = \varnothing$. Then there is a continuous, linear map $T : V \to F$ and $s, t \in \mathbb{R}$ such that $\text{Re}(T(x)) < t < s < \text{Re}(T(y))$ for all $x \in K, y \in H$.

**Theorem E.3** (Riesz Representation)**:** Let $X$ be a locally compact, Hausdorff space. For any positive linear functional $\varphi$ on $C_c(X)$ there is a *unique* regular Borel measure $\mu$ on $X$ such that

$$\phi(f) = \int_X f \, d\mu$$

for all $f \in C_c(X)$.

**Definition E.4:** We define a collection $\varnothing \subset S \subseteq \mathcal{P}(X)$ of subsets of $X$ to be a $\pi$-system if for each $A, B \in S$ we have $A \cap B \in S$.

**Definition E.5:** We define a collection $S \subseteq \mathcal{P}(X)$ of subsets of $X$ to be a $\lambda$-system if it possesses the following three properties:

(i) $X \in S$

(ii) If $A, B \in S$ with $A \subseteq B$ then $B \setminus A \in S$

(iii) if $\{A_n\}$ is a nested sequence of elements of $S$, then $\lim_{n\to\infty} A_n = \bigcup_n A_n \in S$

**Theorem E.6** ($\lambda - \pi$)**:** If $\Lambda$ is a $\lambda$ system containing a $\pi$-system, $\Pi$, then $\Lambda$ also contains $\sigma(\Pi)$, the $\sigma$-algebra generated by $\Pi$ .

**Definition E.7** (Schauder Basis)**:** A countable subset $\{\mathbf{e}_n\}$ of a Banach space, $X$ over a field $\mathbb{F}$–either $\mathbb{C}$ or $\mathbb{R}$, is said to be a *Schauder basis* of $X$ if for all $\mathbf{v} \in X$ we have that there is a *unique* sequence $c_n \in \mathbb{F}$ such that $\lim_{N\to\infty} \left\| \sum_{n=1}^{N} c_n \mathbf{e}_n - \mathbf{v} \right\| = 0$.

**Theorem E.8** (Krein, Milman, Rutman)**:** If $X$ is a separable Banach space with dense, countable subset $Q$, then there exists a Schauder basis of $X$ of elements of $Q$.

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# Bibliography


[1] Daniel Allcock and Jeffrey D. Vaaler. A Banach space determined by the Weil height. *Acta Arithmetica*, 136:279–298, 2009.

[2] A. Baker. Linear forms in the logarithms of algebraic numbers (iv). *Mathematika*, 15(2):204–216, dec 1968.

[3] E. Bombieri and W. Gubler. *Heights in Diophantine Geometry*. New Mathematical Monographs. Cambridge University Press, 2007.

[4] J.M. de Koninck and F. Luca. *Analytic Number Theory: Exploring the Anatomy of Integers*. Graduate studies in mathematics. American Mathematical Soc.

[5] J. Diestel and A. Spalsbury. *The Joys of Haar Measure*. Graduate studies in mathematics. Amer Mathematical Society, 2014.

[6] Arash Ghaani Farashahi. Convolution and Involution on Function Spaces of Homogeneous Spaces. *Bulletin of the Malaysian Mathematical Sciences Society*, 36:1109–1122, 2013.

[7] Paul Fili and Zachary Miner. Orthogonal decomposition of the space of algebraic numbers and lehmers problem. *Journal of Number Theory*, 133(11):3941–3981, 2013.

[8] Michael Kelly and Thái Hoàng Lê. Uniform dilations in higher dimensions. *Journal of the London Mathematical Society*, 88(3):925–940, 2013.

[9] S. Lang. *Introduction to Transcendental Numbers.* Addison-Wesley series in mathematics. Addison-Wesley Pub. Co., 1966.

[10] S. Lang. *Fundamentals of Diophantine Geometry.* Springer, 1983.

[11] S. Lang. *Algebraic Number Theory.* Applied Mathematical Sciences. Springer, 1994.

[12] D. Milman M. Krein and D. Rutman. A note on bases in Banach space. *Comm. Inst. Sci. Math. Méc. Univ. Kharkoff*, 16:106–110, 1940.

[13] J. Neukirch. *Algebraic Number Theory.* Grundlehren der mathematischen Wissenschaften. Springer, 2010.

[14] D. G. Northcott. An inequality in the theory of arithmetic on algebraic varieties. *Proc. Cambridge Philos. Soc.*, 45:502–509, 1949.

[15] H. Reiter and J.D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups.* London Mathematical Society monographs. Clarendon Press, 2000.

[16] W. Rudin. *Real and complex analysis.* Mathematics series. McGraw-Hill, 1987.

[17] J.P. Serre. *Galois Cohomology.* Eso Astrophysics Symposia. Springer Berlin Heidelberg, 2002.

[18] Carl Ludwig Siegel. Abschätzung von Einheiten. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II*, 1969:71–86, 1969.

[19] Jeffrey D. Vaaler. Heights on groups and small multiplicative dependencies. *Trans. Amer. Math. Soc.*, 366(6):3295–3323, 2014.

[20] A. Weil. *Basic Number Theory.* Classics in Mathematics. Springer London, Limited, 1995.

[21] R.L. Wheeden and A. Zygmund. *Measure and Integral: An Introduction to Real Analysis*. Chapman & Hall/CRC Pure and Applied Mathematics. Taylor & Francis, 1977.

# Vita

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