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Monodromies of Torsion D-Branes on Calabi-Yau Manifolds: Extending the Douglas, et al., Program

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Monodromies of Torsion D-Branes on Calabi-Yau Manifolds: Extending the Douglas, et al., Program

by

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Dedicated to my father, Swadesh Mahajan, who taught me to question everything, and who always had the highest expectations for me.

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Monodromies of Torsion D-Branes on Calabi-Yau Manifolds: Extending the Douglas, et al., Program

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Attempts are made to extend to torsion D-branes the Douglas, et al. program of trying to understand questions like the existence of lines of marginal stability by comparing D-branes at different points in the moduli space of (2,2) compactifications through use of monodromies, the intersection form, and a general comparison between exact worldsheet descriptions at the Gepner point and exact geometric descriptions at large volume. The extension made is far from complete, but is not insubstantial.

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Chapter 1

General Background on D-Branes

1.1 Origins

D-branes have come a long way from their humble beginnings in Austin, Texas. They were first discovered in 1989 by Dai, Leigh, and Polchinski [3] while exploring the $R \leftrightarrow \alpha'/R$ T-duality of closed string compactifications on a circle. This duality, which interchanges winding and momentum modes, acts oppositely on left- and right-moving modes on the string ($X_L \rightarrow X_L, X_R \rightarrow -X_R$). When applied to a system containing open strings in addition to the closed, the Neumann boundary conditions

$$\partial_\sigma X_{L\mu} = \partial_\sigma X_{R\mu} \tag{1.1}$$

become Dirichlet boundary conditions

$$\partial_\sigma X_{L\mu} + \partial_\sigma X_{R\mu} = \partial_\sigma X_\mu = 0 \tag{1.2}$$

where the subscript σ indicates that the derivative is with respect to the space coordinate on the worldsheet.

These boundary conditions imply the existence of a hyperplane to which the ends of the open string are "stuck" – the D-brane. Turning on a Wilson line (flat but topologically nontrivial background for the open-string gauge field) creates the possibility of multiple D-branes, with strings stretching between different ones. The massless excitations of these open strings correspond to the collective coordinates of the D-branes.

Even though it was clear from the beginning that D-branes were dynamical objects, and could bend and stretch in addition to translation (through the coupling to the closed-string graviton), they languished in undeserved obscurity until late 1995.

It was then, in the middle of the Second Superstring Revolution, that Polchinski discovered that D-branes were a necessary consequence of string duality. The perturbative spectrum of the type II superstring contains two kinds of gauge fields, one from the Neveu-Schwarz Neveu-Schwarz (NSNS) sector, and the other from the Ramond-Ramond (RR) sector, with respective vertex operators that look like

$$j\bar{\partial}X^{\mu}A_{\mu}(X) \tag{1.3}$$

$$\bar{Q}\Gamma^{[\mu_1}\ldots\Gamma^{\mu_n]}QF_{\mu_1\ldots\mu_n}(X). \tag{1.4}$$

where A is a gauge potential and F is a gauge field. Since there is no vertex operator in the RR sector coupling to the RR gauge potential, instead of the field, there are no RR-charged states in the perturbative string spectrum. Type II string duality, however, suggested that corresponding to the NSNS-charged states there must be RR-charged states.

In [5], Polchinski showed that D-branes are, in fact, the missing RR-charged states. There are many excellent reviews, e.g. [6],[7], of these basic facts.

This discovery then ignited an explosion of results, obtained by considering D-branes as geometrical objects – embedded submanifolds (that are also homology cycles) of the spacetime manifold – with worldvolume gauge theories (in particular, with a worldvolume Chan-Paton gauge bundle, a point to which we shall return). In particular, the worldvolume theories range from a U(1) theory for a single brane to U(n) for n stacked coincident identical branes to So(n) or Sp(n) for more exotic configurations involving both D-branes and orientifolds (similar, though non-dynamical, objects that simultaneously implemented a Z2 orbifold projection and a reversal of orientation on the worldsheet), with matter multiplets entering in a way also dictated by the geometry of the configuration.

Elaborations along these lines led to results as diverse as dramatic new formulations of Type IIB theory compactifications with a varying axion-dilaton and using spacetime geometry to shed light on nonperturbative features of supersymmetric gauge theories, including the $N = 2$ Seiberg-Witten results and the $N = 1$ Seiberg dualities. Most of these developments are beyond the scope of this work, although some will bear crucially, if tangentially, on it.

The natural view of D-brane charge at this point was as an element of cohomology – ignoring time for the moment, the charge of a point particle in an n-manifold is naturally expressed as an n-form which, when evaluated on an n-cycle containing the point particle, gives a number (what we usually think of as the charge). Similarly, for a p-dimensional extended object, the charge is an

$(n-p)$-form that, when evaluated on an $(n-p)$-cycle that "contains" the object by spanning all the transverse direction, gives a number. Since D-branes obey the Dirac quantization condition (see [5]), their charges are naturally thought of as elements of integral cohomology. A D-brane whose charge is a torsion element of that cohomology would be termed a torsion D-brane.

1.2 Boundary States

Parallel to this progression in the geometric view of D-branes there has been development of a formulation tied to the worldsheet picture. Although the analysis based on spacetime geometry proved tremendously illuminating, it seemed to go against the spirit of earlier discoveries. The great lesson of late-1980's string theory, from Gepner models to the final elucidation of mirror symmetry, was that what is important is not spacetime geometry, but spacetime geometry as seen by the string worldsheet, a very different beast - and that analysis on the worldsheet itself is not only most fruitful, but also most general, since string theories need not have a geometric interpretation.

This formulation involves the description of D-branes as *boundary states*, states in the closed-string Hilbert space that encode the boundary conditions felt by open strings.

1.2.1 Worldsheet Duality and Coherent States

The key to this formalism is the well known concept of worldsheet duality, a topological relationship between open-string and closed-string diagrams that involves an exchange of time and space coordinates on the worldsheet. As an example, consider the standard cylinder diagram – it represents either an open-string loop, with the string propagating in time through a circle, or a closed-string tree, with the string propagating in time in a straight line.

The interaction between two D-branes is given by the modes of fluctuation of an open string stretching between them – in perfect analogy with the Casimir effect, where the interaction between two conducting plates is given by the fluctuations of the electromagnetic field between them. This interaction, an open-string "free energy" (with appropriate boundary conditions), is again given by an annulus, or cylinder. Analyzing this in the closed-string channel gives again a closed string tree, but with the amplitude, instead of vacuum-vacuum, being between two "boundary states" representing the different D-branes.

Oddly, this formalism was created, and explicit formulas for boundary states obtained by Callan, et al., [10] in the simplest (free field) case, before D-branes were discovered. Our exposition will closely follow the general pattern and notation of [8][9].

For definiteness, consider a one-loop diagram with an open string circulating and stretching between two parallel Dp-branes with coordinates given by $(y^{p+1}, ..., y^{d-1})$ and $(w^{p+1}, ..., w^{d-1})$, resp. We'll start by considering the bosonic string.

The open string satisfies Neumann boundary conditions both at $\sigma = 0$

and $\sigma = \pi$ along the world-volume directions of the brane, and satisfies the following Dirichlet conditions in the transverse directions:

$$X^i|_{\sigma=0} = y^i \qquad X^i|_{\sigma=\pi} = w^i \qquad i = p+1, ..., d-1 \ , \tag{1.5}$$

where we take σ and τ in the two intervals $\sigma \in [0, \pi]$ and $\tau \in [0, T]$.

We need a conformal transformation taking the boundary conditions for the open string to those for a closed string propagating between the two Dp-branes. In terms of the complex coordinate $\zeta \equiv \sigma + i\tau$, a conformal transformation simply transforms $\zeta \to f(\zeta)$, where $f(\zeta)$ is an arbitrary holomorphic function of ζ. Consider the following conformal transformation

$$\zeta = \sigma + i\tau \to -i\zeta = \tau - i\sigma \ . \tag{1.6}$$

This transformation simply amounts to the inversion $\sigma \to -\sigma$ followed by exchanging σ with τ and vice versa

$$(\sigma, \tau) \to (\tau, \sigma) \ . \tag{1.7}$$

Finally in order to have the closed string variables σ and τ varying in the intervals $\sigma \in [0, \pi]$ and $\tau \in [0, \hat{T}]$ corresponding to a closed string propagating between the two D branes one must perform the following conformal rescaling

$$\sigma \to \frac{\pi}{T}\sigma \qquad \tau \to \frac{\pi}{T}\tau \ , \tag{1.8}$$

with $\hat{T} = \pi^2/T$.

The corresponding boundary conditions at $\tau = 0$ are:

$$\partial_\tau X^\alpha|_{\tau=0}|B_X\rangle = 0 \qquad \alpha = 0, ..., p \ , \tag{1.9}$$

$$X^i|_{\tau=0}|B_X\rangle = y^i \qquad i = p+1, ..., d-1 \ . \tag{1.10}$$

The previous equations can be rewritten in terms of closed-string oscillators by using the standard mode expansions, obtaining

$$(\alpha_n^\alpha + \widetilde{\alpha}_{-n}^\alpha)|B_X\rangle = 0 \ ; \ (\alpha_n^i - \widetilde{\alpha}_{-n}^i)|B_X\rangle = 0 \ \ \forall n \neq 0$$
$$\hat{p}^\alpha|B_X\rangle = 0 \qquad (\hat{q}^i - y^i)|B_X\rangle = 0 \ . \tag{1.11}$$

Introducing the matrix

$$S^{\mu\nu} = (\eta^{\alpha\beta}, -\delta^{ij}) \ , \tag{1.12}$$

the equations for the non-zero modes can be rewritten as

$$(\alpha_n^\mu + S^\mu{}_\nu \widetilde{\alpha}_{-n}^\nu)|B_X\rangle = 0 \qquad \forall \ n \neq 0 \ . \tag{1.13}$$

Callan, et al., showed that the natural prescription of constructing boundary states as coherent-state eigenstates of the boundary conditions is in fact that correct one. The state satisfying the previous equations is

$$|B_X\rangle = N_p \delta^{d-p-1}(\hat{q}^i - y^i) \left(\prod_{n=1}^{\infty} e^{-\frac{1}{n}\alpha_{-n} S \cdot \widetilde{\alpha}_{-n}} \right) |0\rangle_\alpha |0\rangle_{\widetilde{\alpha}} |p=0\rangle \ , \tag{1.14}$$

where N_p is a normalization constant to be fixed.

Since coherent states are notoriously unnormalizable, the construction of the inner product requires some care, and we will come back to it in a later section. The previous boundary state describes only the degrees of freedom corresponding to the string coordinate X. We also need a piece for the ghost degrees of freedom, so the full boundary state looks like

$$|B\rangle = |B_X\rangle |B_{gh}\rangle \ . \qquad (1.15)$$

BRST invariance requires that the total boundary state satisfy the equation

$$(Q + \widetilde{Q})|B\rangle = 0 \ , \qquad (1.16)$$

where the BRST charge is

$$Q = \sum_n c_n L^X_{-n} + \sum_{n=-1}^{\infty} c_{-n} L^{gh}_n + \sum_{n=2}^{\infty} L^{gh}_{-n} c_n \qquad (1.17)$$

$\tilde{Q}$ is given by an analogous expression in terms of the tilded variables. Eq.(1.11) implies that the boundary state for the bosonic coordinate satisfies the following:

$$(L^X_m - \widetilde{L}^X_{-m})|B_X\rangle = 0 \ . \qquad (1.18)$$

Inserting the expression for Q and the corresponding expression for $\tilde{Q}$ in (1.16) and using (1.18) we can see that (1.16) implies the following conditions for the ghost boundary state:

$$(c_n + \widetilde{c}_{-n})|B_{gh}\rangle = 0 \ ; \quad (b_n - \widetilde{b}_{-n})|B_{gh}\rangle = 0 \ . \qquad (1.19)$$

The second condition in the previous equation follows from the first and from the analogue of (1.18) for the ghost boundary state:

$$(L^{gh}_m - \widetilde{L}^{gh}_{-m})|B_{gh}\rangle = 0 \ . \qquad (1.20)$$

Eqs.(1.19) are satisfied by the state

$$|B_{gh}\rangle = e^{\sum_{n=1}^{\infty}(c_{-n}\widetilde{b}_{-n}-b_{-n}\widetilde{c}_{-n})}\left(\frac{c_0+\widetilde{c}_0}{2}\right)|q=1\rangle|\widetilde{q=1}\rangle, \qquad (1.21)$$

where $|q=1\rangle$ is the state that obeys the following conditions:

$$c_n|q=1\rangle = 0 \quad \forall n \geq 1; \quad ; \quad b_m|q=1\rangle = 0 \quad \forall m \geq 0\ . \qquad (1.22)$$

Of more direct relevance is the construction of boundary states in Type II theories, which add spin-$\frac{1}{2}$ fermions and superghosts to the above. We must determine $|B_\psi\rangle$ and $|B_{sgh}\rangle$.

The analogue of the Neumann boundary conditions in (**??**) is the set of equations

$$\begin{cases} \psi_-(0,\tau) = \eta_1\psi_+(0,\tau) \\ \psi_-(\pi,\tau) = \eta_2\psi_+(\pi,\tau) \end{cases} \qquad (1.23)$$

where η_1 and η_1 can be ± 1. Their relation determines the spin structure on the worldsheet - if $\eta_1 = \eta_2$, we get the Ramond sector and if $\eta_1 = -\eta_2$, we get the Neveu-Schwarz sector.

Just as with the bosonic modes, when we have Dirichlet boundary conditions, we get instead

$$\begin{cases} \psi_-^\mu(0,\tau) = \eta_1 S^\mu{}_\nu \psi_+{}^\nu(0,\tau) \\ \psi_-^\mu(\pi,\tau) = \eta_2 S^\mu{}_\nu \psi_+{}^\nu(\pi,\tau) \end{cases} \qquad (1.24)$$

where the matrix S is as previously defined.

In addition to the space periodicity of the fermions, we must, of course give the time periodicity. Again, we get a twofold choice:

$$\begin{cases} \psi_-(\sigma,0) = \eta_3\psi_-(\sigma,T) \\ \psi_+(\sigma,0) = \eta_4\psi_+(\sigma,T) \end{cases} \tag{1.25}$$

where η_3 and η_4 can take the values ± 1. A simple check of consistency between the two sets (space and time) of conditions shows that, in fact, $\eta_3 = \eta_4$

There is one additional complication in determining the fermionic boundary state. Since the coordinates ψ_+ (ψ_-) have conformal weight $h = 1/2$ ($\bar{h} = 1/2$), our worldsheet duality conformal transformation gives an extra factor of $\pm i^{1/2}$.

Putting it all together, we find first that since $\eta_3 = \eta_4$ and these are now the parameters that determine the spin structure for the closed strings (remember, we have interchanged space and time), the closed string boundary state contains only RR and NSNS sectors, and second that the conditions on the fermionic boundary state can be expressed as

$$\left(\psi^\mu_t - i\eta S^\mu{}_\nu \widetilde{\psi}^\nu_{-t}\right)|B_\psi,\eta\rangle = 0 \tag{1.26}$$

where the index t is integer (half-integer) in the RR (NSNS) sector.

The solution in the NSNS sector is straightforward, and just like the bosonic piece:

$$|B_\psi,\eta\rangle = -i\prod_{r=1/2}^{\infty}\left(e^{i\eta\psi_{-r}\cdot S\cdot\widetilde{\psi}_{-r}}\right)|0\rangle \tag{1.27}$$

The RR sector is almost identical, except for the zero-mode piece. If we write

$$|B_\psi,\eta\rangle = -\prod_{m=1}^{\infty} e^{i\eta\psi_{-m}\cdot S\cdot\widetilde{\psi}_{-m}}|B_\psi,\eta\rangle^{(0)}, \tag{1.28}$$

then the piece $|B_\psi, \eta\rangle^{(0)}$ must satisfy the condition

$$\left(\psi_0^\mu - i\eta S^\mu{}_\nu \widetilde{\psi}_0^\nu\right) |B_\psi, \eta\rangle^{(0)} = 0 \tag{1.29}$$

This condition is solved by

$$|B_\psi, \eta\rangle^{(0)} = \mathcal{M}_{AB} |A\rangle |\widetilde{B}\rangle \tag{1.30}$$

where

$$\mathcal{M}_{AB} = \left(C\Gamma^0 ... \Gamma^p \frac{1 + i\eta\Gamma^{11}}{1 + i\eta}\right)_{AB} \tag{1.31}$$

Here C is the charge conjugation matrix and the Γ^μ are Dirac matrices in 10-dimensional spacetime.

After going through an argument about BRST invariance similar to the case for the bosonic ghost, except that it involves the super-Virasoro generators, not just those of the Virasoro algebra, we get the formula for the superghost piece:

$$|B_{sgh}, \eta\rangle_{\mathrm{NS}} = \exp\left[i\eta \sum_{r=1/2}^{\infty} (\gamma_{-r}\tilde{\beta}_{-r} - \beta_{-r}\tilde{\gamma}_{-r})\right] |P = -1\rangle \, |\tilde{P} = -1\rangle \ , \tag{1.32}$$

in the NS sector in the $(-1, -1)$ picture and

$$|B_{sgh}, \eta\rangle_R = \exp\left[i\eta \sum_{m=1}^{\infty} (\gamma_{-m}\tilde{\beta}_{-m} - \beta_{-m}\tilde{\gamma}_{-m})\right] |B_{sgh}, \eta_R\rangle^{(0)} \ , \tag{1.33}$$

in the R sector in the $(-1/2, -3/2)$ picture. The superscript $^{(0)}$ denotes the zero-mode contribution that (where $|P = -1/2\rangle \, |\tilde{P} = -3/2\rangle$ denotes the superghost vacuum that is annihilated by β_0 and $\tilde{\gamma}_0$), is given by [11]

$$|B_{sgh}, \eta\rangle_R^{(0)} = \exp\left[i\eta\gamma_0\tilde{\beta}_0\right] \, |P = -1/2\rangle \, |\tilde{P} = -3/2\rangle \ . \tag{1.34}$$

These exact expressions are of particular importance because we are not done yet – in order to obtain the genuine boundary state we have to perform the GSO projection.

In the NSNS sector the GSO projected boundary state is

$$|B\rangle_{NS} \equiv 1 + (-1)^{\frac{F+G}{2}} \; 1 + (-1)^{\frac{\tilde{F}+\tilde{G}}{2}} \; |B,+\rangle_{NS}, \tag{1.35}$$

where F and G are the fermion and superghost number operators

$$F = \sum_{m=1/2}^{\infty} \psi_{-m} \cdot \psi_m - 1 \quad , \quad G = - \sum_{m=1/2}^{\infty} \left(\gamma_{-m}\beta_m + \beta_{-m}\gamma_m\right) \quad . \tag{1.36}$$

Their action on the boundary state corresponding to the fermionic coordinate ψ and to the superghosts is as follows:

$$(-1)^F|B_\psi,\eta\rangle = -|B_\psi,-\eta\rangle \quad ; \quad (-1)^{\tilde{F}}||B_\psi,\eta\rangle = -|B_\psi,-\eta\rangle \tag{1.37}$$

$$(-1)^G|B_{sgh},\eta\rangle = |B_{sgh},-\eta\rangle \quad ; \quad (-1)^{\tilde{G}}|B_{sgh},\eta\rangle = |B_{sgh},-\eta\rangle \tag{1.38}$$

Using the previous expressions gives, after some manipulation,

$$|B\rangle_{NS} = \frac{1}{2}\Big(|B,+\rangle_{NS} - |B,-\rangle_{NS}\Big) \tag{1.39}$$

In the R-R sector, the GSO projected boundary state is

$$|B\rangle_R \equiv 1 + (-1)^p(-1)^{\frac{F+G}{2}} \; 1 - (-1)^{\frac{\tilde{F}+\tilde{G}}{2}} \; |B,+\rangle_R \tag{1.40}$$

where p is even for Type IIA and odd for Type IIB, and

$$(-1)^F = \psi_{11}(-1)^{\sum_{m=1}^{\infty}\psi_{-m}\cdot\psi_m} \quad , \quad G = -\gamma_0\beta_0 - \sum_{m=1}^{\infty}\left[\gamma_{-m}\beta_m + \beta_{-m}\gamma_m\right] \quad . \tag{1.41}$$

The action of the fermion number operators is given by:

$$(-1)^F|B_\psi,\eta\rangle = (-1)^p|B_\psi,-\eta\rangle \quad ; \quad (-1)^{\widetilde{F}}|B_\psi,\eta\rangle = |B_\psi,-\eta\rangle \tag{1.42}$$

and

$$(-1)^G|B_{sgh},\eta\rangle = |B_{sgh},-\eta\rangle \quad ; \quad (-1)^{\widetilde{G}}|B_{sgh},\eta\rangle = -|B_{sgh},-\eta\rangle \tag{1.43}$$

Finally, one finds that

$$|B\rangle_R = \frac{1}{2}\Big(|B,+\rangle_R + |B,-\rangle_R\Big) \ . \tag{1.44}$$

The GSO projection will loom large in the simplest construction of torsion D-branes, that of Sen.

As is verified in great detail in [8][9], these boundary state descriptions enable one to calculate standard characteristics of D-branes like RR-charge and tension, as well, of course, as brane-brane interactions.

After Polchinski's rediscovery of D-branes in 1995, flat-space boundary states were used in numerous computations.

1.2.2 Boundary Conformal Field Theory – Ishibashi States and the Cardy-Verlinde Formula

The application we will be studying, involving as it does D-branes on Calabi-Yau manifolds, requires that we go well beyond free field conformal theories. It is best to reformulate the whole question of boundary states as part of the general question of arbitrary conformal field theories on 2-manifolds with boundaries. Orientifolds can also be introduced by considering worldsheets with crosscaps as well as boundaries, but they will not concern us in this work.

This subject depends heavily on early work like [12][13] and the seminal work of Cardy [14][15][16][17][18](the last co-authored with Lewellen). The exposition will follow closely that in [81].

D-branes naturally signal the presence of worldsheet boundaries. At tree-level, the appropriate boundary CFT lives on the complex upper half-plane $\mathrm{Im}\, z \geq 0$ (or on the strip). Here, $z = \exp(t + i\sigma)$, where t is the time and σ the space variable.

A conformal theory on the upper half-plane looks in the bulk like a normal conformal field theory, but one requires that conformal transformations should fix the boundary (the real line). This leads to the condition

$$T(z) \; = \; \overline{T}(\bar{z}) \qquad \text{for} \quad z \; = \; \bar{z} \tag{1.45}$$

for the standard chiral fields $T = 2(T_{xx} + iT_{xy})$ and $\overline{T} = 2(T_{xx} - iT_{xy})$.

This condition can also be interpreted as the statement that no energy flows across the boundary. Since these two fields are defined only on the upper half-plane, they cannot be used to construct a Virasoro algebra acting on the space of states. The boundary condition given by (1.45) allows us, however, to weld together T and $\overline{T}$ into a set of generators for a single Virasoro algebra. The individual generators are defined by the standard mode expansion, except using both T and $\overline{T}$:

$$L_n^{(H)} := \frac{1}{2\pi i}\int_C z^{n+1}T(z)dz - \frac{1}{2\pi i}\int_C \bar{z}^{n+1}\overline{T}(\bar{z})d\bar{z} \tag{1.46}$$

where C denotes a semi-circle in the upper half-plane with ends on the real line.

We will need to consider BCFT's in cases where the original chiral algebra is some nontrivial extension of the Virasoro algebra. Here, we can use a slightly more general boundary condition, since other generators of the chiral algebra have a different physical interpretation than does the stress-energy tensor:

$$W(z) = \Omega(\overline{W})(\bar{z}) \qquad \text{for} \quad z = \bar{z} \ , \tag{1.47}$$

where W represents some other generator of the extended chiral algebra and Ω is an automorphism of that algebra that fixed the Virasoro piece. As before, modes can be constructed by

$$W_n^{(H)} := \frac{1}{2\pi i}\int_C z^{n+h_W-1}W(z)dz - \frac{1}{2\pi i}\int_C \bar{z}^{n+h_W-1}\Omega(\overline{W})(\bar{z})d\bar{z} \ . \tag{1.48}$$

The next step is defining boundary states. Again following [81], the basic idea is to redefine variables so that we can talk about a theory on the full plane (remember, the key idea of worldsheet duality relates open-string theories, or theories with boundaries, to closed-string theories, without boundary). Since worldsheet duality also interchanges time and space coordinates, we need to make the time coordinate periodic to carry out this reformulation.

To motivate the notion, consider the finite-temperature correlator (we will use H to label objects of the CFT on the half-plane and P for those on the full plane, which we are trying to construct)

$$\langle\phi_1^{(H)}(z_1,\bar{z}_1)\cdots\phi_N^{(H)}(z_N,\bar{z}_N)\rangle^{\beta_0} \;=\; \mathrm{Tr}_{\mathcal{H}}\big(e^{-\beta_0 H^{(H)}}\phi_1^{(H)}(z_1,\bar{z}_1)\cdots\phi_N^{(H)}(z_N,\bar{z}_N)\big) \tag{1.49}$$

where we assume the arguments z_i to be radially ordered and where $H^{(H)} = L_0^{(H)} - c/24$.

As is familiar from finite-temperature field theory, β_0 will act as a periodic time coordinate. Introduce coordinates $\sigma\in[0,\pi]$ and $t\in[t_0,t_0+\beta_0]$, where $t+i\sigma = \ln z$. After exchanging space and time and rescaling by $\frac{2\pi}{\beta_0}$, we get a cylinder that's periodic in space with period 2π. Next, we map the cylinder to an annulus by

$$\xi \;=\; e^{\frac{2\pi i}{\beta_0}\ln z} \quad\text{and}\quad \bar{\xi} \;=\; e^{-\frac{2\pi i}{\beta_0}\ln\bar{z}}\;, \tag{1.50}$$

and we can finally consider the theory defined on the full ξ-plane, which is what we needed. To map correlators in the old theory to those in the new theory, we need to use

$$\phi(\xi,\bar{\xi}) = (\frac{dz}{d\xi})^h(\frac{d\bar{z}}{d\bar{\xi}})^{\bar{h}}\phi^{(H)}(z,\bar{z})\;,\qquad T(\xi) = (\tfrac{dz}{d\xi})^2\,T^{(H)}(z)+\tfrac{c}{12}\{z,\xi\} \tag{1.51}$$

for primary fields ϕ and the stress-energy tensor resp.; $\{z,\xi\}$ is the Schwartzian derivative.

With these new variables, we can give an explicit definition of the boundary state corresponding to given boundary conditions. The boundary state of the original BCFT we wrote down is a state $|\alpha\rangle$ in the state space (more accurately, in an extension of that space) of the new bulk theory without boundary such that

$$\langle\phi_1^{(P)}(\xi_1,\bar{\xi}_1)\cdots\phi_N^{(P)}(\xi_N,\bar{\xi}_N)\rangle^{\beta_0} \;=\; \langle\Theta\alpha|e^{-\frac{2\pi^2}{\beta_0}H^{(P)}}\phi_1^{(P)}(\xi_1,\bar{\xi}_1)\cdots\phi_N^{(P)}(\xi_N,\bar{\xi}_N)|\alpha\rangle \tag{1.52}$$

Here, $H^{(P)} = L_0 + \overline{L}_0 - c/12$ is the Hamiltonian, Θ the CPT-operator in the bulk theory – and we regard the fields $\phi(\xi,\bar{\xi})$ as living on the full plane.

All this rigmarole then leads us to what we really need to start calculating boundary states – the gluing conditions:

$$\left(L_n^{(P)} - \overline{L}_{-n}^{(P)}\right)|\alpha\rangle_\Omega = 0 \quad \text{and} \quad \left(W_n^{(P)} - (-1)^{h_W}\Omega(\overline{W}_{-n}^{(P)})\right)|\alpha\rangle_\Omega = 0 \tag{1.53}$$

where h_W is the conformal weight of W.

By similarly considering an original theory in which the boundary conditions jump from α to β at the origin, we can look at more general matrix elements, like

$$\langle\phi_1^{(P)}(\xi_1,\bar{\xi}_1)\cdots\phi_N^{(P)}(\xi_N,\bar{\xi}_N)\rangle^{\beta_0} \;=\; \langle\Theta\beta|e^{-\frac{2\pi^2}{\beta_0}H^{(P)}}\phi_1^{(P)}(\xi_1,\bar{\xi}_1)\cdots\phi_N^{(P)}(\xi_N,\bar{\xi}_N)|\alpha\rangle \tag{1.54}$$

between different boundary states.

Next, we have to solve the gluing conditions as the first step in constructing our boundary states. For simplicity, we'll drop the P superscript.

This solution was obtained by Ishibashi [12][13]. Assume for now that the automorphism of the original chiral algebra, Ω, that enters the gluing conditions is trivial. Assume also that the symmetry generators W_n and $\overline{W}_n$ of the bulk CFT generate identical left- and right-moving chiral algebras $\mathcal{A}_{\mathfrak{L}} = \mathcal{A}_{\mathfrak{R}} = \mathcal{A}$, which include the Virasoro algebra. Ishibashi showed that to each irreducible highest weight representation i of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ one can associate a state $|i\rangle\rangle$, unique up to an overall constant, that obeys the gluing conditions. Using $|i, N\rangle$, $N \in \mathbb{Z}_+$, to denote an orthonormal basis of $\mathcal{H}$, we have the expression

$$|i\rangle\rangle = \sum_{N=0}^{\infty} |i, N\rangle \otimes U|i, N\rangle \tag{1.55}$$

where U denotes an anti-unitary operator on the total chiral Hilbert space $\mathcal{H}_{\mathfrak{R}} = \bigoplus \mathcal{H}$ that satisfies the following commutation relations

$$U\,\overline{W}_n = (-1)^{h_W}\,\overline{W}_{-n}\,U \tag{1.56}$$

with the right-moving generators.

Note that, just as coherent states in the free field case are really not in the standard Hilbert space of the string in flat space, the Ishibashi state is not an ordinary state in the bulk Hilbert space of the theory under discussion. In particular, it is not normalizable. The inner product between Ishibashi states will be defined by

$$\langle\langle j|\, q^{L_0 - \frac{c}{24}}\, |i\rangle\rangle = \delta_{i,j}\,\chi_i(q) \tag{1.57}$$

with $\chi_i(q) = \mathrm{Tr}_{\mathcal{H}_i} q^{L_0 - \frac{c}{24}}$ being the conformal character of the irreducible representation i.

Note that in the free field case, with the assumption that Ω is trivial, we can only get Neumann boundary conditions. Dirichlet boundary conditions would correspond to a sign-flip of the U(1) current algebra. Incorporating a nontrivial automorphism of the chiral algebra is quite simple. Ω acts on $\mathcal{H}$ by $A|h\rangle \to \Omega(A)|h\rangle$, mapping $\mathcal{H}$ isomorphically onto its image. Denote the isomorphism by V_Ω and assume that $V_\Omega U = UV_\Omega$. Then the generalized Ishibashi state is given by

$$|i\rangle\rangle_\Omega := \sum_{N=0}^{\infty} |i, N\rangle \otimes V_\Omega U |i, N\rangle \tag{1.58}$$

The Ishibashi states corresponding to a given gluing condition (choice of automorphism of the chiral algebra) form a vector space, but there is a further nonlinear constraint on acceptable boundary states. This constraint derives from the basic definition of boundary states (see (1.54)), where one equates a matrix element in the full-plane (closed-string) picture to a partition function in the half-plane (open-string) picture. Since a partition function involves a sum over states of a given quantity, there is a basic integrality condition.

Cardy was the first to derive this in a beautiful paper [17] which simultaneously involved use of boundary conformal field theory considerations to motivate the celebrated Verlinde formula which relates the fusion rules of a conformal field theory to its behavior under modular transformations.

We will use the notation of [81], since their results will be of great importance later.

Consider once again a boundary CFT with symmetry algebra $\mathcal{A}$, where the boundary condition jumps at $z = 0$ from α to β. Make the time direction periodic with period $\beta_0 = -2\pi i\tau$, and consider the partition function

$$Z_{\alpha\beta} = \mathrm{Tr}_{\mathcal{H}_{\alpha\beta}} q^{H_{\alpha\beta}^{(H)}} \tag{1.59}$$

where $\mathcal{H}_{\alpha\beta}$ is the boundary CFT Hilbert space and $H_{\alpha\beta}^{(H)} = L_0^{(H)} - \frac{c}{24}$ is the Hamiltonian in the coordinate $w = \ln z$ and $q = e^{2\pi i\tau}$.

After the worldsheet duality transformation, we once again look at an annulus diagram propagating between the ends, boundary states $|\alpha\rangle$ and $|\beta\rangle$. Assume also that the Hilbert space of the bulk CFT (i.e. the CFT with no boundaries) decomposes into irreducible representations of the symmetry algebra, now $\mathcal{A} \otimes \mathcal{A}$, as follows:

$$\mathcal{H}_{\mathrm{tot}} = \bigoplus_{\mathrm{J} \in I} \mathcal{H}_j \otimes \mathcal{H}_{j+}, \tag{1.60}$$

corresponding to a diagonal modular invariant partition function.

Specializing further to the case $\Omega = 1$, so the gluing conditions take their simplest form, and appealing back to (1.54), we find that

$$Z_{\alpha\beta}(q) = \langle \Theta\beta | \tilde{q}^{\frac{1}{2}(L_0^{(P)} + \bar{L}_0^{(P)} - \frac{c}{12})} | \alpha \rangle \tag{1.61}$$

with $\tilde{q} = e^{\frac{-2\pi i}{\tau}}$.

This equation actually implies powerful constraints on the boundary states $|\alpha\rangle$ and $|\beta\rangle$, because the left side is not just any matrix element but a partition function. Since the CFT on the half-plane has $\mathcal{A}$ as its symmetry algebra, the Hilbert space $\mathcal{H}_{\alpha\beta}$ decomposes into a sum of irreducible representations $\mathcal{H}_i$ of $\mathcal{A}$, so that the partition function becomes a sum over irreducible characters

$$Z_{\alpha\beta}(q) = \sum_i n^i_{\alpha\beta}\chi_i(q) \tag{1.62}$$

where again $\chi_i(q) = \mathrm{Tr}_{\mathcal{H}_i} q^{L_0 - \frac{c}{24}}$ and, of course, $n^i_{\alpha\beta}$ are nonnegative integers.

Now, consider a boundary state as a general expansion in Ishibashi states

$$|a\rangle = \sum B^i_\alpha |i\rangle\rangle \tag{1.63}$$

Further, define our normalization of the CPT operator Θ as follows:

$$\Theta B^j_\beta |j\rangle\rangle = \bar{B}^j_\beta |j^+\rangle\rangle \tag{1.64}$$

where $|j^+\rangle\rangle$ is the Ishibashi state corresponding to the conjugate representation j^+.

By definition, sandwiching $\tilde{q}^{\frac{1}{2}(L_0^{(P)} + \bar{L}_0^{(P)} - \frac{c}{12})}$ between two Ishibashi states gives an appropriate partition function if the two states are the same and 0 otherwise, so we get

$$\langle \Theta\beta | \tilde{q}^{\frac{1}{2}(L_0^{(P)} + \bar{L}_0^{(P)} - \frac{c}{12})} |\alpha\rangle = \sum_i B^i_\beta B^i_\alpha \chi_i(\tilde{q}) \ . \tag{1.65}$$

In order to express things in terms of the original q variables, we must consider the transformation of the characters under the modular transformation that implements worldsheet duality. At least for rational conformal field theories, where the total number of irreducible representations, and therefore of characters, is finite, modular transformations act linearly on the characters

$$\chi_i(\tilde{q}) = \sum_j S_{ij}\chi_j(q) \tag{1.66}$$

where S is a matrix obeying

$$SS^* = 1, \quad S = S^t, \quad S^2 = C \tag{1.67}$$

(C here is the charge conjugation matrix, $C_{ij} = \delta_{i,j^+}$.

Note that for this to be a unique definition of S, we must assume that characters belonging to distinct irreducible representations are linearly independent. This is not always the case. When this condition does not hold, S should be re-defined as acting on conformal blocks. This procedure is sometimes termed *resolving* the S-matrix (see, e.g., [20]). Continuing, we find that

$$\sum_{ij} B^j_\beta B^j_\alpha S_{ij} \chi_i(q) = \sum_i n^i_{\alpha\beta} \chi_i(q) \tag{1.68}$$

Returning for a moment to (1.60), we see that boundary conditions α and β can be put into rough correspondence with irreducible representations as follows: if we choose α and β so that $\mathcal{H}_{\alpha\beta} = \mathcal{H}_i \otimes \mathcal{H}_{j^+}$, then $\alpha \leftrightarrow i, \beta \leftrightarrow j$.

The crux of Cardy's paper is an ingenious argument, which we won't go into here, that the quantities n^k_{ij} are none other than the fusion-rule coefficients N^k_{ij}. Once this is established, use of the Verlinde formula leads in a straightforward manner to the solution for the boundary states, now labelled by irreducible representations of $\mathcal{A}$. Recall the Verlinde formula [21]

$$N^k_{ij} = \sum_i \frac{S_{il} S_{jl} S^*_{lk}}{S_{0l}} \tag{1.69}$$

where 0 denotes the vacuum representation.

Cardy's solution can be represented as $(a \in I)$

$$|a\rangle = \sum_i \frac{S_{ai}}{S_{0i}{}^{\frac{1}{2}}} |i\rangle\rangle \tag{1.70}$$

Although it is not obvious why this should be, Cardy's solution schema seems to provide enough states to render the matrix of coefficients B (for expansion in Ishibashi states) invertible [17][19], which suffices to establish completeness of the set of states [22].

1.3 Stable non-BPS States – the Sen Construction

The next stop on our tour is the discovery by Sen in a remarkable series of papers [23][24][25][26][27][28][32][30][31] of non-BPS states over which one could retain some analytical control of the charge and mass at strong coupling, and for which one could find stability arguments not dependent on the BPS bound. Given the fact that the BPS relation between charge and mass had been key to most of the nonperturbative statements of the Second String Revolution, this was quite a departure. Sen has reviewed his main results in [33], as have others [35].

The only example that will concern us here is the construction of a state in type I string theory that transforms as a spinor of SO(32) and is manifestly stable as the lowest-mass state with that property, but is also manifestly not BPS. We will discover later that its stability is due to the fact that it carries a $\mathbb{Z}_2$-valued RR charge.

The basic construction is as follows: Take a coincident D1-brane and anti-D1-brane in Type IIA theory, and mod the system out by $(-1)^{F_L}$ (F_L is the left-moving fermion number). In the bulk, this takes Type IIA to Type IIB. Open strings stretching from the brane to the antibrane have Chan-Paton factors defined by 2×2 matrices (each end of the string has two choices), and the "orbifolding" by $(-1)^{F_L}$ essentially keeps CP factors proportional to Id and to σ_1, while projecting out those proportional to σ_3 and $i\sigma_2$ – among the modes projected out is the one that separates the brane and antibrane, so the projection really gives us a single brane [33]. Since the NS sector ground state with CP factor σ_1 has $(-1)^F = -1$, it is physical – i.e., there is a tachyonic mode that survives the projection, thus the brane we get is unstable.

Stability is restored if we further mod out by worldsheet parity. This takes us from Type IIB to Type 1 string theory, and gives us a stable, non-BPS D-particle [28].

To delve deeper into the properties of this state, it helps to compactify on a circle in such a way that the non-BPS D-string we have in Type I is wrapped once. Although the worldsheet parity projection projects out the U(1) gauge field on the D-string, there is a $\mathbb{Z}_2$ subgroup that survives, so we can turn on a $\mathbb{Z}_2$ Wilson line on the S^1. This exposition closely follows [27].

Consider open string states with one end on the D-string and the other on any of the 32 space-filling 9-branes in Type I string theory. These states provide 32 fermionic degrees of freedom on the D-string worldvolume. If there is no Wilson line, they obey antiperiodic boundary conditions, there are no fermionic zero modes, and the states transform according to the scalar representation of SO(32). With a nontrivial Wilson line, each of these 32 fermions has a zero mode, and the quantization of the zero modes gives a ground state (with a corresponding tower of excited states) transforming in the spinor of SO(32). Note that this state doesn't carry any other conserved charges – in particular, in the pre-K-theory understanding of RR charge, it carried none since it descends from a brane-antibrane pair in the Type IIB theory. The lowest mass state of this form that transforms in the spinor of SO(32) should be stable purely by energetic considerations even though it is non-BPS.

Next, consider the configuration of the tachyon in this setup. Since we have turned on a Wilson line, the (bosonic) tachyon must obey antiperiodic boundary conditions and loses its zero mode – the analog of this fact in the infinite-radius limit is that a constant tachyon background (as is typical, finite energy conditions confine the tachyon to a circle at infinity) actually makes the state into the vacuum state (see [26] for the full argument). Thus, in fact, the lowest mode of the tachyon is the one corresponding to the kink at infinite radius – i.e., in the direction of the circle (call it x_0), we have $T(x_0, x) \to -T_0 x \to -\infty$, $T(x_0, x) \to T_0 x \to \infty$. At finite radius, this obeys the antiperiodic boundary conditions required by turning on the Wilson line. Thus, we see that the SO(32) state in type I string theory is identified with the tachyonic kink on the original D-string anti-D-string pair. This notion of tachyon condensation is developed much further in the cited papers and, we shall see, plays a key role in the understanding of D-brane charge as being classified by K-theory.

Note that the stability argument only works for a single state. Since two spinors can combine to form a direct sum of tensor representations, an even number of states of the kind described here could well be unstable to decay into ordinary perturbative states and odd numbers could be unstable to decay into a single spinor state plus perturbative states. This seems to suggest that this state has a charge that is a $\mathbb{Z}_2$ torsion element – but an element of what? Spacetime in this construction, even with compactification on a circle, has no torsion in its cohomology. This puzzle will find resolution in the next section on D-branes and K-theory.

Before we leave this subject, it's worth noting that conformal field theory methods, in particular the boundary state formalism, are very useful in formulating and analyzing the concept of tachyon condensation. This is partly developed in [27] and then more fully in [34], so we'll only sketch it lightly here.

If you compactify at the self-dual radius, $R_c = \sqrt{\frac{\alpha'}{2}}$, you can use the standard conformal field theory trick of "bosonizing" the right-moving and left-moving bosonic coordinates, creating new linear combinations from the fermions one defines, and then "fermionizing" them back into new bosonic coordinates. Performing this procedure on the boundary state description, if you do it right, gives you a new boundary state description in which tachyon condensation is realized and the RR part of the boundary state can be explicitly seen to vanish. We won't go into more detail, since we won't actually need specific expressions.

We will note that another way to describe this state (and similar ones involving higher-dimensional branes) is by a superposition

$$|Dp\rangle + |D\bar{p}'\rangle, \tag{1.71}$$

where p is odd for Type IIB and the ' denotes a $\mathbb{Z}_2$ Wilson line on the antibrane [34]. Though the RR part of such a brane does not vanish trivially, it can be seen by the above-mentioned procedure to vanish on tachyon condensation. Later, we will employ a very similar construction in creating boundary states for other torsion D-branes (which are not restricted only to $\mathbb{Z}_2$ torsion).

1.4 D-Branes and K-Theory

There had always been hints that K-Theory had something to do with D-branes. The most basic was the periodicity in the brane spectrum – 2 for Type II (complex) theories (if a type II theory has a BPS Dp-brane, it also has a BPS $D(p+2)$-brane, and does not have a BPS $D(p+1)$-brane) and 8 for Type I (real) theories (D1- and D9-branes have a worldvolume SO theory, and the D5-brane has a worldvolume Sp theory). This corresponds exactly to Bott periodicity in K-theory.

Another hint came as a result of the geometrical development alluded to earlier. Considering a D-brane wrapping an arbitrary cycle of the spacetime manifold, and using anomaly-inflow arguments, several authors [38][37] independently came up with a formula for D-brane charge that clearly shows its connection with K-theory:

$$Q = \mathrm{ch}\,(f_! E)\,\sqrt{\widehat{A}(\mathcal{TS})} \tag{1.72}$$

where $\mathcal{TS}$ is the tangent bundle to spacetime and $f_!$ is the K-theoretic Gysin map.

Sen's construction/discovery of the type I 0-brane and other stable non-BPS states completed the picture, leading Witten [39] to conjecture that, in fact, D-brane charge takes its values in the K-theory of spacetime (properly defined). His argument for K-theory, which depends only schematically, not in detail, on Sen's construction, is in two steps, and we repeat it below, closely following the original.

First, for ease and definiteness, restrict to Type IIB string theory. Consider a configuration of n 9-branes and n anti-9-branes (the numbers are the same because of the requirement of tadpole cancellation). The 9-branes will carry a U(n) Chan-Paton gauge bundle, say E, and the anti-9-branes will carry a gauge bundle F. Describe this configuration with the ordered pair (E, F).

Next, nucleate m identical brane-antibrane pairs. Basic conservation laws require that the U(m) gauge bundle, call it H, on the branes be equivalent to that on the anti-branes. Furthermore, in terms at least of total conserved D-brane charge, the configuration (E, F) must be identical to $(E \oplus H, F \oplus H)$. This gives us the definition of K-theory – ordered pairs of vector bundles, defined up to stable equivalence. The subgroup of $K(X)$ where E and F have the same rank is known as the *reduced* K-group, $\tilde{K}(X)$.

Thus, D-brane charge naturally lives in K-theory (and for 9-branes, in the K-theory of spacetime), not in cohomology – although the Chern character conveniently provides a natural isomorphism between the torsion-free part of $K(X)$ ($K^0(X)$, to be precise) and the even-dimensional cohomology of X.

Mostly, when dealing with noncompact spacetimes, we will have finite energy or action conditions that restrict us to considering objects equivalent to the vacuum at infinity. Thus, it will usually be more appropriate to consider the K-theory with compact support instead of the normal K-theory.

The next step in the argument is trickier. The 9-brane argument applies to branes of other dimensions to give D-brane charge as an element of the K-theory of the worldvolume, but in order for the approach to be really useful, for us to compare charges and calculate monodromies, we need to have all the D-brane charges living on spacetime, not on the worldvolume.

This is where tachyon condensation comes in, although, as mentioned earlier, in a schematic way. First, consider the construction of a p-brane from a system of a coincident $(p+2)$-brane and anti-$(p+2)$-brane.

On this brane-antibrane pair, we have a U(1)×U(1) gauge field and a tachyon field with charges $(1,-1)$. Consider a "vortex" configuration in which the tachyon field vanishes on a codimension 2 subspace of the worldvolume, and approaches a fixed VEV (up to gauge transformations) at infinity. Such configurations can be classified topologically by their "winding number" about the zero locus. Let's take the basic building block, of winding number 1. This field breaks the U(1)×U(1) to U(1), which means there is a unit of magnetic flux in the broken U(1). That magnetic flux gives the system a Dp-brane charge of 1. The configuration clearly has no net $D(p+2)$-brane charge and, in fact, it looks like the vacuum configuration except in a neighborhood of the tachyon field's zero locus. Thus, we are justified in thinking of it as a p-brane.

We could, of course, repeat this construction stepwise to get lower-dimensional branes, but such a stepwise construction does not preserve the maximum spacetime symmetry. Instead, it is useful to perform an "all-at-once" construction.

Consider a collection of n $(p+2k)$-brane pairs. They carry a $U(n)\times U(n)$ gauge symmetry under which the tachyon field T transforms as $(n,\bar{n})$. T breaks $U(n)\times U(n)$ down to a diagonal $U(n)$, so the gauge orbit of values of T with minimum energy is also $U(n)$. For a p-brane, T should vanish in codimension $2k$ and approaches its vacuum manifold of VEVs at infinity, with some kind of topological twist. These configurations are classified by $\pi_{2k-1}(U(n))$, which, equals $\mathbb{Z}$ for all sufficiently large n, by Bott periodicity.

If we pick $n=2^{k-1}$ (choice of n is arbitrary, of course), we can give a very useful description of the generator of $\pi_{2k-1}(U(n))$. Let S_+ and S_- be the positive and negative chirality spinor representations of $SO(2k)$, of dimension 2^{k-1}. Let $\vec{\Gamma}=(\Gamma_1,\ldots,\Gamma_{2k}$ be Gamma matrices, regarded as maps $S_-\to S_+$. If $\vec{x}=(x_1,\ldots,x_{2k})$ has norm 1 (i.e., is an element of S^{2k-1}, then we can define the tachyon field by

$$T(\vec{x})=\vec{\Gamma}\cdot\vec{x}. \tag{1.73}$$

This map has winding number 1 and generates $\pi_{2^k-1}(U(2^{k-1}))$. It can be verified that this configuration has p-brane charge 1 and that all higher and lower brane charges vanish, so that this is indeed a p-brane.

Now, really, this analysis only applies to flat branes – the homotopy classification, for example, depends on the details of the set where the tachyon field vanishes and on the transverse space. We need to establish this K-theory classification in a more global context in order to use it for our ends. Again, following [90]:

Start by considering Sen's original construction (codimension two) in a global context. Let Z be a closed submanifold of spacetime, of dimension $q = p+1$, and we suppose that Z is contained in Y, a submanifold of spacetime of dimension $q+2$. We assume that Z and Y are orientable, since Type II branes can only wrap on orientable manifolds. Then one can define a complex line bundle Ł over Y, and a section s of Ł that vanishes precisely along Z, with a simple zero. Moreover, one can put a metric on Ł such that, except in a small neighborhood of Z, s has fixed length.

Now, consider a system consisting of a $(p+2)$-brane-antibrane pair, wrapped on Y. We place on the brane a $U(1)$ gauge field that is a connection on $\mathcal{L}$; its p-brane charge is that of a p-brane wrapped on Z. We place on the antibrane a trivial $U(1)$ gauge field, with vanishing p-brane charge. The brane-antibrane system thus has vanishing $(p+2)$-brane charge and p-brane charge the same as that of a p-brane on Z. This suggests that the system could be deformed to a system consisting just of a p-brane wrapped on Z.

As evidence for this, we note that the tachyon field of the brane-antibrane pair, because it has charges $(1, -1)$ under the $U(1) \times U(1)$ that live on the brane and antibrane, should be a section of $\mathcal{L}$. Hence we can take

$$T = c \cdot s, \tag{1.74}$$

with c a constant chosen so that far from Z, $|T|$ is equal to its vacuum expectation value. With this choice of T, the system is in a vacuum state except near Z and can be described by a p-brane wrapped on Z.

A fuller description actually requires the following generalization. Note that a p-brane wrapped on Z has in general in addition to its p-brane charge also r-brane charges with $r = p-2, p-4, \ldots$. Moreover, these depend on the choice of a line bundle $\mathcal{M}$ on Z. Thus, to fully describe all states with a p-brane wrapped on Z in terms of states of a brane-antibrane pair wrapped on Y, we need a way to incorporate $\mathcal{M}$ in the discussion.

If $\mathcal{M}$ extends over Y, we incorporate it in the above discussion just by placing the line bundle $\mathcal{L} \otimes \mathcal{M}$ on the $p+2$-brane and the line bundle $\mathcal{M}$ on the $p \bar{+} 2$-brane. The tachyon field T, given its charges $(1, -1)$, is a section of $(\mathcal{L} \otimes \mathcal{M}) \otimes \mathcal{M}^{-} = \mathcal{L}$, so we can take $T = s$ and flow (presumably) to a configuration containing only a p-brane wrapped on Z. The r-brane charges with $r < p$ now depend on $\mathcal{M}$ in a way that has a simple interpretation: on the p-brane worldvolume there is a $U(1)$ gauge field with line bundle $\mathcal{M}$.

More generally, however, $\mathcal{M}$ may not extend over Y. To deal with this case, we need to use another of the basic constructions of K-theory. First we describe it in mathematical terms. Let Z be a submanifold of a manifold Y, and Z' a tubular neighborhood of Z in Y. Let $\bar{Z}$ be the closure of Z'. Suppose that E and F are two bundles over Z of the same rank (in our example so far, they are line bundles, $E = \mathcal{L} \otimes \mathcal{M}$ and $F = \mathcal{M}$), so that the pair (E, F) defines an element of $K(Z)$. Pull E and F back to $\bar{Z}$, so that (E, F) defines an element of $K(\bar{Z})$. The tachyon field T, which is a section of $E \otimes F^*$, can be regarded as a bundle map $T : F \to E$. Suppose that T is a tachyon field on $\bar{Z}$ which (when viewed in this way as a bundle map)is invertible if restricted to Z^*. Then from this data, one can construct an element of $K(Y)$.

The construction is made as follows. Let $Y' = Y - Z'$; thus Y' consists of Z^* and its "exterior" in Y. If we could extend the bundle F from Z^* over all of Y' then F would be defined over all of Y (since it is defined already on $\bar{Z}$). Since E is isomorphic to F on Z^* (via T), we could extend it over Y' by declaring that it is isomorphic to F over Y'. The pair (E, F) of bundles on Y then give the desired element of $K(Y)$.

If F does not extend over Y', one proceeds as follows. By a standard lemma in K-theory, there is a bundle H over Z such that $F \oplus H$ is trivial over Z, and hence is trivial when pulled back to $\bar{Z}$. Replacing E, F, and T by $E \oplus H$, $F \oplus H$, and $T \oplus 1$, we can extend $F \oplus H$ over Y (as a trivial bundle), and extend $E \oplus H$ by setting it equal to $F \oplus H$ over Y'. The pair $(E \oplus H, F \oplus H)$ then give the desired element of $K(X)$. Note that $E \oplus H$ and $F \oplus H$ are isomorphic over Y' but not over Y. Note the importance of the tachyon in this construction.

This construction is precisely what we need to express in terms of $(p+2)$-branes on Y a p-brane on Z that supports a line bundle $\mathcal{M}$. We find a bundle H over Z such that $\mathcal{M} \oplus \mathcal{H}$ is trivial (and so extends over Y). $\mathcal{L} \otimes \mathcal{M} \oplus \mathcal{H}$ is extended over Y using the fact that (via $T \oplus 1$) it is isomorphic to $\mathcal{M} \oplus \mathcal{H}$ away from Z. Then we consider a collection of $(p+2)$-branes on Y with gauge bundle $\mathcal{L} \otimes \mathcal{M} \oplus \mathcal{H}$, and $(p \bar{+} 2)$-branes on Y with gauge bundle $\mathcal{M} \oplus \mathcal{H}$. The number of branes of each kind is 1 plus the rank of H. The tachyon field is $T \oplus 1$ near Z, and is in the gauge orbit of the vacuum outside of Z'. The system thus describes, under the usual assumptions, a p-brane on Z with gauge bundle $\mathcal{M}$.

In a similar fashion, we could have started with any collection of n p-branes wrapped on Z, with $U(n)$ gauge bundle $\mathcal{W}$, and expressed it in terms of a collection of $(p+2)$-branes and antibranes on Y. One pulls back $\mathcal{W}$ to $\bar{Z}$, uses the tachyon field $\tilde{T} = T \otimes 1$ to identity $\mathcal{W}$ with $\mathcal{L} \otimes \mathcal{W}$ on the boundary of $\bar{Z}$, and then uses the pair of bundles $(\mathcal{L} \otimes \mathcal{W}, \mathcal{W})$ to determine a class in $K(Y)$. Such a class is, finally, interpreted in terms of a collection of branes and antibranes wrapped on Y. The p-brane charge is n; the r-brane charges for $r < p$ depend on $\mathcal{W}$.

We will come back to this question of "globalizing" D-brane charges later, when we will actually have to calculate.

Witten hinted at extension of these results to other string theories, like Type IIA and Type I, as well as to string theory on orbifolds and other objects, hints that were followed up in papers like [40],[41], and others. For Type IIA, the relevant group is not K^0 but K^1, for Type IIB on orbifolds it is the equivariant K-theory, and for Type I, it is KO. This classification finally enables us to make sense of Sen's brane – $KO(S^1)$ or equivalently $KO_c(\mathbb{R})$ (KO with compact support) is $\mathbb{Z}_2$. Later, we will study examples of torsion in Type II branes.

Chapter 2

Generalities on (2,2) Compactifications

In this paper, our primary concern will be with the properties of D-branes in (2,2) compactifications, i.e. compactifications that preserve a world-sheet algebra $\mathcal{A} \otimes \overline{\mathcal{A}}$, where $\mathcal{A}$ ($\overline{\mathcal{A}}$) is the left-moving (right-moving) $N = 2$ superconformal algebra in two dimensions. Such compactifications have long been of special interest because, with the right additional conditions, they preserve $N = 1$ spacetime supersymmetry, thought to be a key phenomenological requirement (although, of course, that supersymmetry must be spontaneously broken at low energies). Among these (2,2) compactifications are those with a simple geometrical intepretation – spacetime is a product $\mathbb{R}^4 \times Y$, where Y is a Calabi-Yau manifold. In general, the moduli space of such compactifications contains regions with geometrical interpretations and those without – we shall be concerned with both.

2.1 The $N = 2$ Superconformal Algebra, the Chiral Ring, and Spectral Flow

In order to proceed, it will be necessary to establish some properties of the $N = 2$ SCA. The formulas in this section were worked out in [42] and are concisely encapsulated in, for example, [45].

In addition to the standard Virasoro generator T, there are two supercurrents G^+,G_- of conformal weight $\frac{3}{2}$ and a U(1) current J of conformal weight 1. We expand them into modes in the standard way for conformal fields. The supercurrents require some care because of the different boundary conditions they can obey:

$$G^{\pm}(z) = \sum_n G^{\pm}_{n\pm a}\, z^{-(n\pm a)-3/2} \ . \qquad (2.1)$$

In practice, we choose a either integral (anti-periodic boundary conditions, Ramond) or half-integral (periodic boundary conditions, Neveu-Schwarz). With these mode expansions, we can write the commutation and anti-commutation relations of the $N = 2$ SCA as follows:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\,\delta_{m+n,0} \ , \\
[J_m, J_n] &= \frac{c}{3} m\,\delta_{m+n,0} \ , \\
[L_n, J_m] &= -m\,J_{m+n} \ , \\
\left[L_n, G^{\pm}_{m\pm a}\right] &= \left(\frac{n}{2} - (m \pm a)\right) G^{\pm}_{m+n\pm a} \ , \\
\left[J_n, G^{\pm}_{m\pm a}\right] &= \pm\, G^{\pm}_{m+n\pm a} \ , \\
\left\{G^+_{n+a}, G^-_{m-a}\right\} &= 2\,L_{m+n} + (n-m+2a)\,J_{n+m} \\
&\quad + \frac{c}{3}\left((n+a)^2 - \frac{1}{4}\right)\delta_{m+n,0} \ .
\end{aligned}$$

In analogy with the definition of conformal primary fields, we can define a superconformal primary as a field ϕ that creates a state $|\phi\rangle$ that obeys the following conditions:

$$L_n|\phi\rangle = 0, \quad G^{\pm}_r|\phi\rangle = 0, \quad J_m|\phi\rangle = 0 \qquad (2.2)$$

for n, r, m strictly positive.

It is useful to make a more restrictive definition as well – define a chiral primary field to be a superconformal primary that also obeys $G^+_{-\frac{1}{2}}|\phi\rangle = 0$; similarly, an antichiral primary is a primary that obeys $G^-_{-\frac{1}{2}}|\phi\rangle = 0$

Of course, one can also consider the antiholomorphic algebra, and define chiral and antichiral primaries similarly. In general, we denote by (c,c) fields that are chiral both in holomorphic and antiholomorphic coordinates, by (c,a) those that are chiral in holomorphic and antichiral in antiholomorphic coordinates, and so on.

Using the anticommutator

$$\{G^-_{1/2}, G^+_{-1/2}\} = 2L_0 - J_0, \tag{2.3}$$

one can see with a little algebra that for any state $|\psi\rangle$ in any unitary representation of the $N = 2$ superconformal algebra, $h_\psi \geq \frac{Q_\psi}{2}$. One can see further that for chiral primaries $h_\psi = \frac{Q_\psi}{2}$ and that, in fact, they are the only states that saturate this bound[42].

Furthermore, one can show that any NS state $|\phi\rangle$ can be decomposed as

$$|\phi\rangle = |\phi_0\rangle + G^+_{-\frac{1}{2}}|\phi_1\rangle + G^-_{\frac{1}{2}}|\phi_2\rangle, \tag{2.4}$$

where $|\phi_0\rangle$ is a chiral primary – an equation that is not accidentally reminiscent of the Hodge decomposition of differential forms on a Kähler manifold.

Even though it appears from (2.2) that we get fundamentally distinct superconformal algebras for different values of a, in fact algebras with different values of a are isomorphic, by an isomorphism known as *spectral flow*. Following [45], let's change our notation slightly by defining $a \equiv \eta + \frac{1}{2}$. Now, define

$$L'_n \equiv L_n + \eta J_n + \frac{c}{6}\eta^2 \delta_{n,0} , \tag{2.5}$$

$$J'_n \equiv J_n + \frac{c}{3}\eta \delta_{n,0} , \tag{2.6}$$

$$G'^{\pm}_r \equiv G^{\pm}_{r\pm\eta} . \tag{2.7}$$

Clearly, the algebra generated by the primed objects is the same as that generated by the unprimed objects. The latter is simply by definition the $N = 2$ SCA with any given value of η, while the former, one can see by explicit calculation, gives the algebra with $\eta = 0$.

If we have a collection of states $|f\rangle$ that provide a representation of the algebra with $\eta = 0$, then there should exist an isomorphic set of states $|f_\eta\rangle$ providing a representation of the algebra with arbitrary η. Then we expect that there exists a map U_η such that

$$|f_\eta\rangle = U_\eta |f\rangle \tag{2.8}$$

$$L'_n = U_\eta L_n U_\eta^{-1} \tag{2.9}$$

$$J'_n = U_\eta J_n U_\eta^{-1} \tag{2.10}$$

This is the spectral flow isomorphism. Next, we see how to construct it explicitly.

Let us define h_η and q_η to be, resp., the conformal weight and charge of the new state $|f_\eta\rangle = U_\eta|f\rangle$, i.e.

$$L_0|f_\eta\rangle \equiv h_\eta|f_\eta\rangle \quad ; J_0|f_\eta\rangle \equiv q_\eta|f_\eta\rangle \tag{2.11}$$

By virtue of (2.8), we have

$$L'_0|f\rangle_\eta \equiv h|f\rangle_\eta \quad ; J'_0|f\rangle_\eta \equiv q|f\rangle_\eta \ . \tag{2.12}$$

Referring back to (2.5),(2.6), we see that

$$h_\eta = h - \eta q + \frac{c}{6}\eta^2 \tag{2.13}$$

$$q_\eta = q - \frac{c}{3}\eta \tag{2.14}$$

The second relationship will allow us to construct the spectral flow operator. First, note that the U(1) current can be bosonized and written as

$$J(z) = i\sqrt{\frac{c}{3}}\,\partial_z\phi \ , \tag{2.15}$$

where ϕ is a free scalar boson. Then, any field f which creates the state $|f\rangle$ with $U(1)$ charge q can be written as

$$f(z) = \hat{f}(z)\, e^{iq\sqrt{\frac{c}{3}}\phi} \ , \tag{2.16}$$

where $\hat{f}$ is a neutral field (verifying this is a standard exercise in conformal normal ordering). Similarly, we can conjecture that the field f_η which creates the state $|f_\eta\rangle$ can be written as

$$f_\eta(z) = \hat{f}(z)\, e^{i\sqrt{\frac{c}{3}}(q-\frac{c}{3}\eta)\phi} \ , \tag{2.17}$$

Again, straightforward algebra shows that this state has the correct conformal weight and charge, in accord with (2.13). To show that the created state is actually in the η-twisted sector, once need only consider the operator product of the field with the supercurrent generators. Comparing (2.17) with (2.16) gives us the spectral flow operator:

$$U_\eta = e^{-i\sqrt{\frac{c}{3}}\eta\phi} \ . \qquad (2.18)$$

As is well known, modular invariance requires that we include both the Neveu-Schwarz sector ($\eta = 0$) and the Ramond sector ($\eta = 1/2$) in the Hilbert space of our theory. Clearly, given one of these sectors, we can obtain the other by spectral flow by 1/2 unit. Since the NS sector gives spacetime bosons and the R sector spacetime fermions, we see that the operation of spectral flow (by 1/2 unit) has a space-time interpretation as the supersymmetry operator, and that therefore there is a fundamental link between $N = 2$ superconformal symmetry on the worldsheet and spacetime supersymmetry. Note that the image of a chiral primary state under spectral flow by 1/2 unit yields a state that is annihilated by $G_0^\pm$ — that is, a *Ramond ground state*. If we flow by another half unit we get an antichiral primary field.

In fact, taking $U_{1/2}$ as the spacetime supersymmetry operator imposes additional constraints on the theory. The SUSY operator should be *semi-local* – it should create a square-root branch cut on the worldsheet, thereby exchanging spacetime bosons and fermions. Considering an operator product of $U_{1/2}$ with an arbitrary field $f(z)$ (see (2.16)), we see immediately that all U(1) charges in the theory must be integral, or we will get worse multivaluedness than a square root, thus destroying any reasonable spacetime interpretation. In order that the operator $U_{1/2}$ explicitly convert bosons to fermions (and vice versa), we must require in addition that all U(1) charges be odd integers (an even integer state does not acquire a branch cut in an OPE with $U_{1/2}$). More thorough analyses have actually established that this is a necessary and sufficient condition for spacetime supersymmetry [59][60].

We should note that this restriction on $U(1)$ charges is on the *whole* theory including the internal and the four-dimensional part. The total central charge, in light cone gauge (so that we do not need to discuss the effects of ghosts) is 12. Hence, when we spectral flow by $\eta = 1/2$ the charge of, say, a NS state is shifted by $c/6 = 2$. Thus, if the original state has odd integral charge, so does its image in the R sector. It will also be helpful to think of the projection onto states of odd integral charge as two steps – first project to integral charge, then orbifold the given $N = 2$ theory by the operator $e^{2\pi i J_0}$ (and by $e^{2\pi i \bar{J}_0}$ – we have been ignoring the antiholomorphic part) [58].

2.2 Gepner Models

In a string of papers from 1986 through 1988 [61][62][63], Gepner introduced an entire schema for producing nongeometrical string compactifications with (2,2) worldsheet supersymmetry and $N = 1$ spacetime supersymmetry. These compactifications, known as Gepner models, have assumed great importance for several reasons. First, since they are based on exactly solvable superconformal minimal models, one can do exact calculations for them, unlike the case with, for example, compactifications on non-toroidal Calabi-Yau manifolds. Second, as we shall see later in some depth, they are connected with Calabi-Yau compactifications since they both live in a larger moduli space of (2,2) compactifications. Third, as we will also see later, they build on the basic correspondence between objects in $N = 2$ superconformal field theory and geometrical objects that we have seen already in the correspondence between chiral primaries and differential forms – they were the crucial initial piece in making the connection between simple automorphisms of the superconformal algebra and complex geometrical correspondences between Calabi-Yau manifolds that is commonly known as mirror symmetry. We will again follow closely the exposition of [81].

In Gepner's approach, one compactifies by replacing $10 - D$ free superfields with some "internal CFT" of central charge $15 - 3D/2$ such that certain conditions, familiar from the exposition of the pevious section, are met:

1. The internal CFT must at least have $N = 2$ world sheet supersymmetry.

2. The total U(1) charges must be odd integers for both left and right movers – here, total refers to internal charges plus charges of the $D - 2$ free external superfields associated to transverse uncompactified directions; this condition implements the generalized GSO projection.

3. The left-moving states must be taken from the NS sectors of each subtheory (external and, in Gepner's models, various internal sub-theories) or

from the R sectors in each subtheory; analogously for the right-moving states.

4. The torus partition function must be modular invariant.

As recalled in e.g. [1], the basic building blocks of the Gepner model are $N = 2$ minimal models. A minimal model is a degenerate (unitary) representation of a conformal algebra, where correlation functions can be calculated because of the existence of a null state at some finite level. The minimal model MM_k has the following representation as a coset theory constructed from Kac-Moody algebras:

$$MM_k = \frac{SU(2)_k \times U(1)_4}{U(1)_{2k+4}} \tag{2.19}$$

Accordingly, the central charge is $c_k = 3k/(k+2)$. The irreducible representations of the theory are labelled by (l, m, s), where l refers to the $SU(2)$, m to the $U(1)_{2k+4}$ in the numerator and s to the $U(1)_4$ in the nominator. These three integers are subject to an additional constraint:

$$l + m + s \quad \text{even.}$$

It is convenient to use the following ranges for these indices:

$$l = 0, 1, \ldots, k \ , \quad m = -k-1, -k, \ldots, k+2 \ , \quad s = -1, 0, 1, 2 \tag{2.20}$$

, subject to the proviso that triples (l, m, s) and $(k-l, m+k+2, s+2)$ give rise to the same representation ("field identification").

The conformal dimension h and charge q of the highest weight state with labels (l, m, s) are given by

$$h^l_{m,s} = l(l+2) - \frac{m^2}{4(k+2)} + \frac{s^2}{8} \pmod 1 , \tag{2.21}$$

$$q^l_{m,s} = \frac{m}{k+2} - \frac{s}{2} \pmod 2 ; \tag{2.22}$$

for many purposes, it is sufficient to know h (and q) up to (even) integers. The exact dimensions and charges of the highest weight state in the representation (l, m, s) can be read off if one first uses the field identification and the transformations $(l, m, s) \mapsto (l, m+k+2, s)$ and $(l, m, s) \mapsto (l, m, s+2)$ to bring (l, m, s) into the standard ranges

$$l = 0, 1, \ldots, k \ , \quad |m - s| \leq l \ , \quad s = -1, 0, 1, 2 \ , \quad l + m + s \ \text{ even} \tag{2.23}$$

or

$$l = 1, \ldots, k \ , \quad m = -l \ , \quad s = -2 \ . \tag{2.24}$$

Representations with an even value of s are part of the NS-sector, while those with $s = \pm 1$ belong to the R-sector. Within the two sectors, representations can be grouped into pairs (l, m, s) and $(l, m, s+2)$ which make up a full representation of the $N = 2$ SCA.

In order to write down partition functions that satisfy all the requirements listed above and therefore describe superstring compactifications, Gepner formed tensor products of $N = 2$ minimal models such that the central charges add up to a multiple of three, adjoined external fermions and bosons, and finally employed an orbifold-like procedure which enforces space-time supersymmetry and modular invariance [45][62][64][65].

We need some further notation before we can state Gepner's result. For a compactification involving r minimal models, we use

$$\lambda := (l_1, \dots, l_r) \quad \text{and} \quad \mu := (s_0; m_1, \dots, m_r; s_1, \dots, s_r) \tag{2.25}$$

to label the tensor product of representations: l_j, m_j, s_j are taken from the range (4.3), and $s_0 = 0, 2, \pm 1$ characterizes the irreducible representations of the $\mathrm{SO}(d)_1$ current algebra that is generated by the d external fermions (the latter also contain an $N = 2$ algebra for each even d and, again, the NS-sector has s_0 even). Accordingly, we write

$$\chi^{\lambda}_{\mu}(q) \; := \chi_{s_0}(q) \chi^{l_1}_{m_1,s_1}(q) \cdots \chi^{l_r}_{m_r,s_r}(q) \tag{2.26}$$

with $\chi^{l}_{m,s}(q) = \mathrm{Tr}_{\mathcal{H}^{l}_{m,s}} q^{L_0 - \frac{c}{24}}$ etc. for the conformal characters of these tensor products of internal minimal model and external fermion representations.

Next, define the $(2r+1)$-dimensional vectors β_0 with all entries equal to 1, and β_j, $j = 1, \dots, r$, having zeroes everywhere except for the 1st and the $(r+1+j)$th entry which are equal to 2. Consider the following products of $2\beta_0$ and β_i with a vector μ as above:

$$2\beta_0 \cdot \mu := -\frac{d}{2}\frac{s_0}{2} - \sum_{j=1}^{r} \frac{s_j}{2} + \sum_{j=1}^{r} \frac{m_j}{k_j + 2} \; , \beta_j \cdot \mu := -\frac{d}{2}\frac{s_0}{2} - \frac{s_j}{2} \; . \tag{2.27}$$

Clearly, $q_{\mathrm{tot}} := 2\beta_0 \cdot \mu$ is just the total $\mathrm{U}(1)$ charge of the highest weight state in $\chi^{\lambda}_{\mu}(q)$, so the projection onto states with odd $2\beta_0 \cdot \mu$ will implement the

GSO projection. Similarly, restricting to states with $\beta_i \cdot \mu$ integer ensures that only states in the tensor product of $r+1$ NS-sectors (or of $r+1$ R-sectors) are admitted (recall that we assumed $d=2$ or $d=6$).

Modular invariance of the partition function can be achieved if the above projections are accompanied by adding “twisted” sectors. To state Gepner's result, we put $K := \mathrm{lcm}(4, 2k_j+4)$ and let $b_0 \in \{0,1,\ldots,K-1\}$, $b_j \in \{0,1\}$ for $j=1,\ldots,r$. Then the partition function of a Gepner model describing a superstring compactification to $d+2$ dimensions is given by

$$Z_G^{(r)}(\tau,\bar\tau) = \frac{1}{2}\frac{(\mathrm{Im}\,\tau)^{-\frac{d}{2}}}{|\eta(q)|^{2d}} \sum_{b_0,b_j} \sum_{\lambda,\mu}{}^{\beta} (-1)^{s_0}\, \chi^{\lambda}_{\mu}(q)\, \chi^{\lambda}_{\mu+b_0\beta_0+b_1\beta_1+\ldots+b_r\beta_r}(\bar q) \qquad (2.28)$$

where $\sum^\beta$ means that we sum only over those λ,μ in the range (4.3) which satisfy $2\beta_0\cdot\mu \in 2\mathbb{Z}+1$ and $\beta_j\cdot\mu\in\mathbb{Z}$. The summations over b_0, b_j introduce the twisted sectors corresponding to the β-restrictions so that, in particular, the Gepner partition function is non-diagonal. The sign is the usual one occurring in (space-time) fermion one-loop diagrams. The τ-dependent factor in front of the sum accounts for the free bosons associated to the d transversal dimensions of flat external space-time, while the $1/2$ is due to the field identification previously mentioned. Using the modular transformation properties of the $\mathrm{SO}(d)_1$ and minimal model characters, whose S-matrices are

$$S^{\mathrm{f}}_{s_0,s_0'} = \frac{1}{2}\, e^{-i\pi\frac{d}{2}\frac{s_0 s_0'}{2}}\,, \qquad (2.29)$$

$$S^{k}_{(l,m,s),(l',m',s')} = \frac{1}{\sqrt{2}(k+2)}\sin\pi\frac{(l+1)(l'+1)}{k+2}\, e^{i\pi\frac{mm'}{k+2}}\, e^{-i\pi\frac{ss'}{2}}\,, \qquad (2.30)$$

Gepner proved that (2.28) is indeed modular invariant.

Gepner's purely algebraic construction is intimately related to geometric string compactifications on certain (complete intersection) Calabi-Yau manifolds. E.g., the number of generations and anti-generations computed from Gepner's partition function agrees with (or can at least be related to) those found in CICY-compactifictions, where they are given by the dimensions of certain Dolbault cohomologies. The connection was made more precise by the work of Greene, Witten and other authors; see [45] for a useful review.

As mentioned before, the connection between compactifications on Calabi-Yau manifolds and the CFT vacua constructed by Gepner led to the concept of mirror symmetry. We will come back to this in a later section where we look at the mirror automorphism of the $N = 2$ SCA.

2.3 Gauged Linear σ Models and Phases of (2,2) Compactifications

In this section, we review Calabi-Yau compactifications and connections with other (2,2) compactifications. Calabi-Yau compactifications have been the subject of an entire industry, including a tremendous body of work by Candelas and a slew of collaborators[66] [67][68][69][70][71][72][73][74], as well as many important papers by others ([75][76][77], others). Even a cursory review of those results is far beyond the scope of this work. We will confine ourselves to reviewing only that which is necessary to understand our analysis of torsion D-branes on Calabi-Yau's. Our exposition will follow the lines of [90][91].

Given a manifold M, with Riemannian metric g and closed 2-form B, we can write the supersymmetric nonlinear σ model in terms of maps $\Phi : \Sigma \to X$, where Σ is the string worldsheet, some orientable 2-manifold. Choosing local coordinates $z, \bar{z}$ on Σ and Φ^I on X, we write

$$S = \int_\Sigma d^2z \Big(\frac{1}{2}(g_{IJ}+iB_{IJ})\partial_z\phi^I\partial_{\bar{z}}\phi^J + \frac{i}{2}g_{IJ}\psi_-^I D_z\psi_-^J + \frac{i}{2}g_{IJ}\psi_+^I D_{\bar{z}}\psi_+^J + \frac{1}{4}R_{IJKL}\psi_+^I\psi_+^J\psi_-^K\psi_-^L\Big), \tag{2.31}$$

where $\Phi^I(z,\bar{z})$ are scalar functions, $\Psi_\pm^I$ (the superpartners of the Φ's) are Grassmann-valued sections of $\psi_\pm^I$ of $K^{\pm 1/2} \otimes \Phi^*(TX)$ (TX is the complexified tangent bundle to X and K is the canonical line bundle of Σ), and the covariant derivatives D are constructed by pulling back the Christoffel connection on TX.

For future reference, it is helpful to formulate this and subsequent theories in superfield notation. In general, we work with two kinds of superfields: Vector superfields V_a with component expansions (here, we work in Wess-Zumino gauge)

$$V = -\sqrt{2}(\theta^-\bar{\theta}^- v_{\bar{z}} + \theta^+\bar{\theta}^+ v_z - \theta^-\bar{\theta}^+\sigma - \theta^+\bar{\theta}^-\bar{\sigma}) + i(\theta^2\bar{\theta}^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}} - \bar{\theta}^2\theta^\alpha\lambda_\alpha) + \frac{1}{2}\theta^2\bar{\theta}^2 D, \tag{2.32}$$

Chiral superfields Φ_i with component expansions

$$\Phi = \phi + \sqrt{2}(\theta^+\psi_+ + \theta^-\psi_-) + \theta^2 F + \cdots\ , \tag{2.33}$$

where $\cdots$ are total derivative terms, as well as their complex conjugates $\bar{\Phi}$.

If X is a Kähler manifold with g a Hermitian metric, then the model is actually $N = 2$ supersymmetric. The condition for conformal invariance (vanishing beta function) of the model, while not known in exact form, is fulfilled if X is a Calabi-Yau manifold with g the unique Ricci-flat metric. While the exact form of this metric is not known for any nontrivial CY manifold, the theory with any other kinetic terms flows to the conformal fixed point, where the kinetic terms corresponds to the Ricci-flat metric. To make an appropriate string theory vacuum, you tensor a Calabi-Yau d-fold (a 2d-dimensional manifold) with $10 - 2d$ dimensions of the $N = 2$ free field theory. This then is what is known as *compactification on a Calabi-Yau manifold.*

Examples of CY manifolds include degree n hypersurfaces in P^{n+1}, complex projective $(n+1)$-space – of much interest to us in particular is the quintic hypersurface in P^4, defined by any quintic homogeneous polynomial that is transverse in the sense that there are no points at which the polynomial and all its derivatives simultaneously vanish. A specific example is the quintic at the Fermat point, with the hypersurface defined simply as the zero set of $\sum_{i=1}^{5} z_i^5$, where z_i are homogeneous coordinates for P^4.

In the above, M was compact, and therefore had no nonconstant holomorphic functions on it – so there were no terms that could be added that would preserve $N = 2$ SUSY. If, instead, one chooses a noncompact M with a nontrivial holomorphic function $W : M \to \mathbb{R}$, one can write a model like this, where Ψ_i are chiral superfields:

$$\int d^2z d^4\theta \, K(\Psi_1, \overline{\Psi}_1, ..., \Psi_n, \overline{\Psi}_n) + \left(\int d^2z d^2\theta \, W(\Psi_1, ..., \Psi_n) + \text{h.c.} \right) \ ,$$

Here, the the function W is known as the *superpotential.* Again, assume that under renormalization group flow this theory reaches a conformal fixed point. It has been proved, at least perturbatively, that the superpotential is invariant under RG flow – or, more accurately, undergoes only wavefunction renormalization (in particular, if W is a quasi-homogeneous function of its arguments, the whole superpotential term changes only by a constant factor). These models are known as *Landau-Ginzburg* models.

In particular, if we consider an LG model with a single chiral superfield Ψ and a superpotential $W = \Psi^{P+2}$, it has been shown [42] that the central charge of this model (at the conformal fixed point) is given by $c_P = \frac{3P}{P+2}$. It so happens that this is also the central charge of the pth minimal model. Since the overall symmetry algebra, central charge, and unitarity together uniquely determine the theory (through, e.g., explicit calculation of null states and then of correlation functions), this particular LG theory is in fact equivalent to the pth minimal model.

As Gepner did, one can then tensor together enough minimal models to make a string compactification – the condition is

$$\sum_{i=1}^{r} \frac{3P_i}{P_i + 2} = 3d \ . \qquad (2.34)$$

Since the Lagrangian of the tensor product of different theories is the sum of the individual Lagrangians, this model is also equivalent to an LG theory.

Since Gepner models involve precisely this procedure, followed by an orbifolding, it follows that Gepner models are equivalent to Landau-Ginzburg orbifolds. And, as Gepner established by careful analysis [62][63], in the case of the so-called 3^5 Gepner model and the quintic hypersurface in P^5, Gepner models are essentially equivalent to certain Calabi-Yau compactifications (they differ only in the existence of certain extra massless modes in the Gepner model, which are generically expected to acquire mass on small perturbations). Thus, one is led to conjecture some correspondence between Calabi-Yau compactifications and Landau-Ginzburg orbifolds. Others then labored to elaborate heuristic arguments for this correspondence [43, **?**]. This work was superseded by a seminal paper of Witten's, published in 1993 [90], that established a much deeper connection between Calabi-Yau compactifications and Landau-Ginzburg orbifolds, revealing both as phases of a larger, more encompassing theory – and simultaneously revealing the phase structure of $N = 2$ compactifications.

The way Witten did this was to introduce a new model, an $N = 2$ supersymmetric gauge theory he called the (gauged) linear σ model, which contains CY and LG theories as phases. In order to write this down, it is necessary to develop some notation. In addition to specifying the various supersymmetric multiplets involved in the construction, it proves to be useful for this and several related calculations of ours to use the language of toric varieties. The details and notation for this are covered in Appendix 1.

In terms of that notation, we will write down in general two models –

a GLσM on a toric variety, V, and another on a Calabi-Yau hypersurface, M, in a toric variety, V (it is well known that no compact Calabi-Yau manifold except the torus is a toric variety).

First, for the toric variety case, we choose $n-d$ abelian gauge superfields, V_a, corresponding to the action of the $(n-d)$-dimensional torus of the toric variety, and n chiral matter multiplets Φ_i with charge vectors Q_i^a under the actions of the gauge fields (these correspond to the different weights with which different coordinates transform under the various 1-parameter subgroups of the torus – all the notation is explained in Appendix 1). Then, we get

$$S = \int_\Sigma d^2z\, d^4\theta \left[\sum_{i=1}^{n} \bar{\Phi}_i \exp\left(2 \sum_{a=1}^{n-d} Q_i^a V_a \right) \Phi_i - \sum_{a=1}^{n-d} \frac{1}{4e_a^2} \overline{\Sigma}_a \Sigma_a - \sum_{a=1}^{n-d} r_a V_a \right] + \int_\Sigma d^2z \sum_{a=1}^{n-d} \frac{\theta_a}{2\pi i} f_a, \tag{2.35}$$

where f_a is the curvature of the gauge connection, and $\Sigma_a = \frac{1}{\sqrt{2}} \overline{D}_+ D_- V_a$ is the (twisted chiral) gauge-invariant field strength associated to the gauge field V_a with component expansion

$$\Sigma = \sigma - i\sqrt{2}(\theta^+ \bar{\lambda}_+ + \bar{\theta}^- \lambda_-) + \sqrt{2}\theta^+ \bar{\theta}^- (D - f) + \cdots . \tag{2.36}$$

The case in which we are actually interested, a Calabi-Yau hypersurface in a toric variety, is only slightly more involved. First, we add another chiral superfield Φ_0 with charges

$$Q_0^a = -\sum_{i=1}^{n} Q_i^a. \tag{2.37}$$

This addition cancels the anomalies in the R-invariance (see [91] for all the details), which guarantees $N=2$ superconformal invariance. Next, we add a superpotential term:

$$L_W = -\int d^2z d\theta^+ d\theta^- W(\Phi)|_{\bar{\theta}^+=\bar{\theta}^-=0} - \text{h.c.}, \tag{2.38}$$

with W a holomorphic, G-invariant function (again, see appendix 1 for notation). A generic choice of W will in fact break the R-symmetry, but if W is homogeneous of some degree in Φ_0 we can recover invariance by accompanying the action of Q_R by a rotation of this superfield. We will choose $W = \Phi_0 P(\Phi)$ where P does not depend on Φ_0. Then (2.37) shows that this is gauge invariant precisely when $P = 0$ determines a Calabi–Yau hypersurface in V. The nonanomalous R-symmetry is thus directly related to the Calabi–Yau condition. This is natural, as the latter is expected to be the condition for the existence of a nontrivial conformal theory in the low-energy limit.

This is a fairly, though not completely, general formulation of the Gauged Linear Sigma Model. One could easily generalize to Calabi-Yau complete intersections in toric varieties, and even, as was done in [90] to varieties, like Grassmannians, that are not toric. This formulation, however, is sufficiently general for us.

In fact, to illumine the basic point of [90], we will specialize to the simplest case Witten considered – the quintic hypersurface in P^4 (or more generally the degree n hypersurface in P^{n-1}) – where the phase structure is particularly simple.

So, using Witten's notation, take a single U(1) gauge field, V,n chiral superfields S_i of charge 1 under that gauge field, and one chiral superfield P of charge $-n$. This satisfies the condition of cancellation of the anomaly in R-symmetry, and we can also add a superpotential of the form $W = PG(S_1, \ldots, S_n)$, where G is homogeneous of degree n. Thus, we see that G can be defined as a polynomial on P^{n-1}, not merely a polynomial on $\mathbb{C}^n$. We also want G to satisfy a strong transversality condition:

$$0 = \frac{\partial G}{\partial S_1} = \cdots = \frac{\partial G}{\partial S_n} \tag{2.39}$$

has no solution except when all $S_i = 0$.

The bosonic potential in this Lagrangian is

$$U = |G(S_i)|^2 + |P|^2 \sum_i \left| \frac{\partial G}{\partial S_i} \right|^2 + \frac{D^2}{2e^2} + 2|\sigma|^2 \left(\sum_i |S_i|^2 + n^2 |P|^2 \right) \tag{2.40}$$

where

$$D = -e^2 \left(\sum_i |S_i|^2 - n|P|^2 - r \right) \tag{2.41}$$

Let's break this down into various cases, depending on the value of r. In each case, we are trying to look at the vacuum solution, with vanishing bosonic potential.

Case 1: $r > 0$. Here, to have $U = 0$, you must have $D = 0$, and the latter implies that at least one $S_i \neq 0$. This, in turn, going back to U, requires that $\sigma = 0$. It also implies, by our transversality condition, that at least one $\frac{\partial G}{\partial S_i} \neq 0$, which in turn implies that $P = 0$. And, of course, $U = 0$ implies that $G(S_i) = 0$. So the S_i's are the only potentially nonzero fields. They all have charge 1, there is an obvious action of the gauge field on them. Simply looking at the $S_i's$, where we have excluded the origin and where $\sum_i |S_i|^2 = r$, we get a manifold that looks like $(S^{2n-1} - \{0\})/U(1) = P^{n-1}$. But we also have $G(S_i) = 0$, so in fact the classical vacuum X is a hypersurface in P^{n-1} defined by a degree n homogeneous polynomial. Such a manifold is well known to be Calabi-Yau. If we take $n = 5$, we get the famous quintic hypersurface in P^4. Most crucially, all modes other than oscillations tangent to X have masses at tree level (the gauge field gets it from the Higgs mechanism, and for the others, it's clear from the bosonic potential plus the VEVs we have just worked out). And, clearly, r plays the role of the Kähler parameter (Calabi-Yau hypersurfaces in P^n have only a single Kähler parameter), setting the size of the vacuum manifold.

Case 2: $r < 0$. Here, in order to set the D-term to zero, we must have $P \neq 0$. This in turn gives $\frac{\partial G}{\partial S_i} = 0$ for all i, which by transversality of G gives us $S_i = 0$ for all i. Thus, P gets a VEV given by $|P| = \sqrt{-\frac{r}{n}}$. By a gauge rotation, we can fix $P = \sqrt{-\frac{r}{n}}$. Since $P \neq 0$, we also find that $\sigma = 0$. Thus, we see that there is a unique classical vacuum. Also, conditions on σ and on $\frac{\partial G}{\partial S_i}$ show that, as long as $n \geq 3$, all modes around the vacuum are massless at tree level. The superpotential, $W = \sqrt{-\frac{r}{n}} G(S_i)$, vanishes to nth order at the origin (a degenerate critical point). These are the classic characteristics of a Landau-Ginzburg theory. Actually, however, the VEV of P doesn't completely destroy the U(1) gauge invariance, because P has charge n – instead, the gauge symmetry is broken to $\mathbb{Z}_n$, the nth roots of unity. Thus, this is really not a Landau-Ginzburg theory but a Landau-Ginzburg orbifold (using the well known fact that orbifolding is the same as dividing by a finite gauge group).

These are not exact results, but for $|r| >> 0$ changes are in detail, not qualitative. Thus, the gauged linear sigma model interpolates between Calabi-Yau and Landau-Ginzburg orbifold phases.

We will work out one more example, which will be of use to use later. This example also reveals the phenomenon of *hybrid* phases.

Consider a GLσM based on the (toric resolution of) the degree-9 hypersurface (say, at the Fermat point) in the weighted projective space $P^{(1,1,1,3,3)}$. As worked out in Appendix 1, this surface has three blowup parameters, although only one is toric. This parameter gives us an extra gauge field, as well as an extra chiral superfield. Thus, we get six chiral superfields with charges under $U(1) \times U(1)$ given as follows:

	q_1	q_2
z_1	1	0
z_2	1	0
z_3	1	0
z_4	-3	1
z_5	0	1
z_6	0	1

Table 2.1: Homogeneous coordinates for the resolved model

To restrict to the hypersurface, as before, add one more field, p, of charge $(0, -3)$ and a gauge-invariant superpotential, $W = pP(z_i)$ to obtain the full GLσM for Y. A choice of gauge-invariant (and $\mathbb{Z}_3$-invariant) superpotential is given by

$$W = pP(z_i) = p(z_1^9 z_4^3 + z_2^9 z_4^3 + z_3^9 z_4^3 + z_5^3 + z_6^3). \qquad (2.42)$$

The possible vacuum configurations have to fulfill the D- and

F-flatness conditions:

$$
\begin{aligned}
F &= |P|^2 + |p|^2 \sum_i \left| \frac{\partial P}{\partial z_i} \right|^2 \\
D_1 &= |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2 - r_1 \\
D_2 &= |z_4|^2 + |z_5|^2 + |z_6|^2 - 3|p|^2 - r_2
\end{aligned}
\tag{2.43}
$$

The model has four phases, depending on the values of the parameters r_i. The limit points of each phase lie at the origin of coordinates for certain readily-defined coordinate patches on the moduli space. We will discuss the structure of the moduli space and define the coordinate patches U_{ij} in §5.3.3. In the meantime, we just label the phases by the corresponding patches:

U_{34} Phase: $r_1 > 0, r_2 > 0$. The excluded gauge orbits in this case are the orbits with $\{z_1 = z_2 = z_3 = 0\}$ and $\{z_4 = z_5 = z_6 = 0\}$. The F-terms require the vanishing of P and p. As a consequence, the low energy modes in this limit are a nonlinear σ-model on the (smooth) Calabi-Yau manifold.

U_{13} Phase: $r_1 < 0, 3r_2 + r_1 > 0$. The orbits $\{z_4 = 0\}$ and $\{z_1 = z_2 = z_3 = z_5 = z_6 = 0\}$ have to be excluded. In a generic D-flat configuration, z_4 is not zero. The Calabi-Yau develops a $\mathbb{Z}_3$ orbifold singularity at the location of the blown-down exceptional divisor.

U_{12} Phase: $r_1 < 0, 3r_1 + r_2 < 0$. To fulfill D-flatness, the orbits $\{z_4 = 0\}$ and $\{p = 0\}$ have to be excluded. The F-terms require that $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = 0$. A gauge transformation by $e^{i\theta q_1}$ leaves p invariant, while rotating z_4. A gauge transformation by $e^{i\theta'(q_1+3q_2)}$ leaves z_4 invariant, while rotating p. We can use these two $U(1)$ actions to fix the values of z_4 and p

completely, so that the vacuum consists of one point. Around this vacuum, there are fluctuations of the fields z_1, z_2, z_3, z_5, z_6. The VEVs for z_4 and p leave unbroken a $\mathbb{Z}_9$ subgroup of the $U(1) \times U(1)$, generated by $e^{2\pi i(q_1+3q_2)/9}$.

U_{24} Phase: $r_1 > 0, r_2 < 0$. The orbits $\{p = 0\}$ and $\{z_1 = z_2 = z_3 = 0\}$ have to be removed. This phase corresponds to a hybrid phase: The fields z_1, z_2, z_3 parametrize a P^2, over which the fluctuations of the fields $z_4, \ldots, z_6$ behave like in a LG theory.

Of course, the correspondence between Gepner models and Landau-Ginzburg orbifolds then shows us that in models with a pure Landau-Ginzburg phase there is a *Gepner point* in the moduli space, a concept that will be critical later.

Next, we need only understand D-branes at some simple points of the (2,2) moduli space to get underway with the program.

Chapter 3

D-Branes in (2,2) Compactifications

Central to the task of studying monodromies of D-branes to understand their physics better will be the question of relating D-branes in Gepner models (at the Gepner point in the (2,2) moduli space) to D-branes on Calabi-Yau manifolds (at a geometric point and, in particular, at the large-volume limit in moduli space). In this chapter, we review the construction of each.

3.1 D-Branes on Calabi-Yau Manifolds

In this section, we will work out the basic geometric interpretation of (BPS) D-branes on Calabi-Yau manifolds, using the techniques developed earlier. This will follow closely the original discussion in [80].

For a Calabi-Yau compactification, the chiral algebra which has to be preserved along the boundary is the $N = 2$ world-sheet supersymmetry algebra. Since we will be using the notation of [80], we will re-state the algebra in terms of operator product expansions of conformal fields, rather than as commutation and anti-commutation relations of modes:

$$\begin{aligned}
T(z)T(w) &= \frac{c/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \cdots \ , \\
T(z)G^{\pm}(w) &= \frac{3/2}{(z-w)^2}\, G^{\pm}(w) + \frac{\partial_w G^{\pm}(w)}{z-w} + \cdots \ , \\
T(z)J(w) &= \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + \cdots \ , \\
G^{+}(z)G^{-}(w) &= \frac{2\,c/3}{(z-w)^3} + \frac{2J(w)}{(z-w)^2} + \frac{2T(w)+\partial_w J(w)}{z-w} + \cdots \ , \\
J(z)G^{\pm}(w) &= \pm\,\frac{G^{\pm}(w)}{z-w} + \cdots \ , \\
J(z)J(w) &= \frac{c/3}{(z-w)^2} + \cdots \ .
\end{aligned}$$

We are working in the context of a Calabi-Yau compactification, i.e. the conformal field theory is the minimal $N = 2$ supersymmetric σ model with target space some Calabi-Yau manifold, M. Thus, there is a standard representation of the fields that generate the $N = 2$ SCA:

Throughout this paper, we set the signs of the left and the right $U(1)$ currents to be

$$J_L = g_{i\bar{j}}\psi^i_L\psi^{\bar{j}}_L, \qquad J_R = g_{i\bar{j}}\psi^i_R\psi^{\bar{j}}_R \ , \tag{3.1}$$

which determines the convention for $G^{\pm}$ as

$$G^+_L \;=\; g_{i\bar{j}}\psi^i_L\partial X^{\bar{j}}, \quad G^-_L = g_{i\bar{j}}\psi^{\bar{j}}_L\partial X^i \ , \tag{3.2}$$

$$G^+_R \;=\; g_{i\bar{j}}\psi^i_R\bar{\partial} X^{\bar{j}}, \quad G^-_R = g_{i\bar{j}}\psi^{\bar{j}}_R\bar{\partial} X^i \tag{3.3}$$

In addition, in order to preserve half of the spacetime supersymmetry, we should take into account the spectral flow operator $e^{i\phi_L}$ defined by

$$\$e^{i\phi_L} = \Omega_{i_1\ldots i_d}\psi^{i_1}\ldots\psi^{i_d} \tag{3.4}$$

Here Ω is the holomorphic d-form on the Calabi-Yau d-fold and $J_L = i\partial\phi_L$. Note that, in this convention, the $N = 1$ supercurrent is generated by

$$G = G_L^+ + G_L^- \tag{3.5}$$

This algebra has a well-known automorphism, the mirror-automorphism, which reverses the sign of the U(1) charge. Also, in order for a boundary state to represent a BPS state in spacetime, the boundary must preserve half of the spacetime supersymmetry – i.e., the boundary state must be invariant under a linear combination of the left and right $N = 2$ algebra extended by the spectral flow operators. Analyzing the possibilities, we get two types of boundary conditions [80]:

A-type boundary condition:[1]

$$J_L = -J_R, \quad G_L^+ = \pm G_R^-, \quad e^{i\phi_L} = e^{-i\phi_R} \tag{3.6}$$

B-type boundary condition:

$$J_L = +J_R, \quad G_L^+ = \pm G_R^+, \quad e^{i\phi_L} = (\pm 1)^d e^{i\theta} e^{i\phi_R} \tag{3.7}$$

The phase factor $e^{i\theta}$ will be determined later. In the A-type boundary condition, it can be absorbed in the definition of Ω.

Both A-type and B-type conditions preserve the $N = 1$ SCA

$$T_L = T_R, \quad G_L = \pm G_R \tag{3.8}$$

[1]Here, boundary conditions are in the notation appropriate for the open string channel.

Now, we need to work through the geometric realization of these two classes of conditions. (3.8) can be solved by

$$\partial X^\mu = R^\mu{}_\nu \bar{\partial} X^\nu, \qquad \psi_L^\mu = \pm R^\mu{}_\nu \psi_R^\nu \ . \tag{3.9}$$

for some matrix R that satisfies

$$g_{\mu\nu} R^\mu{}_\rho R^\nu{}_\sigma = g_{\rho\sigma} \ . \tag{3.10}$$

Eigenvectors of R with eigenvalue -1 give Dirichlet boundary conditions for X and those with eigenvalue +1 give Neumann conditions. In general, R can give rise to mixed Neumann-Dirichlet conditions (which correspond to cases when the D-brane worldvolume U(1) gauge field has nonzero field strength).

In the neighborhood of a p-cycle γ on a Calabi-Yau d-fold, one can choose local coordinates x^A $(A = 1, ..., p)$ (coordinates on the cycle) and y^a $(a = 1, ..., 2d - p)$ (coordinates transverse to the cycle).

Recall the Kähler form is given by an antisymmetrized version of the (Hermitian) metric:

$$k = i g_{i\bar{j}} dz^i \wedge dz^{\bar{j}}. \tag{3.11}$$

Now, impose the B-type boundary condition on J:

$$k_{\mu\nu} R^\mu{}_\rho R^\nu{}_\sigma = k_{\rho\sigma} \ . \tag{3.12}$$

The condition on the spectral flow operator gives:

$$\Omega_{\mu_1 \ldots \mu_d} R^{\mu_1}_{\nu_1} \ldots R^{\mu_d}_{\nu_d} = e^{i\theta} \Omega_{\nu_1 \ldots \nu_d} \ . \tag{3.13}$$

From (3.12) we see that

$$k_{Ab} = 0 \ , \tag{3.14}$$

so the Kähler form must be block diagonal on γ in the tangential and normal directions to γ. Since k is nondegenerate, each block must also be. Since k is a symplectic form, this implies that $dim(\gamma)$ is even (which will also make its codimension even). In particular, k_{ab} can be used to define an almost complex structure on γ and, in fact, this structure is integrable and thus defines a genuine complex structure on γ. Or, in other words, γ is a holomorphic submanifold of M. This clearly tells us that in complex coordinates the non-vanishing components of Ω have $p/2$ holomorphic indices tangent to γ and $d - p/2$ transverse to γ. This fact, in turn, determines for us the phase $e^{i\theta}$ in terms of the gauge field on γ. When that gauge field is flat, so the eigenvalues of R are all ± 1, we find that $e^{i\theta} = (-1)^{d-p/2}$.

For A-type boundary conditions, we get instead

$$k_{\mu\nu} R^{\mu}{}_{\rho} R^{\nu}{}_{\sigma} = -k_{\rho\sigma} \ , \tag{3.15}$$

$$\Omega_{\mu_1 \ldots \mu_d} R^{\mu_1}_{\nu_1} \ldots R^{\mu_d}_{\nu_d} = \bar{\Omega}_{\nu_1 \ldots \nu_d} \ . \tag{3.16}$$

If the background gauge field on γ is flat, then $R^2 = \mathrm{Id}$. Then, (3.15) implies

$$k_{ab} = 0, \qquad k_{AB} = 0 \ . \tag{3.17}$$

Since the full k is nondegenerate, this implies that γ is Lagrangian – in particular, we must have $p = d$ (M is a d-fold). We also find that

$$\Omega_{a_1 \cdots a_m A_{m+1} \cdots A_d} \sim k_{a_1}^{\ A_1} \cdots k_{a_m}^{\ A_m} \Omega_{A_1 \cdots A_d} \tag{3.18}$$

for $m = 1, ..., d$. Thus, we see that $\Omega \wedge \bar{\Omega}$ is proportional to the volume form of the d-fold, which implies that the pull-back of Ω onto γ is proportional to the volume form of γ. A cycle that obeys these two conditions is known as a *special Lagrangian* cycle.

3.2 D-Branes in Gepner models

The other piece we will need to understand the variation of D-branes on moduli spaces of (2,2) compactifications is how to represent D-branes in Gepner models. After the initial work of Recknagel and Schomerus [81], there have been several other papers on the subject [82][83][84][85][86][87][88]. We need only the results of [81].

Note that a Gepner model is *not* a rational conformal field theory, and we have no assurance *ab initio* that the Cardy solution works. Since it is built of minimal models, it is, of course, rational with respect to the tensor product of the individual $N = 2$ superconformal algebras, but it is not with respect to the overall "diagonal" algebra (where, e.g., the stress-energy tensor is the sum of the individual stress-energy tensors).

In order to be able to use the Cardy schema, we will choose to focus only on what might be called "rational" boundary states – states that obey boundary conditions on each of the individual $N = 2$ algebras separately. Although there is some concern about the loss of generality in this approach, there is reason to believe that it is at least more general than it naively appears to be (see [81], section 4.2).

Even with this restriction, we are still not assured that Cardy-type states automatically obey the Cardy conditions, so those must be checked by hand. There are two cases, where each algebra separately obeys A-type boundary conditions or where each obeys B-type boundary conditions (mixing them would obviously mean that the total algebra obeyed neither one nor the other). We are explicitly leaving out the case where a given Gepner model has in it at least two identical minimal models, and we use permutation automorphisms to glue the left-moving generators of one algebra to the right moving generators of the other.

We will simply give the results – calculations are worked out in detail in [81].

For A-type "rational" boundary states, one can make the following ansatz:

$$|\alpha\rangle_A \equiv |S_0; (L_j, M_j, S_j)_{j=1}^r\rangle_A = \frac{1}{\kappa_\alpha^A} \sum_{\lambda,\mu}^{\beta} B_\alpha^{\lambda,\mu} \, |\lambda, \mu\rangle\rangle_A \qquad (3.19)$$

where S_0, L_j, M_j, S_j are integer labels; the summation is over states satisfying the same "β-constraints" as in Gepner's original partition function, κ_α^A is a normalization constant, and

$$B_\alpha^{\lambda,\mu} = (-1)^{\frac{s_0^2}{2}} e^{-i\pi \frac{d}{2} \frac{s_0 S_0}{2}} \prod_{j=1}^{r} \frac{\sin \pi \frac{(l_j+1)(L_j+1)}{k_j+2}}{\sin^{\frac{1}{2}} \pi \frac{l_j+1}{k_j+2}} \, e^{i\pi \frac{m_j M_j}{k_j+2}} \, e^{-i\pi \frac{s_j S_j}{2}} \, . \qquad (3.20)$$

Verification that this ansatz satisfies the Cardy conditions is a straightforward calculation – one simply checks that $Z_{\alpha\tilde{\alpha}}(q) = \langle\Theta\tilde{\alpha}|\, \tilde{q}^{L_0-\frac{c}{24}}\, |\alpha\rangle$ does in fact give an acceptable open string spectrum. The only complication is the constraints in the β sum, which require introduction of Lagrange multipliers to sort out. This calculation also yields the normalization:

$$\kappa_\alpha^A = 2\left(2^{\frac{r}{2}}\frac{(k_1+2)\cdots(k_r+2)}{K}\right)^{\frac{1}{2}} \tag{3.21}$$

where $K = \text{lcm}(4, 2k_j + 4)$.

Since the spin structures of the component theories in $Z_{\alpha\tilde{\alpha}}(q)$ must be coupled as in the closed string case (NSNS or RR – see for example §1.2.1),we must require

$$S_0 - \tilde{S}_0 \equiv S_j - \tilde{S}_j \pmod 2 \tag{3.22}$$

for all $j = 1, \ldots, r$

From the calculation, we also see, as expected, that only states with odd total charge are present. Equivalently,

$$Q(\alpha - \tilde{\alpha}) := -\frac{d}{2}\frac{S_0 - \tilde{S}_0}{2} - \sum_{j=1}^{r}\frac{S_j - \tilde{S}_j}{2} + \sum_{j=1}^{r}\frac{M_j - \tilde{M}_j}{k_j+2}. \tag{3.23}$$

For B-type states, the U(1) charges of left- and right-moving highest weight states must satisfy $q_i = -\bar{q}_{\bar{\imath}}$ (and $h_i = \bar{h}_{\bar{\imath}}$). The condition on the charge puts much more stringent conditions on the B-type states than it did on the A-type – in order for a piece in the Gepner partition function, of the form $\chi_\mu^\lambda(q)\,\chi^\lambda_{\mu+b_0\beta_0+b_1\beta_1+\ldots+b_r\beta_r}(\bar{q})$, to obey this constraint one can show that it is necessary that

$$m_j \equiv b \ (\operatorname{mod} k_j + 2) \tag{3.24}$$

for some $b = 0, 1, \ldots, \frac{K}{2} - 1$ and for all j. This can be shown by recalling that the charge of a highest-weight state in an individual minimal model is given by $q = \frac{m}{k+2} - \frac{s}{2}$ and recalling the definitions of the β's.

So now we can make an ansatz for B-type states:

$$|\alpha\rangle_B \equiv |S_0; (L_j, M_j, S_j)_{j=1}^r\rangle_B = \frac{1}{\kappa_\alpha^B} \sum_{\lambda,\mu}^{\beta,b} B_\alpha^{\lambda,\mu} \, |\lambda, \mu\rangle\rangle_B \tag{3.25}$$

with coefficients $B_\alpha^{\lambda,\mu}$ as before (3.20).

After performing a calculation similar to that with the A-type states [81], we find the normalization to be

$$\kappa_\alpha^B = 2^{\frac{r}{4}} \tag{3.26}$$

and once again we get the condition (3.22).

Now that we have these building blocks as boundary states in solvable conformal field theories, we can begin to make comparisons with D-branes described as geometric objects – among the quantities to be compared are RR-charge and moduli, as we will see in the next chapter.

Chapter 4

Introduction to the Douglas Program

In 1999, Brunner, Douglas, Lawrence, and Römelsberger wrote an important paper [92], opening up a program of inquiry that has been followed up, among others, by [93][94][95][113][99][100][111][112]. This program, in brief, involves taking the existing knowledge about (2,2) compactifications, and in particular about D-branes, at large volume and relating it to that about Gepner models and in particular about D-branes in Gepner models. For a Calabi-Yau manifold with a Gepner point, existing and well understood knowledge of the geometrical phase, following in the tradition of [66][67][68][69][70][71] [72][73][74], can be directly related to the conformal field theory understanding of the Gepner point through consideration of monodromies and of the intersection pairing (about which more later). Once a tentative correspondence is established, it can be tested by looking, for example, at moduli of D-branes (geometric deformations at large volume and marginal operators at the Gepner point). Thus, one can attempt to find nontrivial, nonperturbative results. Does the $D0$-brane live everywhere on the moduli space? Does it always have 3 (complex) moduli (for compactification on a Calabi-Yau 3-fold – here, when we say $D0$ we are referring to extent in the Calabi-Yau directions, ignoring extent in the 4-dimensional Minkowski space that comes along for the ride)? Are there obstructions to the naive geometric moduli space that come from superpotentials on the brane worldvolume (say the CFT calculation shows fewer moduli than the geometric one)? Are there previously unknown bound states (say the geometric calculation shows fewer moduli)?

The line of development of [92] and others cited here has involved considering BPS D-branes on simply-connected Calabi-Yau's. The absence of torsion in the cohomology of the Calabi-Yau's considered has meant that K-theory doesn't play that big a role in these calculations, something that will change in the next section, when we try to apply these same general considerations to torsion D-branes on nonsimply-connected Calabi-Yau's.

First, however, to give the flavor of the considerations and to establish some results and notation that will be indispensable later on, we give a partial exposition of the results of [92]. These results are contained in sections 2 through 5 of the paper.

The main idea is to study the entire moduli space of a particular Calabi-Yau compactification, starting with the exact description of D-brane states we have at the Gepner point and in the large volume limit (LVL) (where instanton corrections vanish). They look in particular at the correspondence between the symplectic intersection form on 3-cycles in the LVL and the index $\mathrm{Tr}_{ab}(-1)^F$ in the open string conformal field theory description, which can easily be calculated at the Gepner point. Since D-brane charges form a discrete group, and the intersection form is a pairing on this discrete group, as we move around the moduli space the only changes in D-brane charge we can see are by automorphisms (we will often refer to them as monodromies) of the discrete group that preserve the pairing. Since the automorphism group of a discrete group is also discrete, the monodromy introduced in going around a homotopically trivial loop in the moduli space is just the identity. In fact, the monodromies will provide a representation of the fundamental group of the appropriate moduli space.

Studying the intersection form enables us to create a mapping between D-branes in the LVL and at the Gepner point. With this mapping, we can do a lot: use geometric methods in one case and CFT methods in the other to look at D-brane moduli, look for lines of marginal stability at which certain states disappear, and much more.

We will give a somewhat cursory review, partly because some of these concepts are developed at length in the classical literature (e.g., [71])and partly because we will reformulate this in a manner more useful for our investigations of torsion branes. This presentation follows closely the lines of [92].

Throughout this investigation, we will be able to obtain insight by going back and forth between a compactification on a given Calabi-Yau manifold and compactification on its mirror. The classification of A-type and B-type branes [80] immediately implies a correspondence between $H^3(M)$ and $H^{ev}(W)$, where W is the mirror of M. The symplectic intersection form on $H^3(M)$ has a counterpart on $H^{ev}(W)$ – we will see in the next section that this is best expressed in the language of K-theory.

Type IIB string compactification on a general Calabi-Yau threefold M leads to an $N = 2$, $d = 4$ supergravity with $b_{2,1} + 1$ vector fields ($b_{2,1}$ vector multiplets plus the graviphoton) and $b_{1,1} + 1$ hypermultiplets (including the 4d dilaton); in IIA, these identifications are reversed. The special geometry of the vector multiplets determines the basic physical observables. This geometry in turn is determined by a prepotential F_K of Kähler deformations in the IIA case, and by the prepotential F_c for complex structure deformations in the IIB case.

F_c can be determined entirely from classical target space geometry; it receives no worldsheet quantum corrections. We can coordinatize the complex structure moduli space as follows. Choose a basis for the 3-cycles $\Sigma^i \in H_3(M, \mathbb{Z})$ (where $i = 0, \ldots, b_{2,1}, b_{2,1} + 1, \ldots, 2b_{2,1} + 2$), so that the intersection form $\eta^{ij} = \Sigma^i \cdot \Sigma^j$ takes the canonical form $\eta^{i,j} = \delta_{j,i+b_{2,1}+1}$ for $i = 0, \ldots, b_{2,1}$ (an a cycle with a b cycle – one can choose such a symplectic basis on any Calabi-Yau threefold). The $b_{2,1} + 1$ vector fields come from reducing the RR potential $C^{(4)}$ on the a cycles, while the b cycles produce their $d = 4$ electromagnetic duals. Thus a three-brane wrapped about the cycle $\Sigma = \sum_i Q_i \Sigma^i$ has (electric,magnetic) charge vector Q_i.

In general, $H_3(X)$ forms a nontrivial bundle over the moduli space $\mathcal{M}$ of complex structures; a given basis in $H_3(X, \mathbb{Z})$ will have monodromy in $Sp(b_3, \mathbb{Z})$ as it is transported around singularities in $\mathcal{M}$.

The description can be done in terms of the periods of the holomorphic three-form,

$$\Pi^i = \int_{\Sigma^i} \Omega.$$

In $N = 2$ language these are the vevs of the scalar fields in the corresponding vector multiplets. The a-cycle Π^i's can be used as projective coordinates on the moduli space; the b-cycle periods then satisfy the relations $\Pi^j = \eta^{ij}\partial\mathcal{F}/\partial$. If we fix (for example) $\Pi^0 = 1$ to pass to inhomogeneous coordinates, the related vector field is the graviphoton. These periods determine the central charge of a three-brane wrapped about the cycle $\Sigma = \sum_i Q_i[\Sigma^i]$:

$$Z = \int_{\Sigma} \Omega = Q_i \Pi^i.$$

Thus the mass of a BPS three-brane is

$$m_Q = c|Z| = c|Q \cdot \Pi| \tag{4.1}$$

where c is independent of Q.

F_K, on the other hand, receives world-sheet instanton corrections to the classical computation – even so, the exact worldsheet result can be obtained by mirror symmetry: F_K for IIA on M is equal to F_c for IIB on the mirror W to M. Of course this requires a map between the periods of M and W.

Before proceeding, we need a few more key concepts. The D-brane charge in the large-volume limit (LVL) is,

$$v(E) = ch(f_! E)\sqrt{\hat{A}(M)}, \tag{4.2}$$

(where E is a vector bundle Σ, the submanifold of M on which the brane in question is wrapped). In examples without torsion, such as the quintic, one may describe the D-brane charge more directly. Assuming the branes give rise to particles in the macroscopic directions, for a $2n$-dimensional worldvolume Σ we can write the D-brane coupling to the RR gauge fields via the "Wess-Zumino term" as [37][38]

$$\int_{\Sigma} C \wedge \mathrm{ch}(F - B) \sqrt{\frac{\hat{A}(M)}{\hat{A}(N)}} \tag{4.3}$$

where

$$C = C^{(2n+1)} + C^{(2n-1)} + \ldots + C^{(1)} \tag{4.4}$$

is a sum over the (k)-form RR potentials that couple to the $2n$-brane.

These RR charges reduce to conventional electric and magnetic charges in the four noncompact dimensions. Given two D-branes which reduce to particles, we can study the Dirac-Schwinger-Zwanziger symplectic inner product on their charges,

$$I(a,b) = Q_{Ea} \cdot Q_{Mb} - Q_{Ma} \cdot Q_{Eb} \ . \tag{4.5}$$

This can be termed the "intersection form," as it is closely related to the topological intersection form for two- and four-branes. For two six-branes, from the formulas above it is

$$I(a,b) = \int \mathrm{ch}(F_a)\ \mathrm{ch}(-F_b)\ \hat{A}(M) \ . \tag{4.6}$$

So far, the exposition has been general. In order to obtain specific results in a case where the special geometry had already been worked out, Douglas, et al., chose to specialize to the quintic in P^4, in many ways the simplest nontrivial example of a Calabi-Yau threefold.

The mirror W to the quintic threefold M can be obtained [75] as a Z_5^3 quotient of a special quintic

$$0 = \sum_{i=1}^{5} z_i^5 - 5\psi z_1 z_2 z_3 z_4 z_5 \ .$$

The transformation $\psi \to \alpha\psi$ with $\alpha^5 = 1$ can be undone by the coordinate transformation $z_1 \to \alpha^{-1} z_1$ and thus the complex moduli space of W's can be parameterized by ψ^5.

The moduli space $\mathcal{M}$ has three singularities, about which the three-cycles in W will undergo monodromy. Each singularity has physical significance. First, $\psi^5 \to \infty$ is the "large complex structure limit" mirror to the large volume limit. In this limit [115],

$$(5\psi)^{-5} \to e^{2\pi i(B+iJ)}, \tag{4.7}$$

where B is the NS B-field flux around the 2-cycle forming a basis of $H_2(M)$, and J is the size of that 2-cycle. Next, $\psi^5 \to 1$ is a conifold singularity; here a wrapped three-brane becomes massless [118][119]. This turns out to be mirror to the "pure" six-brane [120][121]. Finally, at $\psi^5 = 0$ the model obtains an additional Z_5 global symmetry; this is an orbifold singularity of moduli space. The Gepner model $(3)^5$ lives at this point in Kähler moduli space of M [90].

Each singularity in $\mathcal{M}$ gives a noncontractible loop, which is associated with a monodromy on the basis of 3-cycles in W (or, by mirror symmetry, even homology in M) and thus on the periods. We let A be the monodromy induced by $\psi \to \alpha\psi$ around $\psi = 0$; clearly $A^5 = 1$. T will be the monodromy induced by going once around the conifold point, and B will be the monodromy induced by taking $\psi \to \alpha^{-1}\psi$ around infinity. These satisfy the relation $B = AT$ (just look at the fundamental group of the triply-punctured sphere, which is equivalent to the doubly-punctured plane). One may make the physics associated with a given singularity manifest by choosing variables (the periods) for which the associated monodromy is simple.

In our case the periods Π_i satisfy a Picard-Fuchs differential equation of fourth order, since $b^3 = 4$. There will be four independent solutions. Some natural bases include the following: the large volume basis which we will denote $(\Pi_6, \Pi_4, \Pi_2, \Pi_0)^t$. Up to an upper triangular transformation this is determined by the asymptotics as $\psi^5 \to \infty$

$$\begin{pmatrix} \Pi_6 \\ \Pi_4 \\ \Pi_2 \\ \Pi_0 \end{pmatrix} \to \begin{pmatrix} -\frac{5}{6}(B+iJ)^3 \\ -\frac{5}{2}(B+iJ)^2 \\ B+iJ \\ 1 \end{pmatrix}. \tag{4.8}$$

The coefficients correspond to the classical volumes of the cycles. The signs were chosen so that the supersymmetric brane configurations have positive relative charges.

Another natural basis, the Gepner basis, makes the monodromy A simple. If we choose a solution $\Pi_0^G(\psi)$ analytic near $\psi = 0$, the set of solutions

$$\Pi_i^G(\psi) = \Pi_0^G(\alpha^i \psi) \tag{4.9}$$

will provide a basis with the single linear relation $0 = \sum_{i=0}^4 \Pi_i^G$. It turns out that the 0-brane period Π_0 is analytic near $\psi = 0$ and thus we can set $\Pi_0^G = \Pi_0$

and define the others using (4.9). We then (as in [71] choose the period vector $(\Pi_2^G, \Pi_1^G, \Pi_0^G, \Pi_4^G)^t$. In this basis, the three monodromy matrices are

$$A^G = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} T^G = \begin{pmatrix} 1 & 4 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 4 & -4 & 1 \end{pmatrix} B^G = \begin{pmatrix} -1 & -7 & 5 & -1 \\ 1 & 4 & -4 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \tag{4.10}$$

In [71], the Gepner and large volume (LV) bases are related through a third basis in which the conifold monodromy takes a simple form: if we label the elements of the basis Π_i^3,the conifold monodromy takes the form $\Pi_i^3 \to \Pi_i^3 + \delta_{i,2}\Pi_4^3$. So Π_4^3 is the vanishing cycle at the conifold. Using this device, they calculate the matrix relating the Gepner and LV bases to be

$$\Pi = M\Pi^G \qquad Q = Q^G M^{-1} \qquad A = MA^G M^{-1} \ldots \quad M = L \begin{pmatrix} 0 & -1 & 1 & 0 \\ -\frac{3}{5} & -\frac{1}{5} & \frac{21}{5} & \frac{8}{5} \\ \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Here Q and Q^G are the charge vectors in the large-radius and Gepner basis respectively. (In the notation of [71], $M = KNm$: with K a matrix taking the vector $(Q_4, -Q_6, Q_2, Q_0)$ of their conventions to our conventions; and N taken with $a' = b' = c' = 0$.) The matrix L is an as-yet undetermined $Sp(4, Z)$ ambiguity in the Q_2 and Q_0 charges of the six- and four-branes. Given the classical intersection form η in the large-radius limit, we can now determine the intersection form in the Gepner basis:

$$\eta_g = M^{-1}\eta(M^{-1})^t = \begin{pmatrix} 0 & -1 & 3 & -3 \\ 1 & 0 & -1 & 3 \\ -3 & 1 & 0 & -1 \\ 3 & -3 & 1 & 0 \end{pmatrix} \tag{4.11}$$

L does not enter since it is symplectic, and so preserves η. η_g has determinant 25 and thus the Gepner basis is not canonically normalized.

4.1 A-branes on the Quintic

We will work through only one example from the geometric part of the moduli space. We know A-branes on Calabi-Yau's correspond to special Lagrangian cycles.

$$\begin{aligned} \omega|_\Sigma &= 0 \\ \mathrm{Re} e^{i\theta}\Omega|_\Sigma &= 0 \ . \end{aligned}$$

Here Ω is the holomorphic 3-form of the Calabi-Yau and θ is an arbitrary phase. Equivalently to the second equation, we can require that Ω pulls back to a constant multiple of the volume element on Σ. Furthermore the gauge field on this manifold must be flat.

The simplest example of supersymmetric 3-cycles on the quintic are the real surfaces $\mathrm{Im}\omega_j z_j = 0$ with $\omega_j^5 = 1$. These cycles are determined by the five phases $(\omega_1, \omega_2, \omega_3, \omega_4, \omega_5)$ up to the diagonal $\mathbb{Z}_5$ action $\omega_i \to \omega\omega_i$ (which is just a remnant of the equivalence of homogeneous coordinates under complex multiplication), so they come in a 625-dimensional (obviously reducible) representation of the discrete symmetry $S_5 \times \mathbb{Z}_5^4$. Restricting consideration just to the $\mathbb{Z}_5^4$ subgroup, we see that it acts on the given set of cycles in a simply transitive manner – thus, the cycles furnish the *regular representation* of this subgroup.

The equation $\sum(\omega_j x_j)^5 = 0$, where $\omega_i x_i \in \mathbb{R}$, always has a unique solution for x_k in terms of the other real coordinates; thus the cycle is the real projective space $\mathbb{R}P^3$. The first homotopy group is $\pi_1(\mathbb{R}P^3) = \mathbb{Z}_2$; by a result of Hitchin's [126], the wrapped 3-branes cannot have any continuous moduli, but they can support a discrete Z_2-valued Wilson line.

To compare these cycles with Gepner boundary states it will be useful to find their intersection matrix. Choose the coordinate system $z_1 = 1$ on P^4, so that $\omega_1 = 1$. Regard the cycle $(1,1,1,1,1)$ as an embedding of the coordinates x_2,x_3 and x_4 into the quintic with positive orientation. The other surfaces are obtained by Z_5^4 rotation from this one, $\prod_{i=1}^5 g_i^{k_i}(1,1,1,1,1)$. Since the action of Z_5^4 generates the representation (i.e., it acts in a simply transitive manner) and since the intersection matrix must respect the Z_5^4 symmetry, the intersection matrix can actually be written as a polynomial in the generators g_i and is determined by the matrix elements

$$\langle(1,1,1,1,1)|(1,\omega_2,\omega_3,\omega_4,\omega_5)\rangle = \langle(1,1,1,1,1)|g_2^{k_2}g_3^{k_3}g_4^{k_4}g_5^{k_5}|(1,1,1,1,1)\rangle \tag{4.12}$$

where $g_i^{k_i} : z \to \omega^{k_i} z$. S_5 symmetry also constrains the problem in an obvious way.

To find the intersection product, consider the following simpler problem – there are two coordinates instead of five and there is one $\mathbb{Z}_5$ instead of four. For the trivial element (g^0), we get a null intersection, since we can just displace the submanifold parallel to itself. For g^1 or g^2, we get $+1$, since the angle from the first state to the second is less than π (remember, we have an a symplectic product). For g^3 or g^4, we get -1 since that angle is now greater than π. Thus, this suggests that $g + g^2 - g^3 - g^4$ should be a generating function for the intersection product, and

$$I_{\mathbb{R}P^3} = \prod_{i=1}^{5}(g_i + g_i^2 - g_i^3 - g_i^4) = \prod_{i=1}^{5}(1 + g_i - g_i^2 - g_i^3) \qquad (4.13)$$

(since $\prod_{i=1}^{5} g_i = 1$).

4.2 Gepner Model States

To explore the charge lattice of the boundary states, and to find the geometric interpretation of given boundary states, we wish to calculate the intersection product of our branes. The CFT quantity which computes this is $I_\Omega = \mathrm{tr}_R(-1)^F$ in the open string sector [125]. The best way to do this is to start in the closed string sector and to do a modular transformation to the open string sector. In the closed string sector this trace corresponds to the amplitude between the RR parts of the boundary states with a $(-1)^{F_L}$ inserted. The calculation is done in [92] in the appendix.

The result for A-type boundary states is:

$$I_A = \frac{1}{C}(-1)^{\frac{S-\tilde{S}}{2}} \sum_{\nu_0=0}^{K-1} \prod_{j=1}^{r} N_{L_j,\tilde{L}_j}^{2\nu_0+M_j-\tilde{M}_j} \ . \qquad (4.14)$$

For B-type boundary states,

$$I_B = \frac{1}{C}(-1)^{\frac{S-\tilde{S}}{2}} \sum_{m'_j} \delta^{(K')}_{\frac{M-\tilde{M}}{2}+\sum \frac{K'}{2k_j+4}(m'_j+1)} \prod_{j=1}^{r} N_{L_j,\tilde{L}_j}^{m'_j-1}. \qquad (4.15)$$

The intersection matrix depends only on the differences $M - \tilde{M}$ as was required by the discrete symmetry. We also see that the $\mathbb{Z}_2$ action $S \to S + 2$ changes the orientation of a brane.

4.3 A- and B-Type States in the 3^5 model

Let us apply these results to the model $(k = 3)^5$, the Gepner point in the moduli space of the quintic. We will consider boundary states labelled by $L_j \in \{0, 1\}, 0 \leq M_j < (2k+4) = 10$, and $S = 0$. Let the $\mathbb{Z}_5^4$ symmetry be generated by the operators g_j taking $M_j \to M_j + 2$, and satisfying $g_1 \cdots g_5 = 1$. Note that $g_j^{1/2}$ which takes $M_j \to M_j + 1$ is well-defined for these states (using the identifications on LMS, it relates branes to antibranes).

We will be particularly interested in computing the intersection forms (4.14) and (4.15), as we will be able to use them to extract the charges and open string spectrum for a given brane. The main advantage of considering these quantities over the charges themselves is that they are canonically normalized.

Let's look in more detail at the A-type boundary states with $L_j = 0$. Their intersection matrix (4.14) is

$$I_A = (1 - g_1^4)(1 - g_2^4)(1 - g_3^4)(1 - g_4^4)(1 - g_1 g_2 g_3 g_4). \qquad (4.16)$$

To determine the rank of the intersection matrix we can count the number of nonzero eigenvalues. The g_j can be diagonalized as $g_j = \mathrm{diag}(1, e^{\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{\frac{6\pi i}{5}}, e^{\frac{8\pi i}{5}})$. Zero eigenvalues appear if a $g_j = 1$ or if $g_1 g_2 g_3 g_4 = 1$. The combinatorics leads to 204 nonzero eigenvalues, which is the number of independent 3-cycles on the quintic. Thus, the $L_j = 0$ states provide a basis for the charge lattice.

The intersection matrix for the $|11111\rangle_A$ states,

$$\prod_{i=1}^{5}(1 + g_i - g_i^2 - g_i^3) \tag{4.17}$$

coincides with the intersection matrix (4.13) for the three-cycles $\mathrm{Im}\omega_j z_j = 0$, and thus we identify these states with the $\mathbb{R}P^3$'s.

Similar, but slightly more involved CFT computations yield the number of marginal operators associated with a boundary state. In the case we are looking at, this leads to a potential contradiction with the large volume limit in that the $L = 1$ states have one marginal operator, while the $\mathbb{R}P^3$'s of the geometric description do not. Although it might be that this is indeed a contradiction, from what we know at present an equally likely resolution is that the $L = 1$ marginal operator is not strictly marginal; in other words the world-volume theory has a superpotential for the corresponding field. This is an example of the kind of correspondence that can be made between D-branes at different points of the moduli space, with different descriptions. Similar correspondences are made in [92] for B-type branes, but we will not go into them here.

For future use, we set down for future use the intersection form on the Gepner B-type branes. The B-type boundary states at fixed L_j are described by the single integer, $M = \sum M_j$ and the g_j for different j are identified. The intersection matrix (4.15) for $L = 0$ states can be written as:

$$I_B = (1 - g^{-1})^5 = 5g - 10g^2 + 10g^3 - 5g^4. \tag{4.18}$$

The Gepner intersection form in the same notation is:

$$I_g = -g + 3g^2 - 3g^3 + g^4. \tag{4.19}$$

A linear change of basis preserving the action of $\mathbb{Z}_5$ can be written as a polynomial in the operator g as well and a transformation of the form $I \to mIm^t$ will be $I \to Im(g)m(g^{-1})$. The relation

$$I_B = (1 - g)(1 - g^{-1})I_g \tag{4.20}$$

provides this change of basis.

The results of section 3 allow us to write these charges in the large volume basis. The Gepner charge vector Q_G is related to the large volume charge vector Q as

$$Q = Q_G M^{-1} \ . \tag{4.21}$$

Thus $Q_G = \begin{pmatrix} 0 & 1 & -1 & 0 \end{pmatrix}$ becomes $Q = \begin{pmatrix} -1 & 0 & 0 & 0 \end{pmatrix}$ which is a pure (anti)six-brane. The other charges can be found by acting with the operator A_L.

In the next chapter, we will, by use of K-theory and various mathematical methods, including spectral sequences, make this kind of analysis much more systematic.

Chapter 5

Torsion D-Branes on Nonsimply-Connected Calabi-Yau Manifolds

So far, we have analyzed BPS D-branes, which preserve some of the spacetime supersymmetry. We know, however, since Sen's construction[27], that there exist stable non-BPS D-branes. Since the work of Witten[39], we know that such states are stable because they carry a conserved discrete charge – a torsion element of the K-theory[1].

There are two possible approaches to understanding the physics of these stable non-BPS states: First, one can use a microscopic formulation, where D-branes are described as boundary conditions in a conformal field theory. Alternatively, one can use the language of K-theory and vector bundles. In this chapter, we extend (part of) the Douglas, et al., program to the study of non-BPS D-branes in Calabi-Yau compactifications whose K-theory has a torsion part.

In the large-radius limit, where geometrical reasoning applies, we know that D-brane charge is classified by topological K-theory. In the stringy regime,

[1]Not all stable, non-BPS D-branes are explained this way. For instance, among the D-branes of [36][33] are ones which are stable in some *region* of the moduli space, for energetic reason, not because they carry a conserved charge.

on the other hand, boundary conformal field theory provides a powerful tool for the investigation of stringy D-brane physics. While these methods are in principle applicable at generic points in moduli space, they are most useful at rational or “Gepner” points, where the theory is exactly solvable.

Still, while a detailed description of the D-branes may not be possible at a generic point in the moduli space, one might hope to achieve a cruder goal, namely the classification of the allowed D-brane charges. That is, we would like to define a “quantum K-theory” (the phrase appeared in [96], in analogy with quantum cohomology) which would classify the allowed D-brane charges everywhere in the moduli space, and which would reduce to topological K-theory in the geometrical limit. And, wherever possible, we would like to make contact between it and the results of boundary conformal field theory.

What we will focus on here is determining the automorphisms of the quantum K-theory that result from traversing incontractible cycles in the moduli space.

For the free part of the K-theory, these automorphisms are calculable using Mirror Symmetry, as we have seen. The comparison of BPS D-brane charges at different points in moduli spaces was studied in [92][97][99][100][101][102] [95][111][113][112]. Our challenge is to extend these results to include the torsion in the K-theory.

We will also show that, in some examples, it is possible, by generalizing the Sen construction, to construct a stable brane in terms of boundary conformal field theory at the Gepner point, corresponding to a torsion class in the K-theory. This provides some evidence that there is, at least, a path between large-radius and the Gepner point along which the torsion branes are stable. It is not so clear that the torsion branes are stable near the conifold point.

This program was first formulated, in mathematical detail, and applied to the quintic and two orbifolds thereof, in [1].

To their work, we add two examples, first worked out in [2].

First, a rare case of a Calabi-Yau 3-fold, discovered by Beauville [107], with nonabelian fundamental group. Here, we "discover" the existence of stable BPS branes corresponding to higher dimensional irreps of the fundamental group. These states are *not* distinguished by any conserved (discrete) charge from a collection of wrapped 6-branes. Nonetheless, we argue that they must be present as threshold bound states in the multiple 6-brane system in order to account for the monodromies that we find.

Next, discussed in 5.3, is a model whose Kähler moduli space has complex dimension 2. As usual in studying such two-parameter models, the structure of the Kähler moduli space is rather more complicated than in the one-parameter case, and it is interesting to see that one can produce a consistent set of monodromies acting on the K-theory. We do have an advantage in this case; the moduli space contains a Gepner point. So we can compare the results of our topological calculations with the results from CFT.

Section 5.1 is devoted to laying out our guiding assumptions in determining the monodromies in the quantum K-theory as one moves about in the moduli space. In 5.1.3, we review the construction (see [1]) of BPS boundary states on the Gepner orbifolds.

This chapter follows closely the exposition in [1] and [2].

5.1 Review of Brunner-Distler

In this section, we collect some of the results of [1] which will be useful for our present investigations.

5.1.1 Mathematical Results

Every Calabi-Yau manifold, X, whose holonomy group is $SU(3)$, has a finite fundamental group, and is the quotient of a simply-connected Calabi-Yau, Y by a finite group, G, of freely-acting holomorphic automorphisms (which preserve the holomorphic 3-form). The most familiar constructions of Calabi-Yau manifolds – as hypersurfaces or complete intersections in toric varieties – yield simply-connected Calabi-Yau manifolds, which are candidates for the covering space, Y. The K-theory of such a Y is torsion-free. So, to find a suitable Calabi-Yau manifold, X, with torsion in its K-theory, we look for a freely-acting group G to mod out by.

The first problem is to compute the K-theory of $X = Y/G$. For this, one needs a pair of spectral sequences, the Cartan-Leray Spectral Sequence – which computes the *homology* of X from the homology of Y – and the Atiyah-Hirzebruch Spectral sequence, which computes the K-theory of X from the cohomology of X. Some very useful discussions of the AHSS have appeared in the recent physics literature [1][50][51][52].

For a Calabi-Yau manifold (indeed, for a 6-manifold with $H^1(X) = 0$), the AHSS converges at the E_2 term,

$$E_2^{p,q} = \mathrm{H}^p(X, \pi_q(BU)) \tag{5.1}$$

where $\pi_{2n}(BU) = \mathbb{Z}$, $\pi_{2n+1}(BU) = 0$. And, after a bit of computation, one finds that the torsion subgroups of the K-theory (our main interest) fit into exact sequences

$$0 \to \mathrm{H}^4(X)_{tor} \to K^0(X)_{tor} \to \mathrm{H}^2(X)_{tor} \to 0$$
$$0 \to \mathrm{H}^5(X) \to K^1(X)_{tor} \to \mathrm{H}^3(X)_{tor} \to 0$$

(note that $\mathrm{H}^5(X)$ is pure torsion).

The Universal Coefficients Theorem and Poincaré duality determine

$$\mathrm{H}^5(X)_{tor} = \mathrm{H}_1(X)_{tor} = (\mathrm{H}^2(X)_{tor})^*$$
$$\mathrm{H}^4(X)_{tor} = \mathrm{H}_2(X)_{tor} = (\mathrm{H}^3(X)_{tor})^*$$

and the homology groups can be determined from the CLSS, a homology spectral sequence with E^2 term

$$E^2_{p,q} = \mathrm{H}_p(G, \mathcal{H}(\mathcal{Y})) \tag{5.2}$$

the homology with *twisted coefficients.*

Define the *coinvariant quotient*, $\mathrm{H}_2(Y)_G = \mathrm{H}_q(Y)/\mathcal{A}$, where $\mathcal{A}$ is the subgroup of $\mathrm{H}_2(Y)$ generated by elements of the form $x - g \cdot x$. Assuming that $\mathrm{H}_2(Y)_G$ is torsion-free, we find the needed homology groups to be given by

$$\mathrm{H}_1(X) = \mathrm{H}_1(G) = G/[G,G] \tag{5.3}$$

and the exact sequence

$$0 \to \mathrm{H}_2(Y)_G \xrightarrow{\pi_*} \mathrm{H}_2(X) \to \mathrm{H}_2(G) \to 0 \tag{5.4}$$

where π_* is the push-forward by the projection $\pi : Y \to X$. With this, the K-theory of the various examples can be calculated.

5.1.2 Monodromies

As we move away from large-radius, into the interior of the moduli space, the group of D-brane charges is no longer given by topological K-theory. Still, as a discrete abelian group, it is locally constant. When we traverse some incontractible cycle in the moduli space (circle some singular locus), the group of D-brane charges comes back to itself up to an automorphism.

These monodromies must satisfy the following properties

1. They descend to the known action on on $K^\bullet(X)/K^\bullet(X)_{tor}$, which can be computed, say, using Mirror Symmetry.

2. They preserve the skew-symmetric intersection pairing

$$(.,.) : K^0(X) \times K^0(X) \to \mathbb{Z}$$

given by

$$\begin{aligned}(v,w) &= Ind(\overline{\partial}_{v\otimes\overline{w}}) \\ &= \int_X ch(v\otimes\overline{w})Td(X)\end{aligned} \tag{5.5}$$

which annihilates $K^0_{tor}(X)$ and is nondegenerate on $K^0(X)/K^0_{tor}(X)$. They also preserve the corresponding skew-symmetric pairing on $K^1(X)$.

3. They preserve the nondegenerate torsion-pairing [53, **?**, 55, 56]

$$\langle .,. \rangle : K^0(X)_{tor} \times K^1(X)_{tor} \to \mathbb{R}/\mathbb{Z}$$

4. They commute with the quantum symmetry of the Y/G orbifold [57]. Since G acts freely, there are no massless states in the twisted sectors. Still, in the full CFT, we have an action of the quantum symmetry group, G_Q. G_Q is the character group of G which, in turn, is isomorphic to $G_{ab} = G/[G,G]$. Any character χ acts by phase rotation of the states in the g-twisted sector by $\chi(g)$. (If G is nonabelian, the twisted sectors are labeled by conjugacy classes; $\chi(g)$ only depends on the conjugacy class of g.) Such characters correspond to the holonomy of connections on flat line bundles, so the quantum symmetry group acts on the K-theory by

$$v \mapsto v\otimes\mathcal{L} \tag{5.6}$$

for $\mathcal{L}$ a flat line bundle. The flat line bundles form a group under tensor products, isomorphic to G_Q.

Some of the monodromies we encountered can be described rather generally. Near large radius, shifting B by an integral class $\xi \in \mathrm{H}^2(X)$ is a symmetry of the CFT, which acts on the D-brane charges as

$$v \mapsto v \otimes L$$

where L is the line bundle with $c_1(L) = \xi$.

Landau-Ginzburg or orbifold loci are also places where one has a bona fide conformal field theory but may have nontrivial monodromy. Say one finds that, at some locus, the quantum symmetry group G_Q is enlarged to $\hat{G}_Q$. One then finds that this locus is a $\hat{G}_Q/G_Q$ orbifold locus in the moduli space. The monodromy about it has finite order,

$$M^k = 1$$

where $k = |\hat{G}_Q/G_Q|$.

At the (mirror of the) conifold locus, certain wrapped D-branes become massless, giving rise to massless hypermultiplets in the 4-dimensional effective theory. On Y, it is the D6-brane, whose K-theory class is the trivial line bundle, $\mathcal{O}$ which becomes massless. By the Witten effect, circling this locus in the moduli space shifts the charges of all the other D-branes by

$$v \mapsto v - (v, \mathcal{O})\mathcal{O} \tag{5.7}$$

In [1], we saw that, for *abelian* G, all of the flat line bundles (D6-branes carrying torsion charge) become massless, and (5.7) should be replaced by

$$v \mapsto v - |G|(v, \mathcal{O})\mathcal{O} \tag{5.8}$$

This monodromy has a very simple interpretation. Let's say that at some singularity, a brane (a charged particle in the 4D effective theory) corresponding to K-theory class w becomes massless. Circling the singularity shifts the θ-angle – of the $U(1)$ for which this particle is electrically-charged – by 2π. By the Witten effect, the charge of a particle in K-theory class v gets shifted by[2]

$$v \mapsto v - (v, w)w$$

when you shift $\theta \to \theta + 2\pi$.

Now, let 2 different particles, labelled by $\{w_1, w_2\}$, become massless, and compound these different actions (imagine that there are 2 distinct conifold points, very close to each other). You get (performing operation 1 first, then operation 2)

$$v \mapsto v-(v, w_1)w_1-(v-(v, w_1)w_1, w_2)w_2 = v-(v, w_1)w_1-(v, w_2)w_2-(v, w_1)(w_1, w_2)w_2 \tag{5.9}$$

If we reversed the order of operations, of course, we would get a different result. In general, this is not tenable if both particles go massless at a given conifold point (which limiting process do you use?), so one must require that the two particles be mutually local ($(w_1, w_2) = 0$). In that case, we get the simplified

[2]Strictly speaking, this argument involving charges only fixes the action on $K^0(X)/K^0(X)_{tor}$. However, the monodromy must also commute with the action of the quantum symmetry, discussed above. This, together with the action on the quotient, will frequently fix the action on all of $K^0(X)$. Since this monodromy acts trivially on $K^0(X)_{tor}$, the torsion pairing is preserved if it also acts trivially on $K^1(X)$.

form

$$v \mapsto v - (v, w_1)w_1 - (v, w_2)w_2 \tag{5.10}$$

More generally, if several, mutually-local, particles become massless, the shift is

$$v \mapsto v - \sum_i (v, w_i)w_i \tag{5.11}$$

where mutual locality means $(w_i, w_j) = 0$ for all i, j. If k particles which differ from each other only by torsion branes (which give zero in the intersection product) become massless, then the shift is

$$v \mapsto v - k(v, w)w$$

For nonabelian G, the number of flat line bundles (and therefore massless branes corresponding to such) is $|G_{ab}| < |G|$. But, as we shall see in §5.2, at the conifold locus there are other branes, corresponding to flat bundles of higher rank which also become massless, and (5.8) is the correct formula even when G is nonabelian.

In our calculations, we will be specially interested in the K-theory classes that correspond to D-branes wrapped on submanifolds of X – these will, in fact, form a basis for the K-theory. In that regard, we will need the K-theoretic push-forward, as explicated in [1]. We will give here the formula for the special case where Y is a holomorphic submanifold of X. Given $i : Y \hookrightarrow X$, and a K-theory class v on Y,

$$ch(f_! a)Td(X) = f_*(ch(a)Td(Y)), \tag{5.12}$$

where f_* is the cohomology push-forward (Poincare dualize, push forward the homology, then Poincare dualize again).

5.1.3 Construction of Boundary States and Physical Results

Brunner and Distler used these considerations in the particular context of the quintic and two orbifolds thereof. The Fermat quintic,

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$$

where the z_i are the homogeneous coordinates on P^4, is invariant under a freely-acting $\mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry generated by

$$(z_1, z_2, z_3, z_4, z_5) \rightarrow (z_1, \omega z_2, \omega^2 z_3, \omega^3 z_4, \omega^4 z_5) \tag{5.13}$$

where $\omega^5 = 1$ and

$$(z_1, z_2, z_3, z_4, z_5) \rightarrow (z_2, z_3, z_4, z_5, z_1) \tag{5.14}$$

The quintic is simply connected and its K-theory is torsion-free. To form a non-simply connected Calabi-Yau, one can mod out by (5.13) to form the manifold $X = Y/\mathbb{Z}_5$. One can go further and mod out by (5.14) to form $W = X/\mathbb{Z}_5$. X and W were the two examples studied in detail in [1].

In order to do so, it was also necessary to calculate A- and B-type boundary states on Gepner orbifolds, since, of course, X and W correspond to orbifolds of the basic Gepner 3^5 model.

In a general Gepner model (in notation used earlier), the symmetries are given by the subgroup of symmetries of the tensor product theory which preserve worldsheet and space time supersymmetry [128]. These symmetries act as:

$$g(\gamma)\Phi_{\lambda\mu;\bar{\lambda}\bar{\mu}} = \exp i\pi \left(\sum_{j=1}^{r} \frac{\gamma_j(\mu_j + \bar{\mu}_j)}{k_j + 2} \right) \Phi_{\lambda\mu;\bar{\lambda}\bar{\mu}} = e^{\pi i \beta_\gamma \cdot (\mu + \bar{\mu})} \Phi_{\lambda\mu;\bar{\lambda}\bar{\mu}}, \qquad (5.15)$$

where the γ_i in $\gamma = (\gamma_1, \ldots, \gamma_r)$ specify the orbifold action in the individual minimal models. β_γ is the vector $(0; 2\gamma; 0)$. For consistency with the projections, we require $\beta_\gamma \cdot \beta_0 \in \mathbb{Z}$.

To write down the new partition function of the Gepner orbifold [128], one simply takes (2.28) and includes a further vector into the lattice Λ and projects on elements μ with $\beta_\gamma \cdot (\mu + \bar{\mu}) \in \mathbb{Z}$. In lattice language, there are new winding modes with $\mu - \bar{\mu} = n\beta_\gamma$ coming from twisted sectors.

By comparing the intersection form (the Witten index in the open string sector, $tr_R(-1)^F$) and the action of the quantum symmetry with the geometrical calculations, the K-theory classes of these branes could be explicitly identified. By stacking these branes together, in a manner akin to the Sen construction (see 5.3 for a more concrete description) it was possible to construct the torsion branes at the Gepner point.

Brunner and Distler's analysis of the monodromies led to several interesting results, including a complete identification between B-type boundary states in the Gepner orbifold and specific elements of the K-theory of the manifold in question. This leads, among other things, to the assumption that there is at least some open set in the moduli space that contains both the Gepner point and the LVL in which the torsion branes are stable. In the case of W, calculation of the monodromies also led to verification of the conjecture, based on quantum symmetry arguments, that the Gepner point of W's moduli space is actually a smooth point. Our analysis of two more involved examples below will reveal several other points.

5.2 The Beauville Manifold

We have been careful not to assume that the fundamental group of the Calabi-Yau was abelian. There are, in fact, very few known examples of Calabi-Yau's whose fundamental group is a nonabelian finite group.

The main example is due to Beauville [107]. Let Q be the group of unit quaternions,

$$Q = \{\pm 1, \pm I, \pm J, \pm K\} \tag{5.16}$$

with multiplication law

$$\begin{aligned} IJ &= K \qquad \text{(and cyclic)} \\ I^2 &= J^2 = K^2 = -1 \end{aligned} \tag{5.17}$$

Let Q act on $\mathcal{V} = \mathbb{C}$ via the regular representation. This induces an action of Q on the complex projective space, $P^7 = P[\mathcal{V}]$. Let $Y = P^7[2,2,2,2]$ be the intersection of four homogeneous quadrics in P^7. Beauville showed that it is possible to choose quadrics such that Y is smooth and Q acts freely on Y. The quotient, $X = Y/Q$ is a Calabi-Yau manifold with fundamental group $\pi_1(X) = Q$.

Let us recall some facts about the group theory of Q. First, there is an exact sequence,

$$0 \to \mathbb{Z}_2 \to Q \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 0 \tag{5.18}$$

where the commutator subgroup of Q is the $\mathbb{Z}_2$ subgroup, $\{1, -1\}$ and its abelianization, $Q/[Q,Q] = \mathbb{Z}_2 \times \mathbb{Z}_2$.

The irreducible representations of Q are as follows. There are four 1-dimensional irreps: the trivial rep V_1 and the representations V_I, V_J, and V_K. In V_I, ± 1 and $\pm I$ are represented by 1 while $\pm J$ and $\pm K$ are represented by -1 (and similarly for $V_{J,K}$). There is also a 2-dimensional representation, V_2 by Pauli matrices.

The representation ring is

$$\begin{aligned} &V_2 \otimes V_2 = V_1 \oplus V_I \oplus V_J \oplus V_K \\ &V_\alpha \otimes V_2 = V_2 \qquad \alpha = 1, I, J, K \\ &V_I \otimes V_J = V_K \qquad \text{(and cyclic)} \end{aligned} \tag{5.19}$$

The group homology of Q is

$$\mathrm{H}_1(Q) = Q/[Q,Q] = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \qquad \mathrm{H}_2(Q) = 0 \tag{5.20}$$

The Hodge numbers of the covering space, Y, are $h^{1,1}(Y) = 1$, $h^{2,1}(Y) = 65$. In particular, Q must act trivially on $\mathrm{H}_2(Y)$. Plugging into the Cartan-Leray Spectral Sequence,

$$\begin{aligned} \mathrm{H}_1(X) &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \mathrm{H}_2(X)_{tor} &= 0 \end{aligned} \tag{5.21}$$

$\mathrm{H}^{ev}(X)$ is generated by $1, \xi, \chi_1, \chi_2, \eta$ and ρ, with relations

$$\xi^2 = 2\eta, \qquad \xi\eta = \rho$$

The $\chi_i \in \mathrm{H}^2(X)$ are two-torsion, $2\chi_1 = 2\chi_2 = 0$. The total Chern class of X is

$$c(X) = 1 + 8\eta - 16\rho$$

A choice of basis for $K^0(X)$ is

	r	c_1	c_2	c_3
$\mathcal{O}$	1	0	0	0
$a = H - \mathcal{O}$	0	ξ	0	0
$\alpha_1 = \mathcal{L}_1 - \mathcal{O}$	0	χ_1	0	0
$\alpha_2 = \mathcal{L}_2 - \mathcal{O}$	0	χ_2	0	0
$b = \overline{i_!\mathcal{O}_C}$	0	0	$-\eta$	2ρ
$c = i_!\mathcal{O}_p$	0	0	0	2ρ

Here, $i_!\mathcal{O}_p$ is the K-theory class corresponding to a $D0$-brane at point p, and $i_!\mathcal{O}_C$ is the class corresponding to a $D2$-brane wrapped on a rational curve C dual to η. $\mathcal{L}_i$ are the nontrivial flat line bundles on X and H is the hyperplane line bundle. If D is the divisor corresponding to H, then

$$i_!\mathcal{O}_D = \mathcal{O} - \mathcal{O}(-D) \tag{5.22}$$

where D as a subscript indicates a brane wrapping the divisor D considered as a submanifold and $\mathcal{O}(-D)$ indicates the bundle corresponding to the divisor $-D$. The Chern classes of this wrapped 4-brane are as follows:

	r	c_1	c_2	c_3
$i_!\mathcal{O}_D$	0	ξ	-2η	2ρ

In the chosen basis (omitting the α_i, which are torsion elements of the

K-theory) the intersection form, $(v,w) = Ind(\overline{\partial}_{v\otimes\overline{w}})$ is given by the matrix

$$\Omega = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{5.23}$$

The quantum symmetry group, G_Q, is isomorphic to the abelianization of Q

$$G_Q = \mathbb{Z}_2 \times \mathbb{Z}_2 \tag{5.24}$$

and acts on the D-branes by tensoring with a flat line bundle,

$$v \mapsto v \otimes \mathcal{L} \tag{5.25}$$

The Kähler moduli space is the 3-punctured sphere. About the large-radius point, the monodromy is generated by

$$M_r : v \mapsto v \otimes H \tag{5.26}$$

At the conifold point, certain branes become massless. From our previous experience, we expect that these are the flat line bundles. There are four such line bundles, $\mathcal{O}, \mathcal{L}, \mathcal{L}$ and $\mathcal{L}_1 \otimes \mathcal{L}_2$. These do, indeed, become massless at the conifold. But, in addition, there's something else which becomes massless. As we saw above, $\pi_1(X)$ has a 2-dimensional irrep, out of which one can build a rank-2 flat bundle on X. This rank-2 bundle (a threshold bound state of a pair of 6-branes, if you wish) *also* becomes massless at the conifold.

Indeed, we conjecture that this is a general phenomenon. Given any Calabi-Yau manifold, X, whose holonomy group is $SU(3)$ (and not a proper subgroup thereof), we can always write it as $X = Y/G$, for Y a simply-connected Calabi-Yau, and G a finite group. The monodromy about the conifold locus (principal component of the discriminant locus) is always of the form

$$v \to v - \sum_R (v, W_R) W_R \tag{5.27}$$

where the sum is over all irreps, R, of G and W_R is the flat bundle built using the irrep R of G. On the level of K-theory, it is not hard to see that this can be simplified to

$$M_c : v \to v - |G| (v, \mathcal{O}) \mathcal{O} \tag{5.28}$$

(First, $(v, W_R) = (v, n_R \mathcal{O}) = n_R (v, \mathcal{O})$, where $n_R = \dim W_R$, because $W_R - n_R \mathcal{O}$ is a torsion brane. Next, $\sum_R n_R W_R = \sum_R n_r^2 \mathcal{O} = |G| \mathcal{O}$ in K-theory, because $\sum_R n_R W_R$ is the bundle associated with the regular representation of G). This generalizes an old conjecture of Morrison (see [108] and [109, 110]).

In the present case, this is

$$M_c : v \to v - 8 (v, \mathcal{O}) \mathcal{O} \tag{5.29}$$

Finally, the monodromy about the third point,

$$M_h = (M_r M_c)^{-1} \tag{5.30}$$

has, in our example, the property[3] that its square is unipotent of index 4,

$$(M_h^2 - 1)^4 = 0 \tag{5.31}$$

[3] This is a particular example of the monodromy about a "hybrid point" in the moduli space, where the "hybrid theory" has the structure of a Landau-Ginsburg orbifold fibered over a P^k. If the LG orbifold has a $\mathbb{Z}_n$ quantum symmetry, we find, rather generally, that

A more refined characterization is

$$(M_h + 1)^4 = 0 \tag{5.32}$$

i.e. that M_h has a single Jordan block with eigenvalue -1. Note that the multiplicity 8 in (5.29) was crucial to obtaining (5.32).

The existence of the flat rank-2 bundle as a stable single-particle state was not guaranteed by the BPS condition (it is degenerate with a pair of D6-branes), nor by K-theory (it does not carry any K-theory charge by which it might be distinguished from a pair of D6-branes). Nonetheless, we deduced its existence from the consistency of the monodromies that we compute. This is another example of how studying the behavior of string theory near singularities can shed light on many subtle issues (in this case, on the existence of certain threshold bound states).

5.3 A Two-Parameter Example

In this section we will study an example of torsion D-branes in which the Kähler moduli space is 2-dimensional.

The covering space, Y, is the toric resolution of a hypersurface in weighted projective space, $Y = P^4_{1,1,1,3,3}[9]$. Resolving the orbifold singularities of the weighted projective space yields a smooth toric variety, $\mathcal{T}$, which can be realized (in the language of Gauged Linear σ-Models) by six chiral multiplets (the homogeneous coordinates of $\mathcal{T}$), charged under $U(1) \times U(1)$, with charges given in Table 5.1(as worked out in Appendix 1, with a slight

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the monodromy about the hybrid point satisfies

$$(M_h^n \pm 1)^p = 0$$

for some p. In particular, repeating the calculation of the monodromies for the covering space, Y (where there is no question that the conifold monodromy is simply $v \mapsto v - (v, \mathcal{O})\mathcal{O}$), one finds that (5.32) also holds for the monodromy M_h in the moduli space of Y.

rearrangement of coordinates). Adding one more field, p, of charge $(0, -3)$ and a gauge-invariant superpotential, $W = pP(z_i)$, we obtain the GLσM for Y.

To obtain $X = Y/\mathbb{Z}_3$, we mod out by a $\mathbb{Z}_3$ action,

$$(z_1, z_2, z_3, z_4, z_5, z_6) \mapsto (z_2, z_3, z_1, z_4, e^{2\pi i/3} z_5, e^{4\pi i/3} z_6) \quad (5.33)$$

	q_1	q_2
z_1	1	0
z_2	1	0
z_3	1	0
z_4	-3	1
z_5	0	1
z_6	0	1

Table 5.1: Homogeneous coordinates for the resolved model

The Kähler moduli space of Y is 4-dimensional. The exceptional divisor of $\mathcal{T}$, $[z_4]$, intersects the Calabi-Yau hypersurface in three disjoint P^2s, and there is a Kähler modulus corresponding, roughly, to the size of each of the P^2s. Only a 2-dimensional subspace (in which each of the P^2s has the same "size") is represented by toric deformations. This subspace of the moduli space is parametrized by the complexified Fayet-Iliopoulos parameters of the $U(1) \times U(1)$ gauge theory in Table 5.1.

Happily, the orbifold projection (5.33) projects out the nontoric Kähler deformations, and the (2-dimensional) Kähler moduli space of X coincides with the subspace of toric Kähler deformations of Y.

5.3.1 Phases of the model

In 2, we displayed the phase structure of the GLσM on the covering space, Y. We now repeat it, with the addition that we have to mod out by (5.33) by hand.

As described above, the linear sigma model has gauge group $U(1) \times$

$U(1)$ and 7 chiral multiplets $z_1, \dots z_6, p$. A choice of gauge-invariant (and $\mathbb{Z}_3$-invariant) superpotential is given by

$$W = pP(z_i) = p(z_1^9 z_4^3 + z_2^9 z_4^3 + z_3^9 z_4^3 + z_5^3 + z_6^3). \tag{5.34}$$

The possible vacuum configurations have to fulfill the D- and F-flatness conditions:

$$\begin{aligned} F &= |P|^2 + |p|^2 \sum_i \left| \frac{\partial P}{\partial z_i} \right|^2 \\ D_1 &= |z_1|^2 + |z_2|^2 + |z_3|^2 - 3|z_4|^2 - r_1 \\ D_2 &= |z_4|^2 + |z_5|^2 + |z_6|^2 - 3|p|^2 - r_2 \end{aligned} \tag{5.35}$$

The model has four phases, depending on the values of the parameters r_i. The limit points of each phase lie at the origin of coordinates for certain readily-defined coordinate patches on the moduli space. We will discuss the structure of the moduli space and define the coordinate patches U_{ij} in §5.3.3. In the meantime, we just label the phases by the corresponding patches:

U_{34} *Phase:* $r_1 > 0, r_2 > 0$. The excluded gauge orbits in this case are the orbits with $\{z_1 = z_2 = z_3 = 0\}$ and $\{z_4 = z_5 = z_6 = 0\}$. The F-terms require the vanishing of P and p. As a consequence, the low energy modes in this limit are a nonlinear σ-model on the (smooth) Calabi-Yau manifold.

U_{13} *Phase:* $r_1 < 0, 3r_2 + r_1 > 0$. The orbits $\{z_4 = 0\}$ and $\{z_1 = z_2 = z_3 = z_5 = z_6 = 0\}$ have to be excluded. In a generic D-flat configuration, z_4 is not zero. The Calabi-Yau develops a $\mathbb{Z}_3$ orbifold singularity at the location of the blown-down exceptional divisor.

U_{12} *Phase:* $r_1 < 0, 3r_1 + r_2 < 0$. To fulfill D-flatness, the orbits $\{z_4 = 0\}$ and $\{p = 0\}$ have to be excluded. The F-terms require that $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = 0$. A gauge transformation by $e^{i\theta q_1}$ leaves p invariant, while rotating z_4. A gauge transformation by $e^{i\theta'(q_1+3q_2)}$ leaves z_4 invariant, while rotating p. We can use these two $U(1)$ actions to fix the values of z_4 and p completely, so that the vacuum consists of one point. Around this vacuum, there are fluctuation of the fields z_1, z_2, z_3, z_5, z_6. The VEVs for z_4 and p leave unbroken a $\mathbb{Z}_9$ subgroup of the $U(1) \times U(1)$, generated by $e^{2\pi i(q_1+3q_2)/9}$. In addition, we have to mod out the theory by the $\mathbb{Z}_3$ action (5.33). Altogether, we arrive at a $\mathbb{C}^5 / \mathbb{Z}_9 \times \mathbb{Z}_3$ orbifold model. Taking into account the superpotential, the resulting model is a $\mathbb{Z}_9 \times \mathbb{Z}_3$ orbifold of a Landau-Ginzburg model. This Landau-Ginzburg model has an IR description in terms of the Gepner model $(k = 7)^3 (k = 1)^2$, on which (5.33) translates into a permutation of the first three minimal model factors accompanied by a phase multiplication in the two remaining factors.

U_{24} *Phase:* $r_1 > 0, r_2 < 0$. The orbits $\{p = 0\}$ and $\{z_1 = z_2 = z_3 = 0\}$ have to be removed. This phase corresponds to a hybrid phase: The fields z_1, z_2, z_3 parametrize a P^2, over which the fluctuations of the fields $z_4, \ldots, z_6$ behave like in a LG theory. The model has to be modded out by (5.33).

5.3.2 The K-theory

We have worked out in Appendix 1 (A.24) the toric cohomology of Y. As explained there, the toric subspace of $H^2(Y)$ is precisely the subspace that remains invariant under (5.33). Since Y is simply-connected and since (5.33) acts freely, we know that $\mathrm{H}_1(X) = \mathbb{Z}_3$. Since $\mathrm{H}_2(\mathbb{Z}_3) = 0$, the CLSS tells us that , $\mathrm{H}_2(X) = H_2(Y)_G = \mathbb{Z} \oplus \mathbb{Z}$ (recall $H_2(Y)_G$ is the *coinvariant quotient*). Using the Universal Coefficient Theorem ($H^2(X) \equiv \mathrm{Hom}(H_2(X), \mathbb{Z}) \oplus \mathrm{Ext}(H_1(X), \mathbb{Z})$, we find that $H^2(X) \equiv \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$. We can write a basis for $\mathrm{H}^{ev}(X)$, $1, \xi_1, \xi_2, \chi, \eta_1, \eta_2$ and ρ. The ring structure is

$$\begin{gathered} \xi_1^2 = \eta_2, \quad \xi_1\xi_2 = \eta_1 + 3\eta_2, \quad \xi_2^2 = 3(\eta_1 + 3\eta_2) \\ \xi_i\eta_j = \delta_{ij}\rho \end{gathered} \tag{5.36}$$

where $\chi \in \mathrm{H}^2(X)$ is 3-torsion, $3\chi = 0$, and thus it has only trivial products with other elements.

A choice of basis for $K^0(X)$ is

	r	c_1	c_2	c_3
$\mathcal{O}$	1	0	0	0
$a = L_1 - \mathcal{O}$	0	ξ_1	0	0
$b = L_2 - \mathcal{O}$	0	ξ_2	0	0
$\alpha = \mathcal{L} - \mathcal{O}$	0	χ	0	0
$c = \overline{i_!\mathcal{O}_C}$	0	0	$-\eta_1$	2ρ
$d = i_!\mathcal{O}_E$	0	0	$-\eta_2$	0
$e = i_!\mathcal{O}_p$	0	0	0	2ρ

Here, $\mathcal{L}$ is the line bundle whose first Chern class is the torsion element χ. $i_!\mathcal{O}_C$ is the K-theory class corresponding to a $D2$-brane wrapped on the rational curve C dual to η_1, $i_!\mathcal{O}_E$ coresponds to a $D2$-brane wrapped on the elliptic curve E dual to η_2, and $i_!\mathcal{O}_p$ corresponds to a $D0$-brane at point p.

L_1 and L_2 are the line bundles whose first Chern classes are ξ_1 and ξ_2, resp. If D_1, D_2 are the divisors corresponding to these bundles, then

$$i_! \mathcal{O}_{D_i} = \mathcal{O} - \mathcal{O}(-D_i) \tag{5.37}$$

where D as a subscript indicates a brane wrapping the divisor D considered as a submanifold and $\mathcal{O}(-D)$ indicates the bundle corresponding to the divisor $-D$.

The Chern classes of the wrapped 4-branes are as follows:

	r	c_1	c_2	c_3
$i_! \mathcal{O}_{D_1}$	0	ξ_1	$-\eta_2$	0
$i_! \mathcal{O}_{D_2}$	0	ξ_2	$-3(\eta_1 + 3\eta_2)$	9ρ

In the chosen basis (omitting, as always, the torsion class, α), the intersection form is

$$\Omega = \begin{pmatrix} 0 & -1 & -4 & -1 & 0 & -1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 4 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.38}$$

5.3.3 The monodromies

Now, before discussing the monodromies, we need to explain a bit about the structure of the Kähler moduli space, $\mathcal{M}$. $\mathcal{M}$ is, itself, a toric variety. Again, we can describe it most succinctly by giving the GLσM data necessary to construct it: a $U(1) \times U(1)$ gauge theory, with charged fields (homogeneous coordinates for $\mathcal{M}$) listed in Table 5.2. $\mathcal{M}$ is constructed by taking the Fayet-

Iliopoulos parameters $\zeta_{1,2} > 0$, imposing the D-flatness conditions

$$\begin{aligned} 3|s_1|^2 - |s_2|^2 + |s_4|^2 &= \zeta_1 \\ |s_2|^2 + |s_3|^2 &= \zeta_2 \end{aligned} \tag{5.39}$$

and modding out by $U(1) \times U(1)$ gauge transformations. Note that the loci $\{s_1 = s_4 = 0\}$ and $\{s_2 = s_3 = 0\}$ are excluded, as one cannot satisfy (5.39) there. Also note that the locus $\{s_4 = 0\}$ is a $\mathbb{Z}_3$ orbifold locus in $\mathcal{M}$, as $s_1 \neq 0$ leaves an unbroken $\mathbb{Z}_3$ subgroup of the first $U(1)$.

	Q_1	Q_2
s_1	3	0
s_2	-1	1
s_3	0	1
s_4	1	0

Table 5.2: Homogeneous coordinates for Kähler moduli space, $\mathcal{M}$.

$\mathcal{M}$ can be covered by coordinate patches U_{ij} in which s_i and s_j are nonvanishing. Each of these coordinate patches corresponds to a "phase" of the GLσM analysis of X, which we reviewed in §5.3.1.

The boundaries of the moduli space are the four divisors $[s_i]$ as well as the "discriminant locus", which has two components, the conifold locus, $\Delta_0 = [s_1 s_2^3 - \frac{1}{27}(s_2 s_4 - \frac{1}{27}s_3)^3]$, and another locus, $\Delta_1 = [s_1 - \frac{1}{27}s_4^3]$. Δ_0 intersects almost every divisor represented by a horizontal line in the figure in three points. The exceptions are $[s_1]$, which it meets tangentially, and the orbifold locus, $[s_4]$.

The divisors $[s_1]$ and $[s_2]$ correspond, respectively, to the $r_1 \to \infty$ and $r_2 \to \infty$ limits on the Calabi-Yau X. The "large radius limit" is located at the intersection of these two divisors. The monodromies about these divisors are

$$
\begin{aligned}
M_{r_1} &: v \mapsto v \otimes L_1 \\
M_{r_2} &: v \mapsto v \otimes L_2
\end{aligned}
$$

In the basis of (5.38), these are represented by the matrices

$$
M_{r_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \qquad M_{r_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 3 & 9 & 0 & 1 & 0 \\ 0 & 1 & 6 & 0 & 1 & 1 \end{pmatrix} \tag{5.40}
$$

In specifying the monodromies about more "distant" divisors in the moduli space, we need to specify what path we take in circling them. It will be useful to choose a (real) 2-surface homotopic to one of our coordinate divisors, passing through our chosen basepoint near large radius, and specify that the path is constrained to lie *on* that 2-surface.

To this end, we can choose

$$
\begin{aligned}
\mathcal{C}_1 &= \{s_1 - \epsilon_1 s_4^3 = 0\} \\
\mathcal{C}_2 &= \{s_2 - \epsilon_3 \overline{s_4} s_3 = 0\} \\
\mathcal{C}_3 &= \{s_2 s_4 - \epsilon_2 s_3 = 0\}
\end{aligned} \tag{5.41}
$$

The $\mathcal{C}_i$ are chosen to meet at our basepoint near the large radius limit, located at $(q_1, q_2) = (\epsilon_1, \epsilon_2)$, where

$$
q_1 = \frac{s_1}{s_4^3}, \qquad q_2 = \frac{s_2 s_4}{s_3}
$$

Figure 5.1: Schematic depiction of the moduli space of the 2-parameter model. Shown are the divisors $[s_i]$ and $\Delta_{0,1}$, and their mutual intersections, the coordinate patches U_{ij} and the 2-surfaces $\mathcal{C}$ along which our monodromy calculations are done.

are the good local coordinates in the patch U_{34}. Note that $\mathcal{C}_2$ is homotopic to, but is not itself a holomorphic curve in $\mathcal{M}$. In the local coordinates in U_{34}, it is given by

$$\mathcal{C}_2 = \{q_2 = \epsilon_3 |s_4(q_1, q_2)|^2\}$$

where $|s_4(q_1, q_2)|^2$ is a complicated function of q_1, q_2, given by solving the D-flatness condition (5.39) and

$$\epsilon_3 = \frac{\epsilon_2}{|s_4(\epsilon_1, \epsilon_2)|^2}$$

First consider the 2-surface $\mathcal{C}$. It is easy to see that it intersects Δ_0 three times, $[s_2]$ and $[s_3]$ once each, and does not intersect $[s_1]$, $[s_4]$ or Δ_1.

The monodromy around the conifold locus is

$$M_c : v \mapsto v - 3(v, \mathcal{O})\mathcal{O} \tag{5.42}$$

As usual, the "3" is because there are three flat line bundles (6-branes) which become massless at the conifold locus. The monodromies about the other two points of intersection with Δ_0 are related to this by conjugation with M_{r_1},

$$\begin{aligned} M_{c'} &: v \mapsto v - 3(v, L_1)L_1 \\ &= M_{r_1} M_c M_{r_1}^{-1} \\ M_{c''} &: v \mapsto v - 3(v, L_1^2)L_1^2 \\ &= M_{r_1}^2 M_c M_{r_1}^{-2} \end{aligned} \tag{5.43}$$

The monodromy about $[s_3]$, therefore is

$$M_{[s_3]} = M_{r_2} M_c M_{c'} M_{c''} \tag{5.44}$$

and satisfies

$$M_{[s_3]}^3 = 1 \tag{5.45}$$

Similarly, consider $\mathcal{C}_2$. This intersects $[s_1]$, Δ_1 and $[s_4]$. The monodromy about Δ_1 is

$$M_{\Delta_1} : v \mapsto v - (v, x)x \tag{5.46}$$

where $x = \mathcal{O}_{D_2} - 3\mathcal{O}_{D_1} - 3\mathcal{O}_E$. So the monodromy about $[s_4]$ is

$$M_{[s_4]} = M_{r_1} M_{\Delta_1} \tag{5.47}$$

and satisfies

$$M_{[s_4]}^3 = M_{r_2} \tag{5.48}$$

Finally, we turn to $\mathcal{C}_3$. Unlike the previous cases, $\mathcal{C}_3$, or any 2-surface homotopic to it, necessarily crosses $[s_3]$ at the LG point (the intersection of $[s_3]$ and $[s_4]$). Thus it makes sense to talk about the monodromy "about the LG point". $\mathcal{C}_3$ also intersects $[s_1]$, Δ_0 and Δ_1 each once. So we find the monodromy about the LG point is

$$M_{LG} = M_{r_1} M_c M_{\Delta_1} \tag{5.49}$$

which satisfies

$$M_{LG}^9 = 1 \tag{5.50}$$

The quantum symmetry at the LG point is enhanced from $\mathbb{Z}_3$ to $\mathbb{Z}_3 \times \mathbb{Z}_9$. The $\mathbb{Z}_3$ generator is, of course, tensoring with the flat line bundle $\mathcal{L}$. The $\mathbb{Z}_9$ generator is M_{LG}. The $(L = 0)$ B-type fractional branes at the LG point are the orbit under $\mathbb{Z}_3 \times \mathbb{Z}_9$ of the D6-brane, $\mathcal{O}$. They fall into the K-theory classes,

$$\begin{aligned} V_{k,m} &= V_{k,3} - m\alpha \\ V_{k,3} &= M_{LG}^{-k}\, V_{9,3} \\ V_{9,3} &= \mathcal{O} \end{aligned}$$

or, explicitly,

$$
\begin{aligned}
V_{1,3} &= -2\mathcal{O} - \mathcal{O}_{D_1} + \mathcal{O}_E \\
V_{2,3} &= \mathcal{O} + \mathcal{O}_{D_1} = L_1 \\
V_{3,3} &= \mathcal{O} + 3\mathcal{O}_{D_1} - \mathcal{O}_{D_2} + 3\mathcal{O}_E \\
V_{4,3} &= -2\mathcal{O} - 7\mathcal{O}_{D_1} + 2\mathcal{O}_{D_2} + \mathcal{O}_C - 5\mathcal{O}_E \\
V_{5,3} &= \mathcal{O} + 4\mathcal{O}_{D_1} - \mathcal{O}_{D_2} - \mathcal{O}_C + 3\mathcal{O}_E - \mathcal{O}_p \\
V_{6,3} &= -2\mathcal{O} - 3\mathcal{O}_{D_1} + \mathcal{O}_{D_2} - 3\mathcal{O}_E \\
V_{7,3} &= 4\mathcal{O} + 8\mathcal{O}_{D_1} - 2\mathcal{O}_{D_2} - \mathcal{O}_C + 4\mathcal{O}_E \\
V_{8,3} &= -2\mathcal{O} - 5\mathcal{O}_{D_1} + \mathcal{O}_{D_2} + \mathcal{O}_C - 3\mathcal{O}_E + \mathcal{O}_p \\
V_{9,3} &= \mathcal{O}
\end{aligned}
$$

The intersection form

$$(V_{k,m}, V_{k',m'}) = f_{k-l} \tag{5.51}$$

where f_n takes values

n	1	2	3	4	5	6	7	8	9
f_n	-1	1	-1	2	-2	1	-1	1	0

The other limit points of the model are as follows. There is the aforementioned large radius point (at the intersection of $[s_1]$ and $[s_2]$). There's a hybrid point, consisting of a LG model (with a cubic superpotential) fibered over a P^2, at the intersection of $[s_1]$ and $[s_3]$. Circling this point about the $[s_3]$, we detected in (5.45) the enhanced $\mathbb{Z}_3$ quantum symmetry of the LG fiber; circling this point about $[s_1]$, we detect the monodromy, M_{r_1}, associated to shifting the B-field on the P^2 base. Finally, at the intersection of $[s_2]$ and $[s_4]$, the Calabi-Yau develops a $\mathbb{Z}_3$ orbifold singularity. The Calabi-Yau isn't globally a quotient by this $\mathbb{Z}_3$, so we don't really have an enhanced $\mathbb{Z}_3$ quantum symmetry. Rather, circling $[s_4]$ three times is equivalent to shifting the B-field (5.48).

5.3.4 Branes in the small volume phase

According to the above discussion, D-branes in the small volume phase can be investigated by studying D-branes on the orbifold $\mathbb{C}^5/\mathbb{Z}_9 \times \mathbb{Z}_3$. Those can be studied in terms of quiver gauge theory. A basic set of D-branes (with Dirichlet conditions in all directions of the orbifold) is given by the fractional branes, which are labelled by irreducible representations of the orbifold theory. Those form the nodes of the quiver. The chiral matter multiplets, which can be determined in the usual way by projection, give rise to the links of the quiver. Their number can be computed from the index theorem and is equal to the intersection number. This should therefore be compared to the result of a geometric index computation. The continuation of the fractional brane basis to large volume sheaves has been discussed in the literature [95, 111–113]. Here, our focus is on the K theory classes and we'd like to compare the fractional branes to large volume branes whose K-theory classes are $V_{k,m}$.

Let us make this more concrete for the model at hand. Since the $\mathbb{Z}_9 \times \mathbb{Z}_3$ orbifold group is abelian, all irreducible representations are one-dimensional and can be labelled by two phases: $\rho = (e^{2\pi ik/9}, e^{2\pi m/3})$. Working out the representation theory yields the following result: The number of chiral multiplets between a brane (k, m) and a brane (k', m') depends only on the difference $\Delta k = k - k'$. In particular, it is independent of the m label. The dependence on Δk is summarized in the following table:

Δk	1	2	3	4	5	6	7	8	9
	-1	1	-1	2	-2	1	-1	1	0

Comparison with the table in the previous section shows that this exactly reproduces the geometrical intersection numbers of the K-theory classes $V_{k,m}$.

The index computations performed above can be taken to the IR fixed point of the model, which is described by the Gepner model. Let us make the connection to boundary CFT results more explicit.

According to [95], the fractional branes of the quiver discussion should be directly compared to the set of $L = 0$ rational Gepner boundary states. For the covering theory, the Gepner model $(k = 7)^3(k = 1)^2$, these B-type boundary states have been computed in [81]. The Gepner model itself is a $\mathbb{Z}_9$ orbifold of a tensor product of minimal models (+ other projections, which are currently not of importance to us), the $\mathbb{Z}_9$ being the GSO projection. Accordingly, there is a $\mathbb{Z}_9$ quantum symmetry, which we denote g. The boundary states are labelled by a single label M, $M = 0, 2, 4, \ldots 18$, which can be interpreted as discrete $\mathbb{Z}_9$ Wilson lines. This label should be directly compared to the representation label k in the quiver discussion, $M = 2k$. The quantum symmetry acts on the boundary states by $g : M \mapsto M + 2$. In geometrical terms this action maps to the action of the Gepner monodromy on six-branes $\mathcal{O}$.

To determine the intersection matrix, the Witten index $\mathrm{tr}_R(-1)^F$ has to be evaluated in the open string Ramond sector. This is related by a modular transformation to the closed string amplitude $< M_1|(-1)^{F_L}|M_2 >_{RR}$ between boundary states. To compute the intersection matrix on the covering theory, the formulas given in [92] can be used. Due to the symmetry of the model, it can be written in terms of the shift matrix g:

$$I = -3g^{-1} + 3g^{-2} - 3g^{-3} + 6g^{-4} - 6g^{-5} + 3g^{-6} - 3g^{-7} + 3g^{-8} \qquad (5.52)$$

From this, the intersection matrix of the orbifold model can be obtained directly. The boundary states of the covering theory are invariant under the $\mathbb{Z}_3$ orbifold action. To obtain consistent boundary states of the orbifold theory, one adds a twisted sector contribution to the boundary states. This part of the boundary state contains only Ishibashi states built on fields in the twisted sector. In the open string sector they lead to projection operators, since the modular transformation of a twisted sector character leads to an insertion of a group element. The boundary states are distinguished by $\mathbb{Z}_3$ representations, which determine how the projections act in the open string sector.

The index in the orbifold model can be determined without explicit knowledge of that boundary state. It is sufficient to know that the orbifold acts freely, which means that there are no RR ground states in the twisted sector. Therefore, the twisted part of the boundary state cannot give rise to new contributions to the index. All that happens is that there is a projection in the open string sector, picking out an invariant combination of the R-ground states counted in (5.52). To write the new intersection matrix, we introduce the operation h, which is the quantum symmetry corresponding to the $\mathbb{Z}_3$. In terms of the two quantum symmetry operators the intersection matrix reads:

$$I = \left(g^{-1} + g^{-2} - g^{-3} + 2g^{-4} - 2g^{-5} + g^{-6} + g^{-7} + g^{-8}\right)(1 + h + h^2). \quad (5.53)$$

(as before, we replace 3 with $1 + h + h^2$). This matrix is just a different form of presenting the contents of the table in the quiver-based discussion.

The form of the intersection matrix shows that transforming a fractional brane by h cannot change the $\mathbb{Z}$-valued RR charge, but only the torsion charge.

As before, we can construct a torsion brane as a bound state of BPS states.

There are two ways to do so: One way is to take a fractional brane and its h-transformed anti-brane. This system is tachyon-free and represents a classically stable state carrying only torsion charge.

Alternatively, we can take a superposition of fractional branes in the following way:

$$|T> = h|B> + \sum_{n=1}^{8} g^n |B>, \quad (5.54)$$

where $|B>$ is any of the fractional branes. There are tachyons propagating between the individual branes, making this configuration unstable. Mapping the brane charges to large volume shows explicitly that there is a net torsion charge and the decay product is therefore non-trivial.

Chapter 6

Conclusion

We have been able to use the mathematical apparatus of K-theory, in an interesting interplay with the special geometry of Calabi-Yau manifolds, to calculate monodromies of torsion branes on the moduli space of said manifolds, which has then enabled us to predict, *inter alia*, novel forms for the monodromy about a conifold point and the existence of new massless states at a conifold point.

The Douglas program goes much farther than we outlined in our brief introduction. Using the special properties of BPS states, it has branched in two different directions. In [93] and later work, it has led to the prediction of a *stability criterion* for physical D-branes. In [127] and a tremendous proliferation of other papers, instead, what is considered is the *topologically twisted* theory, where brane characteristics are independent of the Kähler moduli. In that case, an attempted classification, not just of D-brane charges, but of all D-brane states (distinguishing, for example, otherwise identical D-branes that have a different location in space), has been made in terms of the *derived category of coherent sheaves* on the Calabi-Yau manifold, with monodromies given by *Fourier-Mukai transforms*. Another approach worth noting, pioneered by several different groups, most notably Govindarajan, et al., and exemplified in [20], attempts to extend the project begun in [90] to compactifications with D-branes by considering GLσM's on worldsheets with boundary.

As yet, none of these approaches has impacted materially on the specific considerations for torsion branes, so they have not been treated in more detail. There are, however, several intriguing directions for potential future study. In particular, the generalized result on conifold monodromies might well be derivable as some extension or version of a Fourier-Mukai transform. As indicated in [2], derived category considerations can also potentially be of use in analyzing stability of physical torsion branes.

Appendix

Appendix A

A Brief Introduction to Toric Varieties

The line of exposition here closely follows [91].

Just as d-dimensional projective space can be described as

$$P^d = (\mathbb{C}^{d+1} - \{0\})/\mathbb{C}^*, \tag{A.1}$$

a general toric variety can be described as

$$V_\Delta = (Y - F_\Delta)/T_\Delta, \tag{A.2}$$

where $Y = \mathbb{C}^n$ and $T_\Delta \sim \mathbb{C}^{*\,(n-d)}$ acts diagonally on the coordinates of Y as

$$g_a(\lambda) : x_i \to \lambda^{Q_i^a} x_i \qquad a = 1, \ldots, (n-d), \quad i = 1, \ldots, n \ , \tag{A.3}$$

and F_Δ is a subset of $Y - \mathbb{C}^{*n}$ that is a union of intersections of coordinate hyperplanes.

To determine F_Δ as well as the Q_i^a requires that we specify the detailed combinatorial data, Δ, that determine the toric variety – this data is known as the fan.

Let $N \sim \mathbb{Z}^d$ be a lattice in $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$. To define V we need a *fan of strongly convex rational polyhedral cones* Δ in $N_\mathbb{R}$. This is a collection of cones with apex at the origin, each of which is spanned by a finite collection ("polyhedral") of elements in N ("rational") and such that the angle subtended by any two of these at the apex is less than π ("strongly convex"). To be a fan the collection must have the property that (i) any two members of the collection intersect in a common face (i.e. a cone of lower dimension bounding each) – note that this includes the origin as a possible intersection – and (ii) for each member of Δ all its faces are also in Δ. This data determines the holomorphic quotient described above as follows.

1. Let $\{v_1, \ldots, v_n\}$ be the integral generators of the one-dimensional cones in Δ, then in (A.2) we set $Y = \mathbb{C}^n$.

2. The set F_Δ is the union of intersections of coordinate hyperplanes $x_{i_1} = \cdots = x_{i_p} = 0$ for each set $1 \leq i_1 \leq \cdots i_p \leq n$ such that $v_{i_1}, \ldots v_{i_p}$ are not contained in any cone of Δ. The irreducible components of F are seen to be determined by collections as above such that any subset of $k < p$ of the vectors in the collection spans a k-dimensional cone in Δ. These are called *primitive collections.*

3. To get T_Δ let $D \subset \mathbb{Z}^n$ be the sublattice of vectors $d = (d_1, \ldots, d_n)$ such that $\sum_i d_i v_i = 0$. Choosing a basis $\{Q^1, \ldots, Q^{n-d}\}$ for D we obtain the T action of (A.8).

The fan encodes a great deal of other information about V. For example, V is compact when and only when the fan is *complete*, i.e. when the cones in Δ

cover $N_{\mathbb{R}}$. And, in case V is *simplicial* (a cone is simplicial when a minimal set of integral generators is linearly independent. A fan is simplicial when every cone in the fan is), the fan also encodes the intersection theory on V.

In general, the cohomology of V is nonzero only in even dimensions – when V is compact, $H^*(V)$ is actually generated by $H^2(V)$ under the intersection product. Thus the complete intersection ring is determined by relations on the elements of $H^2(V)$. These are easily read off from Δ as follows. The group $H^2(V)$ itself is generated by classes $\psi_i \quad i = 1, \ldots, n$ dual to the divisors $\{x_i = 0\}$, subject to linear relations. These essentially express the fact that T-invariant monomials in the homogeneous coordinates x_i (and their inverses) are meromorphic functions on V and hence correspond to trivial divisors. These monomials are parameterized by the lattice $M \sim \mathbb{Z}^d$ dual to N, since 3. above guarantees that for $m \in M$ the monomial $\chi_m = \prod_{i=1}^n x_i^{\langle m, v_i \rangle}$ is T-invariant. Thus we have

$$\sum_{i=1}^{n} \langle m, v_i \rangle \psi_i = 0 \tag{A.4}$$

for every $m \in M$. There are d independent relations of this type, which reduce the dimension of $H^2(V)$ to $n-d$. This determines a basis ξ_a of $H^2(V)$ in which we can write

$$\psi_i = \sum_{a=1}^{n-d} Q_i^a \xi_a \ . \tag{A.5}$$

There are also nonlinear relations in the ring $H^*(V)$. These are read off most easily in the dual picture as excluded intersections of the coordinate hyperplanes. That is, for each irreducible component of F, described as $\{x_a =$

$0 \mid a \in A\}$ for some set $A \subset \{1, \ldots, n\}$, we get a relation

$$\prod_{a \in A} \psi_a = 0 \ . \tag{A.6}$$

(These relations comprise what is known as the *Stanley–Reisner ideal.*)

The relations (A.5) and (A.6) determine the ring structure of $H^*(V)$ completely; the one thing left undetermined is the normalization of the expectation function $\langle \ \rangle_V$ given by evaluation on the fundamental class of V. This can also be determined by the toric data, as follows. Given a collection of d distinct coordinate hyperplanes $\{x_{i_1} = 0\}$, $\ldots$, $\{x_{i_d} = 0\}$ which *do* intersect on V, we have

$$\langle \psi_{i_1} \ldots \psi_{i_d} \rangle_V = \frac{1}{\mathrm{mult}(v_{i_1}, \ldots, v_{i_d})} \ , \tag{A.7}$$

where $\mathrm{mult}(v_{i_1}, \ldots, v_{i_d})$ denotes the index in N of the lattice spanned by these vectors. (This index is always 1 if V is smooth.)

Example:

In some cases, the toric variety we pick is not smooth, and we want to perform a *resolution* before we construct a GLσM on it. The basic idea in constructing resolutions is simple. A simplicial fan in which every cone has volume 1 (in the definition of volume where the n-simplex has volume 1 in n dimensions)leads to a smooth toric variety. If a simplicial fan does not satisfy this condition, one performs the resolution by subdividing cones so that the condition is fulfilled, then looking at the toric variety corresponding to this new fan. In general, weighted projective spaces are not smooth, since they have non-free actions of finite groups on them. The example we pick is of relevance in the main text.

Consider the toric variety $V = P^4_{(1,1,1,3,3)}$. It has a fan, Δ, in which the one-dimensional cones are generated by

$$v_1 = (1,0,0,0), \qquad v_2 = (0,1,0,0), \qquad v_3 = (0,0,1,0),$$
$$v_4 = (0,0,0,1), \quad v_5 = (-1,-1,-3,-3)$$

and the rest of the fan is all possible simplicial cones generated by subsets of these vectors. The volume of the 4-dimensional cones generated by 4-element subsets of this set of vectors can be calculated simply as the absolute value of the determinant of the matrix formed by juxtaposing the given 4 vectors. One can see that the cones generated by $\{v_1, v_2, v_3, v_5\}$ and $\{v_1, v_2, v_4, v_5\}$ have volume 3, not 1.

To perform the resolution, add the vector $v_6 = (0,0,-1,-1)$ to the fan, along with all the simplicial cones that could be generated with v_6 as part of the basis – and replace cones containing v_5 that have volume greater than 1 with cones defined by substituting v_6 for v_5.

In order to write down the GLσM that describes this model, we have to do a few elementary calculations, as outlined above.

1. First, we find F_Δ. Clearly, $\{v_3, v_4, v_6\}$ does not generate a simplicial cone nor can it be contained in any simplicial cone (since, in fact, this set is linearly dependent), so it is not a subset of any cone in the fan. Similarly,

the cone generated by $\{v_1, v_2, v_5\}$ is not simplicial, and cannot be contained in any simplicial cone (v_6 is already contained within this three-dimensional cone, so adding it doesn't help, and adding v_3 or v_4 gives a cone of volume 3). A little consideration will convince one that these are the appropriate primitive collections (any other excluded collection contains one of these) , so $F_\Delta = \{x_1 = x_2 = x_5 = 0\} \bigcup \{x_3 = x_4 = x_6 = 0\}$.

2. Next, we find T_Δ. Let $d = (d_1, \ldots, d_6)$ such that $\sum_{i=1}^n d_i v_i = 0$. This gives us a set of equations:

$$d_1 - d_5 = d_2 - d_5 = 0$$
$$d_3 - 3d_5 - d_6 = d_4 - 3d_5 - d_6 = 0$$

that has solutions dependent on two parameters, which we can choose to be the values of d_5 and d_6. A basis for the solutions can be chosen by taking for the first element $d_5 = 1, d_6 = -3$ and for the second $d_5 = 0, d_6 = 1$. We get two vectors:

$$\vec{d_1} = (1, 1, 0, 0, 1, -3) \quad (A.8)$$

$$\vec{d_2} = (0, 0, 1, 1, 0, 1) \quad (A.9)$$

Along with these vectors, we also obtain the action of the torus $\mathbb{C}^{*2}$ that defines the toric variety. We get

$$
\begin{aligned}
g_1(\lambda) : x_1 \to \lambda x_1, \quad x_2 \to \lambda x_2, \quad x_3 \to x_3 \\
x_4 \to x_4, \quad x_5 \to \lambda x_5, \quad x_6 \to \lambda^3 x_6 \\
g_2(\lambda) : x_1 \to x_1, \quad x_2 \to x_2, \quad x_3 \to \lambda x_3 \\
x_4 \to \lambda x_4, \quad x_5 \to x_5, \quad x_6 \to \lambda x_6
\end{aligned}
$$

Given these quantities, construction of the GLσM is explained in Chapter 2. With appropriate re-ordering of coordinates, we get precisely the results quoted there for the $P^4_{(1,1,1,3,3)}$ model.

Next in the same example, we work out the cohomology. Using the above notation, we can read off from (A.8) the linear relations on the cohomology. We get

$$\psi_1 = \psi_2 = \psi_5 = \xi_1 \tag{A.10}$$

$$\psi_2 = \psi_4 = \xi_2, \quad \psi_6 = \xi_2 - 3\xi_1 \tag{A.11}$$

The nonlinear relations follow from the result for F_Δ. We get

$$\psi_1\psi_2\psi_5 \equiv \xi_1^3 = 0 \tag{A.12}$$

$$\psi_3\psi_4\psi_6 \equiv \xi_2^2(\xi_2 - 3\xi_1) = 0 \tag{A.13}$$

So, $H^2(V)$ is two-dimensional, generated by ξ_1 and ξ_2. Since $H^2(V)$ generated the entire cohomology of V as a ring, we find that $H^4(V)$ is three-dimensional, generated by ξ_1^2, $\xi_1\xi_2$, and ξ_2^2. For $H^6(V)$, we have four potential generators but also two relations, $\xi_1^3 = 0$ and $\xi_2^3 = 3\xi_1\xi_2^2$, so $H^6(V)$ is two-dimensional and generated (as a free abelian group) by $\xi_1^2\xi_2$ and $\xi_1\xi_2^2$. Since V is a compact 8-dimensional manifold,$H^8(V)$ is one-dimensional. We can use (A.7) to delve further. Take the vectors $\{v_1, v_2, v_3, v_4\}$ – they obviously generate the entire lattice N. We thus see that

$$\langle \xi_1^2 \xi_2^2 \rangle = 1 \tag{A.14}$$

, so that $\xi_1^2\xi_2^2$ generates $H^8(V)$ integrally. Using the second relation in (A.12), we see also that

$$\xi_2^4 = 3\xi_1\xi_2^3 = 9\xi_1^2\xi_2^2. \tag{A.15}$$

Our actual example, of course, is not the toric variety itself but the degree 9 hypersurface, M, in it. Here, toric methods do not suffice to calculate the entire cohomology ring (more powerful methods involving spectral sequences and the Koszul resolution do), but they suffice for the part we are actually interested in, which might be called the "toric cohomology" of M, $H_V^*(M)$. In order to calculate this, we need a couple of basic equations. First, since the hypersurface is by construction Calabi-Yau, we know that the divisor it determines is the anticanonical divisor

$$-K = \sum_{i=1}^{n} \psi_i. \tag{A.16}$$

$H_V^*(M)$ is generated as a ring by the second cohomology, as before, with the same linear relations. In addition, one has the *restriction formula* for intersections in M:

$$\langle \eta_{a_1} \cdots \eta_{a_p} \rangle_M = \langle \eta_{a_1} \cdots \eta_{a_p} (-K) \rangle_V \tag{A.17}$$

Now, let's apply this to our example. Looking at our results for the cohomology of V, we see that V has one Kähler modulus associated with a single blowup parameter in addition to the Kähler modulus associated with the overall size. The corresponding exceptional divisor (the hyperplane defined by $z_6 = 0$) intersects the Calabi-Yau hypersurface in three disjoint P^2's, and the full cohomology of M contains three Kähler moduli that correspond roughly to the sizes of the three P^2's – so $H^2(M)$ is 4-dimensional. The toric subspace of that cohomology, however, corresponds to deformations where the sizes of the three P^2's are the same, and thus $H^2_V(M)$ is two dimensional (one Kähler parameter for the sizes of the P^2's and one for the overall size of M). It's not obvious, but in fact $H^4_V(M)$ is also two-dimensional (by Poincare duality, $H^4(M)$ has the same dimension as $H^2(M)$, and the toric restriction applies similarly in each case).

We also have

$$K = -\sum_{i=1}^{n} \psi_i = -3\xi_2 \tag{A.18}$$

.

Thus, we can immediately calculate some intersection products:

$$\langle \xi_1^2 \xi_2 \rangle_M = \langle -3\xi_1^2 \xi_2^2 \rangle_V = 3 \tag{A.19}$$

$$\langle \xi_1 \xi_2^2 \rangle_M = \langle -3\xi_1 \xi_2^3 \rangle_V = 9 \tag{A.20}$$

$$\langle \xi_2^3 \rangle_M = \langle -3xi_2^4 \rangle_V = 27 \tag{A.21}$$

Thus, clearly, the integral generator of $H_V^6(M)$ is $\xi_1^2\xi_2$ – call it ρ. So, $H_V^2(M)$ is generated by ξ_1 and ξ_2, as above. $H_V^4(M)$ is generated by ξ_1^2, ξ_2^2, and $\xi_1\xi_2$, but is only two-dimensional, so there is one relation between them. Let $\chi = A\xi_1^2 + B\xi_1\xi_2 + C\xi_2^2 = 0$. Then, we have $0 = \langle \xi_1 \chi \rangle_M = 3B + 9C = 0$ and $0 = \langle \xi_2 \chi \rangle_M = 3A + 9B + 27C = 0$. We thus find that, in fact,

$$\xi_2^2 = 3\xi_1\xi_2. \tag{A.22}$$

We can see that $\{\xi_1^2, \xi_1\xi_2\}$ is an integral basis for $H^4(M)$. Let's define $\sigma_1 \equiv \xi_1^2$, $\sigma_2 \equiv \xi_1\xi_2$ to be a new basis, and then perform a basis change by the unimodular matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \tag{A.23}$$

to a new basis $\{\eta_1, \eta_2\}$.

Then we find, with a little elementary algebra, that we can express the ring $H_V^*(M)$ as follows:

The basis (as a vector space) is $\{1, \xi_1, \xi_2, \eta_1, \eta_2, \rho\}$ with relations

$$\xi_1^2 = \eta_2, \quad \xi_1\xi_2 = \eta_1 + 3\eta_2, \quad \xi_2^2 = 3(\eta_1 + 3\eta_2) \tag{A.24}$$

$$\xi_i\eta_j = \delta_{ij}\rho. \tag{A.25}$$

Bibliography

[1] J. Distler and I. Brunner, "Torsion D-branes in nongeometrical phases," hep-th/0102018.

[2] I. Brunner, J. Distler, and R. Mahajan , "Return of the Torsion D-branes," hep-th/0106262.

[3] J. Dai, R. G. Leigh and J. Polchinski,"New Connections Between String Theories," *Mod. Phys. Lett.* **A4**, 2073 (1989).

[4] R. G. Leigh, "Dirac-Born-Infeld Action From Dirichlet Sigma Model," *Mod. Phys. Lett.* **A4**, 2767 (1989).

[5] J. Polchinski, "Dirichlet-Branes and Ramond-Ramond Charges,"*Phys. Rev. Lett.* **75** (1995) 4724-4727, hep-th/9510017.

[6] J. Polchinski, S. Chaudhuri, and C. Johnson, "Notes on D-Branes," hep-th/9602052.

[7] J. Polchinski, "TASI Lectures on D-Branes,' hep-th/9611050.

[8] P. Di Vecchia, A. Liccardo, "D-Branes in String Theory, I,' hep-th/9912161.

[9] P. Di Vecchia, A. Liccardo, "D-Branes in String Theory, II,' hep-th/9912275.

[10] C. G. Callan, C. Lovelace, C. R. Nappi and S. A. Yost, "Adding Holes And Crosscaps To The Superstring," *Nucl. Phys.* **B293**, 83 (1987).

[11] S. A. Yost,"Bosonized Superstring Boundary States And Partition Functions," *Nucl. Phys. B* **B321**, 629 (1989).

[12] N. Ishibashi,"The Boundary And Crosscap States In Conformal Field Theories," *Mod. Phys. Lett. A* **A4**, 251 (1989).

[13] T. Onogi, N. Ishibashi, "Conformal Field Theories On Surfaces With Boundaries And Crosscaps," *Mod. Phys. Lett.* **A4**, 161 (1989) [Erratum-ibid. **A4**, 885 (1989)].

[14] J. L. Cardy,"Conformal Invariance And Surface Critical Behavior," Nucl. Phys. B **240**, 514 (1984).

[15] J. L. Cardy,"Effect Of Boundary Conditions On The Operator Content Of Two-Dimensional Conformally Invariant Theories," Nucl. Phys. B **275**, 200 (1986).

[16] J. L. Cardy,"Conformal Invariance In Critical Systems With Boundaries," *In *Bonn 1986, Proceedings, Infinite Lie Algebras and Conformal Invariance In Condensed Matter and Particle Physics*, 81-92.*

[17] J. L. Cardy,"Boundary Conditions, Fusion Rules And The Verlinde Formula," *Nucl. Phys. B* **B324**, 581 (1989).

[18] J. L. Cardy and D. C. Lewellen,"Bulk And Boundary Operators In Conformal Field Theory," *Phys. Lett.***B259**, 274 (1991).

[19] D. C. Lewellen,"Sewing constraints for conformal field theories on surfaces with boundaries," *Nucl. Phys. B* **B372**, 654 (1992).

[20] S. Govindarajan, T. Jayarman, and T. Sarkar,"Worldsheet approaches to D-branes on supersymmetric cycles," *Nucl. Phys. B* **B580**, 519 (200). `hep-th/9907131`.

[21] E. Verlinde,"Fusion Rules And Modular Transformations In 2-D Conformal Field Theory," *Nucl. Phys.* **B300**, 360 (1988).

[22] G. Pradisi, A. Sagnotti and Y. S. Stanev,"Completeness Conditions for Boundary Operators in 2D Conformal Field Theory," *Phys. Lett. B* **B381**, 97 (1996) `hep-th/9603097`.

[23] A. Sen,"Strong coupling dynamics of branes from M-theory," *JHEP* **9710**, 002 (1997) `hep-th/9708002`.

[24] A. Sen,"Stable non-BPS states in string theory," *JHEP* **9806**, 007 (1998) `hep-th/9803194`.

[25] A. Sen,"Stable non-BPS bound states of BPS D-branes," *JHEP* **9808**, 010 (1998) `hep-th/9805019`.

[26] A. Sen,"Tachyon condensation on the brane antibrane system," *JHEP* **9808**, 012 (1998) `hep-th/9805170`.

[27] A. Sen, "SO(32) Spinors of Type I and Other Solitons on Brane-Antibrane Pair," *JHEP* **09** (1998) 023, hep-th/9808141.

[28] A. Sen, "Type I D-particle and its interactions," *JHEP* **10** (1998) 021, hep-th/9809111.

[29] A. Sen,"BPS D-branes on non-supersymmetric cycles," *JHEP* **9812**, 021 (1998) hep-th/9812031.

[30] J. Majumder and A. Sen,"'Blowing up' D-branes on non-supersymmetric cycles," JHEP **9909**, 004 (1999) hep-th/9906109.

[31] A. Sen and B. Zwiebach,"Stable non-BPS states in F-theory," JHEP **0003**, 036 (2000) hep-th/9907164.

[32] A. Sen,"Supersymmetric world-volume action for non-BPS D-branes," JHEP **9910**, 008 (1999) hep-th/9909062.

[33] A. Sen,"Non-BPS states and branes in string theory," hep-th/9904207

[34] M. Frau, L. Gallot, A. Lerda and P. Strigazzi, "Stable non-BPS D-branes in type I string theory," *Nucl. Phys.* **B564**, 60 (2000) hep-th/9903123.

[35] A. Lerda and R. Russo,"Stable non-BPS states in string theory: A pedagogical review," Int. J. Mod. Phys. **A15**, 771 (2000) hep-th/9905006.

[36] O. Bergman and M. R. Gaberdiel, "Stable non-BPS D-particles," *Phys. Lett.* **B441** (1998) 133–140, `hep-th/9806155`.

[37] Y. K. Cheung and Z. Yin,"Anomalies, branes, and currents," *Nucl. Phys.* **B517**, 69 (1998) `hep-th/9710206`.

[38] R. Minasian and G. W. Moore,"K-theory and Ramond-Ramond charge," *JHEP* **9711** (1997) 002 `hep-th/9710230`.

[39] E. Witten,"D-branes and K-theory," *JHEP* **9812** (1998) 019 `hep-th/9810188`.

[40] P. Horava,"Type IIA D-branes, K-theory, and matrix theory," *Adv. Theor. Math. Phys.* **2**, 1373 (1999) `hep-th/9812135`.

[41] H. Garcia-Compean,"D-branes in Orbifold Singularities and Equivariant K-theory," *Nucl. Phys.* **B557**, 480 (1999) `hep-th/9812226`.

[42] W. Lerche, C. Vafa and N. P. Warner,"Chiral Rings In N=2 Superconformal Theories," *Nucl. Phys.* **B324**, 427 (1989).

[43] B. R. Greene, C. Vafa and N. P. Warner, "Calabi-Yau Manifolds And Renormalization Group Flows," *Nucl. Phys.* **B324**, 371 (1989).

[44] E. J. Martinec,"Criticality, Catastrophes And Compactifications," Print-89-0373 (EFI,CHICAGO) *In *Brink, L. (ed.) et al.: Physics and mathematics of strings* 389-433. (see Book Index).*

[45] B. Greene,"String Theory on Calabi-Yau Manifolds," `hep-th/9702155`.

[46] S. Ferrara, J. Harvey, A. Strominger, and C. Vafa, "Second quantized mirror symmetry," *Phys. Lett.* **B361** (1995) 059, `hep-th/9505162`.

[47] J. de Boer *et al.*, "Triples, fluxes, and strings," `hep-th/0103170`.

[48] J. Majumder and A. Sen, "Non-BPS D-branes on a Calabi-Yau orbifold," *JHEP* **09** (2000) 047, `hep-th/0007158`.

[49] M. Mihailescu, K. Oh, and R. Tatar, "Non-BPS branes on a Calabi-Yau threefold and Bose-Fermi degeneracy," *JHEP* **02** (2000) 019, `hep-th/9910249`.

[50] D.-E. Diaconescu, G. Moore, and E. Witten, "E_8 gauge theory, and a derivation of K-theory from M-theory," `hep-th/0005090`.

[51] O. Bergman, E. Gimon, and S. Sugimoto, "Orientifolds, RR torsion, and K-theory," *JHEP* **05** (2001) 047, `hep-th/0103183`.

[52] O. Bergman, E. Gimon, and B. Kol, "Strings on orbifold lines," *JHEP* **05** (2001) 019, `hep-th/0102095`.

[53] M. Atiyah, V. Patodi, and I. Singer, "Spectral asymmetry and Riemannian geometry. I," *Math. Proc. Camb. Phil. Soc.* **77** (1975) 43.

[54] M. Atiyah, V. Patodi, and I. Singer, "Spectral asymmetry and Riemannian geometry. II," *Math. Proc. Camb. Phil. Soc.* **78** (1975) 405.

[55] M. Atiyah, V. Patodi, and I. Singer, "Spectral asymmetry and Riemannian geometry. III," *Math. Proc. Camb. Phil. Soc.* **79** (1976) 71.

[56] J. Lott, "$\mathbb{R}/\mathbb{Z}$ index theory," *Comm. Anal. Geom.* **2** (1994) 279–311.

[57] C. Vafa, "Quantum symmetries of string vacua," *Mod. Phy. Lett.* **A4** (1989) 1615.

[58] C. Vafa, "String Vacua And Orbifoldized L-G Models," *Mod. Phy. Lett.* **A4** (1989) 1615.

[59] A. Sen, *Nucl. Phys.* **278**, 289 (1986).

[60] T. Banks, L. J. Dixon, D. Friedan and E. J. Martinec, "Phenomenology And Conformal Field Theory Or Can String Theory Predict The Weak Mixing Angle?," *Nucl. Phys.* **B299**, 613 (1988).

[61] D. Gepner,"On The Spectrum Of 2-D Conformal Field Theories," *Nucl. Phys.* **B287**, 111 (1987).

[62] D. Gepner,"Space-Time Supersymmetry In Compactified String Theory And Superconformal Models," *Nucl. Phys.* **B296**, 757 (1988).

[63] D. Gepner,"Exactly Solvable String Compactifications On Manifolds Of SU(N) Holonomy," *Phys. Lett.* **B199**, 380 (1987).

[64] J. Fuchs, A. Klemm, C. Scheich and M. G. Schmidt, "Spectra And Symmetries Of Gepner Models Compared To Calabi-Yau Compactifications," *Annals Phys.* **204**, 1 (1990).

[65] J. Fuchs, A. Klemm, C. Scheich and M. G. Schmidt, "Gepner Models With Arbitrary Affine Invariants And The Associated Calabi-Yau Spaces," *Phys. Lett.* **B232**, 317 (1989).

[66] P. Candelas, P. S. Green and T. Hubsch,"Rolling Among Calabi-Yau Vacua," *Nucl. Phys.* **B330**, 49 (1990).

[67] P. Candelas and X. C. de la Ossa,"Comments On Conifolds," *Nucl. Phys.* **B342**, 246 (1990).

[68] P. Candelas, M. Lynker and R. Schimmrigk,"Calabi-Yau Manifolds In Weighted P(4)," *Nucl. Phys.* **B341**, 383 (1990).

[69] P. Candelas and X. de la Ossa,"Moduli Space Of Calabi-Yau Manifolds," *Nucl. Phys.***B355**, 455 (1991).

[70] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, "A Pair Of Calabi-Yau Manifolds As An Exactly Soluble Superconformal Theory," *Nucl. Phys.* **B359**, 21 (1991).

[71] P. Candelas, X. C. De la Ossa, P. S. Green and L. Parkes, "An Exactly Soluble Superconformal Theory From A Mirror Pair Of Calabi-Yau Manifolds," *Phys. Lett.* **B258**, 118 (1991).

[72] P. Candelas, X. De La Ossa, A. Font, S. Katz and D. R. Morrison, "Mirror symmetry for two parameter models. I," *Nucl. Phys.* **B416**, 481 (1994) `hep-th/9308083`.

[73] P. Candelas, A. Font, S. Katz and D. R. Morrison, "Mirror symmetry for two parameter models. II," *Nucl. Phys.* **B429**, 626 (1994) `hep-th/9403187`.

[74] P. Candelas, X. de la Ossa and S. Katz, "Mirror symmetry for Calabi-Yau hypersurfaces in weighted P^4 and extensions of Landau-Ginzburg theory," *Nucl. Phys.* **B450**, 267 (1995) `hep-th/9412117`.

[75] B. R. Greene and M. R. Plesser,"Duality In Calabi-Yau Moduli Space," *Nucl. Phys.* **B338**, 15 (1990).

[76] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, "Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces," *Commun. Math. Phys.* **167**, 301 (1995) `hep-th/9308122`.

[77] S. Hosono, A. Klemm, S. Theisen and S. T. Yau, "Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces," *Nucl. Phys.* **B433**, 501 (1995) `hep-th/9406055`.

[78] S. Katz, D. R. Morrison, and M. R. Plesser, "Enhanced gauge symmetry in type II string theory," *Nucl. Phys.* **B477** (1996) 105–140, `hep-th/9601108`.

[79] A. Klemm and P. Mayr, "Strong coupling singularities and non-abelian gauge symmetries in $N = 2$ string theory," *Nucl. Phys.* **B469** (1996) 37–50, `hep-th/9601014`.

[80] H. Ooguri, Y. Oz and Z. Yin,"D-branes on Calabi-Yau spaces and their mirrors," *Nucl. Phys.* **B477**, 407 (1996) `hep-th/9606112`.

[81] A. Recknagel and V. Schomerus, "D-branes in Gepner models," *Nucl. Phys.* **B531** (1998) 185–225, `hep-th/9712186`.

[82] M. Gutperle and Y. Satoh,"D-branes in Gepner models and supersymmetry," *Nucl. Phys.***B543**, 73 (1999) `hep-th/9808080`.

[83] M. Gutperle and Y. Satoh,"D0-branes in Gepner models and N = 2 black holes," *Nucl. Phys.* **B555**, 477 (1999) `hep-th/9902120`.

[84] M. Naka and M. Nozaki,"Boundary states in Gepner models," JHEP **0005**, 027 (2000) `hep-th/0001037`.

[85] J. Fuchs, C. Schweigert and J. Walcher, "Projections in string theory and boundary states for Gepner models," *Nucl. Phys.* **B588**, 110 (2000) `hep-th/0003298`.

[86] S. Mizoguchi and T. Tani,"Wound D-branes in Gepner models," *Nucl. Phys.* **B611**, 253 (2001) `hep-th/0105174`.

[87] E. Scheidegger, "On D0-branes in Gepner models," `hep-th/0109013`.

[88] I. Brunner and V. Schomerus,"On superpotentials for D-branes in Gepner models," JHEP **0010**, 016 (2000) `hep-th/0008194`.

[89] I. Brunner and V. Schomerus, "D-branes at singular curves of Calabi-Yau compactifications," *JHEP* **04** (2000) 020, `hep-th/0001132`.

[90] E. Witten, "Phases of $N = 2$ Theories in Two Dimensions," `hep-th/9301042`.

[91] D. Morrison, M. Plesser, "Summing the Instantons: Quantum Cohomology and Mirror Symmetry in Toric Varieties," `hep-th/9412236`.

[92] I. Brunner, M. R. Douglas, A. Lawrence, and C. Römelsberger, "D-branes on the quintic," *JHEP* **08** (2000) 015, `hep-th/9906200`.

[93] M. R. Douglas, B. Fiol, and C. Römelsberger, "Stability and BPS branes," `hep-th/0002037`.

[94] M. R. Douglas, B. Fiol, and C. Römelsberger, "The spectrum of BPS branes on a noncompact Calabi-Yau," `hep-th/0003263`.

[95] D.-E. Diaconescu and M. R. Douglas, "D-branes on stringy Calabi-Yau manifolds," `hep-th/0006224`.

[96] D.-E. Diaconescu and J. Gomis, "Fractional branes and boundary states in orbifold theories," *JHEP* **10** (2000) 001, `hep-th/9906242`.

[97] D.-E. Diaconescu and C. Romelsberger, "D-branes and bundles on elliptic fibrations," *Nucl. Phys.* **B574** (2000) 245–262, `hep-th/9910172`.

[98] Y. Oz, T. Pantev, and D. Waldram, "Brane-antibrane systems on Calabi-Yau spaces," hep-th/0009112.

[99] P. Kaste, W. Lerche, C. Lutken, J. Walcher, "D-Branes on K3-Fibrations," *Nucl. Phys.* **B582** (2000) 203, hep-th/9912047

[100] E. Scheidegger,"D-branes on some one- and two-parameter Calabi-Yau hypersurfaces," JHEP **0004**, 003 (2000) hep-th/9912188

[101] S. Govindarajan and T. Jayaraman, "On the Landau-Ginzburg description of boundary CFTs and special Lagrangian submanifolds," *JHEP* **07** (2000) 016, hep-th/0003242.

[102] S. Govindarajan, T. Jayaraman, and T. Sarkar, "Worldsheet approaches to D-branes on supersymmetric cycles," *Nucl. Phys.* **B580** (2000) 519–547, hep-th/9907131.

[103] P. Mayr, "Phases of supersymmetric D-branes on Kähler manifolds and the McKay correspondence," hep-th/0010223.

[104] A. Tomasiello, "D-branes on Calabi-Yau manifolds and helices," hep-th/0010217.

[105] S. Govindarajan and T. Jayaraman, "D-branes, exceptional sheaves and quivers on Calabi-Yau manifolds: From Mukai to McKay," hep-th/0010196.

[106] R. Gopakumar and C. Vafa, "Branes and fundamental groups," *Adv. Theor. Math. Phys.* **2** (1998) 399–411, `hep-th/9712048`.

[107] A. Beauville, "A Calabi-Yau threefold with non-Abelian fundamental group," `alg-geom/9502003`.

[108] D. R. Morrison, "Geometric aspects of mirror symmetry," `math.ag/0007090`.

[109] M. Kontsevich, "Homological algebra of mirror symmetry," *ICM Zurich* (1994) `alg-geom/9411018`.

[110] R. P. Horja, "Hypergeometric functions and mirror symmetry in toric varieties," `math.AG/9912109`.

[111] P. Mayr, "Phases of supersymmetric D-branes on Kähler manifolds and the McKay correspondence," `hep-th/0010223`.

[112] S. Govindarajan and T. Jayaraman, "D-branes, exceptional sheaves and quivers on Calabi-Yau manifolds: From Mukai to McKay," `hep-th/0010196`.

[113] A. Tomasiello, "D-branes on Calabi-Yau manifolds and helices," `hep-th/0010217`.

[114] P. Aspinwall, "An $N = 2$ dual pair and a phase transition," *Nucl. Phys.* **B460** (1996) 057, `hep-th/9510142`.

[115] P. Aspinwall, "The Moduli Space of $N = 2$ Superconformal Field Theories," hep-th/9412115.

[116] J. A. Harvey and G. Moore, "Exact gravitational threshold correction in the FHSV model," *Phys. Rev.* **D57** (1998) 2329–2336, hep-th/9611176.

[117] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD," *Nucl. Phys.* **B431** 484 (1994), hep-th/9408099.

[118] A. Strominger, "Massless black holes and conifolds in string theory," *Nucl. Phys.* **B451**, 96 (1995) hep-th/9504090.

[119] B. R. Greene, D. R. Morrison, and A. Strominger, "Black hole condensation and the unification of string vacua," *Nucl. Phys.* **B451**, 109 (1995) hep-th/9504145.

[120] J. Polchinski and A. Strominger, "New Vacua for Type II String Theory," *Phys. Lett.* **B388**, 736 (1996) hep-th/9510227.

[121] B. R. Greene and Y. Kanter, "Small volumes in compactified string theory," *Nucl. Phys.* **B497**, 127 (1997) hep-th/9612181.

[122] C. Vafa, "Extending mirror conjecture to Calabi-Yau with bundles," hep-th/9804131.

[123] A. Strominger, S.-T. Yau, and E. Zaslow, "Mirror symmetry is T-duality," *Nucl. Phys.* **B479** (1996) 243–259, `hep-th/9606040`.

[124] P. S. Aspinwall and D. R. Morrison, "Chiral rings do not suffice: $N = (2,2)$theories with nonzero fundamental group," *Phys. Lett.* **B334** (1994) 79–86, `hep-th/9406032`.

[125] M. R. Douglas and B. Fiol, "D-branes and discrete torsion. II," `hep-th/9903031`.

[126] N. Hitchin, "Lectures on Special Lagrangian Submanifolds," `math.DG/9907034`.

[127] M. R. Douglas,"D-Branes, Categories And N = 1 Supersymmetry," J. Math. Phys. **42**, 2818 (2001). `hep-th/0011017`.

[128] B. R. Greene and M. R. Plesser, "Duality in Calabi-Yau moduli space," *Nucl. Phys.* **B338** (1990) 15–37.

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