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Topics in Flux Compactifications of Type IIA Superstring Theory

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Topics in Flux Compactifications of Type IIA Superstring Theory

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Dedicated to those who taught me over the years, especially my parents.

Gewidmet all jenen, die mich über die Jahre hinweg gelehrt haben, vor allem meinen Eltern.

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Realistic four-dimensional model building from string theory has been a focus of the string theory community ever since its inception. Toroidal orientifold constructions have emerged as a technically simple class of candidate models. Novel ingredients, such as background fluxes, have been discovered and intensely studied over the past few years. They allow for a (partial) solution of several long standing problems associated with model building in this framework. In this thesis, I summarize progress that has been made in toroidal orientifold constructions in type IIA string theory. This includes a detailed discussion of moduli stabilization and (non-) supersymmetric AdS and Minkowski vacua. Furthermore I commence a systematic study of generalized NSNS, i.e., metric and non-geometric, fluxes. The emergence of novel

D-terms is presented in detail. While most of the discussion applies to generic orientifolds of T^6, most features are exemplified by and studied in terms of a certain orientifold of $T^6/\mathbb{Z}_4$ owing to its somewhat richer structure compared to simpler models studied before. It is also briefly reported on efforts of finding de Sitter vacua and inflation in this class of models.

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Chapter 1

Introduction

Over the past century, two major radically new theories have emerged that provide a very precise description of physics in their respective domain of validity: Quantum mechanics and general relativity.

Our present understanding of particle physics is captured by the so-called standard model of particle physics, a quantum field theoretical model of the electromagnetic, the strong and the weak interactions based on the gauge group $SU(3)\times SU(2)\times U(1)$. This theory has been enormously successful in predicting the physics of elementary particles and their interactions. The agreement with experiments is truly unprecedented in science. However, there are several shortcomings, both from a theoretical and experimental perspective, which indicate that the standard model cannot be considered a complete or final theory. Experiments, e.g., solar neutrino experiments at the Sudbury Neutrino Observatory or MINOS at Fermilab, over the past decade have revealed neutrino oscillations, thus necessitating small but non-zero neutrino masses which are unaccounted for in the standard model. Moreover, evidence from astrophysical observations, leading to the standard model of cosmology (ΛCDM), predict the existence of dark matter in the universe which is absent in the standard model (with the possible exception of axions as dark matter candidates). From

a theoretical point of view, there are several unsatisfactory aspects of the theory, most notably the many unexplained parameters, the failure to explain the observed particle content, the number of families and the hierachy problem, to name a few.
The forth force, gravity, has eluded a successful quantum field theoretical description. The classical theory, general relativity, is a theory based on geometry in the sense that the resulting field equations relate the Ricci curvature tensor of the underlying spacetime to the stress-energy tensor of the matter content. At high enough energy scales (corresponding to small length scales), quantum effects become important and should be incorporated into a theory of quantum gravity. The above mentioned conceptual differencies have, however, precluded quantization of gravity as a conventional quantum field theory.
One of the goals of string theory is to provide a unified theory of all four interactions that correctly reduces to the standard model and general relativity in their respective limits. The fundamentally new idea of string theory is to replace pointlike particles (and their one-dimensional worldlines) by one-dimensional strings (with two-dimensional worldsheets) as the basic constituents of elementary particles and their interactions. In the low energy limit, i.e., at large distances, stringy effects become indetectable (the strings become effectively pointlike) and the effective theory behaves like a quantum field theory. At high energies, the UV divergencies that are associated with pointlike interactions are resolved by the finite length of the strings, thus providing a consistent UV completion. The massless spectra of the five consistent and tachyon-free super-string theories naturally contain a spin-2 particle that can be identified with the graviton.

1.1 A Brief Introduction to String Theory

The so-called critical superstring theories all have 9+1 dimensional space-time as their target space. The prefix 'super' indicates that they are supersymmetric on the worldsheet as well as space-time supersymmetric. There are five different types, which were once believed to be fundamentally distinct. In the 1990s it was discovered that the five superstring theories are not fundamental but related to each other by various string duality transformations, so that they can be considered different ten-dimensional limits or manifestations of one underlying theory, dubbed M-theory: The closed oriented type IIA and IIB superstring theories with $\mathcal{N} = 2$ supersymmetry, the closed oriented heterotic theory with $\mathcal{N} = 1$ supersymmetry and, last but not least, the unoriented $\mathcal{N} = 1$ supersymmetric type I theory which contains both open and closed string excitations. Superstring theories provide a framework to study numerous theoretically appealing ideas such as supersymmetry, Kaluza-Klein reduction, grand unification and many more which can be used in attempting to construct an effective four-dimensional model of realistic physics, i.e., a model correctly reproducing (a generalization of) the standard model of particle physics and the standard model of cosmology.

Despite having a common origin, the different theories have vastly different features which makes some of them more appropriate objects to study in different contexts. For reasons that will become apparent in subsequent chapters, I will almost exclusively focus on type IIA string theory for the purposes addressed in this dissertation. A breakthrough was achieved in the mid-1990s when Polchinski realized that string theories naturally contain extended objects on which open strings can end. These so-called Dp-branes are non-perturbative BPS objects which can support gauge bundles, that is they have gauge theories living on their $(p+1)$-dimensional worldvolume. Phenomenologically this is a very interesting ingredient for model building since a stack of N space-time filling D-branes can give rise to a $U(N)$ gauge group in four dimensions. Furthermore, it is possible to arrange several intersecting stacks of D-branes that give rise to chiral fermions. This is a necessary step in building standard model-like particle representations.

1.2 Flux Compactifications

As mentioned above, the five superstring theories can be consistently written down in $(9+1)$ space-time dimensions. Therefore, the obvious question arises how to make contact with $(3+1)$-dimensional physics that we observe in nature. One possible answer that has been explored in great detail since the beginning of string theory, is to assume a direct product form $M^{(3,1)} \times C^6$ where C^6 is a suitably chosen compact internal space on which the theory is 'compactified'. Performing a Kaluza-Klein reduction on this compact space leads to an effective four-dimensional theory, the details (physics) of which are governed by the geometry of the internal space. This introduces a severe complication to the problem of constructing realistic four-dimensional physics from string theory: While the superstring theories are almost uniquely fixed in ten dimensions (up to dualities), there is a huge number of possible six-dimensional compactification manifolds. The currently prefered paradigm of particle phenomenology states that the compactification should preserve minimal $\mathcal{N}=1$ supersymmetry in four dimensions which is then subsequently broken spontaneously at low energies. This requirement of preserving $\mathcal{N}=1$ supersymmetry restricts our choice of internal space to a certain type of compact Ricci-flat manifolds, known as Calabi-Yau (CY) manifolds, as follows: The unique covariantly constant spinor gives rise to two four-dimensional SUSY parameters and hence the compactification preserves eight supercharges, $\mathcal{N}=2$ in four dimensions. This can be further reduced to $\mathcal{N}=1$ by introducing spacetime-filling D-branes or orientifold planes. However, the number of CY manifolds in six dimensions is of the order 10^5. This automatically leads to an additional and related problem: The problem of moduli. Moduli are parameters associated with the various sizes (Kähler structure) and shapes (complex structure) of cycles of the CY manifold. Classically, there is no potential for these moduli fields, meaning that changing their (expectation) values does not cost any energy, and therefore those values remain undetermined.

In addition, these moduli lead to massless scalar fields in four dimensions and are associated to long range fifth-forces which are not observed in nature, i.e., they are in contradiction with experiments.

However, at the quantum level, there is a quantum vacuum energy. To lowest order, this is just the familiar Casimir energy, but there will be higher order perturbative and non-perturbative contributions as will be explained in detail below. In a compactified theory, this energy would be expected to depend on the size and shape moduli parameters of the compactification manifold. It is also possible to turn on background field strengths which contribute to the vacuum energy. All these effects define an effective potential, a concept that is well known from other areas of physics. It is useful to compute the possible (meta)stable vacua of the theory under consideration which will be local minima of the effective potential. The procedure of finding suitable local minima is known as moduli stabilization. For example, the hope would be to find a local minima that leads to a vacuum with small positive cosmological constant, in accordance with astrophysical observations. Such vacua are necessarily metastable but typically very long-lived (in terms of cosmological time scales).

The huge number of admissable Calabi-Yau manifolds and associated moduli and the structure of the resulting effective potentials lead to a so-called 'landscape of string vacua'. This concept is well known from evolutionary biology and has caused a lot of heated discussion over the past three or four years between its proponents, who call it a 'paradigm shift' in string theory, and its opponents who claim that the landscape of string vacua equates to a complete loss of predictability and falsifiability of string theory. This dissertation does not attempt to treat any questions pertaining to the statistical analysis of the landscape. Rather, I will describe progress made in constructing realistic models of four-dimensional particle physics and cosmology within the framework of toroidal type IIA orientifolds with generalized fluxes.

In the following, I briefly summarize some of the important points described above. Realistic model building in string theory, presented here in the framework of type IIA superstring theory, has to address a number of previously unsolved problems associated with CY compactifications, namely

- supersymmetry breaking,
- the moduli problem,
- a small, but nonvanishing cosmological constant $\Lambda > 0$ [62, 63, 64], indicating an asymptotic de Sitter (dS) type universe. Moreover, $w < -\frac{1}{3}$ indicates an accelerated expansion.

Especially the last point seems to pose a serious challenge for string theory, because (eternal) de Sitter type universes, due to the existence of event horizons, are believed to necessitate a finite number of physical degrees of freedom (resulting in finite dimensional Hilbert spaces) [65, 66], which appears impossible to reconcile with string theory. Moreover, as mentioned above, compactifications of string theory on Calabi-Yau 3-folds to four spacetime dimensions generically produce a large number of massless moduli (scalar fields) which we do not observe in nature. However, a recent proposal by Kachru, Kallosh, Linde and Trivedi (KKLT) [67] manages to address all of the above stated difficulties at once. The authors outline a way to produce a nontrivial (scalar) potential for *all* CY moduli, resulting in supersymmetric anti-de Sitter (AdS) vacua in which all moduli are stabilized. To achieve this, the authors start with a warped compactification of a type IIB orientifold with background fluxes as discussed in [68]. There it was shown that by turning on appropriate R-R and NS-NS 3-form fluxes $\hat{F}_{(3)}$ and $\hat{H}_{(3)}$, it is possible to fix both the complex structure moduli z^α and the axiodilaton $\tau := C_{(0)} + \mathrm{i}e^{-\hat{\phi}}$. However, owing to the fact that the flux-induced superpotential[1] $W_0^{IIB} = \int_{CY_3} \hat{G}_{(3)} \wedge \Omega$ [69] does

[1]Here we introduce the complexified 3-flux $\hat{G}_{(3)} := \hat{F}_{(3)} - \tau \hat{H}_{(3)}$.

not depend on the Kähler moduli of the compactification manifold, one is forced to include nonperturbative corrections to W in order to generate a potential for those moduli. KKLT argue that this can be achieved generically in their class of models by one of two effects: Euclidean $D3$-brane instantons wrapping divisors of arithmetic genus equal to one [70] or gaugino condensation in the gauge theory living on a stack of coinciding $D7$-branes wrapping 4-cycles of the internal CY [71, 72]. Both effects can be shown to lead to stabilization of the remaining Kähler moduli. As a matter of fact, the condition on the arithmetic genus of the divisors can be relaxed in the presence of fluxes, as was discovered recently by several authors (see e.g. [73]). In the final step of the KKLT construction it is argued that by adding $\overline{D3}$-branes to the setup in a suitable fashion, it is possible to break supersymmetry in such a way that the vacuum is lifted to a dS vacuum with a discretely tunable cosmological constant[2]. It is, however, important to note that the dS vacua in question are only local minima of the $\mathcal{N} = 1$ supergravity scalar potential for the relevant moduli. There always exists a global minimum, the Dine-Seiberg runaway vacuum in the large volume or decompactification limit. Therefore the dS vacua are only metastable, albeit at cosmological time scales, thus evading the above mentioned problems concerning eternal de Sitter spacetimes.

The program outlined by KKLT triggered a myriad of work within the framework of type IIB orientifold compactifications [75, 76, 77, 78]. Several important refinements to the original KKLT proposal were made, e.g., V. Balasubramanian, F. Quevedo and collaborators [79, 80, 81] realized that it is inconsistent (at least generically) to neglect the perturbative α'-corrections to the Kähler potential. Stated differently, by including these corrections, one can prove the existence of AdS vacua (even non-supersymmetric ones) and the validity of the construction for a much broader range of parameters as compared to the original proposal without perturbative corrections.

[2]This tuning can be achieved by turning on appropriate fluxes through cycles in the internal manifold.

In recent months, several authors have studied various aspects of the KKLT program in the framework of type IIA orientifold compactifications [82, 83, 84, 85, 86]. One important difference compared to the type IIB case is that here, as we shall see below, the flux-induced superpotential W_0^{IIA} contains contributions both from the complex structure as well as the Kähler moduli. Therefore it is possible to stabilize both types of moduli[3] without having to consider nonperturbative instanton corrections. Another worthwhile observation is that whereas in the type IIB scenario the fluxes are highly constrained by the tadpole cancelation condition for the $\hat{C}_{(4)}$-field, this is not true in the IIA setup, where some of the fluxes, namely $\hat{F}_{(2)}$ and $\hat{F}_{(4)}$, are left unaffected and thus unconstrained by the $\hat{C}_{(7)}$-tadpole cancelation condition [82, 83, 85].

[3] One can stabilize all the complex structure moduli but only one linear combination of the axions.

Chapter 2

Toroidal Orientifold Compactifications in Type IIA with generalized Fluxes

Much recent interest and research activity has been devoted to understanding the space of string theory vacua, especially those which can be described using the formalism of four-dimensional $\mathcal{N} = 1$ supergravity. As different constructions and compactifications have been explored, the number of tools in the model builder's kit has grown, even as our understanding of how they can be used and combined has sometimes diluted. For instance, string theory admits field strengths of various cohomological degree, and turning on fluxes of these field strengths through compact cycles of the internal space can often help stabilize the moduli of the compactification. These include R-R fluxes and fluxes of the NS-NS three form field strength H_3. This situation is fairly well understood. Under T-duality, H-flux can sometimes be converted into twists of the internal space metric, which are known as geometric fluxes. Further T-dualities can introduce so-called non-geometric fluxes, which ruin the global geometric description of the internal space, but still seem to give a consistent picture in the four-dimensional effective theory. In fact, in the effective theory, one can in principle combine all of these fluxes, up to certain constraints and consistency conditions, but it has not been demonstrated that a ten-dimensional construction can necessarily always be found.

Our goal in this chapter is to carefully explore how all of these ingredients can be combined in the context of $\mathcal{N} = 1$ toroidal orientifolds of type IIA, though we believe that many of our methods can be applied in broader contexts. To this end we will follow two different approaches, examining these constructions from the effective field theory point of view and also trying to present honest ten-dimensional constructions of as broad a class as possible. Throughout the chapter we will illustrate each method by referring to the example of an orientifold of $T^6/\mathbb{Z}_4$, whose structure is rich enough to illustrate many of the phenomena and techniques that we will describe.

In the effective field theory approach our primary goal is to classify the possible (untwisted sector) fluxes and translate them into the 4D $\mathcal{N} = 1$ language. We will find that the general NS-NS fluxes are most naturally parametrized by their action on the untwisted cohomology, along the lines described in [1, 2, 3, 4]. So just as one can replace a discussion of the individual components H_{ijk} of H-flux with coefficients p_K in the expansion $H_3 = p_K b_K$, where b_K are the untwisted three-forms which are anti-invariant under the orientifold action, one can also replace metric flux components ω^i_{jk} by coefficients r_{aK} and $\widehat{r}_{\alpha K}$, where K again runs over three forms, and a (α) runs over invariant two-forms which are odd (even) under the orientifold involution. Similarly, the nongeometric flux components Q^{ij}_k and R^{ijk} can be replaced by q_{aK}, $\widehat{q}_{\alpha K}$, and s_K, respectively. In terms of these parameters it is then straightforward to describe the data of the four-dimensional theory, and in particular we find the Kähler potential, the superpotential, and the holomorphic gauge couplings and D-terms. There are additional consistency constraints that such general fluxes must satisfy; one of these is the R-R tadpole condition (to which the orientifold six-planes contribute), and there are also Bianchi identities, which are a set of constraints, quadratic in the NS-NS fluxes. The tadpole condition can be elegantly expressed in terms of our cohomological flux parameters, but unfortunately the Bianchi identities only seem to be cleanly expressed using the original flux components. In any given example, however, we may certainly express the Bianchi identities in our cohomological parameters, but the structure seems complicated and ad-hoc.

The presence of D-terms arising from general NS-NS fluxes is a phenomenon that has not, to our knowledge, been previously discussed in the literature. We describe how adding certain metric fluxes (which are never simply T-duals of H-flux) can lead to electric charges for some of the four-dimensional scalar fields. It will also turn out that certain non-geometric fluxes correspond to magnetic charges for the same fields, making them electric-magnetic dyons in general. However, making use of the Bianchi identities one can show that the dyonic charges are necessarily mutually local, and there is always a consistent Lagrangian description of the effective theory.

As we introduce more general types of fluxes into our story, we will see how they enter in the particular example of $T^6/\mathbb{Z}_4$ with a certain orientifold action, and in particular we will look for supersymmetric solutions with as many moduli as possible stabilized. For some simple cases, such as having only H-flux, or including certain classes of metric fluxes, we are able to find all supersymmetric solutions, but are unable to stabilize all moduli in these contexts. For generic fluxes, subject to a naive quantization condition, we are able to numerically find supersymmetric solutions with all moduli stabilized. Unfortunately, we will later learn that the naive quantization condition was, in fact, naive. Using the correct quantization we can still stabilize all moduli, but are unable to satisfy the tadpole condition. It seems likely, however, that this is not a result of a fundamental obstacle, but simply relates to a lack of understanding of the correct quantization of R-R fluxes in the presence of general NS-NS fluxes (or at least from not using the correct representatives for the K-theory or integral cohomology when using the twisted torus language). We will also prove that fully stabilized supersymmetric Minkowski vacua (as opposed to AdS) require us to at least turn on non-geometric fluxes.

After exhausting the playground of effective field theory, we then attempt to directly construct as many of these models as possible starting from ten dimensions, and following the approach of [5]. We do this by splitting our T^6 into a base and a fiber, and then allowing the fiber to vary over the base. The NS-NS fluxes are then encoded as twists of the fiber theory as we go around closed, non-contractible loops in the base. We outline how to classify such splittings and twists for a given orientifold action, we show that consistency of our picture implies the Bianchi identities, and we also see clearly how to determine the correct quantization conditions on the NS-NS fluxes. Simple integral quantization of the flux components or cohomological parameters turns out to be correct only in a sub-class of cases (which of course includes all situations with only H-flux, and all cases T-dual to those ones).

These constructions enjoy certain advantages; the action of the T-duality group is quite transparent. This approach should easily generalize to many other interesting situations where a well-understood fiber theory is twisted over a toroidal base. Of particular interest, we note that the flux combinations which occur in the low-energy effective theory are naturally described (as noted in [6, 3]) as a sort of covariant derivative on the spin-bundle whose sections are R-R-fields (see also the interesting discussion in [7]).

In the context of our specific example, we will classify all base-fiber splittings and all fluxes which can be obtained in these constructions. We will find that all H-fluxes and almost all metric fluxes can be turned on and a sub-class of non-geometric Q-fluxes can also be turned on. Among the metric flux configurations that we can build are some which are not T-duals of H-flux alone, and in particular we can turn on D-terms and cases with non-standard quantization. We cannot turn on any R-flux, which is not surprising, since there are arguments [6] that any construction giving rise to R-flux cannot have even a locally geometric description in ten dimensions.

The plan of this chapter is as follows. In section 2.1 we lay out our conventions for the $T^6/\mathbb{Z}_4$ orientifold which will be our canonical example throughout. In section 2.2 we delve into effective field theory, starting by describing in general our $\mathcal{N} = 1$ language in section 2.2.1. In section 2.2.2 we add H-flux, first in the general story and then for our example. Next, in section 2.2.3 we include metric fluxes into the story, discussing the general framework in section 2.2.3, the D-terms in section 2.2.3, the induced superpotential in section 2.2.3 and the context of our example in section 2.2.3. Section 2.2.4 introduces the non-geometric fluxes, and we revisit the D-terms in section 2.2.4 and our example in 2.2.4. In 2.2.5 we summarize this approach.

Then we turn to our base-fiber constructions in section 2.3. We introduce the T-duality group in section 2.3.1 and particularly how to discuss the orientifold action in the language of $\mathrm{O}(6,6)$. In 2.3.2 we describe how to encode the NS-NS fluxes as twists of our fibers, starting with a particular example for illustration before moving on to the general case. Section 2.3.3 is devoted to exploring these techniques in the $T^6/\mathbb{Z}_4$ example, including a complete classification of all twists possible with these constructions. Some discussion is presented in 2.3.4.

Finally, two appendices provide some extra detail. Appendix A compares our results with the literature on SU(3)-structure and torsion cycles, providing a nice check on our formulae, as well as a purely geometric interpretation of the D-term constraints (they are equivalent to demanding that the manifold be half-flat). And appendix B provides two different derivations of the Bianchi identities, using the

Jacobi identity for a certain Lie algebra, or alternatively by demanding that the covariant derivative, which encodes the action of the fluxes on the spin-bundle of the R-R-fields, squares to zero.

In the interests of carefully illustrating our techniques (and exploring them ourselves) we return repeatedly to the $T^6/\mathbb{Z}_4$ example in this chapter[1], trying to push the ideas as far as possible in this specific context. Unfortunately, the level of detail necessary in these sections is well beyond what is needed for a basic explanation of our results. Readers only interested in the results and techniques should feel encouraged to skip any section or subsection with the word "example" in the title, namely section 2.1 and sections 2.2.2, 2.2.3, 2.2.4, 2.3.1, 2.3.2, 2.3.3. The other sections should be self-consistent.

2.1 The Basic Example of $T^6/\mathbb{Z}_4$

Our primary example throughout this chapter will be a particular toroidal orientifold described below. Before we dive into detailing this example and our conventions, the reader may be interested to know why we focus on this compactification, rather than one of the other orientifolds in the literature, such as $T^6/\mathbb{Z}_2^2$ or $T^6/\mathbb{Z}_3^2$ which have been more extensively studied and which are in some sense simpler. We certainly believe that our approach here can be applied to these models. One reason for our choice is familiarity, as this example has been studied in the past [8] (see also chapter 3 below) and we are able to build on the solutions found there using T-duality.

However a more important reason to look at this example is that it, unlike the two other examples mentioned above, admits untwisted two-forms which are *even* under the orientifold involution. By reducing the R-R potential C_3 along these forms we find four-dimensional vectors with associated U(1) gauge groups.

[1]Chapter 3 contains a more detailed study of the basic example of the $T^6/\mathbb{Z}_4$ orientifold model, restricted to only NS-NS and R-R fluxes, but including the twisted sector moduli and fluxes, nonsupersymmetric AdS vacua and a related stability analysis.

Later, in section 2.2.3, we will see that with certain metric fluxes turned on, some moduli become charged under the U(1)s, and this gives rise to D-terms in the four-dimensional effective potential, a possibility that has not, to our knowledge, been discussed in the context of these models.

One final interesting property of this particular example that we do not make use of in the present work, is the existence of twisted sector three-forms, which again do not occur in the more well-studied models. In principle these could lead to interesting possibilities for metric and non-geometric twisted-sector fluxes, in the spirit of [4].

2.1.1 Setup

We take the model from [8, 90], namely a certain orientifold of $(T^2)^3$, but several of our conventions will differ, so we review everything here. Let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, and $z_3 = x_3 + (\frac{1}{2} + iU)y_3$ be complex coordinates on the tori (we will see below that U is a real modulus parametrizing the complex structure of the third torus), and the torus identifications are given by integer shifts in each x_i or y_i. The orientifold group is generated by a $\mathbb{Z}_4$ rotation

$$\Theta : (z_1, z_2, z_3) \longrightarrow (iz_1, iz_2, -z_3)\,, \tag{2.1}$$

and the orientifold action is $\Omega_p(-1)^{F_L}\sigma$, where the antiholomorphic involution σ acts as

$$\sigma : (z_1, z_2, z_3) \longrightarrow (\bar{z}_1, i\bar{z}_2, \bar{z}_3)\,. \tag{2.2}$$

Note that $\Theta\sigma = \sigma\Theta^3$, so the full orientifold group is in fact isomorphic to the dihedral group D_4. This model is frequently referred to as an orientifold of the orbifold $T^6/\mathbb{Z}_4$ even though it is not a $\mathbb{Z}_2$ quotient of the orbifold. Rather, the precise statement is that the full orientifold group is a $\mathbb{Z}_2$ extension of the $\mathbb{Z}_4$ orbifold group. We emphasize this point now partly as a warning to the reader, since we will likely be guilty of sloppy language at times in the work below.

This orientifold is the **ABB** model in the classification of [89].

2.1.2 Cohomology

We will begin by describing the untwisted cohomology of $T^6/\mathbb{Z}_4$, dividing further into subspaces which are even or odd under the involution σ. The bases we will present will consist of elements of $H^*(T^6;\mathbb{Z})$ with the correct symmetry properties. In this way we get bases for the untwisted cohomology of the orbifold *over the rationals*. The correct quantization conditions for fluxes in the orientifold are subtle, and should in principle require an understanding of the correct K-theory analog for our model which would go beyond the scope of this thesis [11]. Instead we will point out where such information would be relevant, and explain why we do not believe that it will affect our results significantly.

We start with the even cohomology, implicitly equating classes with their harmonic form representatives. There is one zero form, namely the unit function 1. For two-forms, there are five independent $(1,1)$-forms invariant under the rotations: four odd forms,

$$\begin{aligned}
\omega_1 &= \frac{i}{2} dz_1 \wedge d\bar{z}_1 = dx_1 \wedge dy_1, \\
\omega_2 &= \frac{i}{2} dz_2 \wedge d\bar{z}_2 = dx_2 \wedge dy_2, \\
\omega_3 &= \frac{i}{2U} dz_3 \wedge d\bar{z}_3 = dx_3 \wedge dy_3, \\
\omega_4 &= \frac{1-i}{2}\left(dz_1 \wedge d\bar{z}_2 - i dz_2 \wedge d\bar{z}_1\right) \\
&= dx_1 \wedge dx_2 - dx_1 \wedge dy_2 + dy_1 \wedge dx_2 + dy_1 \wedge dy_2,
\end{aligned} \tag{2.3}$$

and one even form

$$\begin{aligned}\mu &= \frac{1+i}{2}\,(dz_1\wedge d\bar{z}_2 + i dz_2\wedge d\bar{z}_1)\\ &= dx_1\wedge dx_2 + dx_1\wedge dy_2 - dy_1\wedge dx_2 + dy_1\wedge dy_2. \end{aligned} \tag{2.4}$$

Similarly, for four-forms we have four even $(2,2)$-forms

$$\widetilde{\omega}_1 = \omega_2\wedge\omega_3, \qquad \widetilde{\omega}_2 = \omega_1\wedge\omega_3, \qquad \widetilde{\omega}_3 = \omega_1\wedge\omega_2, \qquad \widetilde{\omega}_4 = \omega_3\wedge\omega_4, \tag{2.5}$$

and one odd $(2,2)$-form,

$$\widetilde{\mu} = \omega_3\wedge\mu. \tag{2.6}$$

Finally there is one six-form, which is odd under the involution,

$$\varphi = \omega_1\wedge\omega_2\wedge\omega_3 = dx_1\wedge dy_1\wedge dx_2\wedge dy_2\wedge dx_3\wedge dy_3. \tag{2.7}$$

The nonzero integrals involving these forms over $X = T^6/\mathbb{Z}_4$ are (wedge products are implicit)

$$\begin{gathered}\int_X \varphi = \int_X \omega_1\,\omega_2\,\omega_3 = \frac{1}{4}, \qquad \int_X \omega_3\,\omega_4^2 = \int_X \omega_3\,\mu^2 = -1,\\ \int_X \omega_1\,\widetilde{\omega}_1 = \int_X \omega_2\,\widetilde{\omega}_2 = \int_X \omega_3\,\widetilde{\omega}_3 = \frac{1}{4}, \qquad \int_X \omega_4\,\widetilde{\omega}_4 = \int_X \mu\,\widetilde{\mu} = -1. \end{gathered} \tag{2.8}$$

Next we have the odd cohomology. It turns out that $H^1(X)$ and $H^5(X)$ are empty, so we need only describe the three-forms. The basis we shall use is

$$\begin{aligned} a_1 &= \chi_{xxx} + \chi_{xxy} + \chi_{xyx} + \chi_{yxx} - \chi_{yyx} - \chi_{yyy}, \\ a_2 &= \chi_{xxx} + \chi_{xyx} + \chi_{xyy} + \chi_{yxx} + \chi_{yxy} - \chi_{yyx}, \\ b_1 &= -\chi_{xxx} + \chi_{xyx} + \chi_{xyy} + \chi_{yxx} + \chi_{yxy} + \chi_{yyx}, \\ b_2 &= \chi_{xxx} + \chi_{xxy} - \chi_{xyx} - \chi_{yxx} - \chi_{yyx} - \chi_{yyy}. \end{aligned} \tag{2.9}$$

Here we use notation where $\chi_{xyx} = dx_1 \wedge dy_2 \wedge dx_3$, etc. The forms a_I are even under the involution σ, while b_I are odd. The nonzero integrals are simply $\int_X a_I \wedge b_J = \delta_{IJ}$.

2.1.3 Moduli

With the basis of differential forms given above, we can now describe the various moduli of this model. Most of this work will focus entirely on the untwisted sector, so we shall start there, with a brief description of the twisted sectors at the end of the subsection.

Our choice of complex coordinates has already determined the $(3,0)$-form Ω up to an overall constant factor. We shall fix the phase of this factor by demanding that $\sigma \cdot \Omega = \bar{\Omega}$, and fix the modulus with the requirement that

$$i \int_X \Omega \wedge \bar{\Omega} = 1. \tag{2.10}$$

With these requirements, Ω is determined up to an overall sign which we simply pick by hand, giving

$$\begin{aligned} \Omega &= \frac{1-i}{2\sqrt{U}}\, dz_1 \wedge dz_2 \wedge dz_3 \\ &= \frac{1}{2\sqrt{U}} \left[\left(\frac{1}{2} + U\right) a_1 + \left(\frac{1}{2} - U\right) a_2 + i \left(\frac{1}{2} + U\right) b_1 - i \left(\frac{1}{2} - U\right) b_2 \right]. \end{aligned} \tag{2.11}$$

In this expression, U is the unique untwisted complex structure modulus, and is a real variable in the range $0 < U < \infty$.

For the Kähler form we can write

$$J = v_a \, \omega_a, \tag{2.12}$$

where v_a, $a = 1, \ldots, 4$, are the real Kähler moduli of the metric. The corresponding line element is

$$\begin{aligned} ds^2 &= v_1 \left(dx_1^2 + dy_1^2\right) + v_2 \left(dx_2^2 + dy_2^2\right) \\ &\quad + \frac{v_3}{U} \left(dx_3^2 + dx_3 dy_3 + \left(\frac{1}{4} + U^2\right) dy_3^2 \right) \\ &\quad - 2 v_4 \left(dx_1 dx_2 + dx_1 dy_2 - dy_1 dx_2 + dy_1 dy_2 \right). \end{aligned} \tag{2.13}$$

In order for the metric to have the correct (euclidean) signature, we must have

$$v_1 > 0, \qquad v_2 > 0, \qquad v_3 > 0, \qquad \text{and} \qquad v_1 v_2 - 2 v_4^2 > 0. \tag{2.14}$$

The volume is given by

$$\mathcal{V}_6 = \frac{1}{3!} \int_X J^3 = \frac{1}{4} v_3 \left(v_1 v_2 - 2 v_4^2 \right). \tag{2.15}$$

Note also that having J odd under the anti-holomorphic involution σ implies that the metric is invariant under σ, as required for the orientifold projection. This is why there is no allowed metric deformation corresponding to the even two-form μ.

These moduli pair up with periods of the B-field (which must be odd under σ to survive projection),

$$B = u_a \, \omega_a, \tag{2.16}$$

to give the complex Kähler moduli $t_a = u_a + i v_a$ and the corresponding complexified

Kähler form,

$$J_c = t_a\,\omega_a = B + iJ. \tag{2.17}$$

The untwisted NS-NS sector moduli are then completed by adding in the dilaton ϕ. From the R-R sector, we have only periods of the three form C_3. In order to survive the projection, this form must be even under the action of σ, so we have only two real moduli ξ_I, $I = 1, 2$, where

$$C_3 = \xi_I\, a_I. \tag{2.18}$$

Let us now quickly summarize the twisted sector moduli. The fixed locus of Θ or Θ^3 consists of sixteen points, eight of which are fixed by σ, plus four pairs of points that get swapped by σ. The four pairs will give rise to four even and four odd $(1,1)$-forms and equal numbers of even and odd $(2,2)$-forms. The remaining eight points will each contribute an odd $(1,1)$-form and an even $(2,2)$-form. All together, then, these twisted sectors contribute twelve new complex Kähler moduli, with four moduli of the orbifold being projected out by the orientifold.

The fixed locus of Θ^2 consists of sixteen two-tori. Θ invariance gives four copies of $T^2/\mathbb{Z}_2$, and six pairs of T^2 that get interchanged. Each of these six pairs automatically contributes one even and one odd form. Of the six pairs, two pairs are at σ fixed points and each contributes an odd $(1,1)$-form and an even $(2,2)$-form[2], two more pairs have σ act the same as Θ and hence also act as if they were σ fixed points, and the final two pairs are interchanged by σ, leading to one odd and one even of each $(1,1)$- and $(2,2)$-forms. Similarly the remaining four $T^2/\mathbb{Z}_2$ are each at fixed points of σ, and each contribute an odd $(1,1)$-form and an even $(2,2)$-form. In total then, this sector contains ten two-forms, nine of which are odd,

[2]To see that a σ fixed point gives rise to an odd $(1,1)$-form, note that locally it looks like $(\mathbb{C}^2/\mathbb{Z}_2) \times T^2$ which resolves to $(\mathcal{O}_{\mathbb{P}^1}(-2)) \times T^2$. An explicit Kähler metric can be written down for the latter geometry and one can check that the unique normalizable $(1,1)$-form is odd under an antiholomorphic involution.

twelve three-forms, which split into six odd and six even, and ten four-forms, nine of which are even.

2.1.4 Fluxes

Finally, we turn to the allowed fluxes which we can turn on in our model. As mentioned above, the correct classification of R-R fluxes in this model would involve a careful discussion of K-theory in this setting, and would go beyond the scope of this thesis. We will instead stick to cohomology. Moreover, we will be primarily interested in so-called "bulk fluxes" - fluxes whose image in rational cohomology has a nonzero projection onto the untwisted sector. For this reason, we will write our fluxes as (recalling that F_0 and F_4 need to be even under σ, while F_2 and F_6 need to be odd)

$$\begin{aligned} F_0 &= m_0, \\ F_2 &= m_a\, \omega_a, \\ F_4 &= e_a\, \widetilde{\omega}_a, \\ F_6 &= e_0 \varphi, \end{aligned} \tag{2.19}$$

where quantization conditions say that[3]

$$\frac{\sqrt{2}}{(2\pi)^{p-1}(\alpha')^{(p-1)/2}} \int F_p \in \mathbb{Z}, \tag{2.20}$$

with the integral taken over any p-cycle in X. This is of course not completely correct; proper quantization requires combining untwisted and twisted sector fluxes, but it is not quite as bad as one might fear. Indeed, one can argue (see e.g. [13, 83]) that any bulk flux can be written as one of the above, plus twisted sector

[3]Note the unusual factor of $\sqrt{2}$ in this expression. This is a consequence of the form of the R-R kinetic terms in our conventions. See also the discussion in [83].

contributions which correspond to fractional fluxes at fixed points (modulo again certain K-theoretic subtleties).

The NS-NS three-form flux is in some sense simpler. It must simply lie in $H^3(X;\mathbb{Z}) \cap H^*_{\text{odd}}(X)$. In principle this can be completely worked out - the difficult step of working out the integral cohomology has already been done in [90] - but again we won't need the full detail and we shall again project onto the untwisted sector cohomology. This allows us to write

$$H_3 = p_I\, b_I, \tag{2.21}$$

and impose the simple quantization $p_I/(4\pi^2\alpha') \in \mathbb{Z}$. We will also set $\alpha' = 1/4\pi^2$ unless otherwise noted.

At any rate, we will not make too much use of the underlying quantizations of the R-R fluxes (though we will encounter a puzzle related to this later on), and the quantization of NS-NS fluxes will be treated much more carefully in section 2.3.

2.1.5 Orientifold planes

Orientifold planes will lie at the fixed locus of each orientation-reversing element of the orientifold group. For instance, the fixed locus of the involution σ is described by the set

$$\left(T^6\right)^\sigma = \left\{0 \le x_1 \le 1, y_1 \in \{0, \frac{1}{2}\}; 0 \le x_2 = y_2 \le 1; 0 \le x_3 \le 1, y_3 = 0\right\} \subset T^6. \tag{2.22}$$

In homology, this cycle can be written as

$$\left[\left(T^6\right)^\sigma\right] = 2\pi_{xxx} + 2\pi_{xyx}, \tag{2.23}$$

where π_{xxx} is the cycle represented by y_i fixed, x_i variable and winding once. We also have, e.g.,

$$\int_{\pi_{xxx}\subset T^6} \chi_{xxx} = 1, \tag{2.24}$$

with other choices of integrand giving vanishing results. Using this, we have also picked the orientation of this cycle to be such that it is positively calibrated by $\mathrm{Re}\,\Omega$.

Similar consideration for the other anti-holomorphic involutions of the orientifold group give

$$\begin{aligned} \left[\left(T^6\right)^{\Theta\sigma}\right] &= 2\pi_{xyx} - 4\pi_{xyy} + 2\pi_{yyx} - 4\pi_{yyy}, \\ \left[\left(T^6\right)^{\Theta^2\sigma}\right] &= 2\pi_{yxx} - 2\pi_{yyx}, \\ \left[\left(T^6\right)^{\Theta^3\sigma}\right] &= -2\pi_{xxx} + 4\pi_{xxy} + 2\pi_{yxx} - 4\pi_{yxy}. \end{aligned} \tag{2.25}$$

The total class of the O6-plane is thus

$$[O6] = 4\left(\pi_{xxy} + \pi_{xyx} - \pi_{xyy} + \pi_{yxx} - \pi_{yxy} - \pi_{yyy}\right). \tag{2.26}$$

It is then easy to verify that

$$\int_{[O6]\subset X} a_1 = 4, \qquad \int_{[O6]\subset X} a_2 = \int_{[O6]\subset X} b_1 = \int_{[O6]\subset X} b_2 = 0, \tag{2.27}$$

where we must be careful to divide by four relative to the result on T_6 (this doesn't make sense for individual cycles like π_{xxx} which are not invariant under the orbifold action, but does make sense for the total class $[O6]$).

The reason that it is important to know where the orientifold plane lies is that it can contribute to the tadpole for the seven-form R-R potential, C_7. Explicitly, the equation of motion for C_7 includes a contribution proportional to δ_{O6}, the delta-three form supported on the orientifold plane. From the computations above, we see that in cohomology,

$$[\delta_{\mathrm{O6}}] = 4b_1. \tag{2.28}$$

2.2 Effective Field Theory Approach

Much of the work that has been done on toroidal orientifolds with fluxes turned on has been in the context of a four-dimensional effective field theory description. This is not at all surprising; in backgrounds with R-R fluxes alone there is a lack of satisfactory world-sheet descriptions. Similarly, NS-NS fluxes can be tricky to deal with in a world-sheet formalism, and in some of the more exotic cases (such as R-fluxes [14], which will be discussed below) no ten-dimensional description is known.

So the approach has been to start with ten-dimensional situations in which there is a description and reduce to a four-dimensional effective theory. We will be working in $\mathcal{N} = 1$ language, and so starting from a given set of fluxes, we will need to give a four-dimensional superpotential, a Kähler potential, and holomorphic gauge couplings. From this data we obtain our effective theory. Using the T-duality group, which is also a duality group of the low-energy theory, people have been able to guess the four-dimensional data for more general sets of NS-NS fluxes [2, 3].

In the sections below we will go over these arguments for increasingly more general sets of fluxes, giving the general result and then applying it to our specific example.

2.2.1 $\mathcal{N} = 1$ language

Before we turn on any fluxes, it is useful to review the $\mathcal{N} = 1$ supergravity theory we will be constructing in four dimensions and see what we can learn already. Such a theory will generally consist of one gravity multiplet, some number of chiral multiplets including complex scalars ϕ^I and some number of vector multiplets including vectors A^α. The theory is then specified by giving three functions which will depend on the complex scalars, namely a Kähler potential K, a holomorphic superpotential W, and holomorphic gauge-kinetic couplings $f_{\alpha\beta}$. The bosonic part of the effective action is then

$$\begin{aligned} S^{(4)} &= -\int_{M^4} \left\{ -\frac{1}{2} R * 1 + K_{I\bar{J}} d\phi^I \wedge * d\bar{\phi}^{\bar{J}} + V * 1 \right. \\ &\quad \left. + \frac{1}{2} \left(\operatorname{Re} f_{\alpha\beta}\right) F^\alpha \wedge * F^\beta + \frac{1}{2} \left(\operatorname{Im} f_{\alpha\beta}\right) F^\alpha \wedge F^\beta \right\}, \end{aligned} \tag{2.29}$$

where the scalar potential is

$$V = e^K \left(K^{I\bar{J}} D_I W \, D_{\bar{J}} \bar{W} - 3|W|^2 \right) + \frac{1}{2} \left(\operatorname{Re} f\right)^{-1\,\alpha\beta} D_\alpha D_\beta. \tag{2.30}$$

Here, $*$ is the four-dimensional Hodge star, $K_{I\bar{J}} = \partial_I \bar{\partial}_{\bar{J}} K$, $K^{I\bar{J}}$ is its inverse, $F^\alpha = dA^\alpha$, and $D_I W = \partial_I W + (\partial_I K) W$. D_α is the D-term for the U(1) gauge group corresponding to A^α, i.e.

$$D_\alpha = \partial_I K \, \left(T_\alpha\right)^I{}_J \phi^J + \zeta_\alpha, \tag{2.31}$$

where T_α is the generator of the gauge group, and ζ_α is the Fayet-Iliopoulos term.

Let us now consider how this effective theory is obtained, following [93], from the ten-dimensional models in which we are interested. We will take the situation (as in our example) where besides the constant zero-form 1 and the volume form ϕ, we have odd two-forms ω_a, even two-forms μ_α, even three-forms a_K, odd three-forms b_K, even four-forms $\widetilde{\omega}_a$, and odd four-forms $\widetilde{\mu}_\alpha$, where the index a runs from 1 to $h^{1,1}_{-\,(\text{untwisted})}$, α runs from 1 to $h^{1,1}_{+\,(\text{untwisted})}$, and K runs from 1 to $h^{2,1}_{(\text{untwisted})} + 1$. The intersection numbers are taken to be

$$\int_X \varphi = \frac{1}{|\Gamma|}, \qquad \int_X \omega_a \, \omega_b \, \omega_c = \kappa_{abc}, \qquad \int_X \omega_a \widetilde{\omega}_b = d_{ab}, \tag{2.32}$$

$$\int_X \mu_\alpha \, \mu_\beta \, \omega_a = \widehat{\kappa}_{\alpha\beta a}, \qquad \int_X \mu_\alpha \widetilde{\mu}_\beta = \widehat{d}_{\alpha\beta}, \qquad \int_X a_J \wedge b_K = \delta_{JK},$$

where wedge products are implicit between even forms and where $|\Gamma|$ is the order of the orbifold group (four, for our example).

The four-dimensional chiral fields will be related to moduli of the ten-dimensional theory. First there are the Kähler moduli, $t_a = u_a + iv_a$, from

$$B + iJ = J_c = t_a \omega_a, \tag{2.33}$$

(the complexified Kähler form J_c should be an odd two-form). To describe the complex moduli, let us write the holomorphic three-form as

$$\Omega = \mathcal{Z}_K a_K - \mathcal{F}_K b_K, \tag{2.34}$$

and we take the conventions (as in section 2.1.3) that

$$i \int_X \Omega \wedge \bar{\Omega} = 1, \qquad \sigma \cdot \Omega = \bar{\Omega}, \tag{2.35}$$

so that the $\mathcal{Z}_K$ are real functions of the complex structure moduli of the metric, while the $\mathcal{F}_K$ are pure imaginary, and together they satisfy the constraint $\mathcal{Z}_K \mathcal{F}_K = -i/2$. For comparison with section 2.1.3, we have in our example

$$\begin{aligned} \mathcal{Z}_1 &= \frac{1}{2\sqrt{U}} \left(\frac{1}{2} + U \right), \qquad & \mathcal{Z}_2 &= \frac{1}{2\sqrt{U}} \left(\frac{1}{2} - U \right), \\ \mathcal{F}_1 &= \frac{-i}{2\sqrt{U}} \left(\frac{1}{2} + U \right), \qquad & \mathcal{F}_2 &= \frac{i}{2\sqrt{U}} \left(\frac{1}{2} - U \right). \end{aligned} \tag{2.36}$$

We then define a complexified version

$$\Omega_c = C_3 + 2ie^{-D} \, \mathrm{Re}\, \Omega = \left(\xi_K + 2ie^{-D} \mathcal{Z}_K \right) a_K, \tag{2.37}$$

where $e^{-D} = \mathcal{V}_6^{1/2} e^{-\phi}$ contains the dilaton, and $\mathcal{V}_6 = \frac{1}{6}\kappa_{abc} v_a v_b v_c$ is the volume. The complex moduli $N_K = \frac{1}{2}\xi_K + i e^{-D} \mathcal{Z}_K$ are simply given by the expansion

$$\Omega_c = 2 N_K a_K. \tag{2.38}$$

Similarly, the four-dimensional vectors will come from reducing C_3 against the forms μ^α, so that the total field C_3 (before turning on fluxes), is

$$C_3 = \xi_K a_K + A^\alpha \wedge \mu_\alpha. \tag{2.39}$$

We would next like to derive the functions K, W, and $f_{\alpha\beta}$ by reducing the ten-dimensional action for type IIA, which in the Einstein frame reads

$$\begin{aligned} S^{(10)} \quad = \quad & -\frac{1}{2}\int_{M^4 \times X} \Big\{ -R * 1 + \frac{1}{2} d\phi \wedge * d\phi + \frac{1}{2} e^{-\phi} H_3 \wedge * H_3 + e^{\frac{5}{2}\phi} F_0 * F_0 \\ & + e^{\frac{3}{2}\phi} F_2 \wedge * F_2 + e^{\frac{1}{2}\phi} F_4 \wedge * F_4 + B_2 \wedge dC_3 \wedge dC_3 \\ & + B_2^2 \wedge dC_3 \wedge dC_1 + \frac{1}{3} B_2^3 \wedge dC_1 \wedge dC_1 + \frac{1}{3} F_0 B_2^3 \wedge dC_3 \\ & + \frac{1}{4} F_0 B_2^4 \wedge dC_1 + \frac{1}{20} F_0^2 B_2^5 \Big\}, \end{aligned} \tag{2.40}$$

where in the absence of fluxes,

$$B_2 = u_a \omega_a, \quad H_3 = du_a \wedge \omega_a, \quad F_0 = 0, \quad C_1 = 0, \quad F_2 = dC_1 + F_0 B_2 = 0, \tag{2.41}$$

$$C_3 = \xi_K a_K + A^\alpha \wedge \mu_\alpha, \quad F_4 = dC_3 + C_1 \wedge H_3 + \frac{1}{2} F_0 B_2^2 = d\xi_K \wedge a_K + F^\alpha \wedge \mu_\alpha.$$

Plugging these into (2.40) and then integrating over X we can compare the resulting four-dimensional action with (2.29). For example, comparing the coeffecient of $F^\alpha \wedge F^\beta$ we find that

$$\mathrm{Im}\, f_{\alpha\beta} = u_a \int_X \omega_a \wedge \mu_\alpha \wedge \mu_\beta = \widehat{\kappa}_{\alpha\beta\, a} u_a, \tag{2.42}$$

and the coefficient of $F^\alpha \wedge *F^\beta$ gives

$$\mathrm{Re}\, f_{\alpha\beta} = e^{\frac{1}{2}\phi} \int_X \mu_\alpha \wedge *\mu_\beta = -\widehat{\kappa}_{\alpha\beta\, a} v_a, \tag{2.43}$$

where we have converted to string frame and used an expression for $\int_X \mu_\alpha \wedge *\mu_\beta$ in terms of intersection numbers, as found e.g. in [93]. Alternatively, we could have just used $\mathrm{Im}\, f_{\alpha\beta}$ and our knowledge of the holomorphicity of $f_{\alpha\beta}$. Either way we conclude

$$f_{\alpha\beta} = i\widehat{\kappa}_{\alpha\beta\, a} t_a. \tag{2.44}$$

Similarly (though with more effort) we find that $W = 0$ and the Kähler potential is given by

$$K = 4D - \ln(8\mathcal{V}_6) = 4D - \ln\left(\frac{4}{3}\kappa_{abc} v_a v_b v_c\right), \tag{2.45}$$

and this expression should of course be thought of as a real function of the complex fields t_a and N_K, defined implicitly through its dependence on v_a and D.

We will see that the effect of turning on fluxes will be to introduce a nonzero superpotential W, but that the kinetic terms for the four-dimensional fields will not be affected, and hence neither $f_{\alpha\beta}$ nor K will change.

Finally, one finds that the D-term contribution to the scalar potential also vanishes, and we conclude that all the FI parameters are vanishing and that all complex scalars are neutral under each gauge group. It will be useful for later contexts to think briefly about how one checks the neutrality under gauge transformations here. Note that a gauge transformation

$$A^\alpha \to A^\alpha + d\lambda^\alpha, \tag{2.46}$$

is inherited from the ten-dimensional gauge transformation

$$C_3 \to C_3 + d\left(\lambda^\alpha \mu_\alpha\right), \tag{2.47}$$

which preserves the form of the expansion of C_3. Under this transformation, A^α is the only field which changes. In sections below, we will find that the two-forms μ_α are no longer necessarily closed, and so the ten-dimensional gauge transformation above will also be felt by some of the scalar fields. This effect will be interpreted as a charge on a given field, and so in this case D-terms will be generated (though we will find that the FI parameters will continue to vanish in our setup).

2.2.2 Including only H-flux

To begin, we start by turning on arbitrary R-R fluxes and only the most familiar sort of NS-NS flux, namely H-flux. This situation has been studied extensively, and we primarily follow the work of [93, 83], (though the conventions we will use differ slightly, and are engineered to agree with [2, 14, 6]).

General results

As in section 2.1.4, we expand our fluxes in our cohomological basis as

$$F_0 = m_0, \qquad F_2 = m_a \omega_a, \qquad F_4 = e_a \widetilde{\omega}_a, \qquad F_6 = e_0 \varphi, \tag{2.48}$$

and

$$H_3 = p_K b_K. \tag{2.49}$$

These are in addition to the contributions from the moduli as seen in (2.41). As mentioned above, K and $f_{\alpha\beta}$ are given as before, and the gauge transformation argument proceeds unchanged, showing that there are no charged scalars. Finally, performing an explicit reduction shows that the FI parameters continue to vanish (so there are no D-terms), and there is now a superpotential given by

$$W = W^Q + W^K, \tag{2.50}$$

with

$$W^Q = \int_X \Omega_c \wedge H_3, \qquad W^K = \int_X e^{J_c} \wedge F_{\mathrm{RR}}, \tag{2.51}$$

where

$$e^{J_c} = 1 + J_c + \frac{1}{2} J_c \wedge J_c + \frac{1}{6} J_c \wedge J_c \wedge J_c, \tag{2.52}$$

and

$$F_{\mathrm{RR}} = F_0 + F_2 + F_4 + F_6, \tag{2.53}$$

are formal sums of forms. Performing the integrals over X, we find

$$W = 2N_K p_K + \frac{1}{|\Gamma|} e_0 + d_{ab} t_a e_b + \frac{1}{2} \kappa_{abc} t_a t_b m_c + \frac{m_0}{6} \kappa_{abc} t_a t_b t_c. \tag{2.54}$$

Another very important point which we have ignored up to now is the presence of a tadpole for the R-R field C_7. This field is nondynamical, explaining why we have not included it above, but its tadpole must nonetheless be cancelled. Indeed, the ten-dimensional action has a piece

$$\int_{M^4 \times X} \left\{ -\frac{1}{2} (F_2 + m_0 B) \wedge * (F_2 + m_0 B) + C_7 \wedge \left[\frac{1}{\sqrt{2}} \delta_{D6} - \sqrt{2} \delta_{O6} \right] \right\}, \tag{2.55}$$

where the δs are delta-function three forms representing the localized sources. Since $*(F_2 + m_0 B) = \widetilde{F}_8 = dC_7 + \cdots$, the C_7 equation of motion then implies that

$$-m_0 p_K b_K + \frac{1}{\sqrt{2}} [\delta_{D6}] = \sqrt{2} [\delta_{O6}], \tag{2.56}$$

(though note that the tadpole condition is actually stronger than this cohomological version). Because of the freedom to use D-branes (or anti-D-branes if necessary) to satisfy the tadpole condition, we will attempt first to find vacua without worrying about the tadpole condition, and then see what, if anything, we then need to add.

To find supersymmetric vacua, we need to solve the F-term equations $D_a W = 0$ and $D_K W = 0$. From the results above, and using the useful fact that

$$\partial_K D = -e^D \mathcal{F}_K, \tag{2.57}$$

we find the real and imaginary parts of these equations to be

$$\begin{aligned} 0 &= \operatorname{Re} D_a W = d_{ab} e_b + \kappa_{abc} u_b m_c + \frac{m_0}{2} \kappa_{abc} \left(u_b u_c - v_b v_c \right) \\ &\quad - \frac{3}{2} \frac{\kappa_{abc} v_b v_c}{\kappa_{def} v_d v_e v_f} \operatorname{Im} W, && (2.58) \\ 0 &= \operatorname{Im} D_a W = \kappa_{abc} v_b m_c + m_0 \kappa_{abc} u_b v_c + \frac{3}{2} \frac{\kappa_{abc} v_b v_c}{\kappa_{def} v_d v_e v_f} \operatorname{Re} W, && (2.59) \\ 0 &= \operatorname{Re} D_K W = 2 p_K - 4 i e^D \mathcal{F}_K \operatorname{Im} W, && (2.60) \\ 0 &= \operatorname{Im} D_K W = 4 i e^D \mathcal{F}_K \operatorname{Re} W. && (2.61) \end{aligned}$$

Since not all of the $\mathcal{F}_K$ can vanish (recall the condition $i \int_X \Omega \wedge \bar{\Omega} = 1$), (2.61) requires $\operatorname{Re} W = 0$. One can quickly check that a Minkowski solution (one in which $\operatorname{Im} W$ also vanishes) will force all of the fluxes (R-R and NS-NS) to vanish; in this case the superpotential vanishes, no moduli are stabilized, and the tadpole must be saturated by adding D6-branes.

Suppose now that we are not in a Minkowski, but rather an AdS solution, in which $\operatorname{Im} W \neq 0$. In order for the metric to be positive definite, the matrix $(\kappa v)_{ab} = \kappa_{abc} v_c$ should be invertible, and so equation (2.59) tells us that either $m_0 = 0$ and $m_a = 0$, or

$$u_a = -\frac{m_a}{m_0}. \tag{2.62}$$

The former case reduces to the unstabilized Minkowski vacuum mentioned above.

Also, if the F-term equations hold, then one can subtract

$$e^{-D}\mathcal{Z}_K \operatorname{Re} D_K W + v_a \operatorname{Re} D_a W \tag{2.63}$$

from the imaginary part of the right hand side of (2.54) to show that

$$\operatorname{Im} W = -\frac{2m_0}{15}\kappa_{abc}v_a v_b v_c. \tag{2.64}$$

One can proceed somewhat further in the general case, but since we would like to add more ingredients to our construction, we will refer the reader to [83], and restrict ourselves instead to our specific example.

Example

Now focus on our $T^6/\mathbb{Z}_4$ orientifold, assuming an AdS solution to the F-term equations. Then since we must have $\mathcal{F}_1 \neq 0$ for a nondegenerate solution, equations (2.60) tell us that $p_1 \neq 0$ and that

$$\frac{\mathcal{F}_2}{\mathcal{F}_1} = \frac{p_2}{p_1} \qquad \Longrightarrow \qquad U = \frac{1}{2}\frac{p_1 + p_2}{p_1 - p_2}, \tag{2.65}$$

From this we see that a sensible solution requires $|p_1| > |p_2|$.

Next, we use (2.58) and (2.64) to obtain a set of four quadratic equations for the v_a. The equations are simplest if we write them in terms of quantities

$$\widehat{e}_1 = e_1 - \frac{m_2 m_3}{m_0}, \widehat{e}_2 = e_2 - \frac{m_1 m_3}{m_0}, \widehat{e}_3 = e_3 - \frac{m_1 m_2 - 2m_4^2}{m_0}, \widehat{e}_4 = e_4 - \frac{m_3 m_4}{m_0}. \tag{2.66}$$

It turns out that a sensible solution (i.e. one in which v_1, v_2, v_3 are all positive and $v_1 v_2 > 2v_4^2$) exists if and only if m_0, $\widehat{e}_1$, $\widehat{e}_2$, $\widehat{e}_3$ are all the same sign and if $\widehat{e}_1\widehat{e}_2 > 2\widehat{e}_4^2$.

If these conditions are met, then we have a sensible, physical solution given by

$$v_1 = |\widehat{e}_2| \sqrt{\frac{5}{3m_0} \frac{\widehat{e}_3}{\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2}}, \qquad v_2 = |\widehat{e}_1| \sqrt{\frac{5}{3m_0} \frac{\widehat{e}_3}{\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2}}, \tag{2.67}$$

$$v_3 = \sqrt{\frac{5}{3m_0} \frac{\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2}{\widehat{e}_3}}, \qquad v_4 = \widehat{e}_4 \sqrt{\frac{5}{3m_0} \frac{\widehat{e}_3}{\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2}} \, (\mathrm{sign}\, m_0) \,.$$

Next we can solve for the dilaton. It turns out that $e^D > 0$ implies that p_1 must have the opposite sign of m_0, and then

$$e^D = \left[\frac{27 m_0}{10} \frac{p_1^2 - p_2^2}{\widehat{e}_3 \left(\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2 \right)} \right]^{1/2}, \tag{2.68}$$

or,

$$e^\phi = \frac{3}{2} \sqrt{p_1^2 - p_2^2} \left[\frac{12}{5} m_0 \widehat{e}_3 \left(\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2 \right) \right]^{-1/4}. \tag{2.69}$$

Finally, we can use $\mathrm{Re}\, W = 0$ to solve for one linear combination of the axions, $p_K \xi_K$; the scalar potential is independent of the other linear combination and so this other combination remains a flat direction perturbatively.

$$p_1 \xi_1 + p_2 \xi_2 = -\frac{1}{4} e_0 + \frac{1}{4 m_0} \left(m_1 e_1 + m_2 e_2 + m_3 e_3 - 4 m_4 e_4 \right) - \frac{m_3 \left(m_1 m_2 - 2 m_4^2 \right)}{2 m_0^2}. \tag{2.70}$$

Thus, for a given general set of fluxes (satisfying certain inequalities) we have found the unique solution to the F-term equations and have found that all but one of the moduli are fixed. We still, however, need to satisfy the tadpole constraint. Indeed, since $[\delta_{O6}] = 4 b_1$, we find

$$-\sqrt{2} m_0 p_1 + N_1 = 8, \qquad -\sqrt{2} m_0 p_2 + N_2 = 0, \tag{2.71}$$

where N_1 and N_2 are the number of D-branes wrapping the cycle dual to b_1 or b_2

respectively. Actually one needs to be a bit careful here; a supersymmetric D-brane should have a positive volume as calibrated by $\mathrm{Re}\,\Omega$, but for the cycle dual to b_2 the orientation picked out by this condition depends on whether U is less than or greater than one half. If $U < \frac{1}{2}$, as is the case when $m_0 p_2 > 0$, then we should have $N_2 > 0$ D6 branes, in agreement with the above. On the other hand, if $U > \frac{1}{2}$, then the cycle dual to b_2 is negatively calibrated and N_2 counts the number of anti-D6 branes. In this case $N_2 < 0$ for a SUSY solution, but we also have $m_0 p_2 < 0$, so the tadpole condition can still be satisfied.

Note that to find a physical solution above, we required that $m_0 p_1 < 0$, and hence immediately $N_1 < 8$. In fact, since we also needed $|p_1| \geq |p_2|$, we have that $N_1 + |N_2| < 8$; the total number of D6-branes is bounded. Hence, we see that in some sense the fluxes here contribute to the tadpole with the same sign as the D-branes. We are not allowed to add as many D-branes as we like to saturate the tadpole, but rather (within SUSY) our gauge groups have bounded rank.

Before moving on, let us note that we could have worked directly with the scalar potential of (2.30) and looked for extrema of the potential. Recall that if we have an extremum at which the value of the potential is negative, so that we have AdS_4, then stability does not require that the the extremum be an actual minimum. It is enough that each field Φ^I have a mass squared that is greater than the Breitenlohner-Freedman bound,

$$m_I^2 > -\frac{3}{4}\left|V_{\text{extremum}}\right|, \tag{2.72}$$

where we assume that Φ^I has a canonically normalized kinetic term. Indeed, for the supersymmetric solution above, it turns out that there is one mode with a negative mass squared, but it is above the BF bound.

2.2.3 Adding metric fluxes

The next ingredient we will be adding is known as *metric flux*. It is well known that by T-dualizing one circle of a torus with H-flux, one can swap the H-flux for some nonconstant metric components. One finds that some of the original globally defined one-forms of the torus, dx^i, are no longer globally defined, but need to be replaced by a set of one-forms η^i,[4] which are no longer necessarily closed (see [16, 105] for a discussion of the cohomology of twisted tori), but rather satisfy

$$d\eta^i = -\frac{1}{2}\omega^i_{jk}\eta^j \wedge \eta^k, \tag{2.73}$$

where ω^i_{jk} are constant coefficients, antisymmetric in the lower two indices. These coeffecients are known as metric (or sometimes geometric) fluxes, and arise from the NS-NS sector of the theory, just as the H-flux does.

In fact, one needs not necessarily obtain such solutions by T-duality, but rather one can start from (2.73) directly. An effective four-dimensional theory can still be obtained by performing a generalized Scherk-Schwarz reduction. This has been done in some other models in [82, 14, 6], and some general work has also been done [19, 20, 21], but we will point out a couple of novel features, such as the appearance of nonvanishing D-terms, which have not been explored in these models before.

There are some subtle issues here about the general consistency of this program which we will discuss more in sections 2.2.5 and 2.3, but for now we shall forge ahead.

[4]To be precise, the space of globally defined smooth one-forms on the torus is spanned by the dx^i with coefficients that are smooth, globally defined functions on the torus (e.g. 1, $\cos x$, $2\sin 5x$, etc.). Similarly on the twisted torus the smooth, globally defined one-forms are spanned by the η^i with smooth global functions as coefficients. In fact, the entire ring $\Lambda^\bullet T^*(X)$ is generated from the η^i over smooth functions in this way.

General framework

First, let us note that by taking the exterior derivative of (2.73), we find that $d^2 = 0$ provides a consistency condition,

$$\omega^m_{[ij}\omega^n_{k]m} = 0, \ \forall n, i, j, k, \tag{2.74}$$

We will refer to this condition, along with similar conditions for the other fluxes, as Bianchi identities which the NS-NS fluxes will need to satisfy.

Another perspective on these fluxes which is sometimes useful is that if we write

$$\eta^i = N^i_j(x)dx^j, \tag{2.75}$$

then we can construct vector fields

$$Z_i = \left(N^{-1}\right)^j_i \frac{\partial}{\partial x^j}. \tag{2.76}$$

These turn out to be Killing vectors of the twisted torus, and they form a Lie algebra,

$$[Z_i, Z_j] = \omega^k_{ij} Z_k. \tag{2.77}$$

The Jacobi identity for the Lie algebra simply reproduces (2.74). This algebra is somewhat useful to keep in mind and we can sometimes relate properties of the system with metric fluxes to properties of the algebra. We will discuss these matters in sections 2.2.5 and 2.3.

Another identity which must be satisfied is that the H-flux, which we will now write as[5]

$$H_3 = H_{ijk}\eta^i \wedge \eta^j \wedge \eta^k, \tag{2.78}$$

[5]Note that it is important that H_3 be a globally defined three-form.

must still be closed, leading to the Bianchi identity

$$\omega^i_{[jk} H_{\ell m]i} = 0. \tag{2.79}$$

There is one more constraint that we will impose, namely that traces $\omega^i_{ij} = 0$ for all j. One can obtain this constraint for instance by demanding that the volume form of the torus not be exact, since

$$d\left(\frac{1}{5!}\epsilon_{ij_1\cdots j_5}\eta^{j_1}\wedge\cdots\wedge\eta^{j_5}\right) = \omega^j_{ji}\eta^1\wedge\cdots\wedge\eta^6. \tag{2.80}$$

Happily, this condition will be automatically true anytime we are in a space X with $H^1(X) = 0$; the orbifold projection will not allow any object with a single free index. All of these Bianchi identities will in some sense be unified below in section 2.2.4 and in Appendix B.

As just mentioned, since our interest here is in toroidal orientifolds of type IIA, we must restrict our choices of ω^i_{jk} so that they are invariant under the full orientifold group. We implicitly followed the same procedure for the components H_{ijk} of H-flux, only there we required that they be invariant under the orbifold group and odd under the involution, since the worldsheet parity operator Ω which accompanies the involution flips the sign of B_2. For the metric fluxes the story is similar, except that since the metric is even under worldsheet parity, we find that ω^i_{jk} should be even under the involution. This story can be told more cleanly in the base-fiber approach in section 2.3.

In the case of the H_{ijk}, it was then natural to parametrize our choices of flux not by the individual components that remained after projection, but by the coefficients p_K in the expansion $H_3 = p_K b_K$. A similar choice can be made for the metric fluxes which will vastly simplify our discussion of the effective action and the tadpole constraint. To this end, consider a general p-form,

$$A^{(p)} = \frac{1}{p!} A_{i_1 \cdots i_p} \eta^{i_1} \wedge \cdots \wedge \eta^{i_p}, \tag{2.81}$$

where we will assume here that the coefficients $A_{i_1 \cdots i_p}$ are constants. Then we will define a $(p+1)$-form $\omega \cdot A = -dA$, which in components as above reads[6]

$$(\omega \cdot A)_{i_1 \cdots i_{p+1}} = \binom{p+1}{2} \omega^{j}_{[i_1 i_2} A_{|j| i_3 \cdots i_{p+1}]}, \tag{2.82}$$

and where we are using conventions such that $\binom{n}{m} = 0$ unless $0 \leq m \leq n$.

Then we can now define coefficients r_{aK} and $\widehat{r}_{\alpha K}$ from the expansions

$$\omega \cdot \omega_a = r_{aK} b_K, \qquad \omega \cdot \mu_\alpha = \widehat{r}_{\alpha K} a_K. \tag{2.83}$$

Integration by parts then also furnishes the expansions

$$\omega \cdot a_K = \left(d^{-1}\right)^{ab} r_{bK} \widetilde{\omega}_a, \qquad \omega \cdot b_K = -\left(\widehat{d}^{-1}\right)^{\alpha\beta} \widehat{r}_{\beta K} \widetilde{\mu}_\alpha. \tag{2.84}$$

Note that we are abusing notation here slightly, since we are using the same symbols to denote our forms, which are now expanded in the η basis, e.g. $\omega_1 = \eta^{x_1} \wedge \eta^{y_1}$. We will expand our fluxes in this new basis of forms as before, using integers m_0, m_a, e_a, and e_0 for F_0, F_2, F_4, and F_6. The lack of invariant one- and five-forms ensures that the fluxes associated to F_0, F_4, F_6 remain closed, and the Bianchi identity ensures that H_3 is closed, but we now see that the flux $m_a \omega_a$ corresponding to F_2 is not.

Indeed, by looking at (2.55), we see that there will be a new contribution to the C_7 tadpole,

$$-\sqrt{2} \left(m_0 p_K - m_a r_{aK}\right) b_K + [\delta]_{D6} = 2 \, [\delta]_{O6} \, . \tag{2.85}$$

[6]The reason for introducing this notation here rather than simply using the exterior derivative is so that we can more easily unify the results with nongeometric fluxes introduced below.

Actually, it is worthwhile to briefly rephrase some of these results with an eye toward later sections. As has long been noted in more general geometric setups with H-flux, such as this one, it can be useful to define a modified formal derivative

$$d_H = d + H\wedge = H \wedge \cdot - \omega \cdot, \tag{2.86}$$

The requirement that H be closed can be obtained from imposing $d_H^2 = 0$, and in our case this recovers all of our quadratic Bianchi identities. Many of the Bianchi identities can be found by applying d_H^2 to our cohomological basis,

$$\begin{aligned} d_H^2\, 1 &= -p_K \left(\widehat{d}^{-1}\right)^{\alpha\beta} \widehat{r}_{\beta K} \widetilde{\mu}_\alpha \quad \Longrightarrow \quad p_K \widehat{r}_{\alpha K} = 0,\ \forall\alpha, \\ d_H^2\, \omega_a &= r_{aK} \left(\widehat{d}^{-1}\right)^{\alpha\beta} \widehat{r}_{\beta K} \widetilde{\mu}_\alpha \quad \Longrightarrow \quad r_{aK} \widehat{r}_{\beta K} = 0,\ \forall a, \beta, \end{aligned} \tag{2.87}$$

but sadly, as we shall see in our specific example, this does not capture all of the Bianchi identities. And finally, the contribution to the tadpole is naturally proportional to

$$- d_H F_{RR}|_{3-\text{form}} = -\left(H F_0 - \omega \cdot F_2\right), \tag{2.88}$$

in precise agreement with what we have found.

D-terms

Now let us revisit the gauge transformations for the four-dimensional vectors A_α. Recall that the vectors descended from the three-form potential,

$$C_3 = A^\alpha \wedge \mu_\alpha + \xi_K a_K, \tag{2.89}$$

where we ignore the local piece of C_3 which contributes to the four-form flux $e_a \widetilde{\omega}_a$. In order to generate the required gauge transformation $A^\alpha \to A^\alpha + d\lambda^\alpha$, we perform a gauge transformation

$$C_3 \longrightarrow C_3 + d\left(\lambda^\alpha \mu_\alpha\right) = C_3 + d\lambda^\alpha \wedge \mu_\alpha - \lambda^\alpha \widehat{r}_{\alpha K} a_K, \tag{2.90}$$

or in terms of the four dimensional fields,

$$A^\alpha \longrightarrow A^\alpha + d\lambda^\alpha, \qquad \xi_K \longrightarrow \xi_K - \lambda^\alpha \widehat{r}_{\alpha K}, \tag{2.91}$$

We thus see that our scalar fields are no longer all invariant under the gauge transformations! In particular, if we define a field

$$\Xi_K = \exp\left[iN_K\right], \tag{2.92}$$

then Ξ_K is electrically charged under the gauge group $\mathrm{U}(1)_\alpha$ with charge $-\frac{1}{2}\widehat{r}_{\alpha K}$. Using (2.57) and (2.31), we can then calculate

$$D_\alpha = -2ie^D \mathcal{F}_K \widehat{r}_{\alpha K}, \tag{2.93}$$

(recall that our $\mathcal{F}_K$'s were pure imaginary, so that D_α is real). So for a supersymmetric vacuum we must have, in addition to the F-term equations that we will derive below, that $\mathcal{F}_K \widehat{r}_{\alpha K} = 0$ for each gauge group α. Note that no Fayet-Iliopoulos terms have been generated.

On the other hand, if we are willing to break SUSY, we note that the scalar potential now has a piece

$$V_D = \frac{1}{2}\left(\mathrm{Re}\, f\right)^{-1\,\alpha\beta} D_\alpha D_\beta = -2e^{2D}\left(\mathrm{Re}\, f\right)^{-1\,\alpha\beta}\left(\mathcal{F}_K \widehat{r}_{\alpha K}\right)\left(\mathcal{F}_J \widehat{r}_{\beta J}\right), \tag{2.94}$$

where we recall that

$$\mathrm{Re}\, f_{\alpha\beta} = -\widehat{\kappa}_{\alpha\beta a} v_a. \tag{2.95}$$

Since $(\mathrm{Re}\, f)$ is positive definite, $V_D \geq 0$. In appendix A we give a geometric interpretation for the nonvanishing of the D-terms.

Such D-term contributions have been the subject of much phenomenological interest, as a possible means to uplift the potential to a metastable deSitter vacuum [98, 102, 24, 25, 26], or as a mechanism for generating inflationary potentials [27, 28]. The latter possibility is usually done in a context with FI parameters turned on, but with a minimal holomorphic coupling $f_{\alpha\beta} = \delta_{\alpha\beta}$; it would be interesting to see if the class of models we are discussing in this paper could lead to phenomenologically useful potentials. We are currently investigating these possibilities.

Superpotential

One can obtain the superpotential in the presence of metric fluxes in a number of ways. One can perform an explicit generalized Scherk-Schwarz reduction, or the formula can be deduced from T-duality arguments, as has been done by various authors [14, 6, 29, 2]. Using the formalism of generalized complex geometry is also an interesting approach (see [30, 3]).

The result is that the superpotential can still be written as $W = W^Q + W^K$, with $W^K = \int \exp[J_c] \wedge F_{RR}$, exactly as before, but with W^Q now modified into

$$W^Q = \int_X \Omega_c \wedge (H_3 + \omega \cdot J_c) = \int_X \Omega_c \wedge d_H \left(e^{-J_c}\right), \tag{2.96}$$

where Ω_c and J_c are as before, so that doing the integration,

$$W^Q = 2N_K \left(p_K + r_{aK} t_a\right). \tag{2.97}$$

In particular, W^Q no longer depends only on the complex moduli, but there is now a mixing term $2N_K r_{aK} t_a$.

Given that we discovered in section 2.2.3 above that some of our scalar fields now transform under the gauge groups, an immediate worry is whether the superpotential is neutral, as it must be for consistency. Computing, we find

$$\delta W = -\lambda^{\alpha} \widehat{r}_{\alpha K} \left(p_K + r_{aK} t_a \right) = 0, \tag{2.98}$$

where in the final step we have used the Bianchi identities (2.87) that arise from applying d_H^2 to our cohomological basis. So our setup seems consistent.

We turn now to the F-term equations that result from this superpotential.

$$\begin{aligned} 0 &= \operatorname{Re} D_a W = r_{aK}\xi_K + d_{ab}e_b + \kappa_{abc}u_b m_c + \frac{m_0}{2}\kappa_{abc}\left(u_b u_c - v_b v_c\right) \\ &\quad -\frac{3}{2}\frac{\kappa_{abc}v_b v_c}{\kappa_{def}v_d v_e v_f}\operatorname{Im} W, \end{aligned} \tag{2.99}$$

$$\begin{aligned} 0 &= \operatorname{Im} D_a W = 2e^{-D} r_{aK}\mathcal{Z}_K + \kappa_{abc}v_b m_c + m_0 \kappa_{abc} u_b v_c \\ &\quad +\frac{3}{2}\frac{\kappa_{abc}v_b v_c}{\kappa_{def}v_d v_e v_f}\operatorname{Re} W, \end{aligned} \tag{2.100}$$

$$0 = \operatorname{Re} D_K W = 2p_K + 2r_{aK}u_a - 4ie^{D}\mathcal{F}_K \operatorname{Im} W, \tag{2.101}$$

$$0 = \operatorname{Im} D_K W = 2r_{aK}v_a + 4ie^{D}\mathcal{F}_K \operatorname{Re} W. \tag{2.102}$$

It is once again true in this case that one can use the F-term equations to show

$$\operatorname{Im} W = -\frac{2m_0}{15}\kappa_{abc}v_a v_b v_c. \tag{2.103}$$

Thus, if we would like to find a supersymmetric Minkowski vacuum, we must have $m_0 = 0$. In such a vacuum, (2.102) says that $r_{aK}v_a = 0$, and then contracting (2.100) with v_a we learn that $\kappa_{abc}v_a v_b m_c = 0$. With a couple more manipulations one can then show that the F-term equations now reduce to

$$M \cdot \begin{pmatrix} u_a \\ \xi_K \end{pmatrix} + \begin{pmatrix} (de)_a \\ p_K \end{pmatrix} = 0, \qquad M \cdot \begin{pmatrix} \widetilde{v}_a \\ 2\mathcal{Z}_K \end{pmatrix} = 0, \tag{2.104}$$

where

$$M = \begin{pmatrix} (\kappa m)_{ab} & r_{aK} \\ r^T_{Jb} & 0 \end{pmatrix}, \tag{2.105}$$

and where $\widetilde{v}_a = e^D v_a$. Two more equations must also be satisfied - there is one relation among the $\mathcal{Z}_K$ ($\mathcal{Z}_1^2 - \mathcal{Z}_2^2 = \frac{1}{2}$ in our example), and from $\mathrm{Re}\, W = 0$ we have

$$\frac{1}{|\Gamma|} e_0 + d_{ab} u_a e_b + \frac{1}{2} \kappa_{abc} u_a u_b m_c = 0. \tag{2.106}$$

Since the v_a and D only occur in the combination $\widetilde{v}_a$, there will always be one combination which remains unfixed (this result was also derived by [3]). Explicitly, the mode which scales $e^\phi = g_s \to \lambda g_s$ and $v_a \to \lambda^2 v_a$ leaves $\widetilde{v}_a$ unchanged, so this mode will remain massless. The scaling here is unfortunate; it means that as we go far out along this flat direction, either the string coupling blows up or the volume becomes very small, and our whole framework is expected to break down. This means that we cannot expect parametric control of such an example.

The general situation for supersymmetric AdS vacua, or even more generally for extrema of the full scalar potential, is quite complicated, and we don't have much to say about it here. We do believe that with metric fluxes it should be possible to stabilize all moduli in a supersymmetric AdS vacuum, though a puzzle regarding R-R quantization will interfere with our attempts to provide a fully consistent example here.

Let us now examine the situation in our specific example more closely.

Example

Imposing invariance under the orientifold group, we find that we are left with ten independent metric fluxes,

$$\begin{aligned}
2\quad \omega^1_{16} &= \omega^1_{15} = -\omega^2_{25} = -2\,\omega^2_{26}, \\
\omega^1_{26} &= \omega^2_{16}, \\
\omega^1_{36} &= -\omega^2_{46}, \\
\omega^1_{46} &= \omega^2_{36}, \\
&\qquad \omega^1_{35} = \omega^1_{45} = \omega^2_{35} = -\omega^2_{45} = \omega^1_{36} + \omega^1_{46}, \\
\omega^3_{16} &= -\omega^4_{26}, \\
\omega^3_{26} &= \omega^4_{16}, \\
&\qquad \omega^3_{15} = \omega^3_{25} = \omega^4_{15} = -\omega^4_{25} = \omega^3_{16} + \omega^3_{26}, \\
\omega^3_{36} &= -\omega^4_{46}, \\
2\quad \omega^3_{46} &= \omega^3_{45} = \omega^4_{35} = 2\,\omega^4_{36}, \\
\omega^5_{13} &= -\omega^5_{24}, \\
\omega^6_{13} &= -\omega^6_{14} = -\omega^6_{23} = -\omega^6_{24}, \\
&\qquad \omega^5_{14} = \omega^5_{23} = \omega^5_{13} + \omega^6_{13},
\end{aligned} \tag{2.107}$$

where we can use the ten fluxes in the left-hand column as representatives, and where (here and elsewhere in the paper) we order our coordinates $(x_1, y_1, x_2, y_2, x_3, y_3)$, so that an index $2i-1$ refers to x_i and an index $2j$ refers to y_j.

In terms of r-matrices, we find

$$r_{aK} = \begin{pmatrix} \omega^1_{36} & -\omega^1_{46} \\ -\omega^3_{16} & \omega^3_{26} \\ \omega^5_{13} + \omega^6_{13} & \omega^5_{13} \\ \omega^1_{16} - \omega^1_{26} - \omega^3_{36} - \omega^3_{46} & -\omega^1_{16} - \omega^1_{26} - \omega^3_{36} + \omega^3_{46} \end{pmatrix}, \tag{2.108}$$

$$\widehat{r}_K = \begin{pmatrix} -\omega^1_{16} + \omega^1_{26} - \omega^3_{36} - \omega^3_{46} & -\omega^1_{16} - \omega^1_{26} + \omega^3_{36} - \omega^3_{46} \end{pmatrix}. \tag{2.109}$$

Note that there is a one-to-one correspondence between the independent fluxes ω^i_{jk} and the entries of r and $\widehat{r}$.

Let us now impose the Bianchi identities $\omega^m_{[jk}\omega^i_{\ell]m} = 0$ and $\omega^m_{[ij}H_{kl]m} = 0$. It turns out that the general solution can be divided into four cases.

$$\begin{aligned}
&(a) \quad r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & \beta \\ 0 & 0 \end{pmatrix}, \qquad \widehat{r} = 0, && \forall p_1, p_2 \\
&(a') \quad r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & -\alpha \\ 0 & 0 \end{pmatrix}, \quad \widehat{r} = \begin{pmatrix} \beta & \beta \end{pmatrix}, && p_1 + p_2 = 0, \\
&(a'') \quad r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \alpha & \alpha \\ 0 & 0 \end{pmatrix}, \qquad \widehat{r} = \begin{pmatrix} \beta & -\beta \end{pmatrix}, && p_1 - p_2 = 0, \\
&(b) \quad r = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \\ 0 & 0 \\ \varepsilon & \varphi \end{pmatrix}, \qquad \widehat{r} = \begin{pmatrix} \chi & \kappa \end{pmatrix}, && \chi p_1 + \kappa p_2 = 0,
\end{aligned} \tag{2.110}$$

and where case (b) must additionally satisfy the equations

$$\alpha\chi + \beta\kappa = \gamma\chi + \delta\kappa = \varepsilon\chi + \varphi\kappa = 0, \qquad 8\alpha\gamma - \varepsilon^2 - \chi^2 = 8\beta\delta - \varphi^2 - \kappa^2. \tag{2.111}$$

The first set of these equations is simply $r_{aK}\widehat{r}_K = 0$, as we derived above in (2.87). The one remaining equation, however, cannot be obtained from acting d_H^2 on any element of our orbifold-invariant cohomology, though it can be derived by demanding $d_H^2 = 0$ even on non-invariant forms.

Let us try to find supersymmetric solutions to these models. First of all, note that in cases (a') and (a'') the D-term is proportional to

$$\beta\,(\mathcal{F}_1 \pm \mathcal{F}_2)\,, \tag{2.112}$$

which is always nonvanishing since $|\mathcal{F}_1| > |\mathcal{F}_2|$ in nondegenerate $(0 < U < \infty)$ vacua. Hence, these two cases can never be supersymmetric. So we shall instead examine case (a) more carefully. Here the D-term equations are automatically satisfied since $\widehat{r} = 0$.

Now using (2.102), and assuming that at least one of α and β is nonzero, we find that $\alpha\mathcal{F}_2 = \beta\mathcal{F}_1$, so

$$U = \frac{1}{2}\frac{\alpha+\beta}{\alpha-\beta}. \tag{2.113}$$

For a physical solution, we need $|\alpha| > |\beta|$. Then from (2.101) we learn that in order to solve the F-term equations we have an extra condition on the fluxes, namely that

$$\alpha p_2 = \beta p_1. \tag{2.114}$$

There are a couple of immediate consequences of this. Firstly, observe that if $p_1 \neq 0$, then U actually has the same form (2.65) as before, and we again have that $|p_1| > |p_2|$. Also, note that the axions ξ_1 and ξ_2 appear in the F-term equations only in the combinations $r_{aK}\xi_K$ and $p_K\xi_K$, but thanks to (2.114), both of these are proportional to $(\alpha\xi_1 + \beta\xi_2)$; the equations are independent of the other linear combination, and hence one of the axions remains unfixed. Below, we will argue that this will happen generically if the rank of the matrix r_{aK} (one, for case (a)) is less than the number of axions in the problem (two).

We can now express the general solution to the F-term equations. First we define some useful quantities,

$$\widehat{e}_1 = e_1 - \frac{m_2 m_3}{m_0}, \qquad \widehat{e}_2 = e_2 - \frac{m_1 m_3}{m_0}, \qquad \widehat{e}_3 = e_3 - \frac{m_1 m_2 - 2m_4^2}{m_0},$$

$$\widehat{e}_4 = e_4 - \frac{m_3 m_4}{m_0}, \qquad \widehat{e}_0 = e_0 - \frac{e_1 e_2 - 2e_4^2}{m_3}. \tag{2.115}$$

Then we find that in addition to U given above, we have

$$\begin{aligned}
u_1 &= -\frac{m_1}{m_0} - \frac{\alpha \widehat{e}_2}{\alpha m_3 - p_1 m_0}, \\
u_2 &= -\frac{m_2}{m_0} - \frac{\alpha \widehat{e}_1}{\alpha m_3 - p_1 m_0}, \\
u_3 &= -\frac{m_3}{m_0} + \frac{5\alpha}{m_0} \frac{\left(\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2\right)(\alpha m_3 - p_1 m_0)}{3(\alpha m_3 - p_1 m_0)(\alpha \widehat{e}_0 - p_1 \widehat{e}_3) + \alpha \left(5\alpha - 3p_1 \frac{m_0}{m_3}\right)\left(\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2\right)}, \\
u_4 &= -\frac{m_4}{m_0} - \frac{\alpha \widehat{e}_4}{\alpha m_3 - p_1 m_0}.
\end{aligned} \tag{2.116}$$

In terms of u_3 above, we then have

$$v_3 = \sqrt{-\frac{1}{\alpha m_0}(m_3 + m_0 u_3)(p_1 + \alpha u_3)}, \tag{2.117}$$

and

$$\begin{aligned}
\alpha \xi_1 + \beta \xi_2 &= -\frac{\alpha}{4}\left[\frac{m_3 \widehat{e}_3 - m_0 \widehat{e}_0}{\alpha m_3 - p_1 m_0} + \frac{m_0 (\alpha m_3 + p_1 m_0)\left(\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2\right)}{m_3 (\alpha m_3 - p_1 m_0)^2}\right], \\
v_1 &= -\frac{5}{3v_3} \frac{p_1 + \alpha u_3}{\alpha m_3 - p_1 m_0} \widehat{e}_2, \\
v_2 &= -\frac{5}{3v_3} \frac{p_1 + \alpha u_3}{\alpha m_3 - p_1 m_0} \widehat{e}_1, \\
v_4 &= -\frac{5}{3v_3} \frac{p_1 + \alpha u_3}{\alpha m_3 - p_1 m_0} \widehat{e}_4,
\end{aligned} \tag{2.118}$$

$$e^{\phi} = \frac{3\sqrt{\alpha^2 - \beta^2}\,(\alpha m_3 - p_1 m_0)}{2\sqrt{2}\,|\alpha m_0|} \sqrt{\frac{v_3}{\widehat{e}_1 \widehat{e}_2 - 2\widehat{e}_4^2}}. \tag{2.119}$$

A good solution requires a number of inequalities and conditions to hold; $\alpha m_3 > p_1 m_0$, $\widehat{e}_1 \widehat{e}_2 > 2\widehat{e}_4^2$, and $(\alpha m_3 - p_1 m_0)(\alpha \widehat{e}_0 - p_1 \widehat{e}_3) > \alpha p_1 m_0 (\widehat{e}_1 \widehat{e}_1 - 2\widehat{e}_4^2)/m_3$, and the quantities $\widehat{e}_1$, $\widehat{e}_2$, and m_0 must have the same sign.

As long as these conditions are respected, we can take various limits of the above solution. For instance, one can check that taking the limit $\alpha, \beta \to 0$ (and using (2.114)) recovers the solution from section 2.2.2. For future reference, let us list also the limit $p_1, p_2 \to 0$. In this case the conditions are that α, m_3, $\widehat{e}_0$ must have the same sign, $\widehat{e}_1$, $\widehat{e}_2$, and m_0 must have the same sign, and $\widehat{e}_1\widehat{e}_2 > 2\widehat{e}_4^2$.

$$u_1 = -\frac{e_2}{m_3},\ u_2 = -\frac{e_1}{m_3},\ u_3 = -\frac{3m_3^2\widehat{e}_0}{m_0\left(3m_3\widehat{e}_0 + 5\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)\right)},\ u_4 = -\frac{e_4}{m_3},$$

$$v_1 = |\widehat{e}_2|\sqrt{\frac{5\widehat{e}_0}{3m_3\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}}, \qquad v_2 = |\widehat{e}_1|\sqrt{\frac{5\widehat{e}_0}{3m_3\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}},$$

$$v_3 = \frac{\sqrt{\frac{15m_3^3}{m_0^2}\widehat{e}_0\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}}{3m_3\widehat{e}_0 + 5\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}, \qquad v_4 = \widehat{e}_4\sqrt{\frac{5\widehat{e}_0}{3m_3\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}}\,(\operatorname{sign} m_0)\,, \quad (2.120)$$

$$\alpha\xi_1 + \beta\xi_2 = -\frac{1}{4}\left[\widehat{e}_3 - \frac{m_0\widehat{e}_0}{m_3} + \frac{m_0}{m_3^2}\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)\right],$$

$$e^{\phi} = \frac{3\sqrt{\alpha^2-\beta^2}}{2\sqrt{2}}\left|\frac{m_3}{m_0}\right|\frac{\left[\frac{15m_3^3}{m_0^2}\frac{\widehat{e}_0}{\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2}\right]^{1/4}}{\sqrt{3m_3\widehat{e}_0 + 5\left(\widehat{e}_1\widehat{e}_2 - 2\widehat{e}_4^2\right)}}.$$

We will see later that these two limits are T-duals of each other.

Returning to the general case, note that the tadpole condition is now

$$\sqrt{2}\,(\alpha m_3 - p_1 m_0) + N_1 = 8, \qquad \sqrt{2}\,(\beta m_3 - p_2 m_0) + N_2 = 0. \qquad (2.121)$$

If we are looking for a supersymmetric solution, then we want N_1 to be greater than zero, and the sign of N_2 to be fixed by U (as discussed in section 2.2.2), and from solving the F-term equations we have $\alpha m_3 > p_1 m_0$, $\alpha p_2 = \beta p_1$, and $|\alpha| > |\beta|$, so we find, as before, that $N_1 + |N_2| < 8$.

To help understand why we were unable to find solutions with all moduli stabilized in these examples, note that in the general situation the $h^{2,1}+1$ real fields ξ_K appear in the F-term equations only through the combinations $p_K\xi_K$ and $r_{aK}\xi_K$.

We see immediately that these provide at most $\mathrm{rk}(r)+1$ independent combinations, so that we only have any chance to stabilize all of the axions in the case that $\mathrm{rk}(r) \geq h^{2,1}$ (see also the discussion in [3]). In fact, if $\mathrm{Re}\, W \neq 0$, then we can do even better. In this case we can use (2.102) to show that

$$\mathcal{F}_K = \frac{i}{2} e^{-D} \frac{r_{aK} v_a}{\mathrm{Re}\, W}, \tag{2.122}$$

and then (2.101) implies that

$$p_K = \left(-u_a - \frac{\mathrm{Im}\, W}{\mathrm{Re}\, W} v_a \right) r_{aK}, \tag{2.123}$$

thus reducing us to just $\mathrm{rk}(r)$ independent combinations of axions. In our example, this means that if $\mathrm{Re}\, W \neq 0$, we need an r-matrix of rank two, which was obviously impossible in the context of case (a) above. On the other hand, trying to set $\mathrm{Re}\, W = 0$ seems to typically lead to degenerate solutions, where either the complex structure modulus or a Kähler modulus runs off to the edge of physically allowed values.

So finally, let us turn to case (b), with the hopes of finding an $\mathcal{N} = 1$ vacuum with all moduli fixed. Suppose first that $\widehat{r} \neq 0$. In this case, the D-term equations require $|\chi| < |\kappa|$ and fix $\mathcal{F}_2/\mathcal{F}_1 = -\chi/\kappa$. Then the various Bianchi identities enforce $r_{a2} = -(\chi/\kappa) r_{a1}$ and $p_2 = -(\chi/\kappa) p_1$. It is immediately clear in this case that one combination of the ξ's again remains unfixed.

Hence, let $\widehat{r} = 0$. By the argument above, we should look for solutions in which $\mathrm{rk}(r) = 2$. As we will see in section 2.3, the quantization conditions on metric fluxes is in general not the naive quantization in terms of (even) integers, but is somewhat more complicated. In fact, we will see that we cannot find a correctly quantized set of metric fluxes which both give $\mathrm{rk}(r) = 2$ and make it possible to satisfy the tadpole condition, however we think that this is a reflection of our ignorance of the correct R-R quantization conditions under these circumstances. For now, let us willfully ignore these subtleties and pick a set of NS-NS fluxes with the naive quantization, namely

$$p_1 = p_2 = 0, \qquad r = \begin{pmatrix} 4 & 2 \\ 2 & 0 \\ 0 & 0 \\ 8 & 0 \end{pmatrix}. \tag{2.124}$$

This choice respects the Bianchi identities. Let us then also choose R-R fluxes

$$m_0 = m_1 = m_2 = 0, \quad e_0 = e_1 = e_2 = -e_3 = e_4 = \sqrt{2}, \quad -m_3 = m_4 = \frac{1}{\sqrt{2}}, \tag{2.125}$$

which all satisfy that they are in $\mathbb{Z}/\sqrt{2}$. One can check that these choices satisfy the tadpole conditions with no extra branes. Then one can solve the F-term equations with these fluxes, first finding exact solutions for the u_a and ξ_K

$$u_1 = 0, \qquad u_2 = 2, \qquad u_3 = -\frac{9}{2}, \qquad u_4 = -\frac{1}{2},$$

$$\xi_1 = -\frac{1}{4\sqrt{2}}, \qquad \xi_2 = \frac{1}{2\sqrt{2}}, \tag{2.126}$$

and then solving numerically for the rest of the moduli,

$$v_1 \approx 2.58227, \qquad v_2 \approx 4.26420, \qquad v_3 \approx 3.46108, \qquad v_4 \approx -0.50562,$$

$$U \approx 1.03530, \qquad e^{\phi} \approx 11.956. \tag{2.127}$$

Note that the volumes here are not particularly large, though all the cycle volumes are positive, and the string coupling is definitely not small. This solution should be viewed more as an in principle proof that the F-term equations can stabilize all of the moduli at physical values.

2.2.4 General NS-NS fluxes

As we shall see when we consider T-dualities below, by T-dualizing twice on a torus with H-flux, one can find oneself in a *non-geometric* situation, where there is a local geometric decription, but globally, one must patch torus fibers together with non-geometric elements of the T-duality group. All of this will hopefully be elucidated more cleanly in the next section, but for now note that at least some such considerations are really forced upon us by T-duality. To this end we will introduce objects Q_k^{ij}, analogous to H_{ijk} and ω^i_{jk}.

If one believes that the full $\mathrm{O}(6,6;\mathbb{Z})$ T-duality group acts on the fluxes which appear in the effective four-dimensional description (this is not obviously correct from a ten-dimensional perspective; choosing a flux trivialization reduces the number of available isometries and correspondingly the size of the duality group), then one also should include fluxes which come from dualizing all three legs of H-flux as well, which will be denoted R^{ijk}.

General approach

As mentioned, we introduce fluxes Q_k^{ij} and R^{ijk}. The Q-fluxes, being two T-dualities from H-flux, should be invariant under the orbifold group and odd under the orientifold involution, while the R-fluxes, being two T-dualities away from metric fluxes, should be invariant under the full orientifold group (a pair of T-dualities should preserve the eigenvalue under world-sheet parity; alternatively, for Q-flux we will see this requirement emerge from the base-fiber approach).

It turns out that one can define a natural action of these fluxes on globally defined forms. So we will again take a p-form

$$A^{(p)} = \frac{1}{p!} A_{i_1 \cdots i_p} \eta^{i_1} \wedge \cdots \wedge \eta^{i_p}, \tag{2.128}$$

with constant coefficients[7], and we will define a map to $(p-1)$-forms,

$$(Q \cdot A)_{i_1 \cdots i_{p-1}} = \frac{1}{2} \binom{p-1}{1} Q^{jk}_{[i_1} A_{|jk| i_2 \cdots i_{p-1}]}, \tag{2.129}$$

and to $(p-3)$-forms,

$$(R \cdot A)_{i_1 \cdots i_{p-3}} = \frac{1}{6} \binom{p-3}{0} R^{jk\ell} A_{jk\ell i_1 \cdots i_{p-3}}, \tag{2.130}$$

where we have written the numerical factors in such a way so as to make clear that the forms must be sufficiently high degree (≥ 2 for Q, ≥ 3 for R) to give a nonzero result.

As before, there will be Bianchi identities restricting the NS-NS fluxes. There are a number of different approaches to deriving these identities, and we will discuss some of these in Appendix B; here we will simply list the results. We assume that there are no invariant vector fields, so that a tracelessness condition $Q^{ij}_j = 0$ is satisfied automatically, analogously to the $\omega^j_{ij} = 0$ that we demanded previously. We also assume that there are no zero-forms which are invariant under the orbifold group and odd under the involution, so that $H_{ijk} R^{ijk} = \omega^i_{jk} Q^{jk}_i = 0$ automatically

[7]We are of course cheating here; when there is no good sense of global geometry, there are no sensible definitions of global forms. In fact, we hope the reader will view these for now as schematic short-cuts to obtain expressions for the four-dimensional effective theory. We hope that the role of these constructions becomes clearer in the next section.

as well. With these assumptions, the Bianchi identities read

$$\begin{aligned}
H_{m[ij}\omega^{m}_{k\ell]} &= 0, \\
H_{m[ij}Q^{m\ell}_{k]} - \omega^{m}_{[ij}\omega^{\ell}_{k]m} &= 0, \\
H_{ijm}R^{k\ell m} + \omega^{m}_{ij}Q^{k\ell}_{m} - 4\omega^{[k}_{m[i}Q^{\ell]m}_{j]} &= 0, \\
\omega^{[j}_{mi}R^{k\ell]m} - Q^{[jk}_{m}Q^{\ell]m}_{i} &= 0, \\
Q^{[ij}_{m}R^{k\ell]m} &= 0,
\end{aligned} \tag{2.131}$$

It turns out to be very natural to define a sort of covariant differential [3], in analogy to the twisted differential d_H in the case with metric fluxes,

$$\mathcal{D} = H \wedge \cdot - \omega \cdot + Q \cdot - R \cdot . \tag{2.132}$$

We postpone a full discussion of T-duality until section 2.3, but the main argument in favor of this formulation is that it appears in the correct T-duality-invariant formulation of the tadpole condition, which will now read as[8]

$$-\sqrt{2}\mathcal{D}F_{RR} + [\delta_{D6}] = 2\,[\delta_{O6}]\,. \tag{2.133}$$

Furthermore, as discussed more carefully in Appendix B, the Bianchi identities above follow from imposing $\mathcal{D}^2 = 0$ (along with our assumptions). Finally, the superpotential can also be obtained by duality arguments [2] (or see the approach of [30]) and is given by

$$W = \int_X e^{J_c} \wedge F_{RR} + \int_X \Omega_c \wedge \mathcal{D}\left(e^{-J_c}\right), \tag{2.134}$$

[8]Of course, in the absence of a global (or even local) ten-dimensional geometry it is difficult to interpret this expression, which essentially is derived by T-duality, and in particular the interpretation of branes must be subtle [31, 32].

as expected by comparing with (2.96).

Now we shall represent our fluxes more succinctly by expansions similar to those we had previously,

$$Q \cdot \widetilde{\omega}_a = q_{aK} b_K, \qquad Q \cdot \widetilde{\mu}_\alpha = \widehat{q}_{\alpha K} a_K, \tag{2.135}$$

$$R \cdot \phi = s_K b_K. \tag{2.136}$$

Though we won't prove it here, we also have

$$Q \cdot a_K = -\left(d^{-1}\right)^{ab} q_{bK} \omega_a, \qquad Q \cdot b_K = \left(\widehat{d}^{-1}\right)^{\alpha\beta} \widehat{q}_{\beta K} \mu_\alpha, \tag{2.137}$$

$$R \cdot a_K = |\Gamma|\, s_K\, 1, \qquad R \cdot b_K = 0. \tag{2.138}$$

Many of the Bianchi identities can be obtained by demanding that $\mathcal{D}^2 = 0$ on our cohomological basis, namely

$$\begin{aligned}
\widehat{r}_{\alpha K} p_K = \widehat{r}_{\alpha K} s_K = \widehat{q}_{\alpha K} p_K = \widehat{q}_{\alpha K} s_K &= 0, \quad \forall \alpha, \\
\widehat{r}_{\alpha K} r_{bK} = \widehat{r}_{\alpha K} q_{bK} = \widehat{q}_{\alpha K} r_{bK} = \widehat{q}_{\alpha K} q_{bK} &= 0, \quad \forall \alpha, b, \\
|\Gamma| p_{[K} s_{J]} + \left(d^{-1}\right)^{ab} r_{a[K} q_{|b|J]} = \left(\widehat{d}^{-1}\right)^{\alpha\beta} \widehat{r}_{\alpha[K} \widehat{q}_{|\beta|J]} &= 0, \quad \forall K, J.
\end{aligned} \tag{2.139}$$

Unfortunately, as before, there are some Bianchi identities which are not captured by these equalities.

With these definitions, the tadpole condition reads

$$-\sqrt{2}\left(p_K m_0 - r_{aK} m_a + q_{aK} e_a - s_K e_0\right) + N_K^{(D6)} = 2N_K^{(O6)}, \tag{2.140}$$

and the superpotential is

$$
\begin{aligned}
W &= \frac{1}{|\Gamma|}e_0 + d_{ab}t_a e_b + \frac{1}{2}\kappa_{abc}t_a t_b m_c + \frac{1}{6}m_0\kappa_{abc}t_a t_b t_c \\
&\quad +2N_K\left(p_K + r_{aK}t_a + \frac{1}{2}\kappa_{abc}\left(d^{-1}\right)^{ce} q_{eK}t_a t_b + \frac{|\Gamma|}{6}s_K\kappa_{abc}t_a t_b t_c\right).
\end{aligned} \tag{2.141}
$$

The Kähler potential K and holomorphic couplings $f_{\alpha\beta}$ remain unchanged.

The corresponding F-terms are

$$
\begin{aligned}
D_a W &= d_{ab}e_b + \kappa_{abc}m_b t_c + \frac{1}{2}m_0\kappa_{abc}t_b t_c + 2N_K r_{aK} + 2N_K\kappa_{abc}\left(d^{-1}\right)^{cd} q_{dK}t_b \\
&\quad +|\Gamma|N_K s_K\kappa_{abc}t_b t_c + \frac{3i}{2}\frac{\kappa_{abc}v_b v_c}{\kappa_{def}v_d v_e v_f}W, \qquad (2.142)\\
D_K W &= 2p_K + 2r_{aK}t_a + \kappa_{abc}\left(d^{-1}\right)^{cd} q_{dK}t_a t_b + \frac{|\Gamma|}{3}s_K\kappa_{abc}t_a t_b t_c - 4e^D\mathcal{F}_K W.
\end{aligned}
$$

Note that the superpotential only depends on the NS-NS fluxes p_K, r_{aK}, q_{aK}, and s_K, and not on the hatted fluxes $\widehat{r}_{\alpha K}$ and $\widehat{q}_{\alpha K}$. These appear only in the D-terms, which we turn to next.

D-terms revisited

Recall that in our previous discussion of D-terms, we noted that a gauge transformation of the three-form potential could also, in the presence of certain metric fluxes, shift the axion fields ξ_K, because the forms μ_α, on which the three-form is reduced to give the four-dimensional vectors, were no longer closed. This analysis should still hold in the presence of non-geometric fluxes, but it seems natural to rewrite the extra contribution to the gauge transformation as

$$
\mathcal{D}\left(\lambda^\alpha \mu_\alpha\right) = -\lambda^\alpha \widehat{r}_{\alpha K} a_K. \tag{2.143}
$$

For this gauge transformation, replacing the usual differential d by $\mathcal{D}$ makes no difference; $d\mu_\alpha = \mathcal{D}\mu_\alpha = -\widehat{r}_{\alpha K} a_K$.

However, we can also consider gauge transformations of the *dual* gauge fields in four dimensions. These fields are obtained by reducing the five-form potential against a four-form. Since the orientifold action requires C_5 to be odd, we must reduce it against an odd four-form $\widetilde{\mu}_\alpha$ to get an invariant vector in four-dimensions, $C_5 = \widetilde{A}^\alpha \wedge \widetilde{\mu}_\alpha$. Then we claim that the dual gauge transformations are generated by gauge transformations of the five-form, and that in principle we can pick up an extra piece,

$$C_{RR} \longrightarrow C_{RR} + d\widetilde{\lambda}^\alpha \wedge \widetilde{\mu}_\alpha + \widetilde{\lambda}^\alpha \widehat{q}_{\alpha K} a_K, \tag{2.144}$$

where $C_{RR} = C_1 + C_3 + C_5 + C_7 + C_9$ is a formal sum of R-R potentials.

Thus the fields ξ_K are not invariant under the dual gauge transformation. In other words, the fields Ξ_K defined in (2.92) have magnetic charge $\frac{1}{2}\widehat{q}_{\alpha K}$ under the gauge group $\mathrm{U}(1)_\alpha$. These fields can in fact be dyons, that is have both electric and magnetic charges. It is interesting to ask whether our collection of charged scalars are then mutually local, in the sense of [33] (if they weren't then we would despair of having any Lagrangian description for our effective physics). The condition that two dyons labelled K and J with these charges be mutually local is simply that

$$\left(\widehat{d}^{-1}\right)^{\alpha\beta} \left(\widehat{r}_{\alpha K}\widehat{q}_{\beta J} - \widehat{r}_{\alpha J}\widehat{q}_{\beta K}\right) = 0. \tag{2.145}$$

But this is precisely one of the Bianchi identities of (2.139), so our fields are guaranteed to be mutually local.

This in turn implies that all charges can be made electric charges by a symplectic transformation $M \in \mathrm{Sp}(2n_V;\mathbb{Z})$, where $n_V = h_+^{1,1}$ is the number of vectors, and where the symplectic group is defined here by

$$M \begin{pmatrix} 0 & \widehat{d} \\ -\widehat{d}^T & 0 \end{pmatrix} M^T = \begin{pmatrix} 0 & \widehat{d} \\ -\widehat{d}^T & 0 \end{pmatrix}, \tag{2.146}$$

and M acts on the K^{th} charge vector as

$$M \begin{pmatrix} -\widehat{r}_{\alpha K} \\ \widehat{q}_{\alpha K} \end{pmatrix} = \begin{pmatrix} -\widehat{r}'_{\alpha K} \\ 0 \end{pmatrix}. \tag{2.147}$$

We then have the D-term being

$$D_\alpha = -2ie^D \mathcal{F}_K \widehat{r}'_{\alpha K}. \tag{2.148}$$

We have already seen that the holomorphic couplings for the electric gauge groups are given by $f^{(\text{electric})}_{\alpha\beta} = i\,(\widehat{\kappa}t)_{\alpha\beta}$, where we have made the obvious definition $(\widehat{\kappa}t)_{\alpha\beta} = \widehat{\kappa}_{\alpha\beta a} t_a$. What about for the magnetic gauge groups? In other words, if the $\widehat{r}$ fluxes are taken to vanish, but the $\widehat{q}$ fluxes are nonvanishing then we need to use the dual field strengths in our $\mathcal{N} = 1$ description and this will give a different answer for the holomorphic coupling constants. The new constants can be obtained by rewriting the ten-dimensional action (2.40) (with NS-NS fluxes turned off) in terms of the potential C_5 rather than its dual C_3. The result is

$$f^{(\text{magnetic})}_{\alpha\beta} = -i\,(\widehat{\kappa}t)^{-1\,\gamma\delta}\,\widehat{d}_{\gamma\alpha}\widehat{d}_{\delta\beta}. \tag{2.149}$$

This result is also consistent with T-duality. For the general case, we should take the matrix

$$f = \begin{pmatrix} f^{(\text{electric})} & 0 \\ 0 & f^{(\text{magnetic})} \end{pmatrix}, \tag{2.150}$$

and transform it under our symplectic transformation to get $f' = MfM^T$, and finally take our holomorphic couplings to be $f'^{\,(\mathrm{electric})}_{\alpha\beta}$ (i.e. the top-left block of the matrix f').

By using a combination of $\widehat{r}$ and $\widehat{q}$ fluxes, it is apparent that we can get the D-term piece of the potential,

$$\begin{aligned} V_D &= -2e^{2D} \left(\mathrm{Re}\, f'\right)^{-1\,\alpha\beta} \mathfrak{F}_K \widehat{r}'_{\alpha K} \mathfrak{F}_J \widehat{r}'_{\beta J} \\ &= -2e^{2D} \left[\left(\mathrm{Re}\, f^{(\mathrm{electric})}\right)^{-1\,\alpha\beta} \widehat{r}_{\alpha K} \widehat{r}_{\beta J} + \left(\mathrm{Re}\, f^{(\mathrm{magnetic})}\right)^{-1\,\alpha\beta} \widehat{q}_{\alpha K} \widehat{q}_{\beta J} \right] \mathfrak{F}_K \mathfrak{F}_J, \end{aligned} \qquad (2.151)$$

to have fairly complicated dependence on the Kähler moduli. It would be worthwhile to investigate whether this allows us to achieve some phenomenologically interesting models, with the D-terms either allowing meta-stable deSitter solutions, or possibly even nice inflationary potentials, and we are currently looking at these issues.

Example

Let us briefly see how some of these results work in our example. Unlike in previous subsections, we won't expend much effort trying to solve the equations, but will content ourselves simply with classifying the fluxes permitted by the orientifold action, and stating the equations that we would like to solve.

The Q-fluxes which survive the orientifold projection are

$$
\begin{aligned}
Q_5^{13} &= -Q_5^{14} = -Q_5^{23} = -Q_5^{24}, \\
Q_6^{13} &= -Q_6^{24}, \\
& \quad Q_6^{14} = Q_6^{23} = -Q_5^{13} + Q_6^{13}, \\
2 \quad Q_1^{15} &= -Q_1^{16} = -2\, Q_2^{25} = Q_2^{26}, \\
Q_2^{15} &= Q_1^{25}, \\
Q_3^{15} &= -Q_4^{25}, \\
Q_3^{16} &= Q_4^{16} = Q_3^{26} = -Q_4^{26}, \\
& \quad Q_4^{15} = Q_3^{25} = -Q_3^{15} - Q_3^{16}, \\
Q_1^{35} &= -Q_2^{45}, \\
Q_1^{36} &= Q_2^{36} = Q_1^{46} = -Q_2^{46}, \\
& \quad Q_2^{35} = Q_1^{45} = -Q_1^{35} - Q_1^{36}, \\
Q_3^{35} &= -Q_4^{45}, \\
2 \quad Q_4^{35} &= -Q_4^{36} = 2\, Q_3^{45} = -Q_3^{46},
\end{aligned}
\tag{2.152}
$$

where we take the ten fluxes in the left-hand column as independent. Similarly there are two independent R-fluxes,

$$
\begin{aligned}
R^{135} &= -R^{245}, \\
R^{136} &= -R^{146} = -R^{236} = -R^{246}, \\
& \quad R^{145} = R^{235} = R^{135} + R^{136}.
\end{aligned}
\tag{2.153}
$$

In more succinct terms,

$$q = \begin{pmatrix} -Q_1^{35} & -Q_1^{35} - Q_1^{36} \\ Q_3^{15} & Q_3^{15} + Q_3^{16} \\ -Q_5^{13} + Q_6^{13} & Q_6^{13} \\ Q_1^{15} - Q_2^{15} - Q_3^{35} - Q_4^{35} & -Q_1^{15} - Q_2^{15} - Q_3^{35} + Q_4^{35} \end{pmatrix}, \tag{2.154}$$

$$\widehat{q} = \begin{pmatrix} -Q_1^{15} + Q_2^{15} - Q_3^{35} - Q_4^{35} & -Q_1^{15} - Q_2^{15} + Q_3^{35} - Q_4^{35} \end{pmatrix}, \tag{2.155}$$

$$s = \begin{pmatrix} R^{135} + R^{136} & R^{135} \end{pmatrix}. \tag{2.156}$$

The Bianchi identities are unfortunately quite complicated and unenlightening. In addition to the identities from (2.139), we have the following extra conditions:

$$\begin{gathered}
-8r_{31}s_1 + 8q_{11}q_{21} - (q_{41})^2 - (\widehat{q}_1)^2 = -8r_{32}s_2 + 8q_{12}q_{22} - (q_{42})^2 - (\widehat{q}_2)^2 , \\
\kappa_{3ab}\left(d^{-1}\right)^{cb} r_{c(K}s_{J)} = q_{3(K}q_{|a|J)}, \qquad \text{for } a = 1,2,4;\ \forall K,J, \\
s_1\widehat{r}_2 + s_2\widehat{r}_1 = \widehat{q}_1 q_{32} + \widehat{q}_2 q_{31}, \\
4q_{11}r_{11} + 4q_{21}r_{21} - q_{41}r_{41} - \widehat{q}_1\widehat{r}_1 - 8q_{31}r_{31} = 4q_{12}r_{12} + 4q_{22}r_{22} - q_{42}r_{42} - \widehat{q}_2\widehat{r}_2 - 8q_{32}r_{32}, \\
r_{a(K}\widehat{q}_{J)} = \kappa_{3ab}\left(d^{-1}\right)^{cb} q_{c(K}\widehat{r}_{J)}, \qquad \text{for } a = 1,2,4;\ \forall K \neq J, \\
\kappa_{3ac}\left(d^{-1}\right)^{dc} q_{d(K}r_{|b|J)} = \kappa_{3bc}\left(d^{-1}\right)^{dc} q_{d(K}r_{|a|J)}, \qquad \text{for } a,b \in \{1,2,4\},\ \forall K,J, \\
q_{3(K}r_{|3|J)} + p_{(K}s_{J)} = 0, \\
-8p_1q_{31} + 8r_{11}r_{21} - (r_{41})^2 - (\widehat{r}_1)^2 = -8p_2q_{32} + 8r_{12}r_{22} - (r_{42})^2 - (\widehat{r}_2)^2 , \\
\kappa_{3ab}\left(d^{-1}\right)^{cb} q_{c(K}p_{J)} = r_{3(K}r_{|a|J)}, \qquad \text{for } a = 1,2,4;\ \forall K,J, \\
p_1\widehat{q}_2 + p_2\widehat{q}_1 = \widehat{r}_1 r_{32} + \widehat{r}_2 r_{31}.
\end{gathered} \tag{2.157}$$

The tadpole conditions are just as listed in (2.140), the D-term equations require (2.148) to vanish, and the F-term equations are as given in (2.142).

2.2.5 Summary and Puzzles

We have laid out an approach to studying a class of four-dimensional $\mathcal{N} = 1$ effective theories. Starting from toroidal orientifolds of IIA string theory with NS-NS H-flux turned on, we followed in the footsteps of many authors before us and argued for a more general class of NS-NS fluxes. The arguments proceed roughly by showing at each step that a T-duality induces the possibility of a new type of flux, and then we generalize to a framework capable of accomodating these new fluxes as well as the old ones (and thus allowing configurations that are not simply T-dual to previous ones). In this way we introduced metric fluxes ω^i_{jk}, then non-geometric fluxes Q^{ij}_k, and finally R^{ijk}.

However, these arguments were really made at the level of the effective field theory. In terms of ten-dimensional constructions, there would seem to be some obstacles to this program. For instance, beginning with H-flux on a torus, say $h\, dx \wedge dy \wedge dz$, to perform a ten-dimensional T-duality, one first picks a trivialization of the B-field such as $B = hx\, dy \wedge dz$. Then the Buscher rules [34] allow one to T-dualize along either the y or z directions, resulting in metric flux ω^y_{xz} or ω^z_{xy}, or T-dualize in y and z, resulting in Q^{yz}_x, but it is not obvious how to perform the third T-duality here to get R^{xyz}; our trivialization broke the third isometry, and the Buscher rules no longer apply. Indeed, there are general arguments that any ten-dimensional origin for R-flux cannot even have a local description [6, 35]. So it is very much of interest to ask which configurations can be constructed from ten dimensions.

We have also tried to formulate everything in a language that moves away from the toroidal context. So, instead of phrasing everything in terms of flux components, H_{ijk}, ω^i_{jk}, Q^{ij}_k and R^{ijk}, we rewrite our formulae (thereby serendipitously simplifying the $\mathcal{N} = 1$ expressions at the same time) in terms of matrices p_K, r_{aK}, $\widehat{r}_{\alpha K}$, q_{aK}, $q_{\alpha K}$ and s_K which referred only to the (untwisted) cohomology of the orientifold. Our hope is that this language will also allow the study of general NS-NS fluxes on arbitrary type IIA Calabi-Yau orientifold constructions, resulting in a greatly enriched tool-kit for model building. The major flaw right now in this plan is the Bianchi identities, which we were unable, in general, to recast in terms of the cohomological structure alone. Our hope, however, is that this difficulty can be overcome by studying explicit examples.

The most obvious extension in this direction would be to address another fairly prominent gap in our analysis, namely the incorporation of the twisted sectors. We have ignored twisted sector fluxes and moduli throughout our analysis, since we are more interested, in the present work, in elucidating the general structures that one encounters. In other contexts [83], it has been shown that, at least in specific models, it should be possible to stabilize the twisted sector moduli in such a way as to maintain a separation of scales with the bulk physics, but still trust the analysis. We hope that such considerations will still hold in many of the models discussed here. It would be very interesting to incorporate the twisted-sector cohomology into our general flux analysis (indeed if our approach is valid beyond toroidal examples, then it should be able to treat all of the cohomology democratically), possibly along the lines of [4]. In section 2.3.4 we will mention some ideas in this direction.

Another key point to emphasize here is the quantization of general NS-NS fluxes. For H-fluxes alone, the situation is well understood; the H-flux should be understood as an element of $H^3(X;\mathbb{Z})$, or in our terms, the p_K should be integers[9]. In situations related to these by T-duality the answers are just as straightforward;

[9]Actually, related to our willful ignorance of the twisted sectors, we have glossed over the fact that in our example, p_K should in fact be even integers [90]; our b_K alone are not elements of the integral cohomology, but rather we must take either $nb_1 + mb_2$ with $n + m$ even, or we may take b_K + (twisted). It would be interesting to provide a more complete analysis.

all of the fluxes p_K, r_{aK}, etc. must be integers. It is natural to assume that this is generally the correct condition, especially when we are describing our fluxes in terms of integral cohomology. However, as we shall see in the next section, this naive quantization is not generally correct. There will be examples we can construct (which are not simply T-dual to H-flux) where the quantization condition is much more complicated (though still simple from the point of view of our constructions). This still leaves the question of how fluxes are quantized in those models that we will not succeed in constructing from a ten-dimensional point of view. In that case we do not know what the correct quantization conditions should be. It is possible that those models simply have no legitimate ten-dimensional origin. If they do, we see no route to determining the correct quantization conditions without actual constructions.

In section 2.2.3 we presented one example of a model where all moduli were stabilized at a supersymmetric AdS vacuum and the tadpole condition was saturated without the need for extra D-branes. Unfortunately, this example used only our naive quantization conditions. Using the correct quantization on NS-NS fluxes which we will derive below we will find that it is no longer possible to stabilize all moduli while also satisfying both the F-term equations and the tadpole, the latter because the flux contributions in this case appear to be non-integral! We suspect that this problem with the tadpole is simply an artifact of our not understanding how the generalized NS-NS fluxes affect the correct quantization of R-R fluxes. It would be extremely gratifying to have a better grasp of these issues so as to be able to construct fully realized stable $\mathcal{N} = 1$ vacua[10].

Finally, let us turn to the issue of the regime of validity of this effective field

[10] Of course in section 2.2.3, since we have only turned on H-flux and metric flux, we do still have a global geometric description, and there should be nothing exotic about the quantization of R-R fluxes. Our suspicion, however is that we have run into trouble by trying to use the language of the twisted torus, i.e. in using fluxes defined by forms inherited from T^6. For the types of metric flux used here (and similar examples in the literature), the resulting space is quite different from the original T^6, and so the quantization conditions in our chosen basis will seem quite non-standard.

theory. As in [83], we are able to find models (by taking some of our R-R fluxes to be parametrically large, for instance $\widehat{e}_a$ in our solutions with H-flux only) in which the string coupling is small, and in which the compact directions are large enough to trust supergravity, but still much smaller than the AdS radius (which also characterizes the masses of the stabilized moduli), so that the solution would seem to be effectively four-dimensional[11]. However, just as in that situation, our models generally suffer from the concerns expressed by Banks and van den Broek [29]. Namely, due to the presence of the orientifold singularity, there are regions of our compact manifold in which the string coupling diverges (but see also [36]) and we should turn to eleven-dimensional supergravity instead. In this picture, the large flux integers translate into a large stack of M2-branes at the orientifold locus, and so the larger the flux integers, the more backreaction one has to deal with (and is ignoring in the effective description). We have not repeated this analysis in detail in our models, partly because the ten-dimensional (or eleven-dimensional) physics becomes more obscure for us, but the issue undoubtedly persists. We hope however, that our richer structure of fluxes might provide more corners in which to hide.

2.3 Base-Fiber Approach

In this section we will attempt to put a subset of our class of models on firmer ground by presenting ten-dimensional constructions. These constructions are very much in the spirit of [5] (see also [37, 38, 39, 40, 41, 42]) and are built by allowing a torus fiber to vary over a torus base, but in a way that still admits a generalized Scherk-Schwarz reduction. The NS-NS fluxes will be represented by the global twists in the fibers as one transports them around non-contractible cycles in the base. We will find that the Bianchi identities come out naturally, that dualities are implemented very easily, and that the correct quantization conditions are both obvious in this

[11] Note that these conditions are not preserved by T-duality.

context, and also much more subtle than one would have guessed.

2.3.1 The T-duality group $O(6, 6; \mathbb{Z})$

The T-duality group of type II superstring theory compactified on a d-dimensional torus T^d is denoted $\mathrm{O}(d, d; \mathbb{Z})$ and is defined as follows

$$\mathrm{O}(d, d; \mathbb{Z}) = \left\{ M \in \mathrm{Mat}_{2d\times 2d}(\mathbb{Z}) | MLM^T = L \right\}, \tag{2.158}$$

where

$$L = \begin{pmatrix} 0 & \mathbf{1}_d \\ \mathbf{1}_d & 0 \end{pmatrix}. \tag{2.159}$$

We will in fact focus primarily on elements with determinant one, which correspond to dualities from IIA to itself (or IIB to itself); elements with determinant minus one interchange solutions of IIA and IIB.

To understand the action of this group on the NS-NS sector, it is convenient to combine the torus metric and B-field into a single $d \times d$ matrix $E = G + B$. We assume implicitly here that our coordinate basis is chosen such that each coordinate is periodic with unit period. Let us take an $\mathrm{O}(d, d; \mathbb{Z})$ matrix M and write it in terms of $d \times d$ blocks,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.160}$$

Then the action of M on the NS-NS sector is

$$E \mapsto E' = (aE + b)\,(cE + d)^{-1}, \qquad e^{\phi} \mapsto e^{\phi'} = e^{\phi} \left(\frac{\det G'}{\det G} \right)^{1/4}. \tag{2.161}$$

There is a useful alternative phrasing of this transformation. From G and B we can define a symmetric $2d \times 2d$ matrix

$$H = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{2.162}$$

Then an element $M \in \mathrm{O}(d, d; \mathbb{Z})$ simply acts by

$$H \mapsto H' = M^T H M. \tag{2.163}$$

From this we can identify certain important elements of $\mathrm{O}(d, d; \mathbb{Z})$. For instance, it includes changes of basis for the lattice which defines T^d. These basis changes lie in the subgroup $\mathrm{GL}(d; \mathbb{Z}) \subset \mathrm{O}(d, d; \mathbb{Z})$ of matrices with the form

$$\hat{g} = \begin{pmatrix} \left(g^T\right)^{-1} & 0 \\ 0 & g \end{pmatrix}, \qquad g \in \mathrm{GL}(d; \mathbb{Z}). \tag{2.164}$$

Similarly, we also have constant integral shifts in the periods of the B-field given by matrices

$$\begin{pmatrix} \mathbf{1}_d & b \\ 0 & \mathbf{1}_d \end{pmatrix}, \qquad b^T = -b. \tag{2.165}$$

Finally there is one more type of element which will be of interest to us, corresponding simply to T-dualizing a sub-torus T^k of T^d, for example that corresponding to the first k coordinates. Then the relevant M is

$$M_k = \begin{pmatrix} 0 & 0 & \mathbf{1}_k & 0 \\ 0 & \mathbf{1}_{d-k} & 0 & 0 \\ \mathbf{1}_k & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{d-k} \end{pmatrix}. \tag{2.166}$$

From this and the transformation rules (2.161) above, one can compute the usual

Buscher rules.

It will also be useful to know how elements of $\mathrm{O}(d,d)$ act on the R-R fluxes and potentials, which can be thought of as sections of the spin bundle $\mathrm{Spin}(d,d)$. The action in this case cannot be expressed as simply as in the cases above [43], but it is not hard to write down for certain simple cases which can then be used to generate all of $\mathrm{O}(d,d)$ [44]. In particular, we have three cases.

If B_{ij} is an antisymmetric $d \times d$ matrix, so that $B = \frac{1}{2}B_{ij}dx^i \wedge dx^j$ is a two-form on the torus, then the element

$$g = \begin{pmatrix} \mathbf{1}_d & B \\ 0 & \mathbf{1}_d \end{pmatrix} \tag{2.167}$$

acts as

$$g \cdot F_{RR} = \exp(B) \wedge F_{RR} = F_{RR} + B \wedge F_{RR} + \frac{1}{2} B \wedge B \wedge F_{RR} + \cdots , \tag{2.168}$$

where $F_{RR} = \sum_a F_a$ is the sum of R-R-fluxes of various degrees.

Similarly, if β^{ij} is antisymmetric, $\beta = \frac{1}{2}\beta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$ an antisymmetric bivector, then

$$\begin{pmatrix} \mathbf{1}_d & 0 \\ \beta & \mathbf{1}_d \end{pmatrix} \cdot F_{RR} = \exp(\iota_\beta) \cdot F_{RR}, \tag{2.169}$$

where $\iota_\beta = \frac{1}{2}\beta^{ij}\iota_{\partial_i}\iota_{\partial_j}$ and ι_v acts by contracting a form with a vector v.

Finally, if $g \in \mathrm{GL}(d)$, then

$$\begin{pmatrix} \left(g^T\right)^{-1} & 0 \\ 0 & g \end{pmatrix} \cdot F_{RR} = |\det g|^{1/2} \left(g^{-1}\right)^* F_{RR}, \tag{2.170}$$

where $(g^{-1})^* F_{RR}$ denotes the pullback of F_{RR} by the map g^{-1}.

We are primarily interested in studying toroidal orientifolds, so it is important to understand how to discuss the orientifold group action in this language. For elements of the orbifold group, this is fairly clear; for any lattice preserving diffeomorphism $g \in \mathrm{GL}(d;\mathbb{Z})$ which acts on our torus, such as a rotation, we simply need to construct the corresponding element $\hat{g} \in \mathrm{O}(d,d;\mathbb{Z})$ as in (2.164) above. To describe the full orientifold action, we also need to know how the world-sheet parity operator Ω acts. It turns out that Ω can also be expressed as a $2d \times 2d$ matrix,

$$\Omega = \begin{pmatrix} -\mathbf{1}_d & 0 \\ 0 & \mathbf{1}_d \end{pmatrix}, \tag{2.171}$$

with the understanding that this operator also acts on the remaining $10-d$ coordinates (so that, e.g. it does not exchange IIA and IIB, even if d is odd). This is not an element of $\mathrm{O}(d,d;\mathbb{Z})$, since it satisfies that $\Omega L \Omega^T = -L$ rather than (2.158), but it can be thought of as an element of $\mathrm{Spin}(d,d;\mathbb{Z})$ and we can understand its action on NS-NS moduli simply by following (2.163), i.e. $\Omega \cdot G = G$, $\Omega \cdot B = -B$. We can also work out the action of Ω on R-R fields by following [44], but we in fact know the answer; C_3 and C_7, as well as F_0 and F_4 should be even, while C_1 and C_5, as well as F_2 and F_6, should be odd. In this way, we see that the entire orientifold group can be understood as a finite subgroup of $\mathrm{Spin}(d,d;\mathbb{Z})$ (for instance in our example this subgroup would be generated in this notation by $\hat{\Theta}$ and $\Omega\hat{\sigma}$).

In this notation, the untwisted moduli are simply those which are fixed by the orientifold subgroup $\widehat{\Gamma} \subset \mathrm{Spin}(d,d;\mathbb{Z})$. Note that so far we have not discussed any NS-NS fluxes. R-R fluxes can be accomodated, and both the R-R fluxes and R-R potentials are understood to transform according to the rules described above.

Now given any element $h \in \mathrm{SO}(d,d;\mathbb{Z}) \subset \mathrm{Spin}(d,d;\mathbb{Z})$,[12] we can relate a given orientifold with subgroup $\widehat{\Gamma}$ and moduli given by H, etc., to a dual orientifold

[12]Though we won't use them here, we can certainly in general consider dualities h which lie in $\mathrm{O}(d,d;\mathbb{Z}) \subset \mathrm{Pin}(d,d;\mathbb{Z})$ and take us from IIA to IIB and vice versa. It will still be true that $h\widehat{\Gamma}h^{-1} \subset \mathrm{Spin}(d,d;\mathbb{Z})$.

with subgroup $h\widehat{\Gamma}h^{-1}$ and moduli given by $h^T H h$, etc. Note that in general the elements in $h\widehat{\Gamma}h^{-1}$ need not be block diagonal; the dual orientifold group can be an asymmetric orientifold.

We would like to actually use dualities as a solution generating technique. In this case we focus on elements which do not modify the orientifold group, i.e. the set of $h \in \mathrm{SO}(d,d;\mathbb{Z})$ that satisfy $h\widehat{\Gamma} = \widehat{\Gamma}h$. We can consider such h as simply a map on the moduli and fluxes.

Example

Let us see how this works in our example. There our orientifold is generated by $\hat{\Theta}$ and $\Omega\hat{\sigma}$, where

$$\Theta = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & 0 & 0 \\ 0 & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & 0 \\ 0 & 0 & -\mathbf{1}_2 \end{pmatrix}, \qquad \sigma = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} & 0 & 0 \\ 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & 0 \\ 0 & 0 & \begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix} \end{pmatrix}, \tag{2.172}$$

are both elements of $\mathrm{GL}(6;\mathbb{Z})$.

We would like to identify how to perform a T-duality on (for example) the third two-torus. Unfortunately, by just using (2.166), one finds that $\hat{\Theta}$ is invariant, but $\Omega\hat{\sigma}$ is not. However, one can repair this by combining the standard T-duality with a further rotation $(x_3, y_3) \mapsto (y_3, -x_3)$, defining instead the element

$$M_{T(3)} = \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \\ 0 & 0 & \mathbf{1}_4 & 0 \\ 0 & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & 0 & 0 \end{pmatrix}. \tag{2.173}$$

This version of T-duality does indeed preserve the full orientifold group.

On the NS-NS moduli one can check that it acts by sending $t_3 \mapsto -1/t_3$,

$e^{\phi} \mapsto e^{\phi}/|t_3|$, and all other moduli remain fixed. To get the action on the R-R fields, it is useful to decompose $M_{T(3)}$ as

$$M_{T(3)} = \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_4 & 0 \\ 0 & {\scriptstyle\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 & 0 & {\scriptstyle\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} \\ 0 & 0 & \mathbf{1}_4 & 0 \\ 0 & 0 & 0 & \mathbf{1}_2 \end{pmatrix} \begin{pmatrix} \mathbf{1}_4 & 0 & 0 & 0 \\ 0 & \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_4 & 0 \\ 0 & {\scriptstyle\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}} & 0 & \mathbf{1}_2 \end{pmatrix}. \tag{2.174}$$

Now if one writes, for example,

$$F_{RR} = F^{\perp} + dx_3 \wedge F^x + dy_3 \wedge F^y + dx_3 \wedge dy_3 \wedge F_{\|}, \tag{2.175}$$

one finds the T-duality relation,

$$M_{T(3)} \cdot F_{RR} = -F^{\|} + dx_3 \wedge F^x + dy_3 \wedge F^y + dx_3 \wedge dy_3 \wedge F^{\perp}. \tag{2.176}$$

As a consequence, we find that the ξ_K are invariant, while the R-R fluxes map according to

$$m_0' = -m_3, \qquad m_1' = -e_2, \qquad m_2' = -e_1, \qquad m_3' = m_0, \qquad m_4' = -e_4$$

$$e_1' = m_2, \qquad e_2' = m_1, \qquad e_3' = -e_0, \qquad e_4' = m_4, \qquad e_0' = e_3, \tag{2.177}$$

where primed quantities represent the fluxes in the T-dual solution and unprimed ones are from the original solution. We will discuss duality of NS-NS fluxes after introducing them in the next subsection.

We can now similarly introduce T-dualities $M_{T(1)}$ and $M_{T(2)}$, corresponding to dualizing either the first or second two-torus, by simply permuting the two-by-two blocks of $M_{T(3)}$. Under $M_{T(1)}$ we have $t_1 \mapsto -1/t_1$, $t_2 \mapsto t_2 - 2t_4^2/t_1$, $t_4 \mapsto -t_4/t_1$, and $e^\phi \mapsto e^\phi/|t_1|$, while

$$m'_0 = -m_1, \qquad m'_1 = m_0, \qquad m'_2 = -e_3, \qquad m'_3 = -e_2, \qquad m'_4 = m_4,$$

$$e'_1 = -e_0, \qquad e'_2 = m_3, \qquad e'_3 = m_2, \qquad e'_4 = e_4, \qquad e'_0 = e_1, \tag{2.178}$$

with all other moduli left invariant. The action of $M_{T(2)}$ can be obtained from $M_{T(1)}$ by interchanging one and two throughout.

2.3.2 NS-NS Fluxes

In order to get a feeling for how we would like to encode general NS-NS fluxes, let us start with the example of $T^6/\mathbb{Z}_4$ with only H-flux, from section 2.2.2.

Example

Here we have $H_3 = p_1 b_1 + p_2 b_2$. In order to represent this flux, let us first pick a trivialization that depends only on the coordinates x_1 and y_1 (these coordinates will then be our *base*). Our B-field is thus

$$\begin{aligned} B = p_1 &[-(x_1 - y_1)\, dx_2 \wedge dx_3 + y_1 dx_2 \wedge dy_3 + (x_1 + y_1)\, dy_2 \wedge dx_3 + x_1 dy_2 \wedge dy_3] \\ &+ p_2 [(x_1 - y_1)\, dx_2 \wedge dx_3 + x_1 dx_2 \wedge dy_3 - (x_1 + y_1)\, dy_2 \wedge dx_3 - y_1 dy_2 \wedge dy_3]. \end{aligned} \tag{2.179}$$

If we let E_0 be the combination of the metric and B-field at the point $x_1 = y_1 = 0$ (including values of the moduli t_a), then we can write $E(x_1, y_1) = g(x_1, y_1) \cdot E_0$, where $g(x_1, y_1)$ is a map of the base T^2 into $\mathrm{O}(4,4) \subset \mathrm{O}(6,6)$ given explicitly by

$$g(x_1,y_1) = \begin{pmatrix} \mathbf{1}_4 & \begin{matrix} 0 & 0 & (p_2-p_1)(x-y) & p_1y+p_2x \\ 0 & 0 & (p_1-p_2)(x+y) & p_1x-p_2y \\ (p_1-p_2)(x-y) & (p_2-p_1)(x+y) & 0 & 0 \\ -p_1y-p_2x & -p_1x+p_2y & 0 & 0 \end{matrix} \\ 0 & \mathbf{1}_4 \end{pmatrix}$$
$$= \exp\left[xM_x + yM_y\right], \tag{2.180}$$

where we have suppressed the subscript 1 on x and y, and where in the final step we have defined

$$M_x = \begin{pmatrix} 0 & \begin{matrix} 0 & 0 & p_2-p_1 & p_2 \\ 0 & 0 & p_1-p_2 & p_1 \\ p_1-p_2 & p_2-p_1 & 0 & 0 \\ -p_2 & -p_1 & 0 & 0 \end{matrix} \\ 0 & 0 \end{pmatrix}, \qquad M_y = \begin{pmatrix} 0 & \begin{matrix} 0 & 0 & p_1-p_2 & p_1 \\ 0 & 0 & p_1-p_2 & -p_2 \\ p_2-p_1 & p_2-p_1 & 0 & 0 \\ -p_1 & p_2 & 0 & 0 \end{matrix} \\ 0 & 0 \end{pmatrix}, \tag{2.181}$$

which are mutually commuting constant elements of the Lie algebra $\mathfrak{so}(4,4)$. Note that the map g is not single-valued, but that upon going around a closed cycle in the base the transformation needs to be a symmetry, i.e. we must have

$$g(n,m) \in \mathrm{O}(4,4;\mathbb{Z}), \forall n,m \in \mathbb{Z} \qquad \Leftrightarrow$$
$$\exp\left(M_x\right) \in \mathrm{O}(4,4;\mathbb{Z}),\ \exp\left(M_y\right) \in \mathrm{O}(4,4;\mathbb{Z}). \tag{2.182}$$

This is indeed satisfied by these matrices for integer values of p_I (both satisfy $M^2 = 0$ and hence $\exp(M) = 1 + M$).

We see that $g(x_1, y_1)$, or equivalently M_x and M_y, encodes our H-fluxes. What about metric fluxes? We would like to see how these fluxes map when we T-dualize on the third two-torus. Since we know how the metric and B-field transform, we have

$$M_{T(3)} \cdot E(x,y) = \left(M_{T(3)} g(x,y) M_{T(3)}^{-1}\right) \cdot \left(M_{T(3)} \cdot E_0\right). \tag{2.183}$$

So we see that we should replace our twist $g(x_1, y_1)$ by a new twist in $\mathrm{O}(4,4) \subset$

$O(6,6)$,

$$\begin{aligned} g'(x,y) &= M_{T(3)} g M_{T(3)}^{-1} \\ &= \begin{pmatrix} \mathbf{1}_2 & \begin{smallmatrix} p_1y+p_2x & (p_1-p_2)(x-y) \\ p_1x-p_2y & (p_2-p_1)(x+y) \end{smallmatrix} & 0 & 0 \\ 0 & \mathbf{1}_2 & 0 & 0 \\ 0 & 0 & \mathbf{1}_2 & 0 \\ 0 & 0 & \begin{smallmatrix} -p_1y-p_2x & -p_1x+p_2y \\ (p_2-p_1)(x-y) & (p_1-p_2)(x+y) \end{smallmatrix} & \mathbf{1}_2 \end{pmatrix}, \end{aligned} \tag{2.184}$$

or equivalently,

$$M'_x = \begin{pmatrix} 0 & \begin{smallmatrix} p_2 & p_1-p_2 \\ p_1 & p_2-p_1 \end{smallmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{smallmatrix} -p_2 & -p_1 \\ p_2-p_1 & p_1-p_2 \end{smallmatrix} & 0 \end{pmatrix}, M'_y = \begin{pmatrix} 0 & \begin{smallmatrix} p_1 & p_2-p_1 \\ -p_2 & p_2-p_1 \end{smallmatrix} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{smallmatrix} -p_1 & p_2 \\ p_1-p_2 & p_1-p_2 \end{smallmatrix} & 0 \end{pmatrix}. \tag{2.185}$$

Again these are two commuting elements of $\mathfrak{so}(4,4)$ which exponentiate to elements of $O(4,4;\mathbb{Z})$, though this time rather than shifting the B-field, they act as diffeomorphisms of the fibered T^4. Note that if we write

$$g'(x,y) = \begin{pmatrix} \left(h^T\right)^{-1} & 0 \\ 0 & h \end{pmatrix}, \qquad h \in SL(4), \tag{2.186}$$

then we have

$$\eta^i = \left(h^{-1}\right)^i_{\ j} dx^j = \left(h^{-1}\right)^* dx^i. \tag{2.187}$$

These are the proper, globally-defined one-forms, since as we traverse the base, we are forced to transport our fiber one-forms by the map g'.

In the case just described, we can then compute the metric flux components, namely

$$\omega^5_{13} = -\omega^5_{24} = -p_2,$$
$$\omega^5_{14} = \omega^5_{23} = -p_1,$$
$$\omega^6_{13} = -\omega^6_{14} = -\omega^6_{23} = -\omega^6_{24} = p_2 - p_1, \tag{2.188}$$

or in terms of an r-matrix (compare with (2.108) and (2.109))

$$r = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -p_1 & -p_2 \\ 0 & 0 \end{pmatrix}, \qquad \widehat{r} = 0. \tag{2.189}$$

Thus we see that this particular T-duality has simply sent $p_K \mapsto -r_{3K}$ (also there is no H-flux in this new solution described by g').

By combining this map with the map of moduli and R-R fluxes under T-duality from the previous subsection, one can verify that the solution of section 2.2.2 and that of (2.120) are precisely T-dual to each other.

Let us now perform one more T-duality using $M_{T(2)}$. This sends us to

$$M''_x = \begin{pmatrix} 0 & 0 \\ \begin{smallmatrix} 0 & 0 & p_1 & p_2-p_1 \\ 0 & 0 & -p_2 & p_2-p_1 \\ -p_1 & p_2 & 0 & 0 \\ p_1-p_2 & p_1-p_2 & 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}, \qquad M''_y = \begin{pmatrix} 0 & 0 \\ \begin{smallmatrix} 0 & 0 & -p_2 & p_2-p_1 \\ 0 & 0 & -p_1 & p_1-p_2 \\ p_2 & p_1 & 0 & 0 \\ p_1-p_2 & p_2-p_1 & 0 & 0 \end{smallmatrix} & 0 \end{pmatrix}. \tag{2.190}$$

This will correspond to nongeometric Q-flux. We will argue below in the general case how one should convert these to particular components of Q-flux; for now we merely state the results.

$$Q^{35}_1 = -Q^{45}_2 = -p_1,\ Q^{36}_1 = Q^{36}_2 = Q^{46}_1 = -Q^{46}_2 = p_1-p_2,\ Q^{35}_2 = Q^{45}_1 = p_2, \tag{2.191}$$

or

$$q = \begin{pmatrix} p_1 & p_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \widehat{q} = 0. \tag{2.192}$$

By applying the $M_{T(2)}$ map to the moduli and fluxes of (2.120) we can generate a new solution with only Q-flux (no H-flux or metric flux). It is straightforward to check that the F-term equations are satisfied.

General situation

Let us attempt to generalize this situation. Let $\widehat{\Gamma}$ be the subgroup of $\mathrm{Spin}(6,6;\mathbb{Z})$ which generates our orientifold group, and suppose we have a splitting of our T^6 into a base of dimension n and a fiber of dimension $6-n$ such that $\widehat{\Gamma}$ acts block diagonally (i.e. such that both the base and fiber form real representations of the orientifold group, which is D_4 in our example). We will also assume that $\widehat{\Gamma}$ acts symmetrically on the base, with each element giving rise to a $\mathrm{GL}(n;\mathbb{Z})$ action. Then for every element $h \in \widehat{\Gamma}$ we can decompose

$$h = h_b \oplus h_f \in \mathrm{GL}(n;\mathbb{Z}) \times \mathrm{Spin}(6-n, 6-n;\mathbb{Z}) \subset \mathrm{Spin}(6,6;\mathbb{Z}). \tag{2.193}$$

We would like to classify the elements $g(\vec{x}_b) \in \mathrm{SO}(6-n, 6-n) \subset \mathrm{SO}(6,6)$, depending on the base coordinates $\vec{x}_b$, by which we can twist our fibers. Such twists will have to satisfy a number of conditions.

First of all, they need to be invariant under the orientifold group action, i.e. we require

$$h_f g(\vec{x}_b) h_f^{-1} = g(h_b \cdot \vec{x}_b), \qquad \forall h \in \widehat{\Gamma}. \tag{2.194}$$

Secondly, we require path independence, in the sense that moving in different directions in the base should commute, i.e.

$$[\partial_i g(\vec{x}_b), \partial_j g(\vec{x}_b)]\,, \qquad \forall i, j \in \{1, \ldots, n\}. \tag{2.195}$$

And finally, moving around a closed path must correspond to an element of the duality group, i.e.

$$g(\vec{x}_b + \vec{\lambda})g(\vec{x}_b)^{-1} \in \mathrm{SO}(6,6;\mathbb{Z}), \qquad \forall \vec{\lambda} \in \mathbb{Z}^6,\ \vec{x}_b \in \mathbb{R}^n \subset \mathbb{R}^6. \tag{2.196}$$

A very natural simplifying ansatz for the form of $g(\vec{x}_b)$ is to take

$$g(\vec{x}_b) = \exp\left[\vec{x}_b \cdot \vec{M}\right], \tag{2.197}$$

where each component of $\vec{M}$ is an element of the Lie algebra $\mathfrak{so}(6-n, 6-n)$. With this ansatz, the conditions above become

$$h_f M_i h_f^{-1} = (h_b)^j{}_i M_j, \qquad \forall i, \quad \forall h \in \widehat{\Gamma}, \tag{2.198}$$

$$[M_i, M_j] = 0, \qquad \forall i, j, \tag{2.199}$$

and a quantization condition

$$\exp\left[\lambda^i M_i\right] \in \mathrm{SO}(6,6;\mathbb{Z}), \qquad \forall \vec{\lambda} \in \mathbb{Z}^6, \tag{2.200}$$

which can in general be a bit subtle if our base-fiber splitting is not a good splitting of the lattice. Even in those cases however, the correct quantization condition can be worked out without too much trouble. In the simpler case where the splitting does respect the lattice identifications, the quantization condition is simply that

$$\exp M_i \in \mathrm{SO}(6-n, 6-n;\mathbb{Z}), \qquad \forall i. \tag{2.201}$$

To understand how the matrices M_i translate into general NS-NS flux components, we will consider R-R fields which can be thought of as sections of the spin bundle $\mathrm{Spin}(6-n, 6-n)$. The map $g(\vec{x}_b)$ tells us how to transport sections of this bundle as we move around the base, providing us with the correct globally defined R-R fields. For instance, in the case with only H-flux, where $g(\vec{x}_b)$ consists solely of linear shifts in the B-field, we saw that the globally defined R-R-fluxes are given by $F_{RR} = \exp(B) \wedge F_{RR}^{(0)}$. In this case we have

$$dF_{RR} = \exp(B) \wedge \left(dF_{RR}^{(0)} + H \wedge F_{RR} \right) = \exp(B) \wedge d_H F_{RR}^{(0)}. \tag{2.202}$$

In other words, d_H is a covariant derivative for this bundle [3], and by differentiating our globally defined sections we can deduce the form of d_H and hence the components of H-flux. We will now show that the same story is true more generally. The globally defined R-R-fluxes are given by $g(\vec{x}_b) \cdot F_{RR}^{(0)}$, and

$$dF_{RR} = g(\vec{x}_b) \cdot \mathcal{D} F_{RR}^{(0)}. \tag{2.203}$$

This observation is what allows us to compute the flux components from the M_i.

To get a better feeling for these matters, let's look at some basic cases. Note that a general matrix in $\mathfrak{so}(6-n, 6-n)$ has the form

$$M = \begin{pmatrix} -A^T & B \\ C & A \end{pmatrix}, \tag{2.204}$$

where A is a general $(6-n) \times (6-n)$ matrix, and B and C are antisymmetric $(6-n) \times (6-n)$ matrices.

Suppose first that all of our M_i are nonvanishing only in the top-right block

B above, say[13]

$$M_i = \begin{pmatrix} 0 & (B_i)_{ab} \\ 0 & 0 \end{pmatrix} \qquad \Longrightarrow \qquad g(\vec{x}_b) = \begin{pmatrix} \mathbf{1} & \vec{x}_b \cdot \vec{B} \\ 0 & \mathbf{1} \end{pmatrix}. \tag{2.205}$$

Then

$$\begin{aligned} dF_{RR} &= d\left(\exp\left[\vec{x}_b \cdot \vec{B}\right] \wedge F_{RR}^{(0)}\right) \\ &= \exp\left[\vec{x}_b \cdot \vec{B}\right] \wedge \left(dF_{RR}^{(0)} + \frac{1}{2}\left(B_i\right)_{ab} dx^i \wedge dx^a \wedge dx^b \wedge F_{RR}^{(0)}\right), \end{aligned} \tag{2.206}$$

so

$$H_{iab} = \left(B_i\right)_{ab}. \tag{2.207}$$

Note that for a given base-fiber splitting we can only obtain H-flux with precisely one leg on the base.

Similarly, suppose that the M_i are all block diagonal,

$$M_i = \begin{pmatrix} -A_i^T & 0 \\ 0 & A_i \end{pmatrix} \qquad \Longrightarrow \qquad g(\vec{x}_b) = \begin{pmatrix} e^{-\vec{x}_b \cdot \vec{A}^T} & 0 \\ 0 & e^{\vec{x}_b \cdot \vec{A}} \end{pmatrix}. \tag{2.208}$$

Then

$$\begin{aligned} dF_{RR} = d\left(\exp\left[\frac{1}{2}\mathrm{Tr}\left(\vec{x}_b \cdot \vec{A}\right)\right]\left(e^{-\vec{x}_b \cdot \vec{A}}\right)^* F_{RR}^{(0)}\right) \\ = e^{\frac{1}{2}\mathrm{Tr}\left(\vec{x}_b \cdot \vec{A}\right)}\left(e^{-\vec{x}_b \cdot \vec{A}}\right)^*\left(dF_{RR}^{(0)} + \frac{1}{2}\mathrm{Tr}\left(\vec{A}\right) \cdot d\vec{x}_b \wedge F_{RR}^{(0)} - dx^i \wedge \left(A_i \cdot F_{RR}^{(0)}\right)\right), \end{aligned} \tag{2.209}$$

where A_i acts on a p-form via

$$A_i \cdot \zeta^{(p)} = \binom{p}{1}\left(A_i\right)^b{}_{[a_1}\zeta_{|b|a_2 \cdots a_p]}\frac{1}{p!}dx^{a_1} \wedge \cdots dx^{a_p}. \tag{2.210}$$

[13] We now start using conventions where i, j, etc. refer to base coordinates, while a, b, etc. refer to fiber coordinates.

Comparing with (B.3)[14], we deduce that

$$\omega_{ib}^{a} = (A_i)^a{}_b. \tag{2.211}$$

Again we find that ω must have exactly one lower index along the base, with the other two indices along the fiber. Note that here we do not require A_i to be traceless, though any nonvanishing trace piece would require a base one-form $\mathrm{Tr}(A_i)dx^i$ which would have to be invariant under the orientifold group.

And also,

$$M_i = \begin{pmatrix} 0 & 0 \\ (C_i)^{ab} & 0 \end{pmatrix} \qquad \Longrightarrow \qquad g(\vec{x}_b) = \begin{pmatrix} \mathbf{1} & 0 \\ \vec{x}_b \cdot \vec{C} & \mathbf{1} \end{pmatrix}. \tag{2.212}$$

So from

$$\begin{aligned} dF_{RR} &= d\left(\exp\left[\frac{1}{2}x^i C_i^{ab}\iota_a\iota_b\right]\cdot F_{RR}^{(0)}\right) \\ &= \exp\left[\frac{1}{2}x^i C_i^{ab}\iota_a\iota_b\right]\cdot\left(dF_{RR}^{(0)} + \frac{1}{2}C_i^{ab}dx^i \wedge \left(\iota_a\iota_b F_{RR}^{(0)}\right)\right), \end{aligned} \tag{2.213}$$

we find

$$Q_i^{ab} = -\left(C_i\right)^{ab}. \tag{2.214}$$

Once again, the lower index must be on the base, while the other two (upper) indices lie along the fiber.

Finally, since the exponent of g is linear in the base coordinates, these derivatives simply add, and we find that the map between the matrices M_i and the fluxes is simply,

$$M_i = \begin{pmatrix} -\omega_{ia}^{b} & H_{iab} \\ -Q_i^{ab} & \omega_{ib}^{a} \end{pmatrix}. \tag{2.215}$$

[14] In doing such a comparison, we may assume that the components of $F_{RR}^{(0)}$ are constant, so $dF_{RR}^{(0)} = 0$.

Let us see what we can learn from the constraints (2.198) and (2.199). Consider an element of $\widehat{\Gamma}$ of the form

$$h = \Omega\hat{\sigma} = \begin{pmatrix} -\left(\sigma^T\right)^{-1} & 0 \\ 0 & \sigma \end{pmatrix} = \sigma_b \oplus \begin{pmatrix} -\left(\sigma_f^T\right)^{-1} & 0 \\ 0 & \sigma_f \end{pmatrix}. \tag{2.216}$$

Then substituting (2.215) into (2.198) leads to

$$-\left(\sigma_f^T\right)^{-1} H_i\sigma_f^{-1} = \left(\sigma_b\right)^j{}_i H_j,\ \sigma_f Q_i \sigma_f^T = -\left(\sigma_b\right)^j{}_i Q_j,\ \sigma_f \omega_i \sigma_f^{-1} = \left(\sigma_b\right)^j{}_i \omega_j, \tag{2.217}$$

which can be rephrased as the statement that the metric fluxes ω should be even under the involution σ, while the H- and Q-fluxes should both be odd under σ.

Now substituting (2.215) into (2.199) leads to the conditions

$$\omega^a_{c[i}\omega^c_{j]b} + Q^{ac}_{[i}H_{j]cb} = 0, \qquad H_{ac[i}\omega^c_{j]b} - H_{bc[i}\omega^c_{j]a} = 0, \qquad Q^{c[a}_{[i}\omega^{b]}_{j]c} = 0. \tag{2.218}$$

But it is easy to check that these are precisely the Bianchi identities (B.2) for the situation at hand, namely when each flux has exactly one lower index on the base and all other indices lie along the fiber.

We would like to discuss the quantization condition (2.200), but it is quite complicated in the general case, so let us first see how these base-fiber constructions work in our favorite example.

2.3.3 Example

To classify the possible base-fiber splittings of our $T^6/\mathbb{Z}_4$ orientifold, we need to know how the coordinates of the T^6 split into representations of the orientifold group D_4. As a real vector space (i.e. forgetting the shift identifications of the torus), it can be checked that this $\mathbb{R}^6$ splits into two isomorphic two-dimensional irreducible real representations and two one-dimensional real representations which are not isomorphic. The latter two are given by the span of y_3 and the span of $\hat{x}_3 = x_3 + \frac{1}{2}y_3$. Because of the isomorphism between the two-dimensional representations, there is a two real parameter family of ways to split up the first four coordinates into irreducible real representations. Indeed, if we define

$$\begin{aligned} \hat{x}_1 &= x_1 + a\,(x_2 + y_2)\,, & \hat{y}_1 &= y_1 + a\,(-x_2 + y_2)\,, \\ \hat{x}_2 &= b\,(x_1 - y_1) + x_2, & \hat{y}_2 &= b\,(x_1 + y_1) + y_2, \end{aligned} \tag{2.219}$$

then $\{\hat{x}_1, \hat{y}_1\}$ can be taken to span one invariant subspace, while $\{\hat{x}_2, \hat{y}_2\}$ span the other. The only constraint is that $2ab \neq 1$, so that this change of basis is invertible.

We can now classify all of the possible bases, dimension by dimension.

One-dimensional bases

Here there are two cases; either the base is parametrized by y_3, or by $\hat{x}_3 = x_3 + \frac{1}{2}y_3$. Suppose that the base is y_3. Invariance under Θ^2 ensures that M_{y_3} has the form

$$M_{y_3} = \begin{pmatrix} A & 0 & B & 0 \\ 0 & a & 0 & 0 \\ C & 0 & -A^T & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}, \tag{2.220}$$

for 4×4 matrices A, B, C, and a real number a. But now invariance under Θ implies that $a = 0$, and that M_{y_3} in fact lies in $\mathfrak{so}(4,4)$. But then this one-dimensional case is really a restriction of the case with two-dimensional base T^2_3 where only y_3 dependence is allowed (i.e. $M_{\hat{x}_3} = 0$). This case is treated below without restriction.

Since we did not use the action of σ in the argument above, and since this action is the only difference between y_3 and $\hat{x}_3$, we conclude that an $\hat{x}_3$ base also gives nothing new.

Two-dimensional bases

Here we consider four-dimensional fibers and two-dimensional bases. There are three possibilities.

1) <u>T^2_1 base</u>

In this case the base is spanned by $\{\hat{x}_1, \hat{y}_1\}$, and the fiber is spanned by $\{\hat{x}_2, \hat{y}_2, x_3, y_3\}$. Our orientifold group is represented as

$$\Theta_b = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.221}$$

$$\hat{\Theta}_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{smallmatrix} \end{pmatrix}, \qquad (\Omega\hat{\sigma})_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{smallmatrix} \end{pmatrix}. \tag{2.222}$$

The constraints (2.198) imply that

$$\hat{\Theta}^2_f M_{\hat{x}_1} \hat{\Theta}^2_f = -M_{\hat{x}_1} \qquad \Longrightarrow \qquad M_{\hat{x}_1} = \begin{pmatrix} 0 & a & 0 & c \\ b & 0 & -c^T & 0 \\ 0 & d & 0 & -b^T \\ -d^T & 0 & -a^T & 0 \end{pmatrix}, \tag{2.223}$$

where a, b, c, and d are two-by-two matrices. Next,

$$(\Omega\hat{\sigma})_f \, M_{\hat{x}_1} \, (\Omega\hat{\sigma})_f = M_{\hat{x}_1} \Longrightarrow M_{\hat{x}_1} = \begin{pmatrix} 0 & \begin{smallmatrix} \alpha & \beta \\ \alpha+\beta & -\beta \end{smallmatrix} & 0 & \begin{smallmatrix} \varepsilon & \varphi \\ -\varepsilon & \varphi-\varepsilon \end{smallmatrix} \\ \begin{smallmatrix} \gamma & \gamma \\ \delta & \gamma-\delta \end{smallmatrix} & 0 & \begin{smallmatrix} -\varepsilon & \varepsilon \\ -\varphi & \varepsilon-\varphi \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} \chi & \kappa \\ -\chi-\kappa & \kappa \end{smallmatrix} & 0 & \begin{smallmatrix} -\gamma & -\delta \\ -\gamma & \delta-\gamma \end{smallmatrix} \\ \begin{smallmatrix} -\chi & \chi+\kappa \\ -\kappa & -\kappa \end{smallmatrix} & 0 & \begin{smallmatrix} -\alpha & -\alpha-\beta \\ -\beta & \beta \end{smallmatrix} & 0 \end{pmatrix}. \tag{2.224}$$

Then all of the constraints (2.198) are satisfied if we define

$$M_{\hat{y}_1} = \hat{\Theta}_f M_{\hat{x}_1} \hat{\Theta}_f^{-1} = \begin{pmatrix} 0 & \begin{smallmatrix} \alpha+\beta & -\beta \\ -\alpha & -\beta \end{smallmatrix} & 0 & \begin{smallmatrix} -\varepsilon & \varphi-\varepsilon \\ -\varepsilon & -\varphi \end{smallmatrix} \\ \begin{smallmatrix} \gamma & -\gamma \\ \gamma-\delta & -\delta \end{smallmatrix} & 0 & \begin{smallmatrix} \varepsilon & \varepsilon \\ \varepsilon-\varphi & \varphi \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} -\chi-\kappa & \kappa \\ -\chi & -\kappa \end{smallmatrix} & 0 & \begin{smallmatrix} -\gamma & \delta-\gamma \\ \gamma & \delta \end{smallmatrix} \\ \begin{smallmatrix} \chi+\kappa & \chi \\ -\kappa & \kappa \end{smallmatrix} & 0 & \begin{smallmatrix} -\alpha-\beta & \alpha \\ \beta & \beta \end{smallmatrix} & 0 \end{pmatrix}. \tag{2.225}$$

By imposing the requirement that these matrices commute, we find three extra conditions, namely

$$\beta\gamma + \varepsilon\kappa = 0, \qquad \alpha\gamma + \beta\delta = \varepsilon\chi + \varphi\kappa, \qquad (\alpha+\beta)\,\delta + (\varphi - \varepsilon)\,\chi = 0. \tag{2.226}$$

From the entries of $M_{\hat{x}_1}$ and $M_{\hat{y}_1}$ one can read off the flux components in that basis. One then uses the transformation (2.219) to convert these fluxes back to the lattice compatible basis from before. The resulting fluxes are

$$p = \Delta \begin{pmatrix} \varphi - \varepsilon & \varphi \end{pmatrix},$$

$$r = \begin{pmatrix} 2a^2\Delta^{-1}\delta & 2a^2\Delta^{-1}(\delta-\gamma) \\ \Delta^{-1}\delta & \Delta^{-1}(\delta-\gamma) \\ -\Delta(\alpha+\beta) & -\Delta\alpha \\ 4a\Delta^{-1}\delta & 4a\Delta^{-1}(\delta-\gamma) \end{pmatrix}, \qquad q = \Delta^{-1} \begin{pmatrix} \chi & \chi+\kappa \\ 2a^2\chi & 2a^2(\chi+\kappa) \\ 0 & 0 \\ 4a\chi & 4a(\chi+\kappa) \end{pmatrix}, \tag{2.227}$$

with $\widehat{r} = \widehat{q} = s = 0$, and where a and $\Delta = 1 - 2ab$ are the parameters of the basis transformation. With these definitions, one can check that the constraints (2.226) precisely reproduce the Bianchi identities (2.157) for this case.

Now unless a, b, and Δ are integers, the basis in which the matrices above are expressed is not a basis for our lattice, and so generally the quantization condition is not just that $\exp[M_{\hat{x}_1}]$ and $\exp[M_{\hat{y}_1}]$ are integers. Instead, what we should do is embed these matrices into $\mathfrak{so}(6,6)$, undo the transformation (2.219), and then exponentiate. Following this procedure we find 12×12 matrices M_1, M_2, as well as $M_3 = a(M_1 - M_2)$ and $M_4 = a(M_1 + M_2)$. All four of these matrices turn out to be (three-step) nilpotent, and hence the quantization conditions $\exp[M_i] \in \mathrm{SO}(6,6;\mathbb{Z})$ are simply that the entries of the M_i be integers. Translating back into the matrices above, we learn that the correct quantization condition for this case is nearly the naive one (in fact it is the naive quantization condition in terms of the flux components, ω^i_{jk}, q^{ij}_k, etc.); we must have p_K, r_{cK} and q_{cK} to be integers, but in addition we require $2ar_{3K}$, $2ap_K$, $a(r_{31} - r_{32})$, and $a(p_1 - p_2)$ to be integers. In particular, if a is an integer (for instance if the transformed basis is a lattice basis), then the naive integer quantization is correct.

Let us take a moment and consider the types of solutions that we get if we restrict to the case $q = 0$ ($\chi = \kappa = 0$). Then the Bianchi identities (2.226) force either $\alpha = \beta = 0$, or $\gamma = \delta = 0$. Either way, we are stuck with an r-matrix of rank one, and, following our discussion in section 2.2.3, we cannot stabilize all of the moduli.

2) $\underline{T^2_2 \text{ base}}$

Here we take our base to be spanned by $\{\hat{x}_2, \hat{y}_2\}$. This case works out almost identically to the case described in detail above. In fact, the expression for $M_{\hat{x}_2}$ is precisely the same as that for $M_{\hat{x}_1}$ in (2.224), while $M_{\hat{y}_2}$ is the same as $M_{\hat{y}_1}$ in (2.225). As such, the Bianchi identities are again simply the three equations in (2.226). What does change slightly is the map back to our flux matrices. For this case we have

$$p = \Delta \begin{pmatrix} \varepsilon - \varphi & -\varphi \end{pmatrix},$$

$$r = \begin{pmatrix} -\Delta^{-1}\delta & \Delta^{-1}(\gamma-\delta) \\ -2b^2\Delta^{-1}\delta & 2b^2\Delta^{-1}(\gamma-\delta) \\ \Delta(\alpha+\beta) & \Delta\alpha \\ -4b\Delta^{-1}\delta & 4b\Delta^{-1}(\gamma-\delta) \end{pmatrix}, \qquad q = -\Delta^{-1}\begin{pmatrix} 2b^2\chi & 2b^2(\chi+\kappa) \\ \chi & \chi+\kappa \\ 0 & 0 \\ 4b\chi & 4b(\chi+\kappa) \end{pmatrix}, \tag{2.228}$$

and $\widehat{r} = \widehat{q} = s = 0$. The quantization conditions are exactly as before but with a replaced by b wherever it occurs.

3) T_3^2 base

Finally there is the case in which our base is spanned by $\{x_3, y_3\}$. This case turns out to be richer than the previous cases. The representation of the orientifold group is

$$\Theta_b = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_b = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \tag{2.229}$$

$$\hat{\Theta}_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{smallmatrix} \end{pmatrix}, \qquad (\Omega\hat{\sigma})_f = \begin{pmatrix} \begin{smallmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{smallmatrix} \end{pmatrix}. \tag{2.230}$$

Solving the constraints

$$\begin{aligned} \hat{\Theta}_f M_{x_3} \hat{\Theta}_f^{-1} &= -M_{x_3}, \quad (\Omega\hat{\sigma})_f M_{x_3} (\Omega\hat{\sigma})_f = M_{x_3}, \\ \hat{\Theta}_f M_{y_3} \hat{\Theta}_f^{-1} &= -M_{y_3}, \quad (\Omega\hat{\sigma})_f M_{y_3} (\Omega\hat{\sigma})_f = M_{x_3} - M_{y_3}, \end{aligned} \tag{2.231}$$

leads to a twelve-parameter family of solutions,

$$
\begin{aligned}
M_{x_3} &= \begin{pmatrix}
\begin{smallmatrix}\alpha & 0\\ 0 & -\alpha\end{smallmatrix} & \begin{smallmatrix}\gamma & \gamma\\ \gamma & -\gamma\end{smallmatrix} & 0 & \begin{smallmatrix}\varepsilon & -\varepsilon\\ -\varepsilon & -\varepsilon\end{smallmatrix}\\
\begin{smallmatrix}\beta & \beta\\ \beta & -\beta\end{smallmatrix} & \begin{smallmatrix}0 & \delta\\ \delta & 0\end{smallmatrix} & \begin{smallmatrix}-\varepsilon & \varepsilon\\ \varepsilon & \varepsilon\end{smallmatrix} & 0\\
0 & \begin{smallmatrix}\nu & -\nu\\ -\nu & -\nu\end{smallmatrix} & \begin{smallmatrix}-\alpha & 0\\ 0 & \alpha\end{smallmatrix} & \begin{smallmatrix}-\beta & -\beta\\ -\beta & \beta\end{smallmatrix}\\
\begin{smallmatrix}-\nu & \nu\\ \nu & \nu\end{smallmatrix} & 0 & \begin{smallmatrix}-\gamma & -\gamma\\ -\gamma & \gamma\end{smallmatrix} & \begin{smallmatrix}0 & -\delta\\ -\delta & 0\end{smallmatrix}
\end{pmatrix}, \qquad (2.232)\\
M_{y_3} &= \begin{pmatrix}
\begin{smallmatrix}\alpha/2 & \varphi\\ \varphi & -\alpha/2\end{smallmatrix} & \begin{smallmatrix}\kappa+\gamma/2 & -\kappa+\gamma/2\\ -\kappa+\gamma/2 & -\kappa-\gamma/2\end{smallmatrix} & 0 & \begin{smallmatrix}\mu+\varepsilon/2 & \mu-\varepsilon/2\\ \mu-\varepsilon/2 & -\mu-\varepsilon/2\end{smallmatrix}\\
\begin{smallmatrix}\chi+\beta/2 & -\chi+\beta/2\\ -\chi+\beta/2 & -\chi-\beta/2\end{smallmatrix} & \begin{smallmatrix}\lambda & \delta/2\\ \delta/2 & -\lambda\end{smallmatrix} & \begin{smallmatrix}-\mu-\varepsilon/2 & -\mu+\varepsilon/2\\ -\mu+\varepsilon/2 & \mu+\varepsilon/2\end{smallmatrix} & 0\\
0 & \begin{smallmatrix}\pi+\nu/2 & \pi-\nu/2\\ \pi-\nu/2 & -\pi-\nu/2\end{smallmatrix} & \begin{smallmatrix}-\alpha/2 & -\varphi\\ -\varphi & \alpha/2\end{smallmatrix} & \begin{smallmatrix}-\chi-\beta/2 & \chi-\beta/2\\ \chi-\beta/2 & \chi+\beta/2\end{smallmatrix}\\
\begin{smallmatrix}-\pi-\nu/2 & -\pi+\nu/2\\ -\pi+\nu/2 & \pi+\nu/2\end{smallmatrix} & 0 & \begin{smallmatrix}-\kappa-\gamma/2 & \kappa-\gamma/2\\ \kappa-\gamma/2 & \kappa+\gamma/2\end{smallmatrix} & \begin{smallmatrix}-\lambda & -\delta/2\\ -\delta/2 & \lambda\end{smallmatrix}
\end{pmatrix}.
\end{aligned}
$$

Note that the solution is much simpler in terms of the matrix $M_{\hat{y}_3} = M_{y_3} - \frac{1}{2}M_{x_3}$, but that the solution as given corresponds to the basis for the lattice.

Enforcing $[M_{x_3}, M_{y_3}] = 0$ gives six equations

$$
\begin{gathered}
(\alpha+\delta)\,\chi - (\varphi-\lambda)\,\beta = (\alpha+\delta)\,\pi + (\varphi-\lambda)\,\nu = 0,\\
(\alpha+\delta)\,\kappa - (\varphi-\lambda)\,\gamma = (\alpha+\delta)\,\mu + (\varphi-\lambda)\,\varepsilon = 0, \qquad (2.233)\\
\alpha\varphi - \gamma\chi - \beta\kappa - \mu\nu - \varepsilon\pi = 0, \qquad \alpha\varphi + \delta\lambda = 0.
\end{gathered}
$$

Translating into flux matrices, we find

$$
p = \begin{pmatrix}\mu - \varepsilon/2 & \mu + \varepsilon/2\end{pmatrix}, \qquad (2.234)
$$

$$
r = \begin{pmatrix}
\chi+\beta/2 & \chi-\beta/2\\
-\kappa-\gamma/2 & -\kappa+\gamma/2\\
0 & 0\\
-\varphi-\lambda+\frac{1}{2}(\alpha-\delta) & -\varphi-\lambda-\frac{1}{2}(\alpha-\delta)
\end{pmatrix},\ q = \begin{pmatrix}
0 & 0\\
0 & 0\\
-\pi+\nu/2 & -\pi-\nu/2\\
0 & 0
\end{pmatrix},
$$

$$
\widehat{r} = \begin{pmatrix}\varphi - \lambda - \frac{1}{2}(\alpha+\delta) & -\varphi + \lambda - \frac{1}{2}(\alpha+\delta)\end{pmatrix}, \qquad \widehat{q} = s = 0.
$$

In the case $q = 0$ ($\pi = \nu = 0$), this provides the complete case (b) of section 2.2.3.

One can easily verify that (2.233) gives the correct set of Bianchi identities for this case. In fact the solution to these equations can be broken into four cases,

$$\begin{aligned}
&\text{(i)} && M_{x_3} = 0, \\
&\text{(ii)} && M_{y_3} = \tfrac{1}{2} M_{x_3}, \\
&\text{(iii)} && \alpha + \delta = \varphi - \lambda = 0, \\
& && \alpha\varphi - \gamma\chi - \beta\kappa - \mu\nu - \varepsilon\pi = 0, \\
&\text{(iv)} && \alpha + \delta \neq 0,\ \varphi - \lambda \neq 0, \\
& && \chi = \left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)\beta,\ \pi = -\left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)\nu,\ \kappa = \left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)\gamma,\ \mu = -\left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)\varepsilon, \\
& && \alpha - \delta = \pm\left[(\alpha+\delta)^2 - 8\beta\gamma + 8\varepsilon\nu\right]^{1/2},\ \varphi + \lambda = -\left(\tfrac{\varphi-\lambda}{\alpha+\delta}\right)(\alpha - \delta).
\end{aligned} \tag{2.235}$$

In terms of flux matrices, case (i) has $p_1 = p_2$, $r_{a1} = r_{a2}$, $q_{31} = q_{32}$, and $\widehat{r}_1 = -\widehat{r}_2$. Case (ii) corresponds to $p_1 = -p_2$, $r_{a1} = -r_{a2}$, $q_{31} = -q_{32}$, and $\widehat{r}_1 = \widehat{r}_2$. Case (iii) is simply $\widehat{r} = 0$, with the other components arbitrary (up to one additional Bianchi identity). And case (iv) is the case with arbitrary $\widehat{r}$, but where the conditions $\widehat{r}_K p_K = \widehat{r}_K r_{aK} = \widehat{r}_K q_{3K} = 0$ put constraints on the other fluxes.

In every case we must finally solve the quantization conditions

$$\exp[M_{x_3}], \exp[M_{y_3}] \in \mathrm{SO}(4,4;\mathbb{Z}). \tag{2.236}$$

In each of the four cases this condition is potentially nontrivial because at least one of the two matrices may not be nilpotent. For example, the general expression for the exponentiated version of M_{x_3} includes entries such as

$$e^{\frac{1}{2}(\alpha+\delta)}\left[\cosh\frac{C}{2} + \frac{\alpha-\delta}{C}\sinh\frac{C}{2}\right], \quad \text{or} \quad \frac{2\beta}{C} e^{\frac{1}{2}(\alpha+\delta)}\sinh\frac{C}{2}, \tag{2.237}$$

and many others, where

$$C = \sqrt{(\alpha-\delta)^2 + 8\beta\gamma - 8\varepsilon\nu}. \tag{2.238}$$

Finding the generic situation in which all of these entries are integers is quite difficult. Let us specialize somewhat.

To make contact with the work we did in section 2.2.3, we will focus on case (iii), where $\widehat{r} = 0$, and assume also that $q = 0$. Under what conditions would the naive, integral quantization be correct? The requirement would be that both M_{x_3} and M_{y_3} would have to be nilpotent, and this in turn requires that two expressions vanish,

$$\alpha^2 + 2\beta\gamma = 0, \qquad \varphi^2 + 2\kappa\chi = 0. \tag{2.239}$$

And it turns out that these equations, along with the extra Bianchi identity $\alpha\varphi - \beta\kappa - \gamma\chi = 0$, imply that the rank of r is one. Thus, by arguments in section 2.2, we cannot hope to stabilize all moduli. In particular, the numerical solution we presented at the end of section 2.2.3 is not correctly quantized. In fact, it is possible to find solutions to the quantization conditions which do give rise to an r-matrix of rank two and a superpotential which stabilizes all moduli. However, we have argued that such cases are not nilpotent, so the entries of the r-matrix are not integers and in fact are irrational numbers. But now we have a puzzle, since if all the NS-NS fluxes are irrational numbers, then it is clearly impossible to satisfy the tadpole condition with R-R flux integers!

One plausible solution is that we do not correctly understand the quantization of R-R fluxes in the presence of general NS-NS fluxes, and in particular in non-nilpotent cases where the NS-NS flux quantization is not the naive one. One approach to this problem would involve viewing both NS-NS and R-R fluxes as twists in a U-duality group of the fiber, in which case understanding the full quantization conditions would simply reduce to understanding the structure of the duality group, e.g. $E_{7(7)}(\mathbb{Z})$. This is an avenue of ongoing investigation.

Three-dimensional bases

There are four possible bases in this case, but it will turn out that they are all contained in previously considered examples, so we will focus just on the case with base $\{\hat{x}_1, \hat{y}_1, \hat{x}_3\}$. The other three cases (with either or both of $\{\hat{x}_2, \hat{y}_2\}$ or $\hat{y}_3$) are similar.

Here we have

$$\Theta_b = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \sigma_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.240}$$

$$\hat{\Theta}_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & & & \\ & -1 & & \\ & & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & \\ & & & -1 \end{pmatrix}, \qquad (\Omega\hat{\sigma})_f = \begin{pmatrix} \begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix} & & & \\ & 1 & & \\ & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & \\ & & & -1 \end{pmatrix}. \tag{2.241}$$

Solving our constraints, we find

$$M_{\hat{x}_1} = \begin{pmatrix} 0 & \begin{smallmatrix} \gamma \\ -\gamma \end{smallmatrix} & 0 & \begin{smallmatrix} -\delta \\ -\delta \end{smallmatrix} \\ \begin{smallmatrix} \alpha & -\alpha \end{smallmatrix} & 0 & \begin{smallmatrix} \delta & \delta \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} -\beta \\ -\beta \end{smallmatrix} & 0 & \begin{smallmatrix} -\alpha \\ \alpha \end{smallmatrix} \\ \begin{smallmatrix} \beta & \beta \end{smallmatrix} & 0 & \begin{smallmatrix} -\gamma & \gamma \end{smallmatrix} & 0 \end{pmatrix}, \qquad M_{\hat{y}_1} = \begin{pmatrix} 0 & \begin{smallmatrix} -\gamma \\ -\gamma \end{smallmatrix} & 0 & \begin{smallmatrix} -\delta \\ \delta \end{smallmatrix} \\ \begin{smallmatrix} -\alpha & -\alpha \end{smallmatrix} & 0 & \begin{smallmatrix} \delta & -\delta \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} -\beta \\ \beta \end{smallmatrix} & 0 & \begin{smallmatrix} \alpha \\ \alpha \end{smallmatrix} \\ \begin{smallmatrix} \beta & -\beta \end{smallmatrix} & 0 & \begin{smallmatrix} \gamma & \gamma \end{smallmatrix} & 0 \end{pmatrix},$$

$$M_{\hat{x}_3} = \begin{pmatrix} \begin{smallmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{smallmatrix} & & & \\ & 0 & & \\ & & \begin{smallmatrix} 0 & -\varepsilon \\ -\varepsilon & 0 \end{smallmatrix} & \\ & & & 0 \end{pmatrix}. \tag{2.242}$$

Now let us check the Bianchi identities. It turns out that they give

$$\alpha\gamma = \beta\delta = 0, \quad \text{and} \quad \varepsilon\alpha = \varepsilon\beta = \varepsilon\gamma = \varepsilon\delta = 0. \tag{2.243}$$

But these then imply that either $M_{\hat{x}_3} = 0$, and we are in a special case of two-dimensional bases, or that $M_{\hat{x}_1} = M_{\hat{y}_1} = 0$, and we are in a special case of a one-dimensional base. Either way, we find nothing new.

The other three possible bases lead to the same conclusions.

Four-dimensional bases

Here there are three possibilities. If our base is given by $\{\hat{x}_1, \hat{y}_1, x_3, y_3\}$, then we would have

$$(-\mathbf{1})M_{\hat{x}_1}(-\mathbf{1}) = \hat{\Theta}_f^2 M_{\hat{x}_1} \hat{\Theta}_f^2 = \left(\Theta_b^2 \cdot M\right)_{\hat{x}_1} = -M_{\hat{x}_1} \qquad \Longrightarrow \qquad M_{\hat{x}_1} = 0, \tag{2.244}$$

and similaraly $M_{\hat{y}_1} = 0$, so our base is equivalent to just having $\{x_3, y_3\}$.

By the same argument, a base of $\{\hat{x}_2, \hat{y}_2, x_3, y_3\}$ would reduce to a previously considered case. This leaves only the possibility of $\{x_1, y_1, x_2, y_2\}$. But then we have, e.g.

$$(\mathbf{1})M_{x_1}(\mathbf{1}) = \hat{\Theta}_f^2 M_{x_1} \hat{\Theta}_f^2 = -M_{x_1}, \tag{2.245}$$

and this case in fact forces all $M_i = 0$.

Five-dimensional bases

This final case is also trivial for the same reasons discussed above. Invariance under Θ^2 forces four of the M_i to vanish, and invariance under Θ takes care of the fifth one.

2.3.4 Advantages and Puzzles

In this section we have presented a ten-dimensional construction of IIA toroidal orientifold models with some general NS-NS fluxes. The class of models which we can construct in this way is a sub-class of all the models discussed in section 2.2. It is not clear how representative a sample this sub-class is. For instance, in the case with only metric flux, the models we can construct do not necessarily correspond to nilpotent algebras, and hence are more general than those obtained by T-dualizing H-flux alone, but they are still a restricted set of algebras, and in particular are all solvable algebras. It would be interesting to compare these properties with the geometric properties evidenced, for example, in the classification of twisted tori in [45]. For our $T^6/\mathbb{Z}_4$ example we can construct nearly all possible metric fluxes (all of cases (a) and (b), but not the cases (a') or (a'')), but if non-geometric fluxes are included then only a fairly small fraction of possible models can be built in this way.

Having any kind of ten-dimensional construction, however, is obviously a huge advantage, as it gives us a great deal more confidence that our models can really arise from string theory compactifications (though we are certainly not claiming that other models can't be obtained from string theory, just not using the methods we have explored in this paper). It can also highlight important subtleties that were not so readily apparent from the effective theory approach. As we have seen, the quantization of general NS-NS fluxes is one such subtlety. In cases where our matrices M_i are all two-step nilpotent (which also implies that the underlying Lie algebra, as described in appendix B, is nilpotent) then the quantization condition is simply the naive one, with all flux components (which correspond to entries of the M_i) being integers. Matrices that are nilpotent after more than two-steps must have entries which are rationals (with denominator no larger than the number of steps minus one). More generally, however the condition is that certain exponentials of matrices be integral. These conditions often can be solved, giving irrational flux components (see also related discussions in [5]).

This in turn leads to a puzzle, since the integral tadpole contribution is given by a bilinear pairing of the NS-NS fluxes with the R-R fluxes. If the former are forced by quantization to be irrational numbers, then the latter cannot be integers or rationals, as was presumed. Either such setups are inconsistent or (more likely, in our belief) we have not correctly understood the quantization of R-R fluxes in general NS-NS flux backgrounds (see also the discussion in footnote 10). Presumably there should be some analog of twisted K-theory for the general flat fiber theories that we are studying, including also the non-geometric fluxes. Matching this onto the work of Mathai and collaborators [46, 47, 48, 49, 50, 35, 51, 52] would be very interesting. Similarly, exploiting the connections between the base-fiber approach described here and spaces with generalized complex structure (see e.g. [44]) could potentially lead to a better understanding of these more general classes of string compactifications. We are currently investigating these directions.

One advantage to following this base-fiber approach is that the constructions should be easy to generalize to any situation with a flat fiber over a flat base (and some aspects of the approach should be applicable to more general smooth bases). In particular, we should certainly be able to accommodate type IIB as well as IIA within the framework presented, and we can also work with orientifold actions which are asymmetric on the fiber (compare for instance with the models of [53]). Furthermore, heterotic string theory on a torus or type II on K3 fibers can also be covered in this framework, since the duality groups in those situations are well-understood. The K3 fibered case in particular could be very interesting, and could be compared to the work of [4]. Finally, one can expand the analysis to include U-duality groups, such as that of M-theory on a torus (see [54, 5, 55, 39, 40, 56, 57, 2]). It would be very interesting to understand how far one could push such a program, and whether one could find interesting solutions with a controlled low energy theory.

We have not yet incorporated any of the twisted sector physics into this story. Beyond getting possible hints by studying K3 fibers at their orbifold points, it would be extremely gratifying to have a more complete picture for how to deal with the twisted sector physics in the presence of fiber twists.

Chapter 3

Examples

3.1 Orientifold of $T^6/\mathbb{Z}_4$

We begin by introducing the basic setup and the construction of the orientifold model in section 3.1.1. Section 3.1.2 contains a detailed discussion of moduli stabilization via flux-induced potentials for the moduli of the untwisted sector. We present two different approaches to this problem: First, starting from ten-dimensional massive type IIA supergravity, we obtain the four-dimensional effective scalar potential by Kaluza-Klein reduction. Second, we solve supersymmetric F-flatness conditions in the language of four-dimensional $\mathcal{N} = 1$ supergravity, yielding supersymmetric AdS vacua. We then extend our considerations to the twisted sector moduli fields in section 3.1.3.

3.1.1 Basic setup

The $T^6/\mathbb{Z}_4$ orientifold

In this section, we outline the properties of the type IIA orientifold model under investigation, namely an orientifolded $T^6/\mathbb{Z}_4$ orbifold that preserves $\mathcal{N} = 1$ supersymmetry. A detailled discussion of this model can be found in [89].

The $T^6/\mathbb{Z}_4$ orbifold. As a first step, we want to compactify type IIA string theory on an $T^6/\mathbb{Z}_4$ orbifold background[1]. Let us start by describing the orbifold construction, following [89, 90]. It is important to use a lattice for the T^6 that implements a crystallographic action of the cyclic group. Therefore one chooses the root lattice of an appropriate Lie algebra. In the $\mathbb{Z}_4$ case under investigation the appropriate choice is $SU(2)^6$. Unlike the more complicated orbifolds with quotient group $\mathbb{Z}_N$ for $N > 6$ [92], in the case of $\mathbb{Z}_4$, the root lattice of the Lie algebra allows a choice of complex structure in such a way that the torus factorizes as $T^6 = T^2_{(1)} \times T^2_{(2)} \times T^2_{(3)}$. We parameterize it by three complex coordinates $z^i,\ i \in \{1, 2, 3\}$, together with the periodic identifications

$$z^i \sim z^i + \pi_{2i-1} \sim z^i + \pi_{2i},\ i \in \{1, 2, 3\}, \tag{3.1}$$

where the π_k denote the fundamental 1-cycles of the three 2-tori. The $\mathbb{Z}_4$ action on the torus T^6 is given by

$$\Theta : (z^1, z^2, z^3) \mapsto (\alpha z^1, \alpha z^2, \alpha^{-2} z^3), \tag{3.2}$$

where $\alpha = e^{\mathrm{i}\pi/2} = \mathrm{i}$ is a fourth root of unity and $\Theta^4 = \mathbb{1}$. This action preserves $\mathcal{N} = 2$ supersymmetry in four dimensions, implying that the orbifold is actually a singular limit of a Calabi-Yau 3-fold. The Hodge numbers are given by $h^{1,1} = 31$ and $h^{2,1} = 7$, yielding the number of Kähler and complex structure moduli before the orientifold projection. Table 1 lists how the complex structure and Kähler moduli appear in the different sectors of the orbifold.

[1]The $T^6/\mathbb{Z}_4$ orbifold is among those studied in [87, 88] and has been shown to admit consistent string propagation, e.g., preserving modular invariance.

sector:	untwisted	Θ, Θ^3-twisted	Θ^2-twisted	$\sum$
fixed points/type:	—	16 $\mathbb{Z}_4$	12 $\mathbb{Z}_2$+ 4 $\mathbb{Z}_4$ ($\mathbb{Z}_2$)	—
complex structure:	1	—	6+0	1+6
Kähler:	5	16	6+4	5+26

Table 1: *List of complex structure and Kähler moduli.*

The Euler characteristic turns out to be

$$\chi(T^6/\mathbb{Z}_4) = 2(h^{1,1} - h^{2,1}) = \frac{1}{|\mathbb{Z}_4|} \sum_{gh=hg} \chi(g,h) = 48, \tag{3.3}$$

where $\chi(g,h)$ denotes the Euler characteristic of the subspace invariant under both g and h. $|\mathbb{Z}_4| = 4$ is the order of the group. The sum runs over all pairs of elements of the Abelian subgroup of the quotient group; here, since $\mathbb{Z}_4$ is Abelian, the sum runs over the sixteen pairings involving all four group elements[2].

The orientifold model. As in [89, 90], we construct a $T^6/\mathbb{Z}_4$ orientifold by modding out by $\mathcal{O} = \Omega_p(-1)^{F_L}\sigma$, where Ω_p denotes worldsheet parity and $(-1)^{F_L}$ stands for left-moving fermion number. There are two distinct choices for the antiholomor-

[2]The actions of $\Theta^1, \Theta^2, \Theta^3$ all yield 16 fixed points. However, four pairs of elements, namely those involving combinations of $\Theta^0 =$ and $\Theta^2 : (z^1, z^2, z^3) \mapsto (\alpha^2 z^1, \alpha^2 z^2, z^3)$, leave at least one of the T^2 factors invariant, thus not contributing to the sum, as $\chi(T^6) = \chi(T^2) = 0$.

phic[3] involution σ on each of the T^2. We choose[4]

$$\sigma : z^1 \mapsto \bar{z}^1, \qquad (3.4)$$
$$\sigma : z^2 \mapsto \alpha\bar{z}^2,$$
$$\sigma : z^3 \mapsto \bar{z}^3.$$

For the first two tori, the complex structure is fixed to be i, so $z^i = x^i + \mathrm{i}y^i$, $i = 1, 2$. On the third torus the $\mathbb{Z}_4$ action does not fix the complex structure $z^3 = x^3 + \mathrm{i}U_2 y^3$. The tori and our choices of fundamental 1-cycles are shown in figure 1. After

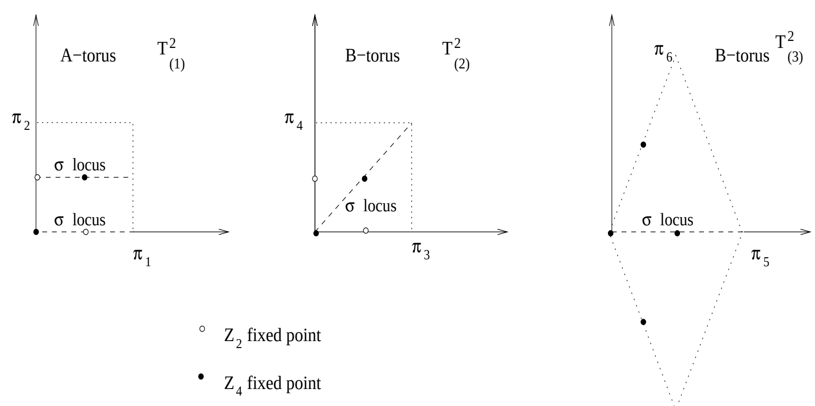

Figure 1: Tori of the **ABB** model.

the orientifold projection we have an $O6$ orientifold plane wrapping the invariant special Lagrangian 3-cycles in $T^6/\mathbb{Z}_4$ and filling the four noncompact dimensions. For reference, we have summarized the invariant cycles in each sector in table 2.

[3]In type IIA superstring theory, the involutive symmetry σ has to be chosen to be antiholomorphic, since the left-moving space-time supercharge corresponds to the holomorphic 3-form, whereas the right-moving space-time supercharge corresponds to the antiholomorphic 3-form. In the type IIB case both supercharges are related to the holomorphic 3-form, thus necessitating a holomorphic involution.[94]

[4]This is the **ABB** model discussed in detail in [90].

There we employ the notation

$$\pi_{ijk} := \pi_i \otimes \pi_j \otimes \pi_k, \tag{3.5}$$

where π_{2i-1} and π_{2i} denote the two fundamental 1-cycles of the three 2-tori T_i^2, $i \in \{1, 2, 3\}$ (see figure 1).

projection	fixed point set
$\mathcal{O}$	$2(\pi_{135} + \pi_{145})$
$\mathcal{O}\Theta$	$2\pi_{145} + 2\pi_{245} - 4\pi_{146} - 4\pi_{246}$
$\mathcal{O}\Theta^2$	$2(\pi_{235} - \pi_{245})$
$\mathcal{O}\Theta^3$	$-2\pi_{135} + 2\pi_{235} + 4\pi_{136} - 4\pi_{236}$

Table 2: Invariant cycles in each sector of the **ABB** model.

The $\mathbb{Z}_4$ action maps the cycles invariant under $\mathcal{O}$ and $\mathcal{O}\Theta^2$ into each other and likewise for the other two cycles. Therefore there are two invariant 3-cycles that are both wrapped once by the $O6$-plane:

$$[a_0] := 2(\pi_{135} + \pi_{145} + \pi_{235} - \pi_{245}) \tag{3.6}$$

$$[a_1] := 4(\pi_{136} - \pi_{146} - \pi_{246} - \pi_{236}) + 2(-\pi_{135} + \pi_{145} + \pi_{245} + \pi_{235}) \tag{3.7}$$

In addition, there will be exceptional 3-cycles related to the blow-ups of the fixed point singularities (cf. section 3.1.3).
The $O6$-plane contributes to a $\hat{C}_{(7)}$-tadpole that has to be canceled either by introducing $D6$-branes or by turning on appropriate fluxes. This issue will be addressed in the next section. It is important to note that both the $O6$-plane and the $D6$-branes can be chosen to preserve/break the same supersymmetry. Thus, we are left

with $\mathcal{N}=1$ supersymmetry in four dimensions.

Moduli and fluxes

Before embarking on the task of generating appropriate potentials by turning on fluxes, let us collect the relevant moduli fields, forms and cycles appearing in our construction. We start out by taking a closer look at the 3-cycles in the game. Since $b^3_{\text{untw.}} = 2+2h^{2,1}_{\text{untw.}} = 4$, we expect four 3-cycles from the untwisted sector. This fits nicely with the observation that the only $(2,1)$-form invariant under the $\mathbb{Z}_4$-action is $dz^1 \wedge dz^2 \wedge d\bar{z}^3$, so that the four 3-cycles are simply the duals of the holomorphic $(3,0)$-form Ω, the antiholomorphic $(0,3)$-form $\overline{\Omega}$, the $\mathbb{Z}_4$-invariant $(2,1)$-form and the associated $\mathbb{Z}_4$-invariant $(1,2)$-form.

The 1-cycles yield the following behavior under the $\mathbb{Z}_4$-action,

$$\begin{aligned}
\Theta^1 : \quad & \pi_1 \mapsto +\pi_2, \pi_3 \mapsto +\pi_4, \pi_5 \mapsto -\pi_5, \\
& \pi_2 \mapsto -\pi_1, \pi_4 \mapsto -\pi_3, \pi_6 \mapsto -\pi_6, \\
\Theta^2 : \quad & \pi_1 \mapsto -\pi_1, \pi_3 \mapsto -\pi_3, \pi_5 \mapsto +\pi_5, \\
& \pi_2 \mapsto -\pi_2, \pi_4 \mapsto -\pi_4, \pi_6 \mapsto +\pi_6, \\
\Theta^3 : \quad & \pi_1 \mapsto -\pi_2, \pi_3 \mapsto -\pi_4, \pi_5 \mapsto -\pi_5, \\
& \pi_2 \mapsto +\pi_1, \pi_4 \mapsto +\pi_3, \pi_6 \mapsto -\pi_6,
\end{aligned} \tag{3.8}$$

leading to the following $\mathbb{Z}_4$-invariant combination of 3-cycles

$$\begin{aligned}
\rho_1 &:= 2(\pi_{135} - \pi_{245}), & \widetilde{\rho}_1 &:= 2(\pi_{136} - \pi_{246}), \\
\rho_2 &:= 2(\pi_{145} + \pi_{235}), & \widetilde{\rho}_2 &:= 2(\pi_{146} + \pi_{236}).
\end{aligned} \tag{3.9}$$

Recall from table 1 that before the orientifold projection there are in addition 5 Kähler moduli from the untwisted sector.

Next, we need to take a closer look at the moduli coming from the twisted sectors. The Θ^1- and the Θ^3-twisted sectors feature 16 $\mathbb{Z}_4$ fixed points, giving rise to 16 additional Kähler moduli. The Θ^2 action leaves the third torus invariant, but acts nontrivially on the first two. Of the sixteen $\mathbb{Z}_2$ fixed points there are four that are also fixed points under the $\mathbb{Z}_4$-action. To each of the sixteen fixed points we associate an exceptional 2-cycle $e_{\alpha\beta},\ \alpha,\beta \in \{1,2,3,4\}$, where $\alpha = 1,4$ denote the $\mathbb{Z}_4$-invariant fixed points and $\alpha = 2,3$ denote the $\mathbb{Z}_2$-invariant fixed points that get mapped into each other under Θ (cf. figure 2). These give a total of 10 Kähler moduli. Certain linear combinations of these 2-cycles may be combined with the

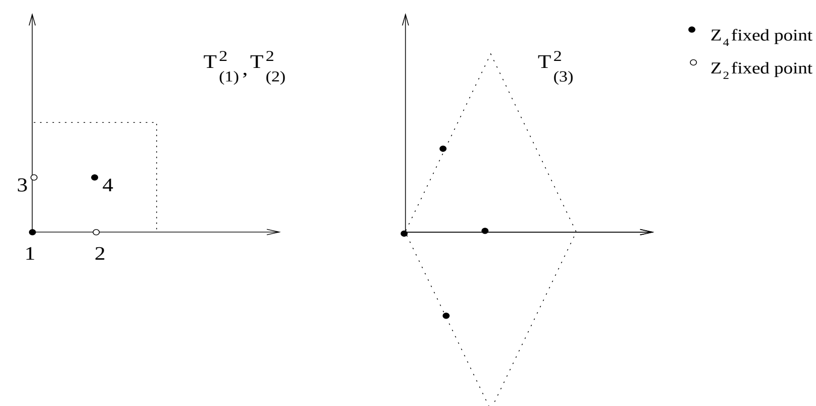

Figure 2: Fixed points of the first two tori and the third torus.

fundamental 1-cycles $\pi_{5,6}$ on the third torus to yield exceptional 3-cycles of topology $S^2 \times S^1$. Demanding invariance of the exceptional 3-cycles under the action of Θ and Θ^3, which is given by [5]

$$\Theta(e_{\alpha\beta} \otimes \pi_{5,6}) = \Theta^3(e_{\alpha\beta} \otimes \pi_{5,6}) = -e_{\eta(\alpha)\eta(\beta)} \otimes \pi_{5,6}, \tag{3.10}$$

[5] The two $_2$ fixed points are interchanged under Θ and Θ^3, while the $_4$ fixed points are invariant (cf. figure 2). The minus sign in (3.10) stems from the reflection of the fundamental 1-cycle of the third torus.

with

$$\eta(1) = 1,\ \eta(4) = 4,\ \eta(2) = 3,\ \eta(3) = 2, \tag{3.11}$$

one finds precisely twelve invariant combinations,

$$\begin{aligned}
\epsilon_1 &:= (e_{12} - e_{13}) \otimes \pi_5, & \widetilde{\epsilon}_1 &:= (e_{12} - e_{13}) \otimes \pi_6, \\
\epsilon_2 &:= (e_{42} - e_{43}) \otimes \pi_5, & \widetilde{\epsilon}_2 &:= (e_{42} - e_{43}) \otimes \pi_6, \\
\epsilon_3 &:= (e_{21} - e_{31}) \otimes \pi_5, & \widetilde{\epsilon}_3 &:= (e_{21} - e_{31}) \otimes \pi_6, \\
\epsilon_4 &:= (e_{24} - e_{34}) \otimes \pi_5, & \widetilde{\epsilon}_4 &:= (e_{24} - e_{34}) \otimes \pi_6, \\
\epsilon_5 &:= (e_{22} - e_{33}) \otimes \pi_5, & \widetilde{\epsilon}_5 &:= (e_{22} - e_{33}) \otimes \pi_6, \\
\epsilon_6 &:= (e_{23} - e_{32}) \otimes \pi_5, & \widetilde{\epsilon}_6 &:= (e_{23} - e_{32}) \otimes \pi_6.
\end{aligned} \tag{3.12}$$

Kaluza-Klein reduction of type IIA theory. The low energy limit of type IIA superstring theory yields ten-dimensional type IIA supergravity. In order to cancel the $\hat{C}_{(7)}$-tadpole, it turns out to be convenient for our purposes to allow for a nonzero $\hat{F}_{(0)}$. This effectively leads to massive type IIA SUGRA with mass $m_0 = \hat{F}_{(0)}$. The corresponding action in the string frame is given by[6]

$$\begin{aligned}
& S^{(10)}_{IIA,m_0} = S_{kin} + S_{CS} + S_{O6} \\
& = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-\hat{g}} \left(e^{-2\hat{\phi}} (\hat{R} + 4\partial_\mu \hat{\phi} \partial^\mu \hat{\phi} - \frac{1}{2} |\hat{H}_3^{\text{tot}}|^2) - (|\hat{F}_2|^2 + |\hat{F}_4|^2 + m_0^2) \right) \\
& - \frac{1}{2\kappa_{10}^2} \int \Big(\hat{B}_{(2)} \wedge d\hat{C}_{(3)} \wedge d\hat{C}_{(3)} + 2\hat{B}_{(2)} \wedge d\hat{C}_{(3)} \wedge \hat{F}^{\text{bg}}_{(4)} + \hat{C}_{(3)} \wedge \hat{H}^{\text{bg}}_{(3)} \wedge d\hat{C}_{(3)} \\
& \qquad - \frac{m_0}{3} \hat{B}_{(2)} \wedge \hat{B}_{(2)} \wedge \hat{B}_{(2)} \wedge d\hat{C}_{(3)} + \frac{m_0^2}{20} \hat{B}_{(2)} \wedge \hat{B}_{(2)} \wedge \hat{B}_{(2)} \wedge \hat{B}_{(2)} \wedge \hat{B}_{(2)} \Big) \\
& + 2\mu_6 \int_{O6} d^7\xi e^{-\hat{\phi}} \sqrt{-\hat{g}} \; - 2\sqrt{2}\mu_6 \int_{O6} \hat{C}_{(7)},
\end{aligned} \tag{3.13}$$

[6] We use hats to indicate that a field is ten-dimensional, following the conventions of [93]. Note also that in our convention for the RR fields we have an additional factor of $\sqrt{2}$.

where $2\kappa_{10}^2 = (2\pi)^7 \alpha'^4$, $\mu_6 = (2\pi)^{-6}\alpha'^{-7/2}$ and the field strengths are given by

$$\hat{H}^{\rm tot}_{(3)} = d\hat{B}_{(2)} + \hat{H}^{\rm bg}_{(3)}, \tag{3.14a}$$

$$\hat{F}_{(2)} = d\hat{C}_{(1)} + m_0 \hat{B}_{(2)}, \tag{3.14b}$$

$$\hat{F}_{(4)} = d\hat{C}_{(3)} + \hat{F}^{\rm bg}_{(4)} - \hat{C}_{(1)} \wedge \hat{H}^{\rm tot}_{(3)} - \frac{m_0}{2}\hat{B}_{(2)} \wedge \hat{B}_{(2)}. \tag{3.14c}$$

In the framework of standard Kaluza-Klein reduction, we expand the ten-dimensional gauge potentials in terms of harmonic forms on the internal space $Y = T^6/\mathbb{Z}_4$, namely

$$\begin{aligned} &\hat{C}_{(1)} = A^0(x),\ \hat{B}_{(2)} = B_{(2)}(x) + b^A(x)\omega_A, \quad A = 1, \ldots, h^{(1,1)}, \\ &\hat{C}_{(3)} = C_{(3)}(x) + A^A(x) \wedge \omega_A + \xi^K(x)\alpha_K - \tilde{\xi}_K(x)\beta^K, \quad K = 0, \ldots, h^{(2,1)}. \end{aligned} \tag{3.15}$$

where $b^A, \xi^K, \tilde{\xi}_K$ are scalars in four dimensions, A^0, A^A are four-dimensional one-forms and $B_{(2)}$ and $C_{(3)}$ are four-dimensional two- and three-forms respectively. The harmonic $(1,1)$-forms ω_A form a basis of $H^{(1,1)}(Y)$ with dual $(2,2)$-forms $\tilde{\omega}_A$, which constitute a harmonic basis of $H^{(2,2)}(Y)$. Moreover, $(\alpha_K, \beta^L) \in H^{(3)}(Y)$ form a real, sympletic basis of harmonic 3-forms on Y with dimension $h^{(3)} = 2h^{(2,1)} + 2$. The intersection numbers are

$$\int_Y \alpha_K \wedge \beta^L = \delta^L_K, \quad \int_Y \omega_A \wedge \tilde{\omega}^B = \delta^B_A. \tag{3.16}$$

Details of the orientifold projection. After modding out by the orientifold projection $\mathcal{O}$, we will be left with an $\mathcal{N} = 1$ supergravity action. To determine the $\mathcal{O}$-invariant states, first recall that the ten-dimensional fields show the following behavior under $(-1)^{F_L}$ and Ω_p (for a review, cf. [95]),

$$(-1)^{F_L}: \text{ odd} : \hat{C}_{(1)}, \hat{C}_{(3)}, \quad \text{even} : \hat{\phi}, \hat{g}, \hat{B}_{(2)}, \tag{3.17}$$

$$\Omega_p: \text{ odd} : \hat{B}_{(2)}, \hat{C}_{(3)}, \quad \text{even} : \hat{\phi}, \hat{g}, \hat{C}_{(1)}. \tag{3.18}$$

Accordingly, states that are $\mathcal{O}$-invariant have to satisfy

$$\begin{aligned} &\sigma^* \hat{\phi} = +\hat{\phi},\ \sigma^* \hat{g} = +\hat{g},\ \sigma^* \hat{B}_{(2)} = -\hat{B}_{(2)}, \\ &\sigma^* \hat{C}_{(1)} = -\hat{C}_{(1)},\ \sigma^* \hat{C}_{(3)} = +\hat{C}_{(3)}. \end{aligned} \tag{3.19}$$

Therefore we want to investigate how the cohomology groups split into even and odd subspaces under the antiholomorphic involution σ,

$$H^p(Y) = H^p_+(Y) \oplus H^p_-(Y). \tag{3.20}$$

The relevant cohomology groups together with their basis elements are summarized in Table 3[7]. Let us begin by studying the behavior of the $(1,1)$-forms in the un-

cohom. gr.	$H^{(1,1)}_+$	$H^{(1,1)}_-$	$H^{(2,2)}_+$	$H^{(2,2)}_-$	$H^{(3)}_+$	$H^{(3)}_-$
dim.	$h^{(1,1)}_+$	$h^{(1,1)}_-$	$h^{(1,1)}_-$	$h^{(1,1)}_+$	$h^{(2,1)}+1$	$h^{(2,1)}+1$
basis	ω_α	ω_a	$\tilde{\omega}^a$	$\tilde{\omega}^\alpha$	a_K	b^K

Table 3: *Cohomology groups and their basis elements.*

twisted sector. We will discuss the twisted sector moduli in chapter 3.1.3. There

[7] Note that the volume form on $T^6/\mathbb{Z}_4$ is odd under σ.

are four σ-odd $\mathbb{Z}_4$-invariant (unnormalized) $(1,1)$-forms, namely

$$\sigma : (dz^i \wedge d\overline{z}^i) \mapsto -(dz^i \wedge d\overline{z}^i), \quad i = 1,2,3, \tag{3.21a}$$

$$\sigma : (dz^1 \wedge d\overline{z}^2 + e^{\mathrm{i}\pi/2} d\overline{z}^1 \wedge dz^2) \mapsto -(dz^1 \wedge d\overline{z}^2 + e^{\mathrm{i}\pi/2} d\overline{z}^1 \wedge dz^2) \tag{3.21b}$$

and one even $(1,1)$-form,

$$\sigma : (dz^1 \wedge d\overline{z}^2 - e^{\mathrm{i}\pi/2} d\overline{z}^1 \wedge dz^2) \mapsto +(dz^1 \wedge d\overline{z}^2 - e^{\mathrm{i}\pi/2} d\overline{z}^1 \wedge dz^2). \tag{3.22}$$

Consequently, $h^{(1,1)}_{+,untw.} = 1$ and $h^{(1,1)}_{-,untw.} = 4$. Moreover, we can combine the $\mathbb{Z}_4$-invariant $(2,1)$-form and the corresponding $(1,2)$-form into an even and an odd combination under σ,

$$\sigma : (dz^1 \wedge dz^2 \wedge d\overline{z}^3 \pm \mathrm{i} d\overline{z}^1 \wedge d\overline{z}^2 \wedge dz^3) \mapsto \pm(dz^1 \wedge dz^2 \wedge d\overline{z}^3 \pm \mathrm{i} d\overline{z}^1 \wedge d\overline{z}^2 \wedge dz^3). \tag{3.23}$$

Fluxes. The following background fluxes of the NS-NS and R-R field strengths are consistent with the orientifold projection and may thus be turned on:

$$\hat{F}_0^{\mathrm{bg}} = m_0,\ \hat{F}_2^{\mathrm{bg}} = -m_a \omega_a,\ \hat{F}_4^{\mathrm{bg}} = e_a \tilde{\omega}^a,\ \hat{H}_3^{\mathrm{bg}} = -p_K b^K, \tag{3.24}$$

where we have taken into account the appropriate behavior of the fluxes under σ. The indices $a = 1, \ldots, h^{(1,1)}_{-,untw.} = 4$ and $K = 0, \ldots, h^{(2,1)}_{untw.} = 1$ label the basis elements of the cohomology groups, as given in table 3, but are restricted to the untwisted sector. More explicitly, we have

$$\omega_1 = \left(\frac{\kappa}{2}\right)^{1/3} \mathrm{i}dz^1 \wedge d\overline{z}^1, \tag{3.25a}$$

$$\omega_2 = \left(\frac{\kappa}{2}\right)^{1/3} \mathrm{i}dz^2 \wedge d\overline{z}^2, \tag{3.25b}$$

$$\omega_3 = \left(\frac{\kappa}{2}\right)^{1/3} \frac{1}{U_2}\mathrm{i}dz^3 \wedge d\overline{z}^3, \tag{3.25c}$$

$$\omega_4 = \left(\frac{\kappa}{2}\right)^{1/3} \frac{(1-\mathrm{i})}{2}(dz^1 \wedge d\overline{z}^2 - \mathrm{i}dz^2 \wedge d\overline{z}^1), \tag{3.25d}$$

and in addition,

$$\tilde{\omega}^1 = \left(\frac{1}{(4\kappa)^{1/3}U_2}\right)(\mathrm{i}dz^2 \wedge d\overline{z}^2) \wedge (\mathrm{i}dz^3 \wedge d\overline{z}^3), \tag{3.26a}$$

$$\tilde{\omega}^2 = \left(\frac{1}{(4\kappa)^{1/3}U_2}\right)(\mathrm{i}dz^3 \wedge d\overline{z}^3) \wedge (\mathrm{i}dz^1 \wedge d\overline{z}^1), \tag{3.26b}$$

$$\tilde{\omega}^3 = \left(\frac{1}{(4\kappa)^{1/3}}\right)(\mathrm{i}dz^1 \wedge d\overline{z}^1) \wedge (\mathrm{i}dz^2 \wedge d\overline{z}^2), \tag{3.26c}$$

$$\tilde{\omega}^4 = -\left(\frac{1}{(4\kappa)^{1/3}U_2}\right)\frac{(1-\mathrm{i})}{2}(dz^1 \wedge d\overline{z}^2 - \mathrm{i}dz^2 \wedge d\overline{z}^1) \wedge (\mathrm{i}dz^3 \wedge d\overline{z}^3), \tag{3.26d}$$

such that

$$\int_Y \omega_1 \wedge \omega_2 \wedge \omega_3 = -\int_Y \omega_3 \wedge \omega_4 \wedge \omega_4 = \kappa \tag{3.27}$$

and

$$\int_Y \omega_a \wedge \tilde{\omega}^b = \delta_a^b. \tag{3.28}$$

We normalize the volume form such that

$$\mathrm{i}\int_Y \Omega \wedge \overline{\Omega} = 1 \implies \Omega = \frac{(1-\mathrm{i})}{2\sqrt{U_2}}dz^1 \wedge dz^2 \wedge dz^3, \tag{3.29}$$

and choose our three forms to be

$$a_0 = \frac{1}{2}(dx^1 \wedge dx^2 - dy^1 \wedge dy^2 + dx^1 \wedge dy^2 + dy^1 \wedge dx^2) \wedge dx^3, \tag{3.30a}$$

$$a_1 = \frac{1}{4}(dx^1 \wedge dx^2 - dy^1 \wedge dy^2 - dx^1 \wedge dy^2 - dy^1 \wedge dx^2) \wedge dy^3, \tag{3.30b}$$

$$b_0 = 2(dx^1 \wedge dx^2 - dy^1 \wedge dy^2 + dx^1 \wedge dy^2 + dy^1 \wedge dx^2) \wedge dy^3, \tag{3.30c}$$

$$b_1 = -4(dx^1 \wedge dx^2 - dy^1 \wedge dy^2 - dx^1 \wedge dy^2 - dy^1 \wedge dx^2) \wedge dx^3. \tag{3.30d}$$

Ω is given in this basis by

$$\Omega = \frac{1}{\sqrt{U_2}}\, a_0 + 2\sqrt{U_2}\, a_1 + \mathrm{i}\frac{\sqrt{U_2}}{4}\, b_0 + \mathrm{i}\frac{1}{8\sqrt{U_2}}\, b_1. \tag{3.31}$$

The mixed-index part of the metric will be parameterized in the following way,

$$g_{i\bar{j}} = \begin{pmatrix} \gamma_1 & \gamma_4 + \mathrm{i}\gamma_5 & 0 \\ \gamma_4 - \mathrm{i}\gamma_5 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}. \tag{3.32}$$

Taking into account the action of σ on g, one finds that $g_{1\bar{2}} = \mathrm{i}g_{2\bar{1}}$, so that $\gamma_4 = \gamma_5$. Therefore, one Kähler modulus of the untwisted sector gets projected out by the orientifold.[8]

3.1.2 Moduli stabilization

We are now ready to calculate the potential for the various moduli fields discussed above. In the next subsection, we will directly calculate the potential from the

[8]Note that there is a non-vanishing metric component of pure type, namely

$$\delta g_{\bar{3}\bar{3}} = -\frac{1}{||\Omega||^2}\overline{\Omega}_{\bar{3}}{}^{kl}(\chi_K)_{kl\bar{3}}(\tilde{z}^K), \tag{3.33}$$

corresponding to the deformations of the complex structure. In our conventions, the untwisted complex structure modulus U_2 also shows up in the effective potential for the untwisted Kähler moduli below.

(massive) IIA supergravity action compactified on the orientifold in the presence of fluxes. Moreover, we will derive several conditions, such as a tadpole cancelation condition and another condition on the 3-form axions ξ^0 and ξ^1 which are related to the complex structure.

Dimensional (Kaluza-Klein) reduction from 10 to 4 dimensions

Again, we shall first restrict ourselves to the untwisted sector of the orientifold model.

Quantization of fluxes. We impose the usual cohomological quantization condition for a canonically normalized field strength,

$$\int \hat{F}_p = 2\kappa_{10}^2 \mu_{8-p} f_p = (2\pi)^{p-1} \alpha'^{(p-1)/2} f_p. \tag{3.34}$$

Accordingly, we have[9]

$$m_0 = \frac{f_0}{2\sqrt{2}\pi\sqrt{\alpha'}},\ m_a = \frac{2\pi\sqrt{\alpha'} f_2^{(a)}}{\sqrt{2}},\ p_K = (2\pi)^2 \alpha' h_3^{(K)},\ e_a = \frac{\kappa^{1/3}}{\sqrt{2}} (2\pi\sqrt{\alpha'})^3 f_4^{(a)}, \tag{3.35}$$

where $f_0, f_2^{(a)}, h_3^{(K)}, f_4^{(a)} \in \mathbb{Z}$.

Tadpole cancelation conditions. The $O6$-plane will generate a tadpole for the $\hat{C}_7$-potential, which we want to cancel solely by background fluxes without adding $D6$-branes. Noting that $*\hat{F}_{(2)} = d\hat{C}_7 - \hat{C}_5 \wedge \hat{H}_3 - \frac{m_0}{24} \hat{B}_2 \wedge \hat{B}_2 \wedge \hat{B}_2 \wedge \hat{B}_2$ contains $\hat{C}_7$, the integrated equations of motion for the $\hat{C}_7$-potential yield

$$\int d\hat{F}_{(2)} = \int m_0 \hat{H}_3^{\text{bg}} \overset{!}{=} 2\sqrt{2}\kappa_{10}^2 \mu_6 = 2(\sqrt{2}\pi\sqrt{\alpha'}). \tag{3.36}$$

[9]Note the additional factor of $\sqrt{2}$ for the RR fields in our conventions.

The $O6$-plane wraps each of the cycles $[a_0] = (\rho_1+\rho_2)$ and $[a_1] = (2(\tilde{\rho}_1-\tilde{\rho}_2)+\rho_2-\rho_1)$ once. Thus we have to integrate (3.36) over $[b_K], K = 0, 1$ leading to

$$m_0 p_K = -2(\sqrt{2}\pi\sqrt{\alpha'}), \quad K = 0, 1. \tag{3.37}$$

Taking into account the quantization condition (3.34), we arrive at the tadpole cancelation conditions

$$\begin{aligned} m_0 p_0 = m_0 p_1 = (\sqrt{2}\pi\sqrt{\alpha'}) f_0 h_3^{(K)} &= -2(\sqrt{2}\pi\sqrt{\alpha'}) \\ \Rightarrow (f_0, h_3^{(K)}) = \pm(2, -1) &\text{ or } \pm(1, -2). \end{aligned} \tag{3.38}$$

For later convenience we define $p \equiv p_0 = p_1$.

Potential for the untwisted complex structure axion. We will begin our discussion of the complex structure moduli by considering the associated axions first. A more detailed examination of the complex structure deformations will be carried out in the next subsection. It actually turns out that the contribution to the superpotential coming from $\hat{H}_{(3)}^{\text{bg}}$ fixes the real part of the complex structure hypermultiplet (namely the geometric complex structure moduli), while it leaves the imaginary part (the axions) unfixed. After the orientifold projection, the remaining axionic modes are[10]

$$\hat{C}_{(3)} = \xi^0 a_0 + \xi^1 a_1, \tag{3.39}$$

noting that $\hat{C}_{(3)}$ has to be even under the involution σ in our construction. The discussion here mostly parallels [83]. The RR field $\hat{C}_{(3)}$ only appears in the Chern-Simons piece of the massive IIA SUGRA action (3.13). It is important to notice that $\hat{C}_{(3)} \wedge \hat{H}_{(3)}^{\text{bg}} \wedge d\hat{C}_{(3)}$ is nonvanishing only if $d\hat{C}_{(3)}$ is polarized in the noncompact directions. Since it does not contain physical degrees of freedom, we will treat it as

[10]We have chosen a symplectic basis for $H^{(3)}(Y)$ such that all the a_K are σ-even and all the b^K are σ-odd.

a Lagrange multiplier $\mathcal{F}_0 := dC_{(3)}$. Plugging its equation of motion back into the action yields

$$S_{\mathcal{F}_0} = -\frac{1}{2\kappa_{10}^2}\int \mathcal{F}_0 \wedge *\mathcal{F}_0. \tag{3.40}$$

Minimizing this contribution to the potential is tantamount to setting $\mathcal{F}_0 = 0$. Doing this and integrating over Y results in an equation involving the 3-form axions, namely

$$p_0\xi^0 + p_1\xi^1 = e_0 + e_a b_a - \kappa m_0 b_3(b_1 b_2 - \frac{b_4^2}{2}), \tag{3.41}$$

with the definition $e_0 := \int \hat{F}^{\mathrm{bg}}_{(6)}$. This means that only one linear combination of the axions is fixed while there is another (independent) one that remains unfixed. This is consistent with the results obtained below from analyzing the superpotential. One could either try to stabilize the remaining axion by introducing nonperturbative effects such as Euclidean $D2$-instantons or by using the unfixed axion(s) to give mass to (potentially anomalous) U(1) brane fields via the Stückelberg mechanism [85].

Equations of motions for the b_a. For simplicity we will set[11] $\hat{F}_2^{bg} = 0$. Since $\hat{C}_1$ has no zero modes, the contributions from the $|\hat{F}_2|^2$ and $|\hat{F}_4|^2$ terms in the action are at least quadratic in the b_a. Since the Chern-Simons term linear in $\hat{B}_2$ has been taken into account above, we find that the action contains no terms linear in b_a. Therefore there is a solution with $b_a = 0, \forall a$. Since we will find supersymmetric and non-supersymmetric vacua some of these solutions might have instabilities. We will further investigate this at the end of this section.

Flux generated potential for the untwisted Kähler and complex structure moduli. In this section we will stabilize the remaining untwisted moduli. We will work in the four dimensional Einstein frame, so we define $g_{(4)\mu\nu} = \frac{e^{\hat{\phi}}}{\sqrt{vol_{(6)}}} g^E_{(4)\mu\nu}$.

[11]Solutions with $\hat{F}_2^{bg} \neq 0$ have qualitatively the same behavior as the $\hat{F}_2^{bg} = 0$ solution as will be shown later.

The effective potential is defined as

$$S = \frac{1}{\kappa_{10}^2} \int d^4x \sqrt{-g_{(4)}^E}(-V_{\text{eff}}). \tag{3.42}$$

For $b_a = 0$ and the ξ^K satisfying their equation of motion we only get contributions from the terms $|\hat{H}_3^{\text{tot}}|^2, |\hat{F}_4|^2, m_0^2$ and the $O6$ Born-Infeld piece. They are

$$\begin{aligned} V_{\text{eff}} =& \frac{e^{2\hat{\phi}}}{vol_{(6)}^2} p^2 (\frac{1}{U_2} + 4U_2) \\ &+ \frac{e^{4\hat{\phi}}}{2vol_{(6)}^3} \left[\sum_{i=1}^{3} e_i^2 v_i^2 + e_4^2 (v_1 v_2 + \frac{v_4^2}{2}) + e_1 e_2 v_4^2 + 2 e_4 v_4 (e_1 v_1 + e_2 v_2) \right] \quad (3.43) \\ &+ \frac{m_0^2}{2} \frac{e^{4\hat{\phi}}}{vol_{(6)}} - 2|m_0 p| \frac{e^{3\hat{\phi}}}{vol_{(6)}^{3/2}} \left(\frac{1}{\sqrt{U_2}} + 2\sqrt{U_2} \right), \quad (3.44) \end{aligned}$$

where

$$\begin{aligned} v_1 &= \frac{1}{2}\left(\frac{2}{\kappa}\right)^{1/3} \gamma_1, \qquad & v_2 &= \frac{1}{2}\left(\frac{2}{\kappa}\right)^{1/3} \gamma_2, \\ v_3 &= \frac{1}{2}\left(\frac{2}{\kappa}\right)^{1/3} U_2\, \gamma_3, \qquad & v_4 &= -\left(\frac{2}{\kappa}\right)^{1/3} \gamma_4, \end{aligned}$$

$$\begin{aligned} vol_{(6)} &= \int_Y dx^1 \wedge dy^1 \wedge dx^2 \wedge dy^2 \wedge dx^3 \wedge dy^3 \sqrt{g_{(6)}} \\ &= U_2 \frac{\gamma_3(\gamma_1\gamma_2 - 2\gamma_4^2)}{4} = \kappa v_3 (v_1 v_2 - \frac{v_4^2}{2}). \end{aligned}$$

Extremizing the potential with respect to the complex structure U_2 fixes it at

$$U_2 = \frac{1}{2}. \tag{3.45}$$

Now we solve

$$v_a \frac{\partial V}{\partial v_a} + \frac{7}{4}\frac{\partial V}{\partial \hat{\phi}} = 0, \tag{3.46}$$

and find

$$e^{\hat{\phi}}\sqrt{vol_{(6)}} = \frac{5}{\sqrt{2}}\left|\frac{p}{m_0}\right|. \tag{3.47}$$

This is (almost) fixed by the tadpole cancelation conditions, cf. equation (3.38) above. This condition ensures that, for minima of the potential, the string coupling automatically becomes small if we tune the fluxes such that the internal volume becomes large enough to trust the supergravity approximation we are using. Relation (3.47) can be used to eliminate the dilaton dependence of the potential. Once the minima have been found, said relation fixes the dilaton w.r.t. a specific set of fluxes. The potential simplifies to

$$V_{\text{eff}} = \frac{25}{8}\frac{p^4}{m_0^2}\left(\frac{-39}{vol_{(6)}^3} + \frac{25}{m_0^2\, vol_{(6)}^5}\left[\sum_{i=1}^{3} e_i^2 v_i^2 + e_4^2(v_1 v_2 + \frac{v_4^2}{2}) + e_1 e_2 v_4^2 + 2e_4 v_4(e_1 v_1 + e_2 v_2)\right]\right).$$

It now only depends on the Kähler moduli v_1, v_2, v_3, v_4. Extremizing with respect to all of the Kähler moduli leads to five sets of solutions. The first is

$$\begin{aligned} v_1 &= \pm e_2\sqrt{\frac{10}{3}}\sqrt{\left|\frac{e_3}{\kappa m_0(2e_1e_2 - e_4^2)}\right|}, \\ v_2 &= \pm e_1\sqrt{\frac{10}{3}}\sqrt{\left|\frac{e_3}{\kappa m_0(2e_1e_2 - e_4^2)}\right|}, \\ v_3 &= \sqrt{\frac{5}{6}}\sqrt{\left|\frac{2e_1e_2 - e_4^2}{\kappa m_0 e_3}\right|}, \\ v_4 &= \mp e_4\sqrt{\frac{10}{3}}\sqrt{\left|\frac{e_3}{\kappa m_0(2e_1e_2 - e_4^2)}\right|}. \end{aligned} \tag{3.48}$$

As we will see below this solution encompasses the supersymmetric solution obtained from minimizing the potential of the 4-d SUGRA action. To allow for a geometrical interpretation of the solution we have to demand that the volume $vol_{(6)}$ and v_3 the area of the third torus are bigger than zero. This implies that $(2e_1e_2 - e_4^2) > 0$ which requires $\text{sign}[e_1e_2] > 0$. The volume is

$$vol_{(6)} = \frac{5}{3}\sqrt{\frac{5}{6}}\sqrt{\left|\frac{e_3(2e_1e_2 - e_4^2)}{\kappa m_0^3}\right|}. \tag{3.49}$$

It can be made parametrically large by tuning the fluxes to large values. The string coupling is determined to be

$$g_s = e^{\hat{\phi}} = |p|\left(\frac{135}{2}\left|\frac{\kappa}{m_0 e_3(2e_1e_2 - e_4^2)}\right|\right)^{1/4}. \tag{3.50}$$

Thus, there is a (countably) infinite number of vacua with small string coupling and large volume.[12]

The value of the potential at the minimum is

$$V_{\min} = -\frac{243}{25}\sqrt{\frac{6}{5}}\sqrt{\left|\frac{\kappa^3 m_0^5}{(e_3(2e_1e_2 - e_4^2))^3}\right|}\, p^4, \tag{3.51}$$

which is always negative so that the vacua are anti-de-Sitter.

The second set of solutions is

$$\begin{aligned}
v_1 &= \pm e_4\sqrt{\frac{5}{3}}\sqrt{\left|\frac{e_2e_3}{\kappa m_0 e_1(2e_1e_2 - e_4^2)}\right|}, \\
v_2 &= \pm e_4\sqrt{\frac{5}{3}}\sqrt{\left|\frac{e_1e_3}{\kappa m_0 e_2(2e_1e_2 - e_4^2)}\right|}, \\
v_3 &= \sqrt{\frac{5}{6}}\sqrt{\left|\frac{2e_1e_2 - e_4^2}{\kappa m_0 e_3}\right|}, \\
v_4 &= \mp 2\sqrt{\frac{5}{3}}\sqrt{\left|\frac{e_1e_2e_3}{\kappa m_0(2e_1e_2 - e_4^2)}\right|}.
\end{aligned} \tag{3.52}$$

[12] If we set for example $e_1 = e_2 = e_3 = e_4 \equiv e \to \infty$, we have $vol \sim e^{3/2}, \quad e^{\hat{\phi}} \sim e^{-3/4}$.

For this case we have to demand that $(2e_1e_2 - e_4^2) < 0$ and $\text{sign}[e_1e_2] > 0$. The volume, the string coupling and the potential at the minimum are the same as above. This is also the case for all the other solutions.

The next set of solutions has v_4 fixed at zero

$$v_1 = \pm\sqrt{\frac{5}{3}}\sqrt{\left|\frac{e_2e_3}{\kappa m_0 e_1}\right|},$$
$$v_2 = \pm\sqrt{\frac{5}{3}}\sqrt{\left|\frac{e_1e_3}{\kappa m_0 e_2}\right|},$$
$$v_3 = \sqrt{\frac{5}{6}}\sqrt{\left|\frac{2e_1e_2 - e_4^2}{\kappa m_0 e_3}\right|},$$
$$v_4 = 0. \tag{3.53}$$

It requires $\text{sign}[e_1e_2] < 0$ which implies $(2e_1e_2 - e_4^2) < 0$.

We furthermore find solutions in which one of the Kähler moduli is unstabilized

$$v_1 = \frac{1}{e_1^2}\left((-e_1e_2 + e_4^2)v_2 \pm e_4\sqrt{|2e_1e_2 - e_4^2|v_2^2 - \left|\frac{10}{3}\frac{e_1^2e_3}{\kappa m_0}\right|}\right), \tag{3.54}$$
$$v_2 = \textbf{unfixed},$$
$$v_3 = \sqrt{\frac{5}{6}}\sqrt{\left|\frac{2e_1e_2 - e_4^2}{\kappa m_0 e_3}\right|},$$
$$v_4 = \frac{1}{e_1}\left((-e_4v_2 \mp \sqrt{|2e_1e_2 - e_4^2|v_2^2 - \left|\frac{10}{3}\frac{e_1^2e_3}{\kappa m_0}\right|}\right).$$

These solutions require $(2e_1e_2 - e_4^2) < 0$ and $v_2^2 > \left|\frac{10}{3}\frac{e_1^2e_3}{\kappa m_0(2e_1e_2 - e_4^2)}\right|$. Since the action is invariant under the simultaneous exchange of $e_1 \leftrightarrow e_2$ and $v_1 \leftrightarrow v_2$, we have corresponding solutions in which v_1 is unfixed.

Although we have turned on the most generic fluxes compatible with the orbifold and orientifold projection, we found solutions that have one unstabilized geometric modulus. As we will see below these solutions are not supersymmetric.

Stability analysis for the b_a. Since we have found vacua that are non-supersymmetric, we have to check that our $b_a = 0$ solution is in fact stable. To do this we consider the terms quadratic in b_a and ξ^K [13]. We find

$$
\begin{aligned}
S_{axion} = &\frac{1}{2\kappa_{10}^2}\int d^4x\sqrt{-g_4^E}\times \\
&\times\left[-\frac{1}{2vol_{(6)}}\partial_\mu b^a\partial^\mu b^b\int_Y(\omega_a\wedge *_6\,\omega_b) - e^{2D}\partial_\mu\xi^K\partial^\mu\xi^L\int_Y(a_K\wedge *_6\,a_L)\right.\\
&- e^{4D}\left(m_0^2b^ab^b\int_Y(\omega_a\wedge *_6\,\omega_b) - m_0b^ab^be_c\int_Y(\omega_a\wedge\omega_b\wedge *_6\,\tilde{\omega}^c)\right.\\
&\left.\left.\quad+\frac{(-p_K\xi^K+e_ab^a)^2}{vol_{(6)}}\right)\right],
\end{aligned}
$$

where we defined the four dimensional dilation as $e^D = \frac{e^{\hat{\phi}}}{\sqrt{vol_{(6)}}}$. Now one has to diagonalize the kinetic energy terms and calculate the mass-squared matrix (Hessian) for each of the solutions described above. To carry out the calculations in full generality is rather tedious. From the action we see that the result will depend on the explicit choices for the fluxes m_0 and e_a. We have calculated the mass-squared matrix for simple sets of fluxes for all of our vacua. In each case, we obtain positive mass eigenvalues with the exception of one zero eigenvalue corresponding to the unstabilized axion $\xi_0 - \xi_1$ (cf. (3.41)). Thus, there exists a stable solution for all vacua (with large fluxes). In conclusion, we see that the solution corresponding to $b_a = 0$, $\forall a$, is a stable minimum of the effective four-dimensional potential, at least for simple choices of the fluxes.

Effective $\mathcal{N} = 1$ SUGRA in $D = 4$

In this subsection we will analyze the problem from the point of view of the effective $\mathcal{N} = 1$ SUGRA theory in four dimensions. One of the virtues of working in this

[13] Remember that (3.41) implies that there is a mixing between the b_a and ξ^K.

framework is that the untwisted and the twisted moduli can be treated on equal footing. As pointed out in [83], another advantage lies in the fact that this type of analysis can be used for general backgrounds since e.g., backreaction and worldsheet instanton corrections are naturally described in terms of the four-dimensional effective theory, whereas they cannot be described in terms of ten-dimensional supergravity. Based on the flux-generated superpotential, as worked out by Grimm and Louis [93] (see also [95]), we will analyze the F-flatness conditions $D_I W = 0$, where I runs over all moduli fields and $D_I = \partial_I + (\partial_I K)$ is the Kähler covariant derivative. Solutions to these equations correspond to supersymmetric minima of the scalar potential,

$$V = e^K \left(\sum_{I\bar{J}} G^{I\bar{J}} D_I W \overline{D_J W} - 3|W|^2 \right) + m_0 e^{K^Q} \mathrm{Im} W^Q, \tag{3.55}$$

namely

$$D_I W = 0 \Rightarrow dV = 0. \tag{3.56}$$

The opposite direction is not true. The structure of the Kähler potential $K = K^K + K^Q$ and the superpotential $W = W^K + W^Q$ will be discussed below.

$\mathcal{N} = 2$ **SUGRA in** $D = 4$**.** The dimensional reduction of (massive) type IIA supergravity from $D = 10$ to $D = 4$ on a Calabi-Yau manifold gives rise to $\mathcal{N} = 2$ supergravity in $D = 4$. The existence of one covariantly constant spinor on the internal CY (with $SU(3)$ holonomy) ensures that there are two four-dimensional SUSY parameters; the compactification therefore preserves eight supercharges, hence $\mathcal{N} = 2$ in $D = 4$. In the presence of fluxes, the resulting effective theory in four dimensions is gauged, i.e., the hypermultiplets are charged under some of the vectormultiplets. For this to be consistent, the metric on the scalar manifold coordinatized by the hypermultiplets, which is in fact a quaternionic manifold, must possess isometries that in turn can be gauged. Table 4 lists the bosonic components of all $\mathcal{N} = 2$

multiplets. There are massless modes coming from deformations of the metric g of

gravity multiplet	1	$(g_{\mu\nu}, A^0)$
vectormultiplets	$h^{(1,1)}$	(A^A, v^A, b^A)
hypermultiplets	$h^{(2,1)}$	$(z^K, \xi^K, \tilde{\xi}_K)$
tensor multiplet	1	$(B_{(2)}, \hat{\phi}, \xi^0, \tilde{\xi}_0)$

Table 4: *Bosonic part of the $\mathcal{N} = 2$ multiplets for Type IIA SUGRA on a CY3.*

the CY manifold that respect the Ricci flatness condition $R_{mn} = 0$. This forces δg to satisfy the Lichnerowicz equation, whose solutions in our case can be identified with harmonic (1,1)- and (2,1)-forms on Y, corresponding to Kähler structure and complex structure deformations, respectively.

Kähler moduli space. Deformations of the Kähler form can be expanded in a basis of harmonic (1,1)-forms,

$$g_{i\bar{j}} + \delta g_{i\bar{j}} = -\mathrm{i}J_{i\bar{j}} = -\mathrm{i}v^A(\omega_A)_{i\bar{j}}, \quad A = 1, \ldots, h^{(1,1)}. \tag{3.57}$$

These deformations can be supplemented by the $h^{(1,1)}$ real scalar fields $b^A(x)$ from the expansion of the B-field, yielding complex fields

$$t^A = b^A + \mathrm{i}v^A, \tag{3.58}$$

that parametrize the complexified Kähler cone. The moduli space of the complexified Kähler structure deformations $\mathcal{M}^{\mathrm{ks}}$ is a special Kähler manifold which can be seen by noting that the metric is given by

$$G_{AB} = \frac{3}{2\kappa}\int_Y \omega_A \wedge *\omega_B = -\frac{3}{2}\left(\frac{\kappa_{AB}}{\kappa} - \frac{3}{2}\frac{\kappa_A \kappa_B}{\kappa^2}\right) = \partial_{t^A}\partial_{\bar{t}^B} K^{\mathrm{ks}}, \tag{3.59}$$

where the intersection numbers are defined as follows

$$\kappa = \int_Y J \wedge J \wedge J = \kappa_{ABC} v^A v^B v^C, \quad \kappa_A = \int_Y \omega_A \wedge J \wedge J = \kappa_{ABC} v^B v^C,$$
$$\kappa_{AB} = \int_Y \omega_A \wedge \omega_B \wedge J = \kappa_{ABC} v^C, \quad \kappa_{ABC} = \int_Y \omega_A \wedge \omega_B \wedge \omega_C.$$

The Kähler potential for the Kähler structure deformations,

$$K^{\rm ks} = -\ln\left(\frac{\rm i}{6}\kappa_{ABC}(t-\bar{t})^A(t-\bar{t})^B(t-\bar{t})^C\right) = -\ln\frac{4}{3}\kappa, \tag{3.60}$$

can be derived from a single holomorphic prepotential $\mathcal{G}(t) = -\frac{1}{6}\kappa_{ABC}t^At^Bt^C$.

Complex structure moduli space. Complex structure deformations are associated with harmonic (1,2)-forms and are parametrized by complex fields $\tilde{z}^K$, $K = 1, \ldots, h^{(2,1)}$, in the following way,

$$\delta g_{ij} = \frac{\rm i}{||\Omega||^2}\bar{\tilde{z}}^K(\overline{\chi}_K)_{i\bar{i}\bar{j}}\Omega^{\bar{i}\bar{j}}{}_j, \tag{3.61}$$

where the χ_K form a harmonic basis of $H^{(2,1)}(Y)$ and $||\Omega||^2 = \frac{1}{3!}\Omega_{ijk}\Omega^{ijk}$. The metric on the complex structure moduli space $\mathcal{M}^{\rm cs}$ is given by

$$G_{K\bar{L}} = -\frac{\int_Y \chi_K \wedge \overline{\chi}_L}{\int_Y \Omega \wedge \overline{\Omega}}. \tag{3.62}$$

Kodaira's formula connects the χ_K to the variation of the harmonic (3,0)-form via

$$\chi_K(\tilde{z}, \bar{\tilde{z}}) = \partial_{\tilde{z}^K}\Omega(\tilde{z}) + \Omega(\tilde{z})\partial_{\tilde{z}^K}K^{\rm cs}, \tag{3.63}$$

where

$$K^{\rm cs}(\tilde{z}, \bar{\tilde{z}}) = -\ln\left[{\rm i}\int_Y \Omega \wedge \overline{\Omega}\right] = -\ln {\rm i}\left[\overline{Z}^K\mathcal{F}_K - Z^K\overline{\mathcal{F}}_K\right]. \tag{3.64}$$

Note that $G_{K\bar{L}} = \partial_{\tilde{z}^K}\partial_{\tilde{\bar{z}}^{\bar{L}}} K^{\mathrm{cs}}$, thus proving that that $\mathcal{M}^{\mathrm{cs}}$ is a Kähler manifold. The holomorphic periods $Z^K, \mathcal{F}_K$ are the expansion coefficients of

$$\Omega = Z^K \alpha_K - \mathcal{F}_K \beta^K, \tag{3.65}$$

so that we have

$$Z^K = \int_Y \Omega \wedge \beta^K, \quad \mathcal{F}_K = \int_Y \Omega \wedge \alpha_K. \tag{3.66}$$

In fact, Ω is only defined up to a complex rescaling with a holomorphic function which changes the Kähler potential by a Kähler transformation. This symmetry can be used to fix a Kähler gauge, in which $Z^0 = 1$. The remaining periods can be identified with the $h^{(2,1)}$ complex structure deformations

$$\tilde{z}^K = \frac{Z^K}{Z^0}. \tag{3.67}$$

Moreover, we find that there exists a prepotential of which $\mathcal{F}_K$ is the first derivative, $\mathcal{F} = \frac{1}{2} Z^K \mathcal{F}_K$. This means that the metric $G_{K\bar{L}}$ is completely determined by $\mathcal{F}$. Therefore $\mathcal{M}^{\mathrm{cs}}$ is in fact a special Kähler manifold.

Supplementing the complex structure deformations $\tilde{z}^K$ with the corresponding axions ξ^K and $\tilde{\xi}_K$ from the RR 3-form $\hat{C}_3$ can be shown to result in a special quaternionic structure of the resulting moduli space. We will refer to this larger manifold, spanned by the scalars in the hypermultiplets, as $\mathcal{M}^Q$. In the next section we will use the fact that $\mathcal{M}^Q$ contains the special Kähler submanifold $\mathcal{M}^{\mathrm{cs}}$ spanned by the complex structure deformations.

Orientifold projection. As already mentioned above, the cohomology groups split into even and odd parts under the antiholomorphic involution σ (cf. (3.20)). The involution must act as [96]

$$\sigma^* J = -J, \quad \sigma^* \Omega = e^{2i\theta} \overline{\Omega}. \tag{3.68}$$

The fixed loci of σ (which the $O6$-plane wraps) are special Lagrangian (sLag) 3-cycles Σ_n fulfilling

$$J\Big|_{\Sigma_n} = 0, \quad \mathrm{Im}(e^{-\mathrm{i}\theta}\Omega)\Big|_{\Sigma_n} = 0. \tag{3.69}$$

Together with the conditions (3.19) we are left with

$$J_c := B + \mathrm{i}J = \sum_{a=1}^{h_-^{(1,1)}} t^a \omega_a. \tag{3.70}$$

Thus, the orientifold projection reduces the Kähler moduli space to a subspace without altering its complex structure and the Kähler potential is inherited directly from $\mathcal{N} = 2$,

$$K^K(t^a) = -\log(\frac{4}{3}\kappa_{abc}v^a v^b v^c). \tag{3.71}$$

For the holomorphic (3,0)-form, we get

$$\Omega(\tilde{z}) = Z^K(\tilde{z})a_K - \mathcal{F}_K(\tilde{z})b^K, \tag{3.72}$$

where we have decomposed $H^{(3)}(Y) = H_+^{(3)}(Y) \oplus H_-^{(3)}(Y)$ as indicated in table 3. As remarked upon earlier, one can always perform a symplectic rotation on the resulting even and odd bases such that all a_K are even and all b^K are odd. Note that the $h_+^{(1,1)}$ vector multiplets do not contain any scalars and will therefore be disregarded. It is customary to package the remaining degrees of freedom in the following way,

$$\Omega_c = \hat{C}_{(3)} + 2\mathrm{i}\mathrm{Re}(C\Omega), \tag{3.73}$$

where we have introduced the complex compensator $C = re^{-\mathrm{i}\theta}$, where $r = e^{-D+K^{\mathrm{cs}}/2}$. r transforms oppositely to the holomorphic 3-form under holomorphic transformations so as to render $C\Omega$ scale-invariant (the compensator replaces the irrelevant scale factor in favor of the physical dilaton field D; for more details see [93, 96]).

The field $\hat{C}_{(3)} = \xi^K a_K$ comprises the surviving axionic modes. Finally, Ω_c can be expanded in a basis of $H^{(3)}_+(Y)$,

$$\Omega_c = 2N^K a_K, \tag{3.74}$$

where

$$N^K = \frac{1}{2}\int_Y \Omega_c \wedge b^K = \frac{1}{2}(\xi^K + 2\mathrm{iRe}(CZ^K)). \tag{3.75}$$

We have now reduced the number of moduli, while preserving the original $\mathcal{N} = 2$ complex structure. Table 5 shows the surviving $\mathcal{N} = 1$ spectrum. The $\mathcal{N} = 1$ Kähler

multiplets	multiplicity	bosonic components
gravity multiplet	1	$g_{\mu\nu}$
vector multiplets	$h_+^{(1,1)}$	A^α
chiral multiplets	$h_-^{(1,1)}$	t^a
chiral multiplets	$h^{(2,1)} + 1$	N^K

Table 5: $\mathcal{N} = 1$ *multiplets after orientifold projection.*

potential is given by

$$K^Q = -2\log\left(2\int \mathrm{Re}(C\Omega) \wedge *\mathrm{Re}(C\Omega)\right) = 4D, \tag{3.76}$$

where

$$\int \mathrm{Re}(C\Omega) \wedge *\mathrm{Re}(C\Omega) = -\mathrm{Re}(CZ^K)\mathrm{Im}(C\mathcal{F}_K) = \frac{e^{-2D}}{2}. \tag{3.77}$$

For the four dimensional dilaton we have

$$e^D = \frac{e^{\hat{\phi}}}{\sqrt{vol}} = \sqrt{8}e^{\hat{\phi}+K^K/2}. \tag{3.78}$$

In conclusion, we have seen that from each quaternionic hypermultiplet only the real part of the complex structure modulus and one axion survives. The degrees of freedom in the universal hypermultiplet are also cut in half, namely the dilaton $\hat{\phi}$ and the axion ξ^0 survive.

Supersymmetric AdS vacua

It was demonstrated by Grimm and Louis [93] that dimensionally reducing massive type IIA supergravity from 10 to 4 dimensions, while neglecting the backreaction of the fluxes and other local sources on the geometry of the compactification manifold, leads to the following scalar potential,

$$V = e^{K^K + K^Q} \left(\sum_{I,J=t^a,N^K} G^{I\bar{J}} D_I W \overline{D_J W} - 3|W|^2 \right) + m_0 e^{K^Q} \mathrm{Im} W^Q. \tag{3.79}$$

The second term cancels with contributions from the $O6$-plane when the tadpole cancelation condition (3.36) is satisfied. The superpotential is given by

$$W(t^a, N^K) = W^Q(N^K) + W^K(t^a), \tag{3.80a}$$

$$W^Q(N^K) = \int_Y \Omega_c \wedge \hat{H}_{(3)} = -2p_K N^K = -p_K \xi^K - 2\mathrm{i} p_K \mathrm{Re}(CZ^K), \tag{3.80b}$$

$$\begin{aligned} W^K(t^a) &= \int_Y e^{-J_c} \wedge \hat{F} \\ &= e_0 + \int_Y J_c \wedge \hat{F}_{(4)} - \frac{1}{2} \int_Y J_c \wedge J_c \wedge \hat{F}_{(2)} - \frac{m_0}{6} \int_Y J_c \wedge J_c \wedge J_c \qquad (3.80c) \\ &= e_0 + e_a t^a + \frac{1}{2} \kappa_{abc} t^a t^b m^c - \frac{m_0}{6} \kappa_{abc} t^a t^b t^c, \qquad (3.80d) \end{aligned}$$

with the definition $\hat{F} = m_0 - \hat{F}^{\mathrm{bg}}_{(2)} - \hat{F}^{\mathrm{bg}}_{(4)} + \hat{F}^{\mathrm{bg}}_{(6)}$ (cf. (3.24)). In the following sections we will first analyze the equations for the moduli from the F-term conditions (3.56) in general and then specialize to the case at hand, namely the $T^6/\mathbb{Z}_4$ orientifold. It is important to note that these equations will be valid for all (untwisted and twisted) moduli. The discussion closely follows the one in [83].

Complex structure equations. Solving for $D_{N^K}W = 0$ yields

$$p_K + 2\mathrm{i}W\mathrm{Im}(C\mathcal{F}_K)e^{2D} = 0. \tag{3.81}$$

We shall study the real and imaginary parts of this equation separately. For the real part one gets

$$p_K - 2e^{2D}\mathrm{Im}(W)\mathrm{Im}(C\mathcal{F}_K) = 0. \tag{3.82}$$

We immediately learn from this equation that $\mathrm{Im}(W) = 0$ is incompatible with non-vanishing $\hat{H}^{\mathrm{bg}}_{(3)}$-flux. Thus assuming $\mathrm{Im}(W) \neq 0$ we find that for each $p_{K_i} = 0$, we have $\mathrm{Im}(C\mathcal{F}_{K_i}) = 0$. For $p_{K_j} \neq 0$, one finds

$$e^{-K^{\mathrm{cs}}/2}\frac{p_{K_j}}{\mathrm{Im}(\mathcal{F}_{K_j})} = 2e^{D}\mathrm{Im}(W) =: Q_0, \tag{3.83}$$

thus fixing all geometric complex structure moduli (including the twisted ones, in our case $K = 0, \ldots, h^{(2,1)} = 7$). As noted above, these equations are invariant under rescalings of Ω and therefore do only depend on the $h^{(2,1)}$ inhomogeneous coordinates of $\mathcal{M}^{\mathrm{cs}}$, yielding $h^{(2,1)}$ equations for the $h^{(2,1)}$ moduli. The dilaton will be stabilized at

$$e^{-\hat{\phi}} = 4\sqrt{2}e^{K^K/2}\frac{\mathrm{Im}(W)}{Q_0}, \tag{3.84}$$

once complex structure and Kähler moduli are fixed.

Turning to the imaginary part of (3.81), we see that, due to the reality of the flux coefficients p_K, all K equations yield the same condition, namely (D and $C = r$ are real[14].)

$$2e^{2D}\mathrm{Re}(W)\mathrm{Im}(C\mathcal{F}_K) = 0 \Rightarrow \mathrm{Re}(W) = 0. \tag{3.85}$$

[14]We absorb θ in the holomorphic 3-form so that it satisfies $\sigma^*\Omega = \overline{\Omega}$

Comparing to the definition of W, this indeed gives the same condition on the axions as derived above (cf. (3.41)),

$$-p_K \xi^K + \mathrm{Re}(W^K) = 0, \tag{3.86}$$

where we have now correctly considered all the axions, including those from the twisted sectors. Another important observation can be made by multiplying (3.81) by $\mathrm{Re}(CZ_K)$ and summing over K. The resulting equation reads

$$-\mathrm{i}W = -p_K \mathrm{Re}(CZ^K) = \frac{1}{2}\mathrm{Im}(W^Q). \tag{3.87}$$

Now since $\mathrm{R}e(W) = 0$ (cf. (3.85)), we find

$$-\mathrm{i}W = \mathrm{Im}(W^K) + \mathrm{Im}(W^Q) = \frac{1}{2}\mathrm{Im}(W^Q) \Rightarrow \mathrm{Im}(W^Q) = -2\,\mathrm{Im}(W^K). \tag{3.88}$$

Therefore we can directly conclude that, provided the complex structure moduli are 'on-shell' (satisfy their equations of motion), the vacuum superpotential can be given solely in terms of the Kähler moduli, i.e.,

$$W(t^a, N^K) = -\mathrm{i}\,\mathrm{Im}(W^K(t^a)), \tag{3.89}$$

thus effectively decoupling the Kähler sector from the complex structure sector.

Kähler structure equations. Let us now consider the Kähler sector in more detail. The corresponding F-flatness conditions $D_{t^a}W = 0$ can be simplified making use of (3.89), yielding

$$\partial_{t^a} W^K - \mathrm{i}\partial_{t^a} K^K \mathrm{Im}(W^K) = 0. \tag{3.90}$$

The imaginary parts of these equations produce conditions on the B-field parameters b_a, due to the fact that K^K only depends on $v^a = \mathrm{Im} t^a$, ensuring the reality of the second term,

$$\mathrm{Im}\partial_{t^a} W^K = \kappa_{abc} v_b (m_c - m_0 b_c) = 0. \qquad (3.91)$$

Therefore, b_c is stabilized at $b_c = \frac{m_c}{m_0}$ and vanishes when $\hat{F}^{\mathrm{bg}}_{(2)} = 0$, as claimed above. Of course, this assumes $m_0 \neq 0$ and also non-vanishing v_b and κ_{abc}. This leads us to the real part of equations (3.90). We will show that these yield $h_-^{(1,1)}$ equations to determine the $h_-^{(1,1)}$ moduli fields v^a or equivalently the γ^a used in the discussion earlier. They read

$$\mathrm{Re}(\partial_{t^a} W^K) + \mathrm{Im}(\partial_{t^a} K^K)\mathrm{Im}(W^K) = 0. \qquad (3.92)$$

More explicitly, we have

$$(4e_a m_0 + 2\kappa_{apq} m^p m^q + 3m_0^2 \kappa_{apq} v^p v^q)\kappa_{def} v^d v^e v^f \qquad (3.93)$$

$$+ (6m_0 e_d v^d + 3\kappa_{def} v^d m^e m^f)\kappa_{apq} v^p v^q = 0, \qquad (3.94)$$

where we made frequent use of the equations for the b_a parameters (see above). Multiplying by v^a and summing over a leads to[15]

$$10m_0 e_d v^d + 5\kappa_{def} v^d m^e m^f + 3m_0^2 \kappa_{def} v^d v^e v^f = 0. \qquad (3.95)$$

This gives us one quadratic equation for every v_a, thus generically fixing all the Kähler structure moduli, namely

$$10m_0 e_a + 5\kappa_{abc} m^b m^c + 3m_0^2 \kappa_{abc} v^b v^c = 0. \qquad (3.96)$$

[15]Solving equation (3.93) directly gives no solution with $vol_{(6)} \neq 0$ and any of the $v_a = 0$.

Application to the $T^6/\mathbb{Z}_4$ model

We start out by neglecting the twisted sector to show that we can reproduce the results found above. Then we discuss the details of the twisted sector and derive the results for all moduli.

Complex structure equations. Combining equations (3.83) and (3.31) we get[16]

$$-\frac{4p_0}{\sqrt{U_2}} = -8\sqrt{U_2}p_1 =: Q_0. \tag{3.97}$$

Assuming that we satisfy the tadpole cancelation conditions $p_0 = p_1 \equiv p$ implies that the complex structure is fixed at $U_2 = \frac{1}{2}$. Since $Q_0 = -4\sqrt{2}p$, the dilaton (cf. (3.84)) gets fixed at

$$e^{-\hat{\phi}} = -\frac{\sqrt{2}}{5}\frac{m_0}{p}\sqrt{vol_{(6)}}. \tag{3.98}$$

Note that this implies that $\text{sign}[m_0 p] = -1$.

The axions as derived above in (3.86) satisfy

$$p_0\xi^0 + p_1\xi^1 = e_0 + e_a b_a + \frac{1}{2}\kappa_{abc}m_a(b_b b_c - v_b v_c) - \frac{m_0}{6}\kappa_{abc}(b_a b_b b_c - 3b_a v_b v_c), \tag{3.99}$$

which agrees with (3.41) for $b_a = \frac{m_a}{m_0}$.

Kähler structure equations. The equations (3.96) yield the following result for

[16] Recall that we have normalized Ω s.t. $\mathrm{i}\int \Omega \wedge \bar{\Omega} = 1$ so that $K^{\text{cs}} = 0$.

the untwisted Kähler moduli,

$$
\begin{aligned}
v_1 &= \pm\sqrt{\frac{10}{3}}\frac{\hat{e}_2\sqrt{\hat{e}_3}}{\sqrt{\kappa m_0}\sqrt{-2\hat{e}_1\hat{e}_2+\hat{e}_4^2}},\\
v_2 &= \pm\sqrt{\frac{10}{3}}\frac{\hat{e}_1\sqrt{\hat{e}_3}}{\sqrt{\kappa m_0}\sqrt{-2\hat{e}_1\hat{e}_2+\hat{e}_4^2}},\\
v_3 &= \mp\sqrt{\frac{5}{6}}\frac{\sqrt{-2\hat{e}_1\hat{e}_2+\hat{e}_4^2}}{\sqrt{\kappa m_0}\sqrt{\hat{e}_3}},\\
v_4 &= \mp\sqrt{\frac{10}{3}}\frac{\hat{e}_4\sqrt{\hat{e}_3}}{\sqrt{\kappa m_0}\sqrt{-2\hat{e}_1\hat{e}_2+\hat{e}_4^2}},
\end{aligned}
$$

where we have defined shifted fluxes invariant under the shifts of t_a[17]

$$\hat{e}_i \equiv e_i + \frac{\kappa_{ijk} m_j m_k}{2m_0}. \tag{3.100}$$

For this solution to have a geometrical interpretation, we have to demand that $\text{sign}\,[m_0(-2\hat{e}_1\hat{e}_2+\hat{e}_4^2)] = \text{sign}\,[\hat{e}_3]$, $v_3 > 0$ and $(2\hat{e}_1\hat{e}_2-\hat{e}_4^2) > 0$. Comparing this with the solution found in (3.48) we see that the additional constraint $\text{sign}\,[m_0 e_3] < 0$ is required for this solution to be supersymmetric.

To look at one explicit supersymmetric large volume and small string coupling example, we use the flux quantization condition (3.34) to express the results in terms of flux integers. Taking the limit $f_1 = f_2 = f_3 = f_4 =: f \gg 1$ leads to $v_1 = v_2 = 2v_3 = -v_4 \sim \frac{72}{\kappa^{1/3}}\frac{\alpha'}{\sqrt{|f_0|}}\sqrt{f}$. Therefore, for the internal volume, the string

[17]Remember that there is a modular transformation that shifts the axions b_a by one.

coupling and the potential we get

$$vol_{(6)} = \kappa v_3(v_1 v_2 - \frac{1}{2}{v_4}^2) \sim 9 \times 10^4 \frac{(\alpha')^3}{|f_0|^{3/2}}\, f^{3/2}, \tag{3.101}$$

$$g_s = e^{\hat{\phi}} \sim 4 \left| \frac{h}{f_0^{1/4}} \right| \, f^{-3/4}, \tag{3.102}$$

$$V_{\text{eff}} = -\sqrt{\frac{3}{10}}\,\frac{243}{1600\pi^8}\sqrt{\left|\frac{f_0^5}{f^9}\right|}\frac{h^4}{(\alpha')^4} \sim -9 \times 10^{-6}\sqrt{|f_0^5|}\frac{h^4}{(\alpha')^4}\, f^{-9/2}. \tag{3.103}$$

Gauge redundancies and counting of solutions. An interesting question is to ask how many physically different solutions there are for different values of the Kähler axions $b_a = \frac{m_a}{m_0}$. There are certain modular transformations of infinite order that act as shifts on the axions and relate equivalent vacua [83]. A integer shift of the Kähler axions

$$b_a \to b_a + u_a, \quad u_a \in \mathbb{Z}, \forall a, \tag{3.104}$$

corresponds to a shift of the $\hat{F}_2$ flux $m_a \to m_a + u_a m_0$. Now, since $|m_0|$ is (almost) fixed by tadpole cancelation, we see that physically inequivalent choices for m_a (and thus b_a) are defined modulo $|m_0|$. Consequently, once m_0 is fixed there are at most two different inequivalent solutions for different values of the b_a.

Validity of approximations. In order for the low energy supergravity approximation (leading order in α') to be valid we have to make sure that the dimensionless expansion parameter

$$\frac{\alpha'}{R^2} \sim f^{-1/2} \ll 1. \tag{3.105}$$

Moreover, we also want the string coupling to be small enough to be in a perturbative regime where we can safely neglect quantum (string loop) corrections. As we have observed above, $g_s \sim f^{-3/4}$. Therefore, by choosing $f \gg 1$ sufficiently large, we can ensure both conditions simultaneously.

Another important issue is the backreaction of the fluxes on the geometry: Namely, in the presence of background fluxes, the internal space is strictly speaking no longer a Calabi-Yau orientifold. However, we want to make sure that the low energy spectrum we assumed is still correct. For this to be true we must check that the mass scale of the (canonically normalized) Kähler moduli is sufficiently small compared to the mass scale of the massive Kaluza-Klein modes ($m_{\rm KK} \sim \frac{1}{R}$) which we neglected. Performing the calculations in the 4D Einstein frame, we find

$$m_{\tilde{v}_a} \sim f^{-9/4} \ll m_{\rm KK} \sim f^{-1/4}, \tag{3.106}$$

where $\tilde{v}_a := \frac{\delta v_a}{\kappa_{10} < v_a >}$ is normalized to give a canonical kinetic term in the Lagrangian. Clearly, their masses will be much smaller than the Kaluza-Klein masses if we choose $f \gg 1$ large.

3.1.3 Moduli stabilization in the twisted sectors

Fixed point structure and exceptional divisors. After having described the moduli stabilization in the untwisted sector, it remains to investigate the stabilization of the blow-up modes in the twisted sectors. Therefore let us briefly summarize the fixed point structure of our orientifold model (table 6).

sector:	untw.	Θ, Θ^3-tw.	Θ^2-tw.	$\sum$
fixed pts.:	—	16 $\mathbb{Z}_4$	12 $\mathbb{Z}_2$+ 4 $\mathbb{Z}_4$ ($\mathbb{Z}_2$)	—
cplx. str.:	1	—	6+0	1+6
Kähler:	$5 \to 4$	$16 \to 12$	$6+4 \to 5+4$	$5+26 \to 4+21$

Table Table 6: *List of moduli before and after orientifold projection.*

The exceptional divisor can be determined as follows: We start by modding out the $T^6 = T^2_{(1)} \times T^2_{(2)} \times T^2_{(3)}$ by the $\mathbb{Z}_2$-action Θ^2. This yields 16 singularities of type $\mathbb{C}^2/\mathbb{Z}_2 \times T^2_{(3)}$, whose blow-up is given by 16 $\mathbb{C}P^1 \times T^2_{(3)}$. In a second step, we mod out this blown-up space $\widetilde{T^6/\mathbb{Z}_2}$ by the $\mathbb{Z}_2$-action Θ. The $\mathbb{C}P^1$s located at $\mathbb{Z}_2$ fixed points of the first two tori (cf. figure 2) get mapped into each other by Θ. Moreover, the two $\mathbb{Z}_2$ fixed points of the second torus are identified under the orientifold involution σ. This leaves us with $6 \to 5$ $\mathbb{C}P^1 \times T^2_{(3)}$ that contribute to the twisted Kähler moduli. Furthermore, the 6 $\mathbb{C}P^1$s at the $\mathbb{Z}_2$ fixed points can be tensored with the two 1-cycles on the third torus to yield 12 twisted 3-cycles of topology $S^2 \times S^1$ (which contribute 6 twisted complex structure moduli). The 4 $\mathbb{C}P^1$s sitting at the $\mathbb{Z}_4$ fixed points of the first two tori remain invariant under this action and contribute 4 Kähler moduli (the sizes of the $\mathbb{C}P^1$s) to the twisted sectors. The 16 fixed loci of the Θ-action are $\mathbb{C}P^1 \times \{\text{point}\}$, where $\{\text{point}\}$ denotes one of the fixed points of the third torus (cf. figure 2). Two of these get identified by σ. Blowing-up results in $16 \to 12$ $\mathbb{C}P^1 \times \mathbb{C}P^1$, which give us the 12 Kähler moduli from the Θ^1, Θ^3 sectors.

Intersection numbers. In order to solve the F-term conditions for the twisted Kähler moduli, we need to calculate the various triple intersection numbers of the blow-up cycles. The results are listed in table 7 below.

divisor	intersection type	intersection number
$T = \mathbb{C}P^1 \times T^2_{(3)}$	$T \circ T \circ T$	0
$T = \mathbb{C}P^1 \times T^2_{(3)}$	$T \circ T \circ \left[U = T^2_{(1)} \times T^2_{(2)}\right]$	$\beta = -2$
$T' = \mathbb{C}P^1 \times \mathbb{C}P^1$	$T' \circ T' \circ T'$	$\alpha = 8$

Table 7: *List of intersection numbers.*

These results can be used to extend the F-term equations discussed above to include the twisted moduli.

It is important to note that there must be a hierarchy between the untwisted and twisted Kähler moduli,

$$|m_0| \ll |e_A| \ll |e_a|, \tag{3.107}$$

in order to remain within the Kähler cone [83]. This is the reason why, although there are non-vanishing intersection numbers linking the twisted sectors to the untwisted sector, the values at which the untwisted Kähler moduli are stabilized will not significantly change compared to the analysis of only the untwisted sector above.

Solutions to Kähler structure equations. For the b_a we have the same solutions as above $b_a = \frac{m_a}{m_0}$ where a now runs from 0 to 26.

For the v_a we have to solve the equations (3.96). The solution is

$$\begin{aligned}
v_1 &= \pm\sqrt{\frac{10}{3}}\frac{\hat{e}_2\sqrt{\hat{e}_3}}{\sqrt{\kappa m_0}\sqrt{(-2\hat{e}_1\hat{e}_2+\hat{e}_4^2)-\frac{\kappa}{\beta}(\hat{e}_5^2+\ldots+\hat{e}_{14}^2)}}, \\
v_2 &= \pm\sqrt{\frac{10}{3}}\frac{\hat{e}_1\sqrt{\hat{e}_3}}{\sqrt{\kappa m_0}\sqrt{(-2\hat{e}_1\hat{e}_2+\hat{e}_4^2)-\frac{\kappa}{\beta}(\hat{e}_5^2+\ldots+\hat{e}_{14}^2)}}, \\
v_3 &= \mp\sqrt{\frac{5}{6}}\frac{\sqrt{(-2\hat{e}_1\hat{e}_2+\hat{e}_4^2)-\frac{\kappa}{\beta}(\hat{e}_5^2+\ldots+\hat{e}_{14}^2)}}{\sqrt{\kappa m_0}\sqrt{\hat{e}_3}}, \\
v_4 &= \mp\sqrt{\frac{10}{3}}\frac{\hat{e}_4\sqrt{\hat{e}_3}}{\sqrt{\kappa m_0}\sqrt{(-2\hat{e}_1\hat{e}_2+\hat{e}_4^2)-\frac{\kappa}{\beta}(\hat{e}_5^2+\ldots+\hat{e}_{14}^2)}}, \\
v_5 &= \pm\sqrt{\frac{10}{3}}\frac{\hat{e}_5\sqrt{\kappa\hat{e}_3}}{\sqrt{m_0}\,\beta\,\sqrt{(-2\hat{e}_1\hat{e}_2+\hat{e}_4^2)-\frac{\kappa}{\beta}(\hat{e}_5^2+\ldots+\hat{e}_{14}^2)}}, \\
&\vdots \\
v_{14} &= \pm\sqrt{\frac{10}{3}}\frac{\hat{e}_{14}\sqrt{\kappa\hat{e}_3}}{\sqrt{m_0}\,\beta\,\sqrt{(-2\hat{e}_1\hat{e}_2+\hat{e}_4^2)-\frac{\kappa}{\beta}(\hat{e}_5^2+\ldots+\hat{e}_{14}^2)}}, \\
v_{15} &= \pm\sqrt{\frac{10}{3}}\sqrt{-\frac{\hat{e}_{15}}{\alpha m_0}}, \\
&\vdots \\
v_{26} &= \pm\sqrt{\frac{10}{3}}\sqrt{-\frac{\hat{e}_{26}}{\alpha m_0}}.
\end{aligned} \tag{3.108}$$

As before, there are some additional conditions on the relative signs of the fluxes. To ensure reality of the Kähler moduli, we need to have

$$\text{sign}\,[\left(\frac{\hat{e}_3}{m_0((-2\hat{e}_1\hat{e}_2+\hat{e}_4^2)-\frac{\kappa}{\beta}(\hat{e}_5^2+\ldots+\hat{e}_{14}^2))}\right)]>0, \tag{3.109}$$

$$\text{sign}\,[\left(\frac{\hat{e}_A}{\alpha m_0}\right)]<0, \quad \forall A=15,\ldots,26. \tag{3.110}$$

The volume and the string coupling constant are

$$vol_{(6)} = \frac{1}{6}\kappa_{abc}v_a v_b v_c \tag{3.111}$$

$$= v_3(\kappa v_1 v_2 - \frac{\kappa}{2}{v_4}^2 + \frac{\beta}{2}\sum_{A=5}^{14} v_A^2) + \frac{\alpha}{6}\sum_{A=15}^{26} v_A^3 \tag{3.112}$$

$$= \frac{5}{3}\sqrt{\frac{5}{6}}\sqrt{\left|\frac{\hat{e}_3((2\hat{e}_1\hat{e}_2-\hat{e}_4^2)+\frac{\kappa}{\beta}(\hat{e}_5^2+\ldots+\hat{e}_{14}^2))}{\kappa m_0^3}\right|} + \frac{\alpha}{6}\sum_{A=15}^{26}\left(-\frac{10\hat{e}_A}{3\alpha m_0}\right)^{3/2}, \tag{3.113}$$

$$g_s = e^{\hat{\phi}} = -\frac{5}{\sqrt{2}}\frac{p}{m_0}\frac{1}{\sqrt{vol_{(6)}}}. \tag{3.114}$$

Due to the hierachy of fluxes mentioned above, the results for the untwisted sector do not deviate substantially from those obtained without taking the twisted sector into account.

Twisted complex structure moduli. As we saw above, including the twisted sector we now have 7 complex structure moduli to stabilize. The holomorphic 3-form is $\Omega(\tilde{z}) = Z^K(\tilde{z})a_K - \mathcal{F}_K(\tilde{z})b^K, K = 0,\ldots,6$. Equation (3.97) is still valid if we fix the normalization of Ω such that $\mathrm{i}\int\Omega\wedge\bar{\Omega} = 1$. For the twisted complex structure the $p_K, K = 2,\ldots,6$ are not constrained by the tadpole conditions. We can for example choose them to be $p_K = 0, K = 2,\ldots,6$ which would fix the corresponding complex structures $\mathrm{Im}(\mathcal{F}_K) = 0$. If we choose any of the $p_K, K = 2,\ldots,6$ to be non zero, the corresponding complex structure is fixed as

$$\mathrm{Im}(\mathcal{F}_K) = -\frac{p_K}{4\sqrt{2}\,p}. \tag{3.115}$$

The axions as derived above in (3.86) satisfy

$$\sum_{h^{(2,1)}=0}^{7} p_i \xi^i = e_0 + \frac{e_a m_a}{m_0} + \frac{\kappa_{abc} m_a m_b m_c}{3m_0^2}, \tag{3.116}$$

where we have used $b_a = \frac{m_a}{m_0}$ and a, b, c run from 1 to 26.

3.2 Toroidal Orientifold of $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ with non-standard involution

Here I present a $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold with non-standard involution, that has not been discussed in the literature, to the best of my knowledge.

We start by introducing the orbifold and orientifold actions and the resulting cohomology. Next we describe the NSNS-, metric and non-geometric fluxes and describe how they map to the parameters of the cohomology. We then discuss the D-terms explicitly.

Let $z^i = x^{2i-1} + \tau_i x^{2i}$, $i = 1, 2, 3$ be complex coordinates on the three 2-tori with the identifications $x^i = x^i + 1$, $i = 1, \ldots, 6$. The orbifold group is generated by two $\mathbb{Z}_2$ rotations

$$\theta_1 : (z^1, z^2, z^3) \mapsto (-z^1, -z^2, z^3), \tag{3.117}$$

$$\theta_2 : (z^1, z^2, z^3) \mapsto (z^1, -z^2, -z^3), \tag{3.118}$$

and the orientifold action is $\Omega_p(-1)^{F_L}\sigma$, where the holomorphic involution is given by

$$\sigma : (z^1, z^2, z^3) \mapsto (\overline{z}^2, \overline{z}^1, \overline{z}^3). \tag{3.119}$$

The mutual consistency of the orbifold and orientifold actions imposes constraints onto the complex structure moduli τ_i. One consistent choice is to set

$$\mathrm{Re}(\tau_1) = -\mathrm{Re}(\tau_2) \equiv U_1,\ \mathrm{Im}(\tau_1) = \mathrm{Im}(\tau_2) \equiv U_2,\ \mathrm{Re}(\tau_3) = 0, \mathrm{Im}(\tau_3) \equiv U_3. \tag{3.120}$$

In the following, we will restrict ourselves to the untwisted sector of the resulting theory. First we can write down the (untwisted) even cohomology of $T^6/\mathbb{Z}_2\times\mathbb{Z}_2$, dividing into even and odd subspaces under the involution σ. The cohomology classes are implicitly equated with their harmonic representatives. In the even subspace, there is one even zero form, namely the unit function 1. There are three independent $(1,1)$-forms: One even form,

$$\mu = dx^1 \wedge dx^2 - dx^3 \wedge dx^4, \tag{3.121}$$

and two odd forms,

$$\omega_1 = dx^1 \wedge dx^2 + dx^3 \wedge dx^4, \tag{3.122}$$

$$\omega_2 = 2dx^5 \wedge dx^6. \tag{3.123}$$

For the four-forms, we find one odd $(2,2)$-form,

$$\widetilde{\mu} = -\mu \wedge \omega_2, \tag{3.124}$$

and two even (2, 2)-forms,

$$\widetilde{\omega}^1 = \omega_1 \wedge \omega_2, \tag{3.125}$$

$$\widetilde{\omega}^2 = \omega_1 \wedge \omega_1. \tag{3.126}$$

The only six-form is odd under the involution σ,

$$\phi = 4dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dx^6. \tag{3.127}$$

Following the conventions introduced in the previous chapter, we find $f = 1$, $\widehat{d} = 1$, and $d_a^b = \text{diag}(1, 1)$ for the intersection numbers. The only non-vanishing components of the totally symmetric triple intersections are $\kappa_{112} = 1$ and $\widehat{\kappa}_{211} = -1$.Specifically, the Kähler form is given by $J = v^1\omega_1 + v^2\omega_2$, the corresponding metric is

$$\begin{aligned}
ds^2 &= \frac{v_1}{U_2}\left(dz^1 \otimes d\overline{z}^1 + dz^2 \otimes d\overline{z}^2\right) + \frac{v_2}{U_3} dz^3 \otimes d\overline{z}^3 \\
&= \frac{v_1}{U_2}\big((dx^1)^2 + (dx^3)^2 + (U_1^2 + U_2^2)((dx^2)^2 + (dx^4)^2) \\
&\qquad + 2U_1(dx^1 dx^2 - dx^3 dx^4)\big) + \frac{v_2}{U_3}(dx^5)^2 + v_2 U_3 (dx^6)^2.
\end{aligned}$$

For the metric to have Euclidean signature, we impose $v_1 > 0$ and $v_2 > 0$. The volume is

$$\mathcal{V}_6 = \frac{1}{2}(v_1)^2 v_2. \tag{3.128}$$

The odd cohomology is quite simple. It turns out that $H^1(X)$ and $H^5(X)$ are empty, so we are left with the three-forms, which again can be devided into even and odd under the involution. The four even three-forms are

$$a_1 = 2dx^1 \wedge dx^3 \wedge dx^6, \tag{3.129}$$

$$a_2 = 2dx^2 \wedge dx^4 \wedge dx^6, \tag{3.130}$$

$$a_3 = \sqrt{2}(dx^1 \wedge dx^4 \wedge dx^5 + dx^2 \wedge dx^3 \wedge dx^5), \tag{3.131}$$

$$a_4 = \sqrt{2}(dx^1 \wedge dx^4 \wedge dx^6 - dx^2 \wedge dx^3 \wedge dx^6), \tag{3.132}$$

and the four odd 3-forms are

$$b_1 = 2dx^2 \wedge dx^4 \wedge dx^5, \tag{3.133}$$

$$b_2 = 2dx^1 \wedge dx^3 \wedge dx^5, \tag{3.134}$$

$$b_3 = \sqrt{2}(dx^1 \wedge dx^4 \wedge dx^6 + dx^2 \wedge dx^3 \wedge dx^6), \tag{3.135}$$

$$b_4 = \sqrt{2}(dx^1 \wedge dx^4 \wedge dx^5 - dx^2 \wedge dx^3 \wedge dx^5). \tag{3.136}$$

The 3-forms satisfy $\int a_K \wedge b^J = \delta_K^J$. The holomorphic 3-form

$$\Omega = \frac{i}{U_2\sqrt{2U_3}} dz^1 \wedge dz^2 \wedge dz^3 = \mathcal{Z}^K a_K - \mathcal{F}_K b^K, \tag{3.137}$$

satisfies $\sigma : \Omega \mapsto \overline{\Omega}$ and is normalized such that $i \int_X \Omega \wedge \overline{\Omega} = 1$. More specifically, we find

$$\mathcal{Z}^1 = -\frac{\sqrt{U_3}}{2\sqrt{2}U_2}, \quad \mathcal{Z}^2 = \frac{\sqrt{U_3}(U_1^2 + U_2^2)}{2\sqrt{2}U_2}, \quad \mathcal{Z}^3 = -\frac{1}{2\sqrt{U_3}}, \quad \mathcal{Z}^4 = \frac{\sqrt{U_3}U_1}{2U_2}, \tag{3.138}$$

$$\mathcal{F}_1 = \frac{\imath(U_1^2 + U_2^2)}{2\sqrt{2U_3}U_2}, \quad \mathcal{F}_2 = \frac{-\imath}{2\sqrt{2U_3}U_2}, \quad \mathcal{F}_3 = \frac{\imath\sqrt{U_3}}{2}, \quad \mathcal{F}_4 = \frac{\imath U_1}{2\sqrt{U_3}U_2}. \tag{3.139}$$

Next we will study the most general NSNS fluxes in the untwisted sector consistent with the orientifold action. The expansion $H = p_K b^K$ yields $p_1 = H_{245}/2$, $p_2 = H_{135}/2$, $p_3 = H_{146}/\sqrt{2} = H_{236}/\sqrt{2}$ and $p_4 = H_{145}/\sqrt{2} = -H_{235}/\sqrt{2}$. Similarly, imposing invariance under the orientifold group, we are left with 12 independent metric fluxes,

$$\omega^1_{35} = \omega^3_{15}, \tag{3.140}$$

$$\omega^1_{36} = -\omega^3_{16}, \tag{3.141}$$

$$\omega^1_{45} = -\omega^3_{25}, \tag{3.142}$$

$$\omega^1_{46} = \omega^3_{26}, \tag{3.143}$$

$$\omega^2_{35} = -\omega^4_{15}, \tag{3.144}$$

$$\omega^2_{36} = \omega^4_{16}, \tag{3.145}$$

$$\omega^2_{45} = \omega^4_{25}, \tag{3.146}$$

$$\omega^2_{46} = -\omega^4_{26}, \tag{3.147}$$

$$\omega^5_{14} = \omega^5_{23}, \tag{3.148}$$

$$\omega^6_{14} = -\omega^6_{23}, \tag{3.149}$$

$$\omega^6_{13}, \tag{3.150}$$

$$\omega^6_{24}. \tag{3.151}$$

The left hand column contains the representatives. In terms of r-matrices we write

$$r^K_a = \begin{pmatrix} \omega^1_{45} & -\omega^2_{35} & (\omega^1_{36} - \omega^2_{46})/\sqrt{2} & (-\omega^1_{35} - \omega^2_{45})/\sqrt{2} \\ -\omega^6_{24} & -\omega^6_{13} & \sqrt{2}\omega^5_{14} & -\sqrt{2}\omega^6_{14} \end{pmatrix}, \tag{3.152}$$

and

$$\widehat{r}_K = \begin{pmatrix} -\omega^2_{36} & \omega^1_{46} & (\omega^1_{35} - \omega^2_{45})/\sqrt{2} & (-\omega^1_{36} - \omega^2_{46})/\sqrt{2} \end{pmatrix}. \tag{3.153}$$

For the Q-fluxes we have

$$Q_5^{14} = -Q_5^{23}, \tag{3.154}$$

$$Q_6^{14} = Q_6^{23}, \tag{3.155}$$

$$Q_3^{15} = -Q_1^{35}, \tag{3.156}$$

$$Q_4^{15} = Q_2^{35}, \tag{3.157}$$

$$Q_3^{16} = Q_1^{36}, \tag{3.158}$$

$$Q_4^{16} = -Q_2^{36}, \tag{3.159}$$

$$Q_3^{25} = Q_1^{45}, \tag{3.160}$$

$$Q_4^{25} = -Q_2^{45}, \tag{3.161}$$

$$Q_3^{26} = -Q_1^{46}, \tag{3.162}$$

$$Q_4^{26} = Q_2^{46}, \tag{3.163}$$

$$Q_5^{13}, \tag{3.164}$$

$$Q_5^{24}. \tag{3.165}$$

Again, we choose the 12 left hand column entries as representatives. In terms of q-matrices, this can be written as

$$q_K^a = \begin{pmatrix} -2Q_4^{16} & 2Q_3^{26} & \sqrt{2}(Q_3^{15} - Q_4^{25}) & \sqrt{2}(Q_3^{16} + Q_4^{26}) \\ -Q_5^{13} & -Q_5^{24} & \sqrt{2}Q_6^{14} & -\sqrt{2}Q_5^{14} \end{pmatrix}, \tag{3.166}$$

and

$$\widehat{q}^K = \begin{pmatrix} 2Q_3^{25} & -2Q^{15_4} & \sqrt{2}(Q_3^{16} - Q_4^{26}) & \sqrt{2}(Q_3^{15} + Q_4^{25}) \end{pmatrix}. \tag{3.167}$$

Finally, we find

$$s_K = \begin{pmatrix} 2R^{136} & 2R^{246} & \sqrt{2}R^{145} = \sqrt{2}R^{235} & \sqrt{2}R^{146} = -\sqrt{2}R^{236} \end{pmatrix}. \tag{3.168}$$

This example allows for D-terms that are properly quantized. The simplest case with H-flux and D-terms ($r_a^K = q_K^a = \widehat{q}^K = s_K = 0$) has the following Bianchi identities

$$\widehat{r}^a \widehat{r}^3 = 0,\ a = 1, 2, 4, \tag{3.169}$$

$$p_K \widehat{r}^K = 0. \tag{3.170}$$

From the base-fiber splitting we obtain the following quantization conditions

$$\widehat{r}^3 = 0, \tag{3.171}$$

$$2p_1, 2p_2, \sqrt{2}p_3, \sqrt{2}p_4 \,\epsilon\, \mathbb{Z}, \tag{3.172}$$

$$\cosh \surd, \frac{\widehat{r}^4}{\sqrt{2}} \frac{\sinh \surd}{\surd}, \widehat{r}^a \frac{\sinh \surd}{\surd} \,\epsilon\, \mathbb{Z},\ a = 1, 2, \tag{3.173}$$

where $\surd = \sqrt{-\widehat{r}^1\widehat{r}^2 - (\widehat{r}^4)^2/2}$. One simple solution is to choose $\pm\widehat{r}^1 = \mp\widehat{r}^2 = \widehat{r}^4/\sqrt{2} = c \,\epsilon\, \mathbb{Z}$. The Bianchi identities then become $p^1 - p^2 \pm \sqrt{2}p^4 = 0$. The general solution to the quantization condition has three classes of solutions

$$I) \quad 1 \leq \cosh \surd = m,\ \widehat{r}^1 = \frac{n_1}{x},\ \widehat{r}^2 = \frac{n_2}{x},\ \widehat{r}^4 = \frac{\sqrt{2}n_4}{x},\ 1 = m^2 + n_1 n_2 + n_4^2,$$

$$\text{with } x = \frac{\sinh \surd}{\surd} \text{ and } m, n_1, n_2, n_4 \,\epsilon\, \mathbb{Z}, \tag{3.174}$$

$$II) \quad \sqrt{\widehat{r}_1 \widehat{r}_2 + \frac{\widehat{r}_4^2}{2}} = k\pi, \quad k \,\epsilon\, \mathbb{Z},\ k > 0, \tag{3.175}$$

$$III) \quad \widehat{r}^a = (k + \frac{1}{2})\pi n_a,\ a = 1, 2,$$

$$\widehat{r}^4 = \sqrt{2}(k + \frac{1}{2})\pi n_4,\ 1 = n_1 n_2 + n_4^2,\ n_1, n_2, n_4, k \,\epsilon\, \mathbb{Z}. \tag{3.176}$$

The Kähler potential and the superpotential are,

$$
\begin{aligned}
K =& 4D = -\log -(N^3 - \bar{N}^3)^2(2(N_1 - \bar{N}_1)(N_2 - \bar{N}_2) + (N_4 - \bar{N}_4)^2) \\
& - \log \frac{i}{2}(t^1 - \bar{t}^1)(t^1 - \bar{t}^1)(t^2 - \bar{t}^2), && (3.177) \\
W =& e_1 t^1 + e_2 t^2 + \frac{m_0}{2} t^1 t^2 t^3 + 2p_1 N^1 + 2p_2 N^2 + 2p_3 N^3 + 2p_4 N^4. && (3.178)
\end{aligned}
$$

These give the following potential for the two Kähler moduli v_a, the dilaton $Q = e^{-D}$ and the three complex structure moduli U_K (the axions are already stabilized or flat directions so that they do not appear in the potential)

$$
\begin{aligned}
V =& \frac{e_1^2}{2Q^4 v_2} + \frac{e_2^2 v_2}{Q^4 v_1^2} + \frac{m_0^2 v_1^2 v_2}{4Q^4} \\
& - \frac{m_0 p_1 \sqrt{U_3}}{\sqrt{2} Q^3 U_2} + \frac{m_0 p_2 (U_1^2 + U_2^2)\sqrt{U_3}}{\sqrt{2} Q^3 U_2} - \frac{m_0 p_3}{Q^3 \sqrt{U_3}} + \frac{m_0 p_4 U_1 \sqrt{U_3}}{Q^3 U_2} \\
& + \frac{p_1^2 U_3}{2Q^2 U_2^2 v_1^2 v_2} + \frac{p_2^2 (U_1^2 + U_2^2)^2 U_3}{2Q^2 U_2^2 v_1^2 v_2} + \frac{p_3^2}{2Q^2 U_3 v_1^2 v_2} + \frac{p_4^2 (2U_1^2 + U_2^2) U_3}{2Q^2 U_2^2 v_1^2 v_2} \\
& - \frac{p_1 p_2 U_1^2 U_3}{Q^2 U_2^2 v_1^2 v_2} - \frac{\sqrt{2} p_1 p_4 U_1 U_3}{Q^2 U_2^2 v_1^2 v_2} + \frac{\sqrt{2} p_2 p_4 U_1 (U_1^2 + U_2^2) U_3}{Q^2 U_2^2 v_1^2 v_2} \\
& + \frac{(\hat{r}^1 (U_1^2 + U_2^2) - \hat{r}^2 + \sqrt{2} \hat{r}^4 U_1)^2}{4Q^2 U_2^2 U_3 v_2}. && (3.179)
\end{aligned}
$$

We need to minimize this potential taking into account the quantization conditions (3.172), (3.173) and the Bianchi identity $\hat{r}^1 p_1 + \hat{r}^2 p_2 + \hat{r}^4 p_4 = 0$. The moduli need to satisfy $v_1, v_2, Q, U_2, U_3 > 0$.

The tadpole condition is

$$
-m_0 p_K b^K + \frac{1}{\sqrt{2}}[\delta_{D6}] = \sqrt{2}[\delta_{O6}]. \qquad (3.180)
$$

Where $[\delta_{O6}] = -2\sqrt{2} b^3$. This means that in the absence of (anti) D-branes $p_1 = p_2 = p_4 = 0$ and thus one probably needs to include (anti) D-branes to get a rich enough potential. This means that the tadpole condition does not give any restrictions since we can just include the right number of branes to satisfy it.

We can eliminate e_1, e_2, m_0 by the following rescaling

$$\begin{aligned} v_1 &\to v_1\sqrt{\frac{e_2}{m_0}}, \\ v_2 &\to v_2\frac{e_1}{\sqrt{e_2 m_0}}, \\ Q &\to Q e_1\sqrt{\frac{e_2}{m_0}}, \\ V &\to V\frac{m_0^{5/2}}{e_1^3 e_2^{3/2}}, \\ \widehat{r}^K &\to \widehat{r}^K\sqrt{\frac{m_0}{e_2}}. \end{aligned}$$

One can furthermore fix the sum of the p_K to for example 1 by rescaling

$$\begin{aligned} Q &\to Q\frac{1}{p}, \\ V &\to V p^4, \\ \widehat{r}^K &\to \widehat{r}^K p, \\ p &= p_1 + p_2 + p_3 + p_4. \end{aligned}$$

Since we also have to satisfy the Bianchi identity $\widehat{r}^1 p_1 + \widehat{r}^2 p_2 + \widehat{r}^4 p_4 = 0$ this leaves us with 5 parameters and 6 moduli.

From the expression for the Kähler potential we can easily calculate the Kähler metric and its inverse. In the complex structure sector we find

$$K_{I\bar{J}} = e^{2D}\begin{pmatrix} \frac{2\left(U_1^2+U_2^2\right)^2}{U_2^2 U_3} & -\frac{2U_1^2}{U_2^2 U_3} & 0 & \frac{2\sqrt{2}U_1\left(U_1^2+U_2^2\right)}{U_2^2 U_3} \\ -\frac{2U_1^2}{U_2^2 U_3} & \frac{2}{U_2^2 U_3} & 0 & -\frac{2\sqrt{2}U_1}{U_2^2 U_3} \\ 0 & 0 & 2U_3 & 0 \\ \frac{2\sqrt{2}U_1\left(U_1^2+U_2^2\right)}{U_2^2 U_3} & -\frac{2\sqrt{2}U_1}{U_2^2 U_3} & 0 & \frac{2\left(2U_1^2+U_2^2\right)}{U_2^2 U_3} \end{pmatrix}, \qquad (3.181)$$

$$K^{\bar{I}J} = e^{-2D} \begin{pmatrix} \frac{U_3}{2U_2^2} & -\frac{U_1^2 U_3}{2U_2^2} & 0 & -\frac{U_1 U_3}{\sqrt{2} U_2^2} \\ -\frac{U_1^2 U_3}{2U_2^2} & \frac{U_3 \left(U_1^2 + U_2^2\right)^2}{2U_2^2} & 0 & \frac{U_1 U_3 \left(U_1^2 + U_2^2\right)}{\sqrt{2} U_2^2} \\ 0 & 0 & \frac{1}{2U_3} & 0 \\ -\frac{U_1 U_3}{\sqrt{2} U_2^2} & \frac{U_1 U_3 \left(U_1^2 + U_2^2\right)}{\sqrt{2} U_2^2} & 0 & \frac{U_3 \left(2U_1^2 + U_2^2\right)}{2U_2^2} \end{pmatrix}. \tag{3.182}$$

The first Kähler derivatives are given by $\partial_I K = -4e^D \mathcal{F}_I$, or explicitly,

$$\partial_1 K = -ie^D \frac{\sqrt{2}\left(U_1^2 + U_2^2\right)}{U_2 \sqrt{U_3}}, \qquad \partial_2 K = ie^D \frac{\sqrt{2}}{U_2 \sqrt{U_3}},$$

$$\partial_3 K = -2ie^D \sqrt{U_3}, \qquad \partial_4 K = -2ie^D \frac{U_1}{U_2 \sqrt{U_3}}. \tag{3.183}$$

As a check, these expressions satisfy the relation $K^{\bar{I}J} \overline{\partial_I K} \partial_J K = 4$.

The Kähler moduli sector can be obtained from the general expressions of section 4.4.

The D-term contribution for $\pm \widehat{r}^1 = \mp \widehat{r}^2 = \widehat{r}^4 / \sqrt{2} = c \,\epsilon\, \mathbb{Z}$ is

$$V_D = \frac{c^2 e^{2D}}{4 v_2 U_2^2 U_3} \left((U_1 \pm 1)^2 + U_2^2 \right)^2 \tag{3.184}$$

In terms of the complex superfields this is

$$V_D = \frac{c^2}{8 v_2} \frac{\left(\operatorname{Im}(N_2) - \operatorname{Im}(N_1) \pm \sqrt{2} \operatorname{Im}(N_4) \right)^2}{\left(2 \operatorname{Im}(N_1) \operatorname{Im}(N_2) + (\operatorname{Im}(N_4)^2)^2\right.}. \tag{3.185}$$

Chapter 4

On the Possibility of Inflation

4.1 Introduction

Inflation is the favored paradigm used to explain the homogeneity, isotropy and flatness of our universe. Two recent papers [108, 107] have investigated the possibility of inflation in type IIA compactifications. The authors study the so-called first slow-roll parameter ϵ and derive a no-go theorem for models featuring a certain important subclass of NSNS and RR fluxes. Moreover, they readily provide suggestions on how to possibly evade the no-go theorem (and similar ones that can be derived for slightly different assumptions). In the following I will elaborate more on this topic in the framework of the models discussed above.

4.2 Inflationary no-go theorems in type IIA flux compactifications

I will begin by briefly reviewing the no-go theorem of [107] and some possible modifications. For inflation to be feasible, we need the first slow-roll parameter $\epsilon \ll 1$ and $V > 0$. Hertzberg et al. derive a *lower bound* on ϵ by looking at only two moduli, namely

$$\hat{\rho} = \sqrt{\frac{3}{2}}\overline{m}_P \ln \rho, \quad \hat{\tau} = \sqrt{2}\overline{m}_P \ln \tau, \tag{4.1}$$

where $\overline{m}_P = \frac{1}{\sqrt{8\pi G}}$ is the reduced Planck mass and $\hat{\rho}, \hat{\tau}$ are redefinitions of the volume and dilaton moduli,

$$\rho = \mathrm{vol}^{1/3}, \quad \tau = e^{-\phi}\mathrm{vol}^{1/2}, \tag{4.2}$$

such that they have canonically normalized kinetic terms in the four-dimensional action. Now, the slow-roll parameter ϵ in multi-field inflation can be expressed as

$$\epsilon = \frac{\overline{m}_P}{2}\left[\left(\frac{\partial \ln V}{\partial \hat{\rho}}\right)^2 + \left(\frac{\partial \ln V}{\partial \hat{\tau}}\right)^2 + \sum_i \left(\frac{\partial \ln V}{\partial \hat{\phi}_i}\right)^2\right], \tag{4.3}$$

where the $\hat{\phi}_i$ denote all other (canonically normalized) moduli in the problem. A lower bound is therefore given by

$$\epsilon \geq \frac{\overline{m}_P}{2}\left[\left(\frac{\partial \ln V}{\partial \hat{\rho}}\right)^2 + \left(\frac{\partial \ln V}{\partial \hat{\tau}}\right)^2\right]. \tag{4.4}$$

To this end, one investigates the different contributions to the scalar potential in four dimensions,

$$V = V_3 + V_{D6} + V_{O6} + \sum_p V_p = \frac{A_3(\phi_i)}{\rho^3\tau^2} + \frac{A_{D6}(\phi_i)}{\tau^3} - \frac{A_{O6}(\phi_i)}{\tau^3} + \sum_p \frac{A_p(\phi_i)}{\rho^{p-3}\tau^4}, \tag{4.5}$$

where V_3 denotes the contribution from the H_3-flux, $V_{D6/O6}$ the contributions from $D6$-branes/$O6$-planes and V_p corresponds to the various RR p-form fluxes F_p. The coefficients $A_j(\phi_i) \geq 0$ are in general complicated functions of the moduli and fluxes. In the absence of more general NSNS fluxes, such as metric and non-geometric fluxes, one can show that the potential (4.5) satisfies

$$-\rho\frac{\partial V}{\partial \rho} - 3\tau\frac{\partial V}{\partial \tau} = 9V + \sum_p pV_p \geq 9V, \tag{4.6}$$

since $V_p \geq 0$. This relation shows that every vacuum in this framework is necessarily AdS, since for $\frac{\partial V}{\partial \rho} = \frac{\partial V}{\partial \tau} = 0$, one obtains $V = -\frac{1}{9}\sum_p pV_p$. Now, assuming $V > 0$, which is a necessary condition for inflation, and rewriting (4.6) in terms of the canonically normalized variables $\hat{\rho}, \hat{\tau}$ yields

$$\overline{m}_P \Big| \sqrt{\frac{3}{2}} \left(\frac{\partial \ln V}{\partial \hat{\rho}}\right) + 3\sqrt{2}\left(\frac{\partial \ln V}{\partial \hat{\tau}}\right) \Big| \geq 9. \tag{4.7}$$

Minimizing ϵ, taking into account this lower bound, reveals

$$V > 0 \Longrightarrow \epsilon \geq \frac{27}{13}. \tag{4.8}$$

The conclusion presented in [107] is that both de Sitter vacua and inflation are ruled out everywhere in field space. This is not very surprising, considering that the generic vacua in these constructions are anti-de Sitter (AdS). Therefore $V > 0$ implies that we are generically not very close to the minima of the potential and naturally the potential should be expected to be steep away from the minima.

It was already pointed out in [107] that a possible way to work around this no-go theorem is to include other fluxes. In the following, we will consider the effects of metric and non-geometric fluxes:

$$V = \left(\frac{A_3(\phi_i)}{\rho^3\tau^2} + \frac{A_f(\phi_i)}{\rho\tau^2} + \frac{A_Q(\phi_i)\rho}{\tau^2} + \frac{A_R(\phi_i)\rho^3}{\tau^2}\right) + \frac{A_{D6/O6}(\phi_i)}{\tau^3} + \sum_p \frac{A_p(\phi_i)}{\rho^{p-3}\tau^4}, \tag{4.9}$$

where we have adopted notation following the conventions in [14]. Restricting the discussion to metric fluxes ($A_Q = A_R = 0$), we find that (4.6) becomes

$$-\rho\frac{\partial V}{\partial\rho} - 3\tau\frac{\partial V}{\partial\tau} = 9V + \sum_p pV_p - 2V_f, \tag{4.10}$$

where $V_f = \frac{A_f(\phi_i)}{\rho\tau^2} \geq 0$ at least for all metric fluxes decending from ten-dimensional constructions (base/fiber splitting [**?**]). Therefore, there is *no lower bound* in this case and it should in principle be possible to find inflation in the class of models including metric (and possibly non-geometric) fluxes. Unfortunately, we have not been able to construct an explicit example yet. Note, however, that this statement is only valid in the case of massive type IIA, $m_0 \neq 0$. Employing a slightly different combination of derivatives, we find

$$-\rho\frac{\partial V}{\partial\rho} - \tau\frac{\partial V}{\partial\tau} = 3V + 2V_3 + \sum_p (p-2)V_p. \tag{4.11}$$

If $m_0 = 0$, we have $V_0 = 0$ and this again implies a lower bound as before, namely

$$\overline{m}_P \Big| \sqrt{\frac{3}{2}}\left(\frac{\partial \ln V}{\partial\hat{\rho}}\right) + \sqrt{2}\left(\frac{\partial \ln V}{\partial\hat{\tau}}\right)\Big| \geq 3, \tag{4.12}$$

yielding $\epsilon \geq \frac{9}{7}$ for the slow-roll parameter, which is still at least of order one so that there is no part of the effective potential that allows for inflation.

4.3 De Sitter vacua

The question of inflation above is intimately tied to the question of whether or not there exist de Sitter (dS) vacua in our class of models. We are especially interested in 'simple' de Sitter vacua, namely those who are local minima of the effective potential considering all of the above mentioned ingredients (fluxes), but nothing else. This would be particularly appealing from an aesthetical view point. Here we review a method for finding dS vacua first proposed by Silverstein [109] in the language appropriate for the type IIA models discussed herein.

4.3.1 Conventions

We will briefly remind the reader of some notations and conventions discussed earlier in the context of the purpose at hand. Consider type IIA string theory on a Calabi-Yau three-fold X, equipped with a $\mathbb{Z}_2$ orientifold action which includes an anti-holomorphic involution σ. The cohomology of X then splits into even and odd parts, depending upon the behavior of each class under σ. We will take the following basis of representative forms:

- The zero-form 1,
- a set of odd (1,1)-forms ω_a, $a = 1, \ldots, h^{1,1}_-$,
- a set of even (1,1)-forms μ_α, $\alpha = 1, \ldots, h^{1,1}_+$,
- a set of even (2,2)-forms $\widetilde{\omega}^a$, $a = 1, \ldots, h^{1,1}_-$,
- a set of odd (2,2-forms $\widetilde{\mu}^\alpha$, $\alpha = 1, \ldots, h^{1,1}_+$,
- a six form φ, odd under σ,
- a set of even three-forms a_K, $K = 1, \ldots, h^{2,1} + 1$,
- and a set of odd three-forms b^K, $K = 1, \ldots, h^{2,1} + 1$.

It turns out that we can always choose the a_K and b^K to form a symplectic basis such that the only non-vanishing intersections are

$$\int_X a_K \wedge b^J = \delta_K^J. \tag{4.13}$$

For the even-degree forms we will allow a bit more freedom of scaling, in order to simplify some explicit computations in the case of toroidal orientifold examples. We will take the intersections to be

$$\int_X \varphi = f, \qquad \int_X \omega_a \wedge \omega_b \wedge \omega_c = \kappa_{abc}, \qquad \int_X \omega_a \wedge \mu_\alpha \wedge \mu_\beta = \widehat{\kappa}_{a\,\alpha\beta},$$

$$\int_X \omega_a \wedge \widetilde{\omega}^b = d_a{}^b, \qquad \int_X \mu_\alpha \wedge \widetilde{\mu}^\beta = \widehat{d}_\alpha{}^\beta. \tag{4.14}$$

If we chose the four-forms to be a basis dual to the two forms, then we would of course set $d_a{}^b = \delta_a^b$, $\widehat{d}_\alpha{}^\beta = \delta_\alpha^\beta$, but we will prefer instead to leave things here more general[1].

Now let us describe the four-dimensional fields of this class of compactifications, restricting ourselves, for simplicity, to the bosonic sector. First we have the Kähler moduli, parametrized by complex scalar fields $t^a = u^a + iv^a$ coming from the expansion

$$B + iJ = J_c = t^a \omega_a, \tag{4.15}$$

where the complexified Kähler form J_c must be odd under σ. Note that the Kähler form $J = v^a \omega_a$ determines the compactification volume (in string frame) via

$$\mathcal{V}_6 = \frac{1}{3!}\int_X J \wedge J \wedge J = \frac{1}{6}\kappa_{abc} v^a v^b v^c. \tag{4.16}$$

To describe the complex moduli, let us write the holomorphic three-form as

$$\Omega = \mathcal{Z}^K a_K - \mathcal{F}_K b^K. \tag{4.17}$$

[1]Note however that Poincaré duality implies in this case that d and $\widehat{d}$ are both invertible matrices. Indeed we will need to use this fact to write explicit expressions below.

We will use conventions in which

$$i \int_X \Omega \wedge \bar{\Omega} = 1, \qquad \sigma^* \Omega = \bar{\Omega}, \tag{4.18}$$

so that the $\mathcal{Z}^K$ are real functions of the complex moduli and $\mathcal{F}_K$ are pure imaginary, and together they satisfy the constraint $\mathcal{Z}^K \mathcal{F}_K = -i/2$. We can now define a complexified version [93]

$$\Omega_c = C_3 + 2ie^{-D}\,\mathrm{Re}\,\Omega = \left(\xi^K + 2ie^{-D}\mathcal{Z}^K\right) a_K, \tag{4.19}$$

where $e^{-D} = \mathcal{V}_6^{1/2} e^{-\phi}$ contains the dilaton and we expand the periods of C_3 (which must be even under σ in order to survive the orientifold projection) as $C_3 = \xi^K a_K$. Note that we abuse notation somewhat here as we ignore other pieces which contribute to the ten-dimensional R-R three-form potential C_3, namely pieces that give rise to four-dimensional vectors and (local) pieces that give the four-form R-R flux, both of which will be discussed below. The complex moduli $N^K = \frac{1}{2}\xi^K + ie^{-D}\mathcal{Z}^K$ are then simply given by the expansion

$$\Omega_c = 2N^K a_K, \tag{4.20}$$

and include the complex structure moduli of the metric, the dilaton, and the R-R three-form periods.

There are also $h_+^{1,1}$ four-dimensional vectors from the decomposition of the R-R three-form potential, which includes a contribution

$$C_3 = A^\alpha \wedge \mu_\alpha. \tag{4.21}$$

We can now consider turning on fluxes. In the R-R sector, this leads us to include

$$F_0 = m_0, \qquad F_2 = m^a \omega_a, \qquad F_4 = e_a \widetilde{\omega}^a, \qquad F_6 = e_0 \varphi. \tag{4.22}$$

From the NS-NS sector, we can include the usual H-flux,

$$H = p_K b^K, \tag{4.23}$$

but metric fluxes can also be considered, which can be described by the $h_-^{1,1}(h^{2,1}+1)$ coefficients r_{aK} and the $h_+^{1,1}(h^{2,1}+1)$ coefficients $\widehat{r}_\alpha^K$ (note that there are $\frac{1}{2}b_2 b_3$ metric fluxes all together, where $b_i = \dim(H^i(X))$ are the Betti numbers of X), and nongeometric fluxes, described by q_K^a, $\widehat{q}^{\alpha K}$, and s_K.

As usual, there will be various constraints on these fluxes. First there will be tadpole conditions,

$$-\sqrt{2}\,(p_K m_0 - raKm^a + q_K^a e_a - s_K e_0) + N_K^{(\mathrm{D6})} = 2N_K^{(\mathrm{O6})}. \tag{4.24}$$

And there will also be Bianchi identities satisfied by the generalized NS-NS fluxes. Some of these identities are universal, namely

$$\begin{aligned} \widehat{r}_\alpha^K p_K = \widehat{r}_\alpha^K s_K = \widehat{q}^{\alpha K} p_K = \widehat{q}^{\alpha K} s_K &= 0, \quad \forall \alpha, \\ \widehat{r}_\alpha^K r_{bK} = \widehat{r}_\alpha^K q_K^b = \widehat{q}^{\alpha K} r_{bK} = \widehat{q}^{\alpha K} q_K^b &= 0, \quad \forall \alpha, b, \\ f^{-1} p_{[K} s_{J]} + \left(d^{-1}\right)_a{}^b r_{b[K} q_{J]}^a = \left(\widehat{d}^{-1}\right)_\alpha{}^\beta \widehat{q}^{\alpha[K} \widehat{r}_\beta^{J]} &= 0, \quad \forall K, J, \end{aligned} \tag{4.25}$$

but constructions from base-fiber splitting indicate that there can be other Bianchi identities as well which can only be seen when considering explicit constructions.

Throughout this section, indices will be dropped whenever their structure can be otherwise apparent, so for example, the above Bianchi identities can be simplified and written in a concise way as

$$\begin{aligned} (\widehat{r}p)_\alpha = (\widehat{r}s)_\alpha = (\widehat{q}p)^\alpha = (\widehat{q}s)^\alpha &= 0, \quad \forall\alpha, \\ (\widehat{r}r)_{\alpha b} = (\widehat{r}q)_\alpha^b = (\widehat{q}r)_b^\alpha = (\widehat{q}q)^{\alpha b} &= 0, \quad \forall\alpha, b, \\ f^{-1}p_{[K}s_{J]} + \left(r(d^{-1}q)\right)_{[KJ]} = \left((\widehat{d}^{-1}\widehat{q})\widehat{r}\right)^{[KJ]} &= 0, \quad \forall K, J. \end{aligned} \tag{4.26}$$

It should be easy to recover the correct index structure whenever necessary

4.4 Effective Potential

The setup with the above described ingredients leads to an effective $\mathcal{N} = 1$ supergravity theory in four dimensions. To describe this effective theory, and particularly the effecitive potential for the complex scalar fields t^a and N^K, one must specify the Kähler potential K, the holomorphic superpotential W, the holomorphic gauge kinetic couplings $f_{\alpha\beta}$, and the gauge transformations of the scalar fields under the different U(1) gauge groups arising from the four-dimensional vectors (i.e. we must provide the electric and magnetic charges of the scalar fields). Then the effective action for the scalars is

$$S = -\int \left\{K_{a\bar{b}} dt^a \wedge *d\overline{t^b} + K_{I\bar{J}} dN^I \wedge *d\overline{N^J} + V * 1\right\}, \tag{4.27}$$

where the scalar potential is

$$V = e^K \left(K^{a\bar{b}} D_a W \overline{D_b W} + K^{I\bar{J}} D_I W \overline{D_J W} - 3|W|^2\right) + \frac{1}{2}\left(\mathrm{Re}\, f\right)^{-1\,\alpha\beta} D_\alpha D_\beta. \tag{4.28}$$

Here, $*$ denotes the four-dimensional Hodge dual, $K_{a\bar{b}} = \frac{\partial}{\partial t^a}\frac{\partial}{\partial \overline{t^b}} K$, $K^{a\bar{b}}$ is its (transpose) inverse, $D_a W = \frac{\partial}{\partial t^a} W + (\frac{\partial}{\partial t^a} K) W$, and similarly for the N^K, and the D-terms are

$$D_\alpha = \frac{i}{W}\left(\delta_\alpha t^a D_a W + \delta_\alpha N^K D_K W\right) = i\left(\delta_\alpha t^a \partial_a K + \delta_\alpha N^K \partial_K K\right) + i\frac{\delta_\alpha W}{W}, \quad (4.29)$$

where $\lambda^\alpha \delta_\alpha \phi$ is the variation of the field ϕ under an infinitessimal gauge transformation $A^\alpha \to A^\alpha + d\lambda^\alpha$. We may also want to discuss D-terms arising from the magnetic gauge groups as well, but the details are similar.

Now for the IIA orientifolds at hand, we can provide this information. The Kähler potential is

$$K = 4D - \ln\left(\frac{4}{3}\left(\kappa v^3\right)\right). \quad (4.30)$$

In the Kähler moduli sector, this leads to

$$\partial_a K = \frac{3i}{2}\frac{\left(\kappa v^2\right)_a}{\left(\kappa v^3\right)}, \quad (4.31)$$

and

$$K_{a\bar{b}} = \frac{9}{4}\left(\kappa v^3\right)^{-2}\left(\kappa v^2\right)_a\left(\kappa v^2\right)_b - \frac{3}{2}\left(\kappa v^3\right)^{-1}\left(\kappa v\right)_{ab}, \quad (4.32)$$

with inverse

$$K^{a\bar{b}} = -\frac{2}{3}\left(\kappa v^3\right)\left(\kappa v\right)^{-1\,ab} + 2v^a v^b. \quad (4.33)$$

Note that there is a no-scale condition

$$K^{a\bar{b}}\partial_a K\overline{\partial_b K} = 3. \quad (4.34)$$

In the complex structure sector, we have that $\partial_I K = -4e^D \mathcal{F}_I$, but there are not explicit expressions for $K_{I\bar{J}}$ and its inverse that are as nice as the expressions in the Kähler sector. However, there is still a useful, no-scale type condition,

$$K^{I\bar{J}}\partial_I K\overline{\partial_J K} = 4, \quad (4.35)$$

which we shall employ below.

The gauge kinetic couplings are

$$f_{\alpha\beta} = i\left(\widehat{\kappa}t\right)_{\alpha\beta}, \tag{4.36}$$

with

$$(\mathrm{Re}\, f)^{-1} = -\left(\widehat{\kappa}v\right)^{-1}. \tag{4.37}$$

We can also introduce the magnetic gauge couplings,

$$\widetilde{f}^{\alpha\beta} = -i\left(\left(\widehat{\kappa}t\right)^{-1}\widehat{d}^2\right)^{\alpha\beta}, \tag{4.38}$$

with

$$\left(\mathrm{Re}\,\widetilde{f}\right)^{-1}_{\alpha\beta} = -\left[\left(\left(\widehat{\kappa}v\right) + \left(\widehat{\kappa u}\right)\left(\widehat{\kappa}v\right)^{-1}\left(\widehat{\kappa}u\right)\right)\widehat{d}^{-2}\right]_{\alpha\beta}. \tag{4.39}$$

The corresponding D-terms are

$$D_\alpha = 2ie^D\left(\widehat{r}\mathcal{F}\right)_\alpha, \qquad \widetilde{D}^\alpha = -2ie^D\left(\widehat{q}\mathcal{F}\right)^\alpha, \tag{4.40}$$

so that the D-term contribution to the potential turns out to be

$$\begin{aligned} V_D &= \frac{1}{2}(\mathrm{Re}\, f)^{-1\,\alpha\beta} D_\alpha D_\beta + \frac{1}{2}\left(\mathrm{Re}\,\widetilde{f}\right)^{-1}_{\alpha\beta}\widetilde{D}^\alpha\widetilde{D}^\beta \\ &= 2e^{2D}\left[\left(\left(\widehat{\kappa}v\right)^{-1}\left(\widehat{r}\mathcal{F}\right)^2\right) + \left(\left(\left(\kappa v\right) + \left(\kappa u\right)\left(\kappa v\right)^{-1}\left(\kappa v\right)\right)\left(\widehat{d}^{-1}\widehat{q}\mathcal{F}\right)^2\right)\right]. \end{aligned} \tag{4.41}$$

Unlike the Kähler potential, the superpotential depends on the fluxes,

$$\begin{aligned} W &= fe_0 + (dte) + \frac{1}{2}\left(\kappa t^2 m\right) + \frac{1}{6}m_0\left(\kappa t^3\right) \\ &\quad + 2\left(Np\right) + 2\left(Nrt\right) + \left(N\kappa(d^{-1}q)t^2\right) + \frac{1}{3}f^{-1}\left(Ns\right)\left(\kappa t^3\right). \end{aligned} \tag{4.42}$$

The covariant derivatives of W are

$$\begin{aligned}
D_a W &= (de)_a + (\kappa m t)_a + \frac{1}{2}m_0\left(\kappa t^2\right)_a + 2\,(Nr)_a + 2\left(N\kappa(d^{-1}q)t\right)_a \\
&\quad + f^{-1}\,(Ns)\left(\kappa t^2\right)_a + \frac{3i}{2}\frac{\left(\kappa v^2\right)_a}{(\kappa v^3)}W, \\
D_K W &= 2p_K + 2\,(rt)_K + \left(\kappa(d^{-1}q)t^2\right)_K + \frac{1}{3}f^{-1}s_K\left(\kappa t^3\right) - 4e^D \mathfrak{F}_K W. \quad (4.43)
\end{aligned}$$

4.5 Dilaton Dependence

To extract the dilaton dependence, write

$$W = W_0 + e^{-D}W_1, \qquad D_a W = A_a + e^{-D}B_a, \qquad D_K W = C_K + e^D D_K, \quad (4.44)$$

where in the most general case,

$$\begin{aligned}
W_0 &= fe_0 + (dte) + \frac{1}{2}\left(\kappa t^2 m\right) + \frac{1}{6}m_0\left(\kappa t^3\right) + (\xi p) + (\xi r t) \\
&\quad + \frac{1}{2}\left(\xi\kappa(d^{-1}q)t^2\right) + \frac{1}{6}f^{-1}\,(\xi s)\left(\kappa t^3\right), \\
W_1 &= 2i\,(\mathcal{Z}p) + 2i\,(\mathcal{Z}rt) + i\left(\mathcal{Z}\kappa(d^{-1}q)t^2\right) + \frac{i}{3}f^{-1}\,(\mathcal{Z}s)\left(\kappa t^3\right), \\
A_a &= (de)_a + (\kappa t m)_a + \frac{1}{2}m_0\left(\kappa t^2\right)_a + (\xi r)_a + \left(\xi\kappa(d^{-1}q)t\right)_a \\
&\quad + \frac{1}{2}f^{-1}\,(\xi s)\left(\kappa t^2\right)_a + \frac{3i}{2}\frac{\left(\kappa v^2\right)_a}{(\kappa v^3)}W_0, \\
B_a &= 2i\,(\mathcal{Z}r)_a + 2i\left(\mathcal{Z}\kappa(d^{-1}q)t\right)_a + if^{-1}\,(\mathcal{Z}s)\left(\kappa t^2\right)_a + \frac{3i}{2}\frac{\left(\kappa v^2\right)_a}{(\kappa v^3)}W_1, \\
C_K &= 2p_K + 2\,(rt)_K + \left(\kappa(d^{-1}q)t^2\right)_K + \frac{1}{3}f^{-1}s_K\left(\kappa t^3\right) - 4\mathfrak{F}_K W_1, \\
D_K &= -4\mathfrak{F}_K W_0. \qquad (4.45)
\end{aligned}$$

The scalar potential is

$$V = \frac{3e^{4D}}{4\,(\kappa v^3)}\left\{K^{a\bar{b}}D_aW\overline{D_bW} + K^{I\bar{J}}D_IW\overline{D_JW} - 3|W|^2\right\} + V_D, \tag{4.46}$$

where

$$K^{a\bar{b}} = -\frac{2}{3}\left(\kappa v^3\right)(\kappa v)^{-1\,ab} + 2v^av^b, \tag{4.47}$$

and where we can extract some more dilaton dependence by writing

$$K^{I\bar{J}} = e^{-2D}\widehat{K}^{I\bar{J}}, \tag{4.48}$$

and

$$V_D = e^{2D}\widehat{V}_D, \tag{4.49}$$

with

$$\widehat{V}_D = 2\left[(\widehat{\kappa}v)^{-1\,\alpha\beta}\,\widehat{r}^I_\alpha\widehat{r}^J_\beta + \left((\widehat{\kappa}v) + (\widehat{\kappa}u)\,(\widehat{\kappa}v)^{-1}\,(\widehat{\kappa}u)\right)_{\alpha\beta}\left(\widehat{d}^{-1}\widehat{q}\right)^{\alpha I}\left(\widehat{d}^{-1}\widehat{q}\right)^{\beta J}\right]\mathcal{F}_I\mathcal{F}_J. \tag{4.50}$$

Then $\widehat{K}^{I\bar{J}}$ and $\widehat{V}_D$ do not depend on the dilaton.

Some useful identities are

$$\mathcal{Z}^K\mathcal{F}_K = -\frac{i}{2}, \qquad \widehat{K}^{I\bar{J}}\mathcal{F}_I\mathcal{F}_J = -\frac{1}{4}. \tag{4.51}$$

In the following we will introduce and discuss the method of Silverstein [109] in order to employ it to try to find de Sitter vacua. To this end, we can write the dilaton dependence of V as

$$V = e^{2D}\left(a - be^D + ce^{2D}\right), \tag{4.52}$$

where the coefficients a, b, and c are given by

$$\begin{aligned} a &= \frac{3}{4\left(\kappa v^3\right)}\left\{K^{a\bar{b}}B_a\overline{B_b}+\widehat{K}^{I\bar{J}}C_I\overline{C_J}-3|W_1|^2\right\}+\widehat{V}_D, \\ b &= \frac{3}{4\left(\kappa v^3\right)}\left\{-K^{a\bar{b}}\left(A_a\overline{B_b}+B_a\overline{A_b}\right)-\widehat{K}^{I\bar{J}}\left(C_I\overline{D_J}+D_I\overline{C_J}\right)\right. \\ &\qquad \left.+3\left(W_0\overline{W_1}+W_1\overline{W_0}\right)\right\}, \\ c &= \frac{3}{4\left(\kappa v^3\right)}\left\{K^{a\bar{b}}A_a\overline{A_b}+\widehat{K}^{I\bar{J}}D_I\overline{D_J}-3|W_0|^2\right\}. \end{aligned} \tag{4.53}$$

Because of the simple form for D_I, c can be simplified even in the general case,

$$c=\frac{3}{4\left(\kappa v^3\right)}\left\{K^{a\bar{b}}A_a\overline{A_b}+|W_0|^2\right\}. \tag{4.54}$$

A simplified case can be considered where there are no non-geometric fluxes (q_K^a, $\widehat{q}^{\alpha K}$, s_K), no two- or six-form R-R flux (m^a, e_0), and only D-term metric flux (only $\widehat{r}_\alpha^K$, no r_{aK}). Then it can be shown that it is at least consistent to eliminate some of the axionic moduli, setting $u^a=0$ and $\xi^K=0$ (consistent in the sense that $u^a=0$ and $\xi^K=0$ ensure a solution to the equations $\frac{\partial}{\partial u^a}V=\frac{\partial}{\partial \xi^K}V=0$). Then we are left with

$$\begin{aligned} W_0 &= i\left(dev\right)-\frac{i}{6}m_0\left(\kappa v^3\right), \\ W_1 &= 2i\left(\mathcal{Z}p\right), \\ A_a &= \left(de\right)_a-\frac{1}{4}m_0\left(\kappa v^2\right)_a-\frac{3}{2}\left(dev\right)\frac{\left(\kappa v^2\right)_a}{\left(\kappa v^3\right)}, \\ B_a &= -3\left(\mathcal{Z}p\right)\frac{\left(\kappa v^2\right)_a}{\left(\kappa v^3\right)}, \\ C_I &= 2p_I-8i\left(\mathcal{Z}p\right)\mathcal{F}_I, \\ D_I &= -4i\left(dev\right)\mathcal{F}_I+\frac{2i}{3}m_0\left(\kappa v^3\right)\mathcal{F}_I, \\ \widehat{V}_D &= 2\left(\left(\widehat{\kappa}v\right)^{-1}\left(\widehat{r}\mathcal{F}\right)^2\right), \end{aligned} \tag{4.55}$$

leading to

$$
\begin{aligned}
a &= \frac{3}{4\left(\kappa v^3\right)}\left\{4\left(\widehat{K}p^2\right)+16\left(\mathcal{Z}p\right)^2\right\}+2\left(\left(\widehat{\kappa}v\right)^{-1}\left(\widehat{r}\mathcal{F}\right)^2\right), \\
b &= \frac{3}{4\left(\kappa v^3\right)}\left\{24\left(dev\right)\left(\mathcal{Z}p\right)-\frac{20}{3}m_0\left(\kappa v^3\right)\left(\mathcal{Z}p\right)-16i\left(dev\right)\left(\widehat{K}p\mathcal{F}\right)\right. \\
&\qquad\left.+\frac{8i}{3}m_0\left(\kappa v^3\right)\left(\widehat{K}p\mathcal{F}\right)\right\}, \\
c &= \frac{3}{4\left(\kappa v^3\right)}\left\{-\frac{2}{3}\left(\kappa v^3\right)\left(\left(\kappa v\right)^{-1}\left(de\right)^2\right)+2\left(dev\right)^2+\frac{1}{9}m_0^2\left(\kappa v^3\right)^2\right\}. \qquad (4.56)
\end{aligned}
$$

To find de Sitter minima, our technique will be to find a model where the combination $4ac/b^2$ has a minimum in the range

$$1 < \frac{4ac}{b^2} < \frac{9}{8}, \qquad (4.57)$$

and then try to find a nearby minimum for V numerically as explained below. The lower bound in (4.57) comes from the fact that in order to obtain a de Sitter vacuum we require $V > 0$, while the upper bound comes from the requirement that the discriminant be greater or equal to zero, which leads to $\frac{2a}{c} \leq \frac{9b^2}{16c^2}$. So in order to have a *small* but positive cosmological constant, we are interested in vacua that almost saturate the lower bound. Defining $c = \frac{b^2}{4a}(1+\delta)$, and eliminating c in favor of $\delta = \frac{4ac}{b^2} - 1$, the potential can be rewritten as

$$V = e^{2D}a\left(1 - \frac{b}{2a}e^D\right)^2 + \delta\frac{b^2}{4a}e^{4D}. \qquad (4.58)$$

The interval (4.57) now corresponds to the interval $0 < \delta < \frac{1}{8}$. The quantity δ generically will be a function of the remaining moduli σ_I and the fluxes. The next step is minimize δ w.r.t. the moduli and tune the fluxes such that $\delta_0 \approx 0$ is very close to zero. If $\delta_0 = 0$, then $(e^D)_0 = \frac{2a_0}{b_0}$ would be a minimum of the potential V, where a_0, b_0 are the functions defined above, evaluated at the values for the moduli $\sigma_{I,0}$

and fluxes at said minimum. The crucial step in the construction is to realize that for δ_0 small, but non-zero, there is still a local de Sitter minimum of the potential, which can be shown to be close to $\sigma_{I,0}, (e^D)_0$ in field space [109].

Below, we will investigate a specific example more closely. We will find that the construction described here cannot be applied to that example because δ will generically be minimized at $\delta_0 = -1$ regardless of the choice of discrete fluxes.

4.5.1 A simple example: A non-standard $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ orientifold

In this example, we shall study the usual $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of T^6 that was briefly introduced above. Remember that instead of using the standard involution $\sigma : z_i \to \bar{z}_i$ (which has $h^{1,1}_+ = 0$, $h^{1,1}_- = 3$, $h^{2,1} = 3$, and only κ_{123} nontrivial), take the involution

$$\sigma : (z_1, z_2, z_3) \to (\bar{z}_2, \bar{z}_1, \bar{z}_3)\,. \tag{4.59}$$

In this case, $h^{1,1}_- = 2$, $h^{1,1}_+ = 1$, $h^{2,1} = 3$. This example was worked out in detail in the previous chapter. Here we will study the functional $\delta(v_1, v_2, U_1, U_2, U_3; f) = \frac{4ac}{b^2} - 1$ for the example at hand. An interesting observation is that here, δ can be recast into the following form,

$$\delta = \frac{\left(F(U_i; f) + G(U_i; f) v_1^2\right)\left(2v_1^2 + (4 + v_1^4) v_2^2\right)}{H(U_i; f) v_1^4 v_2^2} - 1, \tag{4.60}$$

where $F(U_i; f), G(U_i; f), H(U_i; f)$ are functions of only the complex structure moduli $U_i, i = 1, 2, 3$, and the fluxes $f \in \{p_K, m_0, e_1, e_2, \widehat{r}^K\}$ (which are subject to certain constraints, see above). The exact form of these functions is not important for the argument presented below. Performing the first step in Silverstein's construction, i.e., minimizing δ with respect to the moduli, one finds that

$$\frac{\partial \delta}{\partial v_2} = -4 \frac{\left(F(U_i; f) + G(U_i; f) v_1^2\right)}{H(U_i; f) v_1^4 v_2^2} = 0, \tag{4.61}$$

yields either $v_1, v_2 \rightarrow \infty$, which corresponds to the decompactification limit, or $\left(F(U_i;f) + G(U_i;f)v_1^2\right) = 0$ at the minimum. Both situations lead to $\delta = -1$ at the minimum, independent of the fluxes. Therefore Silverstein's method is not applicable here.

This does not rule out de Sitter vacua, but they certainly seem hard to come by in this scenario. But lacking a no-go theorem in the case when generalized fluxes are included, there is still the possibility that one of the models, where D-terms can be consistently quantized, will feature local minima with $V > 0$, i.e, a small but positive cosmological constant. At this point, however, it seems easier to include extra ingredients outside of more general fluxes to achieve de Sitter vacua in type IIA flux compactifications [109].

Appendix A

$SU(3)$ Structure with Metric Fluxes

In this appendix we are analyzing the SU(3) structure of the torus with metric fluxes, as was done in a specific case in [85]. For a general discussion of SU(3) structures, see for example [96].

We start with the Kähler 2-form J and the holomorphic three-form Ω of the geometry, given by

$$J = v_a \omega_a, \tag{A.1}$$

$$\Omega = \mathcal{Z}_K a_K - \mathcal{F}_K b_K, \tag{A.2}$$

where $\mathcal{Z}_K$ are real and $\mathcal{F}_K$ are imaginary. These forms satisfy $J \wedge \Omega = 0$, $J \wedge J \wedge J = 6i\mathcal{V}_6 \Omega \wedge \overline{\Omega}$ and define an SU(3) structure on the twisted torus. The torsion classes are defined by

$$\mathrm{d}J = -12\mathcal{V}_6 \mathrm{Im}(\mathcal{W}_1 \overline{\Omega}) + \mathcal{W}_4 \wedge J + \mathcal{W}_3, \tag{A.3}$$

$$\mathrm{d}\Omega = \mathcal{W}_1 J \wedge J + \mathcal{W}_2 \wedge J + \mathcal{W}_5^* \wedge \Omega. \tag{A.4}$$

$\mathcal{W}_1$ is a complex scalar, $\mathcal{W}_2$ is a complex primitive (1,1)-form i.e., $\mathcal{W}_2 \wedge J \wedge J = 0$, $\mathcal{W}_3$ is a real primitive (2,1)+(1,2)-form i.e., $\mathcal{W}_3 \wedge J = \mathcal{W}_3 \wedge \Omega = 0$, $\mathcal{W}_4$ is a real 1-form and $\mathcal{W}_5$ is a complex (1,0)-form. The prefactor in the first term of dJ is needed to have $\mathrm{d}(J \wedge \Omega) = 0$.

The torsion classes can be read off from

$$\mathrm{d}J = -r_{aK} v_a b_K, \tag{A.5}$$

$$\mathrm{d}\Omega = -\mathcal{Z}_K \left(d^{-1}\right)^{ab} r_{bK} \widetilde{\omega}_a - \mathcal{F}_K \left(\widehat{d}^{-1}\right)^{\alpha\beta} \widehat{r}_{\beta K} \widetilde{\mu}_\alpha. \tag{A.6}$$

Since there are no $\mathbb{Z}_4$ invariant 1-forms we have immediately $\mathcal{W}_4 = \mathcal{W}_5 = 0$. To determine $\mathcal{W}_1$ we use the fact that $\mathcal{W}_2$ is primitive and that $\int_X \omega_a \wedge \tilde{\omega}_b = d_{ab}$.

$$\int_X \mathrm{d}\Omega \wedge J = -\mathcal{Z}_K r_{aK} v_a = \int_X \mathcal{W}_1 J \wedge J \wedge J = \mathcal{W}_1 6 \mathcal{V}_6 \tag{A.7}$$

$$\Rightarrow \mathcal{W}_1 = -\frac{\mathcal{Z}_K r_{aK} v_a}{6\mathcal{V}_6}. \tag{A.8}$$

Now we can read off $\mathcal{W}_3 = \left(2i\mathcal{Z}_L r_{aL} v_a \mathcal{F}_K - r_{aK} v_a\right) b_K$. It is straightforward to calculate $\mathcal{W}_2$. The torsion classes for a generic choice of metric fluxes are

$$\begin{aligned}
\mathcal{W}_1 &= -\frac{\mathcal{Z}_K r_{aK} v_a}{6\mathcal{V}_6}, && \text{(A.9)}\\
\mathcal{W}_2 &= -\left(\mathcal{W}_1 v_a + (\kappa v)^{-1\,ab} \mathcal{Z}_K r_{bK}\right) \omega_a - (\widehat{\kappa} v)^{-1\,\alpha\beta} \mathcal{F}_K \widehat{r}_{\beta K} \mu_\alpha, && \text{(A.10)}\\
\mathcal{W}_3 &= \left(2i\mathcal{Z}_L r_{aL} v_a \mathcal{F}_K - r_{aK} v_a\right) b_K, && \text{(A.11)}\\
\mathcal{W}_4 &= \mathcal{W}_5 = 0, && \text{(A.12)}
\end{aligned}$$

where $(\kappa v)^{-1}$ is the inverse of the matrix $(\kappa v)_{ab} = \kappa_{abc} v_c$, and similarly $(\widehat{\kappa} v)^{-1}$ is the inverse of the matrix $(\widehat{\kappa} v)_{\alpha\beta} = \widehat{\kappa}_{a\alpha\beta} v_a$.

Note that the twisted torus is generically not half-flat. For the twisted torus to be half-flat we would have to demand that $\mathrm{Im}(\mathcal{W}_1) = \mathrm{Im}(\mathcal{W}_2) = \mathcal{W}_4 = \mathcal{W}_5 = 0$.

This is equivalent to $\mathrm{d}J \wedge J = \mathrm{d}(\mathrm{Im}(\Omega)) = 0$ or that $\mathrm{Im}(\mathcal{F}_K)\widehat{r}_{\alpha K} = 0$. But these are precisely the D-term equations that we derived in section 2.2.3. So solving the D-term equations is precisely equivalent to demanding that our manifold be half-flat.

Supersymmetric solutions should also have $\mathcal{W}_3 = 0$ [60]. And indeed we can show this using the F-term equations (2.102). We have

$$0 = \mathcal{Z}_K \left(\mathrm{Im}\, D_K W\right) = 2\mathcal{Z}_K r_{aK} v_a + 2e^D \,\mathrm{Re}\, W, \implies \mathrm{Re}\, W = -e^{-D}\mathcal{Z}_K r_{aK} v_a, \quad \text{(A.13)}$$

and then plugging this back in to (2.102) we find that each component of $\mathcal{W}_3$ must vanish in a supersymmetric solution. Note also that in [61] it was shown that Minkowski vacua of type IIA require $\mathcal{W}_1 = 0$. This fits nicely with the observation that $\mathcal{W}_1 = e^D \,\mathrm{Re}\, W/(6\mathcal{V}_6)$. So we see that our results agree very nicely with the language of SU(3) structure and torsion classes.

Appendix B

Comparison of two different derivations of the Bianchi identities

Here we present two derivations of the Bianchi identities.

The usual derivation [2] is to note that upon reducing on a d-dimensional torus we have (ignoring for now any orientifold group) d vectors from reducing the metric and d vectors from reducing the B-field. Let Z_i and X^i respectively generate the gauge transformations for these two groups of vectors. One then argues, by T-duality or otherwise, that the NS-NS fluxes must appear in the Lie brackets as

$$\begin{aligned} [Z_i, Z_j] &= \omega_{ij}^k Z_k - H_{ijk} X^k, \\ \left[Z_i, X^j\right] &= -\omega_{ik}^j X^k + Q_i^{jk} Z_k, \\ \left[X^i, X^j\right] &= Q_k^{ij} X^k - R^{ijk} Z_k. \end{aligned} \tag{B.1}$$

The Jacobi identities for this Lie algebra then give us the NS-NS Bianchi identities:

$$\begin{aligned} H_{k[i_1 i_2}\omega^k_{i_3 i_4]} &= 0, \\ H_{k[i_1 i_2}Q^{k\,j}_{i_3]} + \omega^j_{k[i_1}\omega^k_{i_2 i_3]} &= 0, \\ H_{k\,i_1 i_2}R^{k\,j_1 j_2} + \omega^k_{i_1 i_2}Q^{j_1 j_2}_k - 4\omega^{[j_1}_{k[i_1}Q^{j_2]k}_{i_2]} &= 0, \\ \omega^{[j_1}_{k\,i}R^{j_2 j_3]k} + Q^{k[j_1}_i Q^{j_2 j_3]}_k &= 0, \\ Q^{[j_1 j_2}_k R^{j_3 j_4]k} &= 0, \end{aligned} \tag{B.2}$$

Let us present an alternative derivation, as suggested by [6]. We have seen that it is natural to replace the exterior derivative d acting on R-R forms by a covariant derivative $\mathcal{D}$. We saw in section 2.2 that such an object was what appeared in the tadpole condition and superpotential, and argued that it should also be used in finding the correct gauge transformations. And in section 2.3 we saw that it could be understood as a covariant derivative for the spin bundle of which R-R fields formed sections. By combining these considerations[1], it is natural to define the general $\mathcal{D}$ by its action on a p-form as

$$\begin{aligned} \mathcal{D}A^{(p)} = &\binom{p+3}{3} H_{[i_1 i_2 i_3}A_{i_4\cdots i_{p+3}]}\,\frac{1}{(p+3)!}dx^{i_1}\wedge\cdots\wedge dx^{i_{p+3}} \\ &-\left\{\binom{p+1}{2}\omega^j_{[i_1 i_2}A_{|j|i_3\cdots i_{p+1}]} + \frac{p+1}{2}\omega^j_{j[i_1}A_{i_2\cdots i_{p+1}]}\right\}\frac{1}{(p+1)!}dx^{i_1}\wedge\cdots\wedge dx^{i_{p+1}} \\ &+\frac{1}{2}\left\{\binom{p-1}{1}Q^{jk}_{[i_1}A_{|jk|i_2\cdots i_{p-1}]} + \binom{p-1}{0}Q^{jk}_j A_{k\,i_1\cdots i_{p-1}}\right\}\frac{1}{(p-1)!}dx^{i_1}\wedge\cdots\wedge dx^{i_{p-1}} \\ &-\frac{1}{6}\binom{p-3}{0}R^{jk\ell}A_{jk\ell\,i_1\cdots i_{p-3}}\,\frac{1}{(p-3)!}dx^{i_1}\wedge\cdots\wedge dx^{i_{p-3}}. \end{aligned} \tag{B.3}$$

For consistency, we need $\mathcal{D}$ to share a key property with the exterior deriva-

[1] From the base-fiber approach we did not require $\omega^a_{ia} = 0$, and can argue for the dependence on this trace, and from the effective field theory approach we can deduce how R must appear.

tive that it is replacing, namely that $\mathcal{D}^2 = 0$ on all forms. Computing,

$$
\begin{aligned}
\mathcal{D}^2 A^{(p)} &= -6\binom{p+4}{4} H_{k\,i_1 i_2}\omega^k_{i_3 i_4} A_{i_5\cdots i_{p+4}}\,\frac{1}{(p+4)!}dx^{i_1}\wedge\cdots\wedge dx^{i_{p+4}} \\
&+\Bigg\{-\binom{p+2}{2}\left(H_{k\ell\,i_1}Q^{k\ell}_{i_2} - \frac{1}{2}Q^{k\ell}_k H_{\ell\,i_1 i_2} - \frac{1}{2}\omega^k_{k\ell}\omega^\ell_{i_1 i_2}\right)A_{i_3\cdots i_{p+2}} \\
&+3\binom{p+2}{3}\left(H_{k\,i_1 i_2}Q^{kj}_{i_3} + \omega^k_{i_1 i_2}\omega^j_{k\,i_3}\right)A_{j\,i_4\cdots i_{p+2}}\Bigg\}\,\frac{1}{(p+2)!}dx^{i_1}\wedge\cdots\wedge dx^{i_{p+2}} \\
&+\Bigg\{\binom{p}{0}\left(-\frac{1}{6}H_{k\ell m}R^{k\ell m} + \frac{1}{4}\omega^k_{k\ell}Q^{\ell m}_m\right)A_{i_1\cdots i_p} \\
&+\frac{1}{2}\binom{p}{1}\left(H_{k\ell\,i_1}R^{k\ell j} - Q^{k\ell}_{i_1}\omega^j_{k\ell} - \omega^k_{k\ell}Q^{\ell j}_{i_1} - Q^{k\ell}_k\omega^j_{\ell\,i_1}\right)A_{j\,i_2\cdots i_p} \\
&-\frac{1}{2}\binom{p}{2}\left(H_{k\,i_1 i_2}R^{k\,j_1 j_2} + 4\omega^{j_1}_{k\,i_1}Q^{k\,j_2}_{i_2} + Q^{j_1 j_2}_k\omega^k_{i_1 i_2}\right)A_{j_1 j_2\,i_3\cdots i_p}\Bigg\}\,\frac{1}{p!}dx^{i_1}\wedge\cdots\wedge dx^{i_p} \\
&+\Bigg\{\frac{1}{2}\binom{p-2}{0}\left(\omega^{j_1}_{k\ell}R^{k\ell\,j_2} + \frac{1}{2}\omega^k_{k\ell}R^{\ell\,j_1 j_2} + \frac{1}{2}Q^{k\ell}_k Q^{j_1 j_2}_\ell\right)A_{j_1 j_2\,i_1\cdots i_{p-2}} \\
&+\frac{1}{2}\binom{p-2}{1}\left(R^{k\,j_1 j_2}\omega^{j_3}_{k\,i_1} + Q^{j_1 j_2}_k Q^{k\,j_3}_{i_1}\right)A_{j_1 j_2 j_3\,i_2\cdots i_{p-2}}\Bigg\}\,\frac{1}{(p-2)!}dx^{i_1}\wedge\cdots\wedge dx^{i_{p-2}} \\
&-\frac{1}{4}\binom{p-4}{0}Q^{j_1 j_2}_k R^{k\,j_3 j_4}A_{j_1 j_2 j_3 j_4\,i_1\cdots i_{p-4}}\,\frac{1}{(p-4)!}dx^{i_1}\wedge\cdots\wedge dx^{i_{p-4}}.
\end{aligned}
\tag{B.4}
$$

From this we find that in order to ensure that $\mathcal{D}^2 = 0$ on all forms, we need precisely the Bianchi identities found above, and one additional one which does not follow by contraction,

$$2H_{k\ell m}R^{k\ell m} + 3\omega^k_{k\ell}Q^{m\ell}_m = 0. \tag{B.5}$$

Note that this final identity is satisfied on *any* IIA orientifold, because there are generally no scalars (zero-forms) which are odd under the involution.

The coefficients of the two trace terms in (B.3) can be argued from the spin bundle transformations of R-R fields, as in section 2.3.2, along with a T-duality argument to get the TrQ term in terms of the Trω term, but there is another nice check as well. If the coefficients of the trace terms were at all different, then $\mathcal{D}^2 = 0$ would lead to more constraints beyond the single extra constraint we found above.

Though not inconsistent, these additional requirements seem surprising and ad-hoc. With the given coefficients, however, these additional constraints follow simply from the traces of constraints with more free indices.

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