The Dissertation Committee for Kenneth William Taliaferro
certifies that this is the approved version of the following dissertation:

# The Dynamics of Bose Gases

Committee:

Thomas Chen, Supervisor

Francesco Maggi

Natasa Pavlović

Nikolaos Tzirakis

Alexis Vasseur

# The Dynamics of Bose Gases

**by**

**Kenneth William Taliaferro, B.S.**

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**The Dynamics of Bose Gases**

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Supervisor: Thomas Chen

We study the Gross-Pitaevskii (GP) hierarchy, which is an infinite sequence of coupled partial differential equations that models the dynamics of Bose gases and arises in the derivation of the cubic and quintic nonlinear Schrödinger equations from an $N$-body linear Schrödinger equation.

In Chapter 2, we consider the cubic case in $\mathbb{R}^3$ and derive the GP hierarchy in the strong topology corresponding to the spaces used by Klainerman and Machedon in [82]. We also prove that positive semidefiniteness of solutions is preserved over time and use this result to prove global well-posedness of solutions to the GP hierarchy. This is based on a joint work with Thomas Chen [24].

In Chapters 3 and 4, we prove uniqueness of solutions to the GP hierarchy in $\mathbb{R}^d$ in a low regularity Sobolev type space in the cubic and quintic cases, respectively. These chapters are an extension of the work of Chen-Hainzl-Pavlović-Seiringer [17] and are based on joint works with Younghun Hong and Zhihui Xie [70, 71].

# Table of Contents

# Chapter 1

# Introduction

The Gross-Pitaevskii (GP) hierarchy emerges in the limit as $N \to \infty$, via the associated Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy, from an $N$-body Schrödinger equation describing an interacting Bose gas under Gross-Pitaevskii scaling. Factorized solutions of the GP hierarchy are defined by the nonlinear Schrödinger equation (NLS). Thereby, one obtains a rigorous derivation of those nonlinear dispersive PDE's, describing the mean field dynamics of a particle in a Bose-Einstein condensate.

Lanford used the BBGKY hierarchy in his study of classical mechanical systems in the infinite particle limit [86, 88]. Shortly afterward, Spohn derived mean field theories via the BBGKY hierarchy [113]. In a very influential series of works, Erdös, Schlein and Yau have, in this framework, derived the cubic NLS and NLH in $\mathbb{R}^3$, [42, 43, 46], for a very broad class of systems. This problem is closely related to the study of Bose-Einstein condensation, where fundamental progress was made in recent years in the works of Lieb, Seiringer, and Yngvason, [2, 93, 94, 96]. An alternative approach to the proof of uniqueness for solutions of GP hierarchies was introduced by Klainerman and Machedon in [82], using techniques from nonlinear dispersive PDE's. This inspired a series of works by various authors using the Klainerman-Machedon framework [82], on the derivation of NLH and NLS, including Kirkpatrick-Schlein-Staffilani [77], T.Chen-Pavlović [19, 23], X.Chen [25, 27], and X.Chen-Holmer [29, 30], and on the Cauchy problem for GP hierarchies, including T.Chen-Pavlović and C-P-Tzirakis [18, 20, 21]. Furthermore, the rate of convergence to mean field equations has been investigated by Rodnianski and Schlein in [105], based on the approach of Hepp [67], which led to many further developments, including works of Grillakis-Machedon and G-M-Margetis [61–64], X.Chen [26], and Lee-Li-Schlein [14]. For related works and other approaches to the derivation of NLH and NLS, see also [1, 5, 41, 48, 49, 51, 102]. A few more details are addressed in the discussion below.

## 1.1 The Gross-Pitaevkii limit for Bose gases

Before we state our results, we give a brief summary of the derivation of the cubic NLS in $\mathbb{R}^d$ based on the BBGKY hierarchy for a gas of interacting bosons proceeds along the following lines, following [42, 43, 46].

### 1.1.1 From $N$-body Schrödinger to BBGKY hierarchy

We model $N$ bosons in $\mathbb{R}^d$ with a wave function $\Phi_N \in L^2(\mathbb{R}^{dN})$ that satisfies the $N$-body Schrödinger equation

$$i\partial_t \Phi_N = H_N \Phi_N, \tag{1.1.1}$$

where the Hamiltonian $H_N$ is the self-adjoint operator on $L^2(\mathbb{R}^{dN})$ given by

$$H_N = \sum_{j=1}^{N}(-\Delta_{x_j}) + \frac{1}{N}\sum_{1\leq i<j\leq N} V_N(x_i - x_j). \tag{1.1.2}$$

The potential $V_N$ satisfies $V_N(x) = N^{d\beta}V(N^\beta x)$, where $V \geq 0$ is spherically symmetric and sufficiently regular. The parameter $\beta$ typically has values in $(0,1]$ (see the discussion below equation (1.1.11)).

According to Bose-Einstein statistics, $\Phi_N$ is invariant under the permutation of particle variables,

$$\Phi_N(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(N)}) = \Phi_N(x_1, x_2, ..., x_N) \qquad \forall \pi \in S_N\,, \tag{1.1.3}$$

where $S_N$ is the $N$-th symmetric group. We note that permutation symmetry (1.1.3) is preserved by the $N$-body Schrödinger equation (1.1.1).

Since the $N$-body Schrödinger equation (1.1.1) is linear and $H_N$ is self-adjoint, the global well-posedness of solutions in $L^2(\mathbb{R}^{dN})$ follows immediately. For $1 \leq k \leq N$, one defines the density matrices

$$\gamma^{(k)}_{\Phi_N}(t, \underline{x}_k, \underline{x}'_k) := \int \Phi_N(t, \underline{x}_k, \underline{x}_{N-k})\overline{\Phi_N(t, \underline{x}'_k, \underline{x}_{N-k})}\, d\underline{x}_{N-k},$$

where $(\underline{x}_k, \underline{x}_{N-k}) \in \mathbb{R}^{dk} \times \mathbb{R}^{d(N-k)}$. Clearly, the property of *admissibility* holds,

$$\gamma^{(k)}_{\Phi_N} = \mathrm{Tr}_{k+1}(\gamma^{(k+1)}_{\Phi_N}), \quad k = 1, \ldots, N-1\,, \tag{1.1.4}$$

for $1 \leq k \leq N-1$, and that $\mathrm{Tr}\gamma^{(k)}_{\Phi_N} = \|\Phi_N\|^2_{L^2_s(\mathbb{R}^{dN})} = 1$ for all $N$, and all $k = 1, 2, ..., N$.

Moreover, the operator on $L^2(\mathbb{R}^{kd})$ with integral kernel $\gamma^{(k)}_{\Phi_N}$ is positive semidefinite.

It follows from the $N$-body Schrödinger equation (1.1.1) that $\gamma_{\Phi_N}$ satisfies the Heisenberg equation

$$i\partial_t \gamma_{\Phi_N}(t) = [H_N, \gamma_{\Phi_N}(t)]\,, \tag{1.1.5}$$

which is explicitly given by

$$\begin{aligned} i\partial_t \gamma_{\Phi_N}(t, \underline{x}_N, \underline{x}'_N) = &- (\Delta_{\underline{x}_N} - \Delta_{\underline{x}'_N})\gamma_{\Phi_N}(t, \underline{x}_N, \underline{x}'_N) \\ &+ \frac{1}{N} \sum_{1\le i<j\le N} [V_N(x_i - x_j) - V_N(x'_i - x'_j)]\gamma_{\Phi_N}(t, \underline{x}_N, \underline{x}'_N)\,. \end{aligned} \tag{1.1.6}$$

Accordingly, the $k$-particle marginals satisfy the BBGKY hierarchy

$$\begin{aligned} i\partial_t \gamma^{(k)}_{\Phi_N}(t, \underline{x}_k; \underline{x}'_k) = &-(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})\gamma^{(k)}_{\Phi_N}(t, \underline{x}_k, \underline{x}'_k) \\ &+ \frac{1}{N} \sum_{1\le i<j\le k} [V_N(x_i - x_j) - V_N(x'_i - x'_j)]\gamma^{(k)}_{\Phi_N}(t, \underline{x}_k; \underline{x}'_k) \end{aligned} \tag{1.1.7}$$

$$\begin{aligned} &+ \frac{N-k}{N} \sum_{i=1}^{k} \int dx_{k+1} [V_N(x_i - x_{k+1}) - V_N(x'_i - x_{k+1})] \\ &\qquad \gamma^{(k+1)}_{\Phi_N}(t, \underline{x}_k, x_{k+1}; \underline{x}'_k, x_{k+1}) \end{aligned} \tag{1.1.8}$$

where $\Delta_{\underline{x}_k} := \sum_{j=1}^{k} \Delta_{x_j}$, and similarly for $\Delta_{\underline{x}'_k}$. We note that the number of terms in (1.1.7) is $O(\frac{k^2}{N}) \to 0$, and the number of terms in (1.1.8) is $\frac{k(N-k)}{N} \to k$ as $N \to \infty$. Accordingly, for fixed $k$, (1.1.7) disappears in the limit $N \to \infty$ described below, while (1.1.8) survives.

### 1.1.2 Derivation of the GP from the BBGKY hierarchy.

In [42, 43, 46], the authors consider asymptotically factorized initial data, and prove convergence in the weak-* topology on the space of trace class marginal density matrices. By definition, asymptotically factorized initial data is approximately of the form

$$\gamma_0^{(k)}(\underline{x}_k;\underline{x}_k') = \prod_{j=1}^{k} \phi_0(x_j)\overline{\phi_0(x_j')}\,, \tag{1.1.9}$$

where $\phi_0 \in H^1(\mathbb{R}^{dk})$. In this case, they extract convergent subsequences $\gamma_{\Phi_{N_j}}^{(k)} \to \gamma^{(k)}$ as $j \to \infty$, for $k \in \mathbb{N}$, and show that those satisfy the the infinite limiting hierarchy

$$\begin{aligned} i\partial_t \gamma^{(k)}(t,\underline{x}_k;\underline{x}_k') = &- (\Delta_{\underline{x}_k} - \Delta_{\underline{x}_k'})\gamma^{(k)}(t,\underline{x}_k;\underline{x}_k') \\ &+ \kappa_0 \sum_{j=1}^{k} \left(B_{j,k+1}\gamma^{k+1}\right)(t,\underline{x}_k;\underline{x}_k')\,, \end{aligned} \tag{1.1.10}$$

which is referred to as the *Gross-Pitaevskii (GP) hierarchy.* Here,

$$\begin{aligned} &(B_{j,k+1}\gamma^{k+1})(t,\underline{x}_k;\underline{x}_k') \\ &:= \int dx_{k+1}dx_{k+1}'[\delta(x_j - x_{k+1})\delta(x_j - x_{k+1}') - \delta(x_j' - x_{k+1})\delta(x_j' - x_{k+1}')] \\ &\qquad \gamma^{(k+1)}(t,\underline{x}_k, x_{k+1};\underline{x}_k', x_{k+1}')\,. \end{aligned} \tag{1.1.11}$$

The coefficient $\kappa_0$ is the *scattering length* if $\beta = 1$ (see [42, 94] for the definition), and $\kappa_0 = \int V(x)dx$ if $\beta < 1$ (corresponding to the Born approximation of the scattering length). For $\beta < 1$, the interaction term is obtained from the weak limit $V_N(x) \rightharpoonup \kappa_0\delta(x)$ in (1.1.8) as $N \to \infty$. The proof for the case $\beta = 1$

is much more difficult, and the derivation of the scattering length in this context is a breakthrough result obtained in [42, 43]. For notational convenience, we will mostly set $\kappa_0 = 1$ in the sequel.

We note that (1.1.10) is the *cubic* GP hierarchy. In Chapter 4, we will investigate the *quintic* GP hierarchy. In general, one can consider the $p$-GP hierarchy, where $p = 2$ and $p = 4$ correspond to the cubic and quintic cases, respectively.

We emphasize the following key properties of solutions of the GP hierarchy that hold in the context of [42, 43]:

- Solutions of the GP hierarchy that are obtained from a limit of asymptotically factorizing solutions to the BBGKY hierarchy (as in [42, 43]) inherit from the latter *global* in $t$ existence of the solutions, and positive semidefiniteness.

- It preserves the property of admissibility,

$$\gamma^{(k)} = \mathrm{Tr}_{k+1}(\gamma^{(k+1)}), \quad \forall\, k \in \mathbb{N}\,, \tag{1.1.12}$$

  which is inherited from the system at finite $N$. See Proposition 2.C.1 in the appendix.

- In [42, 43, 46], solutions of the GP hierarchy are studied in spaces of $k$-particle marginals $\{\gamma^{(k)} \,|\, \|\gamma^{(k)}\|_{\mathfrak{h}^1} < \infty\}$ with norms

$$\|\gamma^{(k)}\|_{\mathfrak{h}^\alpha} := \mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|)\,, \tag{1.1.13}$$

where

$$S^{(k,\alpha)} := \prod_{j=1}^{k} (1-\Delta_{x_j})^{\alpha/2}(1-\Delta_{x_j'})^{\alpha/2} \, . \tag{1.1.14}$$

### 1.1.3 NLS from factorized solutions of GP

In the case of factorized initial data (1.1.9), one can easily verify that

$$\gamma^{(k)}(t,\underline{x}_k;\underline{x}_k') = \prod_{j=1}^{k} \phi(t,x_j)\overline{\phi(t,x_j')}$$

is a solution (referred to as a factorized solution) of the GP hierarchy (1.1.10) with $\kappa_0 = 1$, if $\phi(t) \in H^1(\mathbb{R}^d)$ solves the defocusing cubic NLS,

$$i\partial_t \phi = -\Delta_x \phi + |\phi|^2 \phi \, , \tag{1.1.15}$$

for $t \in I \subseteq \mathbb{R}$, and $\phi(0) = \phi_0 \in H^1(\mathbb{R}^d)$. It is in this sense that the NLS emerges as a mean field description of the dynamics of Bose-Einstein condensates.

### 1.1.4 Uniqueness of solutions of GP hierarchies.

The question of whether solutions to the GP hierarchy are unique is more difficult to answer. In [42, 43, 46], Erdös, Schlein and Yau proved uniqueness in the space $\{\gamma^{(k)} \,|\, \|\gamma^{(k)}\|_{\mathfrak{h}^1} < \infty\}$ using Feynman graph expansion methods.

Subsequently, Klainerman and Machedon [82] found an alternative method for proving uniqueness in a space of density matrices defined by the Hilbert-

Schmidt type Sobolev norms

$$\|\gamma^{(k)}\|_{H^1_k} := \|S^{(k,1)}\gamma^{(k)}\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} < \infty\,. \tag{1.1.16}$$

While this is a different (strictly larger) space of marginal density matrices than the one considered by Erdös, Schlein, and Yau, [42, 43], the authors of [82] impose an additional a priori condition on space-time norms of the form

$$\|B_{j;k+1}\gamma^{(k+1)}\|_{L^2_{t\in[0,T]}H^1_k} < C^k\,, \tag{1.1.17}$$

for some arbitrary but finite $C$ independent of $k$.

The authors of [82] proved uniqueness of solutions of the GP hierarchy (1.1.10) in $d = 3$ using certain space-time bounds on density matrices and a reformulation of a combinatorial result in [42, 43] into a "board game" argument. This approach lead to a significant reduction of the complexity of the problem, conditionally on (1.1.17).

In the case $d = 2$, Kirkpatrick, Schlein, and Staffilani were able to show that the a priori spacetime bound (1.1.17) is satisfied for solutions of the cubic GP hierarchy derived from an $N$-body Schroedinger equation. The proof of their result made use of conservation of energy in the original $N$-body Schrödinger system, and related a priori $H^1$-bounds for the BBGKY hierarchy in the limit $N \to \infty$ derived in [42, 43], combined with a generalized Sobolev inequality for density matrices.

## 1.2 Cauchy problem for GP hierarchies

The work at hand is closely related to the works of Chen-Pavlović on the well-posedness of the GP hierarchy in [18–21, 23], and C-P-Tzirakis in [15, 22].

In [18], spaces $\mathcal{H}_\xi^\alpha$ of sequences of marginal density matrices $\Gamma \in \bigoplus_{k=1}^\infty L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$ were introduced, for $\xi > 0$, endowed with the norm

$$\| \Gamma \|_{\mathcal{H}_\xi^\alpha} := \sum_{k \in \mathbb{N}} \xi^k \| \gamma^{(k)} \|_{H^\alpha} , \tag{1.2.1}$$

where

$$\|\gamma^{(k)}\|_{H^\alpha} := \|S^{(k,\alpha)} \gamma^{(k)}\|_{L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})} \tag{1.2.2}$$

is the norm (1.1.16) considered in [82]. If $\Gamma \in \mathcal{H}_\xi^\alpha$, then $\xi^{-1}$ is an upper bound on the typical $H^\alpha$-energy per particle, [18]. Those spaces are equivalent to those considered by Klainerman and Machedon in [82].

The GP hierarchy can be compactly written in the form (setting $\kappa_0 = 1$)

$$i\partial_t \Gamma + \widehat{\Delta}_\pm \Gamma = B\Gamma \, , \tag{1.2.3}$$

with $\Gamma(0) = \Gamma_0$, where the components of $\widehat{\Delta}\Gamma$ and $B\Gamma$ are termwise defined via (1.1.10), see Section 2.2 for details. As in [18], we refer to a GP hierarchy as being *cubic*, *quintic*, *focusing*, or *defocusing*, according to the type of NLS obtained from factorized solutions.

Local well-posedness was proven in [18] for the regularities of solutions

$$\alpha \in \mathfrak{A}(d,p) := \begin{cases} (\frac{1}{2}, \infty) & \text{if } d = 1 \\ (\frac{d}{2} - \frac{1}{2(p-1)}, \infty) & \text{if } d \geq 2 \text{ and } (d,p) \neq (3,2) \\ \left[1, \infty\right) & \text{if } (d,p) = (3,2)\,, \end{cases} \tag{1.2.4}$$

where $p = 2$ for the cubic, and $p = 4$ for the quintic GP hierarchy. The result is obtained from a Picard fixed point argument, without any requirement on factorization. The parameter $\xi > 0$ is determined by the initial condition, and it sets the energy scale of the given Cauchy problem. By a 2D estimate first proved by X.Chen [25], and later independently by Beckner [8], one can obtain local existence of solutions to the 2D cubic GP hierarchy with $\alpha = 1/2$.

In [22], a conserved energy functional $E_1(\Gamma(t)) = E_1(\Gamma_0)$ was identified, corresponding to the average energy per particle, together with virial identities on the level of GP hierarchies. This allowed to prove Glassey-type blowup for $L^2$-critical and supercritical focusing GP hierarchies. Subsequently, in [21], an infinite family of multiplicative energy functionals was discovered that is conserved under time evolution. These conserved energy made possible to prove global wellposedness for $H^1$ subcritical defocusing GP hierarchies, and for $L^2$ subcritical focusing GP hierarchies, under the assumption that solutions remain positive semidefinite over time.

In [20], the *existence* of solutions to the GP hierarchy was proven in the Klainerman-Machedon type spaces *without* assuming the condition (1.1.17) used for the uniqueness. The proof employs a suitable truncation of the GP hierarchy (for which existence of solutions can be easily obtained) and control of the limit where the truncation is removed. Generalizing this truncation method in [23], Chen and Pavlović proved that solutions of the $N$-body Schrödinger equation converge to the solution of the GP hierarchy in the Klainerman-Machedon type spaces used in [18, 20, 21], for values $\beta \in (0, 1/4)$. While no factorization of solutions was assumed, the specific case of factorized solutions yielded a new derivation of the cubic, defocusing nonlinear Schrödinger equation (NLS) in dimension $d = 3$.

Recently, a derivation of the GP hierarchy in Klainerman-Machedon type spaces was given by X. Chen and J. Holmer in [29], for values $\beta \in (0, 2/3)$. Assuming a regularity requirement that follows from the condition (2.3.2) in Theorem 2.3.1 of our work, they prove that solutions to the BBGKY hierarchy converge to solutions of the GP hierarchy satisfying the Klainerman-Machedon condition (1.1.17). This convergence is shown in the weak-* topology on the space of trace class marginal density matrices. See also [27].

We will call the uniqueness of solutions to the GP hierarchy *unconditional* if it holds without assuming any a priori bound of the form (1.1.17). Recently, in [17], Chen-Hainzl-Pavlović-Seiringer presented a new, simpler proof of the unconditional uniqueness of solutions to the 3D cubic GP hierarchy, which is equivalent to the uniqueness result of Erdös-Schlein-Yau [43]. The authors employed the quantum de Finetti theorem (Theorem 3.1.2 and 3.1.3) combined with the Erdös-Schlein-Yau combinatorial method [42–45] in board game representation as presented by Klainerman-Machedon in [82].

## 1.3 Preview of Results

We begin by deriving the cubic GP hierarchy in $\mathbb{R}^3$ in the strong topology corresponding to the spaces used by Klainerman and Machedon in [82]. We also prove that positive semidefiniteness of solutions is preserved over time and use this result to prove globel well-posedness of solutions to the GP hierarchy. These results are presented in Chapter 2 and are based on a joint work with Thomas Chen [24].

Next, we prove uniqueness of solutions to the GP hierarchy in a low regularity Sobolev type space. This result is presented in Chapter 3 and is based on a joint work with Younghun Hong and Zhihui Xie [70]. It is an extension of the work of Chen-Hainzl-Pavlović-Seiringer [17].

Finally, in Chapter 4, we introduce the *quintic* GP hierarchy and extend the methods from Chapter 3 in order to prove uniqueness of solutions. This chapter is also based on a joint work with Younghun Hong and Zhihui Xie [71].

# Chapter 2

# Derivation in Strong Topology and Global Well-Posedness of Solutions to the Gross-Pitaevskii Hierarchy

## 2.1 Main results of this chapter

In this chapter, we first derive the cubic defocusing GP hierarchy in $\mathbb{R}^3$ from a bosonic $N$-body Schrödinger system. This chapter is based on a joint work with Thomas Chen [24]. We show that solutions to the corresponding $N$-BBGKY hierarchy with initial data $\Gamma_{0,N}$ converge to those of the GP hierarchy strongly in $C([0,T],\mathcal{H}^1_\xi)$ as $N \to \infty$ when the intial data is in $\mathfrak{H}^1_\xi$, see (2.2.7). In [23], this convergence is obtained with initial data in $\mathcal{H}^{1+\delta}_{\xi'}$, for an arbitrary, small $\delta > 0$ extra regularity; in the work at hand, we eliminate this condition. We note that the work at hand (and [23]) are different from previous convergence results in that convergence is shown in the *strong* topology on $\mathcal{H}^1_\xi$ ($\mathcal{H}^{1+\delta}_\xi$, respectively). In the case where the initial data isn't necessarily a finite sum of factorized states (1.1.9), previous works have shown convergence in the *weak-** topology on the space of trace class marginal density matrices. We note that in the context of Theorem 2.3.1, strong convergence in Hilbert Schmidt norm implies weak-* convergence in the trace norm topology, provided that the limit point is trace class. See Proposition 2.B.1.

Furthermore, we prove that solutions to the cubic defocusing GP hierarchy in $\mathbb{R}^3$ remain positive semidefinite over time if the initial data are positive semidefinite. This is the last ingredient needed for proving global well-posedness in $\mathcal{H}^1_\xi$. Indeed, global well-posedness was proven in [21] under the assumption that solutions remain positive semidefinite over time. For our proof, we invoke the quantum de Finetti theorem as presented by Lewin, Nam and Rougerie in [91] (which was recently used in a new proof of unconditional uniqueness of solutions to the GP hierarchy in [17]).

In Appendix 2.A, we show that our local derivation of the GP hierarchy, and global well-posedness of the GP hierarchy can be combined to achieve a derivation of the GP hierarchy on arbitrarily large time intervals $[0, T]$.

## 2.2 Definition of the model

In this section, we introduce the mathematical model studied in this chapter. Most notations and definitions are adopted from [18], where we refer for additional motivations and details.

We consider the $N$-boson Schrödinger equation

$$i\partial_t \Phi_N = \Big( -\sum_{j=1}^{N} \Delta_{x_j} + \frac{1}{N} \sum_{1 \le j < \ell \le N} V_N(x_j - x_\ell) \Big) \Phi_N \tag{2.2.1}$$

on $L^2_{Sym}(\mathbb{R}^{3N})$, with initial data $\Phi_N(0) = \Phi_{0,N} \in L^2_{Sym}(\mathbb{R}^{3N})$, where $L^2_{Sym}(\mathbb{R}^{3N})$ is the subspace of $L^2(\mathbb{R}^{3N})$ that is invariant under permutations (1.1.3) of the

$N$ particle variables. Here, we will take

$$V_N(x) = N^{3\beta} V(N^\beta x) \tag{2.2.2}$$

for some $0 < \beta < 1/4$, and $V \in \mathcal{S}(\mathbb{R}^3)\backslash\{0\}$ is spherically symmetric and nonnegative.

Let

$$\gamma^{(k)}_{\Phi_N}(t, \underline{x}_k, \underline{x}'_k) := \int \Phi_N(t, \underline{x}_k, \underline{x}_{N-k}) \overline{\Phi_N(t, \underline{x}'_k, \underline{x}_{N-k})}\, d\underline{x}_{N-k}.$$

We consider the spaces of sequences of marginal density matrices $\Gamma = \{\gamma^{(k)}\}_{k=1}^\infty \in \bigoplus_{k=1}^\infty L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k})$ introduced in [18]. For brevity, we will write $\underline{x}_k := (x_1, \cdots, x_k)$, and similarly, $\underline{x}'_k := (x'_1, \cdots, x'_k)$.

We call $\Gamma$ *symmetric* if $\gamma^{(k)}(\underline{x}_k, \underline{x}'_k)$ satisfies

$$\begin{gathered} \gamma^{(k)}(x_{\pi(1)}, ..., x_{\pi(k)}; x'_{\pi'(1)}, ..., x'_{\pi'(k)}) = \gamma^{(k)}(x_1, ..., x_k; x'_1, ..., x'_k) \quad \text{and} \\ \gamma^{(k)}(\underline{x}_k; \underline{x}'_k) = \overline{\gamma^{(k)}(\underline{x}'_k; \underline{x}_k)} \end{gathered} \tag{2.2.3}$$

for all $\pi, \pi' \in S_k$ and $k \in \mathbb{N}$.

We say $\Gamma$ is *admissible* if $\gamma^{(k)} = \mathrm{Tr}_{k+1}\gamma^{(k+1)}$, that is,

$$\gamma^{(k)}(\underline{x}_k; \underline{x}'_k) = \int dx_{k+1}\, \gamma^{(k+1)}(\underline{x}_k, x_{k+1}; \underline{x}'_k, x_{k+1}) \tag{2.2.4}$$

for all $k \in \mathbb{N}$. Clearly, the sequence $(\gamma^{(k)}_{\Phi_N})_{k=1}^N$ is admissible for $k = 1, \ldots, N-1$.

We call $\Gamma$ *positive semidefinite* if the operator on $L^2(\mathbb{R}^{3k})$ with integral kernel $\gamma^{(k)}$ is positive semidefinite for all $k$.

Let $0 < \xi < 1$. We define

$$\mathcal{H}_\xi^\alpha := \Big\{ \text{symmetric } \Gamma \in \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k}) \,\Big|\, \|\Gamma\|_{\mathcal{H}_\xi^\alpha} < \infty \Big\} \tag{2.2.5}$$

where

$$\|\Gamma\|_{\mathcal{H}_\xi^\alpha} = \sum_{k=1}^{\infty} \xi^k \| \gamma^{(k)} \|_{H^\alpha(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \,,$$

with

$$\|\gamma^{(k)}\|_{H^\alpha} := \|S^{(k,\alpha)}\gamma^{(k)}\|_{L^2(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \tag{2.2.6}$$

where $S^{(k,\alpha)} := \prod_{j=1}^k (1-\Delta_{x_j})^{\alpha/2}(1-\Delta_{x_j'})^{\alpha/2}$, and $R^{(k,\alpha)} := \prod_{j=1}^k (1-\Delta_{x_j})^{\alpha/2}$.

We also make use of the spaces

$$\mathfrak{H}_\xi^\alpha := \Big\{ \text{symmetric } \Gamma \in \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}^{3k} \times \mathbb{R}^{3k}) \,\Big|\, \|\Gamma\|_{\mathfrak{H}_\xi^\alpha} < \infty \Big\} \tag{2.2.7}$$

where

$$\|\Gamma\|_{\mathfrak{H}_\xi^\alpha} = \sum_{k=1}^{\infty} \xi^k \| \gamma^{(k)} \|_{\mathfrak{h}^\alpha(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \,,$$

with

$$\|\gamma^{(k)}\|_{\mathfrak{h}^\alpha} := \mathrm{Tr}(|S^{(k,\alpha)}\gamma^{(k)}|)$$

that correspond to the spaces of solutions studied in [42, 43].

### 2.2.1 The GP hierarchy

We adopt the necessary notations and definitions for the GP hierarchy from [18]. The cubic defocusing GP hierarchy is given by

$$i\partial_t \gamma^{(k)} = \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma^{(k)}] + \kappa_0 B_{k+1}\gamma^{(k+1)} \tag{2.2.8}$$

for $k \in \mathbb{N}$. Here, $\kappa_0 = \int V(x)dx > 0$ and

$$B_{k+1}\gamma^{(k+1)} = B_{k+1}^{+}\gamma^{(k+1)} - B_{k+1}^{-}\gamma^{(k+1)}\,, \tag{2.2.9}$$

where

$$B_{k+1}^{+}\gamma^{(k+1)} = \sum_{j=1}^{k} B_{j;k+1}^{+}\gamma^{(k+1)}, \tag{2.2.10}$$

and

$$B_{k+1}^{-}\gamma^{(k+1)} = \sum_{j=1}^{k} B_{j;k+1}^{-}\gamma^{(k+1)}, \tag{2.2.11}$$

with

$$\begin{aligned}&\left(B_{j;k+1}^{+}\gamma^{(k+1)}\right)(t, x_1, \ldots, x_k; x_1', \ldots, x_k')\\ &\quad = \int dx_{k+1}dx_{k+1}'\\ &\qquad \delta(x_j - x_{k+1})\delta(x_j - x_{k+1}')\gamma^{(k+1)}(t, x_1, \ldots, x_{k+1}; x_1', \ldots, x_{k+1}'),\end{aligned}$$

and

$$\begin{aligned}&\left(B_{j;k+1}^{-}\gamma^{(k+1)}\right)(t, x_1, \ldots, x_k; x_1', \ldots, x_k')\\ &\quad = \int dx_{k+1}dx_{k+1}'\\ &\qquad \delta(x_j' - x_{k+1})\delta(x_j' - x_{k+1}')\gamma^{(k+1)}(t, x_1, \ldots, x_{k+1}; x_1', \ldots, x_{k+1}').\end{aligned}$$

As stated in the introductory section, we point out that for factorized initial data,

$$\gamma^{(k)}(0; \underline{x}_k; \underline{x}_k') = \prod_{j=1}^{k} \phi_0(x_j)\, \overline{\phi_0(x_j')}\,, \tag{2.2.12}$$

the corresponding solutions of the GP hierarchy remain factorized,

$$\gamma^{(k)}(t, x_1, \dots, x_k; x_1', \dots, x_k') = \prod_{j=1}^{k} \phi(t, x_j)\, \bar{\phi}(t, x_j') \,. \tag{2.2.13}$$

if the corresponding 1-particle wave function satisfies the defocusing cubic NLS

$$i\partial_t \phi = -\Delta \phi + \kappa_0 |\phi|^2 \phi \,.$$

From here on, we will set $\kappa_0 = 1$.

The GP hierarchy can be rewritten in the following compact manner:

$$\begin{aligned} i\partial_t \Gamma + \widehat{\Delta}_{\pm} \Gamma &= B\Gamma \\ \Gamma(0) &= \Gamma_0 \,, \end{aligned} \tag{2.2.14}$$

where

$$\widehat{\Delta}_{\pm} \Gamma := (\, \Delta_{\pm}^{(k)} \gamma^{(k)} \,)_{k \in \mathbb{N}} \,, \qquad \text{with } \Delta_{\pm}^{(k)} = \sum_{j=1}^{k} \left( \Delta_{x_j} - \Delta_{x_j'} \right) \,,$$

and

$$B\Gamma := (\, B_{k+1} \gamma^{(k+1)} \,)_{k \in \mathbb{N}} \,. \tag{2.2.15}$$

We will also use the notation

$$B^{+}\Gamma := (\, B_{k+1}^{+} \gamma^{(k+1)} \,)_{k \in \mathbb{N}} \quad , \quad B^{-}\Gamma := (\, B_{k+1}^{-} \gamma^{(k+1)} \,)_{k \in \mathbb{N}} \,.$$

Moreover, we define the free evolution operator $U(t)$ by

$$(U(t)\Gamma)^{(k)} = U^{(k)}(t)\gamma^{(k)},$$

where

$$(U^{(k)}(t)\gamma^{(k)})(\underline{x}_k, \underline{x}_k') = e^{it\Delta_{\underline{x}_k}} e^{-it\Delta_{\underline{x}_k'}} \gamma^{(k)}(\underline{x}_k, \underline{x}_k')$$

corresponds to the $k$-th component.

### 2.2.2 The BBGKY hierarchy

A similar compact notation for the cubic defocusing BBGKY hierarchy can be introduced as follows, [23]. We consider the cubic defocusing BBGKY hierarchy in $\mathbb{R}^3$,

$$\begin{aligned} i\partial_t \gamma_N^{(k)}(t) &= \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma_N^{(k)}(t)] + \frac{1}{N} \sum_{1 \le j < k} [V_N(x_j - x_k), \gamma_N^{(k)}(t)] \\ &\quad + \frac{(N-k)}{N} \sum_{1 \le j \le k} \mathrm{Tr}_{k+1} [V_N(x_j - x_{k+1}), \gamma_N^{(k+1)}(t)] \,, \end{aligned} \tag{2.2.16}$$

for $k = 1, \ldots, N$, where we recall that $V_N(x) = N^{3\beta} V(N^\beta x)$ for $0 < \beta < 1/4$, and $V \in \mathcal{S}(\mathbb{R}^3) \backslash \{0\}$ spherically symmetric and nonnegative. We extend this finite hierarchy trivially to an infinite hierarchy by adding the terms $\gamma_N^{(k)} = 0$ for $k > N$. This will allow us to treat solutions of the BBGKY hierarchy on the same footing as solutions to the GP hierarchy.

We next introduce the following compact notation for the BBGKY hierarchy.

$$i\partial_t \gamma_N^{(k)} = \sum_{j=1}^{k} [-\Delta_{x_j}, \gamma_N^{(k)}] + (B_N \Gamma_N)^{(k)} \tag{2.2.17}$$

for $k \in \mathbb{N}$. Here, we have $\gamma_N^{(k)} = 0$ for $k > N$, and we define

$$(B_N \Gamma_N)^{(k)} := \begin{cases} B_{N;k+1}^{main} \gamma_N^{(k+1)} + B_{N;k}^{error} \gamma_N^{(k)} & \text{if } k \le N \\ \\ 0 & \text{if } k > N. \end{cases} \tag{2.2.18}$$

The interaction terms on the right hand side are defined by

$$B_{N;k+1}^{main} \gamma_N^{(k+1)} = B_{N;k+1}^{+,main} \gamma_N^{(k+1)} - B_{N;k+1}^{-,main} \gamma_N^{(k+1)} \,, \tag{2.2.19}$$

and

$$B^{error}_{N;k}\gamma^{(k)}_N = B^{+,error}_{N;k}\gamma^{(k)}_N - B^{-,error}_{N;k}\gamma^{(k)}_N\,, \tag{2.2.20}$$

where

$$B^{\pm,main}_{N;k+1}\gamma^{(k+1)}_N := \frac{N-k}{N}\sum_{j=1}^{k} B^{\pm,main}_{N;j;k+1}\gamma^{(k+1)}_N, \tag{2.2.21}$$

and

$$B^{\pm,error}_{N;k}\gamma^{(k)}_N := \frac{1}{N}\sum_{i<j}^{k} B^{\pm,error}_{N;i,j;k}\gamma^{(k)}_N, \tag{2.2.22}$$

with

$$\begin{aligned}&\Big(B^{+,main}_{N;j;k+1}\gamma^{(k+1)}_N\Big)(t,x_1,\dots,x_k;x'_1,\dots,x'_k)\\ &= \int dx_{k+1}V_N(x_j-x_{k+1})\gamma^{(k+1)}_N(t,x_1,\dots,x_k,x_{k+1};x'_1,\dots,x'_k,x_{k+1})\end{aligned} \tag{2.2.23}$$

and

$$\begin{aligned}&\Big(B^{+,error}_{N;i,j;k}\gamma^{(k)}_N\Big)(t,x_1,\dots,x_k;x'_1,\dots,x'_k)\\ &= V_N(x_i-x_j)\gamma^{(k)}(t,x_1,\dots,x_k;x'_1,\dots,x'_k)\,.\end{aligned} \tag{2.2.24}$$

Moreover,

$$\begin{aligned}&\Big(B^{-,main}_{N;j;k+1}\gamma^{(k+1)}_N\Big)(t,x_1,\dots,x_k;x'_1,\dots,x'_k)\\ &\quad= \int dx_{k+1}V_N(x'_j-x_{k+1})\gamma^{(k+1)}_N(t,x_1,\dots,x_k,x_{k+1};x'_1,\dots,x'_k,x_{k+1}).\end{aligned}$$

and

$$\begin{aligned}&\Big(B^{-,error}_{N;i,j;k}\gamma^{(k)}_N\Big)(t,x_1,\dots,x_k;x'_1,\dots,x'_k)\\ &\quad= V_N(x'_i-x'_j)\gamma^{(k)}(t,x_1,\dots,x_k;x'_1,\dots,x'_k)\,.\end{aligned}$$

This notation has the advantage that we can treat the BBGKY hierarchy and the GP hierarchy on the same footing. We remark that in all of the above definitions, we have that $B_{N;k}^{\pm,main}$, $B_{N;k}^{\pm,error}$, etc. are defined to be given by multiplication with zero for $k > N$.

Indeed, we can write the BBGKY hierarchy compactly in the form

$$\begin{aligned} i\partial_t \Gamma_N + \widehat{\Delta}_{\pm} \Gamma_N &= B_N \Gamma_N \\ \Gamma_N(0) &\in \mathcal{H}_{\xi}^{\alpha} \, , \end{aligned} \tag{2.2.25}$$

where

$$\widehat{\Delta}_{\pm} \Gamma_N := (\, \Delta_{\pm}^{(k)} \gamma_N^{(k)} \,)_{k \in \mathbb{N}} \, , \qquad \text{with } \Delta_{\pm}^{(k)} = \sum_{j=1}^{k} \Big( \Delta_{x_j} - \Delta_{x_j'} \Big) \, ,$$

and

$$B_N \Gamma_N := (\, B_{N;k+1} \gamma_N^{(k+1)} \,)_{k \in \mathbb{N}} \, . \tag{2.2.26}$$

In addition, we introduce the notation

$$\begin{aligned} B_N^{+} \Gamma_N &:= \, (\, B_{N;k+1}^{+} \gamma_N^{(k+1)} \,)_{k \in \mathbb{N}} \\ B_N^{-} \Gamma_N &:= \, (\, B_{N;k+1}^{-} \gamma_N^{(k+1)} \,)_{k \in \mathbb{N}} \end{aligned}$$

which will be convenient.

### 2.2.3 Higher order energy functionals

As in [21], we define the higher order energy functionals for the cubic GP hierarchy,

$$\langle K^{(m)} \rangle_{\Gamma(t)} := \mathrm{Tr}_{1,3,5,\ldots,2m+1} (K^{(m)} \gamma^{(2m)}(t)) \tag{2.2.27}$$

for $m \in \mathbb{N}$, where

$$K_\ell := \frac{1}{2}(1-\Delta_{x_\ell})\mathrm{Tr}_{\ell+1} + \frac{1}{4}B^+_{\ell;\ell+1} \quad , \quad \ell \in \mathbb{N}\,,$$
$$K^{(m)} := K_1 K_3 \cdots K_{2m-1}.$$

In [21], it is shown that these higher order energy functionals are conserved:

**Proposition 2.2.1.** *Suppose that $\Gamma \in \mathfrak{H}^1_\xi$ is symmetric, admissible, and solves the GP hierarchy. Then, for all $m \in \mathbb{N}$, the higher order energy functionals* (2.2.27) *are bounded and conserved, $\langle K^{(m)}\rangle_{\Gamma(t)} = \langle K^{(m)}\rangle_{\Gamma(0)}$.*

## 2.3 Main theorems

In this section, we summarize the main results of this chapter. For $I \subseteq \mathbb{R}$, we denote by

$$\mathcal{W}^\alpha_\xi(I) := \{\Gamma \in C(I, \mathcal{H}^\alpha_\xi) \,|\, B^+\Gamma, B^-\Gamma \in L^2_{loc}(I, \mathcal{H}^\alpha_\xi)\}\,, \qquad (2.3.1)$$

the space of local in time solutions of the GP hierarchy, with $t \in I$, following [18].

**Theorem 2.3.1.** *Let $(\Phi_N)_N$ be a sequence of solutions to the N-body Schrödinger equation* (2.2.1) *for which we have that for some $0 < \xi' < 1$, and every $N \in \mathbb{N}$,*

$$\Gamma^{\Phi_N}(0) = (\gamma^{(1)}_{\Phi_N}(0), \ldots, \gamma^{(N)}_{\Phi_N}(0), 0, \cdots) \in \mathcal{H}^1_{\xi'}$$

*and that*

$$\Gamma_0 := \lim_{N\to\infty} \Gamma^{\Phi_N}(0)$$

*exists in* $\mathcal{H}^1_{\xi'}$*. Suppose also that*

$$\langle \Phi_N(0), H_N^k \Phi_N(0)\rangle < C^k N^k \tag{2.3.2}$$

*and* $\|\Phi_N(0)\|_{L^2} = 1$ *for all* $N \in \mathbb{N}$ *and* $k \leq N$*, where* $C$ *doesn't depend on* $k$ *or* $N$*. Define the truncation operator* $P_{\leq K}$ *by*

$$P_{\leq K(N)}\Gamma = (\gamma^{(1)}, \ldots, \gamma^{(K(N))}, 0, \ldots),$$

*where* $\frac{1}{2} b_1 \log N \leq K(N) \leq b_1 \log N$ *for some* $b_1 > 0$*. Then, for sufficently small* $b_1 > 0$ *(depending only on* $\beta$ *(see (2.2.2))) and sufficiently small* $\xi > 0$ *(depending on only on* $\xi'$*) and sufficiently small* $T > 0$ *(depending only on* $\xi$*), the limit*

$$\Gamma := \lim_{N\to\infty} P_{\leq K(N)}\Gamma^{\Phi_N}$$

*exists in* $L^\infty_{t\in[0,T]}\mathcal{H}^1_\xi$ *and satisfies the GP hierarchy with initial data* $\Gamma_0$*. Moreover,*

$$B\Gamma = \lim_{N\to\infty} B_N P_{K\leq N}\Gamma^{\Phi_N}$$

*holds in* $L^2_{t\in[0,T]}\mathcal{H}^1_\xi$*.*

With this result, we remove an extra regularity condition on the initial data which was assumed in [23], where $\Gamma_0 \in \mathfrak{H}^{1+\delta}_{\xi'}$ was required for an arbitrarily small, but positive $\delta > 0$.

We note that, by combining Theorem 2.3.1 and 2.3.3, one can show that Theorem 2.3.1 actually holds for $T$ arbitrarily large, provided that $\Gamma_0 \in \mathfrak{H}^1_{\xi'}$, $\Gamma_0$ is admissible, and that $\xi$ is sufficiently small. See Appendix 2.A.

**Theorem 2.3.2.** *Suppose that $\Gamma_0 \in \mathfrak{H}^1_{\xi'}$ is positive semidefinite, admissible, and satisfies $\operatorname{Tr} \gamma_0^{(1)} = 1$. Then, for sufficiently small $\xi > 0$ (depending only on $\xi'$) and sufficiently small $T > 0$ (depending only on $\xi$), there is a unique solution $\Gamma \in \mathcal{W}^1_\xi([0,T])$ to the cubic defocusing GP hierarchy* (2.2.8) *in $\mathbb{R}^3$ with initial data $\Gamma_0$. Moreover, $\Gamma(t)$ is positive semidefinite for $t \in [0,T]$.*

Our proof of positive semidefiniteness uses the quantum de Finetti theorem in the formulation presented in a recent paper by Lewin, Nam and Rougerie [91]. We note that we are *not* using the proof of unconditional uniqueness from [17], but combine an application of quantum de Finetti with the local well-posedness theory for GP hierarchies developed in [18].

**Theorem 2.3.3.** *Suppose that $\Gamma_0 = \{\gamma_0^{(k)}\}_{k=1}^\infty \in \mathfrak{H}^1_{\xi'}$ is positive semidefinite, admissible, and satisfies $\operatorname{Tr} \gamma_0^{(1)} = 1$. Then, for sufficiently small $\xi > 0$ (depending only on $\xi'$) and sufficiently small $\xi_1 > 0$ (depending only on $\xi$), there is a unique global solution $\Gamma \in \mathcal{W}^1_{\xi_1}(\mathbb{R})$ to the cubic defocusing GP hierarchy* (2.2.8) *in $\mathbb{R}^3$ with initial data $\Gamma_0$. Moreover, $\Gamma(t)$ is positive semidefinite and satisfies*

$$\|\Gamma(t)\|_{\mathcal{H}^1_{\xi_1}} \leq \|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}}$$

*for all $t \in \mathbb{R}$.*

Although we only address the the cubic defocusing GP hierarchy in $\mathbb{R}^d$ for $d = 3$, we note that Theorems 2.3.2 and 2.3.3 can be proved in the same way for the more general cases considered in Theorem 7.2 of [21]. Let $\kappa_0$ be the constant in (2.2.8), and let $p = 2, 4$ correspond to the cubic and quintic GP hierarchies, respectively. Then, we have global well-posedness for the following cases:

- Energy subcritical, defocusing $p$-GP hierarchy with $p < \frac{4}{d-2}$ and $\kappa_0 = +1$.
- $L^2$ subcritical, focusing $p$-GP hierarchy with $p < \frac{4}{d}$ and $\kappa_0 < 0$ with $|\kappa_0|$ sufficiently small (see Theorem 7.2 in [21] for an explicit bound on $|\kappa_0|$).

For the statement of these main theorems, we are using the following constants. We let $b_1 > 0$ be sufficiently small that Lemma 2.E.2 in the appendix is satisfied for all $K, N$ such that $K \leq b_1 \log N$. Throughout this chapter, we will require that, given $\xi' > 0$, the real, positive constants $\xi$ and $\xi_1$ satisfy

$$\begin{cases} \xi < \eta \min\left\{ \frac{1}{\xi'} \, e^{-2\beta/b_1}, e^{-24\beta/b_1} \right\} & \text{and} \\ 0 < \xi_1 < \theta^3 \xi < \theta^6 \xi', \end{cases} \tag{2.3.3}$$

where $\theta := \min\{\eta, (1 + \frac{2}{5} C_{Sob})^{-2/5}\}$; the constant $\eta > 0$ is defined in Lemma 2.E.2, and $C_{Sob} > 0$ is the constant in the trace Sobolev inequality

$$\Big( \int dx |f(x,x)|^2 \Big)^{\frac{1}{2}} \leq C_{Sob} \Big( \int dx_1 \, dx_2 \, \big| \langle \nabla_{x_1} \rangle \langle \nabla_{x_2} \rangle \, f(x_1, x_2) \big|^2 \Big)^{\frac{1}{2}} \tag{2.3.4}$$

for $x_{1,2} \in \mathbb{R}^3$, see [21]. This will ensure that $\xi_1$ and $\xi$ are small enough so that the results of both [23] and [21] hold.

For $\nu \in \mathbb{R}_+$, we define

$$T_0(\nu) := \nu^2/c_0, \tag{2.3.5}$$

where the constant $c_0 > 0$ is defined in Lemma 2.E.1.

## 2.4 Derivation of GP from BBGKY hierarchy

In this section, we prove Theorem 2.3.1.

### 2.4.1 Local well-posedness of the BBGKY hierarchy in $\mathcal{H}^1_\xi$

Taking $\delta = 0$ in Lemma 4.1 of [23] gives us local well-posedness of the $K$-truncated $N$-BBGKY hierarchy:

**Lemma 2.4.1.** *Let $K < b_1 \log N$, for some constant $b_1 > 0$. Let $\Gamma^K_{0,N} := P_{\le K}\Gamma_{0,N} = \{\gamma^{(k)}_{0,N}\}_{k=1}^K \in \mathcal{H}^1_{\xi'}$. Then, for $0 < \xi' < 1$ and $\xi$ satisfying* (2.3.3), *and for $0 < T < T_0(\xi)$ (see* (2.3.5)*), and for $b_1 > 0$ sufficiently small (depending only on $\beta$ (see* (2.2.2)*)), there exists a unique solution $\Gamma^K_N \in L^\infty_{t\in I}\mathcal{H}^1_\xi$ of the BBGKY hierarchy* (2.2.16) *for $I = [0, T]$ such that $B_N\Gamma^K_N \in L^2_{t\in I}\mathcal{H}^1_\xi$. Moreover,*

$$\|\Gamma^K_N\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} \le C_0(T, \xi, \xi')\,\|\Gamma^K_{0,N}\|_{\mathcal{H}^1_{\xi'}} \tag{2.4.1}$$

*and*

$$\|B_N \Gamma_N^K\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \;\leq\; C_0(T,\xi,\xi')\,\|\Gamma_{0,N}^K\|_{\mathcal{H}^1_{\xi'}} \tag{2.4.2}$$

*hold. The constant $C_0 = C_0(T,\xi,\xi')$ is independent of $N$.*

*Furthermore, $(\Gamma_N^K)^{(k)} = 0$ for all $K < k \leq N$, and $t \in I$.*

### 2.4.2 From $(K,N)$-BBGKY to $K$-truncated GP hierarchy

In this section, we show that solutions to the $(K,N)$-BBGKY hierarchy approach those of the $K$-Truncated GP hierarchy as $N \to \infty$.

*Proposition* 2.4.1. Suppose that $\Gamma_0 = \{\gamma_0^{(k)}\}_{k=1}^{\infty} \in \mathfrak{H}^1_{\xi'}$. Let $\Gamma^K \in \{\Gamma \in L^\infty_{t\in[0,T]}\mathcal{H}^1_\xi | B\Gamma \in L^2_{t\in[0,T]}\mathcal{H}^1_\xi\}$ be the solution of the GP hierarchy (2.2.8) with truncated initial data $\Gamma_0^K = P_{\leq K}\Gamma_0$ constructed in [20], where $0 < \xi' < 1$ and $\xi$ satisfy (2.3.3), and $0 < T < T_0(\xi)$ (see (2.3.5)). Let $\Gamma_N^K$ solve the $(K,N)$-BBGKY hierarchy (2.2.16) with the same initial data $\Gamma_{0,N}^K := P_{\leq K}\Gamma_0$. Let

$$K(N) \leq b_1 \log N \tag{2.4.3}$$

as in Lemma 2.4.1. Then,

$$\lim_{N\to\infty} \|\,\Gamma_N^{K(N)} - \Gamma^{K(N)}\,\|_{L^\infty_{t\in[0,T]}\mathcal{H}^1_\xi} = 0 \tag{2.4.4}$$

and

$$\lim_{N\to\infty} \| \, B_N \Gamma_N^{K(N)} \, - \, B\Gamma^{K(N)} \, \|_{L^2_{t\in[0,T]}\mathcal{H}^1_\xi} = 0. \tag{2.4.5}$$

*Proof.* In [20], the authors constructed a solution $\Gamma^K$ of the full GP hierarchy with truncated initial data, $\Gamma(0) = \Gamma_0^K \in \mathcal{H}^1_{\xi'}$, such that for an arbitrary fixed $K$, $\Gamma^K$ satisfies the GP-hierarchy in integral representation,

$$\Gamma^K(t) = U(t)\Gamma_0^K + i \int_0^t U(t-s)\, B\Gamma^K(s)\, ds\,. \tag{2.4.6}$$

and, in particular, $(\Gamma^K)^{(k)}(t) = 0$ for all $k > K$.

Accordingly, we have

$$\begin{aligned}
&B_N\Gamma_N^K - B\Gamma^K \\
&= B_N U(t)\Gamma_{0,N}^K - BU(t)\Gamma_0^K \\
&\quad + i \int_0^t \big( B_N U(t-s) B_N \Gamma_N^K - BU(t-s)B\Gamma^K \big)(s) ds \\
&= (B_N - B)U(t)\Gamma_{0,N}^K + BU(t)(\Gamma_{0,N}^K - \Gamma_0^K) \\
&\quad + i \int_0^t \big( B_N - B\big) U(t-s) B\Gamma^K(s)\, ds \\
&\quad + i \int_0^t B_N U(t-s) \big( B_N\Gamma_N^K - B\Gamma^K \big)(s)\, ds\,.
\end{aligned} \tag{2.4.7}$$

Here, we observe that we can apply Lemma 2.E.2 with

$$\widetilde{\Theta}_N^K := B_N\Gamma_N^K - B\Gamma^K \tag{2.4.8}$$

and

$$\begin{aligned}
\Xi_N^K &:= (B_N - B)U(t)\Gamma_{0,N}^K + BU(t)(\Gamma_{0,N}^K - \Gamma_0^K) \\
&\quad + i \int_0^t \big( B_N - B\big) U(t-s) B\Gamma^K(s)\, ds\,.
\end{aligned} \tag{2.4.9}$$

Given $\xi'$, we introduce parameters $\xi, \xi'', \xi'''$ satisfying

$$\xi \,<\, \theta\, \xi'' \,<\, \theta^2\, \xi''' \,<\, \theta^3 \xi' \tag{2.4.10}$$

where the constant $\theta$ is defined as in (2.3.3), so that $0 < \theta \leq \eta$, where $\eta$ is defined as in Lemma 2.E.2. Accordingly, Lemma 2.E.2 implies that

$$\begin{aligned}
&\|B_N \Gamma_N^K - B\Gamma^K\|_{L^2_{t\in[0,T]}\mathcal{H}^1_\xi} \\
&\leq C_0(T,\xi,\xi'')\Big(\, \|BU(t)(\Gamma_{0,N}^K - \Gamma_0^K)\|_{L^2_{t\in[0,T]}\mathcal{H}^1_{\xi''}} + R_1(N) + R_2(N)\,\Big) \\
&\leq C_1(T,\xi,\xi',\xi'')\Big(\, \|\Gamma_{0,N}^K - \Gamma_0^K\|_{L^2_{t\in[0,T]}\mathcal{H}^1_{\xi'}} + R_1(N) + R_2(N)\,\Big)\,,
\end{aligned} \tag{2.4.11}$$

where we used Lemma A.1 in [23] to pass to the last line. Here,

$$R_1(N) := \|(B_N - B)U(t)\Gamma_{0,N}^K\|_{L^2_{t\in[0,T]}\mathcal{H}^1_{\xi''}} \tag{2.4.12}$$

and

$$R_2(N) := \Big\| \int_0^t \big( B_N - B\big) U(t-s) B\Gamma^K(s)\, ds \Big\|_{L^2_{t\in[0,T]}\mathcal{H}^1_{\xi''}}. \tag{2.4.13}$$

Next, we consider the limit $N \to \infty$ with $K(N)$ as given in (2.4.3).

We have

$$\begin{aligned}
\lim_{N\to\infty} \|\Gamma_{0,N}^{K(N)} - \Gamma_0^{K(N)}\|_{\mathcal{H}^1_{\xi'}} &= \lim_{N\to\infty} \|\, P_{\leq K(N)}\,(\,\Gamma_{0,N} - \Gamma_0\,)\,\|_{\mathcal{H}^1_{\xi'}} \\
&\leq \lim_{N\to\infty} \|\,\Gamma_{0,N} - \Gamma_0\,\|_{\mathcal{H}^1_{\xi'}} \\
&= 0.
\end{aligned} \tag{2.4.14}$$

By Lemmas 2.4.2 and 2.4.3 below, we have that

$$\lim_{N\to\infty} R_1(N) = 0$$

and

$$\lim_{N\to\infty} R_2(N) = 0.$$

Thus (2.4.11) $\to 0$ as $N \to \infty$, and hence the limit (2.4.5) holds. To prove (2.4.4), we observe that

$$\begin{aligned}
&\Gamma_N^{K(N)}(t) - \Gamma^{K(N)}(t) \\
&= U(t)\left(\Gamma_N^{K(N)}(0) - \Gamma^{K(N)}(0)\right) + i\int_0^t U(t-s)\left(B_N\Gamma_N^{K(N)}(s) - B\Gamma^{K(N)}(s)\right) ds,
\end{aligned}$$

and hence, for $0 < t < T$,

$$\begin{aligned}
&\|\Gamma_N^{K(N)}(t) - \Gamma^{K(N)}(t)\|_{\mathcal{H}_\xi^1} \\
&\leq \|U(t)\left(\Gamma_N^{K(N)}(0) + \Gamma^{K(N)}(0)\right)\|_{\mathcal{H}_\xi^1} \\
&\qquad\qquad + t^{1/2}\|U(t-s)\left(B_N\Gamma_N^{K(N)}(s) - B\Gamma^{K(N)}(s)\right)\|_{L^2_{s\in[0,t]}\mathcal{H}_\xi^1} \\
&= \|\Gamma_N^{K(N)}(0) + \Gamma^{K(N)}(0)\|_{\mathcal{H}_\xi^1} + t^{1/2}\|B_N\Gamma_N^{K(N)} - B\Gamma^{K(N)}\|_{L^2_{[0,t]}\mathcal{H}_\xi^1} \\
&\leq \|\Gamma_N^{K(N)}(0) + \Gamma^{K(N)}(0)\|_{\mathcal{H}_\xi^1} + T^{1/2}\|B_N\Gamma_N^{K(N)} - B\Gamma^{K(N)}\|_{L^2_{[0,T]}\mathcal{H}_\xi^1} \qquad (2.4.15) \\
&\to 0 \text{ as } N \to \infty \text{ by (2.4.5).}
\end{aligned}$$

Since the last line (2.4.15) is independent of $t$, the result (2.4.4) follows. □

**Lemma 2.4.2.** *Under the same assumptions as in Proposition 2.4.1,*

$$\lim_{N\to\infty} \left\|(B_N - B)U(t)\,\Gamma_{0,N}^{K(N)}\right\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^1_{\xi''}} = 0. \qquad (2.4.16)$$

*Proof.* We recall that for $g : \mathbb{R}^n \to \mathbb{C}$ of the form $g(x) = f(x, x)$ for some Schwartz class fuction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, one has

$$\widehat{g}(\xi) = \int \widehat{f}(\xi - \eta, \eta)\, d\eta. \tag{2.4.17}$$

We note that the Fourier transform of $\left(U^{(k+1)}(t)\gamma_0^{(k+1)}\right)(\underline{x}_{k+1}, \underline{x}'_{k+1})$ with respect to the variables $(t, \underline{x}_{k+1}, \underline{x}'_{k+1})$ is given by

$$\delta(\tau + |\underline{\xi}_{k+1}|^2 - |\underline{\xi}'_{k+1}|^2)\widehat{\gamma}_0^{(k+1)}(\underline{\xi}_{k+1}, \underline{\xi}'_{k+1}). \tag{2.4.18}$$

Recall that $B^{+,main}_{N,1,k+1}U(t)(\gamma_{0,N}^{(k+1)})$ is given by

$$\int V_N(x_1 - x_{k+1})\gamma_N^{(k+1)}(t, x_1, ..., x_k, x_{k+1}; x'_1, ..., x'_k, x_{k+1})dx_{k+1}$$

and hence its Fourier transform with respect to the variables $(t, \underline{\mathrm{x}}_k, \underline{\mathrm{x}}'_k)$ is given by

$$\begin{aligned}
&\int e^{-ix_{k+1}u_1}\widehat{V}_N(u_1) *_{u_1} (F\gamma_N^{(k+1)})(\tau, u_1, ..., u_k, x_{k+1}; u'_1, ..., u'_k, x_{k+1})\, dx_{k+1} \\
&= \int\int e^{-ix_{k+1}\eta}\widehat{V}_N(\eta)(F\gamma_N^{(k+1)})(\tau, u_1 - \eta, u_2, ..., u_k, x_{k+1}; u'_1, ..., u'_k, x_{k+1})\, d\eta\, dx_{k+1} \\
&= \int\int \widehat{V}_N(\eta)\widehat{\gamma}_N^{(k+1)}(\tau, u_1 - \eta, u_2, ..., u_k, \eta - \nu; u'_1, ..., u'_k, \nu)\, d\nu\, d\eta \qquad \text{(by (2.4.17))} \\
&= \int\int \widehat{V}_N(\eta + \nu)\widehat{\gamma}_N^{(k+1)}(\tau, u_1 - \eta - \nu, u_2, ..., u_k, \eta; u'_1, ..., u'_k, \nu)\, d\nu\, d\eta
\end{aligned}$$

where we substituted $\eta \to \eta + \nu$. Thus, the above equals

$$\begin{aligned}
&= \int\int \widehat{V}_N(u_{k+1} + u'_{k+1}) \\
&\qquad \widehat{\gamma}_N^{(k+1)}(\tau, u_1 - u_{k+1} - u'_{k+1}, u_2, ..., u_k, u_{k+1}; u'_1, ..., u'_k, u'_{k+1})\, du'_{k+1}\, du_{k+1} \\
&= \int\int \widehat{V}_N(u_{k+1} + u'_{k+1})\delta(\cdots) \\
&\qquad \widehat{\gamma}_{0,N}^{(k+1)}(u_1 - u_{k+1} - u'_{k+1}, u_2, ..., u_k, u_{k+1}; \underline{u}'_{k+1})\, du'_{k+1}\, du_{k+1}
\end{aligned} \tag{2.4.19}$$

where the operator $F$ is the Fourier transform with respect to the variables $(t, \underline{\mathrm{x}}_k, \underline{\mathrm{x}}'_k)$ and

$$\delta(...) := \delta(\tau + |u_1 - u_{k+1} - u'_{k+1}|^2 + |\underline{u}_{k+1}|^2 - |u_1|^2 - |\underline{u}'_{k+1}|^2).$$

Equation (2.4.18) was used to pass to the last line (2.4.19). Similarly, the Fourier transform of $B^+_{k+1}U(t)(\gamma^{(k+1)}_{0,N})$ with respect to the variables $(t, \underline{\mathrm{x}}_k, \underline{\mathrm{x}}'_k)$ is given by

$$\int\int \delta(...)\widehat{\gamma}^{(k+1)}_{0,N}(u_1 - u_{k+1} - u'_{k+1}, u_2, ..., u_{k+1}, \underline{u}'_{k+1})du_{k+1}du'_{k+1}.$$

Thus,

$$\begin{aligned}
&\|(B^{+,main}_{N;1;k+1} - B^+_{1;k+1})U(t)\gamma^{(k+1)}_{0,N}\|^2_{L^2_t H^1} \\
&= \int\int\int \prod_{j=1}^{k}\langle u_j\rangle^2 \prod_{j=1}^{k}\langle u'_j\rangle^2 \Big(\int\int (1 - \widehat{V}_N(u_{k+1} + u'_{k+1}))\delta(...) \\
&\qquad \widehat{\gamma}^{(k+1)}_{0,N}(u_1 - u_{k+1} - u'_{k+1}, u_2, ..., u_k, u_{k+1}; \underline{u}'_{k+1})\, du'_{k+1}\, du_{k+1}\Big)^2 d\underline{u}_k\, d\underline{u}'_k\, d\tau \\
&\le \int\int\int J(\tau, \underline{u}_k, \underline{u}'_k) \int\int \delta(...)\langle u_1 - u_{k+1} - u'_{k+1}\rangle^2 \langle u_{k+1}\rangle^2 \langle u'_{k+1}\rangle^2 \\
&\qquad \prod_{j=2}^{k}\langle u_j\rangle^2 \prod_{j'=1}^{k}\langle u'_j\rangle^2 |1 - \widehat{V}_N(u_{k+1} + u'_{k+1})|^2 \\
&\qquad\qquad |\widehat{\gamma}^{(k+1)}_{0,N}(u_1 - u_{k+1} - u'_{k+1}, u_2, ..., u_k, u_{k+1}; \underline{u}'_{k+1})|^2 \\
&\qquad\qquad\qquad du'_{k+1}\, du_{k+1}\, d\underline{u}_k\, d\underline{u}'_k\, d\tau
\end{aligned} \tag{2.4.20}$$

where

$$J(\tau, \underline{u}_k, \underline{u}'_k) := \int\int \frac{\delta(...)\langle u_1\rangle^2}{\langle u_1 - u_{k+1} - u'_{k+1}\rangle^2 \langle u_{k+1}\rangle^2 \langle u'_{k+1}\rangle^2}\, du_{k+1}du'_{k+1}$$

and $J(\tau, \underline{u}_k, \underline{u}'_k)$ is bounded uniformly in $\tau, \underline{u}_k, \underline{u}'_k$, see Proposition 2.1 of [82].

Let $\delta$ satisfy $0 < \delta < \beta$. Recall that $\widehat{V}_N(u) = \widehat{V}(N^{-\beta}u)$. The integral (2.4.20) can now be separated into the regions $\{|u_{k+1} + u'_{k+1}| < N^\delta\}$ and $\{|u_{k+1} + u'_{k+1}| \geq N^\delta\}$.

The portion of the integral (2.4.20) over $\{|u_{k+1} + u'_{k+1}| < N^\delta\}$ is bounded by

$$C_V N^{4(\delta-\beta)} \|\gamma_{0,N}^{(k+1)}\|_{H^1}^2 \tag{2.4.21}$$

because $\nabla\widehat{V}(0) = 0$ and $\widehat{V} \in C^2$, so by Taylor's Theorem,

$$\begin{aligned}
&\sup_{|u_{k+1}+u'_{k+1}|<N^\delta} |1 - \widehat{V}_N(u_{k+1} + u'_{k+1})|^2 \\
&= \sup_{|u_{k+1}+u'_{k+1}|<N^\delta} |1 - \widehat{V}(N^{-\beta}(u_{k+1} + u'_{k+1}))|^2 \\
&\leq \sup_{|u_{k+1}+u'_{k+1}|<N^\delta} C_V(N^{-\beta}(u_{k+1} + u'_{k+1}))^4 \\
&\leq C_V N^{4(\delta-\beta)},
\end{aligned}$$

where $C_V$ is the $L^\infty$ norm of the second derivative of $V$.

The portion of the integral (2.4.20) over $\{|u_{k+1} + u'_{k+1}| \geq N^\delta\}$ is bounded by

$$
\begin{aligned}
&a_{k,N}^2 \\
&:= \int\int\int J(\tau, \underline{u}_k, \underline{u}'_k) \int_{|u_{k+1}| \geq N^\delta} \int \delta(...) \langle u_1 - u_{k+1} - u'_{k+1} \rangle^2 \\
&\qquad \langle u_{k+1} \rangle^2 \langle u'_{k+1} \rangle^2 \prod_{j=2}^{k} \langle u_j \rangle^2 \prod_{j'=1}^{k} \langle u'_j \rangle^2 (1 + \|\widehat{V}\|_\infty)^2 \\
&\qquad\qquad \left| \widehat{\gamma}_{0,N}^{(k+1)}(u_1 - u_{k+1} - u'_{k+1}, u_2, ..., u_k, u_{k+1}; \underline{u}'_{k+1}) \right|^2 \\
&\qquad\qquad\qquad du'_{k+1}\, du_{k+1}\, d\underline{u}_k\, d\underline{u}'_k\, d\tau \qquad (2.4.22) \\
&\leq C \|\gamma_{0,N}^{(k+1)}\|_{H^1}^2 \\
&= C \|\gamma_0^{(k+1)}\|_{H^1}^2. \qquad (2.4.23)
\end{aligned}
$$

We are now ready to bound the desired qauntity (2.4.16) in the statement of the lemma.

Let $\Omega_{k,N} = \{|u_{k+1} + u'_{k+1}| < N^\delta\}$. Then,

$$
\begin{aligned}
&\|(B_N^+ - B^+)U(t)\Gamma_{0,N}^K\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^1_{\xi''}} - \underbrace{\|B_N^{+,error}U(t)\Gamma_{0,N}^K\|_{L^2_{t\in\mathbb{R}}\mathcal{H}^1_{\xi''}}}_{\to 0 \text{ as } N\to\infty \text{ by Proposition A.2 in [23]}} \\
&\leq \sum_{k=1}^K \sum_{j=1}^k (\xi'')^k \|(B^{+,main}_{N;j;k+1} - B^+_{j,k+1})U^{(k+1)}(t)\gamma_{0,N}^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H^1} \\
&\leq \sum_{k=1}^K k(\xi'')^k \|(B^{+,main}_{N;1;k+1} - B^+_{1;k+1})U^{(k+1)}(t)\gamma_{0,N}^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H^1} \\
&\leq \sum_{k=1}^K k(\xi')^k(\xi''/\xi')^k \|(B^{+,main}_{N;1;k+1} - B^+_{1;k+1})U^{(k+1)}(t)\gamma_{0,N}^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H^1} \\
&\leq \left(\sup_k k(\xi''/\xi')^k\right)\left(\sum_{k=1}^K (\xi')^k \|(B^{+,main}_{N;1;k+1} - B^+_{1;k+1})U^{(k+1)}(t)\gamma_{0,N}^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H^1(\Omega_{k,N})}\right. \\
&\qquad \left. + \sum_{k=1}^K (\xi')^k \|(B^{+,main}_{N;1;k+1} - B^+_{1;k+1})U^{(k+1)}(t)\gamma_{0,N}^{(k+1)}\|_{L^2_{t\in\mathbb{R}}H^1(\Omega^c_{k,N})}\right) \\
&\leq \left(\sup_k k(\xi''/\xi')^k\right)\left(C_V(1+\|\widehat{V}\|_\infty)\sum_{k=1}^K (\xi')^k N^{2(\delta-\beta)}\|\gamma_{0,N}^{k+1}\|_{H^1} + \sum_{k=1}^K (\xi')^k a_{k,N}\right),
\end{aligned} \tag{2.4.24}
$$

where (2.4.21) and (2.4.23) were used to pass to the last line (2.4.24).

Now, for $(k, N) \in \mathbb{N}\times\mathbb{N}$, we define

$$
\begin{aligned}
&\widetilde{a_{k,N}}^2 \\
&:= \int\int\int J(\tau, \underline{u}_k, \underline{u}'_k)\int_{|u_{k+1}+u'_{k+1}|\geq N^\delta}\int \delta(...)\langle u_1 - u_{k+1} - u'_{k+1}\rangle^2\langle u_{k+1}\rangle^2\langle u'_{k+1}\rangle^2 \\
&\qquad \prod_{j=2}^k \langle u_j\rangle^2 \prod_{j'=1}^k \langle u'_j\rangle^2 (1+\|\widehat{V}\|_\infty)^2 \\
&\qquad\qquad \left|\widehat{\gamma}_0^{(k+1)}(u_1 - u_{k+1} - u'_{k+1}, u_2, ..., u_k, u_{k+1}; \underline{u}'_{k+1})\right|^2 \\
&\qquad\qquad\qquad du'_{k+1}\, du_{k+1}\, d\underline{u}_k\, d\underline{u}'_k\, d\tau,
\end{aligned} \tag{2.4.25}
$$

and observe that $\widetilde{a_{k,N}} = a_{k,N}$ (as defined in (2.4.22)), for $k \leq N$, because $\gamma_{0,N}^{(k)} = \gamma_0^{(k)}$ for $k \leq N$. Thus we have that

$$\begin{aligned}(2.4.24) \leq \left(\sup_k k(\xi''/\xi')^k\right) \\ \left(C_V(1+\|\widehat{V}\|_\infty)\sum_{k=1}^{\infty}(\xi')^k N^{2(\delta-\beta)}\|\gamma_0^{(k+1)}\|_{H^1} + \sum_{k=1}^{\infty}(\xi')^k a_{k,N}\right) \\ \leq \left(\sup_k k(\xi''/\xi')^k\right)\left(C_V(1+\|\widehat{V}\|_\infty)N^{2(\delta-\beta)}\|\Gamma_0^{(k+1)}\|_{\mathcal{H}^1_{\xi'}} + \sum_{k=1}^{\infty}(\xi')^k a_{k,N}\right).\end{aligned} \tag{2.4.26}$$

It follows from the definition (2.4.25) of $a_{k,N}^2$ that $\sum_{k=1}^{\infty}(\xi')^k a_{k,N} \leq C\|\Gamma_0\|_{\mathcal{H}^1_{\xi'}}$ and that, for fixed $k$, $a_{k,N} \searrow 0$ monotonically as $N \to \infty$. This is because $a_{k,N}^2$ is an integral where the integrand is independent of $N$ and the region of integration shrinks as $N$ grows. Thus, by the monotone convergence theorem, $\sum_{k=1}^{\infty}(\xi')^k a_{k,N} \searrow 0$ as $N \to \infty$. Therefore (2.4.26) $\to 0$ as $N \to \infty$. □

**Lemma 2.4.3.** *Under the same assumptions as in Proposition 2.4.1,*

$$\lim_{N\to\infty}\left\|\int_0^t (B_N - B)U(t-s)B\Gamma^K(s)\,ds\right\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}} = 0.$$

*Proof.* We have that

$$\begin{aligned}&\left\|\int_0^t (B_N - B)U(t-s)B\Gamma^K(s)\,ds\right\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}} \\ &\leq \int_0^T \left\|(B_N - B)U(t-s)B\Gamma^K(s)\right\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}}\,ds.\end{aligned}$$

By the same arguments as in the proof of Lemma 2.4.2 above, the integral above goes to zero as $N \to \infty$ provided that $\|U(t-s)B\Gamma^K(s)\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}}$ is uniformly bounded in $N$. See [20] for a proof of the boundedness of $\|U(t-s)B\Gamma^K(s)\|_{L^2_{t\in I}\mathcal{H}^1_{\xi''}}$. $\square$

### 2.4.3 Control of $\Gamma^{\Phi_N}$ and $\Gamma^K_N$ as $N \to \infty$

We begin by stating an energy estimate used by Erdös, Schlein, and Yau in [43]. Recall that $R^{(k,\alpha)} := \prod_{j=1}^k (1-\Delta_{x_j})^{\alpha/2}$.

**Proposition 2.4.2.** *Suppose that $\psi$ is symmetric with respect to permutations of its $N$ variables. Fix $k \in \mathbb{N}$ and $0 < C < 1$. Then there is $N_0 = N_0(k, C)$ such that*

$$\langle \psi, (H_N + N)^k \psi \rangle \geq C^k N^k \langle \psi, R^{(k,2)} \psi \rangle$$

*for all $N > N_0$.*

**Proposition 2.4.3.** *Suppose that $b_1 > 0$, $b_1 \log(N) \geq K(N) \geq \frac{1}{2} b_1 \log(N)$, and that $\xi > 0$ satisfies*

$$\xi < \eta \min\left\{ \frac{1}{C}\, e^{-8\beta/b_1}, e^{-24\beta/b_1} \right\}, \tag{2.4.27}$$

*where*

$$\operatorname{Tr} S^{(k,1)} \gamma_N^{(K)}(0) < C^K. \tag{2.4.28}$$

*Then*

$$\lim_{N\to\infty} \|B_N \Gamma_N^{K(N)} - P_{\leq K(N)-1} B_N \Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_{\xi}} = 0.$$

*Proof.* From Lemma 6.1 in [23], we have that

$$\|B_N\Gamma_N^K - P_{\leq K-1}B_N\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \leq C(T,\xi)(\eta^{-1}\xi)^K K\|(B_N\Gamma^{\Phi_N})^{(K)}\|_{L^2_{t\in I}H^1} \tag{2.4.29}$$

holds for a finite constant $C(T,\xi)$ independent of $K$, $N$.

It follows immediately from the definition of $V_N$ that

$$\|\widehat{\nabla V_N}\|_{L^1} \leq CN^{4\beta}.$$

Thus we have that

$$\begin{aligned}
&\|(B_N^+\Gamma^{\Phi_N})^{(K)}\|^2_{L^2_{t\in I}H^1} \\
&= \int_I dt \int d\underline{\mathrm{x}}_K\, d\underline{\mathrm{x}}'_K \Bigg| \sum_{\ell=1}^K \int \left[\prod_{j=1}^k \langle\nabla_{x_j}\rangle\langle\nabla_{x'_j}\rangle\right] V_N(x_\ell - x_{K+1}) \\
&\qquad\qquad \Phi_N(t,\underline{\mathrm{x}}_N)\overline{\Phi_N(t,\underline{\mathrm{x}}'_K, x_{K+1},\dots,x_N)}\, dx_{K+1}\dots dx_N\Bigg|^2 \\
&\leq CT(\|V_N\|^2_{L^\infty} + \|\widehat{\nabla V_N}\|^2_{L^1})K^2 \sup_{t\in I}\left(\|R^{(k,1)}\Phi_N\|_{L^2}\|R^{(k,1)}\Phi_N\|_{L^2}\right)^2 \\
&= CTN^{8\beta}K^2 \sup_{t\in I}\left(\mathrm{Tr}(S^{(K,1)}\gamma_N^{(K)}(t))\right)^2.
\end{aligned} \tag{2.4.30}$$

Since $\langle\Phi_N(0), H_N^K, \Phi_N(0)\rangle < C^k N^K$, it follows from Proposition 2.4.2, that

$$
\begin{aligned}
\operatorname{Tr} S^{(K,1)}\gamma_N^{(K)}(t) &= \langle \Phi_N(t), R^{(K,2)}\Phi_N(t)\rangle \\
&\leq \frac{1}{N^k C^k}\langle \Phi_N(t), (H_N+N)^k \Phi_N(t)\rangle \\
&= \frac{1}{N^k C^k}\langle \Phi_N(0), (H_N+N)^k \Phi_N(0)\rangle \\
&\leq \frac{1}{N^k C^k}(2^k\langle \Phi_N(0), H_N^k \Phi_N(0)\rangle + 2^k N^k \langle \Phi_N(0), \Phi_N(0)\rangle) \\
&\leq C^k. \qquad (2.4.31)
\end{aligned}
$$

Combining (2.4.29), (2.4.30), and (2.4.31) yields

$$
\begin{aligned}
&\|B_N\Gamma_N^K - P_{\leq K-1}B_N\Gamma^{\Phi_N}\|_{L^2_{t\in I}H^1_\xi} \\
&\leq C(T,\xi)(\eta^{-1}\xi)^K K\|(B_N\Gamma^{\Phi_N})^{(K)}\|_{L^2_{t\in I}H^1} && \text{by (2.4.29)} \\
&\leq C(T,\xi)(\eta^{-1}\xi)^K KCT^{1/2}N^{4\beta}K\sup_{t\in I}\operatorname{Tr}(S^{(K,1)}\gamma_N^{(K)}(t)) && \text{by (2.4.30)} \\
&\leq C(T,\xi)(\eta^{-1}\xi)^K KCT^{1/2}N^{4\beta}C^K && \text{by (2.4.31)} \\
&\leq \widetilde{C}(T,\xi)(\eta^{-1}\xi)^K KN^{4\beta}C^K \\
&\to 0 \text{ as } N\to\infty
\end{aligned}
$$

because $K(N) \geq \frac{1}{2}b_1\log(N)$ and $\xi$ satisfies (2.4.27). □

### 2.4.4 Proof of Theorem 2.3.1

We are now ready to conclude the proof of Theorem 2.3.1. To this end, we recall again the solution $\Gamma^K$ of the GP hierarchy with truncated initial data, $\Gamma^K(t=0) = P_{\leq K}\Gamma_0 \in \mathcal{H}^1_\xi$. In [20], the authors proved the existence of a solution $\Gamma^K$ that satisfies the $K$-truncated GP-hierarchy in integral form,

$$\Gamma^K(t) = U(t)\Gamma^K(0) + i\int_0^t U(t-s)\, B\Gamma^K(s)\, ds \tag{2.4.32}$$

where $(\Gamma^K)^{(k)}(t) = 0$ for all $k > K$. Moreover, it is shown in [20] that this solution satisfies $B\Gamma^K \in L^2_{t\in I}\mathcal{H}^1_\xi$, where $I := [0,T]$.

Additionally, the following convergence was proved in [20]:

(a) The limit

$$\Gamma := \lim_{K\to\infty} \Gamma^K \tag{2.4.33}$$

exists in $L^\infty_t \mathcal{H}^1_\xi$.

(b) The limit

$$\Theta := \lim_{K\to\infty} B\Gamma^K \tag{2.4.34}$$

exists in $L^2_t \mathcal{H}^1_\xi$, and in particular,

$$\Theta = B\Gamma\,. \tag{2.4.35}$$

(c) The limit $\Gamma$ in equation (2.4.33) satisfies the full GP hierarchy with initial data $\Gamma_0$.

Clearly, we have that

$$\begin{aligned}
&\|B\Gamma - B_N P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \\
&\leq \|B\Gamma - B\Gamma^{K(N)}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} && (2.4.36)\\
&\quad + \|B\Gamma^{K(N)} - B_N\Gamma_N^{K(N)}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} && (2.4.37)\\
&\quad + \|B\Gamma_N^{K(N)} - B_N P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi}. && (2.4.38)
\end{aligned}$$

In the limit $N \to \infty$, we have that (2.4.36) $\to 0$ from (2.4.34) and (2.4.35). By Proposition 2.4.1, (2.4.37) $\to 0$. (2.4.38) $\to 0$ follows from Proposition 2.4.3. This is because $\Gamma_0 \in \mathfrak{H}^1_{\xi'}$ and hence (2.4.28) holds. Therefore,

$$\lim_{N\to\infty} \|B\Gamma - B_N\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} = 0.$$

Moreover, we have that

$$\begin{aligned}
&\|P_{\leq K(N)}\Gamma^{\Phi_N} - \Gamma\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} \\
&\leq \|P_{\leq K(N)}\Gamma^{\Phi_N} - \Gamma_N^{K(N)}\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} && (2.4.39)\\
&\quad + \|\Gamma^{K(N)} - \Gamma\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} && (2.4.40)\\
&\quad + \|\Gamma_N^{K(N)} - \Gamma^{K(N)}|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} && (2.4.41)
\end{aligned}$$

By the Duhamel formula, and applying the Cauchy-Schwarz inequality in time, we have

$$\begin{aligned}
(2.4.39) &= \|\int_0^t U(t-s)B_N(P_{\leq K(N)}\Gamma^{\Phi_N} - \Gamma_N^{K(N)})(s)\,ds\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} \\
&\leq T^{1/2}\|B_N\Gamma_N^{K(N)} - B_N P_{\leq K(N)}\Gamma^{\Phi_N}\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \\
&\to 0 \text{ as } N \to \infty \text{ by Proposition 2.4.3.}
\end{aligned}$$

(2.4.40) $\to 0$ as $N \to \infty$ by (2.4.33). Finally, (2.4.41) $\to 0$ as $N \to \infty$ follows from proposition 2.4.1. Thus

$$\lim_{N\to\infty} \|P_{\leq K(N)}\Gamma^{\Phi_N} - \Gamma\|_{L^\infty_{t\in I}\mathcal{H}^1_\xi} = 0.$$

This completes the proof of Theorem 2.3.1. □

## 2.5 Positive Semidefiniteness and Global Well-Posedness

We now prove Theorems 2.3.2 and 2.3.3. We prove positive semidefiniteness of solutions to the GP hieararchy, and global well posedness of the GP hierarchy, respectively. For the convenience of the reader, we restate these theorems as Theorem 2.5.2 and Theorem 2.5.3 below.

To prove positive semidefiniteness of solutions to the GP hierarchy, we will use the quantum de Finetti theorem below, as stated in [91].

**Theorem 2.5.1.** *(Quantum de Finetti theorem) Let $\mathcal{H}$ be any separable Hilbert space and let $\mathcal{H}^k = \otimes^k_{sym}\mathcal{H}$ denote the corresponding bosonic k-particle space. Let $\Gamma$ denote a collection of admissible bosonic density matrices on $\mathcal{H}$, i.e.,*

$$\Gamma = (\gamma^{(1)}, \gamma^{(2)}, \dots)$$

*with $\gamma^{(k)}$ a non-negative trace class operator on $\mathcal{H}^k$, and $\gamma^{(k)} = \mathrm{Tr}_{k+1}\gamma^{(k+1)}$, where $\mathrm{Tr}_{k+1}$ denotes the partial trace over the $(k+1)$-th factor. Then, there exists a unique Borel probability measure $\mu$, supported on the unit sphere $S \subset \mathcal{H}$,* *and invariant under multiplication of $\phi \in \mathcal{H}$ by complex numbers of modulus one, such that*

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad , \quad \textit{for all } k \in \mathbb{N}.$$

We will also use the following lemma from [17].

**Lemma 2.5.1.** *Let $\mu$ be a Borel probability measure in $L^2(\mathbb{R}^3)$, and assume that*

$$\int d\mu(\phi) \|\phi\|_{H^1}^{2k} \leq M^{2k}$$

*holds for some finite constant $M > 0$, and all $k \in \mathbb{N}$. Then,*

$$\mu\big(\big\{\phi \in L^2(\mathbb{R}^3)\big|\|\phi\|_{H^1} > M\big\}\big) = 0.$$

*Proof.* From Chebyshev's inequality, we have that

$$\mu\big(\big\{\phi \in L^2(\mathbb{R}^3)\big|\|\phi\|_{H^1} > \lambda\big\}\big) \leq \frac{1}{\lambda^{2k}} \int d\mu(\phi) \|\phi\|_{H^1}^{2k} \leq \frac{M^{2k}}{\lambda^{2k}}$$

for any $k > 0$. For $\lambda > M$, the right hand side tends to zero when $k \to \infty$. $\square$

We recall that, for $I \subseteq \mathbb{R}$,

$$\mathcal{W}_\xi^\alpha(I) = \{\Gamma \in C(I, \mathcal{H}_\xi^\alpha) \,|\, B^+\Gamma, B^-\Gamma \in L^2_{loc}(I, \mathcal{H}_\xi^\alpha)\}.$$

We are now ready to prove positive semidefiniteness of solutions to the GP hierarchy.

**Theorem 2.5.2.** *Suppose that* $\Gamma_0 \in \mathfrak{H}^1_{\xi'}$ *is positive semidefinite, admissible, and satisfies* $\operatorname{Tr}\gamma_0^{(1)} = 1$. *Then, for* $0 < \xi' < 1$ *and* $\xi > 0$ *satisfying* (2.3.3), *and for* $0 < T < \min\{T_0(\xi), T_1(\xi)\}$ *(see* (2.3.5) *and* (2.5.10)*), there is a unique solution* $\Gamma \in \mathcal{W}^1_\xi([0,T])$ *to the cubic defocusing GP hierarchy* (2.2.8) *in* $\mathbb{R}^3$ *with initial data* $\Gamma_0$. *Moreover,* $\Gamma(t)$ *is positive semidefinite for* $t \in [0,T]$.

*Proof.* By [18] and Proposition 2.D.1, there exists a unique solution $\Gamma$ to the GP hierarchy in $\mathcal{W}^1_\xi([0,T])$ with initial data $\Gamma_0$.

By the quantum de Finetti theorem (Theorem 2.5.1) and Lemma 2.5.1, there exists a positive semidefinite Borel probability measure $\mu$ on the unit sphere in $L^2(\mathbb{R}^3)$ such that

$$\gamma_0^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \tag{2.5.1}$$

and $\|\phi\|^2_{H^1} \leq (\xi')^{-1}\|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}}$ $\mu$-almost everywhere. Let $S_t$ be the flow map of the cubic defocusing NLS. Since the NLS is well-posed in $H^1$,

$$\tilde{\gamma}^{(k)}(t) := \int d\mu(\phi)(|S_t\phi\rangle\langle S_t\phi|)^{\otimes k} \tag{2.5.2}$$

is well-defined, positive semidefinite, and $\tilde{\Gamma} := \{\tilde{\gamma}^{(k)}\}_{k=1}^\infty$ satisfies the cubic defocusing GP hierarchy.

Moreover, we claim that $\tilde{\Gamma} \in \mathcal{W}^1_\xi([0,T])$. To prove this fact, let $\langle \mathcal{K}^{(m)} \rangle_{\Gamma(t)}$, $m \in \mathbb{N}$, denote the higher order energy functional defined in equation (2.2.27), and we write

$$E[\phi] := \tfrac{1}{2}\|\phi\|^2_{H^1}\|\phi\|^2_{L^2} + \tfrac{1}{4}\|\phi\|^4_{L^4} = E[S_t\phi] \tag{2.5.3}$$

for the conserved energy of the solution of the NLS. Then, it can be easily checked that

$$\langle \mathcal{K}^{(m)} \rangle_{\tilde{\Gamma}(t)} = \int d\mu(\phi) \Big( \frac{1}{2} + E[S_t \phi] \Big)^m . \tag{2.5.4}$$

We have that the sequence of higher energy functionals $\langle \mathcal{K}^{(m)} \rangle_{\Gamma(t)}$, for $m \in \mathbb{N}$, satisfies

$$\begin{aligned} \|\Gamma(t)\|_{\mathfrak{H}^1_\xi} &\le \sum_{m \in \mathbb{N}} (2\xi)^m \langle \mathcal{K}^{(m)} \rangle_{\Gamma(t)} \\ &= \sum_{m \in \mathbb{N}} (2\xi)^m \langle \mathcal{K}^{(m)} \rangle_{\Gamma(0)} \\ &\le \|\Gamma(0)\|_{\mathfrak{H}^1_{\xi'}}, \end{aligned}$$

by Theorem 6.2 in [21].

As a consequence, we find that

$$\begin{aligned} \|\tilde{\Gamma}(t)\|_{\mathcal{H}^1_\xi} &\le \|\tilde{\Gamma}(t)\|_{\mathfrak{H}^1_\xi} \\ &\le \sum_{m \in \mathbb{N}} (2\xi)^m \langle \mathcal{K}^{(m)} \rangle_{\tilde{\Gamma}(t)} \\ &= \sum_{m \in \mathbb{N}} (2\xi)^m \langle \mathcal{K}^{(m)} \rangle_{\tilde{\Gamma}(0)} \\ &\le \|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}} \\ &< \infty \end{aligned}$$

Moreover,

$$
\begin{aligned}
&\|B\tilde{\Gamma}\|_{L^2_{t\in[0,T]}H^1_\xi} \\
&\leq \sum_{k=1}^{\infty}(\xi)^k \int d\mu(\phi) \|\langle\nabla\rangle(|S_t\phi|^2 S_t\phi)\|_{L^2_{t\in[0,T]}L^2(\mathbb{R}^3)} \|\langle\nabla\rangle S_t\phi\|^{2k-1}_{L^\infty_{t\in[0,T]}L^2(\mathbb{R}^3)} \quad (2.5.5)\\
&\leq \sum_{k=1}^{\infty}(\xi)^k \int d\mu(\phi) \||S_t\phi|^2\|_{L^\infty_t L^3(\mathbb{R}^3)} \|\langle\nabla\rangle S_t\phi\|_{L^2_{t\in[0,T]}L^6(\mathbb{R}^3)} \|\langle\nabla\rangle S_t\phi\|^{2k-1}_{L^\infty_{t\in[0,T]}L^2(\mathbb{R}^3)}\\
&\leq \sum_{k=1}^{\infty}(\xi)^k \int d\mu(\phi) \|S_t\phi\|^2_{L^\infty_{t\in[0,T]}L^6(\mathbb{R}^3)} \|\langle\nabla\rangle S_t\phi\|_{L^2_{t\in[0,T]}L^6(\mathbb{R}^3)} \|\langle\nabla\rangle S_t\phi\|^{2k-1}_{L^\infty_{t\in[0,T]}L^2(\mathbb{R}^3)}.
\end{aligned}
\tag{2.5.6}
$$

Here, we use the bound

$$\|\langle\nabla\rangle S_t\phi\|_{L^2_{t\in[0,T]}L^6(\mathbb{R}^3)} \leq C(T)\|\langle\nabla\rangle\phi\|_{L^2} \leq C(T)\sqrt{1+2E[\phi]}\,, \tag{2.5.7}$$

with $T > 0$ as in (2.5.10), below; see for instance [82] or [12] for details. Moreover,

$$\|\langle\nabla\rangle S_t\phi\|_{L^\infty_{t\in[0,T]}L^2(\mathbb{R}^3)} \leq \sup_{t\in[0,T]} \sqrt{1+2E[S_t\phi]} = \sqrt{1+2E[\phi]} \tag{2.5.8}$$

We then obtain that

$$
\begin{aligned}
(2.5.6) &\leq C\sum_{k=1}^{\infty}(2\xi)^k \int d\mu(\phi)\Big(\frac{1}{2}+E[\phi]\Big)^{k+1}\\
&= C\xi^{-1}\sum_{k=2}^{\infty}(2\xi)^k \langle\mathcal{K}^{(k)}\rangle_{\tilde{\Gamma}(0)}\\
&\leq C\xi^{-1}\|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}}\\
&< \infty.
\end{aligned}
\tag{2.5.9}
$$

Finally, we pick $T_1(\xi) > 0$ sufficiently small that (2.5.7) above holds for

$$0 < T < T_1(\xi)\,, \tag{2.5.10}$$

noting that the constant $C(T)$ in (2.5.7) depends on $\|\phi\|_{H^1} < (\xi')^{-1/2}\|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}}$ and thus on $\xi$, where $\xi$ and $\xi'$ are related as in (2.3.3).

Thus, we have shown that $\tilde{\Gamma} \in \mathcal{W}^1_\xi([0,T])$. By uniqueness of solutions to the GP hierarchy in $\mathcal{W}^1_\xi([0,T])$, we conclude that $\Gamma = \tilde{\Gamma}$. Hence, $\Gamma(t)$ is positive semidefinite for $t \in [0,T]$. □

Now that we have positive semidefinitenss of solutions to the GP hierarchy, we are able to to global well posedness of solutions to the GP hierarchy, using an induction argument as in [21] below.

**Theorem 2.5.3.** *Suppose that* $\Gamma_0 = \{\gamma_0^{(k)}\}_{k=1}^\infty \in \mathfrak{H}^1_{\xi'}$ *is positive semidefinite, is admissible, and satisfies* $\operatorname{Tr}\gamma_0^{(1)} = 1$*. Then, for* $0 < \xi' < 1$ *and* $\xi_1$ *satisfying* (2.3.3)*, there is a unique global solution* $\Gamma \in \mathcal{W}^1_{\xi_1}(\mathbb{R})$ *to the cubic defocusing GP hierarchy* (2.2.8) *in* $\mathbb{R}^3$ *with initial data* $\Gamma_0$*. Moreover,* $\Gamma(t)$ *is positive semidefinite and satisfies*

$$\|\Gamma(t)\|_{\mathcal{H}^1_{\xi_1}} \leq \|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}}$$

*for all* $t \in \mathbb{R}$*.*

*Proof.* Let $I_j$ be the time interval $[jT,(j+1)T]$, where $0 < T < \min\{T_0(\xi_1), T_1(\xi_1)\}$ (see (2.3.5) and (2.5.10)) and $\xi, \xi_1$ satisfy (2.3.3). By [18] and Proposition 2.D.1, we have that there is a unique solution $\Gamma$ to the GP hierarchy in $\mathcal{W}^1_\xi(I_0)$. Moreover, by Theorem 2.5.2, $\Gamma$ is positive semidefinite on $I_0$. It follows as in the proof of Theorem 7.2 in [21] that the higher order energy functionals

$\langle K^{(m)}\rangle_{\Gamma(t)}$, which are defined in equation (2.2.27), are conserved on $I_0$. Thus, as in inequality (7.18) in [21], we have that on $I_0$,

$$\|\Gamma(t)\|_{\mathcal{H}^1_\xi} \leq \|\Gamma(t)\|_{\mathfrak{H}^1_\xi} \tag{2.5.11}$$
$$\leq \sum_{m\in\mathbb{N}} (2\xi)^m \langle K^{(m)}\rangle_{\Gamma(t)} \tag{2.5.12}$$
$$\leq \sum_{m\in\mathbb{N}} (2\xi)^m \langle K^{(m)}\rangle_{\Gamma_0} \tag{2.5.13}$$
$$\leq \|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}}. \tag{2.5.14}$$

Note that positive semidefiniteness of $\Gamma$ is needed to pass from (2.5.11) to (2.5.12) because the definition of $\|\Gamma(t)\|_{\mathfrak{H}^1_\xi}$ involves taking absolute values, but the definition of $\langle K^{(m)}\rangle_{\Gamma(t)}$ does not.

Therefore $\Gamma(T) \in \mathfrak{H}^1_\xi$, and so by [18] and Proposition 2.D.1, there is a unique solution $\Gamma \in \mathcal{W}^1_{\xi_1}(I_1)$ of the GP hierarchy with initial data $\Gamma(T)$. By another application of Theorem 2.5.2 and energy conservation (2.5.11) $\sim$ (2.5.14), $\Gamma$ is positive semidefinite on $I_1$ and $\Gamma(2T) \in \mathfrak{H}^1_\xi$. Thus, we can repeat the argument and find that we have a unique solution $\Gamma \in \mathcal{W}^1_{\xi_1}(\mathbb{R})$. Moreover,

$$\|\Gamma(t)\|_{\mathcal{H}^1_{\xi_1}} \leq \|\Gamma(t)\|_{\mathcal{H}^1_\xi} \leq \|\Gamma_0\|_{\mathfrak{H}^1_{\xi'}}$$

for all $t \in \mathbb{R}$. $\square$

## 2.A Global derivation of the GP hierarchy

In this section, we illustrate how one can show that Theorem 2.3.1 holds for arbitrarily large values of $T$, provided that $\Gamma_0 \in \mathfrak{H}^1_{\xi'}$, $\Gamma_0$ is admissible, and that $\xi$ is sufficiently small. To this end, we make use of both Theorem 2.3.1 and Theorem 2.3.3.

We begin by observing that, in the statement of Theorem 2.3.1, instead of assuming that

$$\Gamma_0 := \lim_{N\to\infty} \Gamma^{\Phi_N}(0)$$

holds in $\mathcal{H}^1_{\xi'}$, we may assume that

$$\Gamma_0 := \lim_{N\to\infty} P_{\leq K(N)}\Gamma^{\Phi_N}(0)$$

holds. Indeed, the proof of Theorem 2.3.1 is unaffected by this replacement.

We also note that initial condition $\langle \Phi_N(0), H_N^k \Phi_N(0)\rangle$ implies that $\Gamma^{\Phi_N}(t) \in \mathcal{H}^1_{\xi'}$ for any $t \in \mathbb{R}$, provided that $\xi' < (4(C+1))^{-1}$. This follows from (2.4.31). In fact, given $\xi'$, we have a bound $\widetilde{C}$, uniform in $N$ and $t$, such that

$$\|\Gamma^{\Phi_N}(t)^{(k)}\|_{H^1} < \widetilde{C}^k. \tag{2.A.1}$$

We also note that, by Theorem 2.3.3, the solution to the GP hierarchy $\Gamma(t) \in \mathcal{H}^1_{\xi_1}$ for all $t \in \mathbb{R}$, provided that $\xi_1$ sufficiently small.

Thus, under the assumptions of Theorem 2.3.1, at time $T$, we have

$$\begin{cases} \Gamma^{\Phi_N}(T) \in \mathcal{H}^1_{\xi_1} \text{ and} \\ \Gamma(T) = \lim_{N\to\infty} P_{\leq K(N)}\Gamma^{\Phi_N}(T) \text{ in } \mathcal{H}^1_{\xi_1}, \end{cases} \tag{2.A.2}$$

provided that $\xi_1$ is sufficiently small (note that we also require $\xi_1 < (4(C+1))^{-1}$). By another application of Theorem 2.3.1, we have that at time $2T$,

$$\begin{aligned} &\Gamma^{\Phi_N}(2T) \in \mathcal{H}^1_{\xi_1} \text{ and} \\ &\Gamma(2T) = \lim_{N\to\infty} P_{\leq K(N)}\Gamma^{\Phi_N}(2T) \text{ in } \mathcal{H}^1_{\xi_2}, \end{aligned} \tag{2.A.3}$$

provided that $\xi_2 < \xi_1$ is sufficiently small. (2.A.3) says that

$$\sum_{k=1}^{\infty} \xi_2^k \|\Gamma(2T)^{(k)} - P_{\leq K(N)}\Gamma^{\Phi_N}(2T)^{(k)}\|_{H^1} \to 0 \text{ as } N \to \infty.$$

However, by (2.A.1) and the dominated convergence theorem for sequences, we actually have the stronger statement

$$\sum_{k=1}^{\infty} \xi_1^k \|\Gamma(2T)^{(k)} - P_{\leq K(N)}\Gamma^{\Phi_N}(2T)^{(k)}\|_{H^1} \to 0 \text{ as } N \to \infty,$$

where we have $\xi_1$ instead of $\xi_2$. Thus, at time $2T$ we actually have

$$\begin{cases} \Gamma^{\Phi_N}(2T) \in \mathcal{H}^1_{\xi_1} \text{ and} \\ \Gamma(2T) = \lim_{N\to\infty} P_{\leq K(N)}\Gamma^{\Phi_N}(2T) \text{ in } \mathcal{H}^1_{\xi_1}. \end{cases} \tag{2.A.4}$$

Note that (2.A.4) is the same as (2.A.2), but with $T$ replaced by $2T$. Thus, we may iterate the argument again, and conclude that Theorem 2.3.1 holds for $T$ arbitrarily large, provided that $\Gamma_0 \in \mathfrak{H}^1_{\xi'}$, $\Gamma_0$ is admissible, and that $\xi$ is sufficiently small.

## 2.B Strong vs weak-* convergence

**Proposition 2.B.1.** *Suppose that $\{\gamma_N^{(k)}\}_{N=1}^{\infty}$ is a sequence of operators on $L^2(\mathbb{R}^k)$ such that $\gamma_N^{(k)} \to \gamma_\infty^{(k)}$ strongly in Hilbert Schmidt norm. Suppose also* *that $\gamma_N^{(k)}$ and $\gamma_\infty^{(k)}$ are trace class operators such* $\mathrm{Tr}|\gamma_N^{(k)}| \leq 1$ *for all $N$. Then $\gamma_N^{(k)} \to \gamma_\infty^{(k)}$ in the weak-* topology induced by the trace norm.*

*Proof.* We follow the usual construction of a metric for the weak-* topology induced by the trace norm, as presented in [43], for example. Let $\mathcal{K}_k$ be the space of compact operators on $L^2(\mathbb{R}^k)$ equipped with the operator norm topology. Let $\mathcal{L}_k^1$ be the space of trace class operators on $L^2(\mathbb{R}^{2k})$. By [103], we have that $\mathcal{L}_k^1 = \mathcal{K}_k^*$. Since $\mathcal{K}_k$ is separable, there exists a sequence $\{J_i^{(k)}\}_{i=1}^\infty \in \mathcal{K}_k$ of Hilbert Schmidt operators, dense in the unit ball of $\mathcal{K}_k$. Note that Hilbert Schmidt operators are dense in the space of compact operators, because, by [103], every compact operator on a Hilbert space is of the form $\lim_{N\to\infty} \sum_{n=1}^N \lambda_n \langle \psi_n, \cdot \rangle \phi_n$, with $\{\psi_n\}_{n=1}^\infty$ and $\{\phi_n\}_{n=1}^\infty$ orthonormal sets, and $\{\lambda_n\}_{n=1}^\infty$ positive real numbers such that $\lambda_n \to 0$. On $\mathcal{L}_k^1$, we define the metric $\eta_k$ by

$$\eta_k(\gamma^{(k)}, \widetilde{\gamma}^{(k)}) := \sum_{i=1}^\infty 2^{-i} \left| \mathrm{Tr} J_i^{(k)} \big( \gamma^{(k)} - \widetilde{\gamma}^{(k)} \big) \right|.$$

By [108], the toplogy induced by the metric $\eta_k$ is equivalent to the weak-* toplology on $\mathcal{L}_k^1$.

Now, since $\{J_i^{(k)}\}_{i=1}^\infty \in \mathcal{K}_k$ are Hilbert Schmidt, we have

$$\begin{aligned} \mathrm{Tr}|J_i^{(k)}(\gamma_N^{(k)} - \gamma_\infty^{(k)})| &\leq (\mathrm{Tr}(|J_i^{(k)}|^2))^{1/2} (\mathrm{Tr}(|\gamma_N^{(k)} - \gamma_\infty^{(k)}|^2))^{1/2} \\ &\to 0 \text{ as } N \to \infty. \end{aligned}$$

Moreover,

$$\begin{aligned}\operatorname{Tr}|J_i^{(k)}(\gamma_N^{(k)} - \gamma_\infty^{(k)})| &\le \|J_i^{(k)}\|_{L^2\to L^2} \operatorname{Tr}|\gamma_N^{(k)} - \gamma_\infty^{(k)}| \\ &\le 1 + \operatorname{Tr}|\gamma_\infty^{(k)}|.\end{aligned}$$

Thus, by the dominated convergence theorem for sequences, $\eta_k(\gamma_N^{(k)}, \gamma_\infty^{(k)}) \to 0$ as $N \to \infty$, and so $\gamma_N^{(k)} \to \gamma_\infty^{(k)}$ in the weak-* topology on $\mathcal{L}_k^1$. □

## 2.C Conservation of admissibility for the GP hierarchy

In this part of the appendix, we prove that the GP hierarchy conserves admisibility. This result has been used in many papers, but we have not found an explicit proof. For the convenience of the reader, we present it here.

**Proposition 2.C.1.** *Suppose that* $\Gamma_0 = \{\gamma_0^{(k)}\}_{k=1}^\infty \in \mathcal{H}_{\xi'}^1$ *is admissible and satisfies* $\operatorname{Tr}\gamma_0^{(\mathrm{k})} = 1$ *for all* $k \in \mathbb{N}$*. Then, for* $0 < \xi' < 1$ *and* $\xi$ *satisfying* (2.3.3)*, the unique solution* $\Gamma \in \mathcal{W}_\xi^1(I)$ *to the GP hierarchy obtained in [18] is admissible for all* $t \in I$*, provided that* $A := \{A^{(k)}\}_{k=1}^\infty \in \mathcal{W}_\xi^1(I)$*, where*

$$A^{(k)}(t, \underline{x}_k; \underline{x}_k') := -\gamma^{(k)}(t, \underline{x}_k; \underline{x}_k') + \int \gamma^{(k+1)}(t, \underline{x}_k, x_{k+1}; \underline{x}_k', x_{k+1})\, dx_{k+1}. \tag{2.C.1}$$

*Proof.* We first note that for $f \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, we have

$$\int ((\Delta_{x_1} - \Delta_{x_2})f)(x, x)\, dx = 0. \tag{2.C.2}$$

Indeed, this follows from

$$
\begin{aligned}
&\int ((\Delta_{x_1} - \Delta_{x_2})f)(x,x)\,dx \\
&= \int\int \delta(x_1 - x_2)(\Delta_{x_1} - \Delta_{x_2})f(x_1,x_2)\,dx_1\,dx_2 \\
&= \int\int\int\int \delta(x_1 - x_2)e^{iu_1x_1 + iu_2x_2}((u_2)^2\hat{f}(u_1,u_2) - u_1^2\hat{f}(u_1,u_2))\,du_1\,du_2\,dx_1\,dx_2 \\
&= \int\int\int e^{ix_1(u_1+u_2)}((u_2)^2\hat{f}(u_1,u_2) - u_1^2\hat{f}(u_1,u_2))\,du_1\,du_2\,dx_1 \\
&= \int\int \delta(u_1 + u_2)((u_2)^2\hat{f}(u_1,u_2) - u_1^2\hat{f}(u_1,u_2))\,du_1\,du_2 \\
&= \int (u_1^2 - u_1^2)\hat{f}(u_1,-u_1)\,du_1 \\
&= 0,
\end{aligned}
$$

which implies (2.C.2).

Next, we note that the definition of admissibility implies that $\gamma^{(k)}$ is admissible at time $t$ if and only if

$$A^{(k)}(t, \underline{x}_k, \underline{x}'_k) = 0.$$

Since Γ satisfies the GP hierarchy, we have that

$$
\begin{aligned}
& i\partial_t A^{(1)}(x_1;x_1') \\
&\quad = (\Delta_{x_1} - \Delta_{x_1'})\gamma^{(1)}(x_1;x_1') && (2.C.3)\\
&\qquad - \kappa_0 \Big[\gamma^{(2)}(x_1,x_1;x_1',x_1) - \gamma^{(2)}(x_1,x_1';x_1',x_1')\Big] && (2.C.4)\\
&\qquad + \int \Big[ \big((-\Delta_{\underline{x}_2} + \Delta_{\underline{x}_2'})\gamma^{(2)}\big)(x_1,x_2;x_1',x_2) && (2.C.5)\\
&\qquad\quad + \kappa_0\gamma^{(3)}(x_1,x_2,x_1;x_1',x_2,x_1)\\
&\qquad\quad - \kappa_0\gamma^{(3)}(x_1,x_2,x_1';x_1',x_2,x_1')\\
&\qquad\quad + \kappa_0\gamma^{(3)}(x_1,x_2,x_2;x_1',x_2,x_2)\\
&\qquad\quad - \kappa_0\gamma^{(3)}(x_1,x_2,x_2;x_1',x_2,x_2)\Big]\,dx_2\\
&\quad = \int (\Delta_{x_1} - \Delta_{x_1'})\gamma^{(2)}(x_1,x_2;x_1',x_2)\,dx_2 && (2.C.6)\\
&\qquad - (\Delta_{x_1} - \Delta_{x_1'})A^{(1)}(x_1;x_1')\\
&\qquad - \kappa_0 \int \Big[\gamma^{(3)}(x_1,x_1,x_2;x_1',x_1,x_2) - \gamma^{(3)}(x_1,x_1',x_2;x_1',x_1',x_2)\Big]\,dx_2 && (2.C.7)\\
&\qquad + \kappa_0 A^{(2)}(x_1,x_1;x_1',x_1) - \kappa_0 A^{(2)}(x_1,x_1';x_1',x_1')\\
&\qquad + \int \Big[ \big((-\Delta_{x_1} + \Delta_{x_1'})\gamma^{(2)}\big)(x_1,x_2;x_1',x_2) && (2.C.8)\\
&\qquad\quad + \kappa_0\gamma^{(3)}(x_1,x_2,x_1;x_1',x_2,x_1)\\
&\qquad\quad - \kappa_0\gamma^{(3)}(x_1,x_2,x_1';x_1',x_2,x_1')\Big]\,dx_2\\
&\quad = -(\Delta_{x_1} - \Delta_{x_1'})A^{(1)}(x_1;x_1'), && (2.C.9)\\
&\qquad + \kappa_0 A^{(2)}(x_1,x_1;x_1',x_1) - \kappa_0 A^{(2)}(x_1,x_1';x_1',x_1')
\end{aligned}
$$

where (2.C.1) was used to pass from (2.C.3) to (2.C.6) and from (2.C.4) to

(2.C.7). Moreover, (2.C.2) and density of $\mathcal{S}$ in $H^1$ was used to pass from (2.C.5) to (2.C.8). Symmetry of $\gamma^{(k)}$ was used to pass to (2.C.9).

Observe that (2.C.9) is precisely the right hand side of the first equation in the GP hierarchy. Thus $A^{(1)}$, and similarly $A^{(k)}$ for $k > 1$, satisfies the GP hierarchy. $A(0) = 0$, so by uniquenss of solutions to the GP hierarchy [18], $A = 0$. □

## 2.D Continuity of solutions to the GP hierarchy

In [18], it is shown that there is a unique soution $\Gamma$ to the GP hierarchy (2.2.8) in $\{\Gamma \in L^\infty_{t\in[0,T]}\mathcal{H}^\alpha_\xi \,|\, B^+\Gamma, B^-\Gamma \in L^2_{t\in[0,T]}\mathcal{H}^\alpha_\xi\}$. In this part of the appendix, we show that this solution $\Gamma$ is an element of $C([0,T], \mathcal{H}^1_\xi)$.

**Lemma 2.D.1.** *If* $\gamma^{(k)} \in L^2(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$*, then*

$$\lim_{t\to 0} \| \left(U^{(k)}(t) - U^{(k)}(0)\right) \gamma^{(k)} \|_{L^2(\mathbb{R}^{dk}\times\mathbb{R}^{dk})} = 0.$$

*Proof.* We recall that $U^{(k)}(t) = e^{-it(-\Delta_{\underline{x}_k}+\Delta_{\underline{x}'_k})}$. Since $-\Delta_{\underline{x}_k} + \Delta_{\underline{x}'_k}$ is a self-adjoint operator, we have from theorem VIII.7 in [103] that $U^{(k)}(t)$ is a strongly continuous one-parameter unitary group, and the lemma follows. □

**Lemma 2.D.2.** *If* $\Gamma \in \mathcal{H}^\alpha_\xi(\mathbb{R}^{dk} \times \mathbb{R}^{dk})$*, then*

$$\lim_{t\to 0} \| \left(U(t) - U(0)\right) \Gamma \|_{\mathcal{H}^\alpha_\xi} = 0.$$

*Proof.*

$$\begin{aligned}
&\| \left(U(t) - U(0)\right) \Gamma\|_{\mathcal{H}_\xi^\alpha} \\
&= \sum_{k=1}^{\infty} \xi^k \| \left(U^{(k)}(t) - U^{(k)}(0)\right) S^{(k,\alpha)}\gamma^{(k)}\|_{L^2} \\
&\to 0 \text{ as } t \to 0
\end{aligned}$$

by Lemma 2.D.1, the fact that $\|U^{(k)}(t)\|_{L^2 \to L^2} \leq 1$, and the dominated convergence theorem for series. □

**Proposition 2.D.1.** *The solution* $\Gamma$ *to the GP hierarchy constructed in [18] lies in* $C([0,T], \mathcal{H}_\xi^1)$.

*Proof.* As proven in [18], the solution $\Gamma$ satisfies

$$\Gamma \in L^\infty_{t\in[0,T]}\mathcal{H}_\xi^1, \tag{2.D.1}$$

$$B\Gamma \in L^2_{t\in[0,T]}\mathcal{H}_\xi^1, \text{ and} \tag{2.D.2}$$

$$\Gamma(t) = U(t)\Gamma_0 + i\kappa_0 \int_0^t U(t-s)B\Gamma(s)\, ds. \tag{2.D.3}$$

Thus, in $\mathcal{H}^1_\xi$, we have that

$$\begin{aligned}
&\lim_{h\to 0}\Big[\Gamma(t+h)-\Gamma(t)\Big]\\
&=\lim_{h\to 0}\Big[U(t+h)\Gamma_0+i\kappa_0\int_0^{t+h}U(t+h-s)B\Gamma(s)\,ds\\
&\qquad -U(t)\Gamma_0-i\kappa_0\int_0^{t}U(t-s)B\Gamma(s)\,ds\Big]\\
&=\lim_{h\to 0}U(h)\Gamma_0 &\text{(2.D.4)}\\
&\qquad+\lim_{h\to 0}(U(h)-U(0))\underbrace{i\kappa_0\int_0^{t}U(t-s)B\Gamma(s)\,ds}_{[*]} &\text{(2.D.5)}\\
&\qquad+\lim_{h\to 0}i\kappa_0\int_t^{t+h}U(t+h-s)B\Gamma(s)\,ds. &\text{(2.D.6)}
\end{aligned}$$

By Lemma 2.D.2, (2.D.4) = 0. By (2.D.1) and (2.D.3), $[*]\in\mathcal{H}^1_\xi$, so it follows from Lemma 2.D.2 that (2.D.5) = 0. Now

$$\begin{aligned}
&\left\|\int_t^{t+h}U(t+h-s)B\Gamma(s)\,ds\right\|_{\mathcal{H}^1_\xi}\\
&\le\int_t^{t+h}\|U(t+h-s)B\Gamma(s)\|_{\mathcal{H}^1_\xi}\,ds\\
&\le\sqrt{h}\|U(t+h-s)B\Gamma(s)\|_{L^2_{s\in[t,t+h]}\mathcal{H}^1_\xi}\\
&=\sqrt{h}\|B\Gamma(s)\|_{L^2_{s\in[t,t+h]}\mathcal{H}^1_\xi}\\
&\to 0\text{ as }h\to 0
\end{aligned}$$

by (2.D.2), so (2.D.6) = 0. □

## 2.E Iterated Duhamel formula and boardgame argument

In this part of the appendix, we recall a technical result from [23] that is used in parts of this chapter. It corresponds to Lemma B.3 in [23].

Let $\Xi = (\Xi^{(k)})_{n\in\mathbb{N}}$ denote a sequence of functions $\Xi^{(k)} \in L^2_{t\in[0,T]} H^1(\mathbb{R}^{3k}\times\mathbb{R}^{3k})$, for $T>0$. Then, we define the associated sequence $\mathrm{Duh}_j(\Xi)$ of *$j$-th level iterated Duhamel terms* based on $B_N^{main}$ (see Section 2.2.2 for notations), with components given by

$$\begin{aligned} &\mathrm{Duh}_j(\Xi)^{(k)}(t) \\ &:= \; i^j \int_0^t dt_1 \cdots \int_0^{t_{j-1}} dt_j B^{main}_{N;k+1} e^{i(t-t_1)\Delta_\pm^{(k+1)}} B^{main}_{N;k+2} e^{i(t_1-t_2)\Delta_\pm^{(k+2)}} \\ &\qquad B^{main}_{N;k+2} \cdots\cdots e^{i(t_{j-1}-t_j)\Delta_\pm^{(k+j)}} (\,\Xi\,)^{(k+j)}(t_j)\,, \end{aligned} \tag{2.E.1}$$

with the conventions $t_0 := t$, and

$$\mathrm{Duh}_0(\Xi)^{(k)}(t) \;:=\; (\,\Xi\,)^{(k)}(t) \tag{2.E.2}$$

for $j=0$. Using the boardgame estimates of [42, 43, 82], one obtains:

**Lemma 2.E.1.** *For $\Xi = (\Xi^{(k)})_{k\in\mathbb{N}}$ as above,*

$$\begin{aligned} &\|\,\mathrm{Duh}_j(\Xi)^{(k)}(t)\,\|_{L^2_{t\in I} H^1(\mathbb{R}^{3k}\times\mathbb{R}^{3k})} \\ &\qquad \le\; k\, C_0^k\,(c_0 T)^{\frac{j}{2}} \|\Xi^{(k+j)}\|_{L^2_{t\in I} H^1(\mathbb{R}^{3(k+j)}\times\mathbb{R}^{3(k+j)})}\,, \end{aligned} \tag{2.E.3}$$

*where the constants $c_0, C_0$ depend only on $d, p$. For this work, the dimension is given by $d=3$ and the nonlinearity is given by $p=2$ (cubic GP hierarchy).*

Lemma 2.E.1 is used for the proof of the next result (by suitably exploiting the splitting $B_N = B_N^{main} + B_N^{error}$), which corresponds to Lemma B.3 in [23].

**Lemma 2.E.2.** *Let $\delta' > 0$ be defined by*

$$\beta \;=\; \frac{1-\delta'}{4}\,. \tag{2.E.4}$$

*Assume that $N$ is sufficiently large that the condition*

$$K \;<\; \frac{\delta'}{\log C_0}\,\log N\,, \tag{2.E.5}$$

*holds, where the constant $C_0$ is as in Lemma 2.E.1.*

*Assume that $\Xi_N^K \in L^2_{t\in I}\mathcal{H}^1_{\xi'}$ for some $0 < \xi' < 1$, and that $\xi$ is small enough that $0 < \xi < \eta\xi'$, with*

$$\eta \,<\, (\max\{1, C_0\})^{-1}\,. \tag{2.E.6}$$

*Let $\Theta_N^K$ and $\Xi_N^K$ satisfy the integral equation*

$$\Theta_N^K(t) \;=\; \Xi_N^K(t) \,+\, i\int_0^t B_N\, U(t-s)\,\Theta_N^K(s)ds \tag{2.E.7}$$

*The superscript "K" in $\Theta_N^K$ and $\Xi_N^K$ means that only the first $K$ components are nonzero, and $B_N = B_N^{main} + B_N^{error}$.*

*Then, the estimate*

$$\|\Theta_N^K\|_{L^2_{t\in I}\mathcal{H}^1_\xi} \;\le\; C_1(T,\xi,\xi')\,\|\Xi_N^K\|_{L^2_{t\in I}\mathcal{H}^1_{\xi'}} \tag{2.E.8}$$

*holds for a finite constant $C_1(T,\xi,\xi') > 0$ independent of $K, N$.*

# Chapter 3

# Unconditional Uniqueness of the Cubic Gross-Pitaevskii Hierarchy with Low Regularity

## 3.1 Main result of this chapter

In this chapter, we investigate the unconditional uniqueness of solutions to the cubic GP hierarchy in a low regularity setting. This chapter is based on a joint work with Younghun Hong and Zhihui Xie [70].

We begin by introducing the notation that we will use for the GP hierarchy in this chapter. The cubic Gross-Pitaevskii (GP) hierarchy in $\mathbb{R}^d$ is an infinite system of coupled linear equations given by

$$i\partial_t \gamma^{(k)} = (-\Delta_{\underline{x}_k} + \Delta_{\underline{x}'_k})\gamma^{(k)} + \lambda B_{k+1}\gamma^{(k+1)}, \quad \forall k \in \mathbb{N}, \tag{3.1.1}$$

where $\gamma^{(k)} = \gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k) : I \times \mathbb{R}^{dk} \times \mathbb{R}^{dk} \to \mathbb{C}$, $I \subset \mathbb{R}$ is a time interval and $\lambda = \pm 1$. Here, we denote $d$-dimensional $k$-spatial variables $(x_1, x_2, ..., x_k)$ by $\underline{x}_k$, and the corresponding Laplace operator by $\Delta_{\underline{x}_k} = \sum_{j=1}^{k} \Delta_{x_j}$, and similarly for the primed variables. For each $k \in \mathbb{N}$, $\gamma^{(k)}$ is a bosonic density matrix on $L^2_{sym}(\mathbb{R}^{dk})$ which is hermitian,

$$\gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k) = \overline{\gamma^{(k)}(t, \underline{x}'_k, \underline{x}_k)},$$

and is symmetric in all components of $\underline{x}_k$, and in all components of $\underline{x}'_k$, respectively,

$$\gamma^{(k)}(t, x_{\sigma(1)}, \cdots, x_{\sigma(k)}, x'_{\sigma'(1)}, \cdots, x'_{\sigma'(k)}) = \gamma^{(k)}(t, \underline{x}_k, \underline{x}'_k)$$

for any permutations $\sigma, \sigma'$ on $k$ elements. The equations in (3.1.1) are coupled by the *contraction operator* $B_{k+1}$,

$$B_{k+1} = \sum_{j=1}^{k} B_{j;k+1} = \sum_{j=1}^{k} (B^{+}_{j;k+1} - B^{-}_{j;k+1}),$$

where each $B^{+}_{j;k+1}$ contracts the triple $x_j, x_{k+1}, x'_{k+1}$,

$$\begin{aligned}\left(B^{+}_{j;k+1}\gamma^{(k+1)}\right)(t, \underline{x}_k, \underline{x}'_k) &= \int dx_{k+1} dx'_{k+1} \delta(x_j - x_{k+1}) \delta(x_j - x'_{k+1}) \gamma^{(k+1)}(t, \underline{x}_{k+1}; \underline{x}'_{k+1}) \\ &= \gamma^{(k+1)}(t, \underline{x}_k, x_j, \underline{x}'_k, x_j)\end{aligned}$$

and each $B^{-}_{j;k+1}$ contracts the triple $x'_j, x_{k+1}, x'_{k+1}$,

$$\begin{aligned}\left(B^{-}_{j;k+1}\gamma^{(k+1)}\right)(t, \underline{x}_k, \underline{x}'_k) &= \int dx_{k+1} dx'_{k+1} \delta(x'_j - x_{k+1}) \delta(x'_j - x'_{k+1}) \gamma^{(k+1)}(t, \underline{x}_{k+1}; \underline{x}'_{k+1}) \\ &= \gamma^{(k+1)}(t, \underline{x}_k, x'_j, \underline{x}'_k, x'_j).\end{aligned}$$

The cubic GP hierarchy is called *focusing* (*defocusing*, respectively) if $\lambda = 1$ ($\lambda = -1$, respectively).

Before we state the main theorem, we also introduce the following definitions. Let $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ be a sequence of bosonic density matrices on $L^2_{sym}(\mathbb{R}^{dk})$. We say that $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ is *admissible* if $\gamma^{(k)}$ is a non-negative trace class operator on $L^2_{sym}(\mathbb{R}^{dk})$ and $\gamma^{(k)} = \mathrm{Tr}(\gamma^{(k+1)})$ for all $k \in \mathbb{N}$. We call a sequence $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ a *limiting hierarchy* if there is a sequence $\{\gamma^{(N)}_N\}_{N\in\mathbb{N}}$ of non-negative density matrices on $L^2_{sym}(\mathbb{R}^{dN})$ with $\mathrm{Tr}(\gamma_N^{(N)}) = 1$ such that $\gamma^{(k)}$ is the weak-* limit of the $k$-particle marginals of $\gamma_N^{(N)}$ in the trace class on $L^2_{sym}(\mathbb{R}^{dk})$, that is,

$$\gamma_N^{(k)} := \mathrm{Tr}_{k+1,\cdots,N}(\gamma_N^{(N)}) \rightharpoonup^* \gamma^{(k)} \text{ as } N \to \infty.$$

For $s \in \mathbb{R}$, we define the function space $\mathfrak{H}^s$ by the collection of sequences $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ of density matrices on $L^2_{sym}(\mathbb{R}^{dk})$ such that

$$\mathrm{Tr}(|S^{(k,s)}\gamma^{(k)}|) < M^{2k} \quad \forall k \in \mathbb{N} \text{ for some constant } M > 0,$$

where

$$S^{(k,s)} := \prod_{j=1}^{k}(1-\Delta_{x_j})^{\frac{s}{2}}(1-\Delta_{x'_j})^{\frac{s}{2}}.$$

We say that $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ is a *mild solution*, in the space $L^\infty_{t\in[0,T)}\mathfrak{H}^s$, to the cubic GP hierarchy (3.1.1) with initial data $\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$ if it solves the integral equation

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\lambda \int_0^t U^{(k)}(t-s)B_{k+1}\gamma^{(k+1)}(s)ds,$$

where $U^{(k)}(t) := e^{it(\Delta_{\underline{x}_k} - \Delta_{\underline{x}'_k})}$, and satisfies the bound

$$\sup_{t\in[0,T)} \mathrm{Tr}(|S^{(k,s)}\gamma^{(k)}(t)|) < M^{2k} \quad \forall k \in \mathbb{N} \text{ for some constant } M > 0.$$

Our main theorem states that any mild solution to the cubic GP hierarchy, which is either admissible or a limiting hierarchy, is unconditionally unique in $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ for small $s$.

**Theorem 3.1.1** (Unconditional uniqueness)**.** *Let*

$$\begin{cases} s \geq \frac{d}{6} & \text{if } d = 1, 2, \\ s > s_c & \text{if } d \geq 3, \end{cases} \tag{3.1.2}$$

*where* $s_c = \frac{d-2}{2}$*. If* $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ *is a mild solution in* $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ *to the (de)focusing cubic GP hierarchy* (3.1.1) *with initial data* $\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$*, which is either admissible or a limiting hierarchy for each* $t$*, then it is the only such solution for the given initial data.*

Our theorem reduces the regularity requirement for unconditional uniqueness for the GP hierarchy in [17]. We remark that the regularity assumption in (3.1.2) is the same as in the currently known unconditional uniqueness results for the cubic NLS

$$i\partial_t\phi + \Delta\phi - \lambda|\phi|^2\phi = 0,\ \phi(0) = \phi_0 \in H^s.$$

For NLS, by unconditional uniqueness, we mean uniqueness of solutions in the Sobolev space $H^s$ itself, while uniqueness in the intersection of the Sobolev space and auxiliary spaces is called conditional. By the contraction mapping argument with auxiliary Strichartz spaces, the conditional uniqueness is proved in $H^s$ for $s \geq \max(s_c, 0)$, where $s_c = \frac{d-2}{2}$ (see [11]). However, the unconditional uniqueness is proved in $H^s$ only for $s$ in (3.1.2), and it is an open problem to push $s$ down to zero in one and two dimensions [54, 66, 73, 107, 123].

Our proof uses the Klainerman-Machedon board game formulation [82] of the combinatorial argument of Erdös-Schlein-Yau [42–45], and the method of Chen-Hainzl-Pavlović-Seiringer [17] via the quantum de Finetti theorem.

The quantum de Finetti theorem is a quantum analogue of the Hewitt-Savage theorem in probability theory. We state its strong and weak versions in the formulation of [91].

**Theorem 3.1.2** (Strong quantum de Finetti theorem). *If a sequence* $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ *of bosonic density matrices on* $L^2_{sym}(\mathbb{R}^{dk})$ *is admissible, then there exists a unique Borel probability measure* $\mu$*, supported on the unit sphere* $S \subset L^2(\mathbb{R}^d)$ *and invariant under multiplication of* $\phi \in L^2(\mathbb{R}^d)$ *by complex numbers of modulus one, such that*

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k} \quad k \in \mathbb{N}. \tag{3.1.3}$$

**Theorem 3.1.3** (Weak quantum de Finetti theorem). *If a sequence* $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ *of bosonic density matrices on* $L^2_{sym}(\mathbb{R}^{dk})$ *is a limiting hierarchy, then there exists a unique Borel probability measure* $\mu$*, supported on the unit ball* $\mathcal{B} \subset L^2(\mathbb{R}^d)$ *and invariant under multiplication of* $\phi \in L^2(\mathbb{R}^d)$ *by complex numbers of modulus one, such that* (3.1.3) *holds.*

The crucial advantage of using the quantum de Finetti theorem is that it provides a factorized representation of solutions to the GP hierarchy in the integral form (see (3.2.10)). This structure allows us to make use of techniques of NLS theory to analyze solutions to the GP hierarchies (see [17] and [16]).

As described in Section 6.1.1 of [17], the main difficulty in lowering regularity is from the last cubic term $\||\phi|^2\phi\|_{L^2} = \|\phi\|^3_{L^6}$ in the distinguished tree. Indeed, this last term can be controlled by the Sobolev norm $\|\phi\|^3_{H^s}$ only for $s \geq 1$ in $\mathbb{R}^3$. We solve this problem by using the dispersive estimate

$$\|e^{it\Delta} f\|_{L^{\frac{6}{1+2\epsilon}}} \lesssim |t|^{-(1-\epsilon)} \|f\|_{L^{\frac{6}{5-2\epsilon}}}$$

in $\mathbb{R}^3$, for instance. Indeed, if one applies the dispersive estimate and the endpoint Strichartz estimate to the factorized representation of the solution in the framework of [17], one gets a better last cubic term $\||\phi|^2\phi\|_{L^{\frac{6}{5-2\epsilon}}} = \|\phi\|^3_{L^{\frac{18}{5-2\epsilon}}}$, and it allows us to reduce $s$ down to $\frac{2}{3}+\epsilon$. The regularity requirement in the classical Kato's work on the unconditional uniqueness for the 3D cubic NLS [73] can be covered in this way. We further push $s$ almost down to the critical regularity by employing negative order Sobolev norms (Lemma 3.A.3), which are well-known tools in the literature on unconditional uniqueness for NLS. Combining the dispersive estimate, the Strichartz estimates and negative Sobolev norms, we formulate the key trilinear estimates (Lemma 3.2.5) in our proof.

**Organization of the chapter**. We prove Theorem 3.1.1 in Section 3.2, by reducing it to the main Lemma 3.2.4. In Section 3.3, we present an example calculation to explain the ingredients involved in the proof of Lemma 3.2.4. In Section 3.4, we introduce tree graphs for the organization of iterated Duhamel expansions, and give properties of the associated kernels. Finally, we prove the main Lemma 3.2.4 in Section 3.5. We prove the crucial trilinear estimates in Lemma 3.2.5 in Appendix 3.A.

## 3.2 Proof of the main theorem

In this section, we prove the main theorem. First, in §3.2.1, we present the setup of the proof. In §3.2.2 we review Klainerman-Machedon's board game formulation [82] of the combinatorial argument of Erdös-Schlein-Yau [42–45]. In §3.2.3, we reduce the proof of the main theorem to the key lemma (Lemma 3.2.4), via the quantum de Finetti theorem. The rest of the chapter is then devoted to the proof of the lemma.

### 3.2.1 Setup of the proof

The setup of the proof is similar to that of Chen-Hainzl-Pavlović-Seiringer [17], but we use a negative order Sobolev type norm to lower the regularity.

Let $\{\gamma_1^{(k)}(t)\}_{k\in\mathbb{N}}$ and $\{\gamma_2^{(k)}(t)\}_{k\in\mathbb{N}}$ be two mild solutions in $L^\infty_{t\in[0,T)}\mathfrak{H}^s$ to the cubic GP hierarchy with the same initial data, which are either admissible or limiting hierarchies. For uniqueness, it is enough to show that their difference $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$, given by

$$\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t), \qquad k \in \mathbb{N},$$

vanishes for all $k$ in a certain norm.

Due to the linearity of the GP hierarchy, it follows that the difference $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ solves the GP hierarchy with zero initial data. Hence, each $\gamma^{(k)}(t)$ satisfies the integral equation

$$\gamma^{(k)}(t) = i\lambda \int_0^t U^{(k)}(t-t_1)B_{k+1}\gamma^{(k+1)}(t_1)dt_1.$$

Now fix $k$. Iterating this integral equation $(n-1)$ times, we write

$$\gamma^{(k)}(t) = (i\lambda)^n \int_{t_n\leq\cdots\leq t_1\leq t} U^{(k)}(t-t_1)B_{k+1}\cdots U^{(k+n-1)}(t_{n-1}-t_n)B_{k+n}\gamma^{(k+n)}(t_n)dt_1\cdots dt_n.$$

For notational convenience, we denote $(k+1)$-temporal variables $(t_0, t_1, \cdots, t_n)$ by $\underline{t}_n$ with $t_0 = t$, and the linear propagator $U^{(i)}(t_j - t_{j'})$ by $U^{(i)}_{j,j'}$. Then, we rewrite $\gamma^{(k)}(t)$ in a compact form as

$$\gamma^{(k)}(t) = (i\lambda)^n \int_{t_n \le \cdots \le t_1 \le t} J^k(\underline{t}_n) d\underline{t}_n, \tag{3.2.1}$$

where

$$J^k(\underline{t}_n) := U^{(k)}_{0,1} B_{k+1} U^{(k)}_{1,2} B_{k+2} \cdots U^{(k+n-1)}_{n-1,n} B_{k+n} \gamma^{(k+n)}(t_n).$$

By density, our uniqueness theorem follows from uniqueness in an even weaker norm.

**Proposition 3.2.1.** *For all $t \in [0, T)$ with $T > 0$ small enough, the trace norm of $S^{(k,-d)}$(3.2.1) vanishes as $n \to \infty$ uniformly in $k$, that is*

$$\mathrm{Tr}(|S^{(k,-d)} \gamma^{(k)}(t)|) = 0, \quad \forall k, \tag{3.2.2}$$

*where $d > 0$ is the dimension.*

### 3.2.2 Erdös-Schlein-Yau Combinatorial method in board-game form

One obstacle in showing uniqueness is the number of terms in $J^k(\underline{t}_n)$. Indeed, each $B_{k+i}$ is a sum of $(k+i-1)$ terms. Thus, in the expansion of $J^k(\underline{t}_n)$, there are a total of $k(k+1)\cdots(k+n-1) = \mathcal{O}(n!)$ terms for fixed $k$. We solve this problem by using the powerful combinatorial methods of Erdös-Schlein-Yau [42–45] in the board-game formulation of Klainerman-Machedon [82].

The key idea of the *board game* arguments is that, by grouping the large number of integral terms into equivalence classes in which we have control, we can avoid estimating the rapidly increasing number of terms one by one. Throughout this section, we present a few lemmas that will help us group these terms and derive bounds on certain equivalence classes.

Let $\mu$ be a map from $\{k+1, k+2, \cdots, k+n\}$ to $\{1, 2, \cdots, k+n-1\}$ such that $\mu(2) = 1$ and $\mu(j) < j$ for all $j$. Denotes by $\mathcal{M}_{k,n}$ the set of all such maps.

We express the operators $B_{k+i}$ and $J^k$ in terms of map $\mu$. We have

$$B_{k+i} = \sum_{j=1}^{k+i-1} B_{j;k+i} = \sum_{\mu \in \mathcal{M}_{k,n}} B_{\mu(k+i);k+i}$$

and

$$J^k(\underline{t}_n) = \sum_{\mu \in \mathcal{M}_{k,n}} J^k(\underline{t}_n; \mu), \tag{3.2.3}$$

where

$$J^k(\underline{t}_n; \mu) = U^{(k)}(t-t_1) B_{\mu(k+1);k+1} U^{(k+1)}(t_1-t_2) \cdots U^{(k+n-1)}(t_{n-1}-t_n) B_{\mu(k+n);k+n} \gamma^{(k+n)}(t_n).$$

By the definition of $\mu$, we can represent $\mu$ by highlighting exactly one nonzero entry $B_{\mu(k+l),k+l}$ ($l$-th column, $\mu(k+l)$-th row) in each column of a $(k+n-1) \times n$ matrix. Since $\mu(k+l) < k+l$, we set the remaining entries of the matrix equal to 0.

$$\begin{pmatrix} \mathbf{B_{1;k+1}} & B_{1;k+2} & \cdots & \mathbf{B_{1;k+n}} \\ B_{2;k+1} & B_{2;k+2} & \cdots & B_{2;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{k;k+1} & \mathbf{B_{k;k+2}} & \cdots & B_{k;k+n} \\ 0 & B_{k+1;k+2} & \cdots & B_{k+1;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+n-1;k+n} \end{pmatrix} \tag{3.2.4}$$

Henceforth, we can rewrite (3.2.1) as

$$\gamma^{(k)}(t) = \int_0^t \cdots \int_0^{t_n} \sum_{\mu \in \mathcal{M}_{k,n}} J^k(\underline{t}_{k+n}; \mu) dt_1 \dots dt_n. \tag{3.2.5}$$

Here the time domain $\{t_n \leq t_{n-1} \leq \cdots \leq t\} \subset [0,t]^n$ is the same for all $\mu \in \mathcal{M}_{k,n}$. We now consider the terms $I(\mu, \sigma)$ in the sum $\gamma^{(k)}(t) = \sum_{\mu \in \mathcal{M}_{k,n}} I(\mu, \sigma)$. We have

$$I(\mu, \sigma) = \int_{t_{\sigma(n)} \leq t_{\sigma(n-1)} \leq \cdots \leq t} J^k(\underline{t}_{k+n}; \mu) dt_1 \dots dt_n, \tag{3.2.6}$$

where $\sigma$ is a permutation of $1, 2, \dots, n$. We associate the integral $I(\mu, \sigma)$ the following $(k+n) \times n$ matrix. We may also use it to visualize $B_{\mu(k+j);k+j}$ that correspond to a highlighted entry.

$$\begin{pmatrix} t_{\sigma^{-1}(1)} & t_{\sigma^{-1}(2)} & \cdots & t_{\sigma^{-1}(n)} \\ \mathbf{B_{1;k+1}} & B_{1;k+2} & \cdots & \mathbf{B_{1;k+n}} \\ B_{2;k+1} & B_{2;k+2} & \cdots & B_{2;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ B_{k;k+1} & \mathbf{B_{k;k+2}} & \cdots & B_{k;k+n} \\ 0 & B_{k+1;k+2} & \cdots & B_{k+1;k+n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{k+n} \end{pmatrix} \tag{3.2.7}$$

The columns of matrix (3.2.7) are labeled 1 through $n$, and the rows are labeled 0 through $k+n-1$.

Each term (3.2.6) corresponds to a unique matrix of form (3.2.7). A key observation is that two matrices of this form can have to the same value for $I(\mu;\sigma)$ given that one matrix can be transformed to another under the so called *acceptable moves*.

In the following paragraph, we will present a few key lemmas to help us with the combinatorial reduction. For the proof of these lemmas, we refer the reader to [19, 42–45, 82, 124].

#### 3.2.2.1 Acceptable moves

If $\mu(k+j+1) < \mu(k+j)$, we take the following steps at the same time

- exchange the highlights in columns $j$ and $j+1$
- exchange the highlights in rows $k+j$ and $k+j+1$
- exchange $t_{\sigma^{-1}(j)}$ and $t_{\sigma^{-1}(j+1)}$

The exchange only happens when there is a highlight, if there is no highlight we will skip that step. The following lemma highlights the necessity to introduce *equivalence classes*.

**Lemma 3.2.1.** *Let* $(\mu,\sigma)$ *be transformed into* $(\mu',\sigma')$ *by an acceptable move. Then, for the corresponding integrals* (3.2.6)*, we have* $I(\mu,\sigma) = I(\mu',\sigma')$

#### 3.2.2.2 Equivalence class

Consider the subset $\{\mu_s\} \subset \mathcal{M}_{k,n}$ of *special upper echelon* matrices in which each highlighted element of a higher row is to the left of each highlighted element of a lower row. An example of a special upper echelon matrix (with $k = 1, n = 4$) is

$$\begin{pmatrix} \mathbf{B_{1;2}} & \mathbf{B_{1;3}} & B_{1;4} & B_{1;5} \\ 0 & B_{2;3} & B_{2;4} & B_{2;5} \\ 0 & 0 & \mathbf{B_{3;4}} & B_{3;5} \\ 0 & 0 & 0 & \mathbf{B_{4;5}} \end{pmatrix}$$

**Lemma 3.2.2.** *For each element of $\mathcal{M}_{k,n}$ there is a finite number of acceptable moves which brings the matrix to upper echelon form.*

**Lemma 3.2.3.** *Let $C_{k,n}$ be the number of $(k+n-1) \times n$ special upper echelon matrices of the type discussed above. Then $C_{k,n} \leq 2^{k+2n-2}$.*

Let $\mu_s$ be a special upper echelon matrix. We say $\mu$ is in the equivalence class of $\mu_s$: $\mu \sim \mu_s$ if $\mu$ can be transformed to $\mu_s$ in finitely many acceptable moves.

**Theorem 3.2.2.** *There exists a subset $D$ of $[0,t]^n$ such that*

$$\sum_{\mu \sim \mu_s} \int_0^t \dots \int_0^{t_{n-1}} J^k(\underline{t}_n; \mu) dt_1 \dots dt_n = \int\limits_D J^k(\underline{t}_n; \mu) dt_1 \dots dt_n. \tag{3.2.8}$$

*Proof.* We perform finitely many acceptable moves on the matrix associated to the integral

$$I(\mu, id) = \int_0^t \dots \int_0^{t_{n-1}} J^k(\underline{t}_n; \mu) dt_1 \dots dt_n.$$

Let $I(\mu, id)$ be the integral associated to the upper echelon matrix obtained. By Lemma 3.2.1

$$I(\mu, id) = I(\mu_s, \sigma).$$

Assume that $(\mu_1, id)$ and $(\mu_2, id)$ with $\mu_1 \neq \mu_2$ yield the same echelon form $\mu_s$. Then the corresponding permutations $\sigma_1$ and $\sigma_2$ must be different. Therefore, $D$ can be chosen to be the union of all $\{t \geq t_{\sigma(1)} \geq t_{\sigma(2)} \geq \cdots \geq t_{\sigma(n)}\}$ for all permutations $\sigma$ which occur in a given equivalence class of some $\mu_s$. $\square$

With the above theorem, we are able to reduce the sum of $\mathcal{O}(n!)$ terms into a sum of $\mathcal{O}(C^n)$ terms:

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}_{k,n}} \int_{D_{\sigma,t}} d\underline{t}_n J^k(\underline{t}_n; \sigma), \tag{3.2.9}$$

which we can afford.

### 3.2.3 Proof of the main theorem

As mentioned above, it suffices to show Proposition 3.2.1. For the proof, we uses the framework of Chen-Hainzl-Pavlović-Seiringer [17] via the quantum de Finetti theorem.

Applying the strong or the weak quantum de Finetti theorem, we write

$$\gamma^{(k)}(t) = \int d\tilde{\mu}_t(\phi)(|\phi\rangle \langle\phi|)^{\otimes k}, \quad \forall k \in \mathbb{N}, \tag{3.2.10}$$

where $\tilde{\mu}_t = \mu_t^{(1)} - \mu_t^{(2)}$ with

$$\gamma_i^{(k)}(t) = \int d\mu_t^{(i)}(\phi)(|\phi\rangle \langle\phi|)^{\otimes k}, \quad i = 1, 2.$$

Plugging (3.2.10) into $J^k(\underline{t}_n;\sigma)$ in the reduced Duhamel expansion (3.2.9), we obtain a new expression

$$\gamma^{(k)}(t) = \sum_{\sigma\in\mathcal{M}_{k,n}} \int_{D_{\sigma,t}} d\underline{t}_n \int d\tilde{\mu}_{t_n}(\phi) J^k(\underline{t}_n;\sigma), \tag{3.2.11}$$

where

$$J^k(\underline{t}_n;\sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1} U_{1,2}^{(k+1)} B_{\sigma(k+2);k+2} \cdots U_{n-1,n}^{(k+n-1)} B_{\sigma(k+n);k+n} (|\phi\rangle\langle\phi|)^{\otimes(k+n)}. \tag{3.2.12}$$

Then, we formulate the following key lemma that implies Proposition 3.2.1 (and thus the main theorem).

**Lemma 3.2.4** (Key lemma)**.** *There exists a uniform constant $C>0$ such that for arbitrarily small $\epsilon>0$, we have*

$$\int_{[0,T)^{n-1}} d\underline{t}_{n-1} \mathrm{Tr}(|S^{(k,-d)} J^k(\underline{t}_n;\sigma)|) \le \begin{cases} (CT^{\epsilon})^{n-1} \|\phi\|_{H^{s_\epsilon}}^{2(k+n)} & \text{if } d\ge 3 \\ (CT^{1/3})^{n-1} \|\phi\|_{H^{1/3}}^{2(k+n)} & \text{if } d=2 \\ (CT^{1/2})^{n-1} \|\phi\|_{H^{1/6}}^{2(k+n)} & \text{if } d=1, \end{cases} \tag{3.2.13}$$

*where $s_\epsilon = \frac{d-2}{2}+\epsilon$.*

*Proof of Theorem 3.1.1, assuming Lemma 3.2.4.* We present the proof for the case $d\ge 3$ only. Indeed, when $d=1$ ($d=2$, resp), it can be proved in an analogous way by replacing the $H^{s_c}$ norm with the $H^{1/6}$ norm (the $H^{1/3}$ norm, resp).

Let $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ be as above. The goal is to show that $\mathrm{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) =$

0 for all $k \in \mathbb{N}$. Applying the triangle inequality and Lemma 3.2.4, we write

$$\begin{aligned} \operatorname{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) &\leq \sum_{i=1,2} \sum_{\sigma \in \mathcal{M}_{k,n}} \int_{[0,T)^n} d\underline{t}_n \int d\mu_{t_n}^{(i)}(\phi) \operatorname{Tr}(|S^{(k,-d)} J^k(\underline{t}_n;\sigma)|) \\ &\leq (CT^\epsilon)^{n-1} T \sum_{i=1,2} \sum_{\sigma \in \mathcal{M}_{k,n}} \sup_{t_n \in [0,T)} \int d\mu_{t_n}^{(i)}(\phi) \|\phi\|_{H^{s_\epsilon}}^{2(k+n)}. \end{aligned} \tag{3.2.14}$$

We claim that there exists $M > 0$ such that

$$\|\phi\|_{H^{s_\epsilon}} \leq M \quad \text{a.s. } \mu_t^{(i)}, \quad \forall t \in [0,T). \tag{3.2.15}$$

Indeed, since $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}} \in L^\infty_{t\in[0,T)}\mathfrak{H}^s$, there exists $M > 0$ such that

$$\int d\mu_t^{(i)}(\phi) \|\phi\|_{H^s}^{2k} = \operatorname{Tr}(|S^{(k,s)}\gamma^{(k)}(t)|) < M^{2k}, \quad \forall k \in \mathbb{N}. \tag{3.2.16}$$

Hence, it follows from the Chebyshev inequality that for $\lambda > M$,

$$\mu_t^{(i)}\big(\{\phi \in L^2 : \|\phi\|_{H^s} > \lambda\}\big) \leq \frac{1}{\lambda^{2k}} \int d\mu_t^{(i)}(\phi) \|\phi\|_{H^s}^{2k} < \left(\frac{M}{\lambda}\right)^{2k} \to 0 \quad \text{as } k \to \infty. \tag{3.2.17}$$

Returning to (2.14), by (3.2.15) and Lemma 3.2.3, we prove that

$$\operatorname{Tr}(|S^{(k,-d)}\gamma^{(k)}(t)|) \leq (CT^\epsilon)^{n-1} T \cdot 2 \cdot 2^{k+2n-2} \cdot M^{2(k+n)} = \frac{M^{2k} 2^{k-1} T}{CT^\epsilon} (4CT^\epsilon M^2)^n \to 0 \text{ as } n \to \infty. \tag{3.2.18}$$

for $T < (4CM^2)^{-1/\epsilon}$. □

The remainder of our chapter will be devoted to proving Lemma 3.2.4. We remark that our proof heavily relies on the following trilinear estimates which combine the dispersive estimate, the Strichartz estimates and negative Sobolev norms. The proof of these trilinear estimates is given in the appendix.

**Lemma 3.2.5** (Trilinear estimates)**.** *We define the trilinear form $T$ by*

$$T(f,g,h) = (e^{i(t-t_1)\Delta} f)(e^{i(t-t_2)\Delta} g)(e^{i(t-t_3)\Delta} h).$$

*(i) $d \geq 3$. For small $\epsilon > 0$, we have*

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)} W_x^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \lesssim T^\epsilon \|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|g\|_{H^{s_\epsilon}} \|h\|_{H^{s_\epsilon}}, \tag{3.2.19}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)} H_x^{s_\epsilon}} \lesssim T^\epsilon \|f\|_{H^{s_\epsilon}} \|g\|_{H^{s_\epsilon}} \|h\|_{H^{s_\epsilon}}, \tag{3.2.20}$$

*where $s_\epsilon = s_c + \epsilon = \frac{d-2}{2} + \epsilon$, $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$.*
*(ii) $d = 2$. For small $\epsilon > 0$, we have*

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)} W_x^{-(\frac{1}{3}-\frac{\epsilon}{2}),\frac{2}{2-\epsilon}}} \lesssim T^\epsilon \|f\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),\frac{2}{2-\epsilon}}} \|g\|_{H^{1/3}} \|h\|_{H^{1/3}}, \tag{3.2.21}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)} H_x^{1/3}} \lesssim T^{1/3} \|f\|_{H^{1/3}} \|g\|_{H^{1/3}} \|h\|_{H^{1/3}}. \tag{3.2.22}$$

*(ii) $d = 1$. We have*

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)} L^1_x} \lesssim T^{1/2} \|f\|_{L^1} \|g\|_{L^2} \|h\|_{L^2}, \tag{3.2.23}$$

$$\|T(f,g,h)\|_{L^1_{t\in[0,T)} L^2_x} \lesssim T^{1/2} \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2}. \tag{3.2.24}$$

We will prove Lemma 3.2.4 in the following sections. To this end, we will proceed as in [17] and use *binary tree graphs* to help organize the terms in $J^k(\underline{t}_n, \sigma)$ (see (3.2.12)). For the reader's convenience, before proving the lemma, we give an example calculation in Section 3.3. We remark that the trilinear estimates in Lemma 3.2.5 are the key estimates, and will be applied recursively in general case (see Section 3.5).

## 3.3 An example

In this section, we illustrate the ideas of the proof of Lemma 3.2.4 via an example.

Let $d \geq 3$, $k = 2$ and $n = 4$ in Lemma 3.2.4. We investigate the example

$$\int_{[0,T)^3} d\underline{t}_3 \mathrm{Tr}(|S^{(2,-d)} J^2(\underline{t}_4;\sigma)|) \tag{3.3.1}$$

with a specific map $\sigma$ represented by the matrix

$$\begin{pmatrix} \mathbf{B_{1;3}} & B_{1;4} & B_{1;5} & B_{1;6} \\ B_{2;3} & \mathbf{B_{2;4}} & B_{2;5} & B_{2,6} \\ 0 & B_{3;4} & \mathbf{B_{3;5}} & \mathbf{B_{3,6}} \\ 0 & 0 & B_{4;5} & B_{4,6} \end{pmatrix}. \tag{3.3.2}$$

In other words,

$$J^2 = J^2(\underline{t}_4;\sigma) = U^{(2)}_{0,1} B_{1,3} U^{(3)}_{1,2} B_{2,4} U^{(4)}_{2,3} B_{3,5} U^{(5)}_{3,4} B_{3,6} (|\phi\rangle\langle\phi|)^{\otimes 6}. \tag{3.3.3}$$

To this end, in §3.1-3.2, we organize the terms in $J^2(\underline{t}_4,\sigma)$. Then, in §3.3, we estimate the example by the trilinear estimates (Lemma 3.2.5).

### 3.3.1 Factorization of $J^2$

We will decompose $J^2$ into two one-particle density matrices by examining the effect of the contraction operators starting with the last one on the RHS of (3.3.3). We denote each factor in the last term $(|\phi\rangle\langle\phi|)^{\otimes 6}$ by $u_i$, ordered by increasing index $i$, so that $(|\phi\rangle\langle\phi|)^{\otimes 6} = \otimes_{i=1}^{6} u_i$.

First of all, in (3.3.3), the last interaction operator $B_{3,6}$ contracts the factor $u_3$ and $u_6$, and leaves all other factors unchanged,

$$B_{3,6}(\otimes_{i=1}^{6} u_i) = u_1 \otimes u_2 \otimes \Theta_4 \otimes u_4 \otimes u_5. \tag{3.3.4}$$

where

$$\Theta_4 := B_{1,2}(u_3 \otimes u_6).$$

The index $\alpha$ in $\Theta_\alpha$ associates $\Theta_\alpha$ to the $\alpha$-th interaction operator from the left in (3.3.3). Since we only run the expansion to the $n$-th level, we have $1 \leq \alpha \leq n$. In this specific case, $n = 4$, the 4th interaction operator is $B_{3,6}$.

Next, $B_{3,5}$ contracts $U_{3,4}^{(1)}\Theta_4$ and $U_{3,4}^{(1)}u_5$,

$$B_{3,5}U_{3,4}^{(5)}((3.3.4)) = (U_{3,4}^{(2)}(u_1 \otimes u_2)) \otimes \Theta_3 \otimes (U_{3,4}^{(1)}u_4), \tag{3.3.5}$$

where

$$\Theta_3 := B_{1,2}((U_{3,4}^{(1)}\Theta_4) \otimes (U_{3,4}^{(1)}u_5)).$$

Then, by the semigroup property, $U_{2,3}^{(i)}U_{3,4}^{(i)} = U_{2,4}^{(i)}$. The operator $B_{2,4}$ contracts $U_{2,4}^{(1)}u_2$ with $U_{2,4}^{(1)}u_4$, which correspond to the 2nd and 5th factors in (3.3.5). The other factors are left invariant.

$$B_{2,4}U_{2,3}^{(4)}((3.3.5)) = (U_{2,4}^{(1)}u_1) \otimes \Theta_2 \otimes (U_{2,3}^{(1)}\Theta_3), \tag{3.3.6}$$

where

$$\Theta_2 = B_{1,2}(U_{2,4}^{(2)}(u_2 \otimes u_4)).$$

Finally, $B_{1,3}$ contracts $(U_{1,4}^{(1)}u_1)$ and $(U_{1,3}^{(1)}\Theta_3)$ and leaves other factors unchanged.

$$B_{1,3}U_{1,2}^{(3)}((3.3.6)) = \Theta_1 \otimes (U_{1,2}^{(1)}\Theta_2), \tag{3.3.7}$$

where

$$\Theta_1 = B_{1,2}((U^{(1)}_{1,4}u_1) \otimes (U^{(1)}_{1,3}\Theta_3)).$$

Therefore, $J^2$ can be factorized as

$$J^2 = (U^{(1)}_{0,1}\Theta_1) \otimes (U^{(1)}_{0,2}\Theta_2) := J^1_1 \otimes J^1_2. \tag{3.3.8}$$

In the above expression we may write the factors $J^1_j$ (for $j \leq k = 2$) as one-particle matrices and substitute with $u_i = |\phi\rangle \langle\phi|$, for $i \leq k + n = 6$. Thus, it follows that

$$J^1_1 = U^{(1)}_{0,1}B_{1,2}U^{(2)}_{1,3}B_{2,3}U^{(3)}_{3,4}B_{2,4}(|\phi\rangle \langle\phi|)^{\otimes 4} \tag{3.3.9}$$

where we relabel the index in operators $B_{\sigma_1(r),r}$ such that the interaction operators in (3.3.9) correspond to $B_{1,3}, B_{3,5}, B_{3,6}$ respectively, and most importantly keep the connectivity structure between them. The relabeling function $\sigma_1$ (see the notation in (3.2.12)) take values: $\sigma_1(2) = 1, \sigma_1(3) = 2, \sigma_1(4) = 3$. Moreover, for $j = 1$, we perform the relabeling in the same spirit find that

$$J^1_2 = U^{(1)}_{0,2}B_{1,2}U^{(2)}_{2,4}(|\phi\rangle \langle\phi|)^{\otimes 2} \tag{3.3.10}$$

where $\sigma_2(2) = 1$.

We note that for any $l < l'$, the interaction operators $B_{\sigma(l),l}$ and $B_{\sigma(l'),l'}$ in $J^2$ (associated to the matrix (3.3.2)) belong to the same factor $J^1_j$ if either $\sigma(l) = \sigma(l')$ or $\sigma(l') = l$. In such cases, we consider them as being *connected*. This connectivity structure is exactly the key point of the Duhamel terms that we want to illustrate using binary tree graphs. Each $\sigma_j$ can be viewed as the restriction of $\sigma$ to $J^1_j$. We call factors that have a free propagator applied to each $\phi$ (like $J_2^1$) *regular* and factors that involve the contractions of $(|\phi\rangle\langle\phi|)^{\otimes 2}$ without free propagator in between (like $J_1^1$) *distinguished*.

### 3.3.2 Recursive determination of contraction structure

Next, repeating the argument in §3.3.1, we express the kernel of each factor explicitly.

Consider the distinguished factor $J_1^1$. For $\alpha = 1, 2, 3$, we denote by $\Theta_\alpha$ the kernel obtained after contracting a two particle density matrix to a one particle matrix via the interaction operator. We will determine $\Theta_\alpha$ recursively in the normal form

$$\Theta_\alpha(x, x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha} \psi^\alpha_{\beta_\alpha}(x) \overline{\chi^\alpha_{\beta_\alpha}}(x'),\ c^\alpha_{\beta_\alpha} = \pm 1 \tag{3.3.11}$$

from the last interaction operator. First, contracting variables by $B_{2,4}$, we get

$$B_{2,4}(|\phi\rangle\langle\phi|)^{\otimes 4} = (|\phi\rangle\langle\phi|) \otimes \Theta_3 \otimes (|\phi\rangle\langle\phi|) \tag{3.3.12}$$

with

$$\Theta_3(x, x') = |\phi|^2\phi(x)\overline{\phi}(x') - \phi(x)\overline{|\phi|^2\phi}(x') = \sum_{\beta_3=1}^{2} c^3_{\beta_3}\psi^3_{\beta_3}(x)\overline{\chi^3_{\beta_3}}(x').$$

Next, contracting variables by $B_{2,3}$,

$$B_{2,3}U^{(3)}_{3,4}(3.3.12) = (|U_{3,4}\phi\rangle\langle U_{3,4}\phi|) \otimes \Theta_2, \tag{3.3.13}$$

where $U_{i,j} := e^{i(t_i - t_j)\Delta}$ and

$$\Theta_2(x,x') = \sum_{\beta_3=1}^{2} c^3_{\beta_3}\Big(U_{3,4}\psi^3_{\beta_3}|U_{3,4}\phi|^2\Big)(x)\overline{U_{3,4}\chi^3_{\beta_3}}(x') - c^3_{\beta_3}U_{3,4}\psi^3_{\beta_3}(x)\Big(\overline{U_{3,4}\psi^3_{\beta_3}}|U_{3,4}\chi|^2\Big)(x')$$
$$=: \sum_{\beta_2=1}^{4} c^2_{\beta_2}\psi^2_{\beta_2}(x)\overline{\chi^2_{\beta_2}}(x').$$

Finally, by the first interaction operator $B_{1,2}$,

$$B_{1,2}U^{(2)}_{1,3}(3.3.13) = B_{1,2}\Big(|U_{1,4}\phi\rangle\langle U_{1,4}\phi| \otimes \sum_{\beta_2=1}^{4} c^2_{\beta_2}|U_{1,3}\psi^2_{\beta_2}\rangle\langle U_{1,3}\chi^2_{\beta_2}|\Big) = \Theta_1,$$

where $\Theta_1(x,x')$ is given by

$$\sum_{\beta_2=1}^{4} c^2_{\beta_2}\Big(U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}\overline{U_{1,3}\chi^2_{\beta_2}}\Big)(x)\overline{U_{1,4}\phi}(x') - c^2_{\beta_2}U_{1,4}\phi(x)\Big(\overline{U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}}U_{1,3}\chi^2_{\beta_2}\Big)(x')$$
$$=: \sum_{\beta_1=1}^{8} c^1_{\beta_1}\psi^1_{\beta_1}(x)\overline{\chi^1_{\beta_1}}(x').$$

Therefore, $J^1_1$ can be represented by

$$J^1_1(x,x') = U^{(1)}_{0,1}\Theta_1(x,x') = \sum_{\beta_1=1}^{8} c^1_{\beta_1}U_{0,1}\psi^1_{\beta_1}(x)\overline{U_{0,1}\chi^1_{\beta_1}}(x'),$$

Similarly, we write the regular factor $J^1_2$ as

$$J^1_2(\sigma_2;t_2,t_4) = U^{(1)}_{0,1}\tilde{\Theta}_1(x,x') = \sum_{\tilde{\beta}_1=1}^{2} \tilde{c}^1_{\tilde{\beta}_1}U_{0,1}\tilde{\psi}^1_{\tilde{\beta}_1}(x)\overline{U_{0,1}\tilde{\chi}^1_{\tilde{\beta}_1}}(x'),$$

where

$$\tilde{\Theta}_1(x,x') = (|U_{2,4}\phi|^2U_{2,4}\phi)(x)\overline{U_{2,4}\phi}(x') - U_{2,4}\phi(x)(|U_{2,4}\phi|^2\overline{U_{2,4}\phi})(x')$$
$$=: \sum_{\tilde{\beta}_1=1}^{2} \tilde{c}^1_{\tilde{\beta}_1}\tilde{\psi}^1_{\tilde{\beta}_1}(x)\overline{\tilde{\chi}^1_{\tilde{\beta}_1}}(x').$$

### 3.3.3 Recursive estimates

Now, we estimate the example (3.3.1) using the structural properties obtained from the previous two subsections. The key tool is the trilinear estimates (Lemma 3.2.5).

Observe that in the example (3.3.1), the distinguished factor $J_1^1$ is independent of $t_2$, and the regular factor $J_2^1$ depends only on $t_2$ and $t_4$ (see (3.3.9) and (3.3.10)). Thus, (3.3.1) can be factored as

$$(3.3.1) = \Big(\int_{[0,T)^2} dt_1 dt_3 \mathrm{Tr}(|S^{(1,-d)} J_1^1|)\Big)\Big(\int_0^T dt_2 \mathrm{Tr}(|S^{(1,-d)} J_2^1|)\Big). \quad (3.3.14)$$

We estimate these two factors separately.

#### 3.3.3.1 Distinguished factor

By §3.3.1 and §3.3.2, we have

$$\int_{[0,T)^2} dt_1 dt_3 \mathrm{Tr}(|S^{(1,-d)} J_1^1|) \leq \sum_{\beta_1=1}^{8} \int_{[0,T)^2} dt_1 dt_3 \|\psi^1_{\beta_1}\|_{H^{-d}} \|\chi^1_{\beta_1}\|_{H^{-d}}, \quad (3.3.15)$$

where for each $\beta_\alpha$, only one out of two terms $\psi^\alpha_{\beta_\alpha}$ and $\chi^\alpha_{\beta_\alpha}$ is cubic. Among the eight integrals on the right hand side of (3.3.15), we estimate the following two cases.

**Case 1.** Consider the integral whose $\psi^\alpha_{\beta_\alpha}$'s are all cubic, precisely

$$\begin{aligned} \psi^1_{\beta_1} &= U_{1,4}\phi U_{1,3}\psi^2_{\beta_2}\overline{U_{1,3}\chi^2_{\beta_2}}, & \chi^1_{\beta_1} &= U_{1,4}\phi, \\ \psi^2_{\beta_2} &= U_{3,4}\psi^3_{\beta_3}|U_{3,4}\phi|^2, & \chi^2_{\beta_2} &= U_{3,4}\chi^3_{\beta_3}, \\ \psi^3_{\beta_3} &= |\phi|^2\phi, & \chi^3_{\beta_3} &= \phi. \end{aligned} \quad (3.3.16)$$

We apply the trilinear estimates (3.2.19) recursively keeping the $W^{-s_c+\frac{\epsilon}{2},r_\epsilon}$ norm on $\psi^{\alpha}_{\beta_\alpha}$. Then, we obtain that

$$\begin{aligned}
\int_{[0,T)^2} dt_1 dt_3 \|\psi^1_{\beta_1}\|_{H^{-d}} \|\chi^1_{\beta_1}\|_{H^{-d}} &\lesssim \int_{[0,T)^2} dt_1 dt_3 \|\psi^1_{\beta_1}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\chi^1_{\beta_1}\|_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq)} \\
&= \int_{[0,T)^2} dt_1 dt_3 \|U_{1,4}\phi U_{1,3}\psi^2_{\beta_2} \overline{U_{1,3}\chi^2_{\beta_2}}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|_{H^{s_\epsilon}} \\
&\le C_0 T^\epsilon \int_0^T dt_3 \|\psi^2_{\beta_2}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\chi^2_{\beta_2}\|_{H^{s_\epsilon}} \|\phi\|^2_{H^{s_\epsilon}} \quad \text{(by (3.2.19))} \\
&= C_0 T^\epsilon \int_0^T dt_3 \|U_{3,4}\psi^3_{\beta_3} |U_{3,4}\phi|^2\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|^3_{H^{s_\epsilon}} \\
&\le (C_0 T^\epsilon)^2 \|\psi^3_{\beta_3}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|^5_{H^{s_\epsilon}} \quad \text{(by (3.2.19))} \\
&= (C_0 T^\epsilon)^2 \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|^5_{H^{s_\epsilon}} \\
&\lesssim (C_0 T^\epsilon)^2 \|\phi\|^8_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq).}
\end{aligned}$$

**Case 2.** Consider the integral whose $\psi^{\alpha}_{\beta_\alpha}$'s are all linear except the last one, that is,

$$\begin{aligned}
&\psi^1_{\beta_1} = U_{1,3}\psi^2_{\beta_2}, \quad \chi^1_{\beta_1} = U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2, \\
&\psi^2_{\beta_2} = U_{3,4}\psi^3_{\beta_3}, \quad \chi^2_{\beta_2} = U_{3,4}\chi^3_{\beta_3}|U_{3,4}\phi|^2, \\
&\psi^3_{\beta_3} = |\phi|^2\phi, \qquad \chi^3_{\beta_3} = \phi.
\end{aligned} \tag{3.3.17}$$

In this case, we first combine linear propagators acting on $\psi^3_{\beta_3}$ so that

$$\psi^1_{\beta_1} = U_{1,3}U_{3,4}(|\phi|^2\phi) = U_{1,4}(|\phi|^2\phi).$$

Then, applying the trilinear estimate (3.2.20) twice, we obtain

$$\begin{aligned}\int_{[0,T)^2} dt_1 dt_3 \|\psi^1_{\beta_1}\|_{H^{-d}} \|\chi^1_{\beta_1}\|_{H^{-d}} &\lesssim \int_{[0,T)^2} dt_1 dt_3 \|U_{1,4}(|\phi|^2\phi)\|_{H^{-d}} \|U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2\|_{H^{s_\epsilon}} \\ &= \int_{[0,T)^2} dt_1 dt_3 \||\phi|^2\phi\|_{H^{-d}} \|U_{1,3}\chi^2_{\beta_2}|U_{1,4}\phi|^2\|_{H^{s_\epsilon}} \\ &\leq C_0 T^\epsilon \int_0^T dt_3 \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\chi^2_{\beta_2}\|_{H^{s_\epsilon}} \|\phi\|^2_{H^{s_\epsilon}} \quad \text{(by (3.2.20))} \\ &\leq (C_0 T^\epsilon)^2 \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|\phi\|^5_{H^{s_\epsilon}} \quad \text{(by (3.2.20))} \\ &\lesssim (C_0 T^\epsilon)^2 \|\phi\|^8_{H^{s_\epsilon}} \quad \text{(by Sobolev ineq)},\end{aligned}$$

which is the same bound as in Example 1.

Similarly, one can show that the other six integrals satisfy the same bound. Then, it follows that

$$\int_{[0,T)^2} dt_1 dt_3 \mathrm{Tr}(|S^{(1,-d)} J^1_1|) \lesssim 8(C_0 T^\epsilon)^2 \|\phi\|^8_{H^{s_\epsilon}}.$$

#### 3.3.3.2 Regular factor

For the regular factor, we have

$$\int_0^T dt_2 \mathrm{Tr}(|S^{(1,-d)} J^1_2|) \leq \sum_{\tilde{\beta}_1=1}^{2} \int_0^T dt_2 \|\tilde{\psi}^1_{\tilde{\beta}_1}\|_{H^{-d}} \|\tilde{\chi}^1_{\tilde{\beta}_1}\|_{H^{-d}}, \tag{3.3.18}$$

where for each $\tilde{\beta}_1$, only one out of two terms $\tilde{\psi}^1_{\tilde{\beta}_1}$ and $\tilde{\chi}^1_{\tilde{\beta}_1}$ is cubic. For instance, when $\tilde{\psi}^1_{\tilde{\beta}_1} = |U_{2,4}\phi|^2 U_{2,4}\phi$ and $\tilde{\chi}^1_{\tilde{\beta}_1} = U_{2,4}\phi$, it follows from the trilinear estimate (3.2.20) that

$$\int_0^T dt_2 \|\tilde{\psi}^1_{\tilde{\beta}_1}\|_{H^{-d}} \|\tilde{\chi}^1_{\tilde{\beta}_1}\|_{H^{-d}} \leq \int_0^T dt_2 \||U_{2,4}\phi|^2 U_{2,4}\phi\|_{H^{s_\epsilon}} \|U_{2,4}\phi\|_{H^{s_\epsilon}} \leq C_0 T^\epsilon \|\phi\|^4_{H^{s_\epsilon}}.$$

Similarly, one can also show that the other integral satisfies the same bound. Therefore, we get

$$\int_0^T dt_2 \mathrm{Tr}(|S^{(1,-d)} J_2^1|) \le 2C_0 T^\epsilon \|\phi\|_{H^{s_\epsilon}}^4$$

#### 3.3.3.3 Conclusion

Going back to (3.14)), we conclude that

$$(3.3.1) \lesssim 2^4 \cdot (C_0 T^\epsilon)^3 \|\phi\|_{H^{s_\epsilon}}^{12}.$$

## 3.4 Binary tree graphs for the general case

In order to prove Lemma 3.2.4 in the general case, we proceed as in [17], and use binary tree graphs. These graphs will help us keep track of the contraction operations applied iteratively in the Duhamel expansion (3.2.11).

### 3.4.1 The binary tree graphs

We begin by recalling that, by (3.2.12), $J^k$ is given by

$$J^k(\underline{t}_n;\sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1} U_{1,2}^{(k+1)} B_{\sigma(k+2);k+2} \cdots U_{n-1,n}^{(k+n-1)} B_{\sigma(k+n);k+n} (|\phi\rangle\langle\phi|)^{\otimes(k+n)},$$

where

$$(|\phi\rangle\langle\phi|)^{\otimes(k+n)}(\underline{x}_{k+n};\underline{x}'_{k+n}) = \prod_{i=1}^{k+n} (|\phi\rangle\langle\phi|)(x_i;x'_i)$$

is a product of one-particle kernels. Since the free evolution operators $U$ and the contraction operators $B$ preserve the product structure, it follows that we can also decompose

$$J^k(t, t_1, \ldots, t_r; \sigma; \underline{x}_k; \underline{x}'_k) = \prod_{j=1}^{k} J^1_j(t, t_{\ell_{j,1}}, \ldots, t_{l_{j,m_j}}; \sigma_j; x_j; x'_j) \tag{3.4.1}$$

into a product of one-particle kernels $J^1_j$. We associate to this decomposition $k$ disjoint binary tree graphs $\tau_1, \tau_2, \ldots, \tau_k$. These graphs appear as *skeleton graphs* in [42–45]. As in [17], we assign *root, internal,* and *leaf* vertices to for each tree $\tau_j$.

- A *root* vertex labeled as $W_j$, $j = 1, 2, \cdots, k$, to represent $J^1_j(x_j, x'_j)$.
- An *internal* vertex labeled by $v_l$, $l = 1, 2, \cdots, n$, corresponding to $B_{\sigma(k+l),k+l}$ and attached to the time variable $t_l$.
- A *leaf* vertex $u_i$, $i = 1, 2, \cdots, k+n$, representing each factor $(|\phi\rangle\langle\phi|)(x_i, x'_i)$.

Next, we connect the vertices with *edges*, as described below.

- If $v_l$ is the smallest value of $l$ such that $\sigma(k+l) = j$, then we connect $v_l$ to the root vertex $W_j$ and write $W_j \sim v_l$ (or equivalently $W_j \sim B_{\sigma(k+l),k+l}$). If there is no internal vertex connected to a root vertex $W_j$, then we connect $W_j$ to the leaf $u_j$, and write $W_j \sim u_j$.
- For any $1 < l \leq n$, if $\exists l' > l$ such that $\sigma(k+l) = \sigma(k+l')$ or $\sigma(k+l') = k+l$, then we connect $v_l$ and $v_{l'}$ and write $v_l \sim v_{l'}$ (or equivalently $B_{\sigma(k+l),k+l} \sim B_{\sigma(k+l'),k+l'}$). In this case, we call $v_l$ the *parent vertex* of $v_{l'}$, and $v_{l'}$ the *child vertex* of $v_l$. We denote the two child vertices of $v_l$ by $v_{k_-(l)}$ and $v_{k_+(l)}$, with $k_-(l) < k_+(l)$.

- When there is no internal vertex with $r' > r$ and $k+l = \sigma(k+l')$, we connect $v_l$ to the leaf vertex $u_{k+l}$ and write $v_l \sim u_{k+l}$ (or equivalently $B_{\sigma(k+l),k+l} \sim u_{k+l}$). If there is no internal vertex with $l' > l$ and $\sigma(k+l) = \sigma(k+l')$, then we connect $v_l$ to the leaf vertex $u_{\sigma(k+l)}$ and write $v_l \sim u_{\sigma(k+l)}$ (or equivalently $B_{\sigma(k+l),k+l} \sim u_{\sigma(k+l)}$).

We remark that it follows from the construction above that each root vertex has only one child vertex, and each internal vertex has exactly two child vertices (which can be internal and leaf). We call the tree $\tau_j$ *distinguished* if $v_n \in \tau_j$, and *regular* if $v_n \notin \tau_j$. The two leaves connected to $v_n$ are called *distinguished leaf vertices*, and all other leaves are called *regular leaf vertices*. Clearly, there are $k-1$ regular trees and one distinguished tree in each binary tree graph.

A sample binary tree graph is given in Figure 3.1, for $J^k$ as in (3.3.3). Each tree $\tau_j$ has root vertex $W_j$, for $j = 1, 2$. The two leaf vertices $u_3$ and $u_6$ and the internal vertex $v_4$ (or $B_{3,6}$) are distinguished. $\tau_1$ is the distinguished tree, and is drawn with thick edges.

### 3.4.2 The distinguished one particle kernel $J_j^1$

Let $\tau_j$ denote the distinguished tree graph. It has $m_j$ internal vertices $(v_{\ell_j,\alpha})_{\alpha=1}^{m_j}$ and $m_j+1$ leaf vertices $(u_{j,i})_{i=1}^{m_j+1}$. We enumerate the internal vertices with $\alpha \in \{1, \dots, m_j\}$ and the leaf vertices with $\alpha \in \{m_j+1, \dots, 2m_j+2\}$. To simplify notation, we refer to the vertex $v_{j,\alpha}$ by its label $\alpha$. We observe that

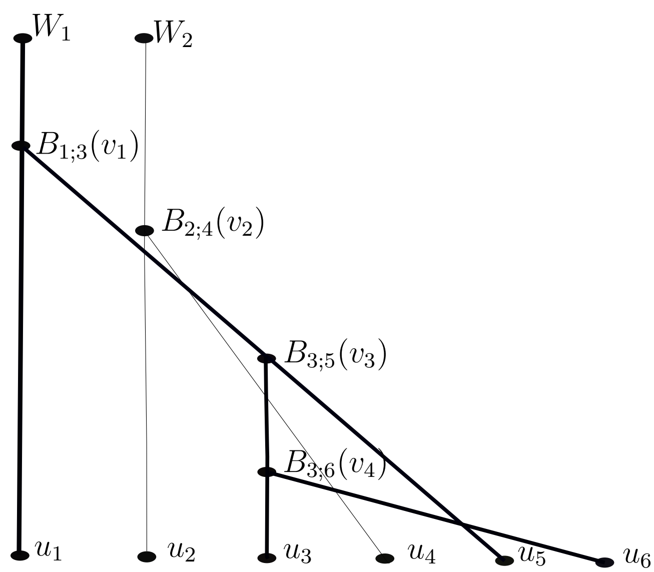


Figure 3.1: An example binary tree graphs of $J^k$. It is a disjoint union of two trees $\tau_1$ and $\tau_2$ with root vertices $W_1$ and $W_2$, respectively. Each tree corresponds to a one-particle kernel in the example in section 3.3, where $k = 2$ and $n = 4$.

$J_j^1$ has the form

$$J_j^1(t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; \sigma_j) \tag{3.4.2}$$
$$= U^{(1)}(t - t_{\ell_{j,1}}) \cdots U^{(1)}(t_{\ell_{j,1}-1} - t_{\ell_{j,1}}) B_{\sigma_j(2),2} \cdots$$
$$\cdots B_{\sigma_j(\alpha),\alpha} U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha-1}+1}) \cdots U^{(\alpha)}(t_{\ell_{j,\alpha}-1} - t_{\ell_{j,\alpha}}) B_{\sigma_j(\alpha+1),\alpha+1} \cdots$$
$$\cdots U^{(m_j)}(t_{\ell_{j,m_j}-1} - t_{l_j,m_j}) B_{\sigma_j(m_j+1),m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)}.$$

By the group property

$$U^{(\alpha)}(t)U^{(\alpha)}(s) = U^{(\alpha)}(t+s),$$

and the fact that $\sigma_j(2) = 1$, (3.4.2) reduces to

$$J_j^1(t, t_{\ell_{j,1}}, \dots, t_{\ell_{j,m_j}}; \sigma_j) \tag{3.4.3}$$
$$= U^{(1)}(t - t_{\ell_{j,1}}) B_{1,2} \cdots$$
$$\cdots B_{\sigma_j(\alpha),\alpha} U^{(\alpha)}(t_{\ell_{j,\alpha-1}} - t_{\ell_{j,\alpha}}) B_{\sigma_j(\alpha+1),\alpha+1} \cdots$$
$$\cdots U^{(m_j)}(t_{\ell_{j,m_j-1}} - t_{l_j,m_j}) B_{\sigma_j(m_j+1),m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)},$$

where $\ell_{j,m_j} = r$.

### 3.4.3 Definition of the kernels $\Theta_\alpha$ at the vertices of the distinguished tree graph

In this section, we proceed as in [17], and recursively assign a kernel $\Theta_\alpha$ to each vertex $\alpha$ of the distinguished tree graph. The kernels at the vertices of the regular tree graph are defined similarly. We begin by assigning the kernel

$$\Theta_\alpha(x; x') := \phi(x)\overline{\phi(x')}$$

to the leave vertex with label $\alpha \in \{m_j+1, \dots, 2m+j+2\}$ (corresponding to $u_{j,\alpha-m_j}$).

Next, we determine $\Theta_{m_j}$ at the distinguished vertex $\alpha = m_j$ from the term on the last line of (3.4.3), given by

$$B_{\sigma_j(m_j+1),m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(m_j+1)} = (|\phi\rangle\langle\phi|)^{\otimes(\sigma_j(m_j+1)-1)} \otimes \Theta_{m_j} \\ \otimes (|\phi\rangle\langle\phi|)^{\otimes(m_j+1-\sigma_j(m_j+1)-1)}$$

where

$$\Theta_{m_j}(x;x') := \tilde{\psi}(x)\overline{\phi(x')} - \phi(x)\overline{\tilde{\psi}(x')} \tag{3.4.4}$$

with $\tilde{\psi} := |\phi|^2\phi$. It is obtained from contracting two copies of $|\phi\rangle\langle\phi|$ at the two leaf vertices $\kappa_-(m_j), \kappa_+(m_j)$ which have $m_j$ as their parent vertex.

Now we are ready to begin the induction. Let $\alpha \in \{1, \dots, m_j-1\}$. Suppose that the kernels $\Theta_{\alpha'}$ have been determined for all $\alpha' > \alpha$. We let $\kappa_-(\alpha), \kappa_+(\alpha)$ label the two child vertices (of internal or leaf type) of $\alpha$,

$$\sigma_j(\alpha) = \sigma_j(\kappa_-(\alpha)) \quad , \quad \alpha = \sigma_j(\kappa_+(\alpha)).$$

Since $\Theta_{\kappa_-(\alpha)}$ and $\Theta_{\kappa_+(\alpha)}$ have already been determined, we can now define

$$\begin{aligned}
&\Theta_\alpha(x;x') \\
&= B_{1,2}((U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)}) \otimes (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)}\Theta_{\kappa_+(\alpha)}))(x;x') \\
&= (U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{k_-(\alpha)})(x;x')[(U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})(x;x) \\
&\qquad - (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)})(x';x')].
\end{aligned}$$

The induction ends when we obtain the kernel $\Theta_1$ at $\alpha = 1$.

### 3.4.4 Key properties of the kernels $\Theta_\alpha$

As in [17], we observe that the kernels $\Theta_\alpha$ satisfy the following properties.

- $\Theta_\alpha$ can be written as a sum of differences of factorized kernels

$$\Theta_\alpha(x;x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha} \chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')} \tag{3.4.5}$$

  with at most $2^{m_j-\alpha}$ nonzero coefficients $c^\alpha_{\beta_\alpha} \in \{1,-1\}$.

- The product $\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')}$ in (3.4.5) above is either of the form

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')} = {} & (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)\overline{(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})}(x') \\ & (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x)\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}(x)\end{aligned} \tag{3.4.6}$$

  or

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}(x')} = {} & (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)\overline{(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})}(x') \\ & (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x')\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}(x')\end{aligned} \tag{3.4.7}$$

  for some values of $\beta_{\kappa_-(\alpha)}, \beta_{\kappa_+(\alpha)}$ that depend on $\beta_\alpha$. Observe that above, the function $\chi^\alpha_{\beta_\alpha}$ is either of the cubic form

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x) = {} & (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x) \\ & (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})(x)\overline{(U_{\alpha;\kappa_+}(\alpha)\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+}})}(x)\end{aligned} \tag{3.4.8}$$

  or the linear form

$$\chi^\alpha_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x). \tag{3.4.9}$$

Accordingly, $\psi^{\alpha}_{\beta_\alpha}$ respectively is either of linear or cubic form, and the product $\chi^{\alpha}_{\beta_\alpha}(x)\overline{\psi^{\alpha}_{\beta_\alpha}(x')}$ always has quartic form (3.4.6) or (3.4.7).

- We call the functions $\chi^{\alpha}_{\beta_\alpha}, \psi^{\alpha}_{\beta_\alpha}$ in the sum (3.4.5) *distinguished* if they are a function of $|\phi|^2\phi$. In the product on the right hand side of (3.4.6), respectively (3.4.7), at most one of the four factors is distinguished. Indeed, this is true for all regular leaf vertices, and for the distinguished vertex (3.4.4). By induction along decreasing values of $\alpha$, it is also true for the internal vertices.

## 3.5 Proof of Lemma 3.2.4

In this section, we prove Lemma 3.2.4. We begin by considering the contribution of each factor $J^1_j$ on the right hand side of (3.4.1) separately. One of these factors is distinguished, and will be dealt with in Proposition 3.5.1 below. Proposition 3.5.2 will be for the regular factors.

We note that the analog of Proposition 3.5.1 in [17] has a shorter proof. This is because, where the authors of [17] work in $L^2$, we work in $W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}$ to achieve lower regularity. In $W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}$, the linear propagators $e^{it\Delta}$ are no longer isometries, and so we have to carefully rearrange them so that they do not interfere with our proof. This occurs in case 2 of our proof of Lemma 3.5.2.

We begin with Proposition 3.5.1, which addresses the contribution of the distinguished factor $J^1_j$. We prove Proposition 3.5.1 by induction. Lemma 3.5.1 will serve as our first induction step, and Lemma 3.5.2 will serve as the remainder of our proof by induction.

**Proposition 3.5.1.** *Let $d \geq 3$. Then, for the distinguished tree $\tau_j$, we have the bound*

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left( \left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\ \leq 2^{m_j-1} C^{m_j-1} T^{\epsilon(m_j-1)} \|\phi\|_{H^{s_\epsilon}}^{2m_j-1} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}. \quad (3.5.1)$$

*Similarly, when $d = 2$, we have the bound*

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left( \left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\ \leq 2^{m_j-1} C^{m_j-1} T^{\frac{1}{3}(m_j-1)} \|\phi\|_{H^{1/3}}^{2m_j-1} \||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}}, \quad (3.5.2)$$

*and, when $d = 1$, we have the bound*

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left( \left| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\ \leq 2^{m_j-1} C^{m_j-1} T^{\frac{1}{2}(m_j-1)} \|\phi\|_{L^2}^{2m_j-1} \||\phi|^2\phi\|_{L^1}. \quad (3.5.3)$$

*Proof.* For $d \geq 3$, Proposition 3.5.1 follows immediately from Lemma 3.5.2 below. Indeed, in the statement of Lemma 3.5.2, there are at most $2^{m_j-1}$ terms in the sum over $\beta_1$.

Observe that in the proofs of Lemmas 3.5.1 and 3.5.2, we use the bounds for $d \geq 3$ presented in Lemma 3.2.5. The proof of Proposition 3.5.1 for $d = 1, 2$ is analogous (we use the corresponding bounds for $d = 1, 2$ presented in Lemma 3.2.5). □

We now prove Lemma 3.5.1, which will serve as the first induction step in our proof of Lemma 3.5.1.

**Lemma 3.5.1.** *Let $d \geq 3$. Then, the distinguished factor*

$$J_j^1(\underline{t}_n;\sigma_j;x,x') = U^{(1)}(t-t_1)\sum_{\beta_1} c^1_{\beta_1}\psi^1_{\beta_1}(x)\chi^1_{\beta_1}(x')$$

*satisfies the following. For each value of $\beta_1$, either there exits a non-negative integer $\ell < m_j - 1$ such that*

$$\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\Big( \Big| S^{(1,-d)} U^{(1)}(t-t_1) c^1_{\beta_1} |\psi^1_{\beta_1}\rangle\langle\chi^1_{\beta_1}| \Big| \Big) \\
&\leq (CT^\epsilon)^\ell \sum_{\beta_1} \int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1}\cdots dt_{m_j-1} \\
&\qquad \|(U_{\ell+2}f^1_{\ell+2})(U_{\ell+2}f^2_{\ell+2})(U_{\ell+2}f^3_{\ell+2})\|_{W^{-s_c+\frac{\epsilon}{2},r_\epsilon}} \|U_{\ell+2}f^2_{\ell+2}\|_{H^{s_\epsilon}} \cdots \|U_{\ell+2}f^{2\ell+4}_{\ell+2}\|_{H^{s_\epsilon}},
\end{aligned} \tag{3.5.4}$$

*where the functions $f$ are defined in terms of the functions $\psi^\alpha_{\beta_\alpha}$ and $\chi^\alpha_{\beta_\alpha}$ as described in the proof below, or*

$$\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\Big( \Big| S^{(1,-d)} U^{(1)}(t-t_1) c^1_{\beta_1} |\psi^1_{\beta_1}\rangle\langle\chi^1_{\beta_1}| \Big| \Big) \\
&\leq C^{m_j-1} T^{\epsilon(m_j-1)} \|\phi\|^{2m_j-1}_{H^{s_\epsilon}} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}.
\end{aligned} \tag{3.5.5}$$

*Moreover, $f^1_{\ell+2}$ is the only distinguished fuction on the right hand side of* (3.5.4).

*Proof.* We recall that $U_{i,j} := e^{i(t_i-t_j)\Delta}$, and let $U_j := U_{j,j+1}$. We have

$$\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\Big( \Big| S^{(1,-d)} U^{(1)}(t-t_1) c^1_{\beta_1} |\psi^1_{\beta_1}\rangle\langle\chi^1_{\beta_1}| \Big| \Big) \\
&\leq \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|\psi^1_{\beta_1}\|_{H^{-d}} \|\chi^1_{\beta_1}\|_{H^{-d}}.
\end{aligned} \tag{3.5.6}$$

Now, we recall from subsection 3.4.4 that one of functions $\psi^1_{\beta_1}, \chi^1_{\beta_1}$ is distinguished. Moreover the distinguished function is either of the cubic form (3.4.8) or of the linear form (3.4.9). We will now label the distinguished function $f^1_1$ and the regular function $f^2_1$.

**Case 1: $f^1_1$ is cubic.** If $f^1_1$ is cubic, then, by (3.4.6) and (3.4.7), $f^1_1$ and $f^2_1$ are of the form

$$\begin{aligned} f^1_1 &= (U_2 f^1_2)(U_2 f^2_2)(U_2 f^3_2), \\ f^2_1 &= U_2 f^4_2. \end{aligned}$$

As in Section 3.3, we apply the $W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}$ norm to the distinguished function $f^1_1$ and the $H^{s_\epsilon}$ norm to the regular function $f^2_1$ and find that

$$\begin{aligned} (3.5.6) &= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f^1_1\|_{H^{-d}} \|f^2_1\|_{H^{-d}} \\ &= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|(U_2 f^1_2)(U_2 f^2_2)(U_2 f^3_2)\|_{H^{-d}} \|U_2 f^4_2\|_{H^{-d}} \\ &\leq C \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|(U_2 f^1_2)(U_2 f^2_2)(U_2 f^3_2)\|_{W^{-(s_c+\frac{\epsilon}{2}), r_\epsilon}} \|U_2 f^4_2\|_{H^{s_\epsilon}}, \end{aligned}$$

which is of the form (3.5.4).

**Case 2: $f^2_1$ is cubic.** In this case, we have that $f^1_1$ and $f^2_1$ are of the form

$$\begin{aligned} f^1_1 &= U_2 f^1_2, \\ f^2_1 &= (U_2 f^2_2)(U_2 f^3_2)(U_2 f^4_2). \end{aligned}$$

Since $f^1_1$ is distinguished, there exists $\ell \geq 1$ such that

$$f^1_2 = U_3 f^1_3, \;\; f^1_3 = U_4 f^1_4, ..., f^1_\ell = U_{\ell+1} f^1_{\ell+1},$$

and

$$f^1_{\ell+1} = (U_{\ell+2} f^1_{\ell+2})(U_{\ell+2} f^2_{\ell+2})(U_{\ell+2} f^3_{\ell+2}) \text{ or } f^1_{\ell+1} = |\phi|^2\phi, \tag{3.5.7}$$

where $f^1_{\ell+2}$ (or $f^2_{\ell+2}$ or $f^3_{\ell+2}$) is a distinguished function. Thus, combining all propagators acting on $f^1_{\ell+1}$, we write

$$f^1_1 = U_{1,\ell+2} f^1_{\ell+1}.$$

Again, we apply the $W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}$ norm to the distinguished function $f^1_1$ and the $H^{s_\epsilon}$ norm to the regular function $f^2_1$ and find that

$$\begin{aligned}
(3.5.6) &= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f^1_1\|_{H^{-d}} \|f^2_1\|_{H^{-d}} \\
&= \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f^1_{\ell+1}\|_{H^{-d}} \|(U_2 f^2_2)(U_2 f^3_2)(U_2 f^4_2)\|_{H^{-d}} \\
&\lesssim \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|f^1_{\ell+1}\|_{W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}} \|(U_2 f^2_2)(U_2 f^3_2)(U_2 f^4_2)\|_{H^{s_\epsilon}}.
\end{aligned} \tag{3.5.8}$$

Since $f_{\ell+1}$ doesn't depend on $t_1, \ldots, t_\ell$, we find that after $\ell$ applications of (3.2.20),

$$(3.5.8) \le (CT^\epsilon)^\ell \int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1} \cdots dt_{m_j-1} \|f^1_{\ell+1}\|_{W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}} \|f^2_{\ell+1}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+4}_{\ell+1}\|_{H^{s_\epsilon}}. \tag{3.5.9}$$

If $f^1_{\ell+1} = |\phi|^2\phi$, then it follows from the binary tree graph structure presented in section 3.4 that $\ell = m_j - 1$ and $f^{\ell''}_{\ell+1} = \phi$ for $\ell'' \ge 2$, and so we have proven (3.5.5). Otherwise, if $f^1_{\ell+1} = (U_{\ell+2} f^1_{\ell+2})(U_{\ell+2} f^2_{\ell+2})(U_{\ell+2} f^3_{\ell+2})$, then we have that

$$
\begin{aligned}
(3.5.9) \leq (CT^{\epsilon})^{\ell} \int_{[0,T)^{m_j-\ell-1}} & dt_{\ell+1} \cdots dt_{m_j-1} \\
& \|(U_{\ell+2} f^1_{\ell+2})(U_{\ell+2} f^2_{\ell+2})(U_{\ell+2} f^3_{\ell+2})\|_{W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}} \|f^2_{\ell+1}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+4}_{\ell+1}\|_{H^{s_\epsilon}} \\
= (CT^{\epsilon})^{\ell} \int_{[0,T)^{m_j-\ell-1}} & dt_{\ell+1} \cdots dt_{m_j-1} \\
& \|(U_{\ell+2} f^1_{\ell+2})(U_{\ell+2} f^2_{\ell+2})(U_{\ell+2} f^3_{\ell+2})\|_{W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}} \|U_{\ell+2} f^2_{\ell+2}\|_{H^{s_\epsilon}} \cdots \|U_{\ell+2} f^{2\ell+4}_{\ell+2}\|_{H^{s_\epsilon}},
\end{aligned}
$$

which is of the form (3.5.4). □

In Lemma 3.5.2, we complete the induction process. Observe that in the proof below, we proceed as in the proof of Lemma 3.5.1. In each induction step, we apply the $W^{s_c+\frac{\epsilon}{2}, r_\epsilon}$ norm to the distinguished function, and the $H^{s_\epsilon}$ norm to the regular functions.

**Lemma 3.5.2.** *Let $d \geq 3$. Then, the distinguished factor*

$$
J^1_j(\underline{t}_n; \sigma_j; x, x') = U^{(1)}(t - t_1) \sum_{\beta_1} c^1_{\beta_1} \psi^1_{\beta_1}(x) \chi^1_{\beta_1}(x')
$$

*satisfies the following. For each value of $\beta_1$,*

$$
\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left( \left| S^{(1,-d)} U^{(1)}(t - t_1) c^1_{\beta_1} |\psi^1_{\beta_1}\rangle\langle\chi^1_{\beta_1}| \right| \right) \\
&\leq C^{m_j-1} T^{\epsilon(m_j-1)} \|\phi\|^{2m_j-1}_{H^{s_\epsilon}} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}), r_\epsilon}}.
\end{aligned} \tag{3.5.10}
$$

*Proof.* By Lemma 3.5.1, we have that for each $\beta_1$, either (3.5.10) holds, or there is a non-negative integer $\ell < m_j - 1$ such that

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left( \left| S^{(1,-d)} U^{(1)}(t-t_1) c^1_{\beta_1} |\psi^1_{\beta_1}\rangle\langle\chi^1_{\beta_1}| \right| \right)$$
$$\leq (CT^\epsilon)^\ell 2^{m_j-1} \int_{[0,T)^{m_j-\ell-1}} dt_{\ell+1} \cdots dt_{m_j-1}$$
$$\|(U_{\ell+2} f^1_{\ell+2})(U_{\ell+2} f^2_{\ell+2})(U_{\ell+2} f^3_{\ell+2})\|_{W^{-s_c+\frac{\epsilon}{2}, r_\epsilon}} \|U_{\ell+2} f^2_{\ell+2}\|_{H^{s_\epsilon}} \cdots \|U_{\ell+2} f^{2\ell+4}_{\ell+2}\|_{H^{s_\epsilon}}, \tag{3.5.11}$$

where $f^1_{\ell+2}$ is the only distinguished function on the right hand side of (3.5.11). We recall from Section 3.4 that $f^1_{\ell+2}$ is either of the cubic form (3.4.8) or the linear for (3.4.9).

Now, we will proceed by induction, and show that in each induction step, we can bound 3.5.11 by an expression of the same form, but with a larger value of $\ell$. In the last induction step, we find that (3.5.16) holds, which completes the proof of (3.5.10). Indeed, this follows from the binary tree graph structure presented in section 3.4.

**Case 1: $f^1_{\ell+2}$ is cubic.** If $f^1_{\ell+2}$ is cubic, then

$$f^1_{\ell+2} = (U_{\ell+3} f^1_{\ell+3})(U_{\ell+3} f^2_{\ell+3})(U_{\ell+3} f^3_{\ell+3}),$$
$$f^2_{\ell+2} = U_{\ell+3} f^4_{\ell+3},\ f^3_{\ell+2} = U_{\ell+3} f^5_{\ell+3}, ...,\ f^{2\ell+4}_{\ell+2} = U_{\ell+3} f^{2\ell+6}_{\ell+3}.$$

Since $f^1_{\ell+2}$ is distinguished, one of $f^1_{\ell+3}, f^2_{\ell+3}, f^3_{\ell+3}$ is distinguished, say $f^1_{\ell+3}$. Then, applying (3.2.19), we get the integral of the form (3.5.11) back:

$$(3.5.11) \lesssim (CT^\epsilon)^{\ell+1} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2} \cdots dt_{m_j-1} \|f^1_{\ell+2}\|_{W^{-(s_c+\frac{\epsilon}{2}), r_\epsilon}} \|f^2_{\ell+2}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+4}_{\ell+2}\|_{H^{s_\epsilon}}.$$
$$= (CT^\epsilon)^{\ell+1} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2} \cdots dt_{m_j-1} \|(U_{\ell+3} f^1_{\ell+3})(U_{\ell+3} f^2_{\ell+3})(U_{\ell+3} f^3_{\ell+3})\|_{W^{-(s_c+\frac{\epsilon}{2}), r_\epsilon}}$$
$$\times \|f^4_{\ell+3}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+6}_{\ell+3}\|_{H^{s_\epsilon}}.$$

**Case 2: $f^2_{\ell+2}$ is cubic.** If $f^1_{\ell+2}$ is cubic, then

$$f^1_{\ell+2} = U_{\ell+3} f^1_{\ell+3},$$
$$f^2_{\ell+2} = (U_{\ell+3} f^2_{\ell+3})(U_{\ell+3} f^3_{\ell+3})(U_{\ell+3} f^4_{\ell+3}),$$
$$f^3_{\ell+2} = U_{\ell+3} f^5_{\ell+3}, \ldots, \; f^{2\ell+4}_{\ell+2} = U_{\ell+3} f^{2\ell+6}_{\ell+3}.$$

Since $f^1_{\ell+2}$ is distinguished, there exists $\ell' \geq 1$ such that

$$f^1_{\ell+3} = U_{\ell+4} f^1_{\ell+4}, f^1_{\ell+4} = U_{\ell+5} f^1_{\ell+5}, \ldots, f^1_{\ell+1+\ell'} = U_{\ell+2+\ell'} f^1_{\ell+2+\ell'},$$

and

$$f^1_{\ell+2+\ell'} = (U_{\ell+3+\ell'} f^1_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^2_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^3_{\ell+3+\ell'}) \text{ or } f^1_{\ell+2+\ell'} = |\phi|^2\phi, \tag{3.5.12}$$

where $f^1_{\ell+3+\ell'}$ is a distinguished function. Thus, combining all linear propagators acting on $f^1_{\ell+2+\ell'}$, we write

$$f^1_{\ell+2} = U_{\ell+2,\ell+3+\ell'} f^1_{\ell+2+\ell'}.$$

Then, applying (3.2.19) and (3.2.20), we obtain

$$\begin{aligned}
(3.5.11) &\leq (CT^\epsilon)^{\ell+1} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2}} dt_{\ell+2} \cdots dt_{m_j-1} \|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+2}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+4}_{\ell+2}\|_{H^{s_\epsilon}} \\
&\leq (CT^\epsilon)^{\ell+2} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-3}} dt_{\ell+3} \cdots dt_{m_j-1} \|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+3}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+6}_{\ell+3}\|_{H^{s_\epsilon}},
\end{aligned} \tag{3.5.13}$$

where, in the second inequality, we applied (3.2.20) to the cubic regular function $f^2_{\ell+2}$. After $\ell' - 1$ applications of (3.2.20), we find that

$$(3.5.13) \leq (CT^\epsilon)^{\ell+1+\ell'} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2-\ell'}} dt_{\ell+2+\ell'} \cdots dt_{m_j-1} \tag{3.5.14}$$

$$\|f^1_{\ell+2+\ell'}\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \|f^2_{\ell+2+\ell'}\|_{H^{s_\epsilon}} \cdots \|f^{2\ell+2\ell'+4}_{\ell+2+\ell'}\|_{H^{s_\epsilon}}. \tag{3.5.15}$$

If

$$f^1_{\ell+2+\ell'} = |\phi|^2\phi, \tag{3.5.16}$$

then it follows from the binary tree graph structure presented in section 3.4 that $\ell+2+\ell' = m_j$ and $f^{\ell''}_{\ell+2+\ell'} = \phi$ for $\ell'' \geq 2$, and so we have completed the proof of (3.5.10). Otherwise, by (3.5.12),

$$\begin{aligned}(3.5.15) = (CT^{\epsilon})^{\ell+1+\ell'} 2^{m_j-1} \int_{[0,T)^{m_j-\ell-2-\ell'}} & dt_{\ell+2+\ell'} \cdots dt_{m_j-1} \\ & \|(U_{\ell+3+\ell'} f^1_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^2_{\ell+3+\ell'})(U_{\ell+3+\ell'} f^3_{\ell+3+\ell'})\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \\ & \times \|U_{\ell+3+\ell'} f^2_{\ell+3+\ell'}\|_{H^{s_\epsilon}} \cdots \|U_{\ell+3+\ell'} f^{2\ell+2\ell'+4}_{\ell+3+\ell'}\|_{H^{s_\epsilon}},\end{aligned}$$

which is of the form (3.5.11).

**Case 3: $f^4_{\ell+2}$ is cubic.** This case can be treated like Case 2. We choose $\ell' \geq 1$ satisfying (3.5.12), and combine linear propagators acting on $f^1_{\ell+2+\ell'}$. Then, we repeat the above procedure to bound (3.5.11) by (3.5.13). □

Next, we consider the contribution of the regular factors $J^1_j$.

**Proposition 3.5.2.** *Let $d \geq 3$. Then, for the regular tree $\tau_j$, we have the bound*

$$\begin{aligned}\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big( \Big| S^{(1,-d)} J^1_j(t, t_1, \cdots, t_{m_j}; \sigma_j) \Big| \Big) \\ \leq 2^{m_j} C^{m_j} T^{\epsilon m_j} \|\phi\|^{2m_j+2}_{H^{s_\epsilon}}.\end{aligned} \tag{3.5.17}$$

*Similarly, when $d = 2$, we have the bound*

$$\begin{aligned}\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big( \Big| S^{(1,-d)} J^1_j(t, t_1, \cdots, t_{m_j}; \sigma_j) \Big| \Big) \\ \leq 2^{m_j} C^{m_j} T^{\frac{1}{3} m_j} \|\phi\|^{2m_j+2}_{H^{1/3}},\end{aligned} \tag{3.5.18}$$

*and, when $d = 1$, we have the bound*

$$\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big( \Big| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \Big| \Big)$$
$$\leq 2^{m_j} C^{m_j} T^{\frac{1}{2} m_j} \|\phi\|_{L^2}^{2m_j+2}. \tag{3.5.19}$$

*Proof.* Again, we consider the case $d \geq 3$, and note that the proof for $d = 1, 2$ is analogous (based on using the bounds for $d = 1, 2$ in Lemma 3.2.5).

We now proceed with the proof for $d \geq 3$.

$$\begin{aligned}
&\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big( \Big| S^{(1,-d)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \Big| \Big) \\
&= \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big( \Big| S^{(1,-d)} U^{(1)}(t - t_1) \Theta_1 \Big| \Big) \\
&\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi_{\beta_1}^1\|_{H^{-d}} \|\chi_{\beta_1}^1\|_{H^{-d}} \\
&\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi_{\beta_1}^1\|_{H^{s_\epsilon}} \|\chi_{\beta_1}^1\|_{H^{s_\epsilon}}
\end{aligned} \tag{3.5.20}$$

By (3.4.6) and (3.4.7), one of $\psi_{\beta_1}^1, \chi_{\beta_1}^1$ is cubic, and the other is linear. We define $f_1^1$ to be the cubic function, and $f_1^2$ to be the linear one. Then, by (3.4.6) and (3.4.7), $f_1^1$ and $f_1^2$ are of the form

$$f_1^1 = (U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3).$$
$$f_1^2 = U_2 f_2^4.$$

By (3.2.20), we have

$$(3.5.20) = \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|(U_2 f_2^1)(U_2 f_2^2)(U_2 f_2^3)\|_{H^{s_\epsilon}} \|U_2 f_2^4\|_{H^{s_\epsilon}} \tag{3.5.21}$$

$$\leq (CT^\epsilon) \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_2 \cdots dt_{m_j} \|f_2^1\|_{H^{s_\epsilon}} \|f_2^2\|_{H^{s_\epsilon}} \|f_2^3\|_{H^{s_\epsilon}} \|f_2^4\|_{H^{s_\epsilon}}. \tag{3.5.22}$$

By construction, only one of the factors $f_2^\ell$ is cubic. Without loss of generality, $f_2^1$ is cubic, and so we have

$$f_2^1 = (U_3 f_3^1)(U_3 f_3^2)(U_3 f_3^3),$$

$$f_2^\ell = U_3 f_3^{\ell+2} \quad \text{for } \ell = 2, 3, 4.$$

Thus,

$$(3.5.22) = (CT^\epsilon) \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_2 \cdots dt_{m_j} \|(U_3 f_3^1)(U_3 f_3^2)(U_3 f_3^3)\|_{H^{s_\epsilon}} \|U_3 f_3^4\|_{H^{s_\epsilon}} \|U_3 f_3^5\|_{H^{s_\epsilon}} \|U_3 f_3^6\|_{H^{s_\epsilon}}$$

which is again of the form (3.5.21). Recall from subsection 3.4.4 that there are at most $2^{m_j}$ terms in the sum over $\beta_1$. Repeating this argument $m_j - 1$ more times yields the desired result (3.5.17). □

Before we proceed with the proof of Lemma 3.2.4, we present a short lemma that we use to bound the term $|\phi|^2\phi$ appearing on the right hand side of (3.5.1).

**Lemma 3.5.3.** *Let $\epsilon > 0$. Then, for $s_c = \frac{d}{2} - 1$, $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$, and $d \geq 3$, we have*

$$\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \lesssim \|\phi\|_{H^{s_\epsilon}}^3. \tag{3.5.23}$$

*Similarly, when $d = 2$, we have*

$$\||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}} \lesssim \|\phi\|^3_{H^{1/3}}. \tag{3.5.24}$$

*Proof.* Let $d \geq 3$. By two applications of the Sobolev inequality, we have

$$\||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} \lesssim \||\phi|^2\phi\|_{L^{\frac{2d}{2d-\epsilon}}} = \|\phi\|^3_{L^{\frac{6d}{2d-\epsilon}}} \lesssim \|\phi\|^3_{H^{\frac{d+\epsilon}{6}}} \leq \|\phi\|^3_{H^{s_\epsilon}}.$$

This establishes (3.5.23). The proof for the case $d = 2$ is similar.

□

We are now ready to conclude the proof of Theorem 3.1.1 by proving Lemma 3.2.4.

*Proof of Lemma 3.2.4.* Recall from (3.4.1) that $J^k$ can be decomposed into a product of $k$ one particle kernels

$$J^k(t, t_1, \ldots, t_n; \sigma) = \prod_{j=1}^{k} J^1_j(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j),$$

where only one of the factors $J^1_j$ distinguished. It now follows from Propositions 3.5.1 and 3.5.2 that

$$\begin{aligned}
&\int_{[0,T)^{n-1}} dt_1 \cdots dt_{n-1} \mathrm{Tr}\left(\left|S^{(k,-d)} J^k(t, t_1, \ldots, t_n; \sigma)\right|\right) \\
&= \int_{[0,T)^{n-1}} dt_1 \cdots dt_{n-1} \prod_{j=1}^{k} \mathrm{Tr}\left(\left|S^{(1,-d)} J^1_j(t, t_{\ell_{j,1}}, \ldots, t_{\ell_{j,m_j}}; \sigma_j)\right|\right) \\
&\leq \begin{cases} 2^n C^{n-1} T^{\epsilon(n-1)} \|\phi\|^{2(k+n)-3}_{H^{s_\epsilon}} \||\phi|^2\phi\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} & \text{if } d \geq 3 \\ 2^n C^{n-1} T^{\frac{1}{3}(n-1)} \|\phi\|^{2(k+n)-3}_{H^{1/3}} \||\phi|^2\phi\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}} & \text{if } d = 2 \\ 2^n C^{n-1} T^{\frac{1}{2}(n-1)} \|\phi\|^{2(k+n)-3}_{L^2} \||\phi|^2\phi\|_{L^1} & \text{if } d = 1. \end{cases}
\end{aligned}$$

Thus, for $t \in [0,T)$, it follows from Lemma 3.5.3 that

$$\int_{[0,T)^{n-1}} d\underline{t}_{n-1} \mathrm{Tr}(|S^{(k,-d)} J^k(\underline{t}_n;\sigma)|)$$
$$\leq \begin{cases} (CT^{\epsilon})^{n-1} \|\phi\|_{H^{s_\epsilon}}^{2(k+n)} & \text{if } d \geq 3 \\ (CT^{1/3})^{n-1} \|\phi\|_{H^{1/3}}^{2(k+n)} & \text{if } d = 2 \\ (CT^{1/2})^{n-1} \|\phi\|_{H^{1/6}}^{2(k+n)} & \text{if } d = 1, \end{cases}$$

which is precisely the statement of Lemma 3.2.4. □

## 3.A Proof of Lemma 3.2.5

We prove Lemma 3.2.5 combining the dispersive estimate, the Strichartz estimates (see [75] for example) and negative order Sobolev norms.

**Lemma 3.A.1** (Dispersive estimates)**.** *For $2 \leq r \leq \infty$, we have*

$$\|e^{it\Delta} f\|_{L^r_x} \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{r})} \|f\|_{L^{r'}_x}. \tag{3.A.1}$$

**Lemma 3.A.2** (Homogeneous Strichartz estimates)**.** *We call a pair of exponents $(q,r)$ Schrödinger admissible if $2 \leq q,r \leq \infty$, $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and $(q,r,d) \neq (2,\infty,2)$. Then for any admissible exponents $(q,r)$ we have the homogeneous Strichartz estimate*

$$\|e^{it\Delta} f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2_x}. \tag{3.A.2}$$

**Lemma 3.A.3** (Negative order Sobolev norms)**.** *Let $\epsilon > 0$ be a small number. Then, for $s \geq s_c + \frac{\epsilon}{2}$, we have*

$$\|fg\|_{W^{-s,r_\epsilon}} \lesssim \|f\|_{W^{-s,r'_\epsilon}} \|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}},$$

*where $r_\epsilon = \frac{2d}{d+2(1-\epsilon)}$.*

*Proof.* By Hölder's inequality, the fractional Leibniz rule and the Sobolev inequality, we have

$$\begin{aligned}\Big|\int f(x)g(x)\overline{h(x)}dx\Big| &\le \|f\|_{W^{-s,r'_\epsilon}}\|g\bar{h}\|_{W^{s,r_\epsilon}}\\ &\lesssim \|f\|_{W^{-s,r'_\epsilon}}\Big(\|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}}\|h\|_{L^{\frac{2d}{\epsilon}}}+\|g\|_{L^{\frac{d}{2(1-\epsilon)}}}\|h\|_{W^{s,r'_\epsilon}}\Big)\\ &\lesssim \|f\|_{W^{-s,r'_\epsilon}}\|g\|_{W^{s,\frac{2d}{d+2-3\epsilon}}}\|h\|_{W^{s,r'_\epsilon}}.\end{aligned}$$

The lemma now follows from the standard duality argument. □

*Proof of Lemma 3.2.5.* $(i)$. For notational convenience, we omit the time interval $[0,T)$ in the norms.

(3.2.19): By Lemma 3.A.3, we get

$$\begin{aligned}\|T(f,g,h)\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}} &\lesssim \|e^{i(t-t_1)\Delta}f\|_{W^{-(s_c+\frac{\epsilon}{2}),r'_\epsilon}}\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}}\\ &\lesssim \tfrac{1}{|t-t_1|^{1-\epsilon}}\|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}\|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}.\end{aligned} \tag{3.A.3}$$

Here, in the second inequality, we use the dispersive estimate:

$$\|e^{i(t-t_1)\Delta}f\|_{W^{-(s_c+\frac{\epsilon}{2}),r'_\epsilon}} \lesssim \tfrac{1}{|t-t_1|^{1-\epsilon}}\|f\|_{W^{-(s_c+\frac{\epsilon}{2}),r_\epsilon}}$$

and the fractional Leibniz rule and the Sobolev inequality:

$$\begin{aligned}&\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}}\\ &\lesssim \|e^{i(t-t_2)\Delta}g\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{L^{\frac{d}{1-\epsilon}}}+\|e^{i(t-t_2)\Delta}g\|_{L^{\frac{d}{1-\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{W^{s_c+\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}}\\ &\lesssim \|e^{i(t-t_2)\Delta}g\|_{H^{s_\epsilon}}\|e^{i(t-t_3)\Delta}h\|_{H^{s_\epsilon}} = \|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}.\end{aligned} \tag{3.A.4}$$

Integrating out the time variable $t$, we prove (3.2.19).

(3.2.20): By the fractional Leibniz rule, we have

$$\begin{aligned}\|T(f,g,h)\|_{L^1_t H^{s_\epsilon}_x} \lesssim \|e^{i(t-t_1)\Delta}f\|_{L^3_t W_x^{s_\epsilon,\frac{6d}{3d-4}}}\|e^{i(t-t_2)\Delta}g\|_{L^3_t L^{3d}_x}\|e^{i(t-t_3)\Delta}h\|_{L^3_t L^{3d}_x} \\ + \|e^{i(t-t_1)\Delta}f\|_{L^3_t L^{3d}_x}\|e^{i(t-t_2)\Delta}g\|_{L^3_t W_x^{s_\epsilon,\frac{6d}{3d-4}}}\|e^{i(t-t_3)\Delta}h\|_{L^3_t W^{3d}_x} \\ + \|e^{i(t-t_1)\Delta}f\|_{L^3_t L^{3d}_x}\|e^{i(t-t_2)\Delta}g\|_{L^3_t L^{3d}_x}\|e^{i(t-t_3)\Delta}h\|_{L^3_t W_x^{s_\epsilon,\frac{6d}{3d-4}}}.\end{aligned}$$

Then, by the Sobolev inequality and the Strichartz estimates, we bound the first term by

$$\begin{aligned}&\lesssim \|e^{i(t-t_1)\Delta}f\|_{L^3_t W_x^{s_\epsilon,\frac{6d}{3d-4}}}\|e^{i(t-t_2)\Delta}g\|_{L^3_t W_x^{s_\epsilon,\frac{6d}{3d-4+6\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{L^3_t W_x^{s_\epsilon,\frac{6d}{3d-4+6\epsilon}}} \\ &\leq T^\epsilon \|e^{i(t-t_1)\Delta}f\|_{L^3_t W_x^{s_\epsilon,\frac{6d}{3d-4}}}\|e^{i(t-t_2)\Delta}g\|_{L_t^{\frac{6}{2-3\epsilon}} W_x^{s_\epsilon,\frac{6d}{3d-4+6\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{L_t^{\frac{6}{2-3\epsilon}} W_x^{s_\epsilon,\frac{6d}{3d-4+6\epsilon}}} \\ &\lesssim T^\epsilon \|f\|_{H^{s_\epsilon}}\|g\|_{H^{s_\epsilon}}\|h\|_{H^{s_\epsilon}}.\end{aligned}$$

Similarly, we bound the other two terms.

*(ii)*. (3.2.21): The proof is similar to that of (3.2.19), but here we use Lemma 3.A.3 with $s = (\frac{1}{3} - \frac{\epsilon}{2})$. Indeed, by the dispersive estimate and Lemma 3.A.3,

$$\begin{aligned}\|T(f,g,h)\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r_\epsilon}} &\lesssim \|e^{i(t-t_1)\Delta}f\|_{W^{-(\frac{1}{3}-\frac{\epsilon}{2}),r'_\epsilon}}\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}} \\ &\lesssim \tfrac{1}{|t-t_1|^{1-\epsilon}}\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}}.\end{aligned}$$

Then, modifying $(A.1)$, we obtain

$$\begin{aligned}&\|(e^{i(t-t_2)\Delta}g)(e^{i(t-t_3)\Delta}h)\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d+2-3\epsilon}}} \\ &\lesssim \|e^{i(t-t_2)\Delta}g\|_{W^{\frac{1}{3}-\frac{\epsilon}{2}\frac{2d}{d-\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{L^{\frac{d}{1-\epsilon}}} + \|e^{i(t-t_2)\Delta}g\|_{L^{\frac{d}{1-\epsilon}}}\|e^{i(t-t_3)\Delta}h\|_{W^{\frac{1}{3}-\frac{\epsilon}{2},\frac{2d}{d-\epsilon}}} \\ &\lesssim \|e^{i(t-t_2)\Delta}g\|_{H^{1/3}}\|e^{i(t-t_3)\Delta}h\|_{H^{1/3}} = \|g\|_{H^{1/3}}\|h\|_{H^{1/3}},\end{aligned}$$

Applying this to the above inequality and Integrating out $t$, we compete the proof.

(3.2.22): Although we set $\epsilon$ to be small and $d \geq 3$ in the proof of (3.2.20), it actually works for $\epsilon = \frac{1}{3}$ and $d = 2$ which is exactly (3.2.22).

*(iii)*. For (3.2.23), by the Hölder inequality and the 1d dispersive estimates, we get

$$\|T(f,g,h)\|_{L^1} \leq \|e^{i(t-t_1)}f\|_{L^\infty}\|e^{i(t-t_2)}g\|_{L^2}\|e^{i(t-t_3)}h\|_{L^2} \lesssim \frac{1}{|t-t_1|^{1/2}}\|f\|_{L^1}\|g\|_{L^2}\|h\|_{L^2}.$$

Integrating out the time variable $t$, we prove (3.2.23).

For (3.2.24), by the Hölder inequality and the Strichartz estimate,

$$\|T(f,g,h)\|_{L^1_t L^2_x} \leq T^{1/2}\|e^{i(t-t_1)}f\|_{L^6_{t,x}}\|e^{i(t-t_2)}g\|_{L^6_{t,x}}\|e^{i(t-t_3)}h\|_{L^6_{t,x}} \lesssim T^{1/2}\|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.$$

□

# Chapter 4

# Uniqueness of Solutions to the 3D Quintic Gross-Pitaevskii Hierarchy

## 4.1 Introduction

In this chapter, we establish uniqueness of small solutions to the three-dimensional quintic Gross-Pitaevskii (GP) hierarchy in the scaling-critical Sobolev type space. This chapter is based on a joint work with Younghun Hong and Zhihui Xie [71].

### 4.1.1 Statement of the main result

The 3D quintic GP hierarchy is an infinite system of coupled linear equations

$$i\partial_t \gamma^{(k)} = (-\Delta_{\underline{x}_k} + \Delta_{\underline{x}'_k})\gamma^{(k)} + \lambda \sum_{j=1}^{k} B_{j;k+1,k+2}\gamma^{(k+2)}, \quad k \in \mathbb{N}, \tag{4.1.1}$$

where $\gamma^{(k)} = \gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) : [0,T)\times\mathbb{R}^{3k}\times\mathbb{R}^{3k} \to \mathbb{C}$, the underlined variables $\underline{x}_k$ and $\underline{x}'_k$ denote $k$-tuples of spacial variables, i.e., $\underline{x}_k = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^{3k}$ and $\underline{x}'_k = (x'_1, x'_2, \cdots, x'_k) \in \mathbb{R}^{3k}$, and the Laplacians are given by $\Delta_{\underline{x}_k} := \sum_{j=1}^{k} \Delta_{x_j}$ and $\Delta_{\underline{x}'_k} := \sum_{j=1}^{k} \Delta_{x'_j}$. We assume that for each $k \in \mathbb{N}$, $\gamma^{(k)}$ is a symmetric marginal density matrix such that

$$\gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) = \overline{\gamma^{(k)}(t, \underline{x}'_k; \underline{x}_k)} \tag{4.1.2}$$

and

$$\gamma^{(k)}(t, x_{\sigma(1)}, \cdots, x_{\sigma(k)}; x'_{\sigma'(1)}, \cdots, x'_{\sigma'(k)}) = \gamma^{(k)}(t, \underline{x}_k; \underline{x}'_k) \tag{4.1.3}$$

for any permutations $\sigma$ and $\sigma'$ on $\{1, 2, \cdots, k\}$. The *contraction operator* $B_{j;k+1,k+2}$ is defined by

$$\begin{aligned}
&B_{j;k+1,k+2}\gamma^{(k+2)}(t, \underline{x}_k; \underline{x}'_k) \\
&:= \int dx_{k+1}dx_{k+2}dx'_{k+1}dx'_{k+2}[\delta(x_j - x_{k+1})\delta(x_j - x_{k+2})\delta(x_j - x'_{k+1})\delta(x_j - x'_{k+2}) \\
&\qquad - \delta(x'_j - x_{k+1})\delta(x'_j - x_{k+2})\delta(x'_j - x'_{k+1})\delta(x'_j - x'_{k+2})]\gamma^{(k+2)}(t, \underline{x}_{k+2}; \underline{x}'_{k+2}) \\
&= \gamma^{(k+2)}(t, \underline{x}_k, x_j, x_j; \underline{x}'_k, x_j, x_j) - \gamma^{(k+2)}(t, \underline{x}_k, x'_j, x'_j; \underline{x}'_k, x'_j, x'_j).
\end{aligned} \tag{4.1.4}$$

The coupling constant is either $-1$ or 1. We call the GP hierarchy (4.1.1) *defocusing* if $\lambda = 1$, and *focusing* if $\lambda = -1$.

To define solutions to the GP hierarchy, we introduce the following definitions (see also [17, 42–45]). For $s \geq 0$, we define the homogeneous Sobolev space $\dot{\mathfrak{H}}^s$ for sequences by

$$\dot{\mathfrak{H}}^s := \left\{ \{\gamma^{(k)}\}_{k\in\mathbb{N}} : \text{Tr } (|R^{(k,s)}\gamma^{(k)}|) < M^{2k} \text{ for some positive constant } M < \infty \right\} \tag{4.1.5}$$

where

$$R^{(k,s)} := \prod_{j=1}^{k} (-\Delta_{x_j})^{\frac{s}{2}} (-\Delta_{x'_j})^{\frac{s}{2}}.$$

Similarly, we define the inhomogeneous Sobolev space $\mathfrak{H}^s$ for sequences by

$$\mathfrak{H}^s := \left\{ \{\gamma^{(k)}\}_{k\in\mathbb{N}} : \text{Tr } (|S^{(k,s)}\gamma^{(k)}|) < M^{2k} \text{ for some constant } M < \infty \right\} \tag{4.1.6}$$

where

$$S^{(k,s)} := \prod_{j=1}^{k}(1-\Delta_{x_j})^{\frac{s}{2}}(1-\Delta_{x'_j})^{\frac{s}{2}}.$$

A sequence $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ is called a *mild solution* in $L^\infty_{t\in[0,T)}\dot{\mathfrak{H}}^s$ (or $L^\infty_{t\in[0,T)}\mathfrak{H}^s$) to the quintic GP hierarchy if it solves the hierarchy of the integral equations

$$\gamma^{(k)}(t) = U^{(k)}(t)\gamma^{(k)}(0) + i\lambda\sum_{j=1}^{k}\int_0^t U^{(k)}(t-s)B_{j;k+1,k+2}\gamma^{(k+2)}(s)ds, \quad \forall k\in\mathbb{N}, \tag{4.1.7}$$

where $U^{(k)}(t) := e^{it(\Delta_{\underline{x}_k}-\Delta_{\underline{x}'_k})}$ is the free evolution operator. A sequence $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ is called *admissible* if for each $k\in\mathbb{N}$ and $t\in[0,T)$, $\gamma^{(k)}$ is a non-negative trace class operator on $L^2_{sym}(\mathbb{R}^{3k}\times\mathbb{R}^{3k})$ (subset of $L^2$ functions that satisfy (4.1.3)) and

$$\gamma^{(k)} = \mathrm{Tr}_{k+1}(\gamma^{(k+1)}) = \int_{\mathbb{R}^3} dx_{k+1}\gamma^{(k+1)}(\underline{x}_k, x_{k+1}; \underline{x}'_k, x_{k+1}). \tag{4.1.8}$$

We call a sequence $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ a *limiting hierarchy* if there is a sequence $\{\gamma_N^{(N)}\}_{N\in\mathbb{N}}$ of non-negative density matrices on $L^2_{sym}(\mathbb{R}^{3N}\times\mathbb{R}^{3N})$ with $\mathrm{Tr}(\gamma_N^{(N)})=1$ such that $\gamma^{(k)}$ is the weak-* limit of the $k$-particle marginals of $\gamma_N^{(N)}$ in the trace class on $L^2_{sym}(\mathbb{R}^{3k}\times\mathbb{R}^{3k})$, that is,

$$\begin{aligned}\gamma_N^{(k)} :&= \mathrm{Tr}_{k+1,\cdots,N}(\gamma_N^{(N)})\\ &= \int_{\mathbb{R}^{3(N-k)}} dx_{k+1}\cdots dx_N\gamma_N^{(N)}(\underline{x}_k, x_{k+1},\cdots x_N; \underline{x}'_k, x_{k+1},\cdots,x_N) \quad (4.1.9)\\ &\rightharpoonup^* \gamma^{(k)} \text{ as } N\to\infty.\end{aligned}$$

In this chapter, we consider mild solutions to the GP hierarchy (4.1.1) that are admissible or limiting hierarchies. Such mild solutions are physically relevant in the theory of derivation of the nonlinear Schrödinger equation (NLS) from the many body linear Schrödinger equation (see Chapter 1).

We now state our main result.

**Theorem 4.1.1** (Uniqueness of small solutions to the quintic GP hierarchy)**.** *Suppose that* $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ *is a mild solution in* $L^{\infty}_{t\in[0,T)}\dot{\mathfrak{H}}^1$ *to the quintic GP hierarchy* (4.1.1) *with initial data* $\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$*, which is either admissible or a limiting hierarchy for each* $t$*. If* $Tr\,(|R^{(k,1)}\gamma^{(k)}|) < M^{2k}$ *for all* $t \in [0,T)$ *for* $M > 0$ *sufficiently small, then* $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ *is the only such solution for the given initial data.*

The quintic GP hierarchy is closely related to the quintic NLS via factorized functions. Indeed, one can check that if $\phi_t$ is a solution to the quintic NLS

$$i\partial_t\phi_t = (-\Delta)\phi_t + \lambda|\phi_t|^4\phi_t, \tag{4.1.10}$$

then a sequence of factorized functions,

$$\gamma^{(k)}(t,\underline{x}_k;\underline{x}'_k) = (|\phi_t\rangle\,\langle\phi_t|)^{\otimes k} := \prod_{j=1}^{k}\phi_t(x_j)\overline{\phi_t(x'_j)}, \tag{4.1.11}$$

solves the GP hierarchy (4.1.1). In this sense, proving uniqueness for the GP hierarchy is more difficult than it is for the quintic NLS.

The quintic GP hierarchy was studied by T. Chen and Pavlović [19] for the derivation of the quintic NLS as the Gross-Pitaevskii field limit of a non-relativistic Bose gas with 3-particle interactions. As a part of their analysis, the authors proved (conditional) uniqueness of solutions to the quintic GP hierarchy in an energy space, that is, a Sobolev type space of order 1, in one and two dimensions. We remark that in all dimensions, proving such uniqueness in an energy space is necessary to derive NLS. However, it is an open problem to prove uniqueness in three dimensions.

Theorem 4.1.1 provides an answer for this open problem under a smallness assumption. We remark that the 3D quintic GP hierarchy is scaling-critical in $\dot{\mathfrak{H}}^1$, and that even with our smallness assumption, our theorem is the first uniqueness theorem for the cubic or quintic GP hierarchy in a scaling-critical space. Moreover, uniqueness in Theorem 4.1.1 is unconditional.

It remains an open problem to remove the smallness assumption. In the case of the 3D quintic NLS, it is known that solutions are unique in the space $H^s$ for $s \geq 1$, without a smallness assumption [11, 34, 66, 73]. However, the proof of unconditional uniqueness in the scaling-critical case $s = 1$ differs from the proof in the subcritical case $s > 1$. In the case of the 3D quintic GP hierarchy, we also expect that an approach different from the one that we use in the scaling-subcritical case is needed to remove the smallness assumption in the scaling-critical case. Currently, the main obstacle to removing the smallness assumption for solutions to the 3D quintic GP hierarchy in the scaling-critical case is the generally infinite cardinality of the support of the measure $\mu$ in the statement of the quantum de Finetti theorem, Theorem 4.2.2.

To compare scaling-critical and subcritical regimes, we provide a uniqueness theorem for the 3D quintic Hartree hierarchy. The 3D quintic Hartree hierarchy is also an infinite hierarchy as (4.1.1). However the contraction operator $B_{j,k+1,k+2}$ in (4.1.4) is replaced by

$$\begin{aligned}
&B_{j;k+1,k+2}\gamma^{(k+2)}(t,\underline{x}_k;\underline{x}'_k)\\
&:=\int dx_{k+1}dx_{k+2}dx'_{k+1}dx'_{k+2}\\
&\qquad V(x_j-x_{k+1},x_j-x_{k+2})V(x_j-x'_{k+1},x_j-x'_{k+2})\gamma^{(k+2)}(t,\underline{x}_{k+2};\underline{x}'_{k+2})\\
&\qquad -\int dx_{k+1}dx_{k+2}dx'_{k+1}dx'_{k+2}\\
&\qquad\qquad V(x'_j-x_{k+1},x'_j-x_{k+2})V(x'_j-x'_{k+1},x'_j-x'_{k+2})\gamma^{(k+2)}(t,\underline{x}_{k+2};\underline{x}'_{k+2}).
\end{aligned} \tag{4.1.12}$$

Note that the 3D quintic Hartree equation is subcritical in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ if the three-particle interaction potential $V$ is less singular than the product of delta functions. This is, if $V(\cdot,\cdot)\in L^r_{x,y}(\mathbb{R}^3\times\mathbb{R}^3)$ for some $r>1$. In this case, we can show unconditional uniqueness for the 3D quintic Hartree hierarchy without a smallness assumption.

**Theorem 4.1.2** (Unconditional uniqueness for the quintic Hartree hierarchy)**.** *Suppose that $V(\cdot,\cdot)\in L^r_{x,y}(\mathbb{R}^3\times\mathbb{R}^3)$ for some $r>1$. Let $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}\in\dot{\mathfrak{H}}^1$ be a mild solution to the quintic Hartree hierarchy* (4.1.7) *with initial data $\{\gamma^{(k)}(0)\}_{k\in\mathbb{N}}$, which is either admissible or a limiting hierarchy for each $t$. If there exists $M>0$ such that Tr $(|R^{(k,1)}\gamma^{(k)}|)<M^{2k}$ for all $t\in[0,T)$, then $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ is the only such solution for the given initial data.*

### 4.1.2 Strategy of the proof

We prove Theorem 4.1.1 and Theorem 4.1.2 in the framework of Chen-Hainzl-Pavlović-Seringer [17]. Due to the linearity of the hierarchy, it suffices to show that solutions solution having a zero initial are the zero solution. In our proof, we iterate the Duhamel formula (4.1.7) with zero initial data $n$ times, resulting in a number of terms that grows factorially in $n$. We reduce the number of terms by the Erdös-Schlein-Yau combinatorial argument in Klainerman-Machedon's formulation [82]. The quintic version of this combinatoric reduction was used by Chen-Pavlovic in [19]. We use it for the 3D quintic GP and Hartee hierarchies without modification. Next, we apply the quantum de Finetti theorem to write each term as an integral sum of factorized states, and reorganize them using a tree-graph structure (see Figure 1 below) which extends the tree-graph in Chen-Hainzl-Pavlović-Seiringer [17]. Then, we iteratively estimate the $n$ integrals. In each step, we apply our multilinear estimates, which can be found in Appendix 4.A. Finally, we send $n \to \infty$ and find that solutions having zero initial data must be the zero solution.

In our previous work [70], we proved unconditional uniqueness for the cubic GP hierarchy in a low regularity setting, using a similar approach. In [70], our key ingredients were the trilinear estimates (2.19), (2.21) and (2.23) in Lemma 2.6. These estimates are based on the dispersive estimates

$$\|e^{it\Delta} f\|_{L^p(\mathbb{R}^d)} \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}(\mathbb{R}^d)}, \quad p \geq 2, \tag{4.1.13}$$

and negative order Sobolev norm estimates (Lemma A.3 in [70]). In the proof, we applied these estimates to the reorganized integrals iteratively together with multilinear estimates based on Strichartz estimates ((2.20), (2.22) and (2.24) in Lemma 2.6). We remark that the use of dispersive estimates is crucial in obtaining the optimal subcritical low regularity uniqueness theorem. The dispersive estimates don't work in the scaling-critical space, however. Roughly speaking, this is due to the failure of integrability (in time) of the bound in (4.1.13). For instance, if one tries to prove uniqueness for the 3D quintic GP hierarchy in $L^\infty_{t\in[0,T)}\mathfrak{H}^1$ by the same approach, one should choose $p = 6$ for the multilinear estimate. Then, the bound in (4.1.13) is not integrable in time.

In the present work, instead of using dispersive estimates, we use multilinear estimates (Proposition 4.A.1 and Propositions 4.A.2) that are based on by Strichartz estimates and a negative order Sobolev norm bound. In the case of the Hartree hierarchy, we also make use of a convolution estimates of W. Beckner [9].

### 4.1.3 Notation

In order to prove Theorem 4.1.1 and Theorem 4.1.2 at the same time, we define

$$V_\infty(y,z) := \begin{cases} V(y,z), & \text{for the Hatree hierarchy.} \\ \lambda\,\delta(y)\delta(z), & \text{for the GP hierarchy.} \end{cases} \tag{4.1.14}$$

With this notation, we can now combine definitions (4.1.4) and (4.1.12) of $B_{j;k+1,k+2}$ for the GP hierarchy and the Hartree hierarchy, respectively, as follows.

$$
\begin{aligned}
&B_{j;k+1,k+2}\gamma^{(k+2)}(t,\underline{x}_k;\underline{x}'_k)\\
&:=\int dx_{k+1}dx_{k+2}dx'_{k+1}dx'_{k+2}\\
&\qquad V_\infty(x_j-x_{k+1},x_j-x_{k+2})V_\infty(x_j-x'_{k+1},x_j-x'_{k+2})\gamma^{(k+2)}(t,\underline{x}_{k+2};\underline{x}'_{k+2})\\
&\qquad -\int dx_{k+1}dx_{k+2}dx'_{k+1}dx'_{k+2}\\
&\qquad\qquad V_\infty(x'_j-x_{k+1},x'_j-x_{k+2})V_\infty(x'_j-x'_{k+1},x'_j-x'_{k+2})\gamma^{(k+2)}(t,\underline{x}_{k+2};\underline{x}'_{k+2}).
\end{aligned}
\tag{4.1.15}
$$

### 4.1.4 Organization of the chapter

This chapter is organized as follows. In section 4.2 we present the road map for the proof of the main theorems and reduce the the main theorems to Proposition 4.2.1. We illustrate with an example how to factorize solutions in section 4.3. In section 4.4, we introduce tree graphs to illustrate our decomposition of each factor, and present properties of the associated kernels. The proof of Proposition 4.2.1 occupies section 4.5. In appendix 4.A, we prove several multilinear estimates that we use in section 4.5.

## 4.2 Outline of the proof

We describe the strategy to prove uniqueness in more detail.

### 4.2.1 Setup

Let $\{\gamma_1^{(k)}(t)\}_{k\in\mathbb{N}}$ and $\{\gamma_2^{(k)}(t)\}_{k\in\mathbb{N}}$ be two mild solutions in $L^\infty_{t\in[0,T)}\dot{\mathfrak{H}}^1$ that solve (4.1.7) with the same initial data, and are either admissible or limiting hierarchies. To prove uniqueness, we will show that their difference $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$, given by

$$\gamma^{(k)}(t) := \gamma_1^{(k)}(t) - \gamma_2^{(k)}(t), \qquad k \in \mathbb{N}, \tag{4.2.1}$$

is zero. By linearity, the difference $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ solves the GP (or Hartree) hierarchy with zero initial data. Therefore, it suffices to prove the following.

**Proposition 4.2.1.** *Suppose that $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ is a mild solution to* (4.1.1) *with zero initial data, and that it is either admissible or a limiting hierarchy.*
*(i) If $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ solves the quintic GP hierarchy and $\|\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}\|_{L^\infty_{t\in[0,T)}\dot{\mathfrak{H}}^1}$ is sufficiently small, then*

$$\mathrm{Tr}(|R^{(k,-1)}\gamma^{(k)}(t)|) = 0, \quad \forall k \in \mathbb{N}. \tag{4.2.2}$$

*(ii) If $\{\gamma^{(k)}(t)\}_{k\in\mathbb{N}}$ solves the quintic Hartree hierarchy and $V \in L^{1+}$, then* (4.2.2) *holds.*

### 4.2.2 Duhamel expansion

To show (4.2.2), we first generate a Duhamel expansion as follows. For each $k \in \mathbb{N}$, $\gamma^{(k)}(t)$ solves

$$\gamma^{(k)}(t) = i\lambda \sum_{j=1}^{k} \int_0^t U^{(k)}(t-t_1) B_{j;k+1,k+2}\gamma^{(k+2)}(t_1)dt_1. \tag{4.2.3}$$

Fix $k \in \mathbb{N}$. Iterating the integral equation (4.2.3) $(n-1)$ times, we write

$$\gamma^{(k)}(t) = (i\lambda)^n \int_{t_n\leq\cdots\leq t_1\leq t} U^{(k)}(t-t_1)B_{k+2}\cdots U^{(k+2n-2)}(t_{n-1}-t_n)B_{k+2n}\gamma^{(k+2n)}(t_n)dt_1\cdots dt_n. \tag{4.2.4}$$

Here, for each $r \geq 1$, the *combined contraction operator* is the sum of $k+2(r-1)$ many operators,

$$B_{k+2r} := \sum_{j=1}^{k+2(r-1)} B_{j;k+2r-1,k+2r}.$$

For notational convenience, we introduce the following notation.

$$U^{(i)}_{j,j'} := U^{(i)}(t_j - t_{j'}),$$

$$\underline{t}_n := (t, t_1, \cdots, t_n), \quad t_0 = t,$$

$$J^k(\underline{t}_n) := U^{(k)}_{0,1} B_{k+2} U^{(k+2)}_{1,2} B_{k+4} \cdots U^{(k+2n-2)}_{n-1,n} B_{k+2n} \gamma^{(k+2n)}(t_n).$$

Then $\gamma^{(k)}(t)$ in (4.2.4) can be expressed in a compact form as

$$\gamma^{(k)}(t) = (i\lambda)^n \int_{t_n \leq \cdots \leq t_1 \leq t} J^k(\underline{t}_n) d\underline{t}_n. \tag{4.2.5}$$

One may have observed that for fixed $k$, the number of terms in $J^k(\underline{t}_n)$ is $k(k+2)\cdots(k+2n-2) \sim \mathcal{O}((2n)!)$. This factorial growth on the number of Duhamel expansion terms is the first difficulty before we proceed with the proof of proposition 4.2.1. As a preparation, we will present a summary of the combinatorial reduction process in section 4.2.3 to reduce $J^k(\underline{t}_n)$ into a smaller number of terms that we can control.

### 4.2.3 Combinatorial reduction

In the celebrated works [42–45], Erdös-Schlein-Yau developed a sophisticated combinatorial arguments to reduce the number of Duhamel terms. Later, Klainerman and Machedon [82] rephrased this as a board game, which was extended to the quintic GP hierarchy by Chen-Pavlović in [19]. Since we will use the same arguments, we only present the notation and key reduction steps in this section. We refer the readers to [19] for the proofs of the related lemmas and theorems.

Let $\sigma$ be a map from $\{k+1, k+2, \cdots, k+2n-1\}$ to $\{1, 2, 3, \cdots, k+2n-2\}$ such that $\sigma(2) = 1$ and $\sigma(j) < j$ for all $j$. $\mathcal{M}_{k,n}$ denotes the set of all such mappings. Then we have that

$$J^k(\underline{t}_n) = \sum_{\sigma \in \mathcal{M}_{k,n}} J^k(\underline{t}_n; \sigma), \tag{4.2.6}$$

where

$$J^k(\underline{t}_n; \sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1,k+2} U_{1,2}^{(k+2)} \cdots U_{n-1,n}^{(k+2n-2)} B_{\sigma(k+2n-1);k+2n-1,k+2n}(\gamma^{(k+2n)}(t_n)) \tag{4.2.7}$$

is a basic term in $J^k(\underline{t}_n)$.

Next, for each $\sigma \in \mathcal{M}_{k,n}$ there is a $(k+2n-1) \times n$ matrix corresponding to it. This matrix can be reduced to a special upper echelon matrix that corresponds to $\sigma_s$ via finite many so called *acceptable moves*. This transformation defines an equivalence relation among all the maps in $\mathcal{M}_{k,n}$. If $\sigma$ and $\sigma_s$ are equivalent, we denote this equivalence by $\sigma \sim \sigma_s$. From each equivalence classes, we pick one map that corresponds to a special upper echelon matrix, denote it by $\sigma_s$. Theorem 7.4 in [19] confirms that there is a subset $D_{\sigma_s,t} \subset [0,t]^n$, such that

$$\sum_{\sigma \sim \sigma_s} \int_0^t \dots \int_0^{t_{n-1}} J^k(\underline{t}_n; \sigma) dt_1 \dots dt_n = \int_{D_{\sigma_s,t}} J^k(\underline{t}_n; \sigma_s) dt_1 \dots dt_n. \tag{4.2.8}$$

Hence we have a new formula for $\gamma^{(k)}(t)$

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}^s_{k,n}} \int_{D_{\sigma,t}} J^k(\underline{t}_n; \sigma) d\underline{t}_n, \tag{4.2.9}$$

where $\mathcal{M}^s_{k,n}$ is the union of all maps that correspond to special upper echelon matrices. By Lemma 7.3 of [19], $\#(\mathcal{M}^s_{k,n}) \leq 2^{k+3n-2}$. [3]

### 4.2.4 Quantum de Finetti theorem

After decomposing $\gamma^{(k)}$ into a sum, we use the *quantum de Finetti* theorems to express each term in a factorized form. The quantum de Finetti theorem has a strong and weak version, and pertains to to bosonic density matrices that are either admissible or obtained as a weak-$*$ limit, respectively. We state both the strong and weak versions [91] below to be used in section 4.2.3.

**Theorem 4.2.2** (Strong quantum de Finetti theorem)**.** *If a sequence* $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ *of bosonic density matrices on* $L^2_{sym}(\mathbb{R}^{3k})$ *is admissible, then there exists a unique Borel probability measure* $\mu$*, supported on the unit sphere* $S \subset L^2(\mathbb{R}^3)$ *and invariant under multiplication of* $\phi \in L^2(\mathbb{R}^3)$ *by complex numbers of modulus one, such that*

$$\gamma^{(k)} = \int d\mu(\phi)(|\phi\rangle\langle\phi|)^{\otimes k}, \quad k \in \mathbb{N}. \tag{4.2.10}$$

**Theorem 4.2.3** (Weak quantum de Finetti theorem)**.** *If a sequence* $\{\gamma^{(k)}\}_{k\in\mathbb{N}}$ *of bosonic density matrices on* $L^2_{sym}(\mathbb{R}^{3k})$ *is a limiting hierarchy, then there*

[3] The multiplier $2^{k+3n-2}$ is affordable to us, since it can be absorbed in $(CT)^n$.

*exists a unique Borel probability measure $\mu$, supported on the unit ball $\mathcal{B} \subset L^2(\mathbb{R}^3)$ and invariant under multiplication of $\phi \in L^2(\mathbb{R}^3)$ by complex numbers of modulus one, such that* (4.2.10) *holds.*

There are different formulations of these theorems that are used in different settings. The formulation for density matrices was presented in a paper Lewin, Nam and Rougerie [91], and in a paper by Ammari and Nier [4]. For additional results related the de Finetti theorems, we refer the reader to Diaconis and Freedman [39], Hudson and Moody [72], and Stormer [116].

To make sure the de Finetti theorems are applicable, we note that if $\{\gamma_1^{(k)}\}_k$ and $\{\gamma_2^{(k)}\}_k$ are admissible, then so is $\{\gamma^{(k)}\}_k$. Similarly, if both $\{\gamma_1^{(k)}\}_k$ and $\{\gamma_2^{(k)}\}_k$ are obtained from a weak-$*$ limit, then so is $\{\gamma^{(k)}\}_k$. Thus by Theorem 4.2.2 and Theorem 4.2.3, we obtain

$$\gamma^{(k)}(t) = \sum_{\sigma \in \mathcal{M}^s_{k,n}} \int_{D_{\sigma,t}} d\underline{t}_n \int d\mu_{\underline{t}_n} J^k(\underline{t}_n;\sigma). \tag{4.2.11}$$

where

$$J^k(\underline{t}_n;\sigma) = U^{(k)}_{0,1} B_{\sigma(k+1);k+1,k+2} U^{(k+2)}_{1,2} \cdots U^{(k+2n-2)}_{n-1,n} B_{\sigma(k+2n-1);k+2n-1,k+2n} (|\phi\rangle\langle\phi|)^{(k+2n)}. \tag{4.2.12}$$

We remark that $J^k(\underline{t}_n;\sigma) = J^k(\underline{t}_n;\sigma;\underline{x}_k;\underline{x}'_k)$ depends on $\underline{x}_k, \underline{x}'_k$. We omit the spatial variables for simplicity. We note that each factor in

$$(|\phi\rangle\langle\phi|)^{(k+2n)}(\underline{x}_{k+2n};\underline{x}'_{k+2n}) = \prod_{i=1}^{k+2n} (|\phi\rangle\langle\phi|)(x_i;x'_i)$$

is a one-particle kernel, and that we can further decompose $J^k(\underline{t}_n;\sigma)$ as

$$J^k(t,t_1,\cdots,t_n;\sigma;\underline{x}_k;\underline{x}'_k)=\prod_{j=1}^{k}J_j^1(t,t_{l_j,1},\cdots,t_{l_j,m_j};\sigma_j;x_j;x'_j). \tag{4.2.13}$$

To better explain the reduction procedure, we present an example in section 4.3, and then go back to the general case in section 4.4.

## 4.3 Example factorization

Consider $k=2,n=4$, and $\rho$ a permutation of $\{1,2,\cdots,n\}$. The map $\sigma_s$ is represented by the following upper echelon matrix (each highlighted entry in a row is to the left of each highlighted entry in a lower row)

$$\begin{pmatrix} t_{\rho^{-1}(1)} & t_{\rho^{-1}(2)} & t_{\rho^{-1}(3)} & t_{\rho^{-1}(4)} \\ \mathbf{B_{1;3,4}} & B_{1;5,6} & B_{1;7,8} & B_{1;9,10} \\ B_{2;3,4} & \mathbf{B_{2;5,6}} & B_{2;7,8} & B_{2,9,10} \\ 0 & B_{3;5,6} & B_{3;7,8} & B_{3;9,10} \\ 0 & B_{4;5,6} & \mathbf{B_{4;7,8}} & \mathbf{B_{4;9,10}} \\ 0 & 0 & B_{5;7,8} & B_{5;9,10} \\ 0 & 0 & B_{6;7,8} & B_{6;9,10} \\ 0 & 0 & 0 & B_{7;9,10} \\ 0 & 0 & 0 & B_{8;9,10} \end{pmatrix} \tag{4.3.1}$$

Then, we have

$$J^2(\underline{t}_4;\sigma)=U_{0,1}^{(2)}B_{1;3,4}U_{1,2}^{(4)}B_{2;5,6}U_{2,3}^{(6)}B_{4;7,8}U_{3,4}^{(8)}B_{4;9,10}. \tag{4.3.2}$$

We will organize the terms in expansion of $J^2(\underline{t}_4;\sigma)$ into two one-particle density matrices by examining the effect of the contraction operators starting with the last one on the RHS of (4.3.2). We denote each factor in the last term $(|\phi\rangle\langle\phi|)^{\otimes 10}$ by $u_i$, ordered by increasing index $i$, so that

$(|\phi\rangle\langle\phi|)^{\otimes 10} = \otimes_{i=1}^{10} u_i$.

First of all, in (4.3.2), the last interaction operator $B_{4;9,10}$ contracts the factor $u_4, u_9$ and $u_{10}$, and leaves all other factors unchanged.

$$B_{4;9,10}(\otimes_{i=1}^{10} u_i) = u_1 \otimes u_2 \otimes u_3 \otimes \Theta_4 \otimes u_5 \cdots \otimes u_8, \tag{4.3.3}$$

where

$$\Theta_4 := B_{1;2,3}(u_4 \otimes u_9 \otimes u_{10}).$$

The index $\alpha$ in $\Theta_\alpha$ associates $\Theta_\alpha$ to the $\alpha$-th interaction operator from the left in (4.3.2). Since we only run the expansion to the $n$-th level, we have $1 \le \alpha \le n$. In this specific case, $n = 4$, and the 4th interaction operator is $B_{4;9,10}$.

Next, $B_{4;7,8}$ contracts $U_{3,4}^{(8)}\Theta_4, U_{3,4}^{(8)}u_7$ and $U_{3,4}^{(8)}u_8$.

$$B_{4;7,8}U_{3,4}^{(8)}((4.3.3)) = (U_{3,4}^{(3)}(u_1 \otimes u_2 \otimes u_3)) \otimes \Theta_3 \otimes (U_{3,4}^{(2)}(u_5 \otimes u_6)), \tag{4.3.4}$$

where

$$\Theta_3 := B_{1;2,3}((U_{3,4}^{(1)}\Theta_4) \otimes (U_{3,4}^{(1)}u_7) \otimes (U_{3,4}^{(1)}u_8)).$$

Then, by the semigroup property, $U_{2,3}^{(i)}U_{3,4}^{(i)} = U_{2,4}^{(i)}$. The operator $B_{2;5,6}$ contracts $U_{2,4}^{(1)}u_2, U_{2,4}^{(1)}u_5$ and $U_{2,4}^{(1)}u_6$, which correspond to the 2nd, 5th, and 6th factors in (4.3.4). The other factors are left invariant.

$$B_{2;5,6}U_{2,3}^{(6)}((4.3.4)) = (U_{2,4}^{(1)}u_1) \otimes \Theta_2 \otimes (U_{2,4}^{(1)}u_3) \otimes (U_{2,3}^{(1)}\Theta_3), \tag{4.3.5}$$

where

$$\Theta_2 = B_{1;2,3}(U_{2,4}^{(3)}(u_2 \otimes u_5 \otimes u_6)).$$

Finally, $B_{1;3,4}$ contracts $U_{1,4}^{(1)}u_1$, $U_{1,4}^{(1)}u_3$, and $U_{1,3}^{(1)}\Theta_3$ and leaves other factors unchanged.

$$B_{1;3,4}U_{1,2}^{(4)}((4.3.5)) = \Theta_1 \otimes (U_{1,2}^{(1)}\Theta_2), \tag{4.3.6}$$

where

$$\Theta_1 = B_{1;2,3}((U_{1,4}^{(1)}u_1) \otimes (U_{1,4}^{(1)}u_3) \otimes (U_{1,3}^{(1)}\Theta_3)).$$

Therefore, $J^2$ can be factorized as

$$J^2 = (U_{0,1}^{(1)}\Theta_1) \otimes (U_{0,2}^{(1)}\Theta_2) := J_1^1 \otimes J_2^1. \tag{4.3.7}$$

Now $J^2$ in (4.3.7) has two factors $J_j^1$ (note $j \leq k = 2$), which are 1-particle matrices. The reason we have such a decomposition is that $B_{\sigma_1(r);r,r+1}$ only affects three $u_i$ each time, and as the contraction processes, all the $u_i$ might be divided into different groups by the contraction connectivity.

For $j = 1$, after replacing back $u_i = |\phi\rangle\langle\phi|$, $i \leq k + 2n = 10$, we have

$$J_1^1 = U_{0,1}^{(1)} B_{1;2,3} U_{1,3}^{(2)} B_{3;4,5} U_{3,4}^{(3)} B_{3;6,7} (|\phi\rangle\langle\phi|)^{\otimes 7} \tag{4.3.8}$$

where we relabel the index in operators $B_{\sigma_1(r);r,r+1}$ such that the interaction operators in (4.3.8) correspond to $B_{1;3,4}, B_{4;7,8}, B_{4;9,10}$ respectively, and leave the connectivity structure among them unchanged. The labeling of function $\sigma_1$ (see the notation in (4.2.13)) takes values $\sigma_1(2) = 1, \sigma_1(4) = 3$, and $\sigma_1(6) = 3$.

For $j = 2$, we perform the relabeling in the same spirit find that

$$J_2^1 = U_{0,2}^{(1)} B_{1;2,3} U_{2,4}^{(3)} (|\phi\rangle\langle\phi|)^{\otimes 3}, \tag{4.3.9}$$

where $\sigma_2(2) = 1$.

We note that for any $\ell < \ell'$, the interaction operators $B_{\sigma(\ell);\ell,\ell+1}$ and $B_{\sigma(\ell');\ell',\ell'+1}$ in $J^2$ (which are highlighted in (4.3.1)) belong to the same factor $J_j^1$ if either $\sigma(\ell) = \sigma(\ell')$ or $\sigma(\ell') = \ell$. In such cases, we consider them as being *connected*. This connectivity structure is exactly the key point of the Duhamel terms that we want to illustrate using *tree graphs*. We include the detailed definitions and descriptions in section 4.4.

We further note that each $\sigma_j$ can be viewed as the restriction of $\sigma$ to $J_j^1$. We call factors that have a free propagator applied to each $\phi$ (like $J_2^1$) *regular*, and factors that have the contractions of $(|\phi\rangle\langle\phi|)^{\otimes 3}$ without free propagator in between (like $J_1^1$) *distinguished*.

## 4.4 Tree graphs for the general case

### 4.4.1 The tree graphs

We begin by recalling that, from (4.2.12), $J^k$ is given by

$$J^k(\underline{t}_n;\sigma) = U_{0,1}^{(k)} B_{\sigma(k+1);k+1,k+2} U_{1,2}^{(k+2)} \cdots U_{n-1,n}^{(k+2n-2)} B_{\sigma(k+2n-1);k+2n-1,k+2n} (|\phi\rangle\langle\phi|)^{\otimes(k+2n)}.$$

where

$$(|\phi\rangle\langle\phi|)^{\otimes(k+2n)}(\underline{x}_{k+2n};\underline{x}'_{k+2n}) = \prod_{i=1}^{k+2n} (|\phi\rangle\langle\phi|)(x_i;x'_i)$$

is a product of one-particle kernels. Since the free evolution operators $U_{j,j'}^{(i)}$ and the contraction operators $B_{\sigma(r);r,r+1}$ preserve the product structure, it follows that we can also decompose

$$J^k(t,t_1,\ldots,t_n;\sigma;\underline{x}_k;\underline{x}'_k) = \prod_{j=1}^{k} J_j^1(t,t_{\ell_{j,1}},\ldots,t_{\ell_{j,m_j}};\sigma_j;x_j;x'_j) \qquad (4.4.1)$$

into a product of one-particle kernels $J_j^1$. We associate to this decomposition $k$ disjoint tree graphs $\tau_1,\tau_2,\ldots,\tau_k$. These graphs appear as *skeleton graphs* in [42–45]. As in [17, 70], we assign *root, internal,* and *leaf* vertices to each tree $\tau_j$.

- A *root* vertex labeled as $W_j$, $j=1,2,\cdots,k$, to represent $J_j^1(x_j;x'_j)$.
- An *internal* vertex labeled by $v_\ell$, $\ell = 1,2,\cdots,n$, corresponding to $B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell}$ and attached to the time variable $t_\ell$.
- A *leaf* vertex $u_i$, $i=1,2,\cdots,k+2n$, representing each factor $(|\phi\rangle\,\langle\phi|)(x_i;x'_i)$.

Next, we connect the vertices with *edges*, as described below.

- If $v_\ell$ is the smallest value of $\ell$ such that $\sigma(k+2\ell-1) = j$, then we connect $v_\ell$ to the root vertex $W_j$ and write $W_j \sim v_\ell$ (or equivalently $W_j \sim B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell}$). If there is no internal vertex connected to a root vertex $W_j$, then we connect $W_j$ to the leaf $u_j$, and write $W_j \sim u_j$.

- For any $1 < \ell \leq n$, if $\exists \ell' > l$ such that $\sigma(k+2\ell-1) = \sigma(k+2\ell'-1)$ or $\sigma(k+2\ell'-1) = k+2\ell-1$, then we connect $v_\ell$ and $v_{\ell'}$ and write $v_\ell \sim v_{\ell'}$ (or equivalently $B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell} \sim B_{\sigma(k+2\ell'-1);k+2\ell'-1,k+2\ell'}$). In this case, we call $v_\ell$ the *parent vertex* of $v_{\ell'}$, and $v_{\ell'}$ the *child vertex* of $v_\ell$. We denote the three child vertices of $v_\ell$ by $v_{k_-(\ell)}$, $v_{k(\ell)}$ and $v_{k_+(\ell)}$, with $k_-(\ell) < k(\ell) < k_+(\ell)$.

- When there is no internal vertex with $\ell' > l$ and $k+2\ell-1 = \sigma(k+2\ell'-1)$, we connect $v_\ell$ to the leaf vertices $u_{k+2\ell-1}$, $u_{k+2\ell}$ and write $v_\ell \sim (u_{k+2\ell-1}, u_{k+2\ell})$ (or equivalently $B_{\sigma(k+2\ell-1);k+2\ell-1,k+2\ell} \sim (u_{k+2\ell-1}, u_{k+2\ell})$).

We remark that it follows from the construction above that each root vertex has only one child vertex, and each internal vertex has exactly three child vertices (which can be either internal and leaf). We call the tree $\tau_j$ *distinguished* if $v_n \in \tau_j$, and *regular* if $v_n \notin \tau_j$. The three leaves connected to $v_n$ are called *distinguished leaf vertices*, and all other leaves are called *regular leaf vertices*. Clearly, there are $k-1$ regular trees and one distinguished tree in

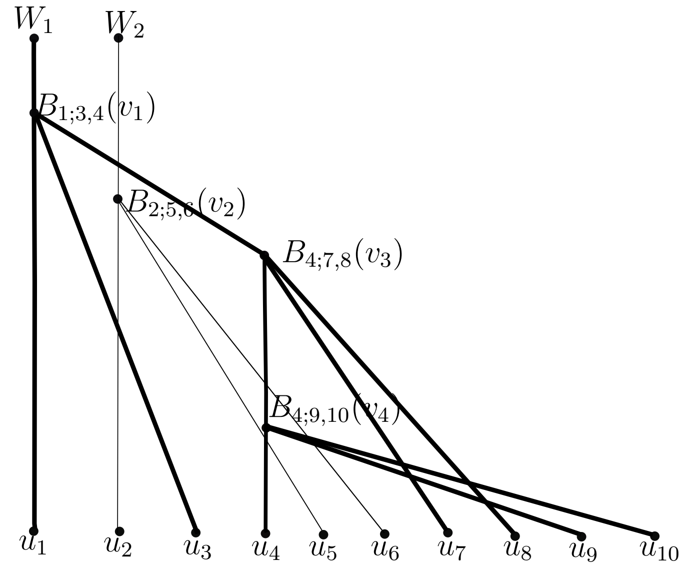


Figure 4.1: An example tree graph for $J^k$. It is a disjoint union of two trees $\tau_1$ and $\tau_2$ with root vertices $W_1$ and $W_2$, respectively. Each tree corresponds to a one-particle kernel in the example in section 4.3, where $k = 2$ and $n = 4$.

each tree graph.

A sample tree graph is given in Figure 4.1, for $J^k$ as in (4.3.2). Each tree $\tau_j$ has root vertex $W_j$, for $j = 1, 2$. The leaf vertices $u_1, u_3, u_4, u_7, u_8, u_9, u_{10}$ and the internal vertices $v_1, v_3, v_4$ (or $B_{1;3,4}, B_{4;7,8}, B_{4;9,10}$) are distinguished. $\tau_1$ is the distinguished tree, and is drawn with thick edges. Tree $\tau_2$ with vertices $W_2, v_2, u_2, u_5, u_6$ is the regular tree, and is drawn with thin edges.

### 4.4.2 The distinguished one particle kernel $J_j^1$

Let $\tau_j$ denote the distinguished tree graph. It has $m_j$ internal vertices $(v_{\ell_j,\alpha})_{\alpha=1}^{m_j}$ and $2m_j+1$ leaf vertices $(u_{j,i})_{i=1}^{2m_j+1}$. We enumerate the internal vertices with $\alpha \in \{1,\dots,m_j\}$ and the leaf vertices with $\alpha \in \{m_j+1,\dots,3m_j+1\}$. To simplify notation, we refer to the vertex $v_{j,\alpha}$ by its label $\alpha$. We observe that $J_j^1$ has the form

$$\begin{aligned} &J_j^1(t,t_{\ell_{j,1}},\dots,t_{\ell_{j,m_j}};\sigma_j) \qquad (4.4.2)\\ &= U^{(1)}(t-t_1)\cdots U^{(1)}(t_{\ell_{j,1}-1}-t_{\ell_{j,1}})B_{\sigma_j(2);2,3}\cdots\\ &\quad \cdots B_{\sigma_j(2\alpha-2);2\alpha-2,2\alpha-1}U^{(2\alpha-1)}(t_{\ell_{j,\alpha-1}}-t_{\ell_{j,\alpha-1}+1})\cdots U^{(2\alpha-1)}(t_{\ell_{j,\alpha}-1}-t_{\ell_{j,\alpha}})B_{\sigma_j(2\alpha);2\alpha,2\alpha+1}\cdots\\ &\quad \cdots U^{(2m_j-1)}(t_{\ell_{j,m_j}-1}-t_{\ell_{j,m_j}})B_{\sigma_j(2m_j),2m_j,2m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(2m_j+1)}. \end{aligned}$$

By the semigroup property

$$U^{(\alpha)}(t)U^{(\alpha)}(s) = U^{(\alpha)}(t+s),$$

and the fact that $\sigma_j(2)=1$, (4.4.2) reduces to

$$\begin{aligned} &J_j^1(t,t_{\ell_{j,1}},\dots,t_{\ell_{j,m_j}};\sigma_j) \qquad (4.4.3)\\ &= U^{(1)}(t-t_{\ell_{j,1}})B_{1;2,3}\cdots\\ &\quad \cdots B_{\sigma_j(2\alpha-2);2\alpha-2,2\alpha-1}U^{(2\alpha-1)}(t_{\ell_{j,\alpha-1}}-t_{\ell_{j,\alpha}})B_{\sigma_j(2\alpha);2\alpha,2\alpha+1}\cdots\\ &\quad \cdots U^{(2m_j-1)}(t_{\ell_{j,m_j}-1}-t_{\ell_{j,m_j}})B_{\sigma_j(2m_j);2m_j,2m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(2m_j+1)}, \end{aligned}$$

where $\ell_{j,m_j}=n$.

### 4.4.3 Definition of the kernels $\Theta_\alpha$ at the vertices of the distinguished tree graph

In this section, we proceed as in [17], and recursively assign a kernel $\Theta_\alpha$ to each vertex $\alpha$ of the distinguished tree graph. The kernels at the vertices of the regular tree graph are defined similarly. We begin by assigning the kernel

$$\Theta_\alpha(x;x') := \phi(x)\overline{\phi}(x')$$

to the leaf vertex with label $\alpha \in \{m_j+1,\dots,3m_j+1\}$.

Next, we determine $\Theta_{m_j}$ at the distinguished vertex $\alpha = m_j$ from the term on the last line of (4.4.3), given by

$$B_{\sigma_j(2m_j);2m_j,2m_j+1}(|\phi\rangle\langle\phi|)^{\otimes(2m_j+1)} = (|\phi\rangle\langle\phi|)^{\otimes(\sigma_j(2m_j)-1)} \otimes \Theta_{m_j} \otimes (|\phi\rangle\langle\phi|)^{\otimes(2m_j+1-\sigma_j(2m_j)-2)}$$

where

$$\Theta_{m_j}(x;x') := \tilde{\psi}(x)\overline{\phi}(x') - \phi(x)\overline{\tilde{\psi}}(x') \tag{4.4.4}$$

with $\tilde{\psi} := |\phi|^4\phi$. It is obtained from contracting three copies of $|\phi\rangle\langle\phi|$ at the three leaf vertices $\kappa_-(m_j), \kappa(m_j), \kappa_+(m_j)$ which have $m_j$ as their parent vertex.

Now we are ready to begin the induction. Let $\alpha \in \{1,\dots,m_j-1\}$. Suppose that the kernels $\Theta_{\alpha'}$ have been determined for all $\alpha' > \alpha$. We let $\kappa_-(\alpha), \kappa(\alpha), \kappa_+(\alpha)$ label the three child vertices (of internal or leaf type) of $\alpha$. Since $\Theta_{\kappa_-(\alpha)}, \Theta_{\kappa(\alpha)}$, and $\Theta_{\kappa_+(\alpha)}$ have already been determined, we can now define

$\Theta_\alpha(x;x')$

$= B_{1;2,3}((U^{(1)}(t_\alpha - t_{\kappa_-(\alpha)})\Theta_{\kappa_-(\alpha)}) \otimes (U^{(1)}(t_\alpha - t_{\kappa(\alpha)})\Theta_{\kappa(\alpha)}) \otimes (U^{(1)}(t_\alpha - t_{\kappa_+(\alpha)})\Theta_{\kappa_+(\alpha)}))(x;x').$

The induction ends when we obtain the kernel $\Theta_1$ at $\alpha = 1$.

### 4.4.4 Key properties of the kernels $\Theta_\alpha$

As in [17], we observe that the kernels $\Theta_\alpha$ satisfy the following properties.

- $\Theta_\alpha$ can be written as a sum of differences of factorized kernels

$$\Theta_\alpha(x;x') = \sum_{\beta_\alpha} c^\alpha_{\beta_\alpha} \chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x') \tag{4.4.5}$$

  with at most $2^{m_j-\alpha}$ nonzero coefficients $c^\alpha_{\beta_\alpha} \in \{1,-1\}$.

- The product $\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x')$ in (4.4.5) above is either of the form

$$\begin{aligned}\chi^\alpha_{\beta_\alpha}(x)\overline{\psi^\alpha_{\beta_\alpha}}(x') = {} & (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)\overline{(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})}(x') \\ & A\Big[V_\infty, (U_{\alpha;\kappa(\alpha)}\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}})\overline{(U_{\alpha;\kappa(\alpha)}\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}})}, \\ & (U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}\Big](x)\end{aligned} \tag{4.4.6}$$

or

$$\chi^{\alpha}_{\beta_\alpha}(x)\overline{\psi^{\alpha}_{\beta_\alpha}}(x') = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)\overline{(U_{\alpha;\kappa_-(\alpha)}\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})}(x')$$
$$A\Big[V_\infty, (U_{\alpha;\kappa(\alpha)}\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}})\overline{(U_{\alpha;\kappa(\alpha)}\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}})},$$
$$(U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}\Big](x') \quad (4.4.7)$$

for some values of $\beta_{\kappa_-(\alpha)}, \beta_{\kappa(\alpha)}, \beta_{\kappa_+(\alpha)}$ that depend on $\beta_\alpha$. The trilinear operator $A$ is defines as

$$A[V_\infty, f, g](x) := \int\int V_\infty(x-y_1, x-y_2)f(y_1)g(y_2)\,dy_1\,dy_2. \quad (4.4.8)$$

Observe that above, the function $\chi^{\alpha}_{\beta_\alpha}$ is either of the quintic form

$$\chi^{\alpha}_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x)$$
$$A\Big[V_\infty, (U_{\alpha;\kappa(\alpha)}\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}})\overline{(U_{\alpha;\kappa(\alpha)}\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}})}, \quad (4.4.9)$$
$$(U_{\alpha;\kappa_+(\alpha)}\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})\overline{(U_{\alpha;\kappa_+(\alpha)}\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}})}\Big](x) \quad (4.4.10)$$

or the linear form

$$\chi^{\alpha}_{\beta_\alpha}(x) = (U_{\alpha;\kappa_-(\alpha)}\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}})(x). \quad (4.4.11)$$

Accordingly, $\psi^{\alpha}_{\beta_\alpha}$ respectively is either of linear or quintic form, and the product $\chi^{\alpha}_{\beta_\alpha}(x)\overline{\psi^{\alpha}_{\beta_\alpha}}(x')$ always has sextic form (4.4.6) or (4.4.7).

- We call the functions $\chi^{\alpha}_{\beta_\alpha}, \psi^{\alpha}_{\beta_\alpha}$ in the sum (4.4.5) *distinguished* if they are a function of $|\phi|^4\phi$. In the product on the right hand side of (4.4.6), respectively (4.4.7), at most one of the six factors is distinguished. Indeed, this is true for all regular leaf vertices, and for the distinguished vertex (4.4.4). By induction along decreasing values of $\alpha$, it is also true for the internal vertices.

As in [17], we make the following assumption, which simplifies the notation without loss of generality.

**Hypothesis 1.** *We assume that only the functions $\psi^1_{\beta_1}$ and $(\psi^{\kappa^q_+(1)}_{\beta_{\kappa^q_+(1)}})$ are distinguished, where we define*

$$\kappa^q_+(1) := \underbrace{\kappa_+(\kappa_+(\dots(\kappa_+}_{q \ times}(1))\dots)).$$

## 4.5 Proof of Proposition 4.2.1

In this section, we prove Proposition 4.2.1. To simplify notation, we denote the time variable $t_{\ell_j,\alpha}$ by $t_\alpha$. We denote the subtree of $\tau_j$ with root at the vertex $\alpha$ by $\tau_{j,\alpha}$, and let

$$\int \Big[\prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'}\Big] := \int_{[0,T)^{d_\alpha}} \Big[\prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'}\Big]$$

be integration with respect to all time variables attached to the internal and root vertices of the subtree $\tau_{j,\alpha}$. Here, the total number of internal and root vertices of the tree $\tau_{j,\alpha}$ is denoted by $d_\alpha$.

**Lemma 4.5.1.** *For $V_\infty \in L^{\frac{1}{1-\epsilon}}(\mathbb{R}^6)$ with small $\epsilon \geq 0$ (or $V_\infty(y,z) = \lambda\delta_0(y)\delta_0(z)$*

*with $\epsilon = 0$ and $\|V_\infty\|_{L^1} := \lambda$), we have the $\dot{H}^{-1}$ bound*

$$\int \Bigg[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \Bigg] \|\psi^{\alpha}_{\beta_\alpha}\|_{\dot{H}^{-1}} \|\chi^{\alpha}_{\beta_\alpha}\|_{\dot{H}^1}$$

$$\leq CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int \Bigg[ \prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)}} dt_{\alpha'} \Bigg] \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1}$$

$$\cdot \int \Bigg[ \prod_{\alpha' \in \tau_{j,\kappa(\alpha)}} dt_{\alpha'} \Bigg] \|\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1}$$

$$\cdot \int \Bigg[ \prod_{\alpha' \in \tau_{j,\kappa_+(\alpha)}} dt_{\alpha'} \Bigg] \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^{-1}} \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1} \tag{4.5.1}$$

*and the $\dot{H}^1$ bound*

$$\int \Bigg[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \Bigg] \|\psi^{\alpha}_{\beta_\alpha}\|_{\dot{H}^{1}} \|\chi^{\alpha}_{\beta_\alpha}\|_{\dot{H}^1}$$

$$\leq CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int \Bigg[ \prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)}} dt_{\alpha'} \Bigg] \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1}$$

$$\cdot \int \Bigg[ \prod_{\alpha' \in \tau_{j,\kappa(\alpha)}} dt_{\alpha'} \Bigg] \|\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1}$$

$$\cdot \int \Bigg[ \prod_{\alpha' \in \tau_{j,\kappa_+(\alpha)}} dt_{\alpha'} \Bigg] \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^{1}} \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1}. \tag{4.5.2}$$

*Proof.* To prove (4.5.1), we apply the bound (4.A.3) (or (4.A.1)) to (4.4.6) and

(4.4.7) and obtain

$$
\begin{aligned}
&\int \left[ \prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'} \right] \|\psi^{\alpha}_{\beta_\alpha}\|_{\dot{H}^{-1}} \|\chi^{\alpha}_{\beta_\alpha}\|_{\dot{H}^{1}} \\
&\leq CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0,T)^{d_\alpha - 1}} \left[ \prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)} \cup \tau_{j,\kappa(\alpha)} \cup \tau_{j;\kappa_+(\alpha)}} dt_{\alpha'} \right] \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \\
&\qquad \cdot \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \|\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^{-1}} \\
&= CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0,T)^{d_{\kappa_-(\alpha)}}} \left[ \prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)}} dt_{\alpha'} \right] \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \\
&\qquad \cdot \int_{[0,T)^{d_{\kappa_-(\alpha)}}} \left[ \prod_{\alpha' \in \tau_{j,\kappa(\alpha)}} dt_{\alpha'} \right] \|\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \\
&\qquad \cdot \int_{[0,T)^{d_{\kappa_+(\alpha)}}} \left[ \prod_{\alpha' \in \tau_{j,\kappa_+(\alpha)}} dt_{\alpha'} \right] \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^{-1}}.
\end{aligned}
$$

In the second step, we performed the $t_\alpha$ integral. In the second step, we used the fact that the terms $\psi^{\alpha}_{\beta_\alpha}, \chi^{\alpha}_{\beta_\alpha}$ depend only on the time variables $t_{\alpha'}$ attached to the vertices of the subtree $\tau_{j,\alpha}$.

To prove (4.5.2), we apply the bound (4.A.4) (or (4.A.2)) to (4.4.6) and

(4.4.7) and obtain

$$
\begin{aligned}
&\int \left[\prod_{\alpha' \in \tau_{j,\alpha}} dt_{\alpha'}\right] \|\psi^{\alpha}_{\beta_\alpha}\|_{\dot{H}^1} \|\chi^{\alpha}_{\beta_\alpha}\|_{\dot{H}^1} \\
&\le CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0,T)^{d_\alpha - 1}} \left[\prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)} \cup \tau_{j,\kappa(\alpha)} \cup \tau_{j;\kappa_+(\alpha)}} dt_{\alpha'}\right] \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \\
&\qquad \cdot \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \|\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1} \\
&= CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0,T)^{d_{\kappa_-(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)}} dt_{\alpha'}\right] \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \\
&\qquad \cdot \int_{[0,T)^{d_{\kappa_-(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j,\kappa(\alpha)}} dt_{\alpha'}\right] \|\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \\
&\qquad \cdot \int_{[0,T)^{d_{\kappa_+(\alpha)}}} \left[\prod_{\alpha' \in \tau_{j,\kappa_+(\alpha)}} dt_{\alpha'}\right] \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1}.
\end{aligned}
$$

□

We now recursively apply the bounds in the statement Lemma 4.5.1 to conclude the proof of uniqueness of solutions to the quintic GP and Hartree hierarchy.

**Proposition 4.5.1.** *For the distinguished tree $\tau_j$, we have the bound*

$$
\begin{aligned}
&\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\left( \left| R^{(1,-1)} J^1_j(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\
&\qquad \le 2^{m_j} C^{m_j-1} T^{3\epsilon(m_j-1)} \|V_\infty\|^{m_j-1}_{L^{\frac{1}{1-\epsilon}}} \|\phi\|^{4m_j-3}_{\dot{H}^1} \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{\dot{H}^{-1}}.
\end{aligned}
\tag{4.5.3}
$$

*Proof.*

$$\int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\Big( \Big| R^{(1,-1)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \Big| \Big)$$
$$= \int_{[0,T)^{m_j-1}} dt_1 \dots dt_{m_j-1} \mathrm{Tr}\Big( \Big| R^{(1,-1)} U^{(1)}(t-t_1)\Theta_1 \Big| \Big)$$
$$\leq \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|\psi^1_{\beta_1}\|_{\dot{H}^{-1}} \|\chi^1_{\beta_1}\|_{\dot{H}^{-1}}$$
$$\leq \sum_{\beta_1} \int_{[0,T)^{m_j-1}} dt_1 \cdots dt_{m_j-1} \|\psi^1_{\beta_1}\|_{\dot{H}^{-1}} \|\chi^1_{\beta_1}\|_{\dot{H}^{1}}$$
$$\leq \sum_{\beta_{\kappa_-(1)}, \beta_{\kappa(1)}, \beta_{\kappa_+(1)}} CT^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int_{[0,T)^{d_{\kappa_-(\alpha)}}} \Big[ \prod_{\alpha' \in \tau_{j,\kappa_-(\alpha)}} dt_{\alpha'} \Big] \|\chi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa_-(\alpha)}_{\beta_{\kappa_-(\alpha)}}\|_{\dot{H}^1} \tag{4.5.4}$$
$$\cdot \int_{[0,T)^{d_{\kappa(\alpha)}}} \Big[ \prod_{\alpha' \in \tau_{j,\kappa(\alpha)}} dt_{\alpha'} \Big] \|\chi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa(\alpha)}_{\beta_{\kappa(\alpha)}}\|_{\dot{H}^1} \tag{4.5.5}$$
$$\cdot \int_{[0,T)^{d_{\kappa_+(\alpha)}}} \Big[ \prod_{\alpha' \in \tau_{j,\kappa_+(\alpha)}} dt_{\alpha'} \Big] \|\chi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^1} \|\psi^{\kappa_+(\alpha)}_{\beta_{\kappa_+(\alpha)}}\|_{\dot{H}^{-1}} \tag{4.5.6}$$

In the last step, we performed the $t_1$ integral using (4.5.1). Now, to bound (4.5.4) and (4.5.5), we iterate the $H^1$ bound (4.5.2). To bound (4.5.6), we iterate both (4.5.1) and (4.5.2). This establishes (4.5.3). □

**Proposition 4.5.2.** *For the regular tree $\tau_j$, we have the bound*

$$\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \mathrm{Tr}\Big( \Big| R^{(1,-1)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \Big| \Big) \leq 2^{m_j} C^{m_j} T^{3\epsilon m_j} \|\phi\|^{4m_j+2}_{\dot{H}^1}. \tag{4.5.7}$$

*Proof.*

$$\begin{aligned}
&\int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \operatorname{Tr}\left( \left| R^{(1,-1)} J_j^1(t, t_1, \cdots, t_{m_j}; \sigma_j) \right| \right) \\
&= \int_{[0,T)^{m_j}} dt_1 \dots dt_{m_j} \operatorname{Tr}\left( \left| R^{(1,-1)} U^{(1)}(t - t_1)\Theta_1 \right| \right) \\
&\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi^1_{\beta_1}\|_{\dot{H}^{-1}} \|\chi^1_{\beta_1}\|_{\dot{H}^{-1}} \\
&\leq \sum_{\beta_1} \int_{[0,T)^{m_j}} dt_1 \cdots dt_{m_j} \|\psi^1_{\beta_1}\|_{\dot{H}^{1}} \|\chi^1_{\beta_1}\|_{\dot{H}^{1}}
\end{aligned}$$

From here, we iterate the $\dot{H}^1$ bound (4.5.2) to obtain (4.5.7). □

**Lemma 4.5.2.** *Suppose that* $V_\infty \in L^{\frac{1}{1-\epsilon}}$. *Then*

$$\|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{\dot{H}^{-1}} \lesssim \begin{cases} \|V_\infty\|_{L^1} \|\phi\|^5_{\dot{H}^1}, & \text{if } \epsilon = 0 \\ \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\phi\|^5_{H^1}, & \text{if } \epsilon > 0. \end{cases}$$

*Notice that when* $\epsilon > 0$, *we measure the norm of* $\phi$ *in the non-homogeneous Sobolev space* $H^1$.

*Proof.* By Strichartz estimates, Sobolev embedding, and Theorem 4.A.3, we have

$$\begin{aligned}
&\|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{\dot{H}^{-1}} \\
&\lesssim \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{L^{\frac{6}{5}}} \\
&\leq \|A[V_\infty, |\phi|^2, |\phi|^2]\phi\|_{L^{\frac{3}{2}}} \|\phi\|_{L^6} \\
&\leq \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \||\phi|^2\|^2_{L^{\frac{3}{1+3\epsilon}}} \|\phi\|_{L^6} \\
&= \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\phi\|^4_{L^{\frac{6}{1+3\epsilon}}} \|\phi\|_{L^6} \\
&\lesssim \begin{cases} \|V_\infty\|_{L^1} \|\phi\|^5_{\dot{H}^1}, & \text{if } \epsilon = 0 \\ \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\phi\|^5_{H^1}, & \text{if } \epsilon > 0. \end{cases}
\end{aligned}$$

□

We are now ready to conclude the proof of Proposition 4.2.1.

*Proof of Proposition 4.2.1.* Recall from (4.4.1) that $J^k$ can be decomposed into a product of $k$ one-particle kernels

$$J^k(t,t_1,\ldots,t_n;\sigma) = \prod_{j=1}^{k} J_j^1(t,t_{\ell_{j,1}},\ldots,t_{\ell_{j,m_j}};\sigma_j),$$

where only one of the factors $J_j^1$ distinguished. It now follows from Propositions 4.5.1 and 4.5.2 that

$$\begin{aligned}
&\int_{[0,T)^{n-1}} dt_1\cdots dt_{n-1}\mathrm{Tr}\left(\left|R^{(k,-1)}J^k(t,t_1,\ldots,t_n;\sigma)\right|\right)\\
&= \int_{[0,T)^{n-1}} dt_1\cdots dt_{n-1}\prod_{j=1}^{k}\mathrm{Tr}\left(\left|R^{(1,-1)}J_j^1(t,t_{\ell_{j,1}},\ldots,t_{\ell_{j,m_j}};\sigma_j)\right|\right)\\
&\leq 2^n C^{n-1}T^{3\epsilon(n-1)}\|V_\infty\|_{L^{\frac{1}{1-\epsilon}}}^{n-1}\|\phi\|_{\dot{H}^1}^{4(k+n)-5}\|A[V_\infty,|\phi|^2,|\phi|^2]\phi\|_{\dot{H}^{-1}}.
\end{aligned}$$

Thus, by Lemma 4.5.2, the difference between two solutions $\gamma := \gamma_1 - \gamma_2$

satisfies

$$\begin{aligned}
&\mathrm{Tr}|R^{(k,-1)}\gamma^{(k)}| \\
&\le (\#\mathcal{M}_{k,n}) \sup_{\sigma\in\mathcal{M}_{k,n}} \sup_{i=1,2} \int_{[0,T)^n} d\underline{t}_n \int d\mu_{t_n}^{(i)}(\phi)\mathrm{Tr}(|R^{(k,-1)}J^k(\underline{t}_n;\sigma)|) \\
&\le \left(CT^{3\epsilon}\|V_\infty\|_{L^{\frac{1}{1-\epsilon}}}\right)^{n-1} \int_0^T dt_n \int d\mu_{t_n}^{(i)}(\phi)\|\phi\|_{\dot{H}^1}^{4(k+n)-5}\|A[V_\infty,|\phi|^2,|\phi|^2]\phi\|_{\dot{H}^{-1}} \\
&\le \begin{cases} \left(C\|V_\infty\|_{L^1}\right)^n \int_0^T dt_n \int d\mu_{t_n}^{(i)}(\phi)\|\phi\|_{\dot{H}^1}^{4(k+n)}, & \text{if } \epsilon=0 \\ \left(CT^{3\epsilon}\|V_\infty\|_{L^{\frac{1}{1-\epsilon}}}\right)^{n-1}\|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \int_0^T dt_n \int d\mu_{t_n}^{(i)}(\phi)\|\phi\|_{H^1}^{4(k+n)}, & \text{if } \epsilon>0 \end{cases} \\
&\le \begin{cases} \left(C\|V_\infty\|_{L^1}\right)^n TM^{4(k+n)}, & \text{if } \epsilon=0 \\ \left(CT^{3\epsilon}\|V_\infty\|_{L^{\frac{1}{1-\epsilon}}}\right)^{n-1}\|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} TM^{4(k+n)}, & \text{if } \epsilon>0 \end{cases} \\
&\to 0 \text{ as } n\to\infty
\end{aligned}$$

for $T$ sufficiently small if $\epsilon > 0$, and for $M$ sufficiently small if $\epsilon = 0$. Thus $\mathrm{Tr}|R^{(k,-1)}\gamma^{(k)}| = 0$. Combining this with the a-priori bound

$$\begin{cases} \mathrm{Tr}|R^{(k,1)}\gamma^{(k)}| < M^{2k}, & \text{if } \epsilon=0 \\ \mathrm{Tr}|S^{(k,1)}\gamma^{(k)}| < M^{2k}, & \text{if } \epsilon>0 \end{cases}$$

yields the desired result. Namely,

$$\begin{cases} \mathrm{Tr}|R^{(k,1)}\gamma^{(k)}| = 0, & \text{if } \epsilon=0 \\ \mathrm{Tr}|S^{(k,1)}\gamma^{(k)}| = 0, & \text{if } \epsilon>0. \end{cases}$$

□

## 4.A Multilinear estimates

In this section, we present the key multilinear estimates that we will use to prove our main theorems. For the GP hierarchy, our key estimates are in Proposition 4.A.1. The key estimates for the Hartree hierarchy are in Propositions 4.A.2.

**Proposition 4.A.1** (Multilinear estimates for GP)**.**

$$\|(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)(e^{it\Delta}f_4)(e^{it\Delta}f_5)\|_{L^1_t\dot{H}^{-1}_x} \lesssim \|f_1\|_{\dot{H}^{-1}}\prod_{j=2}^{5}\|f_j\|_{\dot{H}^1}, \quad (4.A.1)$$

$$\|(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)(e^{it\Delta}f_4)(e^{it\Delta}f_5)\|_{L^1_t\dot{H}^{1}_x} \lesssim \prod_{j=1}^{5}\|f_j\|_{\dot{H}^1}. \quad (4.A.2)$$

*For the proof, we need*

**Lemma 4.A.1** (Negative Sobolev norm estimate)**.**

$$\|fg\|_{\dot{H}^{-1}} \lesssim \|f\|_{\dot{W}^{-1,6}}\|g\|_{\dot{W}^{1,\frac{3}{2}}}.$$

*Proof.* We prove the lemma by the standard duality argument, the product rule and the Sobolev inequality.

$$\begin{aligned}\int fg\overline{h}\,dx &\leq \|f\|_{\dot{W}^{-1,6}}\|gh\|_{\dot{W}^{1,\frac{6}{5}}}\\ &\lesssim \|f\|_{\dot{W}^{-1,6}}\left(\|g\|_{L^3}\|h\|_{\dot{H}^1}+\|g\|_{\dot{W}^{1,\frac{3}{2}}}\|h\|_{L^6}\right)\\ &\lesssim \|f\|_{\dot{W}^{-1,6}}\|g\|_{\dot{W}^{1,\frac{3}{2}}}\|h\|_{H^1}.\end{aligned}$$

□

*Proof.* By Lemma 4.A.1, Sobolev embedding and Strichartz estimates, we prove that

$$
\begin{aligned}
&\|(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)(e^{it\Delta}f_4)(e^{it\Delta}f_5)\|_{L^1_t\dot{H}^{-1}_x}\\
&\lesssim \|e^{it\Delta}f_1\|_{L^2_tW^{-1,6}_x}\left\|\prod_{j=2}^{5}e^{it\Delta}f_j\right\|_{L^2_t\dot{W}^{1,\frac{3}{2}}_x}\\
&\lesssim \|f_1\|_{\dot{H}^{-1}}\left(\|e^{it\Delta}f_2\|_{L^2_t\dot{W}^{1,6}_x}\prod_{j=3}^{5}\|e^{it\Delta}f_j\|_{L^\infty_tL^6_x}+\text{three similar terms (by the product rule)}\right)\\
&\lesssim \|f_1\|_{\dot{H}^{-1}}\prod_{j=2}^{5}\|f_j\|_{\dot{H}^1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(e^{it\Delta}f_1)(e^{it\Delta}f_2)(e^{it\Delta}f_3)(e^{it\Delta}f_4)(e^{it\Delta}f_5)\|_{L^1_t\dot{H}^{1}_x}\\
&\lesssim \|e^{it\Delta}f_1\|_{L^2_t\dot{W}^{1,6}_x}\prod_{j=2}^{5}\|e^{it\Delta}f_j\|_{L^8_tL^{12}_x}+\text{ four similar terms (by the product rule)}\\
&\lesssim \|e^{it\Delta}f_1\|_{L^2_t\dot{W}^{1,6}_x}\prod_{j=2}^{5}\|e^{it\Delta}f_j\|_{L^8_t\dot{W}^{1,\frac{12}{5}}_x}+\text{ four similar terms (by the product rule)}\\
&\lesssim \prod_{j=1}^{5}\|f_j\|_{\dot{H}^1}.
\end{aligned}
$$

□

Recall the definition of the the trilinear operator $A$ in (4.4.8)

$$A[V_\infty,f,g](x):=\int\int V_\infty(x-y_1,x-y_2)f(y_1)g(y_2)\,dy_1\,dy_2.$$

As an analogue of Proposition 4.A.1, we prove:

**Proposition 4.A.2** (Multilinear estimates for Hartree)**.** *Let* $\epsilon \geq 0$*. Then, we have*

$$\|A[V_\infty, (e^{it\Delta}f_1 e^{it\Delta}f_2), (e^{it\Delta}f_3 e^{it\Delta}f_4)] \cdot (e^{it\Delta}f_5)\|_{L^1_t \dot{H}^{-1}_x} \lesssim T^{3\epsilon}\|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_m\|_{\dot{H}^{-1}} \prod_{\substack{\ell=1 \\ \ell \neq m}}^{5} \|f_\ell\|_{\dot{H}^1}, \quad \forall m = 1, \cdots, 5, \tag{4.A.3}$$

*and*

$$\|A[V_\infty, (e^{it\Delta}f_1 e^{it\Delta}f_2), (e^{it\Delta}f_3 e^{it\Delta}f_4)] \cdot (e^{it\Delta}f_5)\|_{L^1_t \dot{H}^1_x} \lesssim T^{3\epsilon}\|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^{5} \|f_\ell\|_{\dot{H}^1}. \tag{4.A.4}$$

We recall the convolution estimates in Beckner [9].

**Theorem 4.A.3.** *For* $1 < p < q < \infty, 1 < s_k < p'/q', k = 1, 2$ *and* $1/q + 2/p' = \sum 1/s_k, 2 < p'/q'$,

$$\|A[V_\infty, f, g]\|_{L^q(\mathbb{R}^d)} \leq \|V_\infty\|_{L^p(\mathbb{R}^{2d})}\|f\|_{L^{s_1}(\mathbb{R}^d)}\|g\|_{L^{s_2}(\mathbb{R}^d)}. \tag{4.A.5}$$

We note that Theorem 4.A.3 also holds for $p = 1$. Indeed, by the change of variables $(x-y, x-z) \to (y, z)$, Minkowski's inequality, and Hölder's inequality, we have

$$\begin{aligned}
\|A[V_\infty, f, g]\|_{L^q} &= \left\| \int \int V_\infty(y, z) f(x-y) g(x-z)\, dy\, dz \right\|_{L^q_x} \\
&\leq \int \int |V_\infty(y, z)| \, \|f(x-y)g(x-z)\|_{L^q_x} \, dy\, dz \\
&\leq \int \int |V_\infty(y, z)| \, \|f(x-y)\|_{L^{s_1}_x} \|g(x-z)\|_{L^{s_2}_x} \, dy\, dz \\
&= \|V_\infty\|_{L^1} \|f\|_{L^{s_1}} \|g\|_{L^{s_2}}.
\end{aligned}$$

*Proof of* (4.A.4). For $j \in \{1,2,3\}$, we have

$$\begin{aligned}
&\left\| \partial_j \left[ A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5) \right] \right\|_{L_t^1 L_x^2} \\
&\le \| A[V_\infty, (\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5) \|_{L_t^1 L_x^2} \\
&\qquad + \text{ four similar terms (by the product rule)} \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

By Theorem 4.A.3, Strichartz estimates, and Sobolev embedding,

$$\begin{aligned}
I_1 &\le \left\| \| A[V_\infty, (\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \|_{L_x^{\frac{12}{5}}} \| (e^{it\Delta} f_5) \|_{L_x^{12}} \right\|_{L_t^1} \\
&\lesssim \| V_\infty \|_{L^{\frac{1}{1-\epsilon}}} \left\| \| \partial_j e^{it\Delta} f_1 e^{it\Delta} f_2 \|_{L_x^{\frac{4}{1+8\epsilon}}} \| e^{it\Delta} f_3 e^{it\Delta} f_4 \|_{L_x^6} \| (e^{it\Delta} f_5) \|_{L_x^{12}} \right\|_{L_t^1} \\
&\le T^{3\epsilon} \| V_\infty \|_{L^{\frac{1}{1-\epsilon}}} \| \partial_j e^{it\Delta} f_1 \|_{L_t^{\frac{2}{1-6\epsilon}} L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^{5} \| e^{it\Delta} f_\ell \|_{L_t^8 L_x^{12}} \\
&\lesssim T^{3\epsilon} \| V_\infty \|_{L^{\frac{1}{1-\epsilon}}} \| \partial_j e^{it\Delta} f_1 \|_{L_t^{\frac{2}{1-6\epsilon}} L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^{5} \| e^{it\Delta} f_\ell \|_{L_t^8 \dot{W}_x^{1,\frac{12}{5}}} \\
&\lesssim T^{3\epsilon} \| V_\infty \|_{L^{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^{5} \| f_\ell \|_{\dot{H}^1}.
\end{aligned}$$

and similarly for $k \in \{2,3,4\}$. For $k = 5$, we have

$$
\begin{aligned}
I_5 &\leq \left\| \|A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)]\|_{L^3_x} \|\partial_j e^{it\Delta} f_5\|_{L^6_x} \right\|_{L^1_t} \\
&\lesssim \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \left\| \|e^{it\Delta} f_1 e^{it\Delta} f_2\|_{L_x^{\frac{6}{1+12\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L^6_x} \|\partial_j e^{it\Delta} f_5\|_{L^6_x} \right\|_{L^1_t} \\
&\leq T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|e^{it\Delta} f_1\|_{L_t^{\frac{8}{1-24\epsilon}} L_x^{\frac{12}{1+24\epsilon}}} \prod_{\ell=2}^{4} \|e^{it\Delta} f_\ell\|_{L^8_t L^{12}_x} \|\partial_j e^{it\Delta} f_5\|_{L^2_t L^6_x} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|e^{it\Delta} f_1\|_{L_t^{\frac{8}{1-24\epsilon}} \dot{W}_x^{\frac{12}{5+24\epsilon}}} \prod_{\ell=2}^{4} \|e^{it\Delta} f_\ell\|_{L^8_t \dot{W}_x^{1,\frac{12}{5}}} \|\partial_j e^{it\Delta} f_5\|_{L^2_t L^6_x} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^{5} \|f_\ell\|_{\dot{H}^1}. \qquad \square
\end{aligned}
$$

Before we proceeds to the proof of (4.A.3), we define $\{P_1, P_2, P_3\}$ to be a conic decomposition of $\mathbb{R}^3$. That is, $P_j$ is a Fourier multiplier with symbol $p_j : \mathbb{R}^3 \to [0,1]$ such that for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$,

$$
\begin{aligned}
&p_j(\xi) = 1 \text{ for } \xi_j^2 \geq 2 \sum_{j' \neq j} \xi_{j'}^2, \\
&p_j(\xi) = 0 \text{ for } \xi_j^2 \leq \frac{1}{2} \sum_{j' \neq j} \xi_{j'}^2, \text{ and} \\
&\sum_j p_j(\xi) = 1 \text{ for all } \xi \in \mathbb{R}^3.
\end{aligned}
$$

Observe that $|\xi_j| \sim |\xi|$ on the support of $p_j$.

*Proof of* (4.A.3) *when* $m = 5$. For $h \in \dot{H}^1(\mathbb{R}^3)$, we have

$$\begin{aligned}
&\int A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)](x)(e^{it\Delta} f_5)(x)\overline{h}(x)\,dx \\
&= \sum_{j=1}^{3} \int\int\int V_\infty(y,z)\partial_j \Big[(e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z)\overline{h}(x)\Big] \\
&\qquad\qquad \times (\partial_j^{-1} P_j e^{it\Delta} f_5)(x)\,dy\,dz\,dx \\
&= \sum_{j=1}^{3} \int\int\int V_\infty(y,z)(\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) \\
&\qquad\qquad \times \overline{h}(x)(\partial_j^{-1} P_j e^{it\Delta} f_5)(x)\,dy\,dz\,dx \\
&\qquad + \text{four similar terms (by the product rule)} \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

By duality, it now suffices to show that

$$\|I_k\|_{L_t^1} \lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_5\|_{\dot{H}^{-1}} \Big(\prod_{\ell=1}^{4} \|f_\ell\|_{\dot{H}^1}\Big) \|h\|_{\dot{H}^1} \tag{4.A.6}$$

holds for $k \in \{1,2,3,4,5\}$. By Theorem 4.A.3, Strichartz estimates, and Sobolev embedding, we have

$$\begin{aligned}
\|I_1\|_{L_t^1} &\leq \sum_{j=1}^{3} \Big\| \Big\| \int\int V_\infty(y,z)(\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z)\,dy\,dz \Big\|_{L_x^{\frac{3}{2}}} \\
&\qquad\qquad \times \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L_x^6} \|h\|_{L_x^6} \Big\|_{L_t^1} \\
&\lesssim \sum_{j=1}^{3} \Big\| \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j e^{it\Delta} f_1 e^{it\Delta} f_2\|_{L^{\frac{3}{1+6\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L^3} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L_x^6} \|h\|_{L_x^6} \Big\|_{L_t^1} \\
&\lesssim \sum_{j=1}^{3} T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j e^{it\Delta} f_1\|_{L_t^{\frac{2}{1-6\epsilon}} L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^{4} \|e^{it\Delta} f_\ell\|_{L_t^\infty L_x^6} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L_t^2 L_x^6} \|h\|_{L_x^6} \\
&\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_5\|_{\dot{H}^{-1}} \Big(\prod_{\ell=1}^{4} \|f_\ell\|_{\dot{H}^1}\Big) \|h\|_{\dot{H}^1},
\end{aligned}$$

and similarly (4.A.6) holds for $k \in \{2,3,4\}$. For $k=5$, we bound $\|I_5\|_{L^1_t}$ by

$$\sum_{j=1}^{3} \Big\| \| \int\int V_\infty(y,z)(e^{it\Delta} f_1 e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z)\, dy\, dz \|_{L^3_x}$$

$$\times \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L^6_x} \|\partial_j h\|_{L^2_x} \Big\|_{L^1_t}$$

$$\lesssim \sum_{j=1}^{3} \Big\| \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|e^{it\Delta} f_1 e^{it\Delta} f_2\|_{L^{\frac{6}{1+12\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L^6} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L^6_x} \|\partial_j h\|_{L^2_x} \Big\|_{L^1_t}$$

$$\lesssim \sum_{j=1}^{3} \Big\| \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^{2} \|e^{it\Delta} f_\ell\|_{L_x^{\frac{12}{1+12\epsilon}}} \prod_{m=3}^{4} \|e^{it\Delta} f_m\|_{L^{12}_x} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L^6_x} \|\partial_j h\|_{L^2_x} \Big\|_{L^1_t}$$

$$\lesssim \sum_{j=1}^{3} T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \prod_{\ell=1}^{2} \|e^{it\Delta} f_\ell\|_{L_t^{\frac{8}{1-12\epsilon}} \dot{W}_x^{1,\frac{12}{5+12\epsilon}}} \prod_{m=3}^{4} \|e^{it\Delta} f_m\|_{L^8_t \dot{W}_x^{1,\frac{12}{5}}} \|\partial_j^{-1} P_j e^{it\Delta} f_5\|_{L^2_t L^6_x} \|\partial_j h\|_{L^2_x}$$

$$\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_5\|_{\dot{H}^{-1}} \left( \prod_{\ell=1}^{4} \|f_\ell\|_{\dot{H}^1} \right) \|h\|_{\dot{H}^1}. \qquad \square$$

*Proof of* (4.A.3) *when* $m \neq 5$. We present the proof for $m=1$, and note that the proof for $m \in \{2,3,4\}$ is similar. i.e. we show that

$$\|A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)] \cdot (e^{it\Delta} f_5)\|_{L^1_t \dot{H}^{-1}_x}$$

$$\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_1\|_{\dot{H}^{-1}} \prod_{\ell=2}^{5} \|f_\ell\|_{\dot{H}^1}.$$

For $h \in \dot{H}^1(\mathbb{R}^3)$, we have

$$\begin{aligned}
&\int A[V_\infty, (e^{it\Delta} f_1 e^{it\Delta} f_2), (e^{it\Delta} f_3 e^{it\Delta} f_4)](x)(e^{it\Delta} f_5)(x)\overline{h}(x)\, dx \\
&= \sum_{j=1}^{3} \int\int\int V_\infty(y,z)(\partial_j^{-1} P_j e^{it\Delta} f_1)(x-y) \\
&\qquad\qquad \times \partial_j \Big[ (e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z)(e^{it\Delta} f_5)(x)\overline{h}(x) \Big]\, dy\, dz\, dx \\
&= \sum_{j=1}^{3} \int\int\int V_\infty(y,z)(\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z) \\
&\qquad\qquad \times (e^{it\Delta} f_5)(x)\overline{h}(x)\, dy\, dz\, dx \\
&\qquad + \text{four similar terms (by the product rule)} \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

By duality, it now suffices to show that

$$\|I_k\|_{L_t^1} \lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_1\|_{\dot{H}^{-1}} \left( \prod_{\ell=2}^{5} \|f_\ell\|_{\dot{H}^1} \right) \|h\|_{\dot{H}^1} \tag{4.A.7}$$

holds for $k \in \{1,2,3,4,5\}$. By Theorem 4.A.3, Strichartz estimates, and

Sobolev embedding, we have

$$
\begin{aligned}
\|I_1\|_{L^1_t} \le \sum_{j=1}^{3} \Big\| \| \int\!\!\int V_\infty(y,z)(\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z)\, dy\, dz \|_{L_x^{\frac{3}{2}}} \\
\times \|e^{it\Delta} f_5\|_{L_x^6} \|h\|_{L_x^6} \Big\|_{L^1_t} \\
\lesssim \sum_{j=1}^{3} \Big\| \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot \partial_j e^{it\Delta} f_2\|_{L^{\frac{3}{1+6\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L^3} \|e^{it\Delta} f_5\|_{L_x^6} \|h\|_{L_x^6} \Big\|_{L^1_t} \\
\lesssim \sum_{j=1}^{3} T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1\|_{L_t^{\frac{2}{1-3\epsilon}} L_x^{\frac{6}{1+6\epsilon}}} \|\partial_j e^{it\Delta} f_2\|_{L_t^{\frac{2}{1-3\epsilon}} L_x^{\frac{6}{1+6\epsilon}}} \prod_{\ell=3}^{5} \|e^{it\Delta} f_\ell\|_{L_t^\infty L^6} \|h\|_{L^6} \\
\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_1\|_{\dot H^{-1}} \left( \prod_{\ell=2}^{5} \|f_\ell\|_{\dot H^1} \right) \|h\|_{\dot H^1},
\end{aligned}
$$

and similarly, (4.A.7) holds for $k \in \{2,3,4\}$. Finally, we bound $\|I_5\|_{L^1_t}$ by

$$
\begin{aligned}
\sum_{j=1}^{3} \| \int\!\!\int V_\infty(y,z)(\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot e^{it\Delta} f_2)(x-y)(e^{it\Delta} f_3 e^{it\Delta} f_4)(x-z)\, dy\, dz \|_{L^1_t L^3_x} \\
\times \|e^{it\Delta} f_5\|_{L_t^\infty L_x^6} \|\partial_j h\|_{L_x^2} \\
\le \sum_{j=1}^{3} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1 \cdot e^{it\Delta} f_2\|_{L_t^{\frac{3}{2}} L_x^{\frac{9}{2+18\epsilon}}} \|e^{it\Delta} f_3 e^{it\Delta} f_4\|_{L^3_t L^9_x} \|e^{it\Delta} f_5\|_{L_t^\infty L_x^6} \|\partial_j h\|_{L_x^2} \\
\le \sum_{j=1}^{3} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1\|_{L_t^2 L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^{4} \|e^{it\Delta} f_\ell\|_{L_t^6 L_x^{18}} \|e^{it\Delta} f_5\|_{L_t^\infty L_x^6} \|\partial_j h\|_{L_x^2} \\
\lesssim \sum_{j=1}^{3} T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|\partial_j^{-1} P_j e^{it\Delta} f_1\|_{L_t^{\frac{2}{1-6\epsilon}} L_x^{\frac{6}{1+12\epsilon}}} \prod_{\ell=2}^{4} \|e^{it\Delta} f_2\|_{L_t^6 \dot W_x^{1,\frac{18}{7}}} \|e^{it\Delta} f_5\|_{L_t^\infty \dot H_x^1} \|\partial_j h\|_{L_x^2} \\
\lesssim T^{3\epsilon} \|V_\infty\|_{L^{\frac{1}{1-\epsilon}}} \|f_1\|_{\dot H^{-1}} \left( \prod_{\ell=2}^{5} \|f_\ell\|_{\dot H^1} \right) \|h\|_{\dot H^1}. \qquad \square
\end{aligned}
$$

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# Vita

Kenneth Taliaferro was born in Bryan, Texas in 1987. Even though he grew up in Texas, his first language was German. His parents spoke German with him until he went to school, where he learned English. While living in Bryan, he played the violin in his schools' orchestras and swam on the city's summer swim team. He received his Bachelor of Science degree in mathematics from Texas A&M University in College Station in August of 2009.

In the Fall of 2009, he began his graduate studies in mathematics at the University of Texas at Austin. Under the guidance of his advisor, Thomas Chen, he studied partial differential equations and mathematical physics. Also during his time at UT Austin, he enjoyed ballroom dancing with Texas Ballroom. He defended his thesis in the Spring of 2015.

Permanent address: ktaliaferro@math.utexas.edu

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