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An Application of The Continuity Method for an Equation on Line Bundles

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An Application of The Continuity Method for an Equation on Line Bundles

by

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Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

August 2002

This work is dedicated to my parents, my sisters and my brother.

Acknowledgments

I thank the Brazilian People who, through the CNPq[1], gave me the financial support to perform this program.

I thank the North American People for what they taught me, gave me, helped me and shared with me. Since i cannot shake everyone's hands i will choose a representative. He is a bus driver. I board the bus with a note, speak no English. At a glance, he understands my situation. He stops the bus, packed with people. He makes a sign, we both drop off, to walk down street one block and a half. He points a spot and says "seven". I understand, and he calmly walks back to his duty. That was my second day in the U.S.

The bus driver emulated the spirit of the priest. I want to do the same.

Endless is the list of people i aknowledge. They are greeted personally... or in more subtle ways.

I am indebted to my Supervisor. So big is my debt that i cannot pay her in this Life. I hope to do it in the next one.

And of course the priest, the priest German.

[1]Under Grant 200456/97-0

An Application of The Continuity Method for an Equation on Line Bundles

Publication No. ______________

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The University of Texas at Austin, 2002

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A system of gauge equations, with 4 real parameters, is introduced on line bundles over closed Riemann surfaces. This system, similarly to the Vortex Equations, leads to a 2^{nd} order non-linear elliptic equation that also appears on the problem of conformally pointwise curvature. A functional whose roots are the solutions to the elliptic equation is defined. Its partial derivative is invertible at a particular root. The Implicit Function Theorem yields a locally defined 1-parameter family of solutions. Uniqueness of the family is proven for a certain range of the parameter. The behaviour of this family beyond that range is discussed.

A coordinate change on the original problem allows the construction of another functional, which is locally convex at its critical points, for a slightly larger range of the parameter.

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Chapter 1

Introduction

This dissertation studies a problem that lies in the interface between Differential Geometry and Analysis. Differential Geometry is the environment where the problem is formulated. From that, we get equations that are translated to an analytical framework. Standard techniques on Analysis on Manifolds are used to either solve or to get clues about the elliptic equations we have just obtained. We then go back to the Geometry and briefly discuss these results, and their implications.

Say that M is a (orientable) closed Riemann surface and two hermitian line bundles over M are defined. We seek holomorphic (1,0) morphisms between them that satisfy two different gauge equations, one for each of the line bundles. The system is

$$\begin{cases} ia_1\Lambda F_1 - i\Lambda\phi^* \wedge \phi = \tau_1 \\ ia_2\Lambda F_2 - i\Lambda\phi \wedge \phi^* = \tau_2 \\ \overline{\partial}_{L_2} \circ \phi + \phi \circ \overline{\partial}_{L_1} = 0 \end{cases} \tag{1.1}$$

where a_1, a_2, τ_1, τ_2 are real parameters. In equations (1.1), F_j $(j = 1, 2)$ are the hermitian curvatures of the bundles and ϕ is the bundle morphism of type $(1, 0)$. The contraction of a form with the Kähler form of M is denoted by Λ. The last equation (1.1) says that ϕ is holomorphic. The holomorphic and hermitian structures of the bundles are not fixed, but allowed to move around.

Natural questions that come up range from the "natural space" for the parameters, uniqueness of solutions on ϕ (if the other elements remain fixed) and assymptotic behaviour of the system when parameters blow up. We get results in the first two of these questions and leave the others for future research.

The study of problem (1.1) leads us to the non-linear elliptic equation

$$\Delta u + bfe^{2u} - \lambda = 0 \ , \tag{1.2}$$

with b, λ real numbers and u a function in the Sobolev Space H_1, that is, u and its first derivatives are square integrable. For convenience, we also assume u has 0 mean value. The function f is smooth, non-negative and enjoys certain properties at its roots, which happen in finite number.

Equation (1.2) has an interest in its own, since it is also derived in the problem of pointwise conformal metrics on surfaces. Fixing f and λ, the existence and uniqueness of solutions in the variables b, u are the posed questions on the study of (1.2). We show that there is a family of solutions defined for all λ less than a positive fixed number, and that uniqueness holds at least if λ is negative.

In this dissertation a huge amount of credit must be given to the work of the following persons: S.Bradlow, O.Garcia-Prada, J.Kazdan and F.Warner. Several of the proofs here are modeled on results from their papers in the reference.

From Bradlow and Garcia-Prada's work we get the basics on gauge equations like (1.1) on Kähler manifolds. In [Br1] we find a good development for this type of problem, with a single equation, for arbitrary dimensions of the base manifold and the bundle. The case of line bundles is discussed in [Gp], [BGP], from what we borrow the idea of fixing all other structures and trying to find the metric that solves (1.1).

In [KW1], [KW2], Kazdan and Warner give us a preliminary insight, and further, very good hints about how to make use of the analytical tools already available to study the elliptic equation. Their papers cover the problem of conformal curvatures on surfaces.

In Chapter 2 we present background on Geometry and Differential Equations. Its goal is more to set up notation than to go thoroughly into the proof of mostly well-known facts. However, we give special attention to topics that seem to be sparse in the literature, specially those related to Sobolev spaces in Riemannian manifolds.

In Chapter 3 we introduce and discuss the equation treated by Kazdan and Warner in [KW1].

Chapter 4 is dedicated to the gauge theory of the problem. Equations (1.1) are reintroduced and rewritten as a single equation for a holomorphic section, in a third line bundle. The following sections study the relationship between the two systems and state results on the existence of their solutions.

Chapter 5 contains the proof on the existence of solutions for (1.2). Here we have attempted for completeness, filling minute details, nevertheless important, that are frequently missing from the literature. An alternative approach to the system of equations (II) presented in Chapter 4 is given, and a theorem on uniqueness of its solutions, after a change of its coordinates, is stated and proven.

Chapter 2

Preliminaries

In this Chapter we state basic definitions and results on the differential geometry of vector bundles and the analysis on Riemannian Manifolds. The reader already familiar with those may wish to jump to Chapter 3.

For the next three sections all maps, manifolds and operators are assumed to be smooth. Denote by M a Riemannian Manifold with a fixed almost complex structure. The dimension of M is arbitrary, unless otherwise stated.

2.1 Geometric Background

Definition 2.1. Let $\mathbb{K}$ be the real ($\mathbb{R}$) or the complex field ($\mathbb{C}$).

1) A $\mathbb{K}$-vector bundle over M is a Manifold E and a projection map $\pi : E \to M$ so that $E_p = \pi^{-1}(p)$ (the fiber of E at p) has the structure of a finite dimensional vector space over $\mathbb{K}$ for all $p \in M$. We also require E to be locally trivial: given $p \in M$, $\exists\ U \subset M$ open, with $p \in U$ and $\pi^{-1}(U) \cong U \times \mathbb{K}^m$ (m is the rank of E), and this diffeomorphism is linear between the fibers. The vector bundle E is also defined by an open cover $\{U_\alpha\}$ of M together with "cocycle functions" $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ (G is a Lie subgroup of the linear group on $\mathbb{K}^m$). The cocycle condition reads as $g_{\alpha\beta}.g_{\beta\gamma}.g_{\gamma\alpha} = g_{\alpha\alpha} = id_G$.

2) A section ξ on a vector bundle E is a map $\xi : M \to E$ such that $\pi \circ \xi = id_M$. Denote the space of sections on E as $\Gamma(E)$. If $U \subset M$ is open

define $\Gamma_U(E)$ as the space of sections of the bundle $\pi^{-1}(U) \to U$. Then $\Gamma(E)$ ($\Gamma_U(E)$) are vector spaces over $\mathbb{K}$.

3) Algebraic operations among vector spaces are transposed to vector bundles (over the same base manifold) in a natural way. For E, F vector bundles define $E \otimes F$, $E \wedge F$, E^* and $\overline{E}$ as the bundles whose fibers are respectively the tensor product, the wedge product, the dual and the complex conjugate (if E is complex) of the fibers of E, F. A rigorous definition of those bundles can be given by using the cocycle functions [GH]. A morphism between vector bundles is a map $E \to F$ commuting with the projections π_E, π_F and linear in the fibers.

4) Notation:

$\mathrm{Hom}(E,F) = F \otimes E^*$	Homomorphisms from E to F
$\mathrm{End}(E) = E \otimes E^*$	Endomorphisms of E
$\Omega^s(E) = \Gamma(\Lambda^s T^*M \otimes E)$	s-forms with values in E
$\Omega^{p,q}(E) = \Gamma(\Lambda^{p,q} T^*M \otimes E)$	(p,q)-forms with values in E .

Definition 2.2. Let E be a complex vector bundle over M. A hermitian metric H is a bundle morphism $H : E \to \overline{E}^*$ such that $H(v)(\overline{w}) = \langle v, w \rangle_H$ defines a hermitian inner-product in the vector space $E_p \ni v, w$ for all $p \in M$. We can think H as a section of $\overline{E}^* \otimes E^*$ that is self-adjoint and positive-definite.

Remark 2.3. The real case of the above definition in the tangent bundle of a Manifold leads us back to the concept of Riemannian metric.

Remark 2.4. A metric $H_E : E \to \overline{E}^*$ induces metrics on the dual and conjugate bundles to E. They are defined by $H_{E^*} = \overline{H}_E^{-1} : E^* \to \overline{E}$ and $H_{\overline{E}} = \overline{H}_E : \overline{E} \to E^*$. Observe that a metric originally defined in either $E, E^*, \overline{E}$ or $\overline{E}^*$ automatically extends to the others in a cannonical way. If E and F are bundles with metrics H_E, H_F their tensor products $E^{\otimes j} \otimes F^{\otimes k}$ induce the tensor metrics $H_E^{\otimes j} \otimes H_F^{\otimes k}$ $(j, k \geq 1)$.

Definition 2.5. A connection on E is an operator $D : \Omega^0(E) \to \Omega^1(E)$ satisfying:

(1)	$D(a\xi + b\eta) = aD(\xi) + bD(\eta)$	for all $\xi, \eta \in \Omega^0(E)$,
(2)	$D(f\xi) = df \otimes \xi + fD(\xi)$	$a, b \in \mathbb{C}$ and $f \in C^\infty(M)$.

The almost complex structure on M yields a splitting of its complexified tangent bundle, $(TM)_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$. The corresponding projections, when composed with D, give rise to the operators $D' : \Omega^0(E) \to \Omega^{1,0}(E)$ and $D'' : \Omega^0(E) \to \Omega^{0,1}(E)$. They satisfy properties similar to (1) and (2) in Definition 2.3. For instance, $D''(a\xi + b\eta) = aD''(\xi) + bD''(\eta)$ and $D''(f\xi) = \overline{\partial} f \otimes \xi + fD''(\xi)$. We call D'' (D') a (0,1) operator ((1,0) operator).

Remark 2.6. The connection D can be extended to an operator $\Omega^k(E) \to \Omega^{k+1}(E)$. The space $\Omega^k(E)$ is locally spanned by products of the form $\omega \otimes \phi$, with $\phi \in \Omega^0_U(E)$ and ω a k-form on U, $U \subset M$ an open set. Then we define $D(\omega \otimes \phi) = d\omega \otimes \phi + (-1)^k \omega \wedge D\phi$, and extend it linearly for the remaining sections. An analogous extension holds for D'', D'.

Remark 2.7. If E, F are vector bundles with connections D_E, D_F respectively, their tensor product $E \otimes F$ induces a connection satisfying Leibniz rule. Sections of the kind $\phi_E \otimes \phi_F$ span $E \otimes F$ locally, with $\phi_E \in \Omega_U(E)$, $\phi_F \in \Omega_U(F)$. We set $D_{E \otimes F}(\phi_E \otimes \phi_F) = D_E(\phi_E) \otimes \phi_F + \phi_E \otimes D_F(\phi_F)$, and extend it linearly to all sections in $\Omega^0(E \otimes F)$. If $F = E^*$, D_{E^*} is chosen so that $D_{E \otimes E^*}(I_E) = 0$ and $c \circ D_{E \otimes E^*} = d \circ c$, where c is the contraction map defined in Section 2.2. The bundle $\overline{E}$ induces D_E by setting $D_{\overline{E}}(\overline{\xi}) = \overline{D_E(\xi)}$ for all $\overline{\xi} \in \Gamma(\overline{E})$. The above procedures can be applied to D', D'' separately , so they are extended to tensor products, duals and conjugates of bundles. Notice that a (1,0) operator in E defines a (0,1) operator in $\overline{E}$, and vice versa.

A (0,1) operator on E can be defined independently of a connection. We are interested in a specific class of such operators:

Definition 2.8. A holomorphic structure on E is a (0,1) operator $\overline{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ satisfying $\overline{\partial}_E \circ \overline{\partial}_E = 0$. The pair $(E, \overline{\partial}_E)$ is called a *holomorphic vector bundle.*

Remark 2.9. The difference $(D_1 - D_2)$ between two connections D_1, D_2 on E is an operator $\Omega^0(E) \to \Omega^1(E)$. It can be shown that $(D_1 - D_2)(\phi)_p$ depends linearly on the value of ϕ at $p \in M$, for $\phi \in \Omega^0(E)$. Hence, $(D_1 - D_2) \in \Omega^1(\mathrm{End}E)$. The space of connections is an affine space modeled by $\Omega^1(\mathrm{End}E)$. The space of holomorphic structures is not in general diffeomorphic to $\Omega^{0,1}(\mathrm{End}E)$, but an affine identification like in the previous case still holds when $\dim_{\mathbb{R}} M = 2$.

Definition 2.10. The curvature (F_D) of a connection D is the operator $F_D = D \circ D : \Omega^k(E) \to \Omega^{k+2}(E)$. A computation shows that F_D is a pointwise operator: at every $p \in M$, $(F_D\phi)_p$ is linear in $\phi(p)$. Hence F_D is identified with a section in $\Omega^2(\mathrm{End}E)$.

Definition 2.11. Let E be a $\mathbb{C}$-vector bundle, $\overline{\partial}_E$ a holomorphic structure and H a hermitian metric on E. We say D is compatible with $\overline{\partial}_E$ if $D^{0,1} = \overline{\partial}_E$. If $D(H) = 0$ we say that D is compatible with H.

In the definition of $\mathbb{K}$-vector bundles we can replace the vector space $\mathbb{K}^m$ by a manifold $\mathcal{F}$ where the gauge group G acts on the left, as a subgroup of the group of diffeomorphisms. We arrive at the more general concept of a *Fiber Bundle* over M, with fiber $\mathcal{F}$. The cocycles $\{g_{\alpha\beta}\}$ then work as transition functions in the change of charts:

$$(x, f) \in U_\alpha \cap U_\beta \times \mathcal{F} \mapsto (x, g_{\alpha\beta}(x)f) \ .$$

Notice that for any fiber bundle F the bundle structure is determined by these changes of charts, since

$$F \cong \coprod_{\alpha} U_\alpha \times \mathcal{F} \,/ (x, f)_\alpha \sim (x, g_{\beta\alpha}(x) f)_\beta \ .$$

We are more interested in the case $\mathcal{F} = G$, for two different actions of G onto itself:

1. The action is given by left multiplication, $g(a) = ga \in G$. The correspondent bundle (P) is called a G-Principal Bundle over M.

2. The action is given by conjugation, $g(a) = gag^{-1} \in G$. We then get the Adjoint Bundle of P, denoted $\mathrm{Ad}P$.

There is a right action of G onto P preserving the fibers. This action is transitive within each fiber, what allows us to obtain any other fiber bundle with structure group G and the same cocycles $\{g_{\alpha\beta}\}$ as a quotient, $(P \times \mathcal{F})/(u, f) \sim (ug, g^{-1} f)$. Then we make more precise the notion of local trivialization: it is just a section $\sigma \in \Gamma_U(P)$ for $U \subset M$ open. If F is an $\mathcal{F}$-fiber bundle constructed as a quotient of $P \times \mathcal{F}$ and $s \in \Gamma_U(F)$, there is a map $f : U \to \mathcal{F}$ so that $s(x) = \sigma(x).f(x) = [\sigma(x), f(x)]$ for all $x \in U$.

Definition 2.12. A gauge transformation g is a section of AdP. In each one of its fibers AdP has a group structure, that is inherited from $\Gamma(\mathrm{Ad}P)$. It is called the group of (global) gauge transformations, denoted $\mathcal{G}$. There is an action of $\mathcal{G}$ onto $\Gamma(E)$. If $g \in \mathcal{G}$, $\xi \in \Gamma(E)$ and $\sigma \in \Gamma_U(P)$ is a local trivialization, let $g = \sigma.h$, $\xi = \sigma.s$, $h : U \to G$ and $s : U \to \mathbb{K}^m$. Then,

$$(g\xi)_{(x)} = \sigma_{(x)}.(hs)_{(x)}$$

for all $x \in U$.

The goal of Gauge Theories is the study of objects which are invariant under the action of gauge transformations. Those include equations, functionals, operators, and others. In the case of rank-1 complex vector bundles the Gauge Group G is either $\mathbb{C} - \{0\}$ or $U(1)$, what characterizes *Abelian Gauge Theories.*

For next two definitions we assume M is compact and Kähler, that is, the Kähler form ω is d-closed.

Definition 2.13. Let $E \to M$ be a complex vector bundle and D be a connection on E with curvature F_D. The first Chern form associated with F_D is

$$c_1(F_D) = \frac{i}{2\pi}\mathrm{tr}(F_D) \quad \in \quad \Omega^2_M(\mathbb{C}) \ . \tag{2.1}$$

It can be shown (see [GH], [MT]) that this form is closed, and in fact, its associated cohomology class does not depend on the choice of D. The first Chern class of E is thus defined by

$$c_1(E) = [c_1(F_D)] \quad \in \quad H^2(M, \mathbb{C}) \ . \tag{2.2}$$

The Chern classes give us information on the C^∞ topology of a complex bundle. In the vortex equations problem ([Br1]) they appear on lower bounds for the Yang-Mills-Higgs functional.

Definition 2.14. Let $n = \dim_{\mathbb{C}} M$. The parameter

$$\deg(E, \omega) = \int_M c_1(E) \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{2.3}$$

is the ω-degree of E. Due to the normalization on Definition 2.13 one can show $\deg(E, \omega) \in \mathbb{Z}$.

When M is a surface the degree of E does not depend on its Kähler structure. For convenience in further computations we define, in this case,

$$c(E) = 2\pi \deg(E) = \int_M 2\pi c_1(E) \quad \in \quad 2\pi\mathbb{Z} \ . \tag{2.4}$$

2.2 Remarks on Coordinates

Most results and definitions in this chapter are well known in the literature. That justifies our goal on stating them in a concise way, without making use of coordinates. Although, some concepts are better exposed and understood with their help. In those cases the reader is encouraged to check the coordinate invariance of the definitions.

Let $U \subset M$ be an open set where we have a basis $\{u^j\}$ for the k-forms on U, that is, $\{u^j(x)\}$ is a basis of $(\wedge^k T^*U)_x$ for every $x \in U$. Then any $\xi \in \Omega^k_U(E)$ can be written in a unique way as

$$\xi = \sum_j u^j \xi_j \tag{2.5}$$

for $\xi_j \in \Gamma_U(E)$.

Definition 2.15. Several pairings can be defined as operators in the space of sections of vector bundles. In general, let $E,\ F$ be $\mathbb{K}$-vector bundles and $\varphi : E \otimes F \to M \times \mathbb{K}$ a bundle morphism. Set the corresponding bilinear pairing

$$(\cdot,\cdot)_\varphi : \Gamma_U(E) \times \Gamma_U(F) \to C^\infty_U(\mathbb{K}) \ .$$

In the case of forms with *coefficients* in a vector bundle we write

$$\wedge : \Omega^k(E) \times \Omega^l(F) \to \Omega^{k+l}(E \otimes F) \qquad \text{for}$$

$$\left(\sum_j u^j \xi_j\right) \wedge \left(\sum_s u^s \eta_s\right) = \sum_{j,s} u^j \wedge u^s \xi_j \otimes \eta_s \ .$$

We make sense of composing $(\cdot,\cdot)_\varphi$ and $\wedge$, for

$$\left(\sum_j u^j \xi_j \wedge \sum_s u^s \eta_s\right)_\varphi = \sum_{j,s} u^j \wedge u^s \ \varphi(\xi_j, \eta_s) \ ,$$

and get an operator $\Omega^k(E) \times \Omega^l(F) \to \Omega^{k+l}(\mathbb{K})$.

Remark 2.16. In the literature there are at least two definitions of wedge product of forms that differ by a factorial. In this thesis we adopt the one compatible with the following: if α, β are 1-forms on M then $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$.

The two important cases of pairings for us are:

1. $\varphi : E \otimes E^* \to \mathbb{K}$ is the contraction map, $\varphi(\xi, \eta) = \eta(\xi)$. Noticing that $E \otimes E^* \cong \mathrm{End}E$, $(\cdot)_\varphi$ is called the *trace operator* ($\mathrm{tr}(\cdot)$).
2. (on a complex bundle) $\varphi : E \otimes \overline{E} \to \mathbb{C}$ is given by a hermitian metric H. In this case we write $\langle \xi, \eta \rangle_H = (\xi, \overline{\eta})_\varphi = H(\xi)(\overline{\eta})$ for $\xi, \eta \in \Gamma_U(E)$.

Generalizing the above for three vector bundles E, F, V we have the composition $\circ : \Gamma_U(\mathrm{Hom}(F, V)) \times \Gamma_U(\mathrm{Hom}(E, F)) \to \Gamma_U(\mathrm{Hom}(E, V))$. It corresponds to the trace of the inner factors in the product $V \otimes F^* \otimes F \otimes E^*$. If all bundles are the same, write

$$\mathrm{tr}(\xi \wedge \eta) = \sum_{j,s} u^j \wedge u^s \ \mathrm{tr}(\xi_j \circ \eta_s) \qquad \text{for} \quad \sum_j u^j \xi_j, \sum_s u^s \eta_s \in \Omega^{\cdot}(EndE) \ .$$

Now consider a Kähler Manifold M of complex dimension n. Let $\{e_j\}_{j=1,\ldots,n}$ be a local orthonormal basis on U for its holomorphic tangent bundle, and $\{e^j\}_{j=1,\ldots,n}$ be the dual basis. The Kähler form is then written $\omega = i \sum_j e^j \wedge \overline{e}^j$. The unitary volum form on U is $\nu = \omega^n / n!$.

Definition 2.17. A hermitian metric in $\Omega_U^{p,q}(\mathbb{C})$ is set by establishing $\{e^J \wedge \overline{e}^K\}$ as an orthonormal basis, where J, K run over the increasing p and q multi-indices on $\{1, \ldots, n\}$. If J is a (increasing) p-index, $J = (j_1, j_2, \ldots, j_p)$, then

$$e^J = e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_p} \ ,$$
$$\overline{e}^J = \overline{e}^{j_1} \wedge \overline{e}^{j_2} \wedge \cdots \wedge \overline{e}^{j_p} \ .$$

For $\xi, \eta \in \Omega_U^{p,q}(\mathbb{C})$ we have

$$\xi = \sum_{JK} \xi_{JK}\ e^J \wedge \overline{e}^K,\ \eta = \sum_{LN} \eta_{LN}\ e^L \wedge \overline{e}^N\ ,$$

$$\langle \xi, \eta \rangle = \sum_{JKLN} \xi_{JK} \overline{\eta}_{LN}\ \delta_{JK,LN} = \sum_{JK} \xi_{JK} \overline{\eta}_{JK}\ .$$

The Hodge star $* : \Omega_U^{p,q}(\mathbb{C}) \to \Omega_U^{n-q,n-p}(\mathbb{C})$ is defined in a way that

$$\alpha \wedge *\overline{\beta} = \langle \alpha, \beta \rangle\ \nu \qquad \text{for} \quad \alpha, \beta \in \Omega_U^{p,q}\ . \tag{2.6}$$

On the basis $\{e^J \wedge \overline{e}^K\}$ we have

$$*(e^J \wedge \overline{e}^K) = \sigma(J,K)\ i^n\ e^{\overline{K}} \wedge \overline{e}^{\overline{J}}\ ,$$

$\overline{K}$ and $\overline{J}$ are the complementary multi-indices of K, J, and $\sigma(J,K) = \pm 1$ is chosen so that equation (2.6) is satisfied for $\alpha = \beta = e^J \wedge \overline{e}^K$.

Remark 2.18. The metric above defined is just the induced metric from the space of $(p+q)$ tensors on M $(\Gamma((TM)_{\mathbb{C}}^{p+q}))$ times a constant factor $(p+q)!$.

Definition 2.19. Let $L : \Omega^{p,q}(\mathbb{C}) \to \Omega^{p+1,q+1}(\mathbb{C})$ be the wedge multiplication by the Kähler form

$$L(\alpha) = \omega \wedge \alpha\ .$$

Define $\Lambda : \Omega^{p,q}(\mathbb{C}) \to \Omega^{p-1,q-1}(\mathbb{C})$ the adjoint of L,

$$\Lambda(\alpha) = L^*(\alpha)\ .$$

Hence for all $\alpha \in \Omega^{p,q},\ \beta \in \Omega^{p-1,q-1}$,

$$\langle \Lambda\alpha, \beta \rangle = \langle \alpha, \omega \wedge \beta \rangle\ .$$

On a surface($\dim_{\mathbb{C}} M = 1$), Λ establishes an isomorphism $\Omega^{1,1}(\mathbb{C}) \to \Omega^{0,0}(\mathbb{C})$ given by

$$\Lambda\alpha = \langle \alpha, \omega \rangle\ .$$

2.3 Holomorphic and Hermitian Bundles

We present a few facts concerning the holomorphic and hermitian structures of a vector bundle. Most of them are standard results that can be found in the reference literature [GH], [KN].

Lemma 2.20. *Let E be a Hermitian vector bundle. There exists only one connection D compatible with the holomorphic ($\overline{\partial}_E$) and hermitian (H) structures on E.*

Proof. Define an operator $D' : \Omega^0(E) \to \Omega^{1,0}(E)$ by

$$\langle D'\xi, \eta\rangle_H = \partial\langle \xi, \eta\rangle_H - \langle \xi, \overline{\partial}_E\eta\rangle_H \qquad \text{for any } \xi, \eta \in \Omega^0(E). \tag{2.7}$$

The non-degeneracy of the form $\langle , \rangle_H$ makes $D'\xi$ well defined. A computation shows us that D' is $\mathbb{C}$-linear on $\Omega^0(E)$ and $D'(f\xi) = \partial f \otimes \xi + fD'\xi$, hence it is a (1,0)-operator on E. Setting $D = D' + \overline{\partial}_E$ we get a connection compatible with the holomorphic structure.

The conjugation of (2.7) corresponds to

$$\langle \eta, D'\xi\rangle_H = \overline{\partial}\langle \eta, \xi\rangle_H - \langle \overline{\partial}_E\eta, \xi\rangle_H. \tag{2.8}$$

Then for arbitrary sections ξ, η, (2.7) and (2.8) imply

$$\begin{aligned} d\langle \xi, \eta\rangle_H &= \partial\langle \xi, \eta\rangle_H + \overline{\partial}\langle \xi, \eta\rangle_H = \\ &= \langle D'\xi, \eta\rangle_H + \langle \xi, \overline{\partial}_E\eta\rangle_H + \langle \overline{\partial}_E\xi, \eta\rangle_H + \langle \xi, D'\eta\rangle_H = \\ &= \langle D\xi, \eta\rangle_H + \langle \xi, D\eta\rangle_H. \end{aligned}$$

The equality of the first and last terms above is equivalent to $D(H) = 0$, so D is compatible with H. Because any connection satisfying the requirements of the Lemma has its (1,0) piece uniquely defined by (2.7), we conclude D is unique. □

The connection obtained in the previous Lemma is called the *hermitian connection* of the bundle $(E, \overline{\partial}_E, H)$. Then, by applying the operator ∂ two times on a coupling $\langle \xi, \eta \rangle_H$ we get, following (2.7),

$$\begin{aligned}\partial^2 \langle \xi, \eta \rangle_H &= \partial(\langle D'\xi, \eta \rangle_H + \langle \xi, \overline{\partial}_E \eta \rangle_H) = \\ &= \langle (D')^2 \xi, \eta \rangle_H - \langle D'\xi, \overline{\partial}_E \eta \rangle_H + \langle D'\xi, \overline{\partial}_E \eta \rangle_H + \langle \xi, (\overline{\partial}_E)^2 \eta \rangle_H = \\ &= \langle (D')^2 \xi, \eta \rangle_H \ .\end{aligned}$$

Notice the flip of the sign of the first term $\langle D'\xi, \overline{\partial}_E \eta \rangle_H$ due to the fact that a differential operator (∂) "jumps over" an odd degree form (see also Remark 2.6). Because $\partial^2 = 0$ we conclude $(D')^2 = 0$. In particular, the curvature of a hermitian connection is of holomorphic type (1,1).

Lemma 2.21. *Let H and K be hermitian metrics on the holomorphic bundle $(E, \overline{\partial}_E)$. Denote by D_H, D_K their hermitian connections, with curvatures F_H, F_K. Then there exists $S \in \Omega^0(EndE)$ that is positive and self-adjoint respect to H, K and such that*

$$F_H = F_K + \overline{\partial}_E (S^{-1} \circ D'_K(S)) \ .$$

Proof. Looking at the metrics as bundle isomorphisms $E \to \overline{E}^*$ we set an endomorphism by $S = K^{-1} \circ H$. Hence $H = K \circ S$, which is equivalent to

$$\langle \xi, \eta \rangle_H = \langle S\xi, \eta \rangle_K = \overline{\langle \eta, \xi \rangle}_H = \langle \xi, S\eta \rangle_K \qquad (2.9)$$

for $\xi, \eta \in \Omega^0(E)$. This shows us $S = S^{*K}$ is self-adjoint respect to K, and by a similar reason is $S^{-1} = (S^{-1})^{*H}$, which implies $S = S^{*H}$. The positiveness of S comes straightforward from $\langle S\xi, \xi \rangle_K = \langle \xi, \xi \rangle_H > 0$ if $\xi \neq 0$.

Now write $D_H = D'_H + \overline{\partial}_E$, $D_K = D'_K + \overline{\partial}_E$. From equations (2.7) and (2.9) we get

$$\begin{aligned}\langle S D'_H \xi, \eta \rangle_K =& \partial \langle S\xi, \eta \rangle_K - \langle S\xi, \overline{\partial}_E \eta \rangle_K = \\ =& \langle D'_K(S\xi), \eta \rangle_K .\end{aligned}$$

The arbitrarity of η then forces $SD'_H\xi = D'_K(S\xi) = D'_K(S)\xi + SD'_K\xi$ and we arrive at

$$D'_H = D'_K + S^{-1} \circ D'_K(S).$$

Being the curvature of a hermitian connection of type (1,1), it can be written for F_H

$$\begin{aligned} F_H = D'_H \circ \overline{\partial}_E + \overline{\partial}_E \circ D'_H = \\ = D'_K \circ \overline{\partial}_E + \overline{\partial}_E \circ D'_K + S^{-1} \circ D'_K(S) \circ \overline{\partial}_E + \overline{\partial}_E \circ S^{-1} \circ D'_K(S) = \\ = F_K + \overline{\partial}_E(S^{-1} \circ D'_K(S)) \ , \end{aligned}$$

what proves the Lemma. $\square$

Let E be a complex vector bundle on M, and denote by $\mathcal{C}$ the space of complex structures and by $\mathcal{H}$ the space of hermitian metrics on E. Define actions of $\mathcal{G}$ onto $\mathcal{C}$, $\mathcal{H}$ and $\Omega^2(\mathrm{End}E)$ in the following way: for $g \in \mathcal{G}$, $\overline{\partial}_E \in \mathcal{C}$, $H \in \mathcal{H}$ and $F \in \Omega^2(\mathrm{End}E)$ set

(2.10) $$g(\overline{\partial}_E) = g \circ \overline{\partial}_E \circ g^{-1} \ ,$$

(2.11) $$g(H)(\xi,\eta) = H(g^{-1}\xi, g^{-1}\eta), \qquad \text{for} \quad \xi,\eta \in \Omega^0(E) \ ,$$

(2.12) $$g(F) = g \circ F \circ g^{-1} \ .$$

These actions are compatible with the one on $\Omega^0(E)$ in the sense that $g(\overline{\partial}_E)(g\xi) = g(\overline{\partial}_E\xi)$, $g(H)(g\xi, g\eta) = H(\xi,\eta)$ and $g(F)(g\xi) = g(F\xi)$. Also, define $F_{(\overline{\partial}_E,H)}$ as the curvature of the hermitian connection in $(E, \overline{\partial}_E, H)$.

Lemma 2.22. *For $g \in \mathcal{G}$ it holds $g(F_{(\overline{\partial}_E,H)}) = F_{(g(\overline{\partial}_E),g(H))}$.*

Proof. Let D be the hermitian connection for $(\overline{\partial}_E, H)$. Then $g(D) = g \circ D \circ g^{-1}$ is compatible with $g(\overline{\partial}_E)$ since $g(D) = g \circ (D'_H + \overline{\partial}_E) \circ g^{-1} = g(D'_H) + g(\overline{\partial}_E)$. Hence $(g(D))^{0,1} = g(\overline{\partial}_E)$.

For $\xi, \eta \in \Omega^0(E)$ we have

$$d\langle \xi, \eta \rangle_{g(H)} = d\langle g^{-1}\xi, g^{-1}\eta \rangle_H =$$
$$= \langle D(g^{-1}\xi), g^{-1}\eta \rangle_H + \langle g^{-1}\xi, D(g^{-1}\eta) \rangle_H$$
$$= \langle g(D)\xi, \eta \rangle_{g(H)} + \langle \xi, g(D)\eta \rangle_{g(H)} \ .$$

Then $g(D)(g(H)) = 0$, and $g(D)$ is compatible with $g(H)$. Computing the curvature of $g(D)$ one obtains

$$F_{g(D)} = F_{(g(\overline{\partial}_E), g(H))} = g \circ D^2 \circ g^{-1} = g(F_{(\overline{\partial}_E, H)}).$$

□

For the next two lemmas assume M is a compact Riemann Surface.

Lemma 2.23. *Let $h \in C^\infty(M)$ such that $\frac{1}{2\pi}\int h \; d\mu = \deg(E)$. Then there is a hermitian metric H on $(E, \overline{\partial}_E)$ with $i\Lambda \mathrm{tr} F_H = h$.*

Proof. Fix any hermitian metric K on E. The idea is to construct a positive self-adjoint endomorphism S on E so that $H = K \circ S$ has the desired property. Then, set $S = e^v I$, where I is the identity endomorphism and $v \in C^\infty(M)$ is a function to be determined. From Lemma 2.21 we can write

$$F_H = F_K + \overline{\partial}_E(e^{-v} D'_K(e^v I)) =$$
$$= F_K + \overline{\partial}_E(e^{-v}\partial(e^v)I) =$$
$$= F_K + \overline{\partial}\partial v I.$$

Since $\dim_{\mathbb{R}} M = 2$ and by Remark 2.38 on the Laplacean operator,

$$\Delta v = -2i\Lambda\overline{\partial}\partial v \Rightarrow F_H = F_K + \frac{\Delta v}{2} i\omega I. \tag{2.13}$$

Let R be the rank of E. Taking the trace of the last equality in (2.13) and contracting with $i\omega$ we get

$$-\frac{R}{2}\Delta v + i\Lambda \mathrm{tr}(F_K) = i\Lambda \mathrm{tr}(F_H). \tag{2.14}$$

According to Lemma 2.39 the equation $\Delta v = \frac{2}{R} i\Lambda \mathrm{tr}(F_K) - \frac{2}{R} h$ has a unique solution up to an additive constant because the right-hand-side has zero mean value. Let v be such a solution, (2.14) then implies that $i\Lambda\mathrm{tr}(F_H) = h$, and the Lemma is proven. □

Now consider a $\mathbb{C}$-line bundle, that is, a rank 1 complex vector bundle over M. The next geometric lemma is necessary to assure the integrability of the logarithm of a holomorphic section's norm.

Lemma 2.24. *Let $L \to M$ be a holomorphic line bundle. If $\xi \in \Omega^0(E)$ is a holomorphic non-trivial section then ξ has finite many roots. In a neighborhood $U \subset M$ of a root it can be written $\xi(z) = h(z).z^k.\eta(z)$, where $z : U \to \mathbb{C}$ is a complex chart on U, $h \in C^\infty(U, \mathbb{C})$ never vanishes and η is a unitary section in $\Gamma_U(L)$.*

Proof. Let p be a root of ξ and $z : U \to \mathbb{C}$ a complex chart at p, with $z(p) = 0$. For η a unitary section in a neighborhood of p we write $\xi(z) = f(z).\eta(z)$. Then $\overline{\partial}_L(\xi) = \overline{\partial} f\eta + f\overline{\partial}_L(\eta)$. Let $\overline{\partial}_L(\eta) = \beta d\overline{z}\ \eta$, with $\beta \in C^\infty(U, \mathbb{C})$. Since $\overline{\partial} f = \overline{\partial}_z f d\overline{z}$ we get

$$
\begin{aligned}
0 = \overline{\partial}_L(\xi) =& (\overline{\partial}_z f + f\beta) d\overline{z}\ \eta \\
&\Rightarrow \overline{\partial}_z f + f\beta = 0.
\end{aligned}
$$

The $\overline{\partial}$-Poincaré Lemma [GH] gives us a smooth solution q defined in an open set $V \subset U$, with $p \in V$, for

$$\overline{\partial}_z q = \beta.$$

Applying the operator $\overline{\partial}_z$ to fe^q we obtain

$$
\begin{aligned}
\overline{\partial}_z(fe^q) = -f\beta e^q + fe^q\overline{\partial}_z q = \\
= e^q(-f\beta + f\beta) = 0.
\end{aligned}
$$

Then $fe^q = \varphi$ is a holomorphic map. In particular the zeros of f and the zeros of φ coincide in the open set of definition for φ. This shows us the set of zeros of f is discrete, and the compactness of M forces this set to be finite.

Noticing that $\varphi(0) = 0$ we derive $\varphi(z) = m(z).z^k$ for $|z|$ small, with m a non-vanishing function. Hence,

$$\xi(z) = e^{-q(z)}.m(z).z^k\ \eta(z) = h(z).z^k\ \eta(z)$$

for $|z|$ small and h a non-vanishing function. □

2.4 Tensors on Riemannian Manifolds

To define the Sobolev Spaces of tensors on M we need workable notions of derivatives of any order. Those are given by the Levi-Civita Connection. We recomend [DC] as a good reference in Riemannian Geometry.

Definition 2.25. The Levi-Civita Connection on M, denoted ∇, is the only torsion-free covariant derivative in TM satisfying $\nabla g = 0$.

The connection (∇) is originally defined as a differential operator $\Gamma(TM) \to \Gamma(TM \otimes T^*M)$. It can be generalized in a unique way for the cotangent bundle T^*M by requesting $\nabla\varphi$ to satisfy

$$d(\varphi(V)) = \nabla\varphi(V) + \varphi(\nabla V) \tag{2.15}$$

for any $\varphi \in \Gamma(T^*M),\ V \in \Gamma(TM)$.

Recall that the metric on M establishes an isometric identification $TM \cong T^*M$, taking V to V^* for any $V \in \Gamma(TM)$. This can be extended to any tensor bundle E over M respecting the order of the factors. Hence, if E is the product of k copies among $TM,\ T^*M$ then

$$E \cong TM^{\otimes k}. \tag{2.16}$$

For $V,\ W \in \Gamma(TM)$ it can be shown that $\nabla_W V^* = (\nabla_W V)^*$. In particular ∇_W commutes with the identifications defined in (2.16). Since ∇_W can be extended to any space of tensors by requesting it to satisfy Leibniz Rule on tensor products, one sees that ∇_W is well defined on the isomorphism classes given by (2.16). Hence we assume from now on that all tensors are contravariant and the connection is an operator

$$\nabla : \Gamma(TM) \longrightarrow \Gamma(TM^{\otimes 2}) \ . \tag{2.17}$$

Denote $T^k = TM^{\otimes k}$, for $k \geq 0$ an integer. The metric on T^k is the tensor metric induced from TM, $g^{\otimes k}$. We have the

Lemma 2.26. *There is only one operator* $\nabla : \Gamma(T^k) \to \Gamma(T^{k+1})$ *such that for any* $T,\ S \in \Gamma(T^k)$ *and* $W \in \Gamma(T^1)$,

$$\langle \nabla T, S \otimes W \rangle = \langle \nabla_W T, S \rangle \ .$$

Proof. It is well defined and unique as a consequence of $S \otimes W \in (T^k \otimes T^1)_x \to (x, \langle \nabla_W T, S \rangle) \in M \times \mathbb{R}$ being a pointwise linear morphism $T^{k+1} \to M \times \mathbb{R}$. Then it corresponds to a section of T^{k+1} by metric duality. □

Notice that when $k = 1$ the (∇) from Lemma 2.26 reduces to the usual Levi-Civita connection as in (2.17).

Denote by S_k the group of permutations of k elements. There is a linear right action of S_k onto $\Gamma(T^k)$. For if $T = t_1 \otimes \cdots \otimes t_k \in \Gamma(T^k)$ and $\pi \in S_k$ we set

$$\pi(T) = T^\pi = t_{\pi(1)} \otimes \cdots \otimes t_{\pi(k)} \ .$$

Observe that for another k-tensor S it holds $\langle T^\pi, S^\pi \rangle = \langle T, S \rangle$. The next Lemma is the best analogous of Leibniz Rule we can have for the covariant derivative (∇).

Lemma 2.27. *If $T \in \Gamma(T^k)$, $S \in \Gamma(T^l)$ then*

$$\nabla(T \otimes S) = (\nabla T \otimes S)^\pi + T \otimes \nabla S \ ,$$

where $\pi = (1, 2, \ldots, k, k+2, k+3, \ldots, k+l+1, k+1) \in S_{k+l+1}$.

Proof. Let $Q_1 \in \Gamma(T^k)$, $Q_2 \in \Gamma(T^l)$ and $W \in \Gamma(T^1)$ be any tensors. By definition,

$$\begin{aligned}
&\langle \nabla(T \otimes S), Q_1 \otimes Q_2 \otimes W\rangle = \langle \nabla_W(T \otimes S), Q_1 \otimes Q_2\rangle = \\
&= \langle \nabla_W T \otimes S + T \otimes \nabla_W S, Q_1 \otimes Q_2\rangle = \\
&= \langle \nabla T, Q_1 \otimes W\rangle\langle S, Q_2\rangle + \langle T, Q_1\rangle\langle \nabla S, Q_2 \otimes W\rangle = \\
&= \langle \nabla T \otimes S, Q_1 \otimes W \otimes Q_2\rangle + \langle T \otimes \nabla S, Q_1 \otimes Q_2 \otimes W\rangle = \\
&= \langle (\nabla T \otimes S)^\pi, Q_1 \otimes Q_2 \otimes W\rangle + \langle T \otimes \nabla S, Q_1 \otimes Q_2 \otimes W\rangle \ ,
\end{aligned}$$

and the result follows. □

Corollary 2.28. *Let T be a k-tensor and S be an l-tensor on M. For $m \geq 0$ an integer there are subsets $S_{j,m} \subset S_{m+k+l}$, $0 \leq j \leq m$ so that*

$$\nabla^m(T \otimes S) = \sum_{j=0}^{m} \sum_{\pi \in S_{j,m}} (\nabla^{m-j} T \otimes \nabla^j S)^\pi \ .$$

Proof. It is done by induction on m with the observation that $\nabla(T^\pi) = (\nabla T)^{(\pi, k+1)}$ for all k-tensors T and $\pi \in S_k$. □

Now consider E an arbitrary vector bundle on M with connection D. With slight modifications the results in this section can be extended to sections of the bundle $E \otimes T^j$. The connection D has range in $\Gamma(E \otimes T^*M)$. Let $\tilde{g}$ be the metric isomorphism $T^*M \to TM$. Then set

$$\tilde{D} = (\mathrm{id}_E \otimes \tilde{g}) \circ D : \Gamma(E) \to \Gamma(E \otimes T^1) \ . \tag{2.18}$$

The analogous for the Levi-Civita connection on the bundle (E, D) is

$$\begin{aligned} &\nabla_E : \Gamma(E \otimes T^j) \to \Gamma(E \otimes T^{j+1}) \\ &\nabla_E(\xi \otimes T) = (\tilde{D}\xi \otimes T)^{\mathrm{id}\otimes\pi} + \xi \otimes \nabla T \ , \end{aligned} \tag{2.19}$$

where $\mathrm{id}\otimes\pi$ is a permutation among the $(j+1)$ contravariant factors of $\tilde{D}\xi\otimes T$. This permutation is necessary in order to make ∇_E coincide with ∇ of Lemma 2.26, when $E = T^l$, $l \geq 0$.

2.5 Sobolev Spaces and Embeddings

With the notion of covariant derivative on tensors we are ready for the introduction of the Sobolev norms.

Let E be a vector bundle with connection D and hermitian metric H. Let $E^j = E \otimes T^j$ $(j \geq 0)$ and denote by $\Gamma^k(E^j)$ the space of sections of E^j which are of class C^k when seen as functions $M \to E^j$. Notice that even though (∇_E) has been previously defined in $\Gamma(E^j) = \Gamma^\infty(E^j)$ it is perfectly well defined, up to order k, on $\Gamma^k(E^j)$.

Definition 2.29. Let $k \geq 0$, $l \geq 0$ integers, $p \geq 1$ a real number. For every $\xi \in \Gamma^k(E^l)$ the (k,p)-Sobolev norm of ξ is defined by

$$\|\xi\|_{k,p} = \left(\int |\xi|^p + |\nabla_E \xi|^p + \cdots + |\nabla_E^k \xi|^p \ d\mu \right)^{\frac{1}{p}} \in \mathbb{R} \ . \tag{2.20}$$

Here $|\nabla_E^j \xi| = \langle \nabla_E^j \xi, \nabla_E^j \xi \rangle^{\frac{1}{2}}$ and $\nabla_E^j \xi \in \Gamma^{k-j}(E^{l+j})$.

When $p = 2$ the norm $\| \cdot \|_{k,2}$ does come from an inner-product in $\Gamma^k(E^l)$, since

$$\int \xi\eta + \langle \nabla_E \xi, \nabla_E \eta \rangle + \cdots + \langle \nabla_E^k \xi, \nabla_E^k \eta \rangle \ d\mu = \langle\!\langle \xi, \eta \rangle\!\rangle_{k,2} \tag{2.21}$$

is a bilinear symmetric positive definite form on $\Gamma^k(E^l) \ni \xi, \eta$.

Definition 2.30. The Sobolev Space $W_{k,p}(E^l)$ is the completion of the vector space $\Gamma^k(E^l)$ under the Sobolev norm $\|\cdot\|_{k,p}$. The Sobolev Space $W_{-k,q}(E^l)$ is by definition $(W_{k,p}(E^l))^*$ where $\frac{1}{q}+\frac{1}{p}=1,\ p,q\geq 1$. When $p=2$ it is customary to write $W_{k,2}(E^l)=H_k(E^l)$.

Remark 2.31. In the non-compact case the completion is taken over the subspace of tensors in $\Gamma^k(E^l)$ where $\|\cdot\|_{k,p}<\infty$.

Remark 2.32. The spaces $\Gamma^k(E^l)$ also endow a norm given by

$$\|\xi\|_{C^k} = \sup_{0\leq j\leq k,\ x\in M} \left\{|\nabla^j_E \xi(x)|\right\} . \tag{2.22}$$

It can be shown that $(\Gamma^k(E^l), \|\cdot\|_{C^k})$ is a Banach space. The space $\Gamma(E^l)$ is a dense subspace of $\Gamma^k(E^l)$ and $W_{k,p}(E^l)$ in their respective norms.

The next results deal with conditions for which the spaces $W_{k,p},\ \Gamma^k$ are included in $W_{l,q}$ and/or Γ^l by a continuous inclusion map. They are stated in [Pl] and their proofs can be found in [Ca]. We stress the importance of the compactness hypothesis for M. Similar embeddings can be proved for non-compact M, but the conditions on k,l,p,q become much more restrictive. Non-compact M results are found in [GT], [Ab].

In the following F denotes any tensor bundle of the form E^j. Assume $\dim_{\mathbb{R}} M = n$.

Theorem 2.33. *Let $1\leq p,q<\infty$ real numbers and $k,l\in\mathbb{Z}$. If $k-\frac{n}{p}\geq l-\frac{n}{q}$ and $k\geq l$ then $W_{k,p}(F)\subseteq W_{l,q}(F)$ and the inclusion map is continuous. If $k-\frac{n}{p}>l-\frac{n}{q}$ and $k>l$ then the inclusion map is compact.*

Theorem 2.34. *If $p\geq 1$ is a real number and $k,l\in\mathbb{Z}$ satisfy $l\geq 0$ and $k-\frac{n}{p}>l$ then $W_{k,p}(F)\subset\Gamma^l(F)$ and this inclusion is compact.*

Associated with these embeddings there are some multiplication theorems which generalize Hölder's inequality to the Sobolev spaces of any order. We present a simple version of them that suffices our needs when $E=T^j$.

Proposition 2.35. *Let $k, l, j, r \geq 0$ be integers, $s, p, q \geq 1$ reals with $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$. Then there is a bilinear continuous map*

$$\begin{array}{ccc} W_{k,p}(T^j) \times W_{l,q}(T^r) & \longrightarrow & W_{m,s}(T^{j+r}) \\ (t, z) & \longmapsto & t \otimes z \end{array}$$

where $m = min\{k, l\}$.

Proof. We show the map is well defined and continuous by estimating $\|t \otimes z\|_{m,s}$. For $0 \leq d \leq m$ the application of Corollary 2.28 gives us

$$\nabla^d (t \otimes z) = \sum_{a=0}^{d} \sum_{\pi \in S_{a,d}} (\nabla^{d-a} t \otimes \nabla^a z)^\pi \ .$$

Because the permutations act by isometries in the space of tensors we get, for almost every $x \in M$,

$$|(\nabla^{d-a} t \otimes \nabla^a z)^\pi_{(x)}| = |(\nabla^{d-a} t \otimes \nabla^a z)_{(x)}| = |\nabla^{d-a} t|_{(x)} |\nabla^a z|_{(x)} \ .$$

The maps $x \mapsto |\nabla^{d-a} t|_{(x)}$ and $x \mapsto |\nabla^a z|_{(x)}$ are in L_p, L_q respectively, and by Hölder's inequality their product is in L_s . Hence,

$$\|\nabla^{d-a} t \otimes \nabla^a z\|_{L_s} \leq \|\nabla^{d-a} t\|_{L_p} \|\nabla^a z\|_{L_q} \leq \|t\|_{k,p} \|z\|_{l,q} \ ,$$

and we conclude

$$\begin{aligned} \|\nabla^d (t \otimes z)\|_{L_s} &\leq C(d, r, j) \|t\|_{k,p} \|z\|_{l,q} \qquad \forall\ 0 \leq d \leq m \\ &\Rightarrow \|t \otimes z\|_{m,s} \leq C(m, j, r) \|t\|_{k,p} \|z\|_{l,q} \ . \end{aligned}$$

□

2.6 Differential Operators on Manifolds

Definition 2.36. Let $k \geq d \geq 0$ be integers. A d^{th} order linear differential operator on M is a map $L : C^k(M) \to C^{k-d}(M)$ given by

$$L(u) = \sum_{j=0}^{d} \langle A^j, \nabla^j u \rangle \tag{2.23}$$

where $A^j \in \Gamma(T^j)$. One requires A^d not to be identically zero.

This definition corresponds, in coordinates, to the usual one (with smooth coefficients) in Euclidean space.

A second order linear differential operator L is called elliptic if at every $x \in M$, $A^2_{(x)}$ is symmetric and positive or negative definite when seen as an endomorphism on T_xM.

Definition 2.37. The Laplacean (Δ) is a second order elliptic operator given by $A^0 = A^1 = 0$, $A^2 = I$.

$$\Delta u = \langle I, \nabla^2 u \rangle \ .$$

Here I is the 2-tensor equivalent to the identity endomorphism in TM.

Remark 2.38. The above definition for the Laplacean is equivalent to taking the trace of the hessian of u, as defined in [DC]. In a Kähler manifold it also holds $\Delta = 2\Delta_{\overline{\partial}}$, where $\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ (see [GH]). For a Riemann surface we can then write $\Delta u = 2\overline{\partial}^*\overline{\partial} u = -2i\Lambda\overline{\partial}\partial u$, for any $u \in C^2(M)$.

The operator L in (2.23) is continuous respect to the Sobolev norms in $C^k(M)$ and $C^{k-d}(M)$. It can then be extended to a linear bounded operator $W_{k,p}(M) \to W_{k-d,p}(M)$ $(p \geq 1)$.

In particular the Laplacean is well defined from $H_k(M)$ to $H_{k-2}(M)$ for $k \geq 2$. It can be extended to an operator $H_1(M) \to H_{-1}(M)$ in the distributional sense, since for $u, v \in C^\infty(M)$,

$$\left| \int \Delta u v \ d\mu \right| = \left| \int -\langle \nabla u, \nabla v \rangle \ d\mu \right| \leq \|u\|_{1,2} \|v\|_{1,2} \ .$$

Hence we get $\Delta : H_k(M) \to H_{k-2}(M) \quad \forall\ k \geq 1$.

The next lemma is a classical result on elliptic operators on closed Riemann manifolds. It combines regularity theory for elliptic operators with existence of weak solutions for Poisson's equation on a Manifold. See, for instance [Ab] theorems 4.7 and 2.34.

Lemma 2.39. *Denote by* $(H_k)_0 \subset H_k$ *the Hilbert subspace of functions with 0 mean value. Then, the operator* $\Delta : (H_k)_0 \to (H_{k-2})_0$ *is a homeomorphism.*

2.7 Fredholm Operators

Let V, W be Banach spaces (over $\mathbb{R}$ or $\mathbb{C}$) and denote $\mathcal{L}(V, W)$ the set of linear continuous maps $V \to W$. Recall that a function $\varphi : V \to W$ is said to be Frechèt differentiable at $x \in V$ if there exists $A(= A_x) \in \mathcal{L}(V, W)$ for which

$$\lim_{z \to 0} \frac{\|\varphi(x+z) - \varphi(x) - Az\|_W}{\|z\|_V} = 0 \ .$$

Write $\varphi'_{(x)} = A$. Then φ is differentiable on V if the above condition holds for all $x \in V$. Like in ordinary Calculus we say φ is of class C^k $(k \geq 1)$ if $\varphi' : V \to \mathcal{L}(V, W)$ is of class C^{k-1}, and $\varphi \in C^0$ if φ is continuous. A map is continuously differentiable if it belongs to $\cap_{k=0}^{\infty} C^k$.

Chain Rule. *Let* V, W, B *be Banach spaces,* $k \geq 0$ *an integer. If* $\varphi : V \to W$ *and* $\psi : W \to B$ *are* C^k *maps then* $\psi \circ \varphi$ *is also* C^k*. If* $k \geq 1$ *and* $x \in V$*, we have*

$$(\psi \circ \varphi)'_{(x)} = \psi'_{(\varphi(x))} \cdot \varphi'_{(x)} \ .$$

Implicit Function Theorem. *Let* $F : V \times S \longrightarrow W$ *be a (Frechèt) differentiable map and* V, S, W *Banach spaces. Suppose* $\exists$ $(v_0, s_0) \in V \times S$ *with* $F(v_0, s_0) = 0$*, and* $\frac{\partial F}{\partial v}_{(v,s)} : V \longrightarrow W$ *a continuous linear isomorphism between Banach spaces, in a neighborhood of* (v_0, s_0)*. Then there is a neighborhood* $U \subset S$ *of* s_0 *and a differentiable map* $q : U \longrightarrow V$ *so that* $q(s_0) = v_0$ *and* $F(q(s), s) = 0 \quad \forall s \in U$*. The map* q *is unique in the sense that any other map defined in some neighborhood of* s_0 *with the same property coincides with* q *in* U*. If* F *is of class* C^k*,* $k \geq 1$*, then* $q \in C^k$*.*

Proof. See Kantorovich [Ka]. □

Definition 2.40. A map $\varphi \in \mathcal{L}(V, W)$ is said to be Fredholm if $\text{im}\varphi$ is closed in W and both $\ker\varphi$, $\text{coker}\varphi = W/\text{im}\varphi$ have finite dimension. Define $\text{index}\varphi = \dim \ker\varphi - \dim \text{coker}\varphi$.

The proof of next two lemmas is found in [La].

Lemma 2.41. *Let $T : V \to V$ be a compact map. Then $I - T : V \to V$ is Fredholm.*

Lemma 2.42. *Denote by* $\text{Fr}(V, W)$ *the Fredholm operators $V \to W$. Then* $\text{Fr}(V, W)$ *is open in* $\mathcal{L}(V, W)$*. The map* index : $\text{Fr}(V, W) \to \mathbb{Z}$ *is continuous, hence constant in the connected components of* $\text{Fr}(V, W)$.

Using these previous results we can prove the next lemma.

Lemma 2.43. *Let f be a smooth function on M. Then for $k \geq 1$ the operator $L : H_k(M) \to H_{k-2}(M)$ given by $L(v) = \Delta v - fv$ is Fredholm of index zero. In particular, for $f \geq 0$, $f \neq 0$, L is isomorphism.*

Proof. Denote the average of a function v by $\overline{v} = \int v \, d\mu$. Then the map $T : H_k \to H_{k-2}$, $T(v) = \Delta v + \overline{v}$ is an isomorphism, since $\Delta : (H_k)_0 \to (H_{k-2})_0$ is one-to-one. Observe that $T^{-1} \circ L(v) = v - \tilde{T}(v)$ with $\tilde{T} : H_k \to H_k$ a compact operator (because of the compact embedding $H_k \subset H_{k-2}$). Hence $T^{-1} \circ L$ is Fredholm by Lemma 2.41, and L is Fredholm.

Consider the continuous family of Fredholm maps $L_s = \Delta - (f + s)$, for all $s \in \mathbb{R}$. Fixing $s > \|f\|_\infty$ the correspondent L_s is an isomorphism. For if $v \in H_k$ satisfies $L_s(v) = 0$ we obtain,

$$0 = \int L_s(v) v \, d\mu = \int -|\nabla v|^2 - (f + s)v^2 \, d\mu \Rightarrow v \equiv 0 \ ,$$

and conclude L_s is injective. For the onto part, notice that $\Delta v - (f+s)v = w$ is the Euler-Lagrange equation for the functional $J(v) = \int \frac{1}{2}|\nabla v|^2 + \frac{1}{2}(f+s)v^2 + wv \; d\mu$, which is convex and bounded from below in H_1. A minimizing sequence converges weakly to a critical point $v_0 \in H_1$, that is, a weak solution of $L_s(v) = w$. If $w \in H_{k-2}$ the regularity of the Laplacean, as in Lemma 2.39, forces $v \in H_k$.

As a consequence we get $\text{index}(L_s) = 0$ when $s > \|f\|_\infty$. Because of the continuity of the maps index and $s \mapsto L_s$, we conclude $\text{index}(L_s) = 0 \quad \forall s \in \mathbb{R}$.

In the case $f \geq 0$ but not identically zero $L = L_0$ is still injective. Then $0 = \text{index}(L) = \dim \ker L - \dim \operatorname{coker} L \Rightarrow \dim \operatorname{coker} L = 0$ and L is onto, so is an isomorphism. □

2.8 Other inequalities

Trudinger Inequality. *There exist constants $\beta,\ C > 0$ such that for any $u \in W_{1,2}(M)$ with $\int u \; d\mu = 0$ and $\|\nabla u\|_2 \leq 1$ it holds*

$$\int e^{\beta u^2} \; d\mu \leq C.$$

Proof. See [Tr]. □

Trudinger's inequality is of capital importance in the study of some non-linear partial differential equations, like the one in [KW1]. It is also necessary for a direct proof of existence of solutions of $\Delta u + bfe^{2u} - \lambda = 0$ when λ lies in some interval of positive numbers. The proof we shall present for this equation does not make use of it but needs one of its corollaries.

Corollary 2.44. *There exist constants $C,\ \gamma > 0$ such that for any $u \in W_{1,2}$ and $\alpha > 0$ it holds*

$$\int e^{\alpha|u|} \; d\mu \leq C \exp(\alpha|\overline{u}| + \frac{\alpha^2\|\nabla u\|_2^2}{4\beta}). \tag{2.24}$$

Proof. Assume $\nabla u \neq 0$. Setting $v = (u - \overline{u})/\|\nabla u\|_2$, v satisfies the conditions of Trudinger's inequality. Hence, $\int e^{\beta v^2} \, d\mu \leq C$. On the other hand it holds

$$\frac{\alpha^2 \|\nabla u\|_2^2}{4\beta} + \beta v^2 \geq \alpha \|\nabla u\|_2 |v| \geq \alpha |u| - \alpha |\overline{u}|.$$

Exponentiating this inequality and integrating over M one obtains

$$C \exp\left(\frac{\alpha^2 \|\nabla u\|_2^2}{4\beta}\right) \geq \int e^{\beta v^2 + \frac{\alpha^2 \|\nabla u\|_2^2}{4\beta}} \, d\mu \geq e^{-\alpha|\overline{u}|} \int e^{\alpha |u|} \, d\mu \ ,$$

from which the Corollary follows. □

Poincaré Inequality. *Let M be a compact Riemannian Manifold. Then there exists a constant $C > 0$ so that for all $u \in H_1(M)$ with $\overline{u} = 0$ it holds*

$$\|u\|_2 \leq C \|\nabla u\|_2 \ .$$

Chapter 3

The Curvature Problem on Surfaces

3.1 Introduction

Let M be a closed surface with Riemannian metric g. A metric $\tilde{g}$ on M is said to be *conformally equivalent* to g if there exists a diffeomorphism φ defined on M and a positive function $\lambda \in C^\infty(M)$ with $\varphi^*(\tilde{g}) = \lambda\, g$. If φ can be chosen to be the identity one says $\tilde{g}$ is *pointwise conformally equivalent* to g. An interesting question is whether a given function $K \in C^\infty(M)$ is the Gaussian curvature of a metric $\tilde{g}$ that is (pointwise) conformally equivalent to g. This problem is more specific than asking whether a given K is the Gaussian curvature of some metric on M.

Kazdan and Warner [KW1] studied this problem obtaining some necessary and sufficient conditions on K for the existence of such $\tilde{g}$. In the pointwise conformal case they derive an elliptic equation of the form

$$\Delta w + K e^{2w} - k = 0 \ , \tag{3.1}$$

with solutions corresponding to the existence of the metrics $\tilde{g}$ (here k is the Gaussian curvature of g). A change of variables makes equation (3.1) equivalent to

$$\Delta u + h e^{u} - c = 0 \ , \tag{3.2}$$

where $c = 2\overline{k}$ is a constant and h is K times a positive function depending on k. In particular, a Gauss-Bonnet condition applies similarly to K and h, that is,

$$\int Ke^{2w}\ d\mu = \int k\ d\mu = 2\pi\chi(M)$$
(3.3)
$$\int he^{u}\ d\mu = \int c\ d\mu = 4\pi\chi(M)\ .$$

These equations give necessary conditions for the signs of K and h, according to the sign of $\chi(M)$, for solvability of (3.1), (3.2). Kazdan and Warner found that the theory of (3.2) is strongly dependent on the sign of c. The three cases of signs are then treated separately. To summarize their results on (3.2) we state the following theorem, concerning existence of solutions.

Theorem 3.1. *Let $h \in C^{\infty}(M)$. Then,*

1. *If $c > 0$ a necessary condition for the existence of a solution is that h be positive somewhere. In this case there exists $0 < c_{+}(h) \leq \infty$ such that (3.2) has a solution if $0 < c < c_{+}(h)$.*

2. *If $c = 0$ and $h \neq 0$ a solution exists if and only if $\overline{h} < 0$ and h is positive somewhere.*

3. *If $c < 0$, a necessary condition is $\overline{h} < 0$. In such case there exists a constant $-\infty \leq c_{-}(h) < 0$ so that (3.2) has solutions if and only if $c_{-}(h) < c < 0$.*

Kazdan and Warner showed that the methods employed to solve (3.2) work for arbitrary dimensions of M and for a function $h \in L_p(M)$, $p > \dim M$, at least for $c < 0$. The equation we will investigate in Chapter 4 and solve by the Implicit Function Theorem is essentially the same as (3.2) with the extra conditions that h does not change sign and has finite many roots. In particular, when $h \leq 0$, Theorem 10.5 (a) of [KW1] gives $c_{-}(h) = -\infty$.

For an h that keeps the sign the condition $c = 0$ is admissible only if $h \equiv 0$. Hence, we study the cases $c \neq 0$, applying variational techniques when $c > 0$ and the upper-lower solutions method when $c < 0$. We remark that solutions for (3.2) in $H_1(M)$ are always smooth by standard Schauder Theory on elliptic operators (Lemma 2.39) and Proposition 5.4.

3.2 The Variational Method

In equation (3.2) assume $c > 0$ and $h \geq 0$, $h \neq 0$. To prove Theorem 3.1 we follow [Bg], [Ms], [KW1] and minimize the functional

$$J(u) = \int \frac{1}{2}|\nabla u|^2 + cu \; d\mu$$

on a suitable submanifold $S \subset H_1(M)$, where S is

$$S = \{u \in H_1(M) \quad | \quad \int he^u \; d\mu = c|M|\} \ .$$

We first show J is bounded from below on S. For $u \in S$ we estimate $\overline{u} = \frac{1}{|M|}\int u \; d\mu$:

$$\begin{aligned} c|M| &= \int he^u \; d\mu = e^{\overline{u}} \int he^{u-\overline{u}} \; d\mu \\ \Rightarrow \; \overline{u} &= \ln\left(\frac{c|M|}{\int he^{u-\overline{u}} \; d\mu}\right) \\ &\geq \ln(c|M|) - \ln\left(\|h\|_\infty \int e^{u-\overline{u}} \; d\mu\right) \ . \end{aligned} \tag{3.4}$$

Using Trudinger's Inequality we obtain

$$\begin{aligned} \overline{u} \geq \ln\left(\frac{c|M|}{\|h\|_\infty}\right) &- \ln\left(C \exp\left(\frac{\|\nabla u\|_2^2}{\beta}\right)\right) \\ &\geq C(c, |M|, \|h\|_\infty) - \frac{\|\nabla u\|_2^2}{\beta} \ . \end{aligned}$$

Hence, for $u \in S$, $J(u)$ becomes

$$(3.5)\qquad \begin{aligned} J(u) &= \int \frac{1}{2}|\nabla u|^2 \, d\mu + c|M|\overline{u} \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{c|M|}{\beta}\|\nabla u\|_2^2 + C \\ &\geq \left(\frac{1}{2} - \frac{c|M|}{\beta}\right)\|\nabla u\|_2^2 + C \ . \end{aligned}$$

Now assume $c < \frac{\beta}{2|M|}$. Inequalities (3.5) imply that the functional J is bounded from below and any minimizing sequence is also bounded in $H_1(M)$. Let $\{u_n\} \subset S$ be a minimizing sequence for J. Since Hilbert spaces are weakly compact, this gives us a subsequence, also called $\{u_n\}$ converging weakly to u_0. It can be shown from the proof of the Sobolev Embedding Theorem that the weak convergence in H_1 implies strong convergence in L_p, $p \geq 1$. By Corollary 5.5, $e^{u_n} \to e^{u_0}$ in L_p, and we conclude that $u_0 \in S$. Moreover,

$$(3.6)\qquad \|u_0\|_p = \lim \|u_n\|_p \ .$$

Weak convergence of the sequence implies

$$(3.7)\qquad \liminf \|u_n\|_{1,2} \geq \|u_0\|_{1,2} \ .$$

From (3.6) and (3.7)

$$\liminf J(u_n) \geq J(u_0) \ ,$$

so u_0 is a point of minimum of $J|_S$.

Observe that S is the preimage of a regular value of the function $T : H_1(M) \to \mathbb{R}$, $T(u) = \int he^u \, d\mu$. We know from differential topology that since u_0 is a critical point of $J|_S$, there exists $s \in \mathbb{R}$ with

$$dJ_{u_0} = s \, dT_{u_0} \ .$$

This equality in $H_{-1}(M)$ is equivalent to

$$-\Delta u_0 + c = she^{u_0} \ ,$$

which after integration yields

$$c|M| = s\int he^{u_0}\ d\mu = sc|M| \ ,$$
$$\text{so} \qquad s = 1 \ .$$

Hence,

$$\Delta u_0 + he^{u_0} - c = 0 \ .$$

Theorem 3.1 (1) is then proved for some constant $c_+(h) \geq \frac{\beta}{2|M|}$. Kazdan and Warner remark that despite $0 < c < c_+(h)$ may not in general be a necessary condition for existence of solutions, there are examples of h on the sphere for which (3.2) has no solution if $c \geq \frac{\beta}{2|M|}$.

3.3 The method of upper and lower solutions

This is an old method that can be found in [SY]. In [KW1] it is used to solve (3.2) when $c < 0$ and an upper solution is known to exist. To be precise, we have

Definition 3.2. An upper solution for the equation $\Delta u + he^u - c = 0$ is a function $u^+ \in H_1(M)$ so that $\Delta u^+ + he^{u^+} - c \leq 0$. A lower solution is a function $u^- \in H_1(M)$ satisfying the opposite inequality.

More generally, we consider the equation

$$\Delta u = f(x,u) \ , \tag{3.8}$$

with $f : M \times \mathbb{R} \to \mathbb{R}$ a continuous function. We also assume an extra condition on f that will imply boundedness for the sequence of upper and lower solutions constructed in the next lemma. Namely, we require f to have a.e. derivatives in u satisfying $|\frac{\partial f}{\partial u}(x,u)| \leq k(x)$, k a continuous positive function.

Lemma 3.3. *Let $f,\ k$ be functions as above. If $u^+,\ u^- \in H_1(M)$ are upper and lower solutions and $u^- \leq u^+$ a.e., then* (3.8) *has a solution $u \in H_2(M)$ with $u^- \leq u \leq u^+$.*

Proof. For any integer $s \geq 1$ set $L : H_s(M) \to H_{s-2}(M)$ by $L(u) = \Delta u - ku$. Even though the function $k(x)$ is just continuous, the statement of Lemma 2.43 is still true, therefore L is a linear homeomorphism. Define inductively $u_0 = u^+$ and $u_j = L^{-1}(f(x, u_{j-1}) - ku_{j-1})$ for $j \geq 1$. Notice that if $u_{j-1} \in H_0$ then

$$\|f(x, u_{j-1})\|_2 \leq \|f(x, 0)\|_2 + \|k\|_\infty \|u_{j-1}\|_2 < \infty \ ,$$

thus $f(x, u_{j-1}) - ku_{j-1} \in H_0$ and $u_j \in H_2$.

We need to show the sequence $\{u_j\}$ is monotonically non-increasing and bounded by u^+ and u^-. This comes from the maximum principle applied to the operator L. Let $v \in H_1$ be a function such that $L(v) \geq 0$, that is,

$$L(v)(w) = \int -\langle \nabla v, \nabla w \rangle - kvw \ d\mu \geq 0 \ ,$$

whenever $w \geq 0$ and $w \in H_1$. Then if $v_+(x) = \max\{0, v(x)\}$ we have

$$\begin{gathered} 0 \leq L(v)(v_+) = \int -|\nabla v_+|^2 - k(v_+)^2 \ d\mu \leq 0 \\ \Rightarrow v_+ \equiv 0 \quad \text{in } H_1(M) \ . \end{gathered}$$

Hence $L(v) \geq 0 \Rightarrow v \leq 0$.

Applying the maximum principle to $(u_1 - u_0)$ we obtain

$$L(u_1 - u_0) = f(x, u_0) - ku_0 - \Delta u_0 + ku_0 \geq 0$$

using the hypothesis that u_0 is an upper solution. We conclude $u_1 \leq u_0$ and similarly, $u_j \leq u_{j-1}$ for all $j \geq 1$. Notice that $\{u_j\}$ are upper solutions.

Now assume $u^- \leq u_{j-1}$. Then, there exists a function $\hat{u}(x)$ between $u^-(x)$ and $u_{j-1}(x)$ with

$$\begin{aligned} L(u^- - u_j) &= \Delta u^- - ku^- - (f(x,u_{j-1}) - ku_{j-1}) = \\ &= \Delta u^- - f(x,u^-) + f(x,u^-) - f(x,u_{j-1}) + k(u_{j-1} - u^-) = \\ &= \Delta u^- - f(x,u^-) + \left[k - \frac{\partial f}{\partial u}(x,\hat{u})\right](u_{j-1} - u^-) \geq 0 \end{aligned}$$

since u^- is a lower solution for (3.8). This proves $u^- \leq u_j \ \forall j$, since it is true for $j = 0$.

Similarly, we start with $v_0 = u^-$ and construct $\{v_j\}$ inductively, $v_j = L^{-1}(f(x,v_{j-1}) - kv_{j-1})$. The sequences $\{u_j\}$ and $\{v_j\}$ enjoy analogous properties, and either one can be used to construct a solution u for (3.8). For this sake let us consider $\{u_j\}$.

Since $v_j \leq u_l$ for $j, l \geq 0$ the sequence $\{u_j\}_{j\geq 1} \subset H_2$ is pointwise bounded above and below by u_1 and $v_1 \in C^0(M)$. Because $\{u_j\}$ is monotonic, the function $u(x) = \lim u_j(x)$ is well defined and $u \in L_\infty(M)$. The Lebesgue Dominated Convergence Theorem implies the L_p convergence for $u_j \to u$, since

$$|u_j - u| = u_j - u \leq u_1 - v_1 \in L_p \ .$$

The fact that $\{u_j\}$ is a Cauchy sequence in L_2 forces $\{f(x,u_j) - ku_j\}$ to be Cauchy as well:

$$|f(x,u_j) - ku_j - (f(x,u_l) - ku_l)| \leq 2k|u_j - u_l| \ ,$$

and since L is a linear homeomorphism we conclude $\{u_j\}$ is Cauchy in H_2. Hence, $u \in H_2$ and by continuity

$$\begin{aligned} L(u) = \lim_j L(u_j) = \lim_j (f(x,u_{j-1}) - ku_{j-1}) = f(x,u) - ku \\ \Rightarrow \ \Delta u = f(x,u) \ , \end{aligned}$$

and u is a solution satisfying $u^- \leq u \leq u^+$. □

Lemma 3.3 has different versions when one changes the hypothesis on the functions k, f, u^+ and u^-. For instance, in [KW1] the function $f(x,u) = c - h(x)e^u$ is not necessarily continuous, because $h(x)$ is just assumed to be in L_p. On the other hand, u^+ and u^- must be in $W_{2,p}$ for some $p > \dim M$. Their proof also uses stronger results on the elliptic theory on L_p spaces.

For our needs the proof of Lemma 3.3 can be done with slightly weaker hypothesis on f, k: it is enough to assume $|\frac{\partial f}{\partial u}(x,u)| \leq k(x)$ whenever $u \leq u^+(x)$ and $u^+ \in H_1(M) \cap C^0(M)$. This is the case when $f(x,u) = c - he^u$ and $k(x) = \max\{1, |h|e^{u^+}\}$ $(h \in C^\infty(M))$.

Corollary 3.4. *Equation $\Delta u + he^u - c = 0$ has a solution $u \in C^\infty(M)$ if there exist upper and lower solutions $u^+, u^- \in H_1(M)$, $u^- \leq u^+$ and u^+ is continuous.*

Observe that Corollary 3.4 does not make hypothesis on the sign of c. Solving equation (3.2) turns into the problem of finding functions u^+ and u^- that solve the two different inequalities we can get out of (3.2). If $c < 0$ and $|c|$ is not too large one can always construct these functions.

Lemma 3.5. *Let $c < 0$ and $\overline{h} < 0$. Then there exists a continuous upper solution u^+ for* (3.2).

Proof. Let v be the smooth solution of $\Delta v = \overline{h} - h$. For $a, b \in \mathbb{R}$ let $w = av + b$. Then

$$\begin{aligned} \Delta w + he^w - c &= a(\overline{h} - h) + he^{av+b} - c \\ &= a\overline{h} + h(e^{av+b} - a) - c \ . \end{aligned} \tag{3.9}$$

One can adjust the constants $a > 0$, b so that

$$|e^{av(x)+b} - a| \leq \frac{a|\overline{h}|}{2\|h\|_\infty} \qquad \forall x \in M \ .$$

Then, (3.9) leads to

$$\Delta w + he^w - c \leq a\overline{h} + \left|\frac{a\overline{h}}{2}\right| - c \leq \frac{a\overline{h}}{2} - c \ ,$$

and if $a\overline{h}/2 \leq c < 0$ then $u^+ = w$ is a continuous upper solution. □

An upper solution of (3.2) is still an upper solution if we substitute c by $\tilde{c}$ when $0 > \tilde{c} \geq c$. Hence we let $c_-(h)$ be the infimum of those c's for which there are upper solutions. Because h is bounded, for any fixed c there are sufficiently large negative constants u^- which are lower solutions and $u^- \leq u^+$. By Corollary 3.4 there exists a solution u for (3.2) if and only if $c_-(h) < c < 0$, and this proves Theorem 3.1 part (3). Notice that when $c < 0$ the condition $\overline{h} < 0$ is actually necessary, since

$$\begin{aligned} 0 &= \int e^{-u}\Delta u + h - ce^{-u} \ d\mu = \overline{h} + \int e^{-u}|\nabla u|^2 - ce^{-u} \ d\mu \\ \Rightarrow \ \overline{h} &= c\int e^{-u} \ d\mu - \int e^{-u}|\nabla u|^2 \ d\mu < 0 \ . \end{aligned}$$

Remark 3.6. When $h \leq 0$ one can show directly from (3.9) that $c_-(h) = -\infty$. It suffices taking $a > 0$ with $a\overline{h} < c$ and b satisfying $e^{-a\|v\|_\infty + b} - a > 0$. Then $w = av + b$ is the upper solution, and the rest of the argument follows likewise.

Chapter 4

A Geometric Problem on a Riemann Surface

4.1 The Equations

Let L_1, L_2 be smooth line bundles over a closed Riemann Surface M. Denote by $\overline{\partial}_{L_j}$ a holomorphic structure, by H_j a hermitian metric in L_j $(j = 1, 2)$ and by ϕ a $(1,0)$ section of $L_2 \otimes L_1^* \cong \mathrm{Hom}(L_1, L_2)$. Following the notation on Chapter 2 write $F_j = F_{(\overline{\partial}_{L_j}, H_j)}$. Then consider the system

$$\text{(I)} \qquad \begin{cases} ia_1\Lambda F_1 - i\Lambda\phi^* \wedge \phi = \tau_1 \\ ia_2\Lambda F_2 - i\Lambda\phi \wedge \phi^* = \tau_2 \\ \overline{\partial}_{L_2} \circ \phi + \phi \circ \overline{\partial}_{L_1} = 0 \end{cases}$$

where a_1, a_2, τ_1, τ_2 are real numbers and $a_1, a_2 \neq 0$. We address the question: Under which conditions on $(a_1, a_2, \tau_1, \tau_2)$ can one find $\overline{\partial}_{L_j}$, H_j and ϕ satisfying (I) ?

The approach we undertake to answer this question is to rewrite system (I) in a more treatable form of one single equation for a holomorphic section ϕ in a tensor product of line bundles associated with L_1, L_2. Then we find a relation between the parameters and the solutions of both problems. Notice that Equations (I) are a special case of the metric equations studied by Bradlow and Garcia-Prada in [BGP] with $n = p = 1$.

Define $\mathcal{L}$ to be the set of isomorphism classes of hermitian line bundles over M. If $(L_a, \overline{\partial}_{L_a}, H_a)$ and $(L_b, \overline{\partial}_{L_b}, H_b)$ are two representatives of the same class then there exists a bundle isomorphism $\varphi : L_a \to L_b$ so that $\varphi^* H_b = H_a$ and $\varphi^* \overline{\partial}_{L_b} = \overline{\partial}_{L_a}$. The next Lemma will be useful in computations.

Lemma 4.1. *The set $\mathcal{L}$ has an abelian group structure with multiplication given by $\otimes$. The identity is represented by $(M \times \mathbb{C}, \overline{\partial}, H_{\mathbb{C}})$. There is a homomorphism $\mathcal{L} \to \Omega^2(\mathbb{C})$ taking $[L, \overline{\partial}_L, H_L] \mapsto F_{(\overline{\partial}_L, H_L)}$.*

Proof. Let $\alpha, \beta \in \mathcal{L}$ be represented by $(L_a, \overline{\partial}_a, H_a)$ and $(L_b, \overline{\partial}_b, H_b)$ respectively. Write $\overline{\partial}_a \otimes \overline{\partial}_b$ and $H_a \otimes H_b$ for the induced holomorphic structure and metric in $L_a \otimes L_b$. One can check that the class $\gamma = [L_a \otimes L_b, \overline{\partial}_a \otimes \overline{\partial}_b, H_a \otimes H_b]$ does not depend on the chosen representatives for α, β, therefore $\alpha.\beta = \gamma$ is well defined.

The transition functions of the trivial bundle can always be chosen to be the identity, so $L_a \otimes (M \times \mathbb{C}) \cong L_a$. Because $\overline{\partial} \otimes \overline{\partial}_a$ and $H_{\mathbb{C}} \otimes H_a$ are isomorphic to $\overline{\partial}_a$ and H_a, by the definition of such operators, we deduce $[M \times \mathbb{C}, \overline{\partial}, H_{\mathbb{C}}]$ is the identity on $\mathcal{L}$.

Now consider $\delta = [L_a^*, \overline{\partial}_a^*, H_a^*]$, where $\overline{\partial}_a^*$ and H_a^* are induced according to Remarks 2.7 and 2.4. It then holds $\overline{\partial}(\eta(\xi)) = \overline{\partial}_a^* \eta(\xi) + \eta(\overline{\partial}_a \xi)$, with $\eta \in \Gamma(L_a^*)$, $\xi \in \Gamma(L_a)$. Concerning the metric it holds

$$H_{\mathbb{C}}(\eta(\xi), \eta'(\xi')) = \overline{\eta'(\xi')}\eta(\xi) = H_a(\xi, \xi') H_a^*(\eta, \eta') \ .$$

Since $\eta(\xi)$ is the same as $\xi \otimes \eta$ under the isomorphism $M \times \mathbb{C} \cong L_a \otimes L_a^*$, we conclude $\delta.\alpha = \alpha.\delta = \mathrm{Id}$, hence $\delta = \alpha^{-1}$.

Let D_a, D_b be the hermitian connections for the bundles $(L_a, \overline{\partial}_a, H_a)$ and $(L_b, \overline{\partial}_b, H_b)$. Then $D_a \otimes D_b$ is the hermitian connection for their tensor product. To prove this, write $D_a = D'_a + \overline{\partial}_a$, $D_b = D'_b + \overline{\partial}_b$, and take $\xi_a \otimes \xi_b \in$

$\Gamma(L_a \otimes L_b)$. Compute

$$
\begin{aligned}
D_a \otimes D_b(\xi_a \otimes \xi_b) &= D_a\xi_a \otimes \xi_b + \xi_a \otimes D_b\xi_b = \\
&= D'_a\xi_a \otimes \xi_b + \overline{\partial}_a\xi_a \otimes \xi_b + \xi_a \otimes D'_b\xi_b + \xi_a \otimes \overline{\partial}_b\xi_b = \\
&= D'_a \otimes D'_b(\xi_a \otimes \xi_b) + \overline{\partial}_a \otimes \overline{\partial}_b(\xi_a \otimes \xi_b) \\
&\Rightarrow \ (D_a \otimes D_b)^{0,1} = \overline{\partial}_a \otimes \overline{\partial}_b \ ,
\end{aligned}
$$

so that $D_a \otimes D_b$ is compatible with $\overline{\partial}_a \otimes \overline{\partial}_b$ (observe that the notation $D_a \otimes D_b$ is misleading. Rigorously, one should instead write $D_a \otimes \mathrm{id}_b + \mathrm{id}_a \otimes D_b$). Similarly, applying $D_a \otimes D_b$ to the product metric yields:

$$
D_a \otimes D_b(H_a \otimes H_b) = D_a(H_a) \otimes H_b + H_a \otimes D_b(H_b) = 0 \ ,
$$

so $D_a \otimes D_b$ is the hermitian connection for $(\overline{\partial}_a \otimes \overline{\partial}_b, H_a \otimes H_b)$.

The curvature then must satisfy

$$
\begin{aligned}
F_{L_a \otimes L_b}\xi_a \otimes \xi_b &= (D_a \otimes D_b)^2 \xi_a \otimes \xi_b \\
&= D_a \otimes D_b(D_a\xi_a \otimes \xi_b + \xi_a \otimes D_b\xi_b) \\
&= D_a^2\xi_a \otimes \xi_b - D_a\xi_a \otimes D_b\xi_b + D_a\xi_a \otimes D_b\xi_b + \xi_a \otimes D_b^2\xi_b \\
&= (F_a \otimes \mathrm{id}_b + \mathrm{id}_a \otimes F_b)\xi_a \otimes \xi_b \ .
\end{aligned} \tag{4.1}
$$

Within line bundles the curvature forms become $\mathbb{C}$-valued forms, hence the last line in (4.1) is equivalent to $(F_a + F_b)\xi_a \otimes \xi_b$. This proves that the map $[L, \overline{\partial}_L, H_L] \mapsto F_{(\overline{\partial}_L, H_L)}$ is a homomorphism. □

Back to system (I) and following the notation in Lemma 4.1, define $[L, \overline{\partial}_L, H_L] = [L_2, \overline{\partial}_{L_2}, H_2].[L_1, \overline{\partial}_{L_1}, H_1]^{-1}$. Multiplying the j^{th} equation ($j = 1, 2$) of (I) by a_j^{-1} and subtracting one from the other we get

$$
i\Lambda F_2 - i\Lambda F_1 - i\frac{1}{a_2}\Lambda\phi \wedge \phi^* + i\frac{1}{a_1}\Lambda\phi^* \wedge \phi = \frac{\tau_2}{a_2} - \frac{\tau_1}{a_1}.
$$

The difference of the curvatures is the curvature of $[L, \overline{\partial}_L, H_L]$, according to the homomorphism defined in Lemma 4.1. The last of equations (I) is just the holomorphicity condition for $\phi \in \Omega^{1,0}(L)$ expressed in terms of the holomorphic structures in the factors L_2, L_1^* of L. Hence, we arrive at

$$\text{(II)} \qquad \begin{cases} i\Lambda F_L - ib_1\Lambda\phi \wedge \phi^* = -b_2 \\ \overline{\partial}_L \phi = 0 \end{cases}$$

for

$$(4.2) \qquad \begin{aligned} b_1 &= \frac{1}{a_1} + \frac{1}{a_2}, \\ b_2 &= \frac{\tau_1}{a_1} - \frac{\tau_2}{a_2} \ . \end{aligned}$$

It is suitable to define $(j = 1, 2)$

$$\begin{aligned} \mathcal{C}_j &= \{\text{holomorphic structures on } L_j\} \\ \mathcal{H}_j &= \{\text{hermitian metrics on } L_j\} \\ \mathcal{C}_{1,2} &= \mathcal{C}_1 \times \mathcal{C}_2 \\ \mathcal{H}_{1,2} &= \mathcal{H}_1 \times \mathcal{H}_2 \ . \end{aligned}$$

and

$$\begin{aligned} S_I &= \{s = (a_1, a_2, \tau_1, \tau_2, \overline{\partial}_{L_1}, \overline{\partial}_{L_2}, \phi, H_1, H_2);\ s \text{ satisfies (I)}\} \\ S_{II} &= \{t = (b_1, b_2, \overline{\partial}_L, \phi, H_L);\ t \text{ satisfies (II)}\} \end{aligned}$$

as the sets of *all* solutions. They are subsets of infinite dimension manifolds,

$$\begin{aligned} S_I &\subset (\mathbb{R} - \{0\})^2 \times \mathbb{R}^2 \times \mathcal{C}_{1,2} \times \Omega^{1,0}(L) \times \mathcal{H}_{1,2} \ , \\ S_{II} &\subset \mathbb{R}^2 \times \mathcal{C}_L \times \Omega^{1,0}(L) \times \mathcal{H}_L \ . \end{aligned}$$

Topological constraints result naturally from (I) and (II) once they are integrated. For convenience, we assume the area of M is 1. Then solvability for (I) and (II) implies the following necessary conditions:

$$(4.3) \qquad \begin{aligned} a_1 c(L_1) - \int i\phi^* \wedge \phi &= \tau_1 \\ a_2 c(L_2) - \int i\phi \wedge \phi^* &= \tau_2, \end{aligned}$$

and

$$c(L) - b_1 \int i\phi \wedge \phi^* = -b_2 \ . \tag{4.4}$$

A last requirement is that

$$c(L) = c(L_2) - c(L_1) \geq -\chi(M) \ , \tag{4.5}$$

where $\chi(M)$ is the evaluation of the first Chern Class of the holomorphic tangent bundle of M (that is, the Euler Characteristic of M). This is necessary to assure the existence of non-trivial holomorphic $(1, 0)$-sections in L (see [Ko]).

The above construction gives us a map $\psi : S_I \to S_{II}$. The relationship between the solutions of the two systems is clarified in a proposition.

Proposition 4.2. *The map* ψ *given by*

$$\psi(a_1, a_2, \tau_1, \tau_2, \overline{\partial}_{L_1}, \overline{\partial}_{L_2}, \phi, H_1, H_2) = (b_1, b_2, \overline{\partial}_L, \phi, H_L)$$

with

$$b_1 = \tfrac{1}{a_1} + \tfrac{1}{a_2} \qquad \overline{\partial}_L = \overline{\partial}_{L_2} \otimes (\overline{\partial}_{L_1})^{-1}$$
$$b_2 = \tfrac{\tau_1}{a_1} - \tfrac{\tau_2}{a_2} \qquad H_L = H_2 \otimes (H_1)^{-1}$$

is onto. For every $v \in S_{II}$*,* $\psi^{-1}(v)$ *is diffeomorphic to three copies of* $\mathbb{R}^2 \times \Omega^{0,1}(\mathbb{C})$ *if* $b_1 \neq 0$ *and two copies of* $\mathbb{R}^2 \times \Omega^{0,1}(\mathbb{C})$ *when* $b_1 = 0$.

Proof. Let $v = (b_1, b_2, \overline{\partial}_L, \phi, H_L) \in S_{II}$. Set $C = C(\phi, H_L) = \int i\phi \wedge \phi^*$. Notice that $C = \int -i\phi^* \wedge \phi \geq 0$. Conditions (4.3) and relations (4.2) imply four equations for a_1, a_2, τ_1, τ_2, namely

$$\begin{aligned} C\frac{1}{a_1} - \frac{\tau_1}{a_1} &= -c(L_1) \\ -C\frac{1}{a_2} - \frac{\tau_2}{a_2} &= -c(L_2) \\ \frac{1}{a_1} + \frac{1}{a_2} &= b_1 \\ \frac{\tau_1}{a_1} - \frac{\tau_2}{a_2} &= b_2 \end{aligned} \tag{4.6}$$

This system is linear in the variables $a_1^{-1}, a_2^{-1}, \tau_1 a_1^{-1}, \tau_2 a_2^{-1}$. In fact, the 4×4 coefficient matrix for it is degenerated, but condition (4.4) can be used to show there is a 1 dimension affine subspace of solutions in those variables. Let $t \in \mathbb{R}$ be a variable that linearly parametrizes this subspace. A computation yields the original parameters in terms of t, v:

$$\begin{aligned} a_1 &= \frac{1}{b_1 - t} & \tau_1 &= \frac{c(L_2) + b_2 - tC}{b_1 - t} \\ a_2 &= \frac{1}{t} & \tau_2 &= \frac{c(L_2) - tC}{t} \end{aligned} \tag{4.7}$$

Now fix $t \in \mathbb{R}$, $t \neq 0, b_1$. The first equation on (4.6) implies that the function $\frac{\tau_1}{a_1} + \frac{i}{a_1}\Lambda(\phi^* \wedge \phi)$ has mean value equal to $c(L_1)$. Then we choose a holomorphic structure $\overline{\partial}_{L_1}$, and Lemma 2.23 provides a metric H_1, uniquely defined up to positive dilation, with

$$i\Lambda F_1 = \frac{\tau_1}{a_1} + \frac{i}{a_1}\Lambda(\phi^* \wedge \phi) \ . \tag{4.8}$$

Therefore, the first equation on (I) is satisfied.

The hermitian bundle L is

$$[L, \overline{\partial}_L, H_L] = [L_2, \overline{\partial}_{L_2}, H_2].[L_1, \overline{\partial}_{L_1}, H_1]^{-1} \ ,$$

hence we get

$$[L_2, \overline{\partial}_{L_2}, H_2] = [L, \overline{\partial}_L, H_L].[L_1, \overline{\partial}_{L_1}, H_1] \ .$$

That shows us the only compatible choices for $\overline{\partial}_{L_2}, H_2$: $\overline{\partial}_{L_2} = \overline{\partial}_L \otimes \overline{\partial}_{L_1}$, $H_2 = H_L \otimes H_1$. Let us show the second of equations (I) is then satisfied. Since the curvature on L_2 is $F_2 = F_L + F_1$, it holds

$$\begin{aligned} & ia_2\Lambda F_2 - i\Lambda\phi \wedge \phi^* = \\ & \qquad = ia_2\Lambda(F_L + F_1 - \left(b_1 - \frac{1}{a_1}\right)\phi \wedge \phi^*) = \\ & \qquad\qquad = a_2\left(-b_2 + \frac{\tau_1}{a_1}\right) = \tau_2 \ , \end{aligned}$$

and we conclude that $s = (a_1, a_2, \tau_1, \tau_2, \overline{\partial}_{L_1}, \overline{\partial}_{L_2}, \phi, H_1, H_2)$ is a preimage of v by ψ. Hence ψ is onto.

The construction of $\psi^{-1}(v)$ has left degrees of freedom in the choices for t, a holomorphic structure and a particular metric whose curvature satisfies a given equation.

The space of holomorphic structures $\mathcal{C}_1$ is diffeomorphic to $\Omega^{0,1}(\mathbb{C})$ by fixing one element $D^{0,1} \in \mathcal{C}_1$ (see Remark 2.9). If a metric $H_f \in \mathcal{H}_1$ is prescribed we get a functional $\sigma : \mathcal{H}_1 \to \mathbb{R}$ given by $H \mapsto (\int |H^{-1} \circ H_f|^2 \, d\mu)^{\frac{1}{2}}$. Since this functional satisfies $\sigma(lH) = l\sigma(H)$ for any positive l, it can be used as a reference in the choice of H_1. In the construction of the metric that yields (4.8) we pick the only H_1 satisfying $\sigma(H_1) = 1$. And since t ranges in three disjoint open intervals when $b_1 \neq 0$ and $t \in \mathbb{R} - \{0\}$ when $b_1 = 0$ we can see how to define a bijective mapping

$$I \times \Omega^{0,1}(\mathbb{C}) \times \mathbb{R} \to \psi^{-1}(v)$$
$$(t, \beta, l) \mapsto (a_1, a_2, \tau_1, \tau_2, \overline{\partial}_{L_1}(\beta), \overline{\partial}_{L_2}, \phi, H_1(l), H_2)$$

with the first four coordinates given by (4.7), $\overline{\partial}_{L_1} = D^{0,1} + \beta$, $H_1(l) = e^l H_1$. The set I is either $\mathbb{R} \cup \mathbb{R}$ or $\mathbb{R} \cup \mathbb{R} \cup \mathbb{R}$ depending on b_1. □

4.2 Gauge Invariance

Our next step is to study the action of the group of gauge transformations on equations (I) and (II).

The group of complex gauge transformations on a line bundle L is identified with the group of smooth maps $M \to \mathbb{C} - \{0\}$. The action of the latter in the line bundle is just fiberwise multiplication by a non-zero complex number. However, to keep consistence with definitions in Chapter 2, we need to make a distinction between actions on L and L^*. Hence, if

$g : M \to \mathbb{C} - \{0\}$ and we think of g as a gauge transformation in L then g acts on L^* by $g(\eta) = \eta \circ g^{-1} = g^{-1}\eta$, for all $\eta \in \Gamma(L^*)$. That makes the action trivial on End$L \cong M \times \mathbb{C}$. Due to a similar reason, the action on $\mathcal{H}$ is given by $g(H) = |g|^{-2}H$ so that $g(H)(g\xi, gv) = H(\xi, v)\ \forall \xi, v \in \Gamma(L)$. These arguments sketch the proof of the next lemma.

Lemma 4.3. *Let $\mathcal{G}_j$ be the group of gauge transformations of L_j. There is an action of $\mathcal{G}_1 \times \mathcal{G}_2$ onto $\mathbb{R}^4 \times \mathcal{C}_{1,2} \times \Omega^{1,0}(L) \times \mathcal{H}_{1,2}$ that is trivial on $\mathbb{R}^4$ and the usual one on $\mathcal{C}_{1,2} \times \Omega^{1,0}(L) \times \mathcal{H}_{1,2}$. Then S_I is closed under this action. There is an action of $\mathcal{G}_L$ onto $\mathbb{R}^2 \times \mathcal{C}_L \times \Omega^{1,0}(L) \times \mathcal{H}_L$ that leaves S_{II} invariant.*

Proof. Assume $s = (a_1, a_2, \tau_1, \tau_2, \overline{\partial}_{L_1}, \overline{\partial}_{L_2}, \phi, H_1, H_2) \in S_I$. One needs to show $(g_1, g_2)s \in S_I$, where

$$(g_1, g_2)s = (a_1, a_2, \tau_1, \tau_2, g_1(\overline{\partial}_{L_1}), g_2(\overline{\partial}_{L_2}), g_2 g_1^{-1}\phi, g_1(H_1), g_2(H_2))\ .$$

Because this action is trivial in EndL_1, Lemma 2.22 yields $F_{(\overline{\partial}_{L_1}, H_1)} = F_{(g_1(\overline{\partial}_{L_1}), g_1(H_1))}$, and since

$$(g\phi)^{*g(H_L)} \wedge (g\phi) = |g|^{-2}\overline{H_L(g\phi)} \wedge (g\phi) = \overline{H_L(\phi)} \wedge \phi = \phi^{*H_L} \wedge \phi$$

for $g = g_2 g_1^{-1}$ and $H_L = H_2 \otimes H_1^*$, one sees that the first of equations (I) is satisfied by $(g_1, g_2)s$. Invariance for the second equation is analogous. Also note that $g_2 g_1^{-1}\phi$ is holomorphic respect to $g_1(\overline{\partial}_{L_1})$ and $g_2(\overline{\partial}_{L_2})$, then $(g_1, g_2)s \in S_I$. This proves S_I is invariant under $\mathcal{G}_1 \times \mathcal{G}_2$ action. The proof for S_{II} and $\mathcal{G}_L$ is similar. □

Next let us define a group morphism by $(g_1, g_2) \in \mathcal{G}_1 \times \mathcal{G}_2 \mapsto g_2 g_1^{-1} \in \mathcal{G}_L$. A computation shows us the map ψ commutes with this morphism, $\psi((g_1, g_2)s) = g_2 g_1^{-1}\psi(s)$. We are then tempted to redefine ψ between the quotient spaces of S_I, S_{II} under the gauge actions. Set

$$\tilde{S}_I = S_I/\mathcal{G}_1 \times \mathcal{G}_2 \qquad \tilde{S}_{II} = S_{II}/\mathcal{G}_L\ ,$$

and $\tilde{\psi} : \tilde{S}_I \to \tilde{S}_{II}$ the quotient obtained from ψ.

Define

$$Hol = \{(\overline{\partial}_L, \phi) \in \mathcal{C}_L \times \Omega^{1,0}(L) \mid \overline{\partial}_L \phi = 0\}$$

as the space of holomorphic pairs and

$$\tilde{Hol} = \frac{Hol}{\mathcal{G}_L} \ ,$$

which is well defined because Hol is closed under $\mathcal{G}_L$ action. The projection

$$\mathbb{R}^2 \times \mathcal{C}_L \times \Omega^{1,0}(L) \times \mathcal{H}_L \to \mathbb{R}^2 \times \mathcal{C}_L \times \Omega^{1,0}(L)$$

descends to

$$S_{II} \to \mathbb{R}^2 \times Hol \ .$$

Under this action, it yields a map:

$$\pi : \tilde{S}_{II} \to \mathbb{R}^2 \times \tilde{Hol} \ . \tag{4.9}$$

Denote by $P_0 \subset \tilde{Hol}$ those elements of the form $[\overline{\partial}_L, 0]$.

Theorem 4.4. *Let $\beta \in \tilde{Hol}$, $\beta \notin P_0$. Then there exists $T_\beta > 0$ so that the following are sufficient conditions for $(b_1, b_2, \beta) \in im\pi$:*

(1) $\quad b_2 < -c(L) \quad$ *and* $b_1 < 0$;

(2) $\quad b_2 = -c(L) \quad$ *and* $b_1 = 0$;

(3) $\quad -c(L) < b_2 < -c(L) + T_\beta \quad$ *and* $b_1 > 0$.

In case $\beta \in P_0$, $(b_1, b_2, \beta) \in im\pi$ if and only if

$$b_2 = -c(L) \ .$$

The proof of this Theorem will be postponed until next chapter. A corollary has a partial answer to the question posed in the beginning of this chapter.

Corollary 4.5. *Given four reals a_1, a_2, τ_1, τ_2, with $a_1, a_2 \neq 0$. Assume they satisfy*

$$\tau_1 - a_1 c(L_1) = a_2 c(L_2) - \tau_2 \geq 0 \ . \tag{4.10}$$

Set

$$\tilde{C} = \tau_1 - a_1 c(L_1) \ , \sigma = a_2 \tau_1 - a_1 \tau_2 + a_1 a_2 c(L) \ .$$

Then, if $\tilde{C} = 0$ there is a solution for (I) with these parameters. Otherwise, there exists $\varepsilon > 0$ (independent of the parameters) and a solution if either

$$sign(\sigma) \neq sign(a_1 a_2) \tag{4.11}$$

or

$$sign(\sigma) = sign(a_1 a_2) \qquad and \qquad \frac{\sigma}{a_1 a_2} < \varepsilon \ . \tag{4.12}$$

Proof. Define b_1, b_2 as usual, given by equations (4.2). We can express:

$$b_2 = \frac{\sigma}{a_1 a_2} - c(L) \ , \tag{4.13}$$

$$\sigma = (a_1 + a_2)\tilde{C} \ . \tag{4.14}$$

Notice that the existence of a solution for (I) with the given parameters implies equations (4.6) with $C = \tilde{C} = \|\phi\|^2_{H_L}$, ϕ being the prospective section-solution. There are three cases to be explored:

(1) $\tilde{C} = 0$.

From (4.14) and (4.13) we get $\sigma = 0$ and $b_2 = -c(L)$. Choose $\beta \in P_0$. Without any constraint on b_1, Theorem 4.4 gives us $(b_1, -c(L), \beta) \in \mathrm{im}\pi$. Then, all solutions in S_I corresponding by $\tilde{\psi}$ to the preimage of $(b_1, -c(L), \beta)$

have $\phi \equiv 0$, and the constant C of Proposition 4.2 is $0 = \tilde{C}$. Take the only $t \in \mathbb{R}$ that satisfies the first two of equations (4.7). Evaluating the following expressions,

$$\frac{c(L_2) + b_2 - tC}{b_1 - t} \quad , \quad \frac{c(L_2) - tC}{t} \tag{4.15}$$

we get:

$$\begin{aligned} \frac{c(L_2) + b_2 - tC}{b_1 - t} &= \left(c(L_2) + \frac{\tau_1}{a_1} - \frac{\tau_2}{a_2} \right) a_1 \\ &= a_1 c(L_2) + \tau_1 - \frac{a_1}{a_2}\tau_2 \\ &= \frac{a_1}{a_2}(a_2 c(L_2) - \tau_2) + \tau_1 \\ &= \frac{a_1}{a_2}\tilde{C} + \tau_1 = \tau_1 \ , \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \frac{c(L_2) - tC}{t} &= a_2 c(L_2) \\ &= a_2 c(L_2) - \tilde{C} \\ &= \tau_2 \ . \end{aligned} \tag{4.17}$$

We conclude that equations (4.7) are fully satisfied for that choice of t. Hence, there is a solution $s \in S_I$ with those parameters.

(2) $\tilde{C} > 0$ and $\operatorname{sign}(\sigma) \neq \operatorname{sign}(a_1 a_2)$.

Let us first show $\operatorname{sign}(b_1) = \operatorname{sign}(b_2 + c(L))$.

$$\begin{aligned} \operatorname{sign}(b_1) &= \operatorname{sign}\left(\frac{a_1 + a_2}{a_1 a_2}\right) \\ &= \operatorname{sign}\left(\frac{\sigma}{\tilde{C} a_1 a_2}\right) = \operatorname{sign}\left(\frac{\sigma}{a_1 a_2}\right) \\ &= \operatorname{sign}(b_2 + c(L)) \ . \end{aligned} \tag{4.18}$$

The sign requirements for any of cases (1) through (3) in Theorem 4.4 are thus satisfied for all $\beta \in \tilde{Hol} - P_0$.

Notice that $\mathrm{sign}(\sigma) \neq \mathrm{sign}(a_1 a_2)$ is equivalent to $b_1 \leq 0$. Hence, we pick any $\beta \notin P_0$. Theorem 4.4 (1) or (2) gives us $(b_1, b_2, \beta) \in \mathrm{im}\pi$. Let $v = (b_1, b_2, \overline{\partial}_L, \phi, H_L) \in S_{II}$ such that $[\overline{\partial}_L, \phi] = \beta$. The constant C of Proposition 4.2 (valid for any solution in S_I corresponding to v) is obtained directly from equation (4.4) if $b_1 < 0$:

$$\begin{aligned} C &= \frac{b_2 + c(L)}{b_1} \\ &= \frac{\frac{\sigma}{a_1 a_2} - c(L) + c(L)}{\frac{1}{a_1} + \frac{1}{a_2}} \\ &= \frac{\sigma}{a_1 + a_2} = \tilde{C} \ , \end{aligned} \tag{4.19}$$

using equation (4.14). If $b_1 = 0$ we can dilate ϕ by $\alpha > 0$ so that

$$(0, b_2, \overline{\partial}_L, \alpha\phi, H_L) \in S_{II}$$

and

$$C = C(\alpha\phi, H_L) = \|\alpha\phi\|^2_{H_L} = \tilde{C} \ .$$

In that case, $\tilde{\beta} = [\overline{\partial}_L, \alpha\phi]$ may be different from β. The important fact is that we get solutions in S_{II} with b_1, b_2 given as in (4.2) and $C = \tilde{C}$. Then let $t = \frac{1}{a_2}$ in (4.7), and computations similar to (4.16) and (4.17) show that equations (4.7) are satisfied. Therefore, $\psi^{-1}(v) \in S_I$ has a solution with the parameters a_1, a_2, τ_1, τ_2. This proves the Corollary when $\tilde{C} > 0$ and (4.11) holds.

(3) $\tilde{C} > 0$ and $\mathrm{sign}(\sigma) = \mathrm{sign}(a_1 a_2)$.

In this case, the computations done in (4.18) and (4.19) apply likewise, hence $b_1, b_2 + c(L)$ are positive and $C = \tilde{C}$. Making $t = 1/a_2$ and evaluating (4.15) lead to the verification of (4.7) by our chosen parameters $(a_1, a_2, \tau_1, \tau_2)$. What remains to be shown is that a suitable choice of $\beta \in \tilde{Hol} - P_0$ yields $b_2 + c(L) < T_\beta$. If we let $\varepsilon = \sup\{T_\beta \mid \beta \in \tilde{Hol} - P_0\}$ then condition $\sigma/(a_1 a_2) < \varepsilon$ forces $b_2 + c(L) = \sigma/(a_1 a_2) < T_\beta$ for some $\beta \in \tilde{Hol} - P_0$. Again Theorem 4.4 (3) is satisfied and the Corollary is proven. $\square$

4.3 The Elliptic Equation

To study the space S_{II} we derive an equation that is the center of our analysis on Chapter 5. The idea is to prescribe a holomorphic pair $(\overline{\partial}_L, \phi) \in Hol$ and to seek for parameters and metrics that solve (II).

Proposition 4.6. *Let $(\overline{\partial}_L, \phi) \in Hol$ with $\phi \neq 0$. Fix $b_1, b_2 \in \mathbb{R}$ with $\mathrm{sign}(b_1) = \mathrm{sign}(b_2 + c(L))$ and choose a metric H_0 corresponding to a constant curvature $F_0 = F_{(\overline{\partial}_L, H_0)}$ on $\mathrm{End}L$. Then, if $b_1 \neq 0$, there is a one-to-one correspondence between hermitian metrics H_L satisfying $(b_1, b_2, \overline{\partial}_L, \phi, H_L) \in S_{II}$ and solutions (b, u) of the equation*

$$\Delta u + bfe^{2u} - \lambda = 0 \ . \tag{4.20}$$

Here $f = |\phi|^2_{H_0} \geq 0$, $\lambda = b_2 + c(L)$, $b \in \mathbb{R}$ and $u : M \to \mathbb{R}$ is smooth and has 0 mean value. If $b_1 = 0$ the above correspondence is still defined but is not injective.

Proof. Any metric $H \in \mathcal{H}_L$ can be written in a unique way as $H = e^{2v}H_0$, for $v : M \to \mathbb{R}$. Define $u = v - \overline{v}$ and $S = H_0^{-1} \circ H = e^{2u+2\overline{v}}$. According to Lemma 2.21 the curvature for the pair $(\overline{\partial}_L, H)$ is

$$\begin{aligned} F_{(\overline{\partial}_L, H)} &= F_0 + \overline{\partial}(e^{-2u-2\overline{v}}\partial(e^{2u+2\overline{v}})) = \\ &= -ic(L)\ \nu + 2\overline{\partial}\partial u = \\ &= -i(c(L) - \Delta u)\ \nu \ . \end{aligned}$$

Since $\phi^{*H} = e^{2u+2\overline{v}}\phi^{*H_0}$, we rewrite the first of equations (II) as

$$\begin{aligned} i\Lambda F_L - ib_1\Lambda\phi \wedge \phi^{*H} &= c(L) - \Delta u - b_1 e^{2\overline{v}}\ fe^{2u} = -b_2 \\ &\Rightarrow \Delta u + bfe^{2u} - \lambda = 0\ , \qquad \text{for } b = b_1 e^{2\overline{v}}\ . \end{aligned}$$

A metric H_L is then taken to a solution (b, u) of (4.20). If $b_1 \neq 0$ the previous steps can be reverted, what shows that the map $H_L \to (b, u)$ is a bijection.

When $b_1 = 0$ then $\lambda = 0$ and the only solution for (4.20) is $(0,0)$, while any positive multiple of a metric of constant curvature is a solution for (II). This proves the Proposition. □

Chapter 5

Analytical methods for geometric equations

Our aim in this chapter is to understand the behaviour of

$$\Delta u + bfe^{2u} - \lambda = 0. \tag{5.1}$$

We fix $f \in C^\infty(M), \quad f \geq 0$, and study the solutions of (5.1) in terms of the real parameters $b, \ \lambda$.

Remark 5.1. The parameter b (when $\neq 0$) in (5.1) can be easily converted in a shift by a constant for a solution u. Since $\Delta u + bfe^{2u} - \lambda = \Delta(u + \ln|b|/2) + (b/|b|)fe^{2(u+\ln|b|/2)} - \lambda$, the sign of b is more important than its actual absolute value. Though we prefer to keep b as a parameter and to assume instead that $\lambda \mapsto u(\lambda)$ ranges on spaces of 0 mean value functions. This will simplify the analysis.

Solutions $u(x)$ for (5.1) are C^∞ by well-known Schauder Theory on elliptic operators. This gives freedom in the choice of the convenient Sobolev spaces to work, since we know before-hand where solutions must live.

In Chapter 3 we have studied an equation analogous to (5.1) with the methods of [KW1]. Translating their results from Theorem 3.1 and Remark 3.6 to equation (5.1), we readily find a maximal open interval $I \supset (-\infty, 0]$ so that $\lambda \in I$ implies the existence of a solution $(b, u) \in \mathbb{R} \times (H_1)_0$. This result can be reached by the continuity method, with the advantage that it gives for free the smooth dependence of (b, u) on λ.

A further development is to figure what happens when $\lambda > 0$ is beyond the bounds of I. Little is known in that case. However, in section 5.2 we are able to prove a type of "uniqueness" when $0 < \lambda < c(L)$.

5.1 The Continuity Method

Theorem 5.2. *There exists $T > 0$ and a C^∞ family of solutions $\lambda \longmapsto (b(\lambda), u(\lambda)) \in \mathbb{R} \times C^\infty(M)$ for equation (5.1) with $\lambda \in (-\infty, T)$ and $\overline{u(\lambda)} = 0$. This family is unique in the interval $\lambda \leq 0$.*

The first corollary of this Theorem is the

proof of Theorem 4.4. Because of Proposition 4.6 we only need to rephrase the sufficient conditions of Theorem 5.2 into those on Theorem 4.4. Let $\beta \in \tilde{Hol}$ and $(\overline{\partial}_L, \phi)$ be a representative for β. If $\phi = 0$ (or $\beta \in P_0$) then $f \equiv 0$ and (5.1) is solvable if and only if $\lambda = 0 \Leftrightarrow b_2 = -c(L)$.

For $\phi \neq 0$, f must be different from zero and the cases (1)-(3) in Theorem 4.4 correspond to $\lambda < 0$, $\lambda = 0$ or $0 < \lambda < T$, since $b_2 = \lambda - c(L)$. If $\mathrm{sign}(b_1) = \mathrm{sign}(b_2 + c(L))$, the constant $\overline{v}$ on Proposition 4.6 can be chosen to make $b(\lambda) = b(b_2 + c(L)) = b_1 e^{2\overline{v}}$ because $\mathrm{sign}(b(\lambda)) = \mathrm{sign}(b_2 + c(L))$. The pair $(b(b_2 + c(L)), u(b_2 + c(L)))$ yields the metric $H_L = e^{2u(b_2+c(L))+2\overline{v}} H_0$ (with b_1, b_2 fixed), hence $(b_1, b_2, \overline{\partial}_L, \phi, H_L) \in S_{II}$ and $(b_1, b_2, \beta) \in \mathrm{im}\pi$.

Notice that the constant T in Theorem 5.2 does not depend on the representative of β, since $f(\overline{\partial}_L, \phi) = |\phi|^2_{H_0} = |g\phi|^2_{g(H_0)} = f(g(\overline{\partial}_L, \phi))$, with g a complex gauge action. Then $T = T_\beta$ works uniformely for any representative of β, and Theorem 4.4 is proven. □

The next result proves the last statement on Theorem 5.2.

Lemma 5.3. *Fix $\lambda \le 0$. There exists at most one solution $(b,u) \in \mathbb{R} \times (H_1)_0$ for $\Delta u + bfe^{2u} - \lambda = 0$.*

Proof. If $\lambda = 0$ then $b = 0$ because Δu has zero mean value. Using Lemma 2.39 we get $u \equiv 0$, and $(0,0)$ is the unique solution in that case.

Now consider $\lambda < 0$. Assuming (b_u, u) and (b_v, v) are solutions, write

$$U = u + \frac{1}{2}\ln|b_u| \ ,$$
$$V = v + \frac{1}{2}\ln|b_v| \ .$$

A negative λ implies $b_u, b_v < 0$. Then, the equations for $U,\ V$ are:

$$\Delta U - fe^{2U} - \lambda = 0 \ ,$$
$$\Delta V - fe^{2V} - \lambda = 0 \ .$$

Subtracting the equations and integrating against $(U - V)$ one obtains

$$\int \Delta(U-V)(U-V) - f(e^{2U} - e^{2V})(U-V)\ d\mu = 0$$
$$\Rightarrow \|\nabla(U-V)\|_2^2 = -\int f(e^{2U} - e^{2V})(U-V)\ d\mu \le 0 \ .$$

This shows $U = V$ a.e. Then $b_u = b_v$ and $u = v$, and uniqueness is proven. □

The proof for the main part of Theorem 5.2 will be done by the Implicit Function Theorem. It will be applied to a functional whose roots are the solutions of (5.1). Define $F : \mathbb{R}^2 \times (H_1)_0 \longrightarrow H_{-1}$ by

$$F(\lambda, b, u)(w) = \int -\langle \nabla u, \nabla w\rangle + bfe^{2u}w - \lambda w\ d\mu \tag{5.2}$$

for every $w \in H_1$. In the distributional form, write $F(\lambda, b, u) = \Delta u + bfe^{2u} - \lambda$. Observe that Corollary 2.44 implies $e^{2u} \in L_q$ for $q > 0$, and hence F is well defined.

For the sake of proving regularity of the solutions respect to the parameter λ it becomes necessary to restrict F to the image of $\mathbb{R}^2 \times (H_k)_0 \to \mathbb{R}^2 \times (H_1)_0$, for $k > 1$. In that case one needs to verify that $F(\mathbb{R}^2 \times (H_k)_0) \subset H_{k-2} \subset H_{-1}$ and the map F is continuously differentiable in the H_k, H_{k-2} norms. All these facts are consequences of the next proposition.

Proposition 5.4. *Let $k \geq 1$ an integer and $q \geq 1$ a real number. The map $\varphi : W_{k,2}(M) \to W_{k-1,q}(M)$ given by $\varphi(u) = e^u$ is continuously differentiable.*

Proof. Let's first prove the continuity of φ (that φ is well defined will come out naturally from this proof). If $u \in C^\infty(M)$ and $j \leq k-1$, define $S_j(u) \in \Gamma(TM^{\otimes j})$ satisfying

$$\nabla^j(e^u) = e^u\ S_j(u).$$

Then $S_1(u) = \nabla u$ and from Lemma 2.27 $S_l(u) = (\nabla u \otimes S_{l-1}(u))^\pi + \nabla S_{l-1}(u)$. One can see $S_j(u)$ is a finite sum of j-tensors, each of them a product of the form $\nabla^{j_1}u \otimes \nabla^{j_2}u \otimes \cdots \otimes \nabla^{j_s}u$, with $j_1 + j_2 + \cdots + j_s = j$. By Proposition 2.35 and the Sobolev Embedding $W_{k,2} \subset W_{k-1,sq'}$ each of these summands is associated to a continuous multilinear map $W_{k,2}^{\times s} \to W_{0,q'}$. Therefore, $u \in W_{k,2} \mapsto S_j(u) \in L_{q'}$ is continuous for all $q' \geq 1$.

Choosing $u, v \in C^\infty(M)$ we estimate the L_q norm on the difference of the j^{th} derivatives of e^u, e^v ,

$$\begin{aligned}\|\nabla^j(e^u) - \nabla^j(e^v)\|_q &= \|(e^u - e^v)S_j(u) + e^v(S_j(u) - S_j(v))\|_q \\ &\leq \|e^u - e^v\|_{2q}\|S_j(u)\|_{2q} + \|e^v\|_{2q}\|S_j(u) - S_j(v)\|_{2q}.\end{aligned} \tag{5.3}$$

If u and v remain bounded in $W_{k,2}$ the same happens to $\|S_j(u)\|_{2q}$ and $\|e^v\|_{2q}$, the last one by Corollary 2.44. We want to show $\nabla^j(e^u)$ approaches $\nabla^j(e^v)$ in L_q when u approaches v in $W_{k,2}$, and for that it suffices proving

$\|e^u - e^v\|_{2q} \to 0$ when $u \to v$. This last fact comes from

$$
\begin{aligned}
\|e^u - e^v\|_{2q} &= \|e^v(e^{u-v} - 1)\|_{2q} \\
&\le \|e^v\|_{4q} \|e^{|u-v|}|u - v|\|_{4q} \\
&\le \|e^v\|_{4q} \|e^{|u-v|}\|_{8q} \|u - v\|_{8q} \\
&\le C(u, v)\|u - v\|_{k,2} \ ,
\end{aligned} \tag{5.4}
$$

and the observation that $C(u, v)$ is bounded when u, v are bounded in $W_{1,2}$. Inequalities (5.3) and (5.4) imply that φ takes Cauchy sequences in $C^\infty(M)$ to Cauchy sequences, respect to the Sobolev norms, hence it can be extended to a continuous map $\varphi : W_{k,2} \to W_{k-1,q}$.

Now we show φ is differentiable. Define $T : W_{k,2} \to \mathcal{L}(W_{k,2}, W_{k-1,q})$ by $T_u(z) = e^u z$. The function T is the composition of a bounded bilinear map with the continuous map $(u, z) \mapsto (\varphi(u), z)$, and hence is continuous. Let's show $\varphi'_u = T_u$ for all u, and then φ is continuously differentiable by applying a bootstrap argument with the chain rule. Fix $u, z \in W_{k,2}$ and define

$$
\psi_s = \frac{e^{u+sz} - e^u}{s} \quad \in \quad W_{k-1,q'} \ ,
$$

for $s \neq 0$. If $j \le k - 1$ it can be checked that $\nabla^j \psi_s$ converges pointwise to $\nabla^j(e^u z)$ as $s \to 0$.

Now let $0 < |s| < 1$. An estimate for $|\nabla^j \psi_s|$ at each point $x \in M$ gives us

$$
\begin{aligned}
|\nabla^j \psi_s| &= \left| \frac{e^{u+sz} S_j(u + sz) - e^u S_j(u)}{s} \right| \\
&\le \left| \left(\frac{e^{u+sz} - e^u}{s} \right) S_j(u + sz) \right| + \left| e^u \left(\frac{S_j(u + sz) - S_j(u)}{s} \right) \right| \\
&\le e^{|z|} |e^u z| Q_1(u, z) + e^u Q_2(u, z) \ .
\end{aligned} \tag{5.5}
$$

Here $Q_1(u, z)$ and $Q_2(u, z)$ are finite sums of terms

$$
|\nabla^{j_1} u| \dots |\nabla^{j_t} u| |\nabla^{l_1} z| \dots |\nabla^{l_m} z| \ ,
$$

the orders satisfying $j_1 + \cdots + j_t + l_1 + \cdots + l_m \leq j$. Since $j \leq k-1$ each one of the factors in (5.5) is in the appropriate $L_{q'}$ space so that their products are in L_q .

In addition with $0 < |s| < 1$, $|\nabla^j \psi_s|$ is pointwise bounded by a fixed L_q integrable function, and the Lebesgue Dominated Convergence Theorem asserts that $\nabla^j \psi_s \to \nabla^j (e^u z)$ in L_q. This is expressed for all $j \leq k-1$ by the limit

$$\lim_{s\to 0} \left\| \frac{\varphi(u+sz) - \varphi(u)}{s} - T_u z \right\|_{k-1,q} = 0 \ ,$$

so φ is continuously differentiable. □

Corollary 5.5. *The map φ is compact, that is, for every bounded subset $V \subset W_{k,2}$, $\varphi(V)$ has compact closure in $W_{k-1,q}$.*

Proof. The proof comes from the observation that $S_j : W_{k,2} \to L_{q'}$ $(j \leq k-1)$ as well as $\varphi : W_{1,2} \to L_q$ are both compact maps. The first because it is the composition of continuous multilinear maps $L_{q_1} \times \cdots \times L_{q_s} \to L_{q'}$ with compact maps $W_{k,2} \to L_{q_1} \times \cdots \times L_{q_s}$ constructed with the operators ∇^{j_t}. The second because of inequality (5.4), and knowing that bounded sequences in $W_{k,2}$ give rise to Cauchy subsequences in L_{8q} . In establishing this within inequality (5.3), we deduce that bounded sequences in $W_{k,2}$ are taken to precompact sequences in $W_{k-1,q}$. □

Corollary 5.6. *For $k \geq 1$ an integer, the map $F : \mathbb{R}^2 \times (H_k)_0 \to H_{k-2}$ defined by $F(\lambda, b, u) = \Delta u + bfe^{2u} - \lambda$ is continuously differentiable.*

Proof. By Proposition 5.4, $u \in H_k$ implies $e^{2u} \in W_{k-1,2} \subset H_{k-2}$, and since $bf \in C^\infty(M)$ it follows $bfe^{2u} \in H_{k-2}$. So F is well defined.

The functional $(\lambda, b, u) \mapsto \Delta u - \lambda \in H_{k-2}$ is linear and bounded. The functional $(\lambda, b, u) \mapsto bfe^{2u}$ is the composition of a continuous bilinear function

$\mathbb{R} \times H_{k-1} \to H_{k-2}$ with the differentiable map $(\lambda, b, u) \mapsto (b, e^{2u}) \in \mathbb{R} \times H_{k-1}$. Being the sum of these two, F is then continuously differentiable. $\square$

Now we fix $k \geq 1$ and compute the derivative of F respect to the last two coordinates. Making use of Proposition 5.4 we set $A_{(b,u)} = \frac{\partial F}{\partial(b,u)}(\lambda, b, u)$ and derive, for $(t, z) \in \mathbb{R} \times (H_k)_0$,

$$A_{(b,u)}(t, z) = \Delta z + tfe^{2u} + bfe^{2u}2z \quad \in \quad H_{k-2} \ .$$

Proposition 5.7. *Let* $(\lambda_0, b_0, u_0) \in \mathbb{R}^2 \times (H_k)_0$ *be a root of* F *and* $\lambda_0 \leq 0$. *Then* $A = A_{(b_0,u_0)} : \mathbb{R} \times (H_k)_0 \to H_{k-2}$ *is an isomorphism.*

Proof. By integrating $\Delta u_0 + b_0 f e^{2u_0} - \lambda_0 = 0$ on M we obtain $b_0 \int fe^{2u_0} \, d\mu = \lambda_0$, hence $b_0 \leq 0$.

We want to prove $A(t, z) = q$ has a unique solution (t, z) for any given $q \in H_{k-2}$. In the case $b_0 = 0$ we have

$$\Delta z + tfe^{2u_0} = q \ .$$

A necessary condition is that $t = \int q \, d\mu / \int fe^{2u_0} \, d\mu$. For such t, the equation $\Delta z = q - tfe^{2u_0} \in (H_{k-2})_0$ has only one solution $z \in (H_k)_0$, by Lemma 2.39.

Now consider $b_0 < 0$. Equation $A(t, z) = q$ corresponds to

$$\Delta \left(z + \frac{t}{2b_0} \right) + 2b_0 fe^{2u_0} \left(z + \frac{t}{2b_0} \right) = q \ .$$

By Lemma 2.43 the operator $v \mapsto \Delta v + 2b_0 fe^{2u_0} v$ is an isomorphism $H_k \to H_{k-2}$. Let v be the unique solution of $\Delta v + 2b_0 fe^{2u_0} v = q$. Setting $t = 2b_0 \int v \, d\mu$ and $z = v - \frac{t}{2b_0}$ we get the only solution of $A(t, z) = q$. This completes the proof of the Proposition. $\square$

The Implicit Function Theorem together with Proposition 5.7 tell us that there is a differentiable family of solutions $(\lambda, b(\lambda), u(\lambda))$ for equation

(5.1) and $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ whenever a particular solution (λ_0, b_0, u_0) is obtained with $\lambda_0 \leq 0$. This is the case of the trivial solution $(0, 0, 0)$.

Consider now a maximal interval $I \ni 0$ and a differentiable map $\lambda \in I \mapsto (b(\lambda), u(\lambda))$ whose graph is contained in the set of roots of F, and satisfying $b(0) = 0,\ u(0) \equiv 0$.

Proposition 5.8. *The maximal interval I defined above is of the form $(-\infty, T)$ for some $0 < T \leq \infty$.*

The proof of Proposition 5.8 is done by estimating the norm of $u_\lambda = u(\lambda)$ when λ is negative. Using the weak compactness of the Hilbert space H_1 one obtains sequences that converge to a solution, when λ approaches a fixed negative value from the right.

Integrating (5.1) times u we get

$$\int -|\nabla u|^2 + bfe^{2u}u \, d\mu = 0$$

$$\Rightarrow \|\nabla u_\lambda\|_2^2 = b(\lambda) \int fe^{2u_\lambda} u_\lambda \, d\mu \ . \tag{5.6}$$

The integral of (5.1) provides a formula for $b(\lambda)$,

$$0 = \int \Delta u_\lambda + b(\lambda) fe^{2u_\lambda} - \lambda \, d\mu$$

$$= b(\lambda) \left(\int fe^{2u_\lambda} \, d\mu \right) - \lambda$$

$$\Rightarrow \quad b(\lambda) = \frac{\lambda}{\int fe^{2u_\lambda} \, d\mu} \ . \tag{5.7}$$

Combining (5.6) and (5.7) we obtain

$$\|\nabla u_\lambda\|_2^2 = \lambda \frac{\int fe^{2u_\lambda} u_\lambda \, d\mu}{\int fe^{2u_\lambda} \, d\mu} \ . \tag{5.8}$$

The particular character of the function f provides a positive lower bound for $\int fe^{2u_\lambda} \, d\mu$. Let "z" be a holomorphic chart locally defined on M, and assume $z=0$ is a root of f. Observe that ϕ is a $(1,0)$-holomorphic section of L if and only if it is a 0-holomorphic section of the bundle $L\otimes T^{1,0}M$. Then Lemma 2.24 can be applied to get an open set $U\subset M$ and a non-vanishing function h with $f(z)=|\phi|^2=|h(z)|^2|z|^{2k}$ for every $z\in U$. That implies $|\ln f|$ is integrable in compact subsets of U, because the dimension of M is 2 . The existence of finite many roots of f and the compactness of M lead to the fact $\ln f\in L_1(M)$. Then,

$$\int fe^{2u}\ d\mu = e^{\overline{\ln(f)}}\int e^{2u+\ln(f)-\overline{\ln(f)}}\ d\mu\ ,$$

and Jensen's Inequality [Rd] can be applied since $2u+\ln(f)-\overline{\ln(f)}$ has 0 mean value. There exists $c>0$ so that

$$\int fe^{2u}\ d\mu \geq c > 0 \qquad \forall \quad u\in (H_1)_0\ . \tag{5.9}$$

A negative λ forces the numerator in (5.8) to be non-positive. Then this inequality yields

$$\left|\int_{u_\lambda\leq 0} fe^{2u_\lambda}u_\lambda\ d\mu\right| \geq \int_{u_\lambda>0} fe^{2u_\lambda}u_\lambda\ d\mu \geq 0\ . \tag{5.10}$$

From Calculus it is known that $|e^{2x}x|\leq\frac{1}{2e}$ if $x\in(-\infty,0]$. Hence,

$$\left|\int_{u_\lambda\leq 0} fe^{2u_\lambda}u_\lambda\ d\mu\right| \leq \int f\frac{1}{2e}\ d\mu \leq \frac{1}{2e}\|f\|_\infty\ .$$

This estimate plus (5.9) and (5.10) gives a bound for (5.8):

$$\|\nabla u_\lambda\|_2^2 \leq |\lambda|\frac{\|f\|_\infty}{ce}. \tag{5.11}$$

Poincaré Inequality is applied and we end up with an H_1 estimate for u_λ,

$$\|u_\lambda\|_{1,2} \leq C(\|f\|_\infty, c)\ |\lambda|^{\frac{1}{2}}\ . \tag{5.12}$$

The next Lemma finishes with the argument to prove Proposition 5.8.

Lemma 5.9. *Let* $J = (\lambda_0, \lambda_0 + s)$ *or* $J = (\lambda_0 - s, \lambda_0)$ *for some* $s > 0$ *so that the map* $\lambda \mapsto (b(\lambda), u(\lambda))$ *is defined on* J*, and* $F(\lambda, b(\lambda), u(\lambda)) = 0$*. If* $|b(\lambda)|$ *and* $\|u(\lambda)\|_{1,2}$ *are bounded in* J *then* $\exists\ (b_0, u_0) \in\ \mathbb{R} \times (H_1)_0$ *satisfying* $F(\lambda_0, b_0, u_0) = 0$ *.*

Proof. Choose any sequence $\{\lambda_n\} \subset J$ with $\lambda_n \longrightarrow \lambda_0$. There is a subsequence so that, after passing to the subsequence indices, $b_n = b(\lambda_n)$ converges to $b_0 \in \mathbb{R}$ and $u_n = u(\lambda_n)$ converges weakly to $u_0 \in H_1$. Because of the compact embedding $H_1 \subset L_q$ for $q \geq 1$, given by the inclusion map, and the map φ of Proposition 5.4, it can be assumed $\{u_n\}$ and $\{e^{2u_n}\}$ also converge strongly in L_2 to $u_0,\ e^{2u_0}$ respectively.

Then, the sequence $\{F(\lambda_n, b_n, u_n)\}$ converges weakly to $F(\lambda_0, b_0, u_0)$ in H_{-1}. Since for an arbitrary $w \in H_1$,

$$\begin{aligned}
\lim_{n\to\infty} F(\lambda_n, b_n, u_n)(w) &= \lim_{n\to\infty} \int -\nabla u_n \nabla w + b_n f e^{2u_n} w - \lambda_n w \ d\mu = \\
&= \lim_{n\to\infty} \int -\nabla u_n \nabla w - u_n w \ d\mu + \lim_{n\to\infty} \int u_n w + b_n f e^{2u_n} w - \lambda_n w \ d\mu = \\
&= \int -\nabla u_0 \nabla w - u_0 w \ d\mu + \int u_0 w + b_0 f e^{2u_0} w - \lambda_0 w \ d\mu = \\
&= F(\lambda_0, b_0, u_0)(w).
\end{aligned}$$

The fact that $\{F(\lambda_n, b_n, u_n)\}$ is identically zero implies $F(\lambda_0, b_0, u_0) = 0$. $\square$

Applying Proposition 5.7 to the point (λ_0, b_0, u_0) above and the I.F. Theorem lead to a smooth family of solutions $(\tilde{b}(\lambda), \tilde{u}(\lambda))$ in the interval $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, for some $\varepsilon > 0$. The uniqueness result from Lemma 5.3 says that this family coincides with the family $(b(\lambda), u(\lambda))$ when restricted to $(\lambda_0, \lambda_0 + \varepsilon)$. This shows that the maximal interval of definition for this family contains $(-\infty, 0]$, what finishes the proof of Theorem 5.2.

5.2 Another approach to System (II)

Recall the map $\pi : \tilde{S}_{II} \to \mathbb{R}^2 \times \tilde{Hol}$ defined in section 4.2. We now want to investigate the uniqueness of solutions for system (II) when $\lambda > 0$. This is formulated in the following sense: given $(b_1, b_2, \beta) \in \mathrm{im}\pi$, what can be said about the cardinality of $\pi^{-1}\{(b_1, b_2, \beta)\}$?

Notice that if $(\overline{\partial}_L, \phi)$ is a holomorphic pair representing β and $\phi \neq 0$, there is a bijection between the set of metrics H_L with $s = (b_1, b_2, \overline{\partial}_L, \phi, H_L) \in S_{II}$ and $\pi^{-1}\{(b_1, b_2, \beta)\}$. For if $\hat{s}$ denotes the first four coordinates of s and $H'_L \neq H_L$ is a metric with $(\hat{s}, H'_L) \in S_{II}$, then it must be $[s] \neq [\hat{s}, H'_L]$ where $[\cdot]$ is the orbit on S_{II} under the $\mathcal{G}_L$ action. Then $\pi([s]) = (b_1, b_2, \beta) = \pi([\hat{s}, H'_L])$, so both $[s]$ and $[\hat{s}, H'_L]$ belong to the preimage of (b_1, b_2, β). On the other hand, assuming $(b_1, b_2, \beta) \in \mathrm{im}\pi$, any point on $\pi^{-1}\{(b_1, b_2, \beta)\}$ has a representative in S_{II} whose first four coordinates are $\hat{s}$. This representative is unique.

Following Proposition 4.6, Theorem 5.2 and Lemma 5.3 we obtain existence and uniqueness of solutions on metrics for (II). In other words, fixing $b_1, b_2, \overline{\partial}_L, \phi$ with $\mathrm{sign}(b_1) = \mathrm{sign}(b_2 + c(L)) = -1$, and $(\overline{\partial}_L, \phi)$ a holomorphic pair with $\phi \neq 0$, there exists only one H_L so that $(b_1, b_2, \overline{\partial}_L, \phi, H_L)$ solves (II). In case $\phi \equiv 0$ the solution H_L is just a metric that yields constant curvature for $\overline{\partial}_L$, hence it is unique up to positive dilation.

In equation (5.1) we can see the uniqueness for $\lambda < 0$ as a consequence of the convexity of the functional $J_\lambda : H_1(M) \to \mathbb{R}$,

$$J_\lambda(v) = \int \frac{1}{2}|\nabla v|^2 + \frac{1}{2} f e^{2v} + \lambda v \; d\mu \; ,$$

whose critical points are the solutions of

$$\Delta v - f e^{2v} - \lambda = 0 \qquad , \qquad v \in H_1(M). \tag{5.13}$$

Writing $b = -e^{2\overline{v}}$, $u = v - \overline{v}$, we see that equation (5.13) is essentially the same as (5.1) with $\lambda < 0$, under the change $v \to (b, u)$.

When $\lambda > 0$ the suitable coordinate change is $v = u + \frac{1}{2}\ln(b)$ with $b > 0$, $u \in (H_1(M))_0$. Equation (5.1) corresponds to

$$\Delta v + fe^{2v} - \lambda = 0 \ , \tag{5.14}$$

which is the Euler-Lagrange equation of

$$J_\lambda(v) = \int \frac{1}{2}|\nabla v|^2 - \frac{1}{2}fe^{2v} + \lambda v \ d\mu \ .$$

For the positive λ it is not clear anymore whether J_λ is or is not convex. It is not even known if J_λ is non-degenerated at its critical points. This illustrates how important is the role played by the sign of the middle term in equations (5.13), (5.14), as it has been already remarked in Chapter 3.

The idea we will employ is the introduction of a new parametrization on the domain of the functional, not necessarily onto, so that for a not large positive λ we still have a convex functional. In fact, we will allow f to change after adding a new variable to the functional.

For instance, making $\tilde{v} = -v$ and substituting into (5.13) we obtain ($\lambda < 0$)

$$\Delta \tilde{v} + fe^{-2\tilde{v}} + \lambda = 0 \ . \tag{5.15}$$

The functional for (5.15) is again convex. The presence of the minus sign in the exponent within a positive middle term gives an idea that will be applied for $\lambda > 0$.

Set a holomorphic structure $\overline{\partial}_L$ and a background metric H_0 in L. Writing $H_u = e^{2u}H_0$ ($u \in C^\infty(M)$) it holds that if $\eta \in \Omega^{0,1}(L^*)$ is fixed then $\eta^{*H_u} = e^{-2u}\eta^{*H_0}$. Setting $\phi = \eta^{*H_u} \in \Omega^{1,0}(L)$, the first of equations (II) turns into

$$\begin{aligned}-i\Lambda F_u + b_1 i\Lambda\phi \wedge \phi^{*H_u} &= b_2 \\ \Rightarrow \Delta u + b_1 i\Lambda\eta^{*H_u} \wedge \eta &= b_2 + c(L) \\ \Rightarrow \Delta u + b_1|\eta|_0^2 e^{-2u} - \lambda &= 0 \ .\end{aligned}$$

The second of equations (II) is equivalent to

$$D'_u\eta = 0 \ , \tag{5.16}$$

D'_u being the antiholomorphic structure in L^* induced by the hermitian connection D_u in L.

Before proceeding further, we set a piece of notation. If ξ, η are (p, q) sections of a line bundle L, with metric H, their L_2 inner-product is

$$\langle\langle \xi, \eta \rangle\rangle_H = \int \langle \xi, \eta \rangle_H \ d\mu \ . \tag{5.17}$$

Similarly, their H_1 inner-product is

$$\langle\langle \xi, \eta \rangle\rangle_{H(1,2)} = \int \langle \xi, \eta \rangle_H \ d\mu + \int \langle D_H\xi, D_H\eta \rangle_H \ d\mu \ . \tag{5.18}$$

If the metric is $H_u = e^{2u}H_0$ we write

$$\langle\langle \xi, \eta \rangle\rangle_u = \langle\langle \xi, \eta \rangle\rangle_{H_u} \ . \tag{5.19}$$

One does not expect in general to solve (5.16), unless η moves together with u. Thus, we need to introduce a new variable in the functional that is about to be constructed, accounting for the (0,1) sections in L^*. The clever choice here is to work on the L_2 closure of a subspace of $\Omega^{0,1}(L^*)$.

From the Hodge Theory on (p,q) sections of L^* (see [GH]) there is an orthogonal decomposition

$$\Omega^0(L^*) = \mathrm{Hol}^{0,0}(L^*) \oplus \mathrm{im}D''^*$$

where $\mathrm{Hol}^{0,0}(L^*)$ are the harmonic sections on L^* and D''^* is the formal adjoint of $D'' : (\Omega^0(L^*), \|\cdot\|_{H(1,2)}) \to (\Omega^{0,1}(L^*), \|\cdot\|_{L_2})$. Let

$$S = H_1\text{-closure of } \mathrm{im} D''^* \ , \tag{5.20}$$

then

$$H_1(\Omega^0(L^*)) = \mathrm{Hol}^{0,0}(L^*) \oplus S \ . \tag{5.21}$$

Observe that S is a closed subspace of a Hilbert space and has finite codimension. The map $D'' : S \to L_2(\Omega^{0,1}(L^*))$ is bounded linear and injective.

From now on it is no longer required that H_0 yields constant curvature for $\overline{\partial}_L$. Also notice that different (smooth) metrics on L^* give rise to equivalent Sobolev norms on the sections of L^*, thanks to the compactness of M. Hence the space S is well defined regardless of the chosen H_0.

Now fix $0 \neq \eta \in \Omega^{0,1}(L^*)$ with $D'_0\eta = 0$. Let $J : H_2(M) \times S \to \mathbb{R}$ be given by

$$J(u,\xi) = \int \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|\eta + D''\xi|_0^2 e^{-2u} + f_0 u \, d\mu \tag{5.22}$$

with $f_0 \in C^\infty(M)$. In spite of $\eta + D''\xi \in L_2$ there is no issue about integrating $\frac{1}{2}|\eta + D''\xi|_0^2 e^{-2u}$ because $u \in H_2$ is continuous. Proposition 5.4 and the observation that $J(u,\xi)$ is quadratic in the second variable justifies that J is continuously differentiable in (u,ξ).

The next theorem was motivated and inspired by the talks we had with Karen Uhlenbeck.

Theorem 5.10. (1) *The critical points of J are the solutions in (u,ξ) for*

$$\begin{cases} \Delta u + i\Lambda(\eta + D''\xi)^{*H_u} \wedge (\eta + D''\xi) - f_0 = 0 \\ D''(\eta + D''\xi)^{*H_u} = 0 \end{cases} \tag{5.23}$$

(2) *Let $H_u = e^{2u}H_0$ and $F_{L_u} = F_{(\overline{\partial}_L, H_u)}$. If (u,ξ) is a critical point for J and $f_0 < i\Lambda F_{L_0}$ then $J''_{(u,\xi)}$ is non-degenerated.*

Proof. Let $z \in H_2(M)$ and compute

$$\begin{aligned} \frac{\partial J}{\partial u}_{(u,\xi)}(z) &= \int \langle \nabla u, \nabla z\rangle - |\eta + D''\xi|_0^2 e^{-2u} z + f_0 z \; d\mu = \\ &= -\int (\Delta u + |\eta + D''\xi|_0^2 e^{-2u} - f_0) z \; d\mu \; . \end{aligned} \tag{5.24}$$

Observe that $|\eta + D''\xi|_0^2 e^{-2u} = \langle \eta + D''\xi, \eta + D''\xi\rangle_u$. Then for $\chi \in S$ the derivative respect to the second coordinate yields

$$\begin{aligned} \frac{\partial J}{\partial \xi}_{(u,\xi)}(\chi) &= \int \frac{1}{2}(\langle \eta + D''\xi, D''\chi\rangle_u + \langle D''\chi, \eta + D''\xi\rangle_u) \; d\mu = \\ &= \mathrm{Re}\left\{\int \langle \eta + D''\xi, D''\chi\rangle_u \; d\mu\right\} = \\ &= \mathrm{Re}\,\{\langle\langle \eta + D''\xi, D''\chi\rangle\rangle_u\} \end{aligned} \tag{5.25}$$

A critical point (u, ξ) then implies (5.24) and (5.25) equal to zero for any choices of z, χ. Hence,

$$\begin{aligned} &\Delta u + |\eta + D''\xi|_u^2 - f_0 = 0 \\ &\mathrm{Re}\{\langle\langle \eta + D''\xi, D''\chi\rangle\rangle_u\} = 0 \; . \end{aligned}$$

Because of the arbitrarity of χ in (5.25), the argument of $\mathrm{Re}\{\cdot\}$ must be identically 0 and we can integrate by parts to get

$$\begin{aligned} 0 &= \int \langle \eta + D''\xi, D''\chi\rangle_u \; d\mu = \int i(D''\chi)^{*u} \wedge (\eta + D''\xi) = \\ &= \int iD'_u(\chi^{*u}) \wedge (\eta + D''\xi) = \\ &= -\int i\chi^{*u} D'_u(\eta + D''\xi) = \\ &= -\int \chi^{*u} i\Lambda D'_u(\eta + D''\xi) \; d\mu \; . \end{aligned} \tag{5.26}$$

Again, the arbitrarity of χ in (5.26) implies $i\Lambda D'_u(\eta + D''\xi) \equiv 0$, from what follows

$$D''(\eta + D''\xi)^{*u} = 0 \; ,$$

and (1) is proven.

The proof of assertion (2) is a long computation. One must show that if $J''_{(u,\xi)}(z,\chi) = 0$ then $(z,\chi) = 0 \in H_2(M) \times S$. For the sake of simplicity, assume the critical point is of the form $(u,0)$. Then write the derivative of J at $(u,0)$ in a column-matrix:

$$J'_{(u,0)} = \begin{pmatrix} -\Delta u - |\eta|_0^2 e^{-2u} + f_0 \\ \varphi_u(\eta) \circ D'' \end{pmatrix} \qquad \in \qquad \begin{matrix} H_{-2}(M) \\ \times \\ S^* \end{matrix}$$

where $\varphi_u(\eta)$ is the functional $(\psi \mapsto \mathrm{Re}\{\langle\langle \eta, \psi \rangle\rangle_u\})$. Then

$$J''_{(u,0)}(z,\chi) = \begin{pmatrix} -\Delta z + 2|\eta|_0^2 e^{-2u} z - 2\mathrm{Re}\{\langle \eta, D''\chi \rangle_0\} e^{-2u} \\ \varphi_u(-2z\eta + D''\chi) \circ D'' \end{pmatrix} .$$

Let I be the first coordinate of $J''_{(u,0)}(z,\chi)$ evaluated at z, or

$$I = \int |\nabla z|^2 + 2|\eta|_0^2 e^{-2u} z^2 - 2\mathrm{Re}\{\langle \eta, D''\chi \rangle_0\} e^{-2u} z \, d\mu \; . \tag{5.27}$$

Assume that the second coordinate of $J''_{(u,0)}(z,\chi)$ is zero. Then a computation similar to (5.26) yields

$$D'_u(-2z\eta + D''\chi) = 0 \; . \tag{5.28}$$

Now let us estimate the last term in (5.27):

$$\begin{aligned} \int 2\mathrm{Re}\{\langle \eta, D''\chi \rangle_0\} e^{-2u} z \, d\mu &= \int \mathrm{Re}\{\langle 2z\eta, D''\chi \rangle_u\} \, d\mu = \\ = \mathrm{Re}\left\{ \int i(D''\chi)^{*u} \wedge (2z\eta) \right\} &= \mathrm{Re}\left\{ -\int i\chi^{*u} D'_u(2z\eta) \right\} \; . \end{aligned} \tag{5.29}$$

The last equation in (5.29) leads to two different expressions. In virtue of (5.28) it holds

$$\begin{aligned} -\int i\chi^{*u} D'_u(2z\eta) &= -\int i\chi^{*u} D'_u D''\chi = \\ &= \int i(D''\chi)^{*u} \wedge D''\chi = \|D''\chi\|_u^2 \; . \end{aligned} \tag{5.30}$$

But it also holds

$$\left|\int i\chi^{*u} D'_u(2z\eta)\right| \le \int 2|\partial z||\chi^{*u}\eta|\ d\mu \le \sqrt{2}\|\nabla z\|_2\|\chi^{*u}\eta\|_2\ , \tag{5.31}$$

after the observation that $|\partial z| = \frac{|\nabla z|}{\sqrt{2}}$, because z is real-valued.

From (5.29) and (5.30) we deduce

$$\langle\!\langle 2z\eta, D''\chi\rangle\!\rangle_u = \|D''\chi\|_u^2\ ,$$

what implies

$$\|2z\eta\|_u^2 \ge \|D''\chi\|_u^2\ . \tag{5.32}$$

Write I as

$$I = \|\nabla z\|_2^2 + 2\|z\eta\|_u^2 - \|D''\chi\|_u^2\ .$$

Hence, using inequality (5.31), get

$$\begin{aligned} I \ge& \|\nabla z\|_2^2 + 2\|z\eta\|_u^2 - \sqrt{2}\|\nabla z\|_2\|\chi^{*u}\eta\|_2 \\ \ge& \left(\|\nabla z\|_2 - \left\|\frac{1}{\sqrt{2}}\chi^{*u}\eta\right\|_2\right)^2 + \frac{1}{2}\|2z\eta\|_u^2 - \frac{1}{2}\|\chi^{*u}\eta\|_2^2\ . \end{aligned} \tag{5.33}$$

The term $\|\chi^{*u}\eta\|_2^2$ is estimated using equality

$$\int iF_{L_u^*}\chi\chi^{*u} = -\|D''\chi\|_u^2 + \|D'_u\chi\|_u^2\ , \tag{5.34}$$

and using the pointwise identity

$$-i\Lambda F_{L_u^*} = i\Lambda F_{L_u} = |\eta|_u^2 - (f_0 - i\Lambda F_{L_0})\ , \tag{5.35}$$

that comes from the first equation on (5.23). Then, if $\chi \ne 0$,

$$\begin{aligned} \|\chi^{*u}\eta\|_2^2 =& \int |\chi|_u^2|\eta|_u^2\ d\mu \\ &< \int |\chi|_u^2(|\eta|_u^2 - f_0 + i\Lambda F_{L_0})\ d\mu = \\ &= \int -i(F_{L_u^*}\chi)\chi^{*u} = \\ &= \|D''\chi\|_u^2 - \|D'_u\chi\|_u^2\ . \end{aligned} \tag{5.36}$$

Inequalities (5.35) and (5.36) applied to (5.33) lead to

$$I > \left(\|\nabla z\|_2 - \left\| \frac{1}{\sqrt{2}} \chi^{*u} \eta \right\|_2 \right)^2 + \frac{1}{2} \|D'_u \chi\|_u^2 \geq 0 \ . \tag{5.37}$$

This last inequality says that if $\chi \neq 0$ then $J''_{(u,0)}(z, \chi)$ does not vanish. In case it does, then $\chi = 0$ and the evaluation of I reduces to

$$I = \int |\nabla z|^2 + |\eta|_0^2 e^{-2u} \ d\mu \tag{5.38}$$

and $I = 0$ if and only if $z = 0$. In conclusion $J''_{(u,0)}$ is not degenerate and hence we have the proof of (2) in the critical point $(u, 0)$. The general case (u, ξ) follows from the case $(u, 0)$ once we set $\tilde{\eta} = \eta + D''\xi$ and $\tilde{J} = \tilde{J}(u, \tilde{\xi})$ defined by (5.22) with η replaced by $\tilde{\eta}$. This ends the proof of the Theorem. $\square$

5.3 Further Remarks

Let us first investigate the behaviour of the family solution $\lambda \mapsto (b(\lambda), u(\lambda))$ from Theorem 5.2, when $\lambda \to T^-$. Assume $T < \infty$. What prevents this family to go over $\lambda \geq T$?

Proposition 5.11. *Let $\lambda \mapsto (b(\lambda), u(\lambda))$ be the family defined in Theorem 5.2. Then either*

$$\lim_{\lambda \to T^-} \|u(\lambda)\|_{1,2} = \infty \tag{5.39}$$

or there exists $(b_T, u_T) \in \mathbb{R} \times (H_1)_0$ so that $F(T, b_T, u_T) = 0$ and the derivative $\frac{\partial F}{\partial(b,u)}(b_T, u_T)$ is not invertible.

Proof. Assume (5.39) is not true. Then there is a sequence $\{\lambda_n\}$ converging to T ($\lambda_n < T$) so that the sequences $\{u_n = u(\lambda_n)\}$ and $\{b_n = b(\lambda_n)\}$ are bounded in H_1 and $\mathbb{R}$, respectively. Applying Lemma 5.9 we obtain a solution (b_T, u_T) for $\lambda = T$.

Let us suppose, by contradiction, that the derivative of F at (b_T, u_T) is invertible. The Implicit Function Theorem gives a family $(\tilde{b}(\lambda), \tilde{u}(\lambda))$ of roots of F defined in a small interval centered at T. The image of this family is contained in an open set $G \subset \mathbb{R} \times (H_1)_0$ with $(b_T, u_T) \in G$, and solutions of $F(\lambda, b, u) = 0$ are unique in G for each λ. It will be shown that the family $(\tilde{b}, \tilde{u})$ coincides with the family (b, u) for all λ sufficiently close to T. Let $(T - \varepsilon, T + \varepsilon)$ be the interval of definition for $(\tilde{b}, \tilde{u})$.

The weak H_1 convergence $u_n \to u_T$ given by Lemma 5.9 implies strong L_p convergence for $p \geq 1$. Hence, by Corollary 5.5, $e^{2u_n} \to e^{2u_T}$ in L_2. Since b_n converges to b_T, the sequence

$$\{\lambda_n - b_n f e^{2u_n}\} = \{\Delta u_n\}$$

is Cauchy in H_0. Therefore, $\{u_n\}$ is Cauchy in H_2 by applying Lemma 2.39. This gives H_1 convergence $u_n \to u_T$.

In particular, (b_n, u_n) belongs to G for all n sufficiently large. Therefore, the set

$$Q = \{\lambda \in (T - \varepsilon, T) \quad | \quad b(\lambda) = \tilde{b}(\lambda),\ u(\lambda) = \tilde{u}(\lambda)\}$$

is not empty. Because both families are continuous in λ and G is open, Q must be open and closed. The traditional argument about connectedeness is applied to Q, and we conclude $Q = (T - \varepsilon, T)$. Then the family $\lambda \mapsto (b(\lambda), u(\lambda))$ can be extended to $T + \varepsilon$, what shows the contradiction. To avoid that it must be

$$\left.\frac{\partial F}{\partial(b, u)}\right|_{(b_T, u_T)}$$

not invertible. □

In fact, we are tempted to conjecture that the family solution always blows up when λ approaches T. Though it has been hard to estimate the behaviour of $\|u(\lambda)\|_{1,2}$ in the positive λ case.

Define $N \subset (H_1)_0$ as the set of cluster points of $\mathrm{im}([0,T) \ni \lambda \mapsto u(\lambda))$.

Corollary 5.12. *Assume $u_0, u_1 \in N$. If for some $t \in \mathbb{R}$ it holds*

$$\|u_0\|_{1,2} < t < \|u_1\|_{1,2} \tag{5.40}$$

then $\exists\ u_T \in N$ with $\|u_T\|_{1,2} = t$. If $\sup_{\lambda\in[0,T)} \|u(\lambda)\|_{1,2} = \infty$, the result still follows if we substitute $\|u_1\|_{1,2}$ by ∞ in (5.40).

Proof. Let $\{\alpha_n\}, \{\beta_n\} \subset [0,T)$ be two sequences converging to T, so that $u(\alpha_n) \to u_0$ and $u(\beta_n) \to u_1$. We can assume $\ldots \alpha_n < \beta_n < \alpha_{n+1} \ldots\ \forall n \in \mathbb{N}$. Then, for all n sufficiently large,

$$\|u(\alpha_n)\|_{1,2} < t < \|u(\beta_n)\|_{1,2}$$

and we can find $\{\gamma_n\} \subset \mathbb{R}$, $\alpha_n < \gamma_n < \beta_n$, with

$$\|u(\gamma_n)\|_{1,2} = t\ .$$

Hence, the sequences $\{u(\gamma_n)\} \subset (H_1)_0$ and $\{b(\gamma_n)\} \subset \mathbb{R}$ are bounded. Following the proof of Proposition 5.11 we obtain a solution (b_T, u_T) with $u_T = \lim_{n\in J} u(\gamma_n)$, where J are the indices of a subsequence. Then, $u_T \in N$ and

$$\|u_T\|_{1,2} = \lim_{n\in J} \|u(\gamma_n)\|_{1,2} = t\ .$$

In the case $\sup_{\lambda\in[0,T)} \|u(\lambda)\|_{1,2} = \infty$ the same reasoning follows if we choose $\{\beta_n\}$ so that $\|u(\beta_n)\|_{1,2} \to \infty$. □

Corollary 5.12 is an interesting way of "breaking" the uniqueness of equation (5.1) when $\lambda = T$. If there are at least two solutions with different norms (one of them can be the "infinity" solution) then there is a whole bunch of solutions with intermediate norms. Even when the solutions u_0, u_1 have the same norm, if $u_0 \neq -u_1$, a variation of the Corollary 5.12 and Proposition 5.11 still applies because the metric in $(H_1)_0$ can be dilated in the direction of one of the solutions. This dilation leaves the orthogonal subspace to the solution with the original metric. Then, in the new metric, u_0 and u_1 have different norms.

So far, we have used that f is the square norm of a holomorphic section in the following facts:

1. $f \geq 0$ and f has finite many roots.
2. Existence of bounds for $b(\lambda)$, from (5.7) plus (5.9).

To have ϕ as a holomorphic section, though, is a very strong condition. For instance, it restricts ϕ to be a linear combination over $\mathbb{C}$ of finite many holomorphic sections on $\Omega^{1,0}(L)$. The candidates for $f = |\phi|^2_{H_0}$ are a lot less arbitrary than the smooth functions allowed by conditions 1 and 2 above. We wonder if this could be used to improve the results already obtained, that is, either extending T to infinity or providing a better knowledge of the family solution behaviour when $\lambda \to T^-$.

This idea is inspired by the results obtained by S-Y. Chang and P.C. Yang on [CY]. They found that on the sphere $S^2 \subset \mathbb{R}^3$ with the induced Euclidean metric ($|S^2| = 4\pi$) the equation

$$\Delta u + Ke^{2u} - 1 = 0 \tag{5.41}$$

has a solution if $K > 0$ and K satisfies conditions on its critical points. The next Theorem is an exact quote of Theorem II in [CY]:

Theorem 5.13. *Let K be a positive smooth function (on S^2) with only non-degenerated critical points, and in addition $\Delta K(Q) \neq 0$ where Q is any critical point. Suppose there are $p+1$ local maximum points of K, and q saddle points* *of K with $\Delta K(Q) < 0$. If $q \neq p$ then K admits a solution to the equation* (5.41).

Local behaviour of f inherited from the holomorphic condition on ϕ seems to be really important. If one fixes the problem that f is not in general strictly positive, one may attempt to find sections ϕ whose square norms have the good hypothesis on their critical points.

Finally, we have Theorem 5.10, that corresponds in equation (5.1) to seek solutions in b, u and also f. When $b_1 > 0$ the solutions for system (II) with a fixed holomorphic pair yield solutions for system (5.23) in the variables (u, ξ), after the change $\eta = (\sqrt{b_1}\phi)^{*H_u}$, $f_0 = \lambda = b_2 + c(L)$. How the local uniqueness for this solution together with the previous remarks change the picture of the problem is the open question we leave for future researchers.

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This Dissertation was typeset with LaTeX[‡] by the author.

[‡]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.