Solving Single Variable Equations in the Algebra Classroom

by

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Abstract

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This report recognizes the common mistakes students make solving single variable equations and attempts to connect these errors to varying levels of developmental readiness. Striving to meet the needs of all students, this report offers an alternative to lecture based mathematics lessons by exploring the benefits of a unit based curriculum. The unit poses an overarching problem for the students to investigate and answer, while the students still receive the instruction on solving algebraic equations. The goal is to move beyond traditional procedures when solving equations to help students become more familiar with symbol manipulation involved in solving real world story problems.

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Chapter 1: Introduction

It is important for high school mathematics teachers to examine the ways in which students are taught to solve single variable equations in Algebra and build their content knowledge for higher level mathematics. Because students are required to master the skill of solving equations in order to achieve success in their proceeding mathematics courses such as Geometry, Algebra 2, Trigonometry, and Pre-Calculus it is important to analyze the methods of instruction and the success of these methods in the classroom.

The typical high school algebra textbook places a strong emphasis on solving single variable equations by following a “step-by-step” process. The process begins by drawing a vertical line through the equal sign, and using inverse operations to move terms from one side of the equal sign to the other in order to solve for a single variable. Once students are successful solving one step equations, advancement is made to word problems requiring students to set up an equation and solve for the variable. Next in the progression, the curriculum moves to two step and multistep equations using the distributive property with variables on both sides of the equation, before moving to word problems of the same nature.

Is this method of learning the procedure and then applying those steps to word and story problems helping the students to comprehend the reasoning behind algebra? Does this algorithmic equation solving process hinder the students’ algebraic understanding?

Chapter 2: Literature Review

The articles included in this report address some of the fundamental problems students encounter when solving algebraic equations and the varying developmental levels required in solving these equations. The report first examines Herscovics and Linchevski's (1994) article on the common misunderstandings and mistakes students make solving algebraic equations. Herscovics and Linchevski also observed student preferences when writing equations and student readiness for using mathematical symbols in place of numeric values. This leads to Adi's (1978) research that defined three distinct levels of developmental readiness and the correlation of higher levels finding more success when solving algebraic equations.

Next, new methods of instruction are considered using the work of Brenner et al. (1997) which compares the standard textbook approach to a new unit based curriculum. The unit approach introduced story and word problems to students before initial lessons on how to solve for a missing variable, and allowed students to find the solution through original methods.

Furthermore, Koedinger and Nathan's (2000) article determined, contrary to teacher and researcher prediction, students had more difficulty solving symbolic equation problems, than verbal story and word problems. Lastly, Arcavi's (1994) paper suggests ways for teachers to introduce mathematical symbols into the classroom to help students realize the important role symbols play in applying mathematics to various types of problems and situations.

Chapter 3: Solving Single Variable Equations

When deciding the best instructional methods for teaching secondary students to solve single variable equations it is important to understand the conceptual struggle students' encounter and the common mistakes made. To do this Herscovics and Linchevski (1994) studied the cognitive gap between arithmetic and algebra in order to identify some of the common mistakes students make while solving equations and why these misunderstandings exist.

Herscovics and Linchevski (1994) interviewed 22 seventh graders, with limited exposure to solving equations to observe the processes used to solve different types of equations. Students were asked to define the word "equation" and to provide the interviewer with an original example of an equation. The researchers were also interested in the use of the equality symbol and found that with statements such as, "$9 = 4 + 5$, many children would refuse to accept it [$9 = 4 + 5$], claim that it was written backwards and re-write as $4 + 5 = 9$" (p. 65). Later in the interview, students struggled with this decomposition when they began to read the problems backward from right to left.

> This was of no consequence when only a sum was involved as in $35 = n + 16$. However, in the case of subtraction such as $364 = 796 - n$, three students changed the inverted problem by reading it from right to left, n minus 796 equals 364. (p. 65)

Herscovics and Linchenski (1994) also focused on the ways in which students followed the rules of order of operations. The study used the following equations to evaluate students' work,

$$5 + 6 \times 10 = ?$$
$$17 - 3 \times 5 = ?$$
$$8 \times (5 + 7) = ?$$

(p. 66)

Of the 22 students interviewed in the study, 17 incorrectly answered the first problem, giving an answer of 110 because addition was performed first, but 19 students correctly answered the second problem, performing the multiplication first and then subtraction. When the interviewer challenged the students that incorrectly answered the first problem with an alternative answer of 65, all students remembered the order of operations and performed the multiplication before the addition when given a second opportunity to solve the problem.

Longer equations, of five or more terms, were also given to determine how the students would solve a string of operations. Given the problem,

$$17 + 59 - 59 + 18 - 18 = ?,$$

(Herscovics and Linchevski, 1994, p. 66)

only five students observed both cancellations, three students observed one of the cancellations, and the remaining 13 did not observe either cancellation. These 13 students performed each operation one at a time moving from the left to right, without noticing the cancellations.

To finish the interview Herscovics and Linchevski (1994) asked students to identify which type of equation they would prefer to work when solving equations. Of the 22 students, "12 chose to work with the concatenated form ($3n$), eight chose the form using the multiplication sign ($3 \times n$), and two preferred to work with placeholders ($3 \times \square$)" (p. 67).

It is important to notice the types of arithmetic errors the students were making when solving equations. The order of operations, the flexibility when writing the equation (i.e. the position of the equal sign), and the understanding of zero pairs (e.g. -18 and 18) are all necessary skills for students to efficiently solve single variable equations. A student's preference for writing an equation helps shed light on the developmental readiness required to handle formal mathematical equations. While Herscovics and Linchevski's (1994) study found there was a group of students ready to begin using variables and coefficients, others still needed to use a box as a missing place holder, as well as the multiplicative symbol of, "×." Knowing this difference in readiness exists, does it affect students' success rates when solving equations? Do certain students find more success in solving algebraic equations because they are more developmentally ready to do so?

Adi (1978) explores the relationship between a student's developmental level and understanding of solving single variable equations. To discover if a relationship exists between developmental level and understanding of solving equations, Adi conducted a study with 75 college students. Using Piaget's theory on intellectual development the students were grouped into three developmental categories as determined by their performance on an assigned paper and pencil task:

> IIA: Early concrete operational thought (49.3% of the group or 37 subjects)
> IIB: Late concrete operational thought (34.7% of the group or 26 subjects)
> IIIA: Early formal operational thought (16% of the group or 12 subjects) (p. 208).

After the students were given an assigned task and grouped according to their developmental level, all subjects attended five 50 minute sessions where they were taught how to correctly solve equations using two different methods. The first two sessions were dedicated solely to the *inversion method* of solving, which means "reversing an action by undoing it" (p. 204). Adi used the example,

$$14 - \frac{15}{7 - x} = 9$$

14 minus what equals nine? (Answer = 5)

15 divided by what equals 5? (3)

7 minus what equals 3? (4)

Solution: 4

(p. 205).

The third and fourth sessions were used to teach students the method of *compensation*, which means "reversing an action by cancelling for its effects" (p. 204). Adi used the same example as before to show,

$$14 - \frac{15}{7 - x} = 9$$

$$14(7 - x) - 15 = 9(7 - x)$$

$$98 - 14x - 15 = 63 - 9x$$

$$83 - 14x = 63 - 9x$$

Add ($14x$) to both members,

$$83 = 63 + 5x$$

Subtract (63) from both members,

$$20 = 5x$$

Divide both members by 5,

$$4 = x$$

(p. 205).

The fifth session was used for a review of both methods before the students took the posttest. The test consisted of 12 items; the first six items tested inversion and the final six tested compensation. Students were graded on their procedure, reasoning, and correctness of the solution to each problem. Students were considered successful if they could correctly answer five out of six questions from each set.

Adi (1978) found, “a positive relationship between the developmental levels and success on the two methods of equation solving” (p. 210).

> According to Piaget’s theory of intellectual development, two different forms of reversibility- negations (inversions) and reciprocities (compensations) - are applicable on the level of concrete operations. The coordination between these two forms of reversibility within the same system was defined to be a trait of formal operations. (p.204)

The late concrete operational group scored better than the early concrete operational group on inversion problems, while the early formal operational group scored higher on the compensation based questions. A one tailed *t*-test revealed a *t* value suggesting, “that reversal equation solving was much easier than formal equation solving for the early concrete operational group as opposed to the early formal operational group” (p. 211).

Having established the different developmental levels associated with learning to solve equations, does it seem realistic that all students will become proficient with solving equations by the traditional methods of compensation and inversion? The step-by-step process model provides a clear set of directions on how to solve any equation for the unknown, but is this the best method for teaching a novice during their first Algebra class? Simply teaching the process of solving an equation does not suggest that a student will be able to transfer this skill to a new problem or a unique situation. After *repetitive practice*, i.e., solving an entire worksheet of single variable equations, will a student be able to read and understand a word problem that requires the construction of an equation and a solution that coincides?

Research conducted by Brenner et al. (1997) suggested organizing curriculum into cohesive units of instruction as opposed to teaching separate lessons each day on a new topic from the textbook. Six Pre-Algebra classes consisting of 128 seventh and eighth graders participated in a study taught by three different teachers at three different schools. Each teacher taught both a conventional Pre-Algebra class following the textbook, and then a second class of different students used the representation based unit on functions and materials as designed by the research team. Students in all classes took four pretests, were given 20 days of instruction, and then took the posttests to compare the results from the representation based unit to the standard textbook method of instruction.

During the 20 days of class time the teachers covered, "functions, variable expressions, and one variable equations," (Brenner et al., 1997, p. 670). The representation based unit intended to determine which pizza company should provide pizza to the school cafeteria for the best value and taste. Students began with a taste test in lesson 1. In lessons 2 through 5 the students used patterns found on order forms and invoices to create tables and graphs of the data. For lesson 6 students used variable expressions to determine the correct location when delivering pizza, and in lessons 7 through 10 developed formulas for area. Nutrition and fat content were used by students to create equations in lessons 11 through 14, and in lessons 15 through 18 the students used tables and graphs to organize the profit and loss statements of the pizza data. Students then spent the last two days generating a report on which pizza company should supply pizza to the cafeteria (p. 675). In comparison, the conventional Pre-Algebra class covered the same topics but as separate lecture based lessons on functions, graphing, and solving equations.

To interpret the results of the study, "…an analysis of covariance (ANCOVA) was conducted on the posttest scores using the pretest as a covariate" (Brenner et al., 1997, p. 677). The group of students using the representation based unit as a curriculum guide scored significantly higher on problem representation and problem solving skills in the posttests, while the conventional curriculum group of students scored significantly higher on equation solving.

> The results demonstrate qualitative differences in the learning outcomes produced by two different instructional treatments. The cognitive consequences of traditional instruction focusing on symbol manipulation are reflected in improvements in students' ability to solve equations and perform arithmetic computations. The cognitive consequences of learning by understanding involve improvements in students' ability to represent functional relations in words, equations, tables and figures and to translate among these representations. (p. 684)

It is important to determine whether the goal is to teach students to become experts at solving equations or to become expert problem solvers that have mastered the skill of solving equations. In current textbook curriculum, word problems are a culminating task, set aside until the very end of a skill set. Does this sequence in the textbook promote the idea to educators that students are unable to solve word problems until after they have mastered a skill? Are students able to solve word problems without mastering a concept, but rather by using critical analysis?

Koedinger and Nathan (2000) developed a study using high school mathematics teachers and mathematics educational researchers to determine if their views would align with how students performed on a given mathematics performance task. The teachers were directed to rank 12 given problems in order of difficulty, beginning with the easiest problems. The 12 problems consisted of three formats: arithmetic story, arithmetic word equation, and arithmetic symbolic equation problems, as shown,

> Problem 4: When Ted got home from his waiter job, he took the $81.90 he earned that day and subtracted the $66 he earned in tips. Then he divided the remaining money by the 6 hours he worked and found his hourly wage. How much per hour does Ted make?
>
> Problem 5: Starting with 81.90, if I subtract 66 and then divide by 6, I get a number. What is it?
>
> Problem 6: Solve for x: $(81.90 - 66)/6 = x$. (p. 170)

There were also two classifications of the problems, *result-unknown* and *start-unknown*. The example problems given above (problems 4-6) are the result-unknown problems, "in which the unknown quantity is the result of the events or mathematical operations described on the problem" (Koedinger and Nathan, 2000, p. 170) In contrast, with the start-unknown problems, "the unknown value refers to a quantity needed to specify a relationship" (p. 170). The three start-unknown problems included in the study were,

> Problem 1: When Ted got home from his waiter job, he multiplied his hourly wage by the 6 hours he worked that day. The he added the $66 he made in tips and found he earned $81.90. How much did Ted make?
>
> Problem 2: Starting with some number, if I multiply by 6 and then add 66, I get 81.90. What did I start with?
>
> Problem 3: Solve for x: $x \cdot 6 + 66 = 81.90$ (p. 170)

The researchers were also asked to rank the problems, but were asked to begin by identifying the problems students would have the most difficulty with and continue to the problems that would be easiest to solve. The researchers and teachers were very similar in that the teachers ranked Problems 6, 5, and 4 as the easiest, Problem 3 as medium-easy, Problem 1 as medium-hard, and Problem 2 as the hardest. The researchers ranked Problems 5, 4, and 6 as the easiest, Problem 3 as medium-easy, and Problems 2 and 1 as the hardest (Koedinger and Nathan, 2000, p. 178).

Koedinger and Nathan (2000) compared the teachers' and researchers' predictions to actual results found in a previous study in which 76 Geometry and Algebra 1 students completed the six problem set. Analyzing the student results showed Problems 4 and 5 were the easiest for the students, Problems 1, 2, and 6 were medium-easy, and Problem 3 was the hardest with only 37% of the students solving it correctly (p. 178).

Side-by-side comparison of students' performance to the teachers' and researchers' expectations of the students, teachers and researchers were correctly able to identify that students would have, "lower performance levels on start-unknown (Algebra) problems than on the result-unknown problems" (Koedinger and Nathan, 2000, p. 179). But, "…contrary to teachers' expectations, students experienced greater difficulties when solving symbolic-equation problems than when solving verbally presented problems" (p. 178). The teachers assumed the word and story problems would be the most difficult for the students to solve, and instead, "30% of the symbolic equations - more than twice the rate for all other problem types - elicited no response from the Algebra students" (p. 176).

This current research demonstrates that teachers may need to evaluate how to teach students to solve single variable equations, moving away from the textbook which first introduces, "…arithmetic computations in symbolic form, followed by the application of these procedures to stories and scenarios" (Koedinger and Nathan, 2000, p. 181). Not all students will be ready to work with variables and symbols in formal equation writing, but some may find success in solving word and story problems when allowed to use their own method.

To expedite the learning of symbols and the manipulations involved in equation solving, teachers need to convey the purpose of symbols in mathematics and the role of symbols in solving equations. Arcavi (1994) introduces eight behaviors exemplifying the use of symbols and the ways in which teachers can encourage *symbol sense* (similar to number sense) to help students become more comfortable with the use of symbols when solving single variable equations. The suggestions is made that,

> Algebraic symbolism should be introduced from the very beginning in situations in which students can appreciate how empowering symbols can be in expressing generalizations and justifications of arithmetical phenomena. (p. 33)

Figure 1 provides an example to demonstrate the powerful use of symbols in solving problems,

> In the following arrangement of *n* tables, *X* indicates a seat for a single person, and "…" indicates a variable number of tables. How many people can be seated? (p. 33)

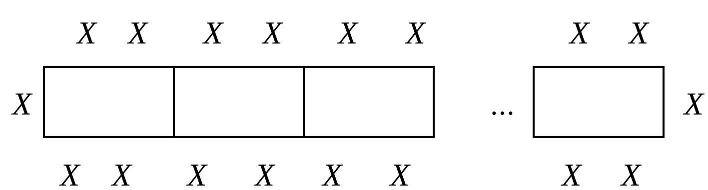

Figure 1: Seating

This problem presents very few numbers to work with and forces the student to find a general formula that can be used once the number of tables is known. The article suggests students may solve the problem by one of two ways. First, students may use the perimeter to find the number of seats, yielding a formula of,

$$2(2n) + 2 \times 1, \text{ namely } 4n + 2.$$

(Arcavi, 1994, p. 33)

The second solution can be found by finding the number of seats per table, (4) multiplying by the number of tables, taking off the 2 end tables that hold 10 people altogether which can be added on at the end, and deriving the formula,

$$4(n - 2) + 10.$$

(p. 33)

Teachers can, in turn, develop a class discussion around the different equations students developed for finding the number of seats at the table. Analyzing the, "…alternatives is usually helpful in establishing connections between symbolic and other approaches" (Arcavi, 1994, p. 34). Working with problems that move beyond basic arithmetical skills will prepare students for the application of symbols used in solving single variable equations in the Algebra classroom.

Chapter 4: Conclusion

Students will use the skills acquired in beginning Algebra classes throughout their mathematical careers. The National Council of Teachers of Mathematics (2000) sets an Algebra standard for students to, "write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency—mentally or with paper and pencil in simple cases and using technology in all cases" (p. 296). In order to meet this National Standard, students must work to become fluent with arithmetic, variables, and mathematical symbols to solve equations, while also having mastered order of operations and the abstract concept of a variable representing a missing numeric value. Developing a solid foundation in solving equations and critical problem solving skills is important because it opens the door to other branches of mathematics for all students.

The articles included in this report highlight the fact that students successfully respond and learn to solve equations for a single variable through various instructional methods. According to Adi (1978) students of the early formal operational group are developmentally ready for formal equation solving, and this group only represented 16 percent of the study. The other 84 percent of students needed an alternative method that Adi (1978) referred to as reversal equation solving.

High school mathematics teachers do not have the flexibility to teach 16 percent of the class one method and the remaining 84 percent in another manner to reach the same goal. Teachers need curriculum that will reach the majority of students in the classroom and capture the attention of each and every student. Therefore, it is important to begin algebra instruction by giving the mathematical symbols meaning that the students are able to understand. Arcavi (1994) suggests symbols be used from the start and given relevance for the students. When the symbols develop meaning, students are now able to apply the symbols to word and story problems by developing a general formula or equation to find a missing value. Students will not be restricted to basic one step equations that ask for the missing value of x. Teachers are also encouraged to begin an instructional unit with a full scale real world problem that will outline the next set of lessons and lead students through a set of mathematical skills. Instead of leaving the word and story problems for the end of a skill set, research shows to use them to introduce and move students through the curriculum.

Through this report teachers can understand the importance of symbolic meaning to the students and the alternative methods that are available to teach students to solve single variable equations. Moving beyond standard “step-by-step” procedures outlined in typical algebra textbooks, teachers can help more students to find success in solving equations and developing critical thinking and analysis skills.

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Vita

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