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The Goodwillie Tower of Free Augmented Algebras over Connective Ring Spectra

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The Goodwillie Tower of Free Augmented Algebras over Connective Ring Spectra

by

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The Goodwillie Tower of Free Augmented Algebras over Connective Ring Spectra

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Let R be a connective ring spectrum and let M be an R-bimodule. In this paper we prove several results that relate the K-theory of $R \ltimes M$ and T_R^M to a"topological Witt vectors" construction $W(R;M)$, where $R \ltimes M$ is the square-zero extension of R by M and T_R^M is the tensor algebra on M. Our main results include a desciption of the Taylor tower of $K(R \ltimes (-))$ and the derived functor of $\widetilde{K}(T_R(-))$ on the category of R-bimodules in terms of the Taylor tower of $W(R;-)$. $W(R;-)$ has an easily described Taylor tower, given explicitly by Lindenstrauss and McCarthy in [17].

Our main results serve as generalizations of the results for discrete rings in [17, 18] and also extend the computations by Hesselholt and Madsen [15] showing that $\pi_0(TR(R;p))$ is isomorphic to the p-typical Witt vectors over R when R a commutative ring.

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Chapter 1

Introduction

Algebraic K-theory has its origins in attempts to understand classical problems in algebraic geometry and geometric topology. Beginning as the study of Grothendieck groups that encoded invariants associated to various geometric or algebraic objects, it has evolved over the years to become a complex, powerful machine. Part of this power lies in the ability for modern constructions of algebraic K-theory to represent the universal (in a sense that can be made precise) functor that "splits short exact sequences" or "stores Euler characteristics".

In this generality, K-theory is a machine that takes in any object $\mathcal{C}$ with a suitable notion of "short exact sequence" and spits out a connective spectrum $K(\mathcal{C})$. The homotopy groups of this spectrum are the *K-groups* of $\mathcal{C}$, and they encode deep information about the structure of exact sequences in $\mathcal{C}$.

Unfortunately, even understanding the K-theory of relatively "down-to-earth" objects such as rings (obtained as the K-theory of the exact category of finitely generated projective modules) is generically a very difficult thing to compute. A posteriori, this can be seen as consequence of the large amount of information about invariants that K-theory tends to pick up:

- Knowing that $K_{4n}(\mathbb{Z}) = 0$ for all n is equivalent to the presently unreseolved Kummer-Vandiver conjecture, which has to do with class numbers of maximal real subfields of cyclotomic fields.

- The K-theory of the category of spaces over a connected manifold contains a factor that reads off stable pseudo-isotopies – something fairly geometric and difficult to compute. This is Waldhausen's $A(X)$ functor, which we will discuss later.

- If X is a finitely-dominated space (a retract of a finite CW complex), then the K-theory of the group ring $\mathbb{Z}[\pi_1(X)]$ contains classical obstructions to X being homotopy equivalent to a finite CW complex.

As a consequence of the overall difficulty of performing K-theory computations, a fairly profitable industry has been created which attempts to approximate K-theory with more computable functors. Our work will fall along these lines, following most immediately in the footsteps of Lindenstrauss and McCarthy, who were trying to understand the (fairly intractable) functor

$$A \mapsto \widetilde{K}(A) = \mathrm{hofib}(K(A) \to K(R))$$

on the category of algebras augmented over a given ring R. Our goal is the same, but with R replaced by a connective *ring spectrum*. The main content of our work is to reduce to the results of Lindenstrauss and McCarthy, so we describe the motivation behind their approach.

As just mentioned, studying the K-theory functor on the whole category of augmented R-algebras is too difficult, so it makes sense to try and attack a simpler problem. One might hope that it would be easier to look at what happens when restricted to free augmented algebras, that is looking at the functor on R-bimodules

$$M \mapsto \widetilde{K}(T_R(M)),$$

where $T_R(M)$ is the tensor algebra on M.

Indeed, one might even go further and try to understand the K-theory of the "linearized" tensor algebra. The Goodwillie derivative of the identity functor on augmented R-algebras is given by

$$D(id)(A) = A/I^2,$$

where I is the augmentation ideal of A. In the case where $A = T_R(M)$ for some R–bimodule M, then $D(id)(T_R(M)) = R \vee M$, where $R \vee M$ is the the square-zero extension of R by M, the ring given by demanding that $M^2 = 0$. In studying the K-theory of square-zero extensions, then, we are also studying the K-theory of the linearization of the tensor algebra functor, which is far simpler.

In the investigations of Dundas and McCarthy in [10] of stable K-theory, they prove the following, which hints at what sort of structure we should be looking for in $K(R \vee M)$:

Theorem 1.1 ([10]). *For R a ring and M a discrete R-bimodule*

$$\widetilde{K}(R; B_\bullet M) \simeq \mathit{hofib}\{K(R \vee M) \to K(R)\} = \widetilde{K}(R \vee M).$$

The left side is defined as follows:

Definition 1.1. Let R be a ring, and let M be an R-bimodule. We define the parametrized K-theory of R with coefficients in M, $K(R; M)$, to be the K-theory of the exact category P_R^M, consisting of pairs (P, f) where P is a finitely-generated projective R-module and $f : P \to P \otimes_R M$ is a map of R-modules. There is a natural map from P_R^M to P_R, the category that computes $K(R)$ which forgets the presence of a map. This induces a map $K(R; M) \to K(R)$, and so we can take the homotopy fiber to define a reduced version:

$$\widetilde{K}(R; M) := \mathrm{hofib}(K(R; M) \to K(R)).$$

If M is simplicial, we extend this definition by geometrically realizing.

Work by Almkvist [1, 2] recognizes $\pi_0(\widetilde{K}(R;M))$ as a dense λ-subring of the Witt vectors of R. A generalization of this fact appears in the papers of Lindenstrauss and McCarthy, where they prove the following:

Theorem 1.2 ([19, Theorem 9.2])**.** *Let R be a discrete ring, and $M_\bullet$ a connected simplicial R-bimodule. Then there is a natural zig-zag of equivalences*

$$\widetilde{K}(R;M_\bullet) \simeq W(R;M_\bullet).$$

Given the Dundas-McCarthy result, we can read this as

$$\widetilde{K}(R \vee M) \simeq W(R;B_\bullet M).$$

The left hand side is the parametrized K-theory that we mentioned above, and the right hand side is a "topological Witt vectors" construction. The spectrum $W(R;M)$ is built out of the C_n fixed points of spectra $U^n(R;M)$ equipped with an action by C_n by taking the homotopy limit over restriction maps that relate these fixed points. The motivation behind this construction is a (surprisingly successful) attempt to mimic the creation of $TR(R)$ from $THH(R)$ in the presence of a bimodule coordinate breaking the cyclic symmetry in $THH(R)$.

The previous result is proven using the techniques of Goodwillie calculus, treating $K(R;M[-])$ as a functor from spaces to spectra. From this perspective, knowing that these functors agree on connected input lets us come to the following conclusion:

Corollary 1.3. *The functors $\widetilde{K}(R;-)$, $W(R;-)$ from simplicial R-bimodules to spectra have the same Taylor tower (in the sense of Goodwillie, see e.g. [12–14]).*

The advantage of this description is that the Taylor tower of $W(R;-)$ is very explicitly described, given by something which looks remarkably similar to the "fundamental cofibration sequences" that we see when dealing with cylotomic spectra:

Theorem 1.4 ([19, Cor. 5.9]). *There is a homotopy fiber sequence*

$$U^n(R;M)_{hC_n} \to W^n(R;M) \to W^{(n-1)}(R;M).$$

In fact, for $M = R$, we have $W(R;R) \simeq TR(R)$ and their sequence reduces to the standard fundamental cofibration sequence. This sequence exhibits the $W^n(R;-)$ (which are "truncated Witt vectors" used to build $W(R;-)$) as the n-th Goodwillie derivatives of $W(R;-)$, with the layers given by $U^n(R;-)_{hC_n}$.

As is the story with $TC(R)$, the existence of this cofiber sequence allows for inductive analysis of $W(R;M)$, which leads to results in [17] and [19].

From Theorem 1.2, Lindenstrauss and McCarthy go on to describe the K-theory of the full tensor algebra in terms of the Witt vectors:

Theorem 1.5 ([18, Corollary 3.3]). *If R is a unital ring, there is a natural zig-zag of equivalences of functors of connected simplicial R-bimodules*

$$\Sigma\widetilde{K}(R;-) \simeq \widetilde{K}(\mathcal{T}_R(-)),$$

where $\mathcal{T}_R(-)$ is the derived tensor algebra functor over R.

Altogther, the Lindenstrauss-McCarthy results provide us with an understanding a good first approximation to the K-theory functor on the full category of augmented R-algebras, and it is this type of understanding that we will try to extend.

Goals and Outline of Main Results

The purpose of this paper is to extend some of the above results to apply to connective ring spectra, the slogan being that we can (in the words of Dundas [8]) "resolve rings up to homotopy by simplicial rings". That is to say, any connective ring spectrum is the homotopy limit of $H\mathbb{Z}$-algebras, which are equivalent to simplicial rings. One might also phrase this slogan as "simplicial rings are dense in Γ-rings". This is made precise later, but a summary is as follows:

Proposition 1.6. *For any Γ-ring R, there is an infinite cube with R as the initial vertex, all of the noninitial vertices canonically equivalent to $H\mathbb{Z}$-algebras, and such that the map from R into the homotopy limit over any punctured, size-k subcube is k-connected.*

As such, we can recover R as the homotopy limit over the punctured infinite cube of $H\mathbb{Z}$-algebras. This resolution is also compatible with a similar resolution of an R-bimodule M.

We then prove that the composite functor obtained by taking the K-theory of square-zero extensions is well-behaved with respect to this sort of resolution cube:

Proposition 1.7. *For any Γ-ring R and R-bimodule M, there is an infinite cube with the reduced (over R) K-theory $\widetilde{K}(R \vee M)$ as the initial vertex, and all of the noninitial vertices are computing the reduced K-theory of square-zero extensions of simplicial rings.*

This cube is such that the map from $\widetilde{K}(R \vee M)$ to the homotopy limit over the punctured infinite is an equivalence.

Using the above, the fact that the K-theory of radical extensions of simplicial rings can be computed levelwise, and the discrete results of Lindenstrauss-McCarthy, we obtain our first main result. This gives us a description of the Taylor tower of the K-theory of square-zero extensions:

Theorem 1.8. *Let R be a Γ-ring (a connective ring spectrum) and let M be an R–bimodule. Then there is a zig-zag of equivalences of simplicial sets*

$$\widetilde{K}(R \vee M) \simeq W(R; \Sigma M),$$

and therefore a zig-zag of equivalences of the associated spaces or spectra. As a result, the functors from R-bimodules to spectra

$$\widetilde{K}(R \vee -) \quad \textit{and} \quad W(R;-)$$

have the same Taylor tower.

The 1st derivative of $W(R;-)$ is known to be $THH(R;-)$, so as a corollary we reproduce the identification of stable K-theory as THH:

Corollary 1.9. *Let R be a Γ-ring. Then the derivative of the functor $\tilde{K}(R \vee \Omega(-))$ from R-bimodules to spectra is naturally equivalent to to $THH(R;-)$:*

$$D(\tilde{K}(R \vee \Omega(-))) \simeq THH(R;-).$$

We then move on to study the tensor algebra functor over a Γ-ring R, which is defined in much the same way that tensor algebras over rings are defined. The strategy for doing this parallels the one for studying the square-zero extension, but there are more technical difficulties to grapple with, coming from the tensor algebras being defined as an infinite colimit. Regardless, we prove the same sort of resolvability result:

Proposition 1.10. *For any Γ-ring R and any cofibrant R-bimodule M, there is an infinite cube with the reduced K-theory $\widetilde{K}(T_R(M))$ as the initial vertex, and all of the noninitial vertices are computing the reduced K-theory of derived tensor algebras of simplicial rings.*

This cube is such that the map from $\widetilde{K}(T_R(M))$ to the homotopy limit over the punctured infinite is an equivalence.

Using this, a reduction to a level-wise computation for simplicial rings, and the discrete results of Lindenstrauss-McCarthy, we relate the K-theory of the tensor algebra $T_R(M)$ to that of the square-zero extension when M is cofibrant. In conjunction with our first main result, we obtain:

Theorem 1.11. *Let R be a Γ-ring, Then there is a zig-zag of natural transformations of functors*

$$\Sigma\widetilde{K}(R \vee -) \leftrightarrow \widetilde{K}(\mathcal{T}_R(\Sigma(-))$$

that is an equivalence when applied to connected, cofibrant input.

As a result, the derived functors of

$$\Sigma\widetilde{K}(R \vee -) \quad \textit{and} \quad \widetilde{K}(\mathcal{T}_R(\Sigma(-))$$

have the same Taylor tower.

Given Corollary 3.30, we could also say the same of

$$\Sigma W(R; -) \quad \textit{and} \quad \widetilde{K}(\mathcal{T}_R(-))).$$

This theorem specializes to the following (discussed in the introduction to [17]), which provides a description of Waldhausen's A-theory on suspensions of connected spaces that provides some explanation for the results of [5]:

Corollary 1.12. *Let X be a connected simplicial set. Then there is an equivalence*

$$\Sigma W(\mathbf{S}; \mathbf{S}X) \simeq \widetilde{K}(T_{\mathbf{S}}\mathbf{S}X) =: A(\Sigma X).$$

While we do not do so here, it seems certain that our results could be used to provide concrete descriptions of the K-theory of some simple square-zero extensions (say, of a sufficiently nice ring spectrum by itself). This program was carried out by Lindenstrauss and McCarthy for discrete rings in [17], and it is likely that many of their techniques would work with connective ring spectra as well.

Outline

The structure of this paper is as follows:

- In Chapter 2 we define the square-zero extension functor and prove some properties of this functor that we will need. This chapter also serves to exhibit the main technical tools that we use throughout the paper – resolving Γ-rings by simplicial rings, using "cubical resolvability", and computing functors levelwise for simpicial rings.
- In Chapter 3 we recall the definition of Lindenstrauss-McCarthy's "topological Witt vectors" functor for connective ring spectra, prove some new properties of this functor, and then use these properties to relate the topological Witt vectors to the K-theory of square-zero extensions.
- In Chapter 4 we describe tensor algebras on bimodules over Γ-rings and relate their K-theory to the K-theory of square-zero extensions, the topological Witt vectors, and Waldhausen's $A(X)$ functor.
- The Appendix serves to provide background material on Γ-spaces and the resolutions we use throughout the paper.

Chapter 2

K-Theory of Square-Zero Extensions

This chapter will introduce the square-zero extension functor and describe some of the properties of this functor that we will need later in order to compare it to the topological Witt vectors.

Convention. Throughout the paper, we will be working with a model of connective ring spectra as Γ-rings, because of the simple point-set relationship between $H\mathbb{Z}$-algebras in Γ-rings and simplicial rings. While $H\mathbb{Z}$-algebras in any model of spectra are Quillen equivalent to simplicial rings via a zig-zag of lax monoidal functors, our goal is to use results for discrete rings, so it is convenient to know a bit more than what we are given by a zig-zag of Quillen equivalences.

The crucial property that Γ-rings enjoy is the existence of a *strong* symmetric monoidal left Quillen functor L that takes $H\mathbb{Z}$-algebras to simplicial rings and modules over $H\mathbb{Z}$-algebras to modules over the corresponding simplicial rings.

Chapter A contains the background information about Γ-spaces, their model structures, and more details about their relationship to simplicial rings than we will provide in the main text.

2.1 Definitions and Some Technical Tools

Note. Throughout the paper we will be assuming knowledge of the basic properties of algebraic K-theory as applied to Γ-rings. There are several equivalent ways that this can be constructed, and the output can be considered as:

- A simplicial set that is the 0th simplicial set in an Ω-spectrum valued in simplicial sets;
- An infinite loop space;
- The spectrum associated to either of the above objects.

For some of the details of these constructions and proofs that they are equivalent, see [9].

Definition 2.1. Let R be a Γ-ring and M an R-bimodule. We define the *square-zero extension* $R \vee M$. This is a Γ-ring, given as a Γ-space by $R \vee M$, and defined by the property that the M factor is given the square-zero multiplication.

To be precise, $R \vee M = R \vee M$ as Γ-spaces, and we define the multiplication

$$(R \vee M) \wedge (R \vee M) \to R \vee M$$

by first using the splitting (as Γ-spaces) of the left hand side smash product

$$(R \vee M) \wedge (R \vee M) \simeq (R \wedge R) \vee (R \wedge M) \vee (M \wedge R) \vee (M \wedge M)$$

and then defining the multiplication on each wedge summand:

$$R \wedge M \to M \quad \text{and} \quad M \wedge R \to M$$

using the R-bimodule structure on M,

$$R \wedge R \to R$$

using the algebra structure on R, and we demand that

$$M \wedge M$$

maps to $*$.

There is a natural map $R \vee M \to R$ of Γ-rings, given by projecting onto the R factor, and this induces a map

$$K(R \vee M) \to K(R),$$

so we can define the *reduced K-theory* by taking the homotopy fiber:

$$\widetilde{K}(R \vee M) := \mathrm{hofib}(K(R \vee M) \to K(R)).$$

While we will not describe in detail the category of Γ-rings presently (this is done in Chapter A), we will disuss some of the technical tools we use in this paper and set notational conventions.

2.1.1 Cubical Diagrams

We first review the theory and notation of cubical diagrams, which appear in the study of Goodwillie calculus and related subjects (see, e.g. [13] or the appendix to [9]).

Definition 2.2. If A is a set, we let $\mathcal{P}A$ be the category of subsets of A with morphisms given by inclusions. We denote $\mathcal{P}\mathbb{N}$ by $\mathcal{P}$ and $\mathcal{P}\{1, \dots, n\}$ by $\mathcal{P}n$. An A-cube (with values in a category $\mathcal{C}$) is a functor

$$\mathcal{X} : \mathcal{P}A \to \mathcal{C}.$$

We use the shorthand $\mathcal{X}^S$ to denote the value $\mathcal{X}(S)$.

If we have an A-cube $\mathcal{X}$, then for any subset $B \in \mathcal{P}A$ we obtain a *B-subcube* by the composite

$$B \to \mathcal{P}A \xrightarrow{\mathcal{X}} \mathcal{C}.$$

If the cardinality of B is n, we call this an *n-subcube*.

Definition 2.3. Let $\mathcal{X}$ be an A-cube with values in any category with well-defined notions of homotopy (co)limits, connectivity, etc. We say that $\mathcal{X}$ is *k-Cartesian* if the map

$$\mathcal{X}^{\varnothing} \to \operatorname*{holim}_{S\neq\varnothing} \mathcal{X}^S$$

is k-connected. We say that $\mathcal{X}$ is *k-coCartesian* if the map

$$\operatorname*{hocolim}_{S\in P\setminus A} \mathcal{X}^S \to \mathcal{X}^A$$

is k-connected.

We say that $\mathcal{X}$ is *(co)Cartesian* if it is k-(co)Cartesian for all k.

Example. An example of a 3-cube is the following, where we have suppressed the maps that factor through other maps in the cube (eg. $\mathcal{X}^{\varnothing} \to \mathcal{X}^{12}$):

$$\begin{tikzcd}
 & \mathcal{X}^{\varnothing} \arrow[rr] \arrow[dl] \arrow[dd] & & \mathcal{X}^{1} \arrow[dl] \arrow[dd] \\
\mathcal{X}^{2} \arrow[rr, crossing over] \arrow[dd] & & \mathcal{X}^{12} & \\
 & \mathcal{X}^{3} \arrow[rr] \arrow[dl] & & \mathcal{X}^{13} \arrow[dl] \\
\mathcal{X}^{23} \arrow[rr] & & \mathcal{X}^{123} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

If we puncture it (again, suppressing many of the maps), we get:

$$\begin{tikzcd}
\mathcal{X}^{\varnothing} \arrow[ddr] \arrow[dr] \arrow[drrr] & & & & \\
 & \operatorname{holim}_{S\in\mathcal{P}3\setminus\varnothing} X^S \arrow[rr] \arrow[d] \arrow[dd, shift left] & & \mathcal{X}^{1} \arrow[dl] \arrow[dd] \\
 & \mathcal{X}^{2} \arrow[r, crossing over] \arrow[dd] & \mathcal{X}^{12} & \\
 & \mathcal{X}^{3} \arrow[rr] \arrow[d] & & \mathcal{X}^{13} \arrow[dl] \\
 & \mathcal{X}^{23} \arrow[r] & \mathcal{X}^{123} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

For this cube to be k-Cartesian we require the map from $\mathcal{X}^{\varnothing}$ to $\operatorname{holim}_{S\in\mathcal{P}3\setminus\varnothing}\mathcal{X}^S$ to be k-connected.

Unfortunately, being k-(co)Cartesian is generally not a strong enough condition on a cube to guarantee that it plays nicely with the functors that we will be considering. We will also need to impose good behavior on subcubes:

Definition 2.4. Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We say that an A-cube $\mathcal{X}$ is *f-(co)Cartesian* if every n-subcube of $\mathcal{X}$ is $[f(n)]$-(co)Cartesian.

Example. For $f(n) = n$ as in the above definition, we say that an f–(co)Cartesian cube $\mathcal{X}$ is (id)-(co)Cartesian. This means that every n-dimensional subcube of $\mathcal{X}$ is n-(co)Cartesian.

The most useful property of (id)-Cartesian cubes that we will exploit (proven later as Theorem 2.9) is that if $\mathcal{X}$ is an (id)-Cartesian $\mathbb{N}$-cube, then there is an equivalence

$$\mathcal{X}^{\varnothing} \to \operatorname*{holim}_{S\neq\varnothing}\mathcal{X}^S.$$

We will be dealing with functors that preserve Cartesian-ness of cubes to varying degrees; we now codify the sort of good behavior that these functos have:

Definition 2.5. Let F be a functor. If F takes (id)-Cartesian k-cubes to cubes that are $f(k)$-Cartesian with $f(k)$ an increasing function of k, we will say that F is *cubically resolvable.*

By Theorem 2.9, applying a cubically resolvable functor to an (id)-Cartesian $\mathbb{N}$-cube $\mathcal{X}$ grants us an equivalence

$$F(\mathcal{X}^{\varnothing}) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}\setminus\varnothing} F(\mathcal{X}^S).$$

2.1.2 Resolutions of Γ-rings

As stated in the introduction, our working slogan is that "simplicial rings are dense in Γ-rings". It is important to know exactly how the category of simplicial rings, Ring_Δ, lives in the category of $H\mathbb{Z}$-algebras, $\mathrm{Alg}_{H\mathbb{Z}}$, for this philosophy to be useful.

For any simplicial ring R, we have an Eilenberg-MacLane functor H that produces a Γ-ring HR. In fact H is lax monoidal, so for all R, HR becomes a module over $H\mathbb{Z}$. This therefore defines a functor

$$H : \mathrm{Ring}_\Delta \to \mathrm{Alg}_{H\mathbb{Z}}\,.$$

There is a left adjoint L to H that has nice properties, which we will use frequently:

Theorem 2.1 ([24, Lemma 1.2, Theorem. 4.4])**.** *There is a Quillen adjoint pair*

$$\begin{tikzcd} \mathrm{Alg}_{H\mathbb{Z}} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\vdash"] & & \mathrm{Ring}_\Delta \arrow[ll, bend left, "H"] \end{tikzcd}$$

with the following properties:

- *H, L define a Quillen equivalence between the stable model structure on $H\mathbb{Z}$-algebras and the model structure on* Ring_Δ *(cf. Chapter A);*
- *H is a lax symmetric monoidal;*
- *L preserves finite products;*
- *L is strong symmetric monoidal.*

In addition, if B is a simplicial ring then the adjoint functors H and L form a Quillen equivalence between the categories of B-modules and HB-modules (again, see Chapter A for a discussion of the model structures involved).

In addition, we can use the functor $H\mathbb{Z} \wedge (-)$ to produce cosimplicial resolutions of Γ-rings by $H\mathbb{Z}$-algebras. The details of this are recorded in Chapter A, but one might think of this as taking iterated homology or performing the Adams resolution. The cubes associated to these resolutions are nice enough to recover the Γ-ring we start with (essentially because of the Hurewicz theorem), and moreover this resolution is compatible with bimodule structure.

Warning. The maps in this cube are *not* maps of $H\mathbb{Z}$-algebras. If this were the case, then we would be able to take this homotopy limit in the category of $H\mathbb{Z}$-algebras (simplicial rings), and this would imply that all connective ring spectra are stably equivalent to simplicial rings. This is not the case, as it can be shown that the sphere spectrum $\mathbf{S}$ is not stably equivalent to a simplicial ring.

We encode these facts in the following proposition, prefaced by a definition that codifies the compatibility of the module structures in our resolution:

Definition 2.6. A cube $S \mapsto (R^S, M^S)$ of Γ-rings and bimodules is *admissible* if for all $U, V \subset \mathcal{P}$:

- the maps $R^U \to R^V$ are maps of Γ-rings;
- the map $M^U \to M^V$ is a map of R^U-modules, where M^V is given an R^U-module structure using the map $R^U \to R^V$.

Proposition 2.2. *For any Γ- ring R and R-bimodule M, there are cubes $S \mapsto R^S$ and $S \mapsto M^S$ that are (id)–Cartesian, and so there are equivalences*

$$R \xrightarrow{\simeq} \operatorname*{holim}_{S \in \mathcal{P} \setminus \varnothing} R^S$$

and

$$M \xrightarrow{\simeq} \operatorname*{holim}_{S \in \mathcal{P} \setminus \varnothing} M^S.$$

Moreover, these cubes are such that:

- *M^S is an $H\mathbb{Z}$-module which is a bimodule for the $H\mathbb{Z}$-algebra R^S;*
- *the cube $S \mapsto (R^S, M^S)$ is admissible.*

We also need the following consequence of the Blakers-Massey Theorem for cubes, relating Cartesian and coCartesian cubes:

Theorem 2.3. *[[9, Lemma A.7.3.2]] Let $k > 0$. An A-cube of spaces is $(id + k)$-Cartesian if and only if it is $(2 * id + k - 1)$-coCartesian.*

Finally, we record an important fact about simplicial sets that we will use quite often. This allows us to pass equivalences of simplicial sets or spaces through homotopy limits – not something one can do in all cases.

Proposition 2.4 ([9, Lemma A.6.2.3])**.** *Let $\eta : F \to G$ be a natural transformation of functors from a small category I to pointed simplicial sets. Then if η is a pointwise equivalence (induces an equivalence $F(i) \to G(i)$ for all $i \in I$), then there are natural equivalences*

$$\operatorname*{holim}_{I} F \xrightarrow{\simeq} \operatorname*{holim}_{I} G$$

$$\operatorname*{hocolim}_{I} F \xrightarrow{\simeq} \operatorname*{hocolim}_{I} G$$

2.2 Properties of the Square-Zero Extension

Now that we have access to the tools described in the last section, our program will be to resolve Γ-spaces by simplicial rings and then compute things levelwise. This will put us in a situation where we can use the discrete results of

Lindenstrauss-McCarthy and Dundas-McCarthy. To do this, we need to make sure that our construction of the square-zero extension is compatible with every part of our program.

2.2.0.1 Simplicial Rings Associated to the Resolution

The first thing we'll need is the ability to use the resolutions of R and M separately to resolve $R \vee M$.

Proposition 2.5. *Let $S \mapsto R^S$ and $S \mapsto M^S$ be the resolution cubes described earlier. Then the cube $S \mapsto R^S \vee M^S$ is (id)-Cartesian, and so provides a resolution of $R \vee M$ by $H\mathbb{Z}$-algebras.*

Proof. First note that as Γ-spaces, $S \mapsto R^S \vee M^S$ is the wedge cube $S \mapsto R^S \vee M^S$, so we consider this cube. The Blakers-Massey theorem (Theorem 2.3) tells us that coCartesian cubes and Cartesian cubes coincide in Γ-spaces, so it suffices to check that this cube is (id)–coCartesian.

Let $\mathcal{X}, \mathcal{Y}$ be A-subcubes of $S \mapsto R^S$ and $S \mapsto M^S$, respectively. Then by assumption we know that the maps

$$x : \operatorname*{hocolim}_{S \in PA \neq A} \mathcal{X}^S \to \mathcal{X}^A$$

and

$$y : \operatorname*{hocolim}_{S \in PA \neq A} \mathcal{Y}^S \to \mathcal{Y}^A$$

are $|A|$-connected. We can then form the wedge cube $\mathcal{X} \vee \mathcal{Y}$, where the value at S is $\mathcal{X}^S \vee \mathcal{Y}^S$. We need to show that this cube is $|A|$-coCartesian. Because the wedge product is a homotopy colimit and homotopy limits commute, we have an equivalence

$$\operatorname*{hocolim}_{S \in PA \neq A} \mathcal{X}^S \vee \mathcal{Y}^S \xrightarrow{\simeq} \operatorname*{hocolim}_{S \in PA \neq A} \mathcal{X}^S \vee \operatorname*{hocolim}_{S \in PA \neq A} \mathcal{X}^S,$$

and moreover under this identification the natural map

$$\operatorname*{hocolim}_{S \in PA \neq A} \mathcal{X}^S \vee \mathcal{Y}^S \to \mathcal{X}^A \vee \mathcal{Y}^A$$

is identified as the wedge of the separate maps

$$\operatorname*{hocolim}_{S \in PA \neq A} \mathcal{X}^S \vee \operatorname*{hocolim}_{S \in PA \neq A} \mathcal{X}^S \xrightarrow{x \vee y} \mathcal{X}^A \vee \mathcal{Y}^A.$$

Again, these are both $|A|$-connected, and again because the wedge is a homotopy colimit, this implies that this map $x \vee y$ is $|A|$-connected. More generally, suppose that we have k-connected maps $f_1 : X_1 \to X_2, f_2 : Y_1 \to Y_2$. Then the homotopy fibers F_1, F_2 of the respective maps are k-connected. Because finite homotopy colimits (wedges, in this case) commute with homotopy limits (fibers, in this case), the fiber of the map

$$f_1 \vee f_2 : X_1 \vee Y_1 \to X_2 \vee Y_2$$

is naturally equivalent to $F_1 \vee F_2$, which is stably equivalent to $F_1 \times F_2$, as we are working with Γ-spaces. As $\pi_*(F_1 \times F_2) \simeq \pi_*(F_1) \times \pi_a st(F_2)$, this implies that $F_1 \vee F_2$ is k-connected, and so the map $f_1 \vee f_2$ is as well. $\square$

Next, we need to know is that the simplicial rings associated to the resolution of $R \vee M$ are of an appropriate form.

Proposition 2.6. *The simplicial ring associated to the $H\mathbb{Z}$-algebra $R^S \vee M^S$ via L is a square-zero extension of the form $L(R^S) \vee L(M^S)$, where $L(R^S)$ has the multiplication induced by L.*

Proof. We first prove a general statement:

Lemma 2.7. *Let $F : (\mathcal{C}, \otimes) \to (\mathcal{D}, \wedge)$ be a strong symmetric monoidal functor between pointed symmetric monoidal categories that preserves coproducts. Then F also* *preserves square-zero extensions of monoids with respect to the symmetric monoidal product.*

Proof of Lemma. Let M be a monoid in $\mathcal{C}$, N an M-bimodule, and $M \vee N$ be the corresponding square-zero extension. Then there are multiplication maps that respect the splitting

$$(M \vee N) \otimes (M \vee N) \cong (M \otimes M) \vee (M \otimes N) \vee (N \otimes M) \vee (N \otimes N).$$

The multiplication on $F(M \otimes N)$ is induced by the multiplication on $(M \otimes N)$, but we defined this using the above splitting and demanding that the multiplication be given by the M multiplication, the M-bimodule structure on N and 0, respectively. $\square$

Knowing this, we can appeal to the properties of L described in Chapter A (namely that it satisfies the conditions of the above lemma) to say that

$$L(R^S \vee M^S) \simeq L(R^S) \vee L(M^S).$$

$\square$

A result we will rely heavily on is the cubical resolvability of K-theory, proven by Dundas:

Theorem 2.8 ([8, Proposition 5.1])**.** *Let $\mathcal{X}$ be an (id)-Cartesian n-cube of Γ-rings. Then*

$$K(\mathcal{X}) : S \mapsto K(\mathcal{X}^S)$$

is $(n+1)$-Cartesian.

Using this, we can obtain the cubical resolvability of the square-zero extension. This will follow from the more general theorem we prove about cubically resolvable functors:

Theorem 2.9. *Let F be a cubically resolvable functor. Then the natural map*

$$F(\mathcal{X}^{\varnothing}) \to \operatorname*{holim}_{S \in \mathcal{P} \setminus \varnothing} F(\mathcal{X}^S)$$

is an equivalence for any infinite (id)-Cartesian cube $\mathcal{X} : S \mapsto \mathcal{X}^S$.

One is tempted to say that it is fine to take the homotopy limit of the increasingly connected finite (id)-Cartesian cubes to obtain an infinte (id)-Cartesian cube, but we need to make sure that the cubes assemble properly. To prove Theorem 2.11, we follow the procedure used in the proof of [9, III.3.2.2] that shows that K-theory is cubically resolvable. This will require the following lemma:

Lemma 2.10 ([9, A.6.2.4])**.** *Let $f : I \subseteq J$ be an inclusion of small categories and F a J-shaped diagram of simplicial sets. Then the natural map*

$$\operatorname*{holim}_{J} F \to \operatorname*{holim}_{I} F$$

is a fibration.

Proof. We repeatedly apply this to the cubes coming from restricting $S \mapsto (R^S, M^S)$ to the nested, increasing finite sets

$$\mathcal{P}1 \subset \mathcal{P}2 \subset \mathcal{P}3 \subset \cdots.$$

This gives us a map from $F(\mathcal{X}^{\varnothing})$ into a tower of fibrations, where we have indicated the connectivity of the maps (obtained from Proposition 2.5):

$$\begin{tikzcd}
 & \vdots \arrow[d, two heads] \\
 & \operatorname{holim}_{S\in\mathcal{P}5\setminus\varnothing} F(\mathcal{X}^S) \arrow[d, two heads] \\
 & \operatorname{holim}_{S\in\mathcal{P}4\setminus\varnothing} F(\mathcal{X}^S) \arrow[d, two heads] \\
 & \operatorname{holim}_{S\in\mathcal{P}3\setminus\varnothing} F(\mathcal{X}^S) \arrow[d, two heads] \\
F(\mathcal{X}^{\varnothing}) \arrow[r, "\simeq_{\leq 2}"] \arrow[ur, "\simeq_{\leq 3}"] \arrow[uur, "\simeq_{\leq 4}"] \arrow[uuur, "\simeq_{\leq 5}"] \arrow[uuuur] & \operatorname{holim}_{S\in\mathcal{P}2\setminus\varnothing} F(\mathcal{X}^S) \arrow[d, two heads] \\
 & *
\end{tikzcd}$$

Using the Bousfield-Kan spectral sequence (cf. [9, A.6.4.3]) on the homotopy fibers of the increasingly connected maps

$$F(\mathcal{X}^{\varnothing}) \to \operatorname*{holim}_{S\in\mathcal{P}n\setminus\varnothing} F(\mathcal{X}^S)$$

we see that the map from $F(\mathcal{X}^{\varnothing})$ into the homotopy limit of this tower is an equivalence:

$$F(\mathcal{X}^{\varnothing}) \xrightarrow{\simeq} \operatorname*{holim}_{n} \operatorname*{holim}_{S\in\mathcal{P}n\setminus\varnothing} F(\mathcal{X}^S).$$

However, because this is a tower of fibrations of fibrant objects, the limit and homotopy limit coincide, and the following natural map is an equivalence:

$$\lim_{n} \operatorname*{holim}_{S\in\mathcal{P}n\setminus\varnothing} F(\mathcal{X}^S) \xrightarrow{\simeq} \operatorname*{holim}_{n} \operatorname*{holim}_{S\in\mathcal{P}n\setminus\varnothing} F(\mathcal{X}^S).$$

We put this together into a commutative diagram, using the universal property of the limit:

$$\begin{tikzcd}
\lim_n \operatorname{holim}_{S\in\mathcal{P}n\smallsetminus\varnothing} F(\mathcal{X}^S) \arrow[r, "\simeq"] & \operatorname{holim}_n \operatorname{holim}_{S\in\mathcal{P}n\smallsetminus\varnothing} F(\mathcal{X}^S) \\
 & F(\mathcal{X}^\varnothing) \arrow[u, "\simeq"'] \arrow[ul, dashed, bend left]
\end{tikzcd}$$

The dashed map is in fact the natural map

$$F(\mathcal{X}^\varnothing) \to \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} F(\mathcal{X}^S)$$

as the target in the previous diagram can be identified as a model for the homotopy limit over the infinite cube. The diagram commuting implies that this map is an equivalence, as desired. $\square$

Theorem 2.11. *Let R be a Γ-ring, M an R-bimodule, $S \mapsto (R^S, M^S)$ the admissible (id)-Cartesian resolution by $H\mathbb{Z}$-algebras/modules. Then there is a weak equivalence*

$$K(R \vee M) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} K(R^S \vee M^S).$$

This is compatible with the reduction maps, so we also have an equivalence

$$\widetilde{K}(R \vee M) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} \widetilde{K}(R^S \vee M^S).$$

Proof. Theorem 2.8 tells us that K-theory is cubically resolvable, and Proposition 2.5 tells us that $S \mapsto R^S \vee M^S$ is (id)-Cartesian. As a result, Theorem 2.9 grants us an equivalence

$$K(R) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} K(R^S \vee M^S).$$

In addition, we have

$$K(R) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} K^{\mathbf{S}}(R^S).$$

These cubes are compatible, as for any $S, U \subseteq \mathcal{P}$ with $S \subseteq U$ there is a commutative diagram

$$\begin{tikzcd} R^S \vee M^S \arrow[r] \arrow[d] & R^U \vee M^U \arrow[d] \\ R^S \arrow[r] & R^U \end{tikzcd}$$

which exhibits the maps $R^S \vee M^S \to R^U \vee M^U$ as compatible with the reductions. Because homotopy limits commute, this gives us the reduced statement

$$\widetilde{K}(R \vee M) \xrightarrow{\simeq} \operatorname*{holim}_{S \in \mathcal{P} \setminus \varnothing} \widetilde{K}(R^S \vee M^S).$$

□

2.2.1 Levelwise Computation for Simplicial Rings

In order to reduce to the results of Lindenstrauss-McCarthy for discrete rings, we are going to need to be able to compute the K-theory of our functors evaluated on simplicial rings levelwise. Moreover, it will be important to make sure that we can include compatibility with taking diagonals ("realization", in the words of Lindenstrauss and McCarthy) of simplicial bimodules.

Convention. Throughout, anytime we have a simplicial object X, we will use $d(X)$ to denote its diagonal. For a multisimplicial set M, the diagonal simplicial set $d(M)$ geometrically realizes to the geometric realization of M, and so we can think of geometric realization as either a functor from multisimplicial sets to simplicial sets or spaces.

In [11], Goodwillie proves that we can actually compute the K-theory of "small" extensions of simplicial rings levelwise. To make this precise, we first recall the following:

Definition 2.7. Let R be a ring. Then the *Jacobson radical* of R is the intersection of all maximal left (or right) ideals of R. If A, B are simplicial rings, then a map $f : A \to B$ is a *radical extension* if the kernels of the maps

$$f_q : A_q \to B_q$$

are in the Jacobson radical of A_q for all q.

Given any map $f : A \to B$ of (simplicial) rings, we can take the relative K-theory, $K(f)$, which is given by the homotopy fiber

$$K(f) := \text{hofib}(K(A) \xrightarrow{f_*} K(B)).$$

The following tells us that we can compute these levelwise:

Theorem 2.12 ([11])**.** *The K-theory of a radical extension of simplicial rings can be computed degreewise. That is, if $f : A \to B$ is a radical extension of simplicial rings, then we have a zig-zag of weak equivalences of bisimplicial sets*

$$K(f) \simeq \left\{[q] \mapsto K(f_q)\right\}.$$

where the left side is the relative K-theory and the right side is the degreewise relative K-theory.

On the level of the K-theory spaces (or spectra), we have

$$K(f) \simeq \left|[q] \mapsto K(f_q)\right|.$$

Proposition 2.13. *Let R be a simplicial ring and $M_\bullet$ a simplicial R-bimodule. Then the relative K-theory $\widetilde{K}(R \vee M_\bullet)$ can be computed levelwise. That is, there is a natural equivalence of bisimplicial sets*

$$\widetilde{K}(R \vee M_\bullet) := \left\{[q] \mapsto \widetilde{K}(R \vee M_q)\right\} \simeq \widetilde{K}(R \vee d(M)).$$

Note. We are going to belabor the description of what is going on in this proof a bit, because it is a model for a technique that we will use several times in the future. In general, we are going to want to both compute K-theory of simplicial rings levelwise *and* carry a simplicial object through that levelwise computation. This sort of argument lets us do so.

Proof. Consider the map $X : \Delta^{op} \times \Delta^{op} \to \mathrm{Set}$:

$$[i, j] \mapsto R_i \vee M_{i,j}.$$

We give this the structure of a bisimplicial ring.

First we note that for a fixed, i, j, this is a ring. We just need to define the simplicial structure maps, which we do on each factor in the obvious way. As an example: let d_i^k be one of the face maps in the i-direction for R_i and $M_{i,j}$. We write $X(i, j) \to X(i-1, j)$ as

$$R_i \vee M_{i,j} \xrightarrow{d_i^k \vee d_i^k} R_{i-1} \vee M_{i-1,j}$$

and letting d_j^l one of the face maps in the j-direction for $M_{i,j}$, we write $X(i, j) \to X(i, j-1)$ as

$$R_i \vee M_{i,j} \xrightarrow{id \vee d_j^l} R_i \vee M_{i,j-1}.$$

The degeneracy maps are defined in the same way, and this defines the structure of a bisimplicial ring, as the i, j simplicial structure maps commute.

In addition, this is compatible with the maps to R. As another example, we have evident commutative diagrams

$$\begin{tikzcd} R_i \vee M_{i,j} \arrow[r, "d_i^k"] \arrow[d] & R_{i-1} \vee M_{i-1,j} \arrow[d] \\ R_{i-1} \arrow[r, "="] & R_{i-1} \end{tikzcd} \qquad \begin{tikzcd} R_i \vee M_{i,j-1} \arrow[r, "d_j^l"] \arrow[d] & R_i \vee M_{i,j-1} \arrow[d] \\ R_i \arrow[r, "="] & R_i \end{tikzcd}$$

as the downward arrows are simply projections onto the first factor, and the simplicial structure maps are defined factor-wise. This tells us that as i varies, we get the structure of an augmented simplicial ring

$$R_\bullet \vee M_{\bullet,j} \to R_\bullet.$$

As j varies, this assembles to an augmented bisimplicial ring.

On the other hand, as j varies, we have an augmented simplicial ring

$$R_i \vee M_{i,\bullet} \to R_i,$$

which assembles to the same augmented bisimplicial ring as we vary i.

This implies that we have the structure needed to write a trisimplicial set

$$[k, j] \mapsto \widetilde{K}(R_j \vee M_{k,j}),$$

where the reductions are implictly happening with respect to R_j:

$$\begin{array}{|cccc}
\vdots & \vdots & \vdots & \\
\widetilde{K}(R_3 \vee M_{0,3}) & \widetilde{K}(R_3 \vee M_{1,3}) & \widetilde{K}(R_3 \vee M_{2,3}) & \dots \\
\widetilde{K}(R_2 \vee M_{0,2}) & \widetilde{K}(R_2 \vee M_{1,2}) & \widetilde{K}(R_2 \vee M_{2,2}) & \dots \\
\widetilde{K}(R_1 \vee M_{0,1}) & \widetilde{K}(R_1 \vee M_{1,1}) & \widetilde{K}(R_1 \vee M_{2,1}) & \dots \\
\widetilde{K}(R_0 \vee M_{0,0}) & \widetilde{K}(R_0 \vee M_{1,0}) & \widetilde{K}(R_0 \vee M_{2,0}) & \dots \\
\hline
\end{array}$$

Taking the vertical bisimplicial sets

$$\widetilde{K}(R_\bullet \vee M_{j,\bullet}),$$

Theorem 2.12 tells us that there is a natural equivalence between these and the simplicial sets

$$\widetilde{K}(R \vee M_j),$$

where each one is the reduced K-theory simplicial of the simplicial ring $R \vee M_j$. The realization lemma assembles this to an equivalence of this trisimplicial set with the bisimplicial set

$$[j] \mapsto \widetilde{K}(R \vee M_j),$$

the above having bisimplicial structure from the simplicial structure on M_j and the functoriality of K-theory. The diagonal of this (for the simplicial set, or the realization for the space) is our definition of $\widetilde{K}(R \vee M_\bullet)$

On the other hand, Theorem 2.12 tells us that there is a natural equivalence

$$\left[[j] \mapsto \widetilde{K}(R_j \vee M_{j,j})\right] \simeq \widetilde{K}(R_\bullet \vee d(M)).$$

which must therefore be naturally equivalent to

$$[j] \mapsto \widetilde{K}(R \vee M_j).$$

□

2.2.2 A Brief Aside on Parametrized K-Theory

We now entertain a discussion of parametrized K-theory, which is the K-theory defined using the Waldhausen category of parametrized endomorphisms described in the introduction. We recall the definition:

Definition 2.8. Let R be a discrete rings and M an R-bimodule. We can consider the exact category P_R^M of finitely-generated R-modules P equipped with R-module maps $P \to P \otimes_R M$. This has an exact structure by forgetting the maps and using the exact structure in the category of finitely-generated R-modules P_R, and so we can define the *parametrized K-theory of R with coefficients in M* to be

$$K(R; M) := K(P_R^M).$$

There is an obvious exact functor $P_R^M \to P_R$ that forgets the maps, and so we get a map

$$K(R;M) \to K(R),$$

which lets us define the *reduced parametrized K-theory* as

$$\widetilde{K}(R;M) := \mathrm{hofib}(K(R;M) \to K(R)).$$

If $M_\bullet$ is a simplicial R-module, we can extend this definition by geometrically realizing (or taking diagonals of) the simplicial spectra (or spaces, or simplicial sets):

$$K(R;M_\bullet) := \big|[q] \mapsto K(R;M_q)]\big| \qquad \widetilde{K}(R;M_\bullet) := \big|[q] \mapsto \widetilde{K}(R;M_q)\big|.$$

Recall the theorem of Dundas-McCarthy [10] mentioned in the introduction:

Theorem 2.14 ([10])**.** *For R a ring and M a discrete R-bimodule, we have a natural equivalence*

$$\widetilde{K}(R;B_\bullet M) \xrightarrow{\simeq} \widetilde{K}(R \vee M).$$

We do not prove a version of this statement for Γ-rings (or simplicial rings), however, we can get a partial result, which says that the K-theory of the square-zero extension is computable by the parametrized K-theory of simplicial rings that arise from our resolution:

Theorem 2.15. *Let R be a Γ-ring, M an R-bimodule, $S \mapsto (R^S, M^S)$ the admissible (id)-Cartesian resolution by $H\mathbb{Z}$-algebras/modules from Construction A.10. Then the the reduced K-theory $\widetilde{K}(R \vee M)$ is resolvable by the parametrized K-theory of discrete rings, i.e.*

$$\widetilde{K}(R \vee M) \simeq \operatorname*{holim}_{S \in \mathcal{P} \smallsetminus \varnothing} |\widetilde{K}(L(R^S)_i; B_\bullet(L(M^S)_i))|.$$

Warning. The maps in this homotopy limit are not maps coming from discrete rings – this is simply a pointwise equivalence.

Proof. Theorem 2.11 gives us an equivalence

$$\widetilde{K}(R \vee M) \simeq \operatorname*{holim}_{S \in \mathcal{P} \setminus \varnothing} \widetilde{K}(R^S \vee M^S).$$

We need only recognize the terms in the homotopy limit on the right hand side of this equality as those we are interested in and make sure that the identification is compatible with the maps in the homotopy limit system.

By Proposition 2.6, $L(R^S \vee M^S)$ is of the form $L(R^S) \vee L(M^S)$, and because K-theory behaves appropriately with respect to the Quillen equivalence between simplicial rings and $H\mathbb{Z}$-algebras, we have the equivalence

$$\widetilde{K}(R^S \vee M^S) \simeq \widetilde{K}(L(R^S) \vee L(M^S)).$$

We can compute this levelwise, by Theorem 2.12, and noting that the levels of a square-zero extension of a simplicial ring are square-zero extensions by discrete rings, we get

$$\widetilde{K}(L(R^S) \vee L(M^S)) \simeq |\widetilde{K}(L(R^S)_i \vee L(M^S)_i)|.$$

Now we apply the discrete Dundas-McCarthy theorem (Theorem 2.14) levelwise to the rightmost term, whereby the realization lemma grants us an equivalence

$$|\widetilde{K}(L(R^S)_i \vee L(M^S)_i)| \simeq |\widetilde{K}(L(R^S)_i; B_\bullet(L(M^S)_i))|.$$

□

Chapter 3

Topological Witt Vectors

In this chapter we introduce a construction which Lindenstrauss-McCarthy refer to as "topological Witt vectors". This nomenclature can be justified both by the similarity between the construction of the Witt vectors as an inverse limit and the fact that the actual Witt vectors appear in computations involving these functors in [18]. Moreover, they can be seen as a generalization of $TR(R)$, where the p-typical Witt vectors appear (cf. [15]).

Our use of the model of connective ring spectra as Γ-rings allows our definitions to coincide with the definitions given in [18], etc. for FWSs and FSPs.

3.1 Definitions

Before we define the Witt vectors, we need to remark on a potential conflict between the objects studied by Lindenstrauss-McCarthy and our Γ-spaces.

Convention. Consistently, we will allow ourselves to evaluate Γ-spaces or Γ-rings on simplicial sets, i.e. think of them as their associated functors with stabilization (FWS) or functors with smash product (FSP) (cf. [4, 17]) that are obtained via prolongation (cf. Section A.4). We will continue to refer to them as Γ-spaces or Γ-rings, however.

In order for this convention to be consistent with both the Lindenstrauss-McCarthy results our use of Γ-spaces earlier, we need a few facts, which are easily obtained by looking at the constructions of the prolongation functor and the way that Lindenstrauss and McCarthy assign FSPs to (simplicial) rings:

- If we prolong a Γ-space X to obtain a FWS, X^{FWS}, the restriction of X^{FWS} to finite sets coincides with the values of X.
- The spectrum associated to a Γ-space is the same as the one associated to the FWS, which is definitional. The spectrum associated to a Γ-space is obtained by prolongation and evaluation on simplicial spheres, which is exactly the spectrum associated to the FWS.
- The FSP associated to a (simplicial) ring R by Lindenstrauss-McCarthy, R^{FSP} coincides with HR^{FSP}, the FSP associated to the Γ-ring HR.
- If M is a bimodule over a (simplicial) ring R, then the R^{FSP}-bimodule M^{FWS} coincides with the HR^{FSP}-bimodule HM^{FWS}.
- Homotopy (co)limits are computed in the projective model structure for FWS. Because all of the FWS we consider have Ω-spectra associated to them and we only end up considering the associated spectra, this amounts to computing homotopy (co)limits of the simplicial sets that form the 0th spaces of our Ω-spectra.

With the above in mind, we now define a simplicial Γ-space $W(R;M)$. It will be built out of the following simplicial Γ-spaces:

Definition 3.1. Let $\mathcal{C}$ be a Γ-ring and $\mathcal{M}$ an R-bimodule. Let I denote the category of finite sets with injections (the indexing category of Bökstedt, as in [4]), S_* the category of pointed simplicial sets. We define $V_k^n(R;M) : I^{n(k+1)} \to S_*$ by

$$V_k^n(R;M)(\mathbf{x}) = \bigwedge_{i=1}^{n} \left[M(S^{x_{i,0}}) \bigwedge_{j=1}^{k} R(S^{x_{i,j}}) \right].$$

where for $k = 0$ we understand the latter smash product as empty (there are only bimodule entries).

We then define, $U^n_k(R;M) : S_* \to S_*$, which is given by

$$U^n_k(R;M)(X) := \operatorname*{hocolim}_{\mathbf{x}\in I^{n(k+1)}} \Omega^{\mathbf{x}}(X \wedge V^n_k(R;M)(\mathbf{x})),$$

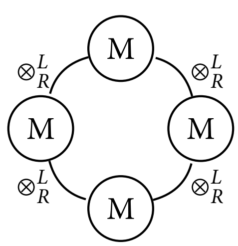
where $\Omega^{\mathbf{x}} = \mathrm{Hom}_{S_*}(\bigwedge_{i,j} S^{x_{i,j}}, \mathrm{sing}(|-|))$ is the derived loops functor indexed by the vector $\mathbf{x}$.

As k varies, we have that these functors have simplicial structure exactly analagous to that of THH: the internal face maps are given by using the action of $\mathcal{C}$ on itself and the first and last face maps use the right/left bimodule structure of $\mathcal{M}$.

In addition to the simplicial structure, there is an evident action of C_n on $U^n_k(R;M)$ that is compatible with the simplicial structure. This comes from permuting the n "blocks" that show up in $V^n_k(R;M)$ (the parts with a fixed j in the above notation) and the loop coordinates that show up in $\Omega^{\mathbf{x}}$. The details of the simplicial structure and the C_n-action are written out in detail in [17].

These $U^n(R;M)_k$ functors serve as models for the derived cyclic tensor product of n copies of the R-bimodule M, which have the evident C_n actions we just described.

Example. $U^1(R;M) = THH(R;M)$ as usually defined for Γ-rings. U_4 looks something like:



and the C_4 action is given by rotating the copies of M.

We wish to keep track of the relationships between the various fixed point spaces $U^m(R;M)^{C_m}$, as when we build TR from THH, and so we are led to the following:

Construction 3.1. As in ([18]), the C_n-fixed points of U^m are given by

$$U_k^m(R;M)^{C_n} \simeq \operatorname*{hocolim}_{x \in I^{k+1}} \Omega^{\mathbf{x}^{\wedge m}}(V_k^m(R;M)(\mathbf{x}^{\times m}))^{C_n}.$$

For $\mathbf{x} \in I^{m(k+1)}$ and if $n|m$, then there are $C_{\frac{m}{n}} \simeq C_m/C_n$-equivariant homeomorphisms

$$((S^{\mathbf{x}})^{\wedge m})^{C_n} \simeq (S^{\mathbf{x}})^{\wedge \frac{m}{n}} \tag{3.1}$$

$$(V^n(R;M)(\mathbf{x}^{\times m}))^{C_n} \simeq V^{\frac{m}{n}}(R;M)(\mathbf{x}^{\times \frac{m}{n}}). \tag{3.2}$$

If G is a group with a normal subgroup H and X, Y are $G-$spaces, then there is a map given by restriction to H-fixed points:

$$res^H : Hom(X,Y)^G \to Hom(X^H, Y)^G = Hom(X^H, Y^H)^{G/H}$$

In the case where we have $n|m$, we can take $G = C_m, H = C_n$, and so (keeping in mind (Eq. (3.1)) and (Eq. (3.2)) we have maps

$$\left[\Omega^{\mathbf{x}^{\wedge m}}(X \wedge V_k^m(R;M)(\mathbf{x}^{\times m}))\right]^{C_m} \to \left[\Omega^{\mathbf{x}^{\wedge \frac{m}{n}}}(X \wedge V_k^{\frac{m}{n}}(R;M)(\mathbf{x}^{\times \frac{m}{n}}))\right]^{C_{\frac{m}{n}}}$$

which induce maps

$$res^n : U^m(R;M)^{C_m} \to U^{\frac{m}{n}}(R;M)^{C_{\frac{m}{n}}}.$$

We can now define $W(R;M)$, using a procedure similar to the construction of TR starting from THH:

Definition 3.2. The *topological Witt vectors of $\mathcal{C}$ (with coefficients in $\mathcal{M}$)* are the simplicial Γ-space

$$W(R;M)(X) := \operatorname*{holim}_{n,res^i} U^n(R;M)(X)^{C_n}.$$

We can also define the "truncated" versions

$$W(R;M)^{(n)}(X) := \operatorname*{holim}_{j\leq n,res_i} U^j(R;M)(X)^{C_j}.$$

With these definitions, we have

$$W(R;M) = \operatorname*{holim}_{n} W^n(R;M).$$

By evaluating on the simplicial set S^0, we obtain bisimplicial sets $U^n(R;M;S^0)$ and $W(R;M;S^0)$. The diagonal of these bisimplicial sets produces simplicial sets which are the 0th spaces of an Ω-spectrum in simplicial sets. Geometrically realizing gives us a space which is the 0th space of an Ω-spectrum in spaces. See [17] for further details.

Convention. As a matter of course, when we refer to $U^n(R;M)$, $W(R;M)$, etc. without any reference to their structure as simplicial Γ-spaces or simplicial spectra, we will be referring to one of the above simplicial sets, spaces, or spectra. We will be clear which we are referring to if it is important, and unclear if a statement is true of all of them.

Example. If $R = M$, then $W(R;R)$ is exactly $TR(R)$ as usually defined. This is a fact that we will need to use later to define a "multi-trace" map from K-theory.

Note. We stop to make some remarks, first about the chosen nomenclature. We can write $W^n(R;M)$ as a homotopy limit

$$W^n(R;M) \simeq \operatorname{holim}(U^n(R;M) \xrightarrow{res} W^{(n-1)}(R;M)),$$

over the map res induced by the restriction maps out of $U^n(R;M)$. Morally, then, an element of $W^n(R;M)$, consists of an element $x \in U^n(R;M)$ and $y \in W^{(n-1)}(R;M)$ such that $res(x) = y$. This looks very much like the construction of the p-typical Witt vectors as a limit over $\mathbb{Z}/p^n\mathbb{Z}$.

Second, we remark on the utility of having $W(R;M)$ related to K-theory. From [19, Proposition 5.2], we have the "fundamental cofibration sequence"-like homotopy fiber sequence of simplicial sets:

$$U^n(R;M;X)_{hC_n} \to W^n(R;M;X) \to W^{(n-1)}(R;M;X). \tag{3.3}$$

One of the remarkable features of this sequence is that it is actually a description of the layers of the Taylor tower (the homotopy fiber sequence $D_n \to P_n \to P_{n-1}$ that picks off the degree n "homogeneous" part of the n-th polynomial approximation we will discuss later). The upshot is that we get tractable models for the spectra in the layers of the Taylor tower and a cofiber sequence that relates them to each other, suggesting the possibility of making inductive arguments to compute K-theory. Lindenstrauss and McCarthy use this strategy to great effect in [19].

3.2 Properties

We now prove the properties of the U^n, W functors that we will need in order to compare them to K-theory. They will largely be the same as those we proved for the square-zero extension earlier.

3.2.1 Degreewise Computation

Lemma 3.2. *Let $R_\bullet$ be a simplicial Γ-ring, and $M_\bullet$ an $R_\bullet$-bimodule. Then $U^n(R_\bullet;M_\bullet)$ can be computed degreewise, i.e. for all simplicial sets X*

$$U^n(d(R);d(M);X) \simeq d\left([q] \mapsto U^n(R_q;M_q;X_q)\right).$$

and therefore we get the equivalence of spectra

$$U^n(d(R); d(M)) \simeq \left|[q] \mapsto U^n(R_q; M_q)\right|.$$

Proof. Let $\mathbf{x} \in I^{n(k+1)}$, then because the smash product is formed degree-wise, we have that

$$X \wedge V_k^n(d(R); d(M))(\mathbf{x}) = [q \mapsto X_q \wedge V_k^n(R_q; M_q)_q(\mathbf{x})].$$

We can recognize the right hand side as the diagonal of the bisimplicial set $[[q, j] \mapsto X_q \wedge V_k^n(R_q; M_q)_j(\mathbf{x})]$. As j varies, the levels $X_q \wedge V_k^n(R_q; M_q)$ are connected, so we can perform loops degreewise (see [9, A.5.0.5], e.g.), and so there is a natural equivalence

$$\Omega^{\mathbf{x}}\left[X \wedge V^n(d(R); d(M))_k(\mathbf{x})\right] \xrightarrow{\simeq} \left[q \mapsto \Omega^{\mathbf{x}}\left[X_q \wedge V_k^n(R_q; M_q)(\mathbf{x})\right]\right].$$

Using that homotopy colimits commute with geometric realization and an application of the Bökstedt approximation theorem, we get a natural equivalence:

$$|U^n(d(R); d(M); X)_k| \xrightarrow{\simeq} \left|[q] \mapsto U^n(R_q; M_q; X_q)_k\right|,$$

and an application of the realization lemma completes the proof. $\square$

Corollary 3.3. *Let R be a simplicial ring, M an R-bimodule. Then for any simplicial set X there is an equivalence*

$$U^n(R; M; X) \simeq d\left([q] \mapsto U^n(R_q; M_q; X_q)\right).$$

Evaluating at $X = S^0$, we get an equivalence

$$U^n(R; M) \simeq d\left([q] \mapsto U^n(R_q; M_q)\right).$$

Proof. This statement does not immediately appear to follow from Lemma 3.2, but the following observation makes this possible: Let R be a simplicial ring, then it can be considered as a Γ-ring in two ways. The first is the normal construction HR, whose value on a simplicial set X is

$$HR(X) := R \wedge X.$$

On the other hand, we can consider the simplicial Γ-ring $HR_\bullet$, whose levels are given by the Γ-rings associated to the discrete rings R_i :

$$HR_i(X) := R_i \wedge X.$$

The important thing is that the diagonal of $HR_\bullet$ is the same as HR :

$$d(HR_\bullet)(X)_i = R_i \wedge X_i,$$

and so

$$d(HR_\bullet)(X) = R \wedge X,$$

as the smash product is performed levelwise. Indeed, the same fact holds for M, and so with this in mind, we can apply Lemma 3.2 to $HR_\bullet$ and $HM_\bullet$ to obtain the desired $\square$

Corollary 3.4. *If $M_\bullet$ is instead a simplicial R-bimodule, then for any simplicial set X there is an equivalence*

$$U^n(R; d(M); X) \simeq d\left([q] \mapsto U^n(R; M_q; X_q)\right),$$

and therefore an equivalence of simplicial sets

$$U^n(R; d(M)) \simeq d\left([q] \mapsto U^n(R; M_q)\right).$$

Proof. We consider R as a constant simplicial Γ-ring $R_\bullet$, and then a simplicial R-bimodule is simply a module over this Γ-ring. In this case, $d(R_\bullet) = R$, and $R_q = R$. Lemma 3.2 tells us that for any simplicial set X we have an equivalence

$$U^n(R; d(M); X) \simeq d\left([q] \mapsto U^n(R; M_q; X_q)\right).$$

Evaluating at $X = S^0$ to get the 0th space of the associated spectrum, we get

$$U^n(R; d(M)) \simeq d\left([q] \mapsto U^n(R; M_q)\right).$$

$\square$

Corollary 3.5. *Combining the above two facts, we have an equivalence of bisimplicial sets*

$$U^n(R; M_i) \simeq U^n(R_i; d(M_i)) \simeq U^n(R_i; M_{i,\bullet})$$

Using the above, one can prove the analogous statement for $W(R; M)$, as in [18]:

Proposition 3.6. *Let $R_\bullet$ be a simplicial FSP, and $M_\bullet$ an $R_\bullet$-bimodule. Then $W(R_\bullet; M_\bullet)$ can be computed degreewise, i.e. for all simplicial sets X*

$$W(d(R); d(M); X) \simeq d\left\{[q] \mapsto W(R_q; M_q; X_q)\right\}.$$

and therefore we get the equivalence of spectra

$$W(d(R); d(M)) \simeq \left|[q] \mapsto W(R_q; M_q)\right|.$$

As before, this implies:

Corollary 3.7. *Let R be a simplicial ring, M an R-bimodule. Then there is an equivalence*

$$W(R; M) \simeq |W(R_i; M_i)|.$$

If $M_\bullet$ is instead a simplicial R-bimodule, then we have an equivalence

$$W(R; d(M)) \simeq |W(R; M_i)|.$$

Combining the above two facts, we have an equivalence

$$|W(R; M_i)| \simeq |W(R_i; d(M_i))| \simeq \|W(R_i; M_{i,\bullet})\|.$$

3.2.2 Cubical resolvability

We now prove the same sort of resolvability statement for $W(R; M)$ that we proved for $K(R \ltimes M)$. This will be slightly more complicated than that, but not too difficult – the proof follows roughly the same strategy that one uses to show the analogous result for topological cyclic homology.

First we need to show several related statements for the U^n functors that we use to build W, and we need to recall the following statement about smashing cubes from [10, IV.1.4.2] or [9, Lemma IV.1.4.1]:

Lemma 3.8. *Let $\mathcal{X}_i$ be $(id + x_i)$-Cartesian cubes for $i = 1, \ldots, n$. Then*

$$\mathcal{X} := \left\{ S \mapsto \bigwedge_{1 \leq i \leq n} \mathcal{X}_i^S \right\}$$

is $(id + \sum_i x_i)$-Cartesian.

Using this, we can show the following:

Proposition 3.9. *Let $S \mapsto (R^S, M^S)$ be an admissible (id)-Cartesian cube of Γ-rings and bimodules and let X be a k−connected simplicial set. Then the cube*

$$S \mapsto U^n(R^S, M^S; X)$$

is $(id+k+1)$*-Cartesian. As a consequence, the cube of simplicial sets, spaces, or spectra*

$$S \mapsto U^n(R^S; M^S)$$

is $(id+1)$*-Cartesian.*

Proof. By fibrantly replacing everything we can assume that all of the nodes in the cubes involved are fibrant. This does not affect the Cartesian-ness of the cube, as the fibrant replacement of the punctured cube (as a diagram) we would use to compute the homotopy limit has the effect of fibrantly replacing the objects of the cube. Fibrant Γ-spaces are very special (cf. Chapter A), and so the (id)-Cartesian conditions hold pointwise (on all the values of the Γ-spaces).

We first show that for all $q \geq 0$, $S \mapsto U^n(R, M; X)_q$ is $(2 * id + k)$-coCartesian. Let $\mathbf{x} = (x_{i,j}) \in I^{n(q+1)}$, then for all q, the previous lemma tells us that

$$S \mapsto X \wedge V_q^n(R^S; M^S)(\mathbf{x}) = X \wedge \bigwedge_{i=1}^{n} \left[M^S(S^{x_{i,0}}) \bigwedge_{j=1}^{q} R^S(S^{x_{i,j}}) \right]$$

is $(id + k + 1 + |\mathbf{x}|)$–Cartesian (the factor of $k+1$ coming from the constant cube $S \mapsto X$). Looping down by using $\Omega^{\mathbf{x}}$, we get that this is $(id + k + 1)$–Cartesian, and taking the associated homotopy colimit to get $U^n(R^S; M^S; X)_q$, we obtain that

$$S \mapsto U^n(R^S; M^S; X)_q$$

is $(id + k + 1)$–Cartesian. Blakers-Massey ([9, A.7.2]) tells us that this is the same as the cube being $(2 * id + k)$-coCartesian.

Now, homotopy colimits commute with realization, so

$$S \mapsto U^n(R^S; M^S; X)$$

is equally coCartesian. Applying Blakers-Massey once more gives us the desired result. $\square$

Again, Theorem 2.9 implies:

Corollary 3.10. *Let (R,M) be an admissible id-Cartesian cube of Γ-rings and bimodules. Then the natural map*

$$U^n(R^{\varnothing};M^{\varnothing}) \to \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} U^n(R^S;M^S)$$

is an equivalence.

Lemma 3.11. *Let (R,M) be an admissible cube of Γ-rings and bimodules such that $U^n(R;M)$ is id–Cartesian for all n. Then $W(R;M)$ is also id–Cartesian.*

Proof. We need to show that $W(R;M)$ is m-Cartesian when restricted to m-cubes; to do this we will use the "fundamental cofibration sequence" Eq. (3.3) and induction.

Let $\mathcal{X} = (\mathcal{X}_R, \mathcal{X}_M)$ be any m-subcube of the admissible cube (R,M), then by assumption the cube $U^n(\mathcal{X}_R,\mathcal{X}_M)$ is m-Cartesian. Because homotopy orbits preserve connectivity and homotopy colimits, we have that $U^n(\mathcal{X}_R,\mathcal{X}_M)_{hC_n}$ is m-Cartesian as well. We have a map of homtopy fiber sequences

$$\begin{array}{ccccc}
\operatorname*{holim}_{S\in\mathcal{X}\smallsetminus\varnothing} U^n(R^S;M^S)_{hC_n} & \longrightarrow & \operatorname*{holim}_{S\in\mathcal{X}\smallsetminus\varnothing} W^n(R^S;M^S) & \longrightarrow & \operatorname*{holim}_{S\in\mathcal{X}\smallsetminus\varnothing} W^{(n-1)}(R^S;M^S) \\
\downarrow & & \downarrow & & \downarrow \\
U^n(R^{\varnothing};M^{\varnothing})_{hC_n} & \longrightarrow & W^n(R^{\varnothing};M^{\varnothing}) & \longrightarrow & W^{(n-1)}(R^{\varnothing};M^{\varnothing})
\end{array}$$

We induct on n to show that $W^n(\mathcal{X}_R,\mathcal{X}_M)$ is m-Cartesian for all n. The base case $n=1$ is the statement for $U^1 = THH$, which is true by Proposition 3.9.

For general n, the same proposition gives us that left hand map is a m-equivalence, and by induction we have that the right hand map is also an m-equivalence. Using the five lemma for homtopy fiber sequences, get that the middle map is an m–equivalence, or that the restricted cube is m-Cartesian, as desired. □

Combining Proposition 3.9, Lemma 3.11, and Theorem 2.9 we obtain:

Corollary 3.12. *Let (R, M) be an admissible id-Cartesian k-cube of Γ-rings and bimodules. Then the cube*

$$S \mapsto W(R^S; M^S)$$

is $(id+1)$-Cartesian.

Theorem 3.13. *Let (R, M) be an admissible id-Cartesian cube of Γ-rings and bimodules. Then the natural map*

$$W(R^{\varnothing}; M^{\varnothing}) \to \operatorname*{holim}_{S \in \mathcal{P} \smallsetminus \varnothing} W(R^S; M^S)$$

is an equivalence.

3.3 K-Theory of Square-Zero Extensions

In this section we relate the K-theory of square-zero extensions ot the Witt vectors discussed in the previous sections.

3.3.1 Construction of a Map Between K-theory and the Witt Vectors

We now define a weak map $\widetilde{K}(R \ltimes M) \to W(R; \Sigma M)$ that restricts to the Lindenstrauss-McCarthy equivalence on discrete rings in a suitable fashion.

Construction 3.14. Recall from [9], there are natural trace maps

$$tr_n : K(A) \to sd_n THH(A)$$

that are compatible with the restriction maps used to build $TR(A)$, so there is a map

$$tr : K(A) \to TR(A).$$

Given that $U^n(A) \cong sd_a THH(A)$ and $W(A) \cong TR(A)$, this gives us maps

$$K(A) \to U^n(A) \quad \text{and} \quad K(A) \to W(A).$$

Using the above, we get the following commutative diagram, where we denote

$$\mathrm{hofib}(W(R \ltimes M) \to W(R))$$

as $\widetilde{W}(R \ltimes M)$

$$\begin{tikzcd}
\widetilde{K}(R \ltimes M) \arrow[r, "\tilde{tr}"] \arrow[d] & \widetilde{W}(R \ltimes M) \arrow[d] \\
K(R \ltimes M) \arrow[r, "tr"] \arrow[d] & W(R \ltimes M) \arrow[d] \\
K(R) \arrow[r, "tr"] & W(R)
\end{tikzcd}$$

There is also a map $\widetilde{W}(R \ltimes M) \to W(R; \Sigma M)$, which we now describe. First, some notation:

Definition 3.3. Let $\vec{j}$ be an n-tuple of non-negative numbers. We denote the sum of the entries of $\vec{j}$ by $|\vec{j}|$.

Let $V^{n,\vec{j}}(R; M)(\mathbf{x})$ denote the part of $V^n(R \ltimes M)$ where M appears j_i times in block i (the "$\vec{j}$-homogeneous" part of $V^n(R \ltimes M)$). To be precise, we define

$$V^{n,j}(R; M)(\mathbf{x})_k := \bigvee_f \left[\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{k+1} F_f^n(i, j)(S^{x_{i,j}}) \right]$$

where $f = \{f_1, \ldots, f_n\}$ ranges over n-tuples of injections $f_i : [j_i] \to [k+1]$ and

$$F_f(i, j) = \begin{cases} R & (i, j) \in \mathrm{im}(f_i) \\ M & \text{otherwise} \end{cases}.$$

Note that $V_k^{n,\vec{0}}(R; M)(\mathbf{x})$ is exactly $V_k^n(R)(\mathbf{x})$, and that these assemble into simplicial objects like $V_\bullet^n(R; M)$.

Recall the exact form of $V_k^n(R \ltimes M)$:

$$V_k^n(R \ltimes M)(\mathbf{x}) = \bigwedge_{i=1}^{n} \bigwedge_{j=0}^{k} (R \ltimes M)(S^{x_{i,j}}).$$

Expanding the latter smash product out and distributing, we see that this can be written as

$$V_k^n(R \ltimes M)(\mathbf{x}) = \bigvee_{\vec{j}} V_k^{n,\vec{j}}(R; M)(\mathbf{x}).$$

We want to promote this decomposition to something for $U^n(R \ltimes M)$, so to this end we define the simplicial Γ-space

$$U_k^{n,\vec{l}}(R; M)(X) := \bigvee_{|\vec{j}|=l} \operatorname*{hocolim}_{\mathbf{x} \in I^{n(k+1)}} \Omega^{\mathbf{x}}(X \wedge V^{n,\vec{j}}(R; M)(\mathbf{x})).$$

These also have a C^n action, as the C^n action on $U^n(R \ltimes M)$ preserves the total number of times the bimodule coordinate shows up. One can think of this C^n action as acting internally on $V^{n,\vec{j}}$ and then acting on the vectors $\vec{j}$ by cyclically permuting the entries.

The C_n fixed points can only involve the vectors $\vec{j}$ that are diagonal (i.e. they have the same number of bimodule coordinates in each block). We denote the special vectors that correspond to these by $\vec{v}_k := (k, k, \ldots, k)$.

In this case, we have that

$$V^{n,\vec{v}_k}(R; M)(\mathbf{x})$$

has a C_n action, and so we can define

$$U_j^{n,k}(R; M)(X) := \operatorname*{hocolim}_{\mathbf{x} \in I^{n(j+1)}} \Omega^{\mathbf{x}}(X \wedge V^{n,\vec{v}_k}(R; M)(\mathbf{x})).$$

Lemma 3.15. *There is a C_n-equivariant decomposition of $U^n(R \ltimes M)$ as*

$$U^n(R \ltimes M) \simeq U^n(R) \times \prod_{|\vec{j}|>0} U^{n,\vec{j}}(R;M)$$

and therefore an equivariant decomposition of $\widetilde{U}^n(R \ltimes M)$ as

$$\widetilde{U}^n(R \ltimes M) \simeq \prod_{|\vec{j}|>0} U^{n,\vec{j}}(R;M),$$

and moreover, both of these decompositions are compatible with the restriction maps, in the sense that

$$\widetilde{res_n} : \widetilde{U}^m(R \ltimes M)^{C_m} \to \widetilde{U}^{\frac{m}{n}}(R \ltimes M)^{C_{\frac{m}{n}}}$$

restricts to a map

$$\widetilde{U}^{m,\vec{j}}(R;M)^{C_m} \to \widetilde{U}^{(\frac{m}{n},\frac{\vec{j}}{n})}(R;M)^{C_{\frac{m}{n}}},$$

defined to be trivial if n does not divide j.

Passing to fixed points, we get decompositions

$$U^n(R \ltimes M)^{C_n} \simeq U^n(R)^{C_n} \times \prod_{j>0} U^{n,j}(R;M)^{C_n}$$

and

$$\widetilde{U}^n(R \ltimes M)^{C_n} \simeq \prod_{j>0} U^{n,j}(R;M)^{C_n}.$$

Proof. The inclusions and projections

$$V_k^{n,i}(R;M)(\mathbf{x}) \subseteq V_k^n(R \ltimes M)(\mathbf{x}) \to V_k^{n,j}(R;M)(\mathbf{x})$$

induce C_n-equivariant maps

$$\bigvee_{j\geq 0} U^{n,j}(R;M)(X) \to U^n(R \ltimes M)(X) \to \prod_{j\geq 0} U^{n,j}(R;M)(X).$$

As in [9], this is an equivalence.

The compatibility with the restriction maps is apparent when one analyzes their definition. □

We can identify the least connected part of the above that could possibly have any C_n-fixed points:

Lemma 3.16. *There is a natural equivalence of $C_n \times S^1$-spaces*

$$U^{n,n}(R;M) \xleftarrow{\simeq} (S^1_+)^n \wedge U^n(R;M),$$

where the latter has a diagonal C_n-action and free, diagonal S^1 action on the left coordinate.

Proof. The inclusion $V^n_k(R;M) \subset V^{n,n}_k(R;M)$ is determined by a choice of one of the $(k+1)$ positions in each of the n factors of $V^{n,n}_k$. This is encoded by an evident equivalence

$$(C_{(k+1)_+})^n \wedge U^n_k(R;M) \to U^{n,n}_k(R;M)$$

that specifies this inclusion.

This determines a map of cyclic C_n-spaces

$$(S^1_+)^n \wedge U^n(R;M) \to U^{n,n}(R;M).$$

□

Because all of the terms in the decomposition of the fixed points $\widetilde{U}^m(R \ltimes M)^{C_n}$ are of the form $U^{n,kn}(R;M)^{C_n}$ which are $2k * \mathrm{conn}(M)$-connected, the following is clear:

Lemma 3.17. *Let M be a k-connected R-bimodule. Then the projection map*

$$\widetilde{U}^m(R \ltimes M)^{C_n} \to U^{n,n}(R;M)^{C_n}$$

and the inclusion map

$$U^{n,n}(R;M)^{C_n} \to \widetilde{U}^m(R \ltimes M)^{C_n}$$

are $2k$-connected.

Construction 3.18. [17] gives us a natural equivalence

$$((S^1)^n \wedge U^n(R;M))^{C_n} \xrightarrow{\simeq} U^n(R;B_\bullet M)^{C_n}.$$

We can piece these maps together in order to get a map on $\widetilde{W}(R \ltimes M)$. There is a projection map $\pi_n : \widetilde{W}(R \ltimes M) \to \widetilde{U}^n(R \ltimes M)^{C_n}$, allowing us to form the composite weak maps

$$\begin{tikzcd}
\widetilde{W}(R \ltimes M) \arrow[r, "\pi_n"] & \widetilde{U}^n(R \ltimes M)^{C_n} \arrow[r] & U^{n,n}(R;M)^{C_n} & (S^1_+) \wedge U^n(R;M)^{C_n} \arrow[l, "\simeq"'] \arrow[d] \\
 & & U^n(R;B_\bullet M)^{C_n} & (S^1) \wedge U^n(R;M)^{C_n} \arrow[l, "\simeq"']
\end{tikzcd}$$

All of the relevant maps are compatible with the restriction maps, so taking the homotopy limit over the restriction maps grants us a weak map

$$\gamma : \widetilde{W}(R \ltimes M) \to W(R;B_\bullet M).$$

Finally, we let T denote the composite weak map

$$\widetilde{K}(R \ltimes M) \xrightarrow{\tilde{tr}} \widetilde{W}(R \ltimes M) \xrightarrow{\gamma} W(R;B_\bullet M).$$

We wish to show that T is an equivalence, but our strategy of proof will require that we know two things:

- T can be computed levelwise on both sides;
- T agrees with the Lindenstrauss-McCarthy map up to homotopy when applied to discrete rings and simplicial bimodules.

We will need the following technical lemma, allowing us to compare the Lindenstrauss-McCarthy map to our own:

Lemma 3.19. *Let R be a discrete ring, M an R-bimodule, X, Y simplicial sets with Y k-connected. Then the map on homotopy groups induced by β*

$$\pi_* \operatorname{hofib}(\widetilde{K}(R; B_\bullet M[X \vee Y]) \to \widetilde{K}(R; B_\bullet M[X])) \to \pi_* \operatorname{hofib}(W(R; B_\bullet M[X \vee Y]) \to W(R; M[X]))$$

is $2k$-equivalent to the map induced by T

$$\pi_* \operatorname{hofib}(\widetilde{K}(R \ltimes M[X \vee Y]) \to \widetilde{K}(R; M[X]) \to \pi_* \operatorname{hofib}(W(R; B_\bullet M[X \vee Y]) \to W(R; M[X])).$$

As of this writing, a full proof of this comparison not complete. In the following, we provide some results that provide direction for how one method of proof should go.

Proposition 3.20. *The Lindenstrauss-McCarthy trace map is compatible with our weak map T.*

Proof. We must be clear about what is meant by the above proposition, but we will produce a decomposition of $\widetilde{W}(R \ltimes M[X \vee Y])$ that lines up with the decomposition of $W(R; B_\bullet M \oplus B_\bullet N)$ that it is mapped to via T. To define T, we looked in $U^n(R \ltimes M)$ and filtered this by the number of occurrences of M. When we have $M[X \vee Y]$ instead, we will filter this further by looking at the separate number of occurrences of X and Y.

Let us first describe the strategy used by Lindenstrauss-McCarthy. For M, N R-bimodules with N k-connected, Lindenstrauss and McCarthy write a decomposition

$$\begin{tikzcd}
hofib \arrow[r] \arrow[d, "\simeq"] & \widetilde{K}(R; B_\bullet M \oplus B_\bullet N) \arrow[r] \arrow[d, "\simeq"] & \widetilde{K}(R; B_\bullet M) \arrow[d, "\simeq"] \\
\widetilde{K}(R \ltimes M; B_\bullet N) \arrow[r] & \widetilde{K}((R \ltimes M) \ltimes N) \arrow[r] & \widetilde{K}(R \ltimes M)
\end{tikzcd}$$

where the rows split. By applying the Dundas-McCarthy trace map [10] to the bottom left corner, they obtain a $2(k+1)$-connected map

$$\widetilde{K}(R; B_\bullet M \vee B_\bullet N) \to THH(R \ltimes M; B_\bullet N) \times \widetilde{K}(R; B_\bullet M).$$

By projecting out, they obtain a $2(k+1)$-connected map

$$\mathrm{hofib}(\widetilde{K}(R; B_\bullet M \vee B_\bullet N) \to \widetilde{K}(R; B_\bullet M)) \to THH(R \ltimes M; B_\bullet N).$$

On the other hand, they provide a $2(k+1)$-connected equivariant map

$$U^n(R; B_\bullet M \vee B_\bullet N) \to U^n(R; B_\bullet M) \times (C_n)_+ \wedge U^n(R; B_\bullet M, \dots, B_\bullet M, B_\bullet N).$$

Passing to C_n-fixed points and taking the homotopy limit used to define W, they obtain a $2(k+1)-1$-connected map

$$W(R; B_\bullet M \vee B_\bullet N) \to W(R; B_\bullet M) \times THH(R \ltimes M; B_\bullet N).$$

By projecting out, this gives a $2(k+1)-1$-connected map

$$\mathrm{hofib}(W(R; B_\bullet M \vee B_\bullet N) \to W(R; B_\bullet M)) \to THH(R \ltimes M; B_\bullet N)$$

This tells us that the two homotopy fibers

$$\mathrm{hofib}(W(R; B_\bullet M \vee B_\bullet N) \to W(R; B_\bullet M))$$

and

$$\mathrm{hofib}(\widetilde{K}(R; B_\bullet M \vee B_\bullet N) \to \widetilde{K}(R; B_\bullet M))$$

are abstractly $(2k)$-equivalent to $THH(R \ltimes M; B_\bullet N)$. The crucial thing they prove is that this abstract equivalence is induced by their map β.

Our weak map T

$$T : \widetilde{K}(R \ltimes [M \vee N]) \leftrightarrow W(R; B_\bullet M \vee B_\bullet N)$$

also induces a map to $THH(R \ltimes M; B_\bullet N)$ by projecting

$$\widetilde{K}(R \ltimes [M \vee N]) \leftrightarrow W(R; B_\bullet M \vee B_\bullet N) \to THH(R \ltimes M; B_\bullet N). \tag{3.4}$$

To complete the proof, it would suffice to show that this weak map coincides with the Lindenstrauss-McCarthy map. A first thing to do is to show that we can shortcut the "weak" part of T, which is the intent of the present proposition.

To do this, we make a further decomposition of $U^n(R \ltimes [M \vee N])$. Recall the decomposition used to define T:

$$U^n(R \ltimes [M \vee N]) \simeq U^n(R) \times \prod_{|\vec{j}|>0} U^{n,\vec{j}}(R; M \vee N)$$

where $\vec{j}$ indexed the number of occurrences of $M \vee N$. Because the smash product distributes over the wedge, we can decompose this further by indexing on the occurrences of M, N separately. In the following, $\vec{j}$ indexes M and $\vec{k}$ counts N, and the terms $U^{n,\vec{j},\vec{k}}(R; M \vee N)$ are defined in the same way we defined $U^{n,\vec{j}}(R; M \vee N)$:

$$U^n(R \ltimes [M \vee N]) \simeq U^n(R) \times \prod_{|\vec{j}|+|\vec{k}|>0} U^{n,\vec{j},\vec{k}}(R; M \vee N).$$

Again, the fixed points can only involve "diagonal" vectors which define C_n-invariant subspaces, so we have

$$U^n(R \ltimes [M \vee N])^{C_n} \simeq U^n(R) \times \prod_{j+k>0} U^{n,j,k}(R; M \vee N)^{C_n}.$$

Our map projected away $U^n(R)$ and mapped onto $U^{n,n}(R; M \vee N)$, and we now show that this is compatible with our decomposition. We see that

$$U^{n,n}(R; M \vee N)^{C_n} \cong U^{n,n,0}(R; M \vee N)^{C_n} \vee \bigvee_{k>0} U^{n,j,k}(R; M \vee N)^{C_n}.$$

We identified $U^{n,n}(R; M \vee N)^{C_n}$ as $(S^1_+) \wedge U^n(R; M \vee N)^{C_n}$, and it is clear that this was done in a way that respects this decomposition (we noted that we needed to keep track of the rotations of $U^n(R; M \vee N)$ in order to insert them into $U^{n,n}(R; M \vee N)$ – the same is true for each piece in the above decomposition of $U^{n,n}(R; M \vee N)$).

The next step in our weak map is the equivalence and then projection

$$\begin{array}{ccc} U^{n,n,0}(R; M \vee N)^{C_n} & \vee & \bigvee_{k>0} U^{n,j,k}(R; M \vee N)^{C_n} \\ \uparrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \simeq} \\ (S^1_+) \wedge U^{n,n,0}(R; M \vee N) & \vee & \bigvee_{k>0} (S^1_+) \wedge U^{n,j,k}(R; M \vee N)^{C_n} \\ \downarrow & & \downarrow \\ (S^1) \wedge U^{n,n,0}(R; M \vee N) & \vee & \bigvee_{k>0} (S^1) \wedge U^{n,j,k}(R; M \vee N)^{C_n} \end{array}$$

The Lindenstrauss-McCarthy map is seen to be projecting away the left factor and projecting on to the right factor

$$\begin{aligned} \bigvee_{k>0} (S^1) \wedge U^{n,j,k}(R; M \vee N)^{C_n} \to (S^1) \wedge U^{n,n-1,1}(R; M \vee N)^{C_n} \simeq S^1 \wedge U^n(R; M^{\otimes^{n-1}}, N) \\ \simeq U^n(R; B_\bullet M^{\otimes^{n-1}}, B_\bullet N) \end{aligned}$$

As n varies, these assemble to a map

$$\widetilde{W}(R \ltimes [M \vee N]) \to \prod_n U^n(R; B_\bullet M^{\otimes^{n-1}}, B_\bullet N) \simeq THH(R \ltimes M; B_\bullet N)$$

that lifts to the fiber above $\widetilde{W}(R \ltimes M)$. Altogether, projection onto the res-invariant subspace (which lives in the fiber over $\widetilde{W}(R \ltimes M)$) of $\widetilde{W}(R \ltimes [M \vee N])$,

$$\prod_n U^{n,n-1,1}(R; M \vee N)^{C_n}$$

gives us a commutative diagram, where the zig-zags are those in the definition of T:

$$
\begin{tikzcd}
 & \prod_n U^{n,n-1,1}(R; M \vee N)^{C_n} \arrow[r, leftrightarrow] & \prod_n U^n(R; B_\bullet M^{\otimes^{n-1}}, B_\bullet N) \\
\widetilde{K}((R \ltimes M) \ltimes N) \arrow[r] \arrow[d] & \widetilde{W}((R \ltimes M) \ltimes N) \arrow[u, "\pi"] \arrow[d] & \\
\widetilde{K}(R \ltimes [M \vee N]) \arrow[r] \arrow[d] & \widetilde{W}(R \ltimes [M \vee N]) \arrow[r, leftrightarrow] \arrow[d] & W(R; B_\bullet M \oplus B_\bullet N)) \arrow[uu] \arrow[d] \\
\widetilde{K}(R \ltimes M) \arrow[r] & \widetilde{W}(R \ltimes M) \arrow[r, leftrightarrow] & W(R; B_\bullet M)
\end{tikzcd}
$$

To prove our main comparison result, it would suffice to compare the map

$$\widetilde{K}((R \ltimes M) \ltimes N) \to \widetilde{W}((R \ltimes M) \ltimes N) \to \prod_n U^{n,n-1,1}(R; M \vee N)^{C_n}$$

to the Dundas-McCarthy trace in a stable range. $\square$

Lemma 3.21. *If $R_\bullet$ is a simplicial ring and $M_\bullet$ is an R-bimodule, then T can be computed levelwise. That is, we have a commutative diagram of weak maps*

$$
\begin{tikzcd}
\widetilde{K}(R_\bullet; M_\bullet) \arrow[r, "T"] \arrow[d, "\simeq"] & W(R_\bullet; BM_\bullet) \arrow[d, "\simeq"] \\
\left|\widetilde{K}(R_i; M_i)\right| \arrow[r, "|T|"] & \left|W^n(R_i; BM_i)\right|.
\end{tikzcd}
$$

Proof. The statement of this fact for $\tilde{tr}$ is implicit in the arguments of McCarthy in [22]. The rest of the zig-zag is passing through maps that can evidently be computed levelwise. $\square$

3.3.2 Goodwillie Calculus

In order to prove the main result of this chapter, we need to to recall some of the basics of Goodwillie calculus.

Definition 3.4. Let $F : C \to D$ be a homotopy functor between two suitable categories C, D. Then F is *n-excisive* if for every strongly coCartesian $(n+1)$-cube $\mathcal{X}$, $F(\mathcal{X})$ is Cartesian. Here, "strongly coCartesian" means that every subcube is coCartesian.

Every such functor F admits a universal n-excisive "polynomial" approximation P_nF (initial amongst all n-excisive functors that F admits a natural transformation to), which fits into a "Taylor tower" of functors and natural transformations

$$\begin{tikzcd}
& \vdots \arrow[d] \\
& P_{n+1}F \arrow[d] \\
F \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] \arrow[ddr] & P_nF \arrow[d] \\
& P_{n-1} \arrow[d] \\
& \vdots
\end{tikzcd}$$

The general goal of Goodwillie calculus is to study functors via their n-excisive approximations, with the hope of recovering properties of F as the homotopy limit of its approximations. Not all functors are approximable in this way, but many are, so this is often a fruitful enterprise.

Example. Let F be a functor between suitably nice pointed categories such that $F(*) = *$. Then we have an explicit formula for the first polynomial approximation $P_1(F)$:

$$P_1(F)(X) \simeq \operatorname*{hocolim}_n \Omega^n F(\Sigma^n X).$$

This formula obviously requires Σ, Ω and homotopy colimits to make sense, but this is the case in all of the situations we will encounter.

If we, say, consider the identity functor on the category of spaces, then we get

$$P_1(id)(X) \simeq \Omega^\infty \Sigma^\infty X = QX.$$

In most cases where the statement makes sense, the polynomial approximations only depend on connected input. This can be seen in the above example, as we ended up suspending X an arbitrary number of times in the colimit defining the 1st derivative. As a result, two functors that agree at all connected input necessarily have the same Taylor tower.

3.3.3 Analyticity

In the context of Goodwillie calclus, we can develop a notion of analyticity.

Definition 3.5. Let F be a functor that preserves homotopy equivalences. Then F is *n-excisitve* if for every $(n+1)$-cube all of whose faces are co-Cartesian ("strongly co-Cartesian") is taken to a Cartesian diagram.

The relevant definition is a "stable" version of the above where we demand that F be excisive on highly connected objects:

Definition 3.6. Let F be a functor that preserves homotopy equivalences. Then F is *stably n-excisive* if the following condition $E_n(c, \kappa)$ is satisfied for some c, κ :

$E_n(c, \kappa)$: If $\mathcal{X}$ is any strongly co-Cartesian $(n+1)$-cube such that for all S the map $\mathcal{X}^{\varnothing} \to \mathcal{X}^S$ is k_s-connected and $k_s \geq \kappa$ then $F(\mathcal{X})$ is $(-c + \sum k_s)$-Cartesian.

This lets us define our notion of analyticity. The condition, as written, seems a bit complicated, but one should have in mind the analagous conditions on a real-valued function that guarantee that it is analytic. These conditions involve the decay rates of the derivatives, and there is a sensible way in which connectivity can be thought of as an inverse measure of distance. Regardless, the definition is as follows:

Definition 3.7. The functor F is *ρ-analytic* if there is some q such that F satisfies $E_n(n\rho - q, \rho + 1)$ for all $n \geq 1$.

The ρ is meant to describe the radius of convergence of the functor, and we say that F is *analytic* if it is ρ-analytic for some ρ.

The fact that we will need to use is that "analytic continuation" is a sensible thing to do (!). This is the method of proof used in [17] to prove their main theorem, so refer to that for a more in-depth discussion.

Proposition 3.22. *Let $f : F \to G$ be a natural transformations of functors between two suitable categories $\mathcal{C} \to \mathcal{D}$. If*

- *F, G are both analytic at a basepoint C_0;*
- *F, G agree at C_0 via f;*
- *f induces an isomorphism of 1st derivatives at all points in $\mathcal{C}$*

then f is an equivalence on all input in $\mathcal{C}$ within the minimum of the radii of convergence of F, G at C_0.

3.3.4 Proof of Main Theorem

Using the results of the previous sections, we now prove a generalized version of the main Lindenstrauss-McCarthy result from [18].

Theorem 3.23. *Let R be a Γ-ring, M an R−bimodule, X a connected space. Then there is a natural zig-zag of equivalences of simplicial sets*

$$\widetilde{K}(R \ltimes M[X]) \xrightarrow{\simeq} W(R; \Sigma M[X]),$$

and therefore a natural zig-zag of equivalences of the associated spaces or spectra.

We can actually promote Theorem 3.23 to a result about all connected R-bimodules:

Corollary 3.24. *Let R be a Γ-ring, M a connected R−bimodule. Then there is a natural zig-zag of equivalences of simplicial sets*

$$\widetilde{K}(R \ltimes M) \xrightarrow{\simeq} W(R; \Sigma M),$$

and therefore a natural zig-zag of equivalences of the associated spaces or spectra.

Proof of Corollary 3.24, given Theorem 3.23. If M is connected, then $M \simeq \Omega M[S^1]$, and Theorem 3.23 gives us a zig-zag of equivalences

$$\widetilde{K}(R \ltimes M) \simeq \widetilde{K}(R \ltimes \Omega M[S^1]) \xrightarrow{\simeq} W(R; \Sigma\Omega M[S^1]) \simeq W(R; \Sigma M).$$

□

Now we embark on our proof of Theorem 3.23. We are going prove this using the Goodwillie technique of "analytic continuation". That, we are going to fix M and compare the two functors from spaces to spectra that are computing the limits over the finite resolution cubes:

$$\operatorname*{holim}_{S \in Pn \smallsetminus \varnothing} \widetilde{K}(R^S \ltimes M^S[-]) \quad \text{and} \quad \operatorname*{holim}_{S \in Pn \smallsetminus \varnothing} W(R^S; \Sigma M^S[-]).$$

As per Proposition 3.22, to show that these functors have equivalent Taylor towers it suffices to show that they

1. agree at the point $*$;
2. are suitably analytic;
3. have the same differentials at every space X.

The first item is obvious, as

$$\widetilde{K}(R \ltimes 0) \simeq * \simeq W(R; 0).$$

The second has mostly been addressed for us, although we will need to be careful with the latter statement, which only holds for discrete rings:

Lemma 3.25 ([17, Proposition 7.14])**.** *For any FSP F and F-bimodule P, the functor*

$$W(F; P \otimes -)$$

is 0-analytic.

In particular, the functor $W(R; M[-])$ is 0-analytic.

Lemma 3.26 ([22, Proposition 3.2])**.** *Let R be a discrete ring and M an R-bimodule. Then the functor*

$$\widetilde{K}(R \ltimes M[-])$$

is 0-analytic.

Because analyticity is preserved by finite homotopy limits, we obtain:

Corollary 3.27. *The functors*

$$\operatorname*{holim}_{S\in Pn\setminus\varnothing} \widetilde{K}(R^S \ltimes M^S[-])$$

and

$$\operatorname*{holim}_{S\in Pn\setminus\varnothing} W(R^S;\Sigma M^S[-])$$

are 0-analytic.

Lemma 3.28. *Let R be a Γ-ring, M an R-bimodule X a connected space. Then for all n, T induces a zig-zag of equivalences*

$$\operatorname*{holim}_{S\in Pn\setminus\varnothing} \widetilde{K}(R^S \ltimes M^S[X]) \simeq \operatorname*{holim}_{S\in Pn\setminus\varnothing} W(R^S;\Sigma M^S[X]).$$

Proof of Theorem 3.23, given Lemma 3.28. We have a commutative diagram, where the horizontal arrows are induced by the natural zig-zag T and the vertical arrows are induced by the projection maps from the homotopy limit over the infinite cube to the homotopy limit over the cube defined by $\mathcal{P}n$:

$$\begin{tikzcd}
\widetilde{K}(R \ltimes M) \arrow[r] \arrow[d] & W(R;\Sigma M[X]) \arrow[d] \\
\operatorname{holim}_{S\in Pn\setminus\varnothing} \widetilde{K}(R^S \ltimes M^S[X]) \arrow[r, "\simeq"] & \operatorname{holim}_{S\in Pn\setminus\varnothing} W(R^S;\Sigma M^S[X])
\end{tikzcd}$$

The vertical maps are increasingly connected as a function of n, which implies the same for the top horizontal arrow. □

Proof of Lemma 3.28. Given the analyticity of both functors from Corollary 3.27 and their agreement at $*$ it suffices to show that their differentials are equivalent through a zig-zag for every space X. We will denote

$$K_n(-) := \operatorname*{holim}_{S\in Pn\setminus\varnothing} \widetilde{K}(R^S \ltimes M^S[-]) \qquad K^S(-) := \widetilde{K}(R^S \ltimes M^S[-])$$

and

$$W_n(-) := \operatorname*{holim}_{S \in Pn \setminus \varnothing} W(R^S; \Sigma M^S[-]) \quad W^S(-) := W(R^S; \Sigma M^S[-]).$$

In this notation, we need that for every space X and every k-connected space Y, the zig-zag of homotopy fibers induced by T

$$\begin{tikzcd}
F_K \arrow[r, leftrightarrow, "\tilde{T}"] \arrow[d] & F_W \arrow[d] \\
K_n(X \vee Y) \arrow[r, leftrightarrow, "T"] \arrow[d] & W_n(X \vee Y) \arrow[d] \\
K_n(X) \arrow[r, leftrightarrow, "T"] & W_n(X)
\end{tikzcd}$$

is at least $2k$-connected. First, note that homotopy limits commute and preserve connectivity, so it would suffice to show that this is true for each node in the cubes defining K_n and W_n. We are then reduced to considering

$$\begin{tikzcd}
F^S_K \arrow[r, leftrightarrow, "\tilde{T}"] \arrow[d] & F^S_W \arrow[d] \\
K^S(X \vee Y) \arrow[r, leftrightarrow, "T"] \arrow[d] & W^S(X \vee Y) \arrow[d] \\
K^S(X) \arrow[r, leftrightarrow, "T"] & W^S(X).
\end{tikzcd} \tag{3.5}$$

For $S \neq \varnothing$, $R^S \ltimes M^S$ is equivalent to a simplicial ring and the map $R^S \ltimes M^S[X \vee Y] \to R^S \ltimes M^S[X]$ is not just a map of Γ-spaces but a map of $H\mathbb{Z}$-algebras. The functor L is compatible with the tensoring of $H\mathbb{Z}$-algebras and simplicial rings over simplicial sets and as a result, we can take the homotopy fibers F^S_K and W^S_K in the category of simplicial rings. We are now reduced to looking at Eq. (3.5) when the underlying R^S is a simplicial ring and M^S is an R^S-bimodule. We will drop the S from now on.

We want to compute this levelwise, and for this we use Proposition 2.13, Corollary 3.7 and Lemma 3.21, giving us T-compatible equivalences

$$K(R \ltimes M[X]) \simeq |K(R_i \ltimes M_i[X])|$$

and

$$W(R; \Sigma M[X]) \simeq |W(R_i; B_\bullet M_i[X])|.$$

Because homotopy fibers of connective simplicial spectra can be computed levelwise and the realization of a levelwise $2k$-connected simplicial spectrum is $2k$-connected, it therefore suffices to show that the zig-zag of fibers $F_K \leftrightarrow F_W$ is $2k$-connected when R is a discrete ring and M is an R-bimodule.

We are then reduced to showing that the zig-zag

$$\mathrm{hofib}(\widetilde{K}(R \ltimes M[X \vee Y]) \to \widetilde{K}(R \ltimes M[X])) \leftrightarrow \mathrm{hofib}(W(R; \Sigma M[X \vee Y]) \to W(R; \Sigma M[X]))$$

is a $2k$-equivalence when both R, M are discrete and Y is k-connected.

If T were the Lindenstrauss-McCarthy map β, then we would be done:

Lemma 3.29. *[17, Technical Lemma 9.4] Let R be a ring and let M, N be simplicial R-bimodules. If N is k-connected, then β induces a $2k$-connected map*

$$\mathrm{hofib}(\widetilde{K}(R; B_\bullet M \oplus B_\bullet N) \to \widetilde{K}(R; B_\bullet M)) \to \mathrm{hofib}(W(R; B_\bullet M \oplus B_\bullet N) \to W(R; B_\bullet M)).$$

Our map is not exactly β, but it is close enough. Lemma 3.19 implies that the discrepancy between the zig-zag coming from T we are interested in and the Lindenstrauss-McCarthy map on fibers lies outside of the stable range we need to check. As the latter is $2k$-connected, the former is as well, which is exactly what we needed to show. □

If M is connected, then M is naturally equivalent to $\Sigma\Omega M$, so we can rewrite Theorem 3.23 as:

Corollary 3.30. *Let R be a Γ-ring and M is a connected R-bimodule. Then there is a natural zig-zag of equivalences of simplicial sets, spaces, or spectra*

$$\widetilde{K}(R \ltimes \Omega M) \simeq W(R; M).$$

It is easy to see how Corollary 3.30 implies the following calculus-theoretic statement:

Corollary 3.31. *The functors from R-bimodules to spectra*

$$\widetilde{K}(R \ltimes \Omega(-)) \quad \textit{and} \quad W(R; -)$$

have the same Taylor tower.

Note that $THH(R; -)$ is $U^1(R; -)$, which, by the above remark, is the 1st derivative of $W(R; -)$. This gives us:

Corollary 3.32 (Stable K-Theory is THH)**.** *Let R be a Γ-ring. Then the derivative of the functor $\tilde{K}(R \ltimes \Omega(-))$ from R-bimodules to spectra is naturally equivalent to to $THH(R; -)$.*

$$P_1(\tilde{K}(R \ltimes \Omega(-))) \simeq THH(R; -).$$

Chapter 4

K-Theory of The Tensor Algebra

In this chapter we define the tensor algebra and derived tensor algebra over a Γ-ring. The goal of this chapter is to reproduce the main result of [18], which computes the K-theory of tensor algebras in terms of the topological Witt vectors construction.

4.1 Definitions

Definition 4.1. Let R be an $\mathbf{S}$-algebra, and M a connected R-bimodule. We define the tensor algebra of M over R to be

$$T_R(M) := R \vee M \vee (M \wedge_R M) \vee (M \wedge_R M \wedge_R M) \vee \cdots = R \vee (\bigvee_i M^{\wedge_R i}).$$

This has the structure of an augmented (over R) Γ-ring, where the multiplication is obtained as follows: The multiplication by R uses the R-bimodule structure on $M^{\wedge_R^i}$, given by

$$R \wedge M^{\wedge_R^i} \to M^{\wedge_R^i}$$

or

$$M^{\wedge_R^i} \wedge R \to M^{\wedge_R^i}.$$

Multiplication on the factors involving powers of M is obtained using the quotient map from the smash product over $\mathbf{S}$ to the smash product over R:

$$M^{\wedge_R^i} \wedge M^{\wedge_R^j} \to M^{\wedge_R^i} \wedge_R M^{\wedge_R^j} \cong M^{\wedge_R^{i+j}}.$$

The augmentation $T_R(M) \to R$ comes from the quotient map obtained by collapsing all of the factors except R. This is clearly split by the inclusion into the first factor $R \to T_R(M)$. An important multiplicatively-closed subspace of $T_R(M)$ is the *augmentation ideal* I, which is the homotopy fiber

$$I \to T_R(M) \to R.$$

We can consider "powers" of the augmentation ideal, I^{n+1}, which are the homotopy fibers of the projection map onto the first n factors:

$$T_R(M) \to R \vee M \vee (M \wedge_R M) \vee \cdots \vee M^{\wedge_R n}.$$

There are two important things to note about this:

- If M is k-connected, then the quotient map $T_R(M) \to T_R(M)/I^{n+1}$ is (nk)-connected. This is evident from the fact that the lowest connectivity of a factor that has been collapsed is that of $M^{\wedge_R n}$, which is (nk)-connected by an application of the Künneth spectral sequence;
- There are inclusions $T_R(M)/I^{n+1} \to T_R(M)/I^{n+2}$ that are compatible with the quotient maps $T_R(M) \to I^{n+1}$, giving us a map from $T_R(M)$ into the limit system $\lim_n T_R(M)/I^{n+1}$ that is increasingly connected.

We also define the derived tensor algebra of M over R to be

$$\mathcal{T}_R(M) := R \vee M \vee (M \wedge_R^L M) \vee (M \wedge_R^L M \wedge_R^L M) \vee \cdots = R \vee (\bigvee_i M^{\wedge_R^L i}).$$

where $\wedge_R^L$ is the derived smash product of R-modules.We obtain this by replacing any factor of the smash product with something cofibrant, and can be made functorial using a functorial choice of cofibrant replacement.

The augmentation of these R-algebras allow us to consider their reduced (over R) K-theory, which we will denote using tildes as before.

4.2 Properties

We now prove some relevant properties of the tensor algebra functor, which are roughly the same as those we proved earlier for the square-zero extension functor.

Proposition 4.1. *The construction of the tensor algebras is natural in pairs. That is, let $S \mapsto (R^S, M^S)$ be an admissible cube of rings and bimodules. Then there is an induced cube of augmented Γ-rings*

$$S \mapsto T_{R^S}(M^S)$$

and therefore a cube of reduced K-theory spectra

$$S \mapsto \widetilde{K}(T_{R^S}(M^S)).$$

If M is a cofibrant R-bimodule and the cube in question is the resolution cube by $H\mathbb{Z}$-modules we have been working with, then this is a cube of derived tensor algebras

$$S \mapsto \mathcal{T}_{R^S}(M^S).$$

Proof. Suppose that we have an admissible pair of maps $f : R_1 \to R_2$ and $g : M_1 \to M_2$ in the cube we are considering. For each factor $(M_1)^{\wedge^n_{R_1}}$ in $T_{R_1}(M_1)$ we will write a map of Γ-spaces

$$\tilde{g}^n : (M_1)^{\wedge^n_{R_1}} \to (M_2)^{\wedge^n_{R_2}}.$$

The construction is as follows: we know that the map $g : M_1 \to M_2$ is a map of R_1-modules, where M_2 has the structure of an R_1-module via f:

$$R_1 \wedge M_2 \xrightarrow{f \wedge id} R_2 \wedge M_2 \to M_2.$$

As such, we get maps

$$g^n : (M_1)^{\wedge^n_{R_1}} \to (M_2)^{\wedge^n_{R_1}},$$

but because the R_1-module structure on M_2 is passing through the R_2-module structure, there is a map

$$q_n : (M_2)^{\wedge^n_{R_1}} \to (M_2)^{\wedge^n_{R_2}}$$

coming from coequalizing the "remaining action" of R_2. The composite map $q_n \circ g^n$ is our map $\tilde{g}^n$.

Using these we can write a map $F : T_{R_1}(M_1) \to T_{R_2}(M_2)$, which is, a priori, not a map at all.

$$\begin{array}{ccccccccc}
R_1 & \vee & M_1 & \vee & (M_1)^{\wedge^2_{R_1}} & \vee & (M_1)^{\wedge^3_{R_1}} & \vee & \dots \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \tilde{g}} & & \downarrow{\scriptstyle \tilde{g}^2} & & \downarrow{\scriptstyle \tilde{g}^3} & & \\
R_2 & \vee & M_2 & \vee & (M_2)^{\wedge^2_{R_2}} & \vee & (M_2)^{\wedge^3_{R_2}} & \vee & \dots
\end{array}$$

This defines a map of Γ-spaces, by virtue of the following simple argument, which just says that we can map between coproducts "factorwise" in a pointed category: Let $\mathcal{C}$ be a pointed category and let $C = \bigvee_i C_i$, $D = \bigvee_j D_j$ be two coproducts of elements of $\mathcal{C}$. Then if we have maps $F_i : C_i \to D_i$, this defines a map $F : C \to D$, using the universal property of C. That is, the inclusion $D_i \to D$ gives us maps

$$C_i \xrightarrow{F_i} D_i \to D.$$

These are compatible with the coproduct in C, as the following diagram commutes

$$\begin{tikzcd}
C_i \arrow[r, "F_i"] \arrow[d, "*"] & D_i \arrow[r] \arrow[d, "*"] & D \arrow[d, "="] \\
C_j \arrow[r, "F_j"] & D_j \arrow[r] & D,
\end{tikzcd}$$

and so we get a map $C \to D$.

Now, because of the simple structure of the multiplication in the tensor algebras, this is actually a map of Γ-rings. With the exception of the multiplication by the R_1 and R_2 factors, the multiplication in the tensor algebras is given by quotienting the smash product and juxtaposition of factors, which is easily seen to be compatible with the map we have defined. We only need to check that this is compatible with the multiplication that comes from R_1 and R_2. But this is just asking for the following diagrams to commute for all n:

$$\begin{tikzcd}
R_1 \wedge (M_1)^{\wedge^n_{R_1}} \arrow[r] \arrow[d, "f \wedge \tilde{g}^n"] & (M_1)^{\wedge^n_{R_1}} \arrow[d, "\tilde{g}^n"] \\
R_2 \wedge (M_2)^{\wedge^n_{R_2}} \arrow[r] & (M_2)^{\wedge^n_{R_2}}
\end{tikzcd}$$

$$\begin{tikzcd}
(M_1)^{\wedge^n_{R_1}} \wedge R_1 \arrow[r] \arrow[d, "\tilde{g}^n \wedge f"] & (M_1)^{\wedge^n_{R_1}} \arrow[d, "\tilde{g}^n"] \\
(M_2)^{\wedge^n_{R_2}} \wedge R_2 \arrow[r] & (M_2)^{\wedge^n_{R_2}}
\end{tikzcd}$$

a fact that follows from the map $M_1 \to M_2$ being a map of R_1-bimodules and the left/right action of R_1 on $(M_1)^{\wedge^n_{R_1}}$ being induced by the left/right action of R_1 on M_1.

Finally, the augmentations are induced by the projection maps onto the first factors:

$$\pi_1 : T_{R_1}(M_1) \to R_1 \qquad \pi_2 : T_{R_2}(M_2) \to R_2.$$

As only R_1 maps to R_2 in $T_{R_2}(M_2)$, it is clear that we have a commutative diagram of Γ-rings

$$\begin{tikzcd}
T_{R_1}(M_1) \arrow[r, "F"] \arrow[d, "\pi_1"] & T_{R_1}(M_1) \arrow[d, "\pi_2"] \\
R_1 \arrow[r, "f"] & R_2
\end{tikzcd}$$

this induces a commutative diagram

$$\begin{tikzcd}
K(T_{R_1}(M_1)) \arrow[r, "F_*"] \arrow[d, "(\pi_1)_*"] & K(T_{R_1}(M_1)) \arrow[d, "(\pi_2)_*"] \\
K(R_1) \arrow[r, "f_*"] & K(R_2)
\end{tikzcd}$$

and therefore an induced map on the homotopy fibers

$$\widetilde{K}(T_{R_1}(M_1)) \to \widetilde{K}(T_{R_2}(M_2)).$$

Because the functor $(H\mathbb{Z} \wedge R) \wedge_R (-)$ is a left Quillen functor (cf. [24]), it preserves cofibrant objects. However,

$$(H\mathbb{Z} \wedge R) \wedge_R M \simeq (H\mathbb{Z}) \wedge (R \wedge_R M) \simeq H\mathbb{Z} \wedge M,$$

because the R-module structure on $H\mathbb{Z} \wedge R$ is trivial on the $H\mathbb{Z}$ factor. As a result, if M is a cofibrant R-bimodule then we have that $H\mathbb{Z} \wedge M$ is a cofibrant $H\mathbb{Z} \wedge R$-bimodule.

This shows that if we are working with our $H\mathbb{Z}$-module resolution cube and M is a cofibrant R-module, then all of M^S are cofibrant R^S-modules, as all of the M^S are obtained by an iteration of this procedure. As a result, we have that the cube

$$S \mapsto T_{R^S}(M^S)$$

is in fact a cube of derived tensor algebras, as each smash product in each factor of each tensor algebra involves the cofibrant R^S-module M^S. □

Now, the following lets us resolve the tensor algebra $T_R(M)$ by using the resolutions for R and M separately:

Proposition 4.2. *Let $\mathcal{X} : S \mapsto (R^S, M^S)$ be the admissible (id)-Cartesian resolution cube of* Γ*-rings and bimodules by $H\mathbb{Z}$-algebras/modules. Then the cube of spectra*

$$S \mapsto T_{R^S}(M^S)$$

is id-Cartesian.

Proof. Throughout, we will be working with a k-dimensional subcube $\mathcal{X}' : S \mapsto (R^S, M^S)$ of $\mathcal{X}$ so in the following, S is a subset of $\mathcal{P}k$. By assumption, this subcube is k-Cartesian. We first show that the cubes $T_{R^S}(M^S)/I_S^{n+1}$ are (id)-Cartesian. Proposition 4.1 tells us that this cube makes sense (and is a cube of Γ-rings) because the maps in the cube $S \mapsto T_{R^S}(M^S)$ are maps of augmented R^S-algebras.

This cube looks like the finite wedge

$$S \mapsto R^S \vee (\bigvee_i^n (M^S)^{\wedge_{R^S} i}).$$

Because wedge products preserve Cartesian-ness, it suffices to prove that each cube $S \mapsto (M^S)^{\wedge_{R^S} n}$ is (id)-Cartesian.

Claim. The cube $S \mapsto (M^S)^{\wedge^n_{R^S}}$ is equivalent (as cube of Γ-spaces) to the cube

$$S \mapsto (\mathbf{S})^{\wedge^{n-|S|}} \wedge (H\mathbb{Z})^{\wedge^{|S|}} \wedge (M)^{\wedge^n_R} \cong (H\mathbb{Z})^{\wedge^{|S|}} \wedge (M)^{\wedge^n_R}$$

where the maps are given by repeated application of the unit $\eta : \mathbf{S} \to H\mathbb{Z}$.

Proof of Claim. To see how this goes, consider the first map in the cube, for the case $n = 2$:

$$M \wedge_R M \to (H\mathbb{Z} \wedge M) \wedge_{H\mathbb{Z} \wedge R} (H\mathbb{Z} \wedge M).$$

This is defined as the composite

$$M \wedge_R M \to (H\mathbb{Z} \wedge M) \wedge_R (H\mathbb{Z} \wedge M) \to (H\mathbb{Z} \wedge M) \wedge_{H\mathbb{Z} \wedge R} (H\mathbb{Z} \wedge M),$$

where the 2nd term is given an R-module structure via the map $R \simeq \mathbf{S} \wedge R \to H\mathbb{Z} \wedge R$.

Because this R-module structure is given on the $H\mathbb{Z}$ part by the composite $S \wedge H\mathbb{Z} \to H\mathbb{Z} \wedge H\mathbb{Z} \to H\mathbb{Z}$, we see that R is acting trivially on this factor. We therefore get an isomorphism of $(H\mathbb{Z} \wedge R)$-modules

$$(H\mathbb{Z} \wedge M) \wedge_R (H\mathbb{Z} \wedge M) \cong (H\mathbb{Z} \wedge H\mathbb{Z}) \wedge (M \wedge_R M),$$

where the $(H\mathbb{Z} \wedge R)$-module structure on the latter is diagonal. As a result, we can write the map

$$(H\mathbb{Z} \wedge M) \wedge_R (H\mathbb{Z} \wedge M) \to (H\mathbb{Z} \wedge M) \wedge_{H\mathbb{Z} \wedge R} (H\mathbb{Z} \wedge M)$$

as

$$(H\mathbb{Z} \wedge H\mathbb{Z}) \wedge (M \wedge_R M) \xrightarrow{q \wedge id} (H\mathbb{Z} \wedge_{H\mathbb{Z}} H\mathbb{Z}) \wedge (M \wedge_R M) \xrightarrow{\mu \wedge id,\, \simeq} H\mathbb{Z} \wedge (M \wedge_R M),$$

where $q : H\mathbb{Z} \wedge H\mathbb{Z} \to H\mathbb{Z} \wedge_{H\mathbb{Z}} H\mathbb{Z}$ is the expected quotient map and μ is the isomorphism $H\mathbb{Z} \wedge_{H\mathbb{Z}} H\mathbb{Z} \simeq H\mathbb{Z}$ given by multiplication.

We can therefore identify the map

$$M \wedge_R M \to (H\mathbb{Z} \wedge M) \wedge_R (H\mathbb{Z} \wedge M)$$

as

$$M \wedge_R M \simeq (\mathbf{S} \wedge \mathbf{S}) \wedge M \wedge_R M \xrightarrow{\eta \wedge \eta \wedge id} (H\mathbb{Z} \wedge H\mathbb{Z}) \wedge (M \wedge_R M).$$

and the composite map

$$M \wedge_R M \to (H\mathbb{Z} \wedge M) \wedge_{H\mathbb{Z} \wedge R} (H\mathbb{Z} \wedge M)$$

as

$$M \wedge_R M \cong \mathbf{S} \wedge (M \wedge_R M) \cong \mathbf{S} \wedge \mathbf{S} \wedge (M \wedge_R M) \xrightarrow{\mu \circ ((q \circ \eta)^2) \wedge id} H\mathbb{Z} \wedge (M \wedge_R M).$$

The composite map on the left factor from the 2nd term to the last term, $\mathbf{S} \to H\mathbb{Z}$, is the unit map, so we finally see that the map in question is given by

$$\mathbf{S} \wedge (M \wedge_R M) \xrightarrow{\eta \wedge id} H\mathbb{Z} \wedge (M \wedge_R M).$$

The case for $n \neq 2$ is the same, mutatis mutandis, telling us that the map

$$M^{\wedge_R^n} \to (H\mathbb{Z} \wedge M)^{\wedge^n_{H\mathbb{Z}\wedge R}}$$

is really

$$M^{\wedge_R^n} \xrightarrow{\eta \wedge id} H\mathbb{Z} \wedge M^{\wedge_R^n}.$$

The claim follows from inductively applying this fact to every node in the cube, starting with the initial vertex. $\square$

Now, using Lemma 3.8, we see that the cube $S \mapsto (M^S)^{\wedge_{R^S} n}$ is k-Cartesian for all n, as it is the smash product of the cubes

$$S \mapsto H\mathbb{Z}^{\wedge^{|S|}}$$

and the constant cube

$$S \mapsto (M)^{\wedge_R^n}.$$

The first is our resolution cube of $\mathbf{S}$ by $H\mathbb{Z}$-algebras and is therefore (id)-Cartesian, and constant cubes are as Cartesian as we like.

This gives us that $S \mapsto T_{R^S}(M^S)/I^{n+1}$ is (id)-Cartesian for all n. That is, we have k-connected maps

$$T_{R^\varnothing}(M^\varnothing)/I^{n+1} \to \operatorname*{holim}_{S \in \mathcal{P}k \smallsetminus \varnothing} T_{R^S}(M^S)/I^{n+1}.$$

These assemble to a k-connected map

$$\operatorname*{hocolim}_{n} T_{R^\varnothing}(M^\varnothing)/I^{n+1} \to \operatorname*{hocolim}_{n} \operatorname*{holim}_{S \in \mathcal{P}k \smallsetminus \varnothing} T_{R^S}(M^S)/I^{n+1}.$$

But the map from the cube $S \mapsto T_{R^S}(M^S)$ into the colimit system of cubes

$$\cdots \to [S \mapsto T_{R^S}(M^S)/I^{n+1}] \to [S \mapsto T_{R^S}(M^S)/I^{n+2}] \to \cdots$$

is (pointwise) as connected as we like. In fact, the connectivity of these cubes increases uniformly, as if M is l-connected then $M^{\wedge_R n}$ is (nl)-connected, and therefore so is $(M^S)^{\wedge^n_{R^S}}$ which is the least connected part of the augmentation ideal I_S^{n+1}. As a result, all of the quotient maps $T_{R^S}^{M^S} \to T_{R^S}(M^S)/I_S^{n+1}$ are (nl)-connected, meaning that the map $T_{R^S}(M^S) \to \operatorname{hocolim}_n T_{R^S}(M^S)/I_S^{n+1}$ is an equivalence for all S. This tells us that $T_{R^\varnothing}(M^\varnothing) \xrightarrow{\simeq} \operatorname{hocolim}_n T_{R^\varnothing}(M^\varnothing)/I^{n+1}$, and so we get natural maps:

$$\begin{tikzcd}
\operatorname{hocolim}_n T_{R^\varnothing}(M^\varnothing)/I^{n+1} \arrow[r, "\simeq_{\leq k}"] & \operatorname{hocolim}_n \operatorname{holim}_{S\in\mathcal{P}k\smallsetminus\varnothing} T_{R^S}(M^S)/I^{n+1} \\
T_{R^\varnothing}(M^\varnothing) \arrow[u, "\simeq"] &
\end{tikzcd},$$

We would like to naturally identify the rightmost term as $\operatorname{holim}_{S\in\mathcal{P}k\smallsetminus\varnothing} T_{R^S}(M^S)$. The following lemma allows us to do so:

Lemma 4.3 ([9, A.7.2.6])**.** *Let I, J be small categories where BJ is a finite space (has only finitely many nondegenerate simplices) and let X be a functor $I\times J$ to spectra. Then the canonical maps*

$$\operatorname*{hocolim}_I \operatorname*{holim}_J \to \operatorname*{holim}_J \operatorname*{hocolim}_I$$

$$\operatorname*{hocolim}_J \operatorname*{holim}_J \to \operatorname*{holim}_I \operatorname*{hocolim}_J$$

are equivalences.

Passing to the associated spectra and letting $I = \mathbb{N}, J = (\mathcal{P}k \smallsetminus \varnothing)$, we can define a functor $X : \mathbb{N} \times (\mathcal{P}k \smallsetminus \varnothing) \to Sp$ by

$$(n, S) \mapsto T_{R^S}(M^S)/I_S^{n+1}.$$

The above lemma tells us that we have equivalences

$$\operatorname*{hocolim}_n \operatorname*{holim}_{S\in\mathcal{P}k\smallsetminus\varnothing} T_{R^S}(M^S)/I_S^{n+1} \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}k\smallsetminus\varnothing} \operatorname*{hocolim}_n T_{R^S}(M^S)/I_S^{n+1} \xleftarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}k\smallsetminus\varnothing} T_{R^S}(M^S).$$

Altogether, we get a commutative diagram

$$\begin{tikzcd}
\operatorname{hocolim}_n T_{R^\varnothing}(M^\varnothing)/I^{n+1} \arrow[r, "\simeq_{\leq k}"] & \operatorname{hocolim}_n \operatorname{holim}_{S\in\mathcal{P}k\smallsetminus\varnothing} T_{R^S}(M^S)/I^{n+1} \arrow[d, "\simeq"] \\
 & \operatorname{holim}_{S\in\mathcal{P}k\smallsetminus\varnothing} \operatorname{hocolim}_n T_{R^S}(M^S)/I_S^{n+1} \\
T_{R^\varnothing}(M^\varnothing) \arrow[uu, "\simeq"] \arrow[r] & \operatorname{holim}_{S\in\mathcal{P}k\smallsetminus\varnothing} T_{R^S}(M^S) \arrow[u, "\simeq"']
\end{tikzcd}$$

which implies that the bottom horizontal map is k-connected, telling us that the subcube we started with is k-Cartesian. □

As with the square-zero extension, this implies the cubical resolvability of the K-theory of the tensor algebra:

Corollary 4.4. *$\tilde{K}(T_R(M))$ is cubically resolvable: there is a natural equivalence*

$$\tilde{K}(T_R(M)) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} \tilde{K}(T_{R^S}(M^S)).$$

Proposition 4.5. *The simplicial rings associated to the noninitial vertices in the cubes*

$$S \mapsto T_{R^S}(M^S) \qquad S \mapsto \mathcal{T}_{R^S}(M^S)$$

are tensor algebras of the same form.

Proof. The left adjoint L of the $H\mathbb{Z}$-algebra/simplicial ring equivalence is strong symmetric monoidal and preserves arbitrary coproducts of modules, so as a simplicial abelian group, $L(T_R^S(M^S))$ is a tensor algebra (L takes $\vee$ to $\oplus$ and $\wedge_{R^S}$ to $\otimes_{L(R^S)}$). The algebra structure is induced from applying L to the structure maps of $T_{R^S}(M^S)$, and because of the particularly simple form of the tensor algebra multiplication (it is the identity on each factor of a tensor product $M^{\wedge_R n}$), this is preserved as well.

To check the statement about the derived tensor algebra, we just need that L preserves cofibrant objects, which it does because it is the left Quillen functor in a Quillen equivalence. This is sufficient, as we obtain the derived tensor algebra $\mathcal{T}_{R^S}(M^S)$ by cofibrantly replacing any of the factors in the various smash products $(M^S)^{\wedge^n_{R^S}}$. This factor gets mapped via L to $(L(M^S))^{\otimes^n_{L(R^S)}}$, and if any of the factors in this tensor product are cofibrant (flat), this is a derived tensor product. $\square$

As with the square-zero extension, K-theory of a simplicial ring tensor algebra is computable levelwise:

Lemma 4.6. *Let R be a simplicial ring and let M be an R-bimodule. Then we can compute the reduced K-theory $\widetilde{K}(T_R(M))$ and $\widetilde{K}(\mathcal{T}_R(M))$ levelwise. That is, we have a natural zig-zag of weak equivalences*

$$\widetilde{K}(T_R(M)) \simeq d(\tilde{K}(T_{R_q}(M_q))) \qquad \widetilde{K}(\mathcal{T}_R(M)) \simeq d(\widetilde{K}(\mathcal{T}_{R_q}(M_q))).$$

On the level of spectra, we have natural zig-zags of equivalences

$$\widetilde{K}(T_R(M)) \simeq |\widetilde{K}(T_{R_q}(M_q))| \qquad \widetilde{K}(\mathcal{T}_R(M)) \simeq |\widetilde{K}(\mathcal{T}_{R_q}(M_q))|.$$

Proof. Let I be the augmentation ideal $T_R(M) \to R$. Then the quotient map

$$T_R(M) \to T_R(M)/I^{n+1}$$

is n-connected, and so the map

$$\widetilde{K}(T_R(M)) \to \widetilde{K}(T_R(M)/I^{n+1})$$

is $(n+1)$-connected. Clearly $T_R(M)/I^{n+1}$ is a nilpotent extension of R, so by Theorem 2.12 there is a natural zig-zag of weak equivalences of bisimplicial sets

$$\widetilde{K}(T_R(M)/I^{n+1}) \simeq \widetilde{K}(T_{R_\bullet}(M_\bullet)/I_\bullet^{n+1}),$$

where we have made the straightforward identification $T_R(M)_n \simeq T_{R_n}(M_n)$. As this equivalence is natural, it is compatible with the systems

$$\cdots \to T_R(M)/I^{n+1} \to T_R(M)/I^{n+2} \to \cdots$$

$$\cdots \to T_{R_i}(M_i)/I^{n+1} \to T_{R_i}(M_i)/I^{n+2} \to \cdots$$

We therefore get an equivalence of homotopy colimits of bisimplicial sets (taken in the injective model structure, so that the homotopy colimits produce levelwise homotopy colimits)

$$\operatorname*{hocolim}_n \widetilde{K}(T_R(M)/I^{n+1}) \simeq \operatorname*{hocolim}_n \widetilde{K}(T_{R_\bullet}(M_\bullet)/I_\bullet^{n+1}).$$

The left hand side is computing the homotopy colimit in simplicial sets. We have that the map $T_R(M) \to T_R(M)/I^{n+1}$ can be made as connected as we like, as the kernel I^{n+1} is $((n+1)(conn(M))$-connected. Because K-theory preserves connectivity, the quotient maps induce a weak equivalence

$$\widetilde{K}(T_R(M)) \xrightarrow{\simeq} \operatorname*{hocolim}_n \widetilde{K}(T_R(M)/I^{n+1})$$

And therefore we get a zig-zag of weak equivalences

$$\widetilde{K}(T_R(M)) \simeq \operatorname*{hocolim}_n \widetilde{K}(T_{R_\bullet}(M_\bullet)/I_\bullet^{n+1}).$$

However, the right hand side homotopy colimit computes levelwise homotopy colimits in our chosen model structure, and each level

$$\operatorname*{hocolim}_n \widetilde{K}(T_{R_i}(M_i)/I_i^{n+1})$$

can be identified with $\widetilde{K}(T_{R_i}(M_i))$ in the same way we just identified $\widetilde{K}(T_R(M))$ with the homotopy colimit of $\widetilde{K}(T_R(M)/I^{n+1})$.

Altogether, this gives us a zig-zag of equivalences of bisimplicial sets

$$\widetilde{K}(T_R(M)) \simeq \widetilde{K}(T_{R_\bullet}(M_\bullet)).$$

Taking diagonals gives the first result.

The above proof also produces the result for the derived tensor algebra, given the identification of the levels

$$\mathcal{T}_R(M)_n \simeq \mathcal{T}_{R_n}(M_n).$$

This follows from the fact that a cofibrant module M over a simplicial ring R is levelwise flat. That is, M_n is a flat module over R_n, and the smash product of modules over simplicial rings is performed levelwise. Therefore, if any of the tensor factors in the various tensor product factors factors $M^{\otimes_R^k}$ are cofibrant (which would happen if we were considering the derived tensor product), then the modules $M_n^{\otimes_{R_n}^k}$ also represent derived tensor products. $\square$

4.3 K-theory of the Tensor Algebra

We now compute the reduced K-theory of the tensor algebra in terms of topological Witt vectors or square-zero extensions. As described in the introduction, this can be thought of as a first approximation to the derived functor of $\widetilde{K}(-)$ on the category of augmented R-algebras. Recall the result of Lindenstrauss and McCarthy from [18]:

Theorem 4.7 ([18, Corollary 3.3]). *If R is a unital ring, there is a natural zig-zag of equivalences of functors of connected simplicial R-bimodules*

$$\Sigma\widetilde{K}(R;-) \simeq \widetilde{K}(\mathcal{T}_R(-)).$$

4.3.1 Construction of The Map

Again, we are left with the problem of defining a map between our objects of interest that suitably restricts to the Lindenstrauss-McCarthy map. Betley and Schlichtkrull [3] have done the job of defining the map for us:

Construction 4.8. Let $\mathbb{Z}_n$ denote the truncated polynomial ring $\mathbb{Z}[x]/x^{n+1}$. We can consider the units

$$u_n = 1 + x + x^2 + \cdots + x^n.$$

Units in $\mathbb{Z}_n$ can be lifted to classes in $\widetilde{K}_1(\mathbb{Z}_n)$ (using the diagonal matrices that correspond to multiplication by units).

Proposition 4.9 ([3, Lemma 3.9])**.** *The classes u_n lift to a stable class in $K_1(\mathbb{Z}[x])$.*

In the same way that tensoring rings with $\mathbb{Z}[x]$ produces polynomial rings, there is a spectral version of this where we instead use the "spherical monoids" $\mathbb{S}[\Pi_n]$, where Π_n is the set of monomials in $\mathbb{Z}[x]/x^{n+1}$. These have the property that $\pi_0\mathbb{S}[\Pi_n] \simeq \mathbb{Z}[x]/x^{n+1}$ and for any Γ-ring R there is a map

$$\mathbb{S}[\Pi_n] \wedge R \to T_R(R)/I^{n+1}.$$

Because $K_1(\mathbb{S}[\Pi_n]) \simeq K_1(\mathbb{Z}[x]/x^{n+1})$ (compatibly with the reductions $\mathbb{S}[\Pi_n] \to \mathbb{S}$ and $\mathbb{Z}[x]/x^{n+1} \to \mathbb{Z}$), this gives us a lift of u_n to $\widetilde{K}_1(\mathbb{S}[\Pi_n])$. The pairing on K-theory gives us a map

$$\widetilde{K}(\mathbb{S}[\Pi_n]) \wedge K(R) \xrightarrow{\simeq} \widetilde{K}(T_R(R)/I^{n+1}),$$

and multiplying by the lift of u_n to $\widetilde{K}_1(\mathbb{S}[\Pi_n])$ gives us a map

$$U_n : \Sigma K(R) \to \widetilde{K}(T_R(R)/I^{n+1}).$$

We can now define a map τ as the zig-zag induced by the limit of the maps τ_n

$$\begin{tikzcd}
\Sigma\widetilde{K}(R \ltimes M) \arrow[r, "\tau_n"] \arrow[d] & hofib \arrow[d] \\
\Sigma K(R \ltimes M) \arrow[r, "U_n"] \arrow[d] & \widetilde{K}(T_{R\ltimes M}(R \ltimes M)/I^{n+1}) \arrow[d] \\
\Sigma K(R) \arrow[r, "U_n"] & \widetilde{K}(T_R(R)/I^{n+1})
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma\widetilde{K}(R \ltimes M) \arrow[r, "\mathrm{holim}_n\, \tau_n"] \arrow[d] \arrow[rr, bend left, leftrightarrow, "\tau"] & \mathrm{holim}\,\mathrm{hofib} \arrow[d] & \mathrm{hofib} \arrow[l, "\simeq"'] \arrow[d] \\
\Sigma K(R \ltimes M) \arrow[r, "\mathrm{holim}_n\, U_n"] \arrow[d] & \mathrm{holim}_n\, \widetilde{K}(T_{R\ltimes M}(R \ltimes M)/I^{n+1}) \arrow[d] & \widetilde{K}(\mathcal{T}_{R\ltimes M}(R \ltimes M)) \arrow[l, "\simeq"'] \arrow[d] \\
\Sigma K(R) \arrow[r, "\mathrm{holim}_n\, U_n"] & \mathrm{holim}_n\, \widetilde{K}(T_R(R)/I^{n+1}) & \widetilde{K}(\mathcal{T}_R(R)) \arrow[l, "\simeq"']
\end{tikzcd}$$

As of the present writing, the proof of the following proposition is incomplete, but work is in progress to complete it. Contemplation of the map defined in [18] suggests that this is possible.

Conjecture 4.10. There is a weak map

$$c : \widetilde{K}(\mathcal{T}_R(\Sigma M)) \leftrightarrow \mathrm{hofib}$$

defined for all Γ-rings R and R-bimodules M. When applied to a discrete ring R and a flat R-bimodule M, this map makes the following diagram commute up to homotopy

$$\begin{tikzcd}
\Sigma\widetilde{K}(R;B_\bullet M) \arrow[r, "\simeq", "LMC"'] & \Sigma W(R;B_\bullet M) \arrow[r, "\simeq", "LMC"'] & \widetilde{K}(T_R(B_\bullet M)) \arrow[d, leftrightarrow, "c"] \\
\Sigma\widetilde{K}(R \ltimes M) \arrow[u, "\simeq", "DMC"'] \arrow[r, "\operatorname{holim}_n \tau_n"] \arrow[d] \arrow[rr, bend left, leftrightarrow, "\tau"] & \operatorname{holim} \operatorname{hofib} \arrow[d] & \operatorname{hofib} \arrow[l, "\simeq"] \arrow[d] \\
\Sigma K(R \ltimes M) \arrow[r, "\operatorname{holim}_n U_n"] \arrow[d] & \operatorname{holim}_n \widetilde{K}(T_{R\ltimes M}(R \ltimes M)/I^{n+1}) \arrow[d] & \widetilde{K}(T_{R\ltimes M}(R \ltimes M)) \arrow[l, "\simeq"] \arrow[d] \\
\Sigma K(R) \arrow[r, "\operatorname{holim}_n U_n"] & \operatorname{holim}_n \widetilde{K}(T_R(R)/I^{n+1}) & \widetilde{K}(T_R(R)) \arrow[l, "\simeq"]
\end{tikzcd}$$

4.3.2 Proof of Main Chapter Theorem

Using the earlier structural results and the conjectured compatibility with a global map, we can prove the main theorem of this chapter:

Theorem 4.11. *Let R be a Γ-ring and let M be a cofibrant R-bimodule. Then there is an equivalence*

$$\Sigma\widetilde{K}(R \vee M) \simeq \widetilde{K}(T_R(\Sigma M)).$$

Proof. As before, we use the cubical resolvability of both functors:

$$\tilde{K}(T_R(\Sigma M)) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} \tilde{K}(T_{R^S}(\Sigma M^S)) \qquad \Sigma\tilde{K}(R \vee M) \xrightarrow{\simeq} \operatorname*{holim}_{S\in\mathcal{P}\smallsetminus\varnothing} \Sigma\tilde{K}(R^S \vee M^S)$$

Our map τ produces a map of the cubes these homotopy limits are being computed over, and so it suffices to show that τ induces a pointwise equivalence

$$\tilde{K}(T_{R^S}(\Sigma M^S)) \simeq \Sigma\tilde{K}(R^S \vee M^S).$$

For $S \neq \varnothing$, $R^S \vee M^S$ and $T_{R^S}(M^S)$ are equivalent to simplicial rings, and Proposition 2.6 and Proposition 4.5 together tell us that these simplicial rings are square-zero extensions and tensor algebras, respectively.

Dropping the S, it therefore suffices to show that τ induces an equivalence

$$\tilde{K}(T_R(B_\bullet M)) \simeq \Sigma \tilde{K}(R \vee M)$$

when R, M are discrete.

Conjecture 4.10 tells us that τ and the Lindenstrauss-McCarthy map are homotopic on discrete rings. The latter is an equivalence, and so we are done. $\square$

Theorem 4.11 implies our penultimate theorem:

Theorem 4.12. *Let R be a Γ-ring, and let C denote the cofibrant replacement functor in the category of R-bimodules. Then there is a zig-zag of natural equivalences of functors from R-bimodules to spectra*

$$\Sigma \widetilde{K}(R \vee (C \circ -)) \simeq \widetilde{K}(\mathcal{T}_R \Sigma (C \circ -)).$$

As a result, these functors have the same Taylor tower.

Given Corollary 3.30, we can also write this as

$$\Sigma W(R; C \circ -) \simeq \widetilde{K}(\mathcal{T}_R(C \circ -)).$$

Finally, we can relate this to computations that appear involving Waldhausen's $A(X)$ functor. We briefly recall the relevant definitions (cf. [7] or [27]):

Definition 4.2. Let X be a connected, based space with the homotopy type of a CW complex. Then we can define:

$$Q(\Omega X_+) := \lim_k \Omega^k \Sigma^k (\Omega X_+),$$

which is the infinite loop space of a connective ring spectrum, the ring structure coming from the product on ΩX_+. We can therefore define:

$$A(X) := K(Q(\Omega X_+)).$$

As any space has a unique map to $*$, we can take a reduced version:

$$\widetilde{A}(X) := \operatorname{hofib}(A(X) \to A(*)).$$

In [7], Carlsson et al. show that there is an equivalence

$$\widetilde{A}(\Sigma X) \xleftarrow{\simeq} \Sigma \prod_{n \geq 1} Q(X^{\wedge n}_{hC_n}).$$

One the level of spectra, we could write this as

$$\widetilde{A}(\Sigma X) \xleftarrow{\simeq} \Sigma \bigvee_{n=1}^{\infty} [(\mathbf{S}X)^n]_{hC_n}.$$

Moreover, if X is connected, the James splitting tells us that we have an equivalence

$$Q(\Omega\Sigma X_+) \simeq T_{\mathbf{S}}\mathbf{S}X,$$

that is compatible with the map $\Sigma X \to *$, and therefore an equivalence

$$\widetilde{A}(\Sigma X) = \widetilde{K}(Q(\Omega\Sigma X_+)) \simeq \widetilde{K}(T_{\mathbf{S}}\mathbf{S}X).$$

On the other hand, tom Dieck splitting actually lets us compute:

$$\Sigma W(\mathbf{S}; \mathbf{S}X) \simeq \Sigma \bigvee_{n=1}^{\infty} [(\mathbf{S}X)^n]_{hC_n}.$$

Putting these facts together we see that there is a string of equivalences

$$\Sigma W(\mathbf{S}; \mathbf{S}X) \simeq \Sigma \bigvee_{n=1}^{\infty} [(\mathbf{S}X)^n]_{hC_n} \simeq \widetilde{A}(\Sigma X) \simeq \widetilde{K}(T_{\mathbf{S}}\mathbf{S}X)$$

Our final result is proof of this fact without appealing to any of these computations:

Theorem 4.13. *Let X be a connected simplicial set. Then there is an equivalence*

$$\Sigma W(\mathbf{S}; \mathbf{S}X) \simeq \widetilde{K}(T_{\mathbf{S}}\mathbf{S}X).$$

Proof. Because X is connected, $\mathbf{S}X$ is connected, and in fact it is a cofibrant $\mathbf{S}$-module (cf. [18]). As such, we can apply Theorem 4.11 to give us an equivalence

$$\widetilde{K}(T_{\mathbf{S}}\mathbf{S}X) \simeq \Sigma\widetilde{K}(\mathbf{S} \vee \Omega\mathbf{S}X). \tag{4.1}$$

On the other hand, Corollary 3.30 gives us an equivalence

$$\Sigma W(\mathbf{S};\mathbf{S}X) \simeq \Sigma\widetilde{K}(\mathbf{S} \vee \Omega\mathbf{S}X),$$

which can be strung together with Eq. (4.1) to produce the stated result. $\square$

Appendices

Appendix A

Γ-spaces

Γ-spaces were introduced by Segal in [26] in order to try and abstract the structure present in topological abelian monoids that allows us to define classical cohomology theories. Γ-spaces provide a diagrammatic description of homotopy coherent commutative topological monoids, and are an example of an "infinite loop space machine". This machine allows us to extract interesting cohomology theories from homotopy coherent commutative topological monoids (as opposed to the sums of classical cohomology theories that we obtain from requiring the on-the-nose commutativity of topological abelian monoids).

In modern language, Γ-spaces provide a model for connective spectra, in that the category of Γ-spaces admits a model structure for which there is a zig-zag of Quillen equivalences (of varying monoidal-ness) to any other reasonable model of connective spectra (EKMM, symmetric spectra, Lewis-May, etc.). In the following sections we will provide the relevant definitions, descriptions of the model structure, and show how arbitrary ring spectra (Γ-rings) can be resolved by Eilenberg-MacLane Γ-rings that come from simplicial rings.

The reason that we will need to use Γ rings (as opposed to consistently using some other, more modern category of spectra) is that we obtain good point-set models for simplicial rings associated to $H\mathbb{Z}$-algebras in the category Γ-rings, allowing us to show that, say, square-zero extensions of $H\mathbb{Z}$-algebras are taken to square-zero extensions of simplicial rings.

In this section, we provide some background material related to Γ-spaces, which we include for reference.

A.1 Basic Definitions and Closed Structure

Definition A.1. Let Γ be the category of finite sets with objects given by

$$ob(\Gamma) = \{[n] = \{0, 1, \ldots, n\} \quad n \geq 0\}$$

and morphisms given by

$$\Gamma([m], [n]) = \{\text{set maps } [m] \to [n] \text{ that preserve } 0\}.$$

Let C be a pointed category with zero object $*$. Then a Γ-*object in* C is a functor from the opposite category Γ^{op} to C that takes $[0]$ to $*$. We denote the category of Γ-objects in C by ΓC.

The most important example of this is the category ΓS_* of Γ-objects in the category of pointed simplicial sets, which we refer to as Γ-*spaces*.

Example. The sphere Γ-space $\mathbf{S}$ is given by the inclusion of Γ^{op} into the category of pointed simplicial sets. More precisely,

$$\mathbf{S}([n]) = [n]$$

where $[n]$ is treated as a discrete pointed simplicial set.

The spectrum associated to $\mathbf{S}$ is the sphere spectrum, as one would hope.

We also have a smash product on the category of Γ-spaces, which we now describe:

Definition A.2. Given two Γ-spaces X, Y, the *external smash product* $X \bar{\wedge} Y$ is the bi-Γ-space

$$(X \bar{\wedge} Y)(k, l) = X(k) \wedge Y(l).$$

We can then define the *smash product* of X, Y as the left Kan extension along the smash product functor on Γ^{op} :

$$(X \wedge Y)(n) = \operatorname*{colim}_{k \wedge l \to n} X(k) \wedge Y(l).$$

This has the desired properties:

Theorem A.1. *[20] The smash product of Γ-spaces is associative and commutative with unit* $\mathbf{S}$*, and moreover there is an isomorphism of Γ-spaces*

$$\mathrm{Hom}(X \wedge Y, Z) \simeq \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)),$$

making ΓS into a closed symmetric monoidal category.

The central objects of our study will be the monoids in the category of Γ-spaces, which model connective ring spectra:

Definition A.3. A Γ-*ring* is a monoid in the symmetric monoidal category $(\Gamma S, \wedge, \mathbf{S})$. More precisely, a Γ-ring is a Γ-space A equipped with maps

$$\mathbf{S} \to A \quad \text{and} \quad A \wedge A \to A$$

(the unit and multiplication maps, respectively) that satisfy the evident associativity and unit diagrams. A map of Γ-rings is a map of Γ-spaces that commutes with the respective unit and multiplication maps

As with rings, we can consider modules and algebras over Γ-rings:

Definition A.4. Let R be a Γ-ring. Then a *left R-module* is a Γ-space M with a map

$$R \wedge N \to N$$

that make the appropriate associativity and unit diagrams commute. We define *right R-modules* similarly, and *R-bimodules* by requiring both left and right R-module structures that commute.

If R is commutative, then all of these categories are equivalent and the category of R-modules obtains the structure of a symmetric monoidal category with the product given by taking the smash product over R, $\wedge_R$. In this situation, we can define an R-algebra to be a monoid in this symmetric monoidal category.

Construction A.2. Let $\mathbb{Z}$ be the *free simplicial abelian group functor*, which takes a pointed simplicial set X to the simplicial set given in degree k as the free abelian group on the k-simplices of X:

$$\mathbb{Z}(X)_k = \mathbb{Z}[X_k].$$

The image of the basepoint of X under $\mathbb{Z}$ produces a sub-simplicial set $\mathbb{Z}(*)$, and we define the *reduced free simplicial abelian group functor* $\widetilde{\mathbb{Z}}$ by

$$\widetilde{\mathbb{Z}}(X) = \mathbb{Z}(X)/\mathbb{Z}(*).$$

Now let $R \in \mathcal{A}b^{\Delta}$ be a simplicial abelian group. Then we can form an element of $\Gamma\mathcal{A}b^{\Delta}$, $\bar{H}R$, by

$$\bar{H}R(k_+) := R \otimes \widetilde{\mathbb{Z}}[k] \simeq R^{\times k},$$

where $\widetilde{\mathbb{Z}}$ is the reduced free simplicial abelian group functor. If we follow this with the forgetful functor from simplicial abelian groups to simplicial sets $U : \mathcal{A}b^{\Delta} \to S_*$ then we obtain a composite functor

$$H := U\bar{H}.$$

We refer to either of these as the *Eilenberg-MacLane objects* associated to R.

As in Lemma A.4, the functor H is lax symmetric monoidal, and so we get the following: If R is a simplicial ring and M is an R-module, then HM is a module for the Γ-ring HR. If in addition R is commutative and M is an R-algebra, then HM is an algebra over HR. In particular, if R is a simplicial ring then HR is both a Γ-ring and an $H\mathbb{Z}$-algebra.

The following construction allows us to produce a simplicial abelian group from an arbitrary Γ-space:

Construction A.3. Let X be a Γ-space, and define $L(X)$ be the cokernel of the map of simplicial abelian groups

$$(p_1)_* + (p_2)_* - \Delta_* : \widetilde{\mathbb{Z}}[X(2_+)] \to \widetilde{\mathbb{Z}}[X(1_+)],$$

where p_1, p_2 are the two projections from 2_+ to 1_+ in Γ and Δ is the map 2_+ to 1_+ such that $\Delta(1) = \Delta(2) = 1$.

If X is a Γ-ring, then $L(X)$ inherits a multiplicative structure that makes it into a simplicial ring from the following more general construction: let X, Y be Γ-spaces, then the universal property of the smash product provides a map

$$X([1]) \wedge Y([1]) \to (X \wedge Y)([1] \wedge [1]) = (X \wedge Y)([1]).$$

Applying $\widetilde{\mathbb{Z}}$ to this and taking cokernels to obtain L gives us an associative, commutative and unital natural transformation

$$L(X) \otimes L(Y) \to L(X \wedge Y).$$

If $X = Y$ and X is a Γ-ring, then this produces a map

$$L(X) \otimes L(X) \to L(X \wedge X)$$

and applying L to the map $X \wedge X \to X$ that defines the Γ-ring structure, we get a map

$$L(X) \otimes L(X) \to L(X)$$

making $L(X)$ into a simplicial ring.

As mentioned earlier, Schwede gives us some nice facts about the functors L and H:

Lemma A.4 ([24, Lemma 1.2])**.** *H is a lax symmetric monoidal functor, and L*

1. *is left adjoint and left inverse to H;*
2. *preserves finite products;*
3. *preserves arbitrary coproducts;*
4. *is strong symmetric monoidal.*

A.2 Model Structures

There are several different model structures (see e.g. [23] for information about model categories) on the category of Γ-spaces, but we will be using the "stable model structure" that provides a nice Quillen equivalence between simplicial rings and $H\mathbb{Z}$-algebras (as modeled in Γ-spaces). We first need to describe the "strict model structure":

Definition A.5. A map of $f : A \to B$ of Γ-spaces is a:

- *strict Q-fibration* if the induced map of simplicial sets $A([n]) \to B([n])$ is a Kan fibration for all n;

- *strict Q-equivalence* if the induced map of simplicial sets $A([n]) \to B([n])$ is a weak equivalence for all n ;
- *strict Q-cofibration* if f satisfies the left lifting property with respect to all acyclic strict Q-fibrations.

This produces a model structure for us:

Theorem A.5. *[24, Thm. 1.4] The strict Q-fibrations, Q-equivalences, and Q-cofibrations endow the category of* Γ*-spaces with the structure of a closed simplicial model category.*

Being concerned with stable phenomena, we are more interested in a slightly different model structure that is obtained from the strict model structure by localizing at the stable equivalences:

Definition A.6. A map of $f : A \to B$ of Γ-spaces is a:

- *stable Q-equivalence* if f induces an isomorphism of homotopy groups;
- *stable Q-cofibration* if f is a strict Q-cofibration;
- *stable Q-fibration* if it has the right lifting property with respect to all acyclic stable Q-cofibrations.

Again, we get a model structure:

Theorem A.6. *[24, Thm. 1.5] The stable Q-fibrations, Q-equivalences, and Q-cofibrations endow the category of* Γ*-spaces with the structure of a cofibrantly-generated closed simplicial model category.*

A Γ*-space A is fibrant in this model structure if and only if:*

1. *A is very-special;*

2. *for all n, $A([n])$ is a fibrant simplicial set (Kan complex).*

We can use this model structure to produce model structures on categories of modules or algebras over a Γ-ring by forgetting to Γ–spaces:

Definition A.7. Let R be a Γ-ring, and let M be any of the following categories: left or right R-modules, R-bimodules, or R-algebras. Then we say that a map in M is a:

- *weak equivalence* if it is a stable Q-equivalence of Γ-spaces;
- *fibration* if it is a stable Q-fibration of Γ-spaces;
- *cofibration* if it has the left lifting property with respect to all acyclic fibrations in M.

Theorem A.7. *[24, Thm. 2.2, Thm. 2.5] The above weak equivalences, fibrations, and cofibrations endow M with the structure of a closed simplicial model category. The categories of (bi)modules are cofibrantly generated, and cofibrant R-algebras or bimodules are cofibrant as R-modules.*

The category of modules over a simplicial ring has a model structure where the fibrations and weak equivalences are created in the underlying category of simplicial abelian groups. L is compatible with this model structure:

Theorem A.8. *[24, Thm. 4.4] Let B be a simplicial ring. Then the adjoint functors H and L form a Quillen equivalence between the categories of B-modules and HB-modules.*

These model categories are where we do a lot of Goodwillie calculus, and therefore we are forced to compute homotopy fibers of cubes consisting of (simplicial) Γ-rings. In the process, we often use the following fact:

Proposition A.9 ([9, Corollary A.7.2.7])**.** *The homotopy limit of simplicial connective spectra over an indexing category with finite cohomological dimension can be computed levelwise.*

In particular, homotopy limits of simplicial connective spectra over $\mathbb{N}$ can be taken levelwise, which implies that homotopy fibers of simplicial connective spectra can be taken levelwise.

A.3 Resolutions

We now describe the main technical device that will be used in this paper, which is a way of resolving Γ-rings and bimodules by simplicial rings and bimodules over simplicial rings.

Construction A.10. We have an adjunction between the category of simplicial abelian groups $\mathcal{A}b^{\Delta}$ and the category of pointed simplicial sets S_*:

$$\widetilde{\mathbb{Z}} : S_* \rightleftarrows \mathcal{A}b^{\Delta} : U$$

where $\widetilde{\mathbb{Z}}$ is the "free simplicial abelian group" functor (as above) and U is the forgetful functor. Applying these functors pointwise to the values of Γ-objects in $\mathcal{A}b^{\Delta}$ and S_*, we get an adjunction

$$\widetilde{\mathbb{Z}} : \Gamma S_* \rightleftarrows \Gamma\mathcal{A}b^{\Delta} : U$$

which can be promoted to an adjunction

$$\widetilde{\mathbb{Z}} : \Gamma\text{-rings} \rightleftarrows H\mathbb{Z}\text{-algebras} : U.$$

Letting R be a Γ-ring, repeated application of the unit of this adjunction gives us an augmented cosimplicial object (with R as the 0 simplex):

$$R \to \left\{[q] \mapsto (U\widetilde{\mathbb{Z}})^{q+1}R\right\}$$

which determines a cube:

$$S \mapsto (U\widetilde{\mathbb{Z}})^{|S|}R$$

(with $\varnothing \mapsto R$). We will denote this cube by $S \mapsto R^S$.

This cube has the useful property that every noninitial vertex is an $H\mathbb{Z}$-algebra, and is therefore canonically stably equivalent to a simplicial ring as discussed earlier.

We record one last fact, which tells us that suspension of $H\mathbb{Z}$-modules is related to the bar construction of simplicial abelian groups:

Lemma A.11. *Let M be an $H\mathbb{Z}$-module. Then there is a natural equivalence $L(\Sigma M) \simeq BL(M)$.*

Proof. The suspension functor is performed space-wise on Γ-spaces, which is to say that $(\Sigma M)(1_+) = \Sigma(M(1_+))$ and $(\Sigma M)(2_+) = \Sigma(M(2_+))$. Recalling the way that L is defined, it is straightforward to verify that taking the coequalizer involved in the definition is compatible with this suspension.

This nets us an equality $L(\Sigma M) = \Sigma L(M)$. In [28], Wu shows that there is a natural map

$$\sigma : \Sigma X \to BX$$

for any simplicial group X. This has the property that the looped map

$$G\sigma : G\Sigma X \to GBX$$

is a weak equivalence, where G is Kan's loop group functor. However, both ΣX and BX are connected, so comparing the long exact sequence of homotopy groups associated to the path/loops fibration tells us that the map $\Sigma X \to BX$ is an equivalence.

A.4 Γ-spaces and Prolongation

To each Γ-space we can construct an FSP in the classical sense via prolongation (cf. [6]):

Construction A.12. Let R be a Γ-space. We can first extend R from a functor $\Gamma^{op} \to S_*$ to a functor from the category of based sets. Let W be a based set, then we define the value of R on W to be the colimit of its values on all of its finite subsets:

$$R(W) := \operatorname*{colim}_{\substack{V \subset W \\ V \in \Gamma}} R(V).$$

Now, we can apply this levelwise to extend this to a functor from S_* to itself. Let $X = \{X_n\}$ be a simplicial set, then we can define

$$R(X)_n := R(X_n) = \operatorname*{colim}_{\substack{V \subset X_n \\ V \in \Gamma}} R(V),$$

with the face and degeneracy maps induced from functoriality and the face and degeneracy maps of X.

This gives us an FSP in simplicial sets and by evaluating on the simplicial spheres $X = S^n$ we get a sequence of simplicial sets $\{R(S^n)\}$, and the maps internal to Γ^{op} can be used to give us the structure maps of a spectrum.

This also has the description as a coend:

$$R(X) = \int^{[n] \in \Gamma^{op}} X^n \wedge R([n]).$$

This prolongation functor $\Gamma S_* \to \text{Spectra}$ produces a Quillen equivalence between any suitable model category of connective spectra (see [21], [25], and [16]).

Appendix B

K-Theory

We now include a brief discussion of both the Waldhausen definition of K-theory and the plus construction that is used in [9].

B.1 Review of Waldhausen Categories and K-Theory

Waldhausen categories were introduced in [27], where they were called "categories with weak equivalences and cofibrations". As the latter name suggests, these are categories equipped with sets of maps deemed to be weak equivalences and cofibrations, and these maps are required to satisfy some reasonable properties along the lines of what we know happens with weak homotopy equivalences and cofibrations in the more familiar category of topological spaces (or more generally, some model category). More precisely:

Definition B.1. A *Waldhausen category* is a pointed category $\mathcal{C}$ equipped with two distinguished subcategories: the *weak equivalences* $w\mathcal{C}$ (denoted with $\xrightarrow{\sim}$) and the *cofibrations* $\operatorname{cof}\mathcal{C}$ (denoted with $\rightarrowtail$). These are required to satisfy the following axioms:

1. The isomorphisms in $\mathcal{C}$ are contained in $w\mathcal{C}$;

2. $w\mathcal{C} \subseteq \operatorname{cof}\mathcal{C}$;

3. the maps from the 0 object are in $\operatorname{cof}\mathcal{C}$;

4. *(Cobase change).* If $A \rightarrowtail B$ is a cofibration and $A \to C$ is any map, then the pushout $C \cup_A B$ exists and moreover the map $C \to C \cup_A B$ is itself a cofibration;

5. *(Gluing).* If we have a commutative diagram

$$\begin{tikzcd} B \arrow[d, "\sim"] & A \arrow[l, tail] \arrow[r] \arrow[d, "\sim"] & C \arrow[d, "\sim"] \\ B' & A' \arrow[l, tail] \arrow[r] & C', \end{tikzcd}$$

then the map

$$B \cup_A C \to B' \cup_{A'} C'$$

is also in $w\mathcal{C}$.

We refer to a sequence $A \rightarrowtail B \twoheadrightarrow A/B$ as an *cofiber sequence*, where A/B is such that it fits into a pushout square

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & B \arrow[d, two heads] \\ 0 \arrow[r] & B/A. \end{tikzcd}$$

We can now define the $S_\bullet$ construction, which serves the purpose of both (homotopically) identifying weakly equivalent objects and splitting cofiber sequences:

Definition B.2. For any category $\mathcal{C}$, the *arrow category* $Ar(\mathcal{C})$ is the category whose objects are the morphisms in $\mathcal{C}$ and whose morphisms are commutative diagrams. We let Ar_n denote the arrow category $Ar([n])$, where $[n]$ is the category associated to the ordered set $[n] = \{0 < 1 < \cdots < n\}$.

Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. We define a simplicial Waldhausen category $S_\bullet\mathcal{C}$ given as follows: in degree n is the category $S_n\mathcal{C}$ of functors $C : Ar_n \to \mathcal{C}$ satisfying:

1. The diagonal objects $C_{i,i}$ are 0;
2. if $i \leq j \leq k$, then $C_{i,j} \to C_{i,k}$ is a cofibration, and $C_{i,j} \rightarrowtail C_{i,k} \twoheadrightarrow C_{j,k}$ is a cofiber sequence.

This has a Waldhausen structure where the weak equivalences and cofibrations are pointwise, and so it makes sense to take the nerve $wS_\bullet\mathcal{C}$, which is a bisimplicial set. We then define the *K-theory space*

$$K(\mathcal{C}) := \mathrm{diag}^* \, wS_\bullet\mathcal{C}.$$

Waldhausen shows in [27] that $\Omega|K(\mathcal{C})|$ is an infinite loop space, and so we can also refer to the associated connective spectrum as $K(\mathcal{C})$; this distinction will be clear from context.

B.2 The Plus Construction

There are a number of ways to define the algebraic K-theory of Γ-rings, all of which are equivalent in a suitable sense. One way is to use the category P_R of finite cell R-modules, but [9] use the following construction:

Definition B.3. Let R be a Γ-ring. Then we define $M_n(R)$ to be the Γ-ring of nxn matrices in R "with only one entry in each column" that is for a simplicial set X

$$M_n(R)(X) := \prod_n \bigvee_n R(X).$$

Let $\widetilde{M}_n(R)$ be a functorial fibrant replacement of $M_n(R)$; then we have that $\pi_0\widetilde{M}_n(R) = M_n(\pi_0(R))$, so we can define $GL_n(R)$ as the homotopy pullback

$$\begin{array}{ccc} GL_n(R) & \longrightarrow & \widetilde{M}_n \\ \big\downarrow & & \big\downarrow \\ GL_n(\pi_0(R)) & \longrightarrow & M_n(\pi_0(R)). \end{array}$$

The wedge induces a map

$$M_n(R) \to M_{n+1}(R)$$

which is compatible with the Whitehead sum on $M_n(\pi_0(R))$ and therefore there is a colimit system

$$\cdots \to GL_n \to GL_{n+1} \to \dots.$$

We can then define $GL(R)$ as the Γ-ring that arises from this limit sytem. Its value on 1 is a simplicial monoid, which we can group complete by taking its classifying space and using the plus construction, and so we define

$$K(R)^{\Delta} := K_0(\pi_0(R)) \times B(GL(R)(1_+))^+$$

which is a simplicial set, and is the 0th simplicial set in an Ω-spectrum.

Note. The standard procedure to define algebraic K-theory gives us a space. That is, we would apply the plus construction to the *topological* monoid $|BGL(R)(1_+)|$ to obtain $|B(GL(R)(1_+))|^+$. We would then define

$$K(R) = K_0(\pi_0(R)) \times |B(GL(R)(1_+))|^+.$$

Because geometric realization preserves both the plus construction (moving from simplicial monoids to topological monoids) and products (!), this is equivalent to the geometric realization of $K(R)^{\Delta}$.

Bibliography

[1] Gert Almkvist. The grothendieck ring of the category of endomorphisms. *Journal of Algebra*, 28(3):375–388, 1974.

[2] Gert Almkvist. K-theory of endomorphisms. *Journal of Algebra*, 55(2):308–340, 1978.

[3] Stanislaw Betley and Christian Schlichtkrull. The cyclotomic trace and curves on k-theory. *Topology*, 44(4):845–874, 2005.

[4] Marcel Bökstedt. Topological hochschild homology. *preprint, Bielefeld*, 1985.

[5] Marcel Bökstedt, Wu Chung Hsiang, and Ib Madsen. The cyclotomic trace and algebraic K-theory of spaces. *Inventiones mathematicae*, 111(1):465–539, 1993.

[6] Aldridge K Bousfield and Eric M Friedlander. Homotopy theory of Γ-spaces, spectra, and bisimplicial sets. In *Geometric applications of homotopy theory II*, pages 80–130. Springer, 1978.

[7] G E Carlsson, R L Cohen, Thomas G Goodwillie, and Wu Chung Hsiang. The free loop space and the algebraic K-theory of spaces. *K-theory*, 1(1):53–82, 1987.

[8] Bjørn Ian Dundas. Relative k-theory and topological cyclic homology. *Acta Mathematica*, 179(2):223–242, 1997.

[9] Björn Ian Dundas, Thomas G Goodwillie, and Randy McCarthy. *The local structure of algebraic K-theory*, volume 18. Springer, 2013.

[10] Bjorn Ian Dundas and Randy McCarthy. Stable k-theory and topological hochschild homology. *Annals of Mathematics*, pages 685–701, 1994.

[11] Thomas G Goodwillie. Relative algebraic k-theory and cyclic homology. *Annals of Mathematics*, pages 347–402, 1986.

[12] Thomas G Goodwillie. Calculus I: The first derivative of pseudoisotopy theory. *K-theory*, 4(1):1–27, 1990.

[13] Thomas G Goodwillie. Calculus II: analytic functors. *K-theory*, 5(4):295–332, 1991.

[14] Thomas G Goodwillie. Calculus III: Taylor Series. *arXiv preprint math/0310481*, 2003.

[15] Lars Hesselholt and Ib Madsen. On the k-theory of finite algebras over witt vectors of perfect fields. *Topology*, 36(1):29–101, 1997.

[16] John A Lind. Diagram spaces, diagram spectra and spectra of units. *Algebraic and geometric topology*, 13:1857–1935, 2013.

[17] Ayelet Lindenstrauss and Randy McCarthy. The algebraic k-theory of extensions of a ring by direct sums of itself. *Urbana*, 51:61801, 2008.

[18] Ayelet Lindenstrauss and Randy McCarthy. On the algebraic K-theory of formal power series. *K-Theory arXiv*, 1010, 2010.

[19] Ayelet Lindenstrauss and Randy McCarthy. On the Taylor tower of relative K–theory. *Geometry and Topology*, 16:685–750, 2012.

[20] Manos Lydakis. Smash products and Gamma-spaces. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 126, pages 311–328. Cambridge Univ Press, 1999.

[21] Michael A Mandell, J Peter May, Stefan Schwede, and Brooke Shipley. Model categories of diagram spectra. *Proceedings of the London Mathematical Society*, 82(2):441–512, 2001.

[22] Randy McCarthy. Relative algebraic k-theory and topological cyclic homology. *Acta Mathematica*, 179(2):197–222, 1997.

[23] Daniel G Quillen. *Homotopical algebra*. Springer-Verlag Berlin, 1967.

[24] Stefan Schwede. Stable homotopical algebra and Gamma-spaces. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 126, pages 329–356. Cambridge Univ Press, 1999.

[25] Stefan Schwede and Brooke Shipley. Equivalences of monoidal model categories. *Algebr. Geom. Topol*, 3:287–334, 2003.

[26] Graeme Segal. Categories and cohomology theories. *Topology*, 13(3):293–312, 1974.

[27] Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and geometric topology*, pages 318–419. Springer, 1985.

[28] Jie Wu. On fibrewise simplicial monoids and milnor–carlsson's constructions. *Topology*, 37(5):1113–1134, 1998.

Vita

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