Effective stochastic behavior in dynamical systems with incomplete information

Michael A. Buice[1,2] and Carson C. Chow[1]

[1]*Laboratory of Biological Modeling, NIDDK, NIH, Bethesda, Maryland 20892, USA*

[2]*Center for Learning and Memory, University of Texas at Austin, Austin, Texas, USA*

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Complex systems are generally analytically intractable and difficult to simulate. We introduce a method for deriving an effective stochastic equation for a high-dimensional deterministic dynamical system for which some portion of the configuration is not precisely specified. We use a response function path integral to construct an equivalent distribution for the stochastic dynamics from the distribution of the incomplete information. We apply this method to the Kuramoto model of coupled oscillators to derive an effective stochastic equation for a single oscillator interacting with a bath of oscillators and also outline the procedure for other systems.

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I. INTRODUCTION

In complex high-dimensional dynamical systems one must cope with many unknown degrees of freedom. One approach to controlling this uncertainty is to construct effective low-dimensional models by marginalizing over the statistical distribution describing the incomplete information of the system (e.g., [1,2]). The marginalized degrees of freedom would be manifested as effective stochastic terms in a reduced model. Here we provide an explicit method to carry out such a procedure on the Kuramoto model of nonlinear coupled oscillators and construct an effective stochastic equation. Our method is generalizable to other systems.

Efforts to derive effective stochastic behavior from deterministic microscopic dynamics date back to the origins of statistical mechanics (for a brief account, see [3]), and in particular kinetic theory [4,5]. In this framework, the one-particle probability density obeys a Boltzmann equation, which is equivalent to a truncation of an infinite moment hierarchy (Bogoliubov-Born-Green-Kirkwood-Yvo (BBGKY) hierarchy [4]) that encodes all of the information in the microstate distribution. We previously applied this conceptual approach to coupled networks [6,7], and derived a response functional field theory for the Kuramoto model that is equivalent to a BBGKY hierarchy where the inverse system size is a loop expansion parameter. The existence of a small parameter obviates the need for any assumptions regarding dynamic averaging or "molecular chaos" for large system size and allows one to compute effects beyond mean field theory. Here we compute the effective stochastic dynamics of individual oscillators embedded in an incompletely specified network, directly from the dynamics of the fully deterministic system.

The approach of marginalizing over a background environment in order to derive effective stochastic behavior is not novel, and has been well explored in the literature in the context of Brownian motion in a bath of interacting particles with a well defined Hamiltonian for the total system [8–11]. Those studies took advantage of the exact integrability of the harmonic oscillator in order to derive exact expressions for the noise distribution of the particle interacting with the bath. In the context of oscillator systems, noise has been represented as externally added Gaussian white noise [12,13] without any established connection to the internal dynamics of the system. In neural applications, investigators have used a self-consistent assumption of Poisson noise [14]. Our method does not rely upon exact integrability, nor does it only apply to systems in the Hamiltonian class, but to any dynamical system. Indeed, the method we demonstrate relies only on the tractability of the BBGKY hierarchy, or equivalently, the tractability of the loop expansion for a density field theory defined as in Ref. [7]. Using this formulation, we show that finite size intrinsic fluctuations for the Kuramoto model cannot be represented using the *ad hoc* tool of additive Gaussian noise.

II. ENSEMBLE AVERAGES AND FINITE SIZE EXPANSIONS

The Kuramoto model of N coupled oscillators obeys

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_j \sin(\theta_j - \theta_i), \tag{1}$$

where ω_i is the driving frequency of the ith oscillator, drawn from a fixed distribution $g(\omega)$. This model exhibits a phase transition from incoherence to synchrony as the coupling K is increased. Here we consider the incoherent state in which the oscillators are approximately uniformly distributed in phase. Strogatz and Mirollo [12] found that in the infinite system-size mean field limit the incoherent state is marginally stable because each individual oscillator decouples from the population. External additive noise stabilizes the incoherent state by adding an explicit diffusion term to the mean field equation. Buice and Chow [7] showed that finite size effects stabilize the marginal modes by generating effective diffusion without additive noise suggesting that the effective dynamics of individual oscillators is a random walk due to interactions with the bath of oscillators in the network. Here we show how incomplete knowledge of the initial data of the Kuramoto system yields an effective stochastic equation for a given oscillator.

Consider a distinguished *measured* oscillator with phase $\phi(t)$ within a bath of *unmeasured* oscillators all obeying the Kuramoto equation (1). Determining the motion of $\phi(t)$ is equivalent to determining which element of a statistical ensemble from which the unmeasured oscillators are drawn is realized. In each small time interval, the oscillator $\phi(t)$ interacts with the bath of unmeasured oscillators, which provides some small amount of knowledge about the specific phase and driving frequency distribution for the unmeasured oscillators.

To an external observer, the ensemble is a distribution from which a stochastic driving term is being drawn during each time interval. There is thus an equivalent distribution for the observed stochastic evolution of $\phi(t)$ derivable from the ensemble distribution of possible networks of oscillators. This distribution is equivalent to the information that the observer possessed about the unmeasured oscillators prior to the experiment.

Let the N unmeasured oscillators obey Eq. (1). The distinguished measured oscillator is labeled by $\phi(t)$, with intrinsic driving frequency Ω. The combined system dynamics obey

$$\dot{\phi}(t) = \Omega + \frac{K}{N}\sum_j \sin(\theta_j - \theta_i),$$

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_j \sin(\theta_j - \theta_i) + \frac{K}{N}\sin[\phi(t) - \theta_i(t)],$$

where the normalization for the entire system remains N for simplicity. We can rewrite the Kuramoto system in terms of a continuity equation, or "Klimontovich" equation [6,7], by defining

$$\eta(\theta,\omega,t) = \frac{1}{N}\sum_i \delta[\theta - \theta_i(t)]\delta(\omega - \omega_i). \tag{2}$$

The Kuramoto model can be written in terms of $\eta(\theta,\omega,t)$ as

$$\dot{\phi}(t) = \Omega + K\int d\theta d\omega \sin[\theta - \phi(t)]\eta(\theta,\omega,t),$$

$$\dot{\theta}_i(t) = \omega_i + K\int d\theta d\omega \sin[\theta - \theta_i(t)]\eta(\theta,\omega,t) + \frac{K}{N}\sin[\phi(t) - \theta_i(t)].$$

The second equation provides a velocity field for the $\theta_i(t)$ variables, which we can use to define the equation for $\eta(\theta,\omega,t)$:

$$\partial_t\eta(\theta,\omega,t) - \omega\partial_\theta\eta(\theta,\omega,t) - \partial_\theta\left(\left\{K\int d\theta' d\omega' \sin(\theta' - \theta)\eta(\theta',\omega',t) + \frac{K}{N}\sin[\phi(t) - \theta]\right\}\eta(\theta,\omega,t)\right) = 0.$$

The full set of dynamical equations is then

$$\dot{\phi}(t) = \Omega + K\int d\theta d\omega \sin[\theta - \phi(t)]\eta(\theta,\omega,t), \tag{3}$$

$$\partial_t\eta(\theta,\omega,t) = \omega\partial_\theta\eta(\theta,\omega,t) + \partial_\theta\left(\left\{K\int d\theta' d\omega' \sin(\theta' - \theta)\eta(\theta',\omega',t) + \frac{K}{N}\sin[\phi(t) - \theta]\right\}\eta(\theta,\omega,t)\right). \tag{4}$$

The density η from Eq. (2) is a distributional solution of Eq. (4) given the equation of motion for $\phi(t)$ and (3) is the equation of motion for oscillator $\phi(t)$ with driving frequency Ω in terms of η.

We suppose that each unmeasured oscillator with initial angle θ and driving frequency ω is drawn independently from the distribution given by $\rho_0(\theta,\omega) = \frac{g(\omega)}{2\pi}$ to yield an ensemble of similarly prepared systems. A given instance of the Kuramoto model corresponds to a given instance of the initial state configuration. We fix the initial condition of ϕ to $\phi(t_0)$ and Ω for *every* instance of the ensemble.

The distribution of population densities η over the ensemble is represented by a *probability density functional* $P[\eta]$, which we use to marginalize the unmeasured oscillator dynamics over the ensemble. We apply a response function formalism developed in nonequilibrium statistical mechanics in which η is transformed to the variables $\tilde{\varphi}$ and φ using a Doi-Pelití-Jansson transformation followed by a shift by the mean $\langle\eta\rangle = g(\omega)/2\pi \equiv \rho_0$ (see [7,15–18]). The variables $\varphi,\tilde{\varphi}$ represent counting processes in the sense that terms of the form $\tilde{\varphi}\Delta^{-1}\varphi$ are Poisson distributions; here the "counting process" is the number of oscillators within a bin of width $d\theta$. A derivation of the generating functional from the system equations of motion is given in the Appendix, including a derivation of the initial condition sampling corrections. This derivation constructs a distribution from the deterministic dynamics acting on the initial distribution. We caution those familiar with standard field theoretic approaches that the resulting response functional formalism has some subtle differences with field theories such as φ^4 which arise in equilibrium statistical mechanics. First, the "free" theory is off-diagonal; it is of the form $\int\tilde{\varphi}\Delta^{-1}\varphi$. The most important difference is that operators within a vertex are normal ordered as a result of the assumption of the "Ito" condition, and thus cannot contract with themselves. This is a consequence of the limit considered when defining the functional integral. The resulting effective stochastic differential equation therefore will be of the Ito type. The logarithm of the density functional, called the action, has the form

$$S = S_{\text{bulk}} + S_{\text{int}} + S_\phi, \tag{5}$$

where

$$\begin{aligned} S_{\text{bulk}} = N\int d\omega d\theta dt\tilde{\varphi}\Big\{&(\partial_t + \omega\partial_\theta)\varphi \\ &+ K\int d\omega' d\theta' \partial_\theta[\sin(\theta' - \theta)(\varphi'\rho_0 + \rho_0'\varphi + \varphi'\varphi)] \\ &+ K\int d\omega' d\theta' \tilde{\varphi}'\partial_\theta[\sin(\theta' - \theta)(\varphi' + \rho_0')(\varphi + \rho_0)]\Big\} \\ &- N\sum_{k=2}^{\infty}\frac{(-1)^{k+1}}{k}\left[\int d\theta d\omega\tilde{\varphi}(\theta,\omega,t_0)\rho_0(\theta,\omega)\right]^k \end{aligned} \tag{6}$$

is the Kuramoto system action in Ref. [7], which includes initial condition terms representing finite size (sampling) corrections to the moments of the initial ensemble,

$$S_{\text{int}} = K\int d\omega d\theta dt\, \tilde{\varphi}\partial_\theta\{\sin[\delta\phi(t) + \Omega(t - t_0) - \theta](\varphi + \rho_0)\} \tag{7}$$

is the coupling of the N unmeasured oscillators to the single measured oscillator $\phi(t)$, where $\delta\phi(t) = \phi(t) - \Omega(t - t_0)$ is the deviation from the mean field value $\langle\phi(t)\rangle_{\text{mf}} = \Omega(t - t_0)$, and

$$S_\phi = \int dt\, \tilde{\phi}(t)\Big\{\delta\dot{\phi}(t) - K\int d\theta d\omega \sin[\theta - \Omega(t - t_0) - \delta\phi(t)](\tilde{\varphi}\varphi + \varphi + \tilde{\varphi}\rho_0)\Big\} \tag{8}$$

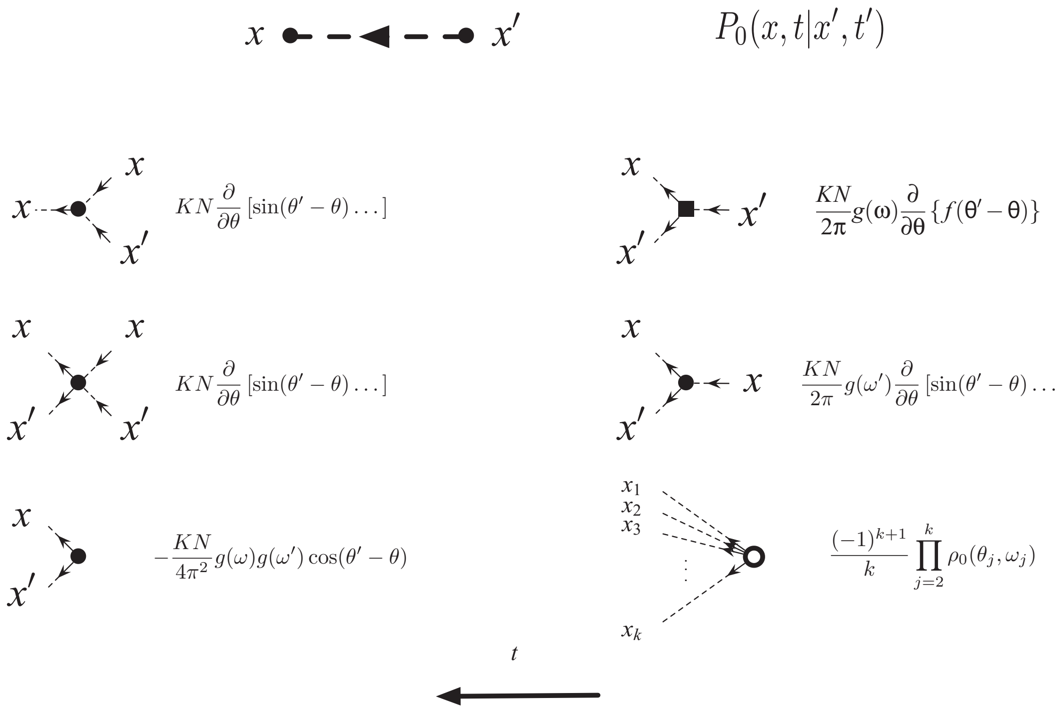

FIG. 1. The vertices and propagator derived from the action S_{bulk}. Compare with the diagrams from Ref. [7]. The open circle vertex represents initial condition terms which are the sampling corrections to the initial state. We label the coordinates of each vertex leg because these connections are nonlocal. $x = (\theta,\omega,t)$ and similarly for x'.

is the action for the measured oscillator. All statistical properties of the system are contained in the action. Any dynamical system can be described by an action in terms of densities for high-dimensional systems (e.g., fields such as φ), directly in terms of the variables themselves [e.g., $\delta\phi(t)$] or with a mixture of the two. The variable $\tilde{\varphi}$ is an imaginary response field.

The vertices from the Feynman rules for the action S are shown in Figs. 1 (S_{bulk}), 2 (S_ϕ), and 3 (S_{int}). To facilitate the following calculations, they include quadratic terms which normally would be included in the propagator. We choose the free propagator to be of the form $\tilde{\varphi}\Delta^{-1}\varphi + \tilde{\phi}\delta\dot{\phi}$. In the present case, the terms of Fig. 3 are $O(1/N)$ with respect to

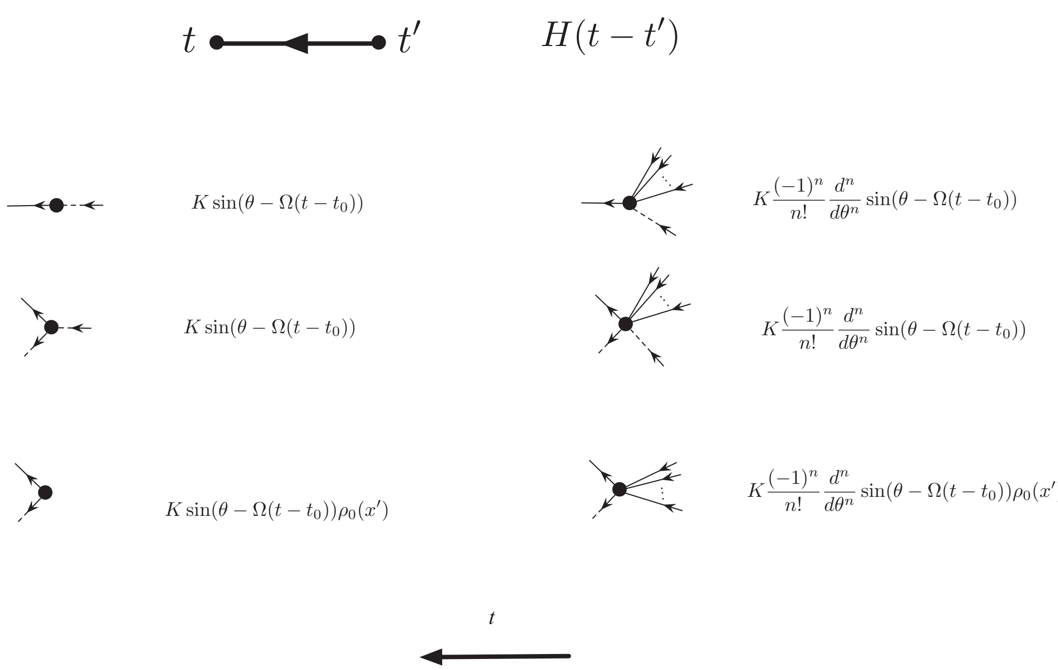

FIG. 2. The vertices and propagator derived from the action S_ϕ. These vertices are nominally $O(1)$, but will contract with vertices from S_{bulk} and S_{int} such that moments will have $1/N$ scaling.

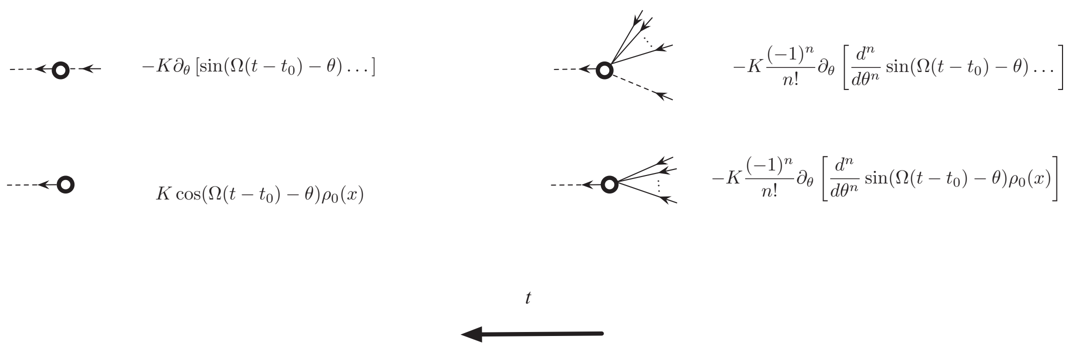

FIG. 3. The vertices derived from $S_{\rm int}$. These are $O(1)$ but source the φ operator, which means they can be regarded as counterterms. They will produce factors of $1/N$ in the overall moment for each such vertex.

those in $S_{\rm bulk}$ and thus can be regarded as "counterterms" in the loop expansion. Because of this, it is simpler to regard all of these terms as vertices. We use open dots to represent initial conditions and these counterterms.

III. EFFECTIVE ACTION

We construct an effective action for the field $\delta\phi(t)$ (i.e., equivalent probability density) by marginalizing over the $\varphi,\tilde\varphi$ fields, which leaves the action

$$S_{\rm EA} = \int dt\, \tilde\phi\delta\dot\phi - W_O\{K'\sin[\theta-\Omega(t-t_0)-\delta\phi(t)]\tilde\phi(t)\}, \tag{9}$$

where $W_O[\lambda(\theta,\omega,t)]$ for $\lambda \equiv K\sin[\theta-\Omega(t-t_0)-\delta\phi(t)]\,\tilde\phi(t)$ is the cumulant or connected generating functional of the operator $O \equiv \tilde\varphi\varphi + \varphi + \tilde\varphi\rho_0$ under the bulk action from which all cumulants or connected moments can be derived. (O is the operator which describes deviations of the density η from the mean field value.) This effective action represents a stochastic process for the evolution of the measured oscillator and is equivalent to the stochastic differential equation

$$\dot\phi(t) = \Omega + \delta\Omega(t) + \xi(t), \tag{10}$$

where $\delta\Omega(t)$ is a frequency shift of the measured oscillator due to interaction with the unmeasured oscillators and $\xi(t)$ is a zero-mean stochastic forcing term, which will be in general non-Gaussian, nonwhite, and multiplicative [amplitude depends on $\phi(t)$]. To lowest order, the expression for $\delta\Omega(t)$ will be given by the coefficient of $\tilde\phi$ in W_0 and the Gaussian contribution to $\xi(t)$ will be given by the coefficient of $\tilde\phi^2$. Non-Gaussian contributions to the noise will come from the coefficients of higher powers of $\tilde\phi$. The computation of these coefficients will involve computing the moments of φ and $\tilde\varphi$ under the action $S_{\rm bulk} + S_{\rm int}$. The tractability of this marginalization will depend upon the tractability of computing the moments of the bulk dynamics.

All of the fluctuations in the combined system given by Eq. (5) stem from the terms in $S_{\rm bulk}$ and $S_{\rm int}$. The distribution represented by the path integral is the distribution over the ensemble of similarly prepared $N+1$ oscillator systems by construction. Therefore, each instance of the stochastic evolution under this equivalent equation corresponds to an instance of the initial configuration of the Kuramoto system. The marginalization procedure is an explicit construction of an equivalent distribution for the possible evolutions of $\phi(t)$. It effectively encodes the initial state of the $N+1$ oscillator network into the stochastic forcing term $\xi(t)$. Determining the entire exact evolution of $\phi(t)$ is equivalent to determining the exact initial configuration of the $N+1$ oscillator system (more precisely, it determines the information about the initial ensemble sufficient to fix the evolution of the order parameter for all time). The connected generating functional $W_O[\lambda(\theta,\omega,t)]$ does not have a finite closed form expression but we can compute it perturbatively using a large N expansion. At order $1/N$,

$$W_O^1[\lambda(\theta,\omega,t)] = W_{O,1}^1 + W_{O,2}^1, \tag{11}$$

where

$$W_{O,1}^1[\lambda(\theta,\omega,t)] = \int d\theta d\omega dt\, \lambda(\theta,\omega,t)\langle O(\theta,\omega,t)\rangle_1, \tag{12}$$

$$W_{O,2}^1[\lambda(\theta,\omega,t)] = \frac{1}{2}\int d\theta d\omega dt d\theta' d\omega' dt'\, \lambda(\theta,\omega,t)\lambda(\theta',\omega',t') \times \langle O(\theta,\omega,t)O(\theta',\omega',t')\rangle_1. \tag{13}$$

The moments of $O(\theta,\omega,t)$ in Eq. (11) are evaluated perturbatively to lowest order in the fields using the full action in Eq. (5) with the relevant diagrams depicted in Fig. 1. The first moment is

$$\begin{aligned}\langle O(\theta,\omega,t)\rangle_1 &= \langle\tilde\varphi(\theta,\omega,t)\varphi(\theta,\omega,t) + \varphi(\theta,\omega,t) + \tilde\varphi(\theta,\omega,t)\rho_0(\theta,\omega)\rangle \\ &= \langle\varphi(\theta,\omega,t)\rangle \\ &= K\int d\theta' d\omega' dt' P_0(\theta,\omega,t|\theta',\omega',t')\cos[\delta\phi(t') \\ &\quad + \Omega(t'-t_0) - \theta']\frac{g(\omega')}{2\pi},\end{aligned} \tag{14}$$

where $P_0(\theta,\omega,t|\theta',\omega',t') = \langle\varphi(\theta,\omega,t)\tilde\varphi(\theta',\omega',t')\rangle$ is the tree level propagator for the bulk action $S_{\rm bulk}$ and satisfies the linearized Klimontovich equation

$$(\partial_t + \omega\partial_\theta)P_0(\theta,\omega,t|\theta',\omega',t') + K\frac{g(\omega)}{2\pi}$$
$$\times\,\partial_\theta\int_{-\infty}^{\infty}\int_0^{2\pi} d\theta_1 d\omega_1 \sin(\theta_1-\theta)P_0(\theta_1,\omega_1,t|\theta',\omega',t')$$
$$= \frac{1}{N}\delta(\theta-\theta')\delta(\omega-\omega')\delta(t-t'). \qquad (15)$$

The mean $\langle\varphi\rangle$ under the bulk action alone is zero [7]. This symmetry is broken by the presence of the interaction action $S_{\rm int}$ (7), which represents the fact that the presence of the measured oscillator induces a degree of synchrony in the unmeasured oscillators, similar to an induced magnetization. The second moment is

$$\langle O(\theta,\omega,t)O(\theta',\omega',t')\rangle_1$$
$$= \langle[\tilde\varphi(\theta,\omega,t)\varphi(\theta,\omega,t) + \varphi(\theta,\omega,t) + \tilde\varphi(\theta,\omega,t)\rho_0(\theta,\omega)]$$
$$\times[\tilde\varphi(\theta,\omega,t)\varphi(\theta,\omega,t) + \varphi(\theta,\omega,t) + \tilde\varphi(\theta,\omega,t)\rho_0(\theta,\omega)]\rangle$$
$$= \langle\varphi(\theta,\omega,t)\tilde\varphi(\theta',\omega',t')\rangle\rho_0(\theta',\omega') + \langle\varphi(\theta,\omega,t)\varphi(\theta',\omega',t')\rangle$$
$$= P_0(\theta,\omega,t|\theta',\omega',t')\frac{g(\omega')}{2\pi} + C(\theta,\omega,t;\theta',\omega',t') \qquad (16)$$

for $t > t'$; the moment is symmetric with respect to $t \leftrightarrow t'$. For the second equality we have removed terms which are either zero or higher order in $1/N$. $C = \langle\varphi(\theta,\omega,t)\varphi(\theta',\omega',t')\rangle$ is the $1/N$ connected correlation function computed previously in Refs. [6,7]. Because the connected generating functional W_O is evaluated at $\lambda(\theta,\omega,t) = K'\sin[\theta-\Omega(t-t_0)-\delta\phi(t)]\tilde\phi(t)$ according to Eq. (9), the normal ordering of the operators $\delta\phi(t)$ and $\tilde\phi(t)$ is not preserved across different factors of λ since each is essentially a new vertex. When calculating moments of $\delta\phi(t)$, it is important to keep this in mind as factors of $\delta\phi(t)$ and $\tilde\phi(t)$ from different λ's will contract with each other. Normal ordering can be restored by using the linear response for $\phi(t)$ term by term in the effective action. Because the connected generating functional W_O is quadratic to first order in $1/N$, the action $S_{\rm EA}$ contains terms which are quadratic in $\tilde\phi(t)$. These represent Gaussian noise, albeit colored noise since the second moment of O, Eq. (16), indicates that O is not δ correlated. The dynamics of the Kuramoto model induce temporal correlations in the evolution of $\phi(t)$.

Inserting (14) into Eq. (12) leads to the expression

$$W^1_{O,1} = \int_{t'}^{\infty} dt\tilde\phi(t)\,K^2\int d\theta d\omega\sin[\theta-\Omega(t-t_0)-\delta\phi(t)]$$
$$\times\int_{t_0}^{t} dt'd\omega'd\theta'\,P_0(\theta,\omega,t|\theta',\omega',t')$$
$$\times\,\cos[\Omega(t'-t_0)+\delta\phi(t')-\theta']\frac{g(\omega')}{2\pi}$$
$$\equiv \int_{t_0}^{\infty} dt\tilde\phi(t)\delta\Omega_A(t). \qquad (17)$$

Inserting (16) into Eq. (13) gives $W^1_{O,2} = W^1_{O,2C} + W^1_{O,2P}$ where

$$W^1_{O,2C}[\lambda(\theta,\omega,t)]$$
$$= K^2\int_{t_0}^{\infty} dt d\theta d\omega$$
$$\times\int_{t_0}^{t} dt'd\theta'd\omega'\tilde\phi(t)\tilde\phi(t')\sin[\theta-\Omega(t-t_0)-\delta\phi(t)]$$
$$\times\,\sin[\theta'-\Omega(t'-t_0)-\delta\phi(t')]P(\theta,\omega,t|\theta',\omega',t')\frac{g(\omega')}{2\pi},$$

$$W^1_{O,2P}[\lambda(\theta,\omega,t)]$$
$$= \frac{1}{2}\int d\theta d\omega dt d\theta' d\omega' dt'\,\lambda(\theta,\omega,t)$$
$$\times\,\lambda(\theta',\omega',t')C(\theta,\omega,t;\theta',\omega',t')$$
$$= K^2\int_{t_0}^{\infty} dt d\theta d\omega\int_{t_0}^{t} dt'd\theta'd\omega'\tilde\phi(t)\tilde\phi(t')$$
$$\times\,\sin[\theta-\Omega(t-t_0)-\delta\phi(t)]$$
$$\times\,\sin[\theta'-\Omega(t'-t_0)-\delta\phi(t')]C(\theta,\omega,t;\theta',\omega',t').$$

To order $1/N$ we can eliminate some of the terms, although we need to be careful of normal ordering. Factors of $\delta\phi(t)$ will contract with factors of $\tilde\phi(t')$ for $t > t'$ because the marginalization does not preserve normal ordering. We use the fact that to lowest order in $1/N$ we have $\langle\delta\phi(t)\tilde\phi(t')\rangle = H(t-t')$ where $H(t)$ is the step function. This yields

$$W^1_{O,2C}[\lambda(\theta,\omega,t)]$$
$$= K^2\int_{t_0}^{\infty} dt d\theta d\omega\int_{t_0}^{t} dt'd\theta'd\omega'\tilde\phi(t)$$
$$\times\,\{\tilde\phi(t')\sin[\theta-\Omega(t-t_0)] - \cos[\theta-\Omega(t-t_0)]\}$$
$$\times\,\sin[\theta'-\Omega(t'-t_0)]C(\theta,\omega,t;\theta',\omega',t')$$
$$\equiv \int_{t_0}^{\infty} dt\,\tilde\phi(t)\left[\int_{t_0}^{t} dt'\tilde\phi(t')G_B(t,t') + \delta\Omega_B(t)\right],$$

$$W^1_{O,2P}[\lambda(\theta,\omega,t)]$$
$$= K^2\int_{t_0}^{\infty} dt d\theta d\omega\int_{t_0}^{t} dt'd\theta'd\omega'\tilde\phi(t)$$
$$\times\,\{\tilde\phi(t')\sin[\theta-\Omega(t-t_0)] - \cos[\theta-\Omega(t-t_0)]\}$$
$$\times\,\sin[\theta'-\Omega(t'-t_0)]P(\theta,\omega,t|\theta',\omega',t')\frac{g(\omega')}{2\pi}$$
$$\equiv \int_{t_0}^{\infty} dt\,\tilde\phi(t)\left[\int_{t_0}^{t} dt'\tilde\phi(t')G_C(t,t') + \delta\Omega_C(t)\right].$$

Hence

$$W^1_O[\lambda(\theta,\omega,t)] = \int_{t_0}^{\infty} dt\,\tilde\phi(t)\left[\int_{t_0}^{t} dt'\tilde\phi(t')G(t,t') + \delta\Omega(t)\right], \qquad (18)$$

where

$$\delta\Omega(t) = \delta\Omega_A(t) + \delta\Omega_B(t) + \delta\Omega_C(t), \qquad (19)$$
$$G(t,t') = G_B(t,t') + G_C(t,t'). \qquad (20)$$

The terms subscripted by A arise from the leftmost graph in Fig. 1, those with subscript B arise from the middle graph, and those with subscript C arise from the rightmost graph. The computation of the effective action reduces to computing (19) and (20).

Taking the Fourier transform of (15) in θ and θ' gives

$$d\Omega_A(t) = \frac{K^2\pi}{2i}\int d\omega \int_{t_0}^{t} dt' d\omega' g(\omega') \times [P_{-1,1}(\omega,t|\omega',t')e^{i[\Omega(t'-t)+\delta\phi(t')-\delta\phi(t)]} - P_{1,-1}(\omega,t|\omega',t')e^{-i[\Omega(t'-t)+\delta\phi(t')-\delta\phi(t)]}], \quad (21)$$

where

$$\int d\omega\, P_{\pm 1,\mp 1} = \frac{1}{2\pi N}\frac{1}{s\pm\omega'}\frac{1}{\Lambda_{\pm}(s)},$$
$$\Lambda_{\pm 1}(s) = 1 - \frac{K}{2}\int d\omega \frac{g(\omega)}{s\pm i\omega}$$

are $O(1/N)$. Using the identity

$$\int d\omega \frac{g(\omega)}{s\pm i\omega} = 2\frac{1-\Lambda_{\pm}(s)}{K}$$

gives

$$d\Omega_A(t) = -\frac{K}{4\pi N}\int_0^{t-t_0} d\tau \int_C ds e^{s\tau}\left[\frac{1-\Lambda_{-1}(s)}{\Lambda_{-1}(s)}e^{-i\Omega\tau} - \frac{1-\Lambda_1(s)}{\Lambda_1(s)}e^{i\Omega\tau}\right].$$

If $g(\omega)$ is even, then

$$\Lambda_{\pm 1}(s) = 1 - \frac{K}{2}\int d\omega \frac{g(\omega)s}{s^2+\omega^2}$$

which results in

$$d\Omega_A(t) = -\frac{K}{4\pi N}\int_0^{t-t_0} d\tau \times \int_C ds e^{s\tau}\left[\frac{1-\Lambda_1(s)}{\Lambda_1(s)}\right](e^{-i\Omega\tau} - e^{i\Omega\tau}).$$

To simplify the expressions, we adopt the Cauchy-Lorentz distribution

$$g(\omega) = \frac{1}{\pi}\frac{\gamma}{\omega^2+\gamma^2}$$

which gives

$$\Lambda_{\pm 1}(s) = \frac{s+\gamma-\frac{K}{2}}{s+\gamma}.$$

Resulting in

$$\begin{aligned} d\Omega_A(t) &= -i\frac{K^2}{4N}\int_0^{t-t_0} d\tau e^{-(\gamma-\frac{K}{2})\tau}(e^{-i\Omega\tau} - e^{i\Omega\tau}) \\ &= -i\frac{K^2}{4N}\left[\frac{1}{i\Omega+\gamma-\frac{K}{2}}\left(1-e^{-(\gamma-\frac{K}{2}+i\Omega)(t-t_0)}\right) - \frac{1}{-i\Omega+\gamma-\frac{K}{2}}\left(1-e^{-(\gamma-\frac{K}{2}-i\Omega)(t-t_0)}\right)\right] \\ &= -\frac{1}{N}\frac{K^2}{2}\left\{\left[\frac{\Omega}{\Omega_f^2+\left(\gamma-\frac{K}{2}\right)^2}\right] - e^{-(\gamma-\frac{K}{2})(t-t_0)} \times \left[\frac{\omega_f\cos\Omega(t-t_0)+\left(\gamma-\frac{K}{2}\right)\sin\Omega(t-t_0)}{\Omega^2+\left(\gamma-\frac{K}{2}\right)^2}\right]\right\}. \end{aligned}$$

Now consider

$$\begin{aligned} \delta\Omega_B(t) &= -K^2\int d\theta d\omega \int_{t_0}^{t} dt' d\theta' d\omega' \cos[\theta-\Omega(t-t_0)] \times \sin[\theta'-\Omega(t'-t_0)]C(\theta,\omega,t;\theta',\omega',t') \\ &= 4\pi^2\frac{K^2}{4i}\int_{t_0}^{t} dt' \int d\omega d\omega' [C_{-1,1}(\omega,t;\omega',t')e^{-i\Omega(t-t')} - C_{1,-1}(\omega,t;\omega',t')e^{i\Omega(t-t')}]. \end{aligned}$$

We can simplify this expression using

$$\begin{aligned} &\int d\omega d\omega' C_{-1,1}(\omega,t;\omega',t') \\ &= KN\int_{t_0}^{t'} dt_1 \int d\omega d\omega' d\omega_1 d\omega_1' P_{1,-1}(\omega,\omega_1,t-t_1) \times P_{-1,1}(\omega',\omega_1',t'-t_1)g(\omega_1)g(\omega_1') \end{aligned}$$

and

$$\int d\omega d\omega_1 P_{\pm 1,\mp 1}(\omega,\omega_1,s)g(\omega_1) = \frac{1}{\pi NK}\left[\frac{1-\Lambda_{\pm}(s)}{\Lambda_{\pm}(s)}\right], \quad (22)$$

which is particularly simple with the Cauchy-Lorentz distribution:

$$\int d\omega d\omega_1 P_{\pm 1,\mp 1}(\omega,\omega_1,t-t_1)g(\omega_1) = \frac{1}{2\pi N}e^{-(\gamma-\frac{K}{2})(t-t_1)} \quad (23)$$

to obtain

$$\int d\omega d\omega' C_{-1,1}(\omega,t;\omega',t') = \frac{K}{4\pi^2 N}\int_{t_0}^{t'} dt_1 e^{-(\gamma-\frac{K}{2})(t+t'-2t_1)}.$$

Thus

$$\begin{aligned} \delta\Omega_B(t) &= -\frac{K^3}{2N}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt_1 e^{-(\gamma-\frac{K}{2})(t+t'-2t_1)}\sin[\Omega(t-t')] \\ &= -\frac{K^3}{2N}\left(\frac{\{-\sin[\Omega(t-t_0)]e^{-(\gamma-\frac{K}{2})(t-t_0)}\}}{\left[\Omega^2+\left(\gamma-\frac{K}{2}\right)^2\right]} + \frac{\Omega}{2\gamma-K}\frac{\left(1-e^{-2(\gamma-\frac{K}{2})(t-t_0)}\right)}{\left[\Omega^2+\left(\gamma-\frac{K}{2}\right)^2\right]}\right). \end{aligned} \quad (24)$$

Now

$$\delta\Omega_C(t) = -K^2\int d\theta d\omega \int_{t_0}^{t} dt' d\theta' d\omega' \cos[\theta-\Omega(t-t_0)] \times \sin[\theta'-\Omega(t'-t_0)]P(\theta,\omega,t;\theta',\omega',t')\frac{g(\omega')}{2\pi}$$

with the Cauchy-Lorentz distribution becomes

$$\delta\Omega_C(t) = -\frac{K^2}{2}\frac{1}{N}\int_0^{t-t_0} d\tau e^{-(\gamma-\frac{K}{2})\tau}\sin(\Omega\tau)$$

which is equal to $\delta\Omega_A(t)$.

Hence

$$\delta\Omega(t) = -\frac{1}{N}\frac{K^2}{2}\Bigg\{\left[\frac{\Omega}{\Omega^2+\left(\gamma-\frac{K}{2}\right)^2}\right]\left(\frac{4\gamma-K}{2\gamma-K}\right) - \frac{e^{-(\gamma-\frac{K}{2})(t-t_0)}}{\Omega^2+\left(\gamma-\frac{K}{2}\right)^2}[2\Omega\cos\Omega(t-t_0) + 2\gamma\sin\Omega(t-t_0)] - \frac{\Omega K}{2\left(\gamma-\frac{K}{2}\right)\left[\Omega^2+\left(\gamma-\frac{K}{2}\right)^2\right]} \times e^{-2(\gamma-\frac{K}{2})(t-t_0)}\Bigg\}. \tag{25}$$

As $t \to \infty$, the initial state terms vanish and we are left with a constant shift in the driving frequency given by

$$\delta\Omega_\infty = -\frac{1}{N}\frac{K^2}{2}\left[\frac{\Omega}{\Omega^2+\left(\gamma-\frac{K}{2}\right)^2}\right]\left(\frac{4\gamma-K}{2\gamma-K}\right). \tag{26}$$

The contribution to the two-point function is given by $G(t,t') = G_B(t,t') + G_C(t,t')$, which is

$$G(t,t') = K^2\int d\theta d\omega d\theta' d\omega' \sin[\theta-\Omega(t-t_0)] \times \sin[\theta'-\Omega(t'-t_0)]\left[C(\theta,\omega,t;\theta',\omega',t') + P(\theta,\omega,t;\theta',\omega',t')\frac{g(\omega')}{2\pi}\right]. \tag{27}$$

This can be computed in closed form in terms of the dielectric function $\Lambda(s)$ (C and P are computed in terms of Λ in Refs. [6,7]). For a Cauchy-Lorentz distribution of frequencies, it reduces to (with $t > t'$)

$$G(t,t') = \frac{K^2}{2N}e^{-(\gamma-\frac{K}{2})(t-t')}\cos[\Omega(t-t')] \times \left(\frac{2\gamma}{2\gamma-K} - \frac{K}{2\gamma-K}e^{-(2\gamma-K)(t'-t_0)}\right). \tag{28}$$

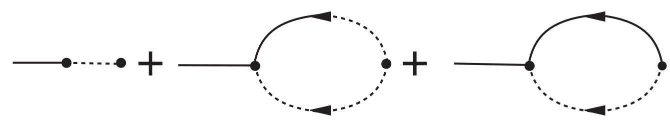

FIG. 4. Feynman diagrams for $\langle\delta\phi(t)\rangle$ derived directly from the action S from Eq. (5). Dashed lines represnt φ, $\tilde{\varphi}$ variables; solid lines represent $\delta\phi(t)$, $\tilde{\phi}(t)$ variables.

Returning to the effective stochastic equation (10), we have derived $\delta\Omega(t)$ and shown that $\xi(t)$ is a noise term such that $\langle\xi(t)\rangle = 0$ and

$$\langle\xi(t)\xi(t')\rangle = G(t,t')H_+(t-t') + G(t',t)H_-(t'-t), \tag{29}$$

where H_+ and H_- are Heaviside step functions with $H_+(0) = 1$ and $H_-(0) = 0$.

A. Moments of $\delta\phi$

We use the action S_{EA} to compute the moments of $\delta\phi(t)$. Under the action (5), the terms correspond to amputated versions of the diagrams shown in Fig. 4. By definition, the initial condition is $\delta\phi(t_0) = 0$ for all instances in the ensemble, which implies that $\langle\delta\phi(t_0)\rangle = 0$. Using the Cauchy-Lorentz distribution, the mean evolution is given by

$$\langle\delta\phi(t)\rangle = -\frac{1}{N}\frac{K^2}{2}\Bigg(\left[\frac{\Omega}{\Omega^2+\left(\gamma-\frac{K}{2}\right)^2}\right]\left(\frac{4\gamma-K}{2\gamma-K}\right)(t-t_0) - \frac{e^{-(\gamma-\frac{K}{2})(t-t_0)}}{\left[\Omega^2+\left(\gamma-\frac{K}{2}\right)^2\right]^2}\left\{2\left[\Omega^2-\gamma\left(\gamma-\frac{K}{2}\right)\right]\sin\Omega(t-t_0) - 2\left[\Omega\left(\gamma-\frac{K}{2}\right)+\gamma\Omega\right]\cos\Omega(t-t_0)\right\} - \frac{\Omega K}{4\left(\gamma-\frac{K}{2}\right)^2\left[\Omega^2+\left(\gamma-\frac{K}{2}\right)^2\right]}\left(1-e^{-2(\gamma-\frac{K}{2})(t-t_0)}\right) - \frac{2\left[\Omega\left(\gamma-\frac{K}{2}\right)+\gamma\Omega\right]}{\left[\Omega^2+\left(\gamma-\frac{K}{2}\right)^2\right]^2}\Bigg)$$

and the nonequal time two-point function is given by (with $t > t'$)

$$\langle\delta\phi(t)\delta\phi(t')\rangle = \int_{t'}^{t} dt_2\int_{t_0}^{t'} dt_1 G(t_2,t_1) + 2\int_{t_0}^{t'} dt_2\int_{t_0}^{t_2} dt_1 G(t_2,t_1).$$

Defining $\bar{\gamma} \equiv \gamma - \frac{K}{2}$ yields

$$\int_{t_0}^{T} dt_1 G(t_2,t_1) = \frac{1}{2}\left(\frac{K^2}{4N}\frac{2\gamma}{2\gamma-K}\right)\frac{1}{\bar{\gamma}-i\Omega}\left(e^{(-\bar{\gamma}+i\Omega)(t_2-T)} - e^{(-\bar{\gamma}+i\Omega)(t_2-t_0)}\right) + \text{c.c.} - \frac{1}{2}\left(\frac{K^2}{4N}\frac{K}{2\gamma-K}\right)\frac{1}{-\bar{\gamma}-i\Omega}\left(e^{-\bar{\gamma}(t_2+T)+i\Omega(t_2-T)+2\bar{\gamma}t_0} - e^{-\bar{\gamma}(t_2-t_0)+i\Omega(t_2-t_0)}\right) + \text{c.c.}$$

Hence

$$\int_{t'}^{t} dt_2 \int_{t_0}^{t'} dt_1 G(t_2,t_1)$$
$$= -\left(\frac{K^2}{4N}\frac{2\gamma}{2\gamma - K}\right)\frac{1}{\bar{\gamma}^2+\Omega^2}[e^{-\bar{\gamma}(t-t')}\cos\Omega(t-t') - e^{-\bar{\gamma}(t-t_0)}\cos\Omega(t-t_0) - 1 + e^{-\bar{\gamma}(t'-t_0)}\cos\Omega(t'-t_0)]$$
$$-\left(\frac{K^2}{4N}\frac{K}{2\gamma - K}\right)\frac{1}{\bar{\gamma}^2+\Omega^2}[e^{-\bar{\gamma}(t+t')+2\bar{\gamma}t_0}\cos\Omega(t-t') - e^{-\bar{\gamma}(t-t_0)}\cos\Omega(t-t_0) - e^{-2\bar{\gamma}(t'-t_0)} + e^{-\bar{\gamma}(t'-t_0)}\cos\Omega(t'-t_0)]$$

and

$$2\int_{t_0}^{t'} dt_2 \int_{t_0}^{t_2} dt_1 G(t_2,t_1) = \frac{K^2}{2N}\left\{\frac{\gamma}{\bar{\gamma}^2+\Omega^2}(t'-t_0) + \frac{\gamma}{\bar{\gamma}}\frac{\bar{\gamma}^2-\Omega^2}{(\bar{\gamma}^2+\Omega^2)^2}[\cos\Omega(t'-t_0)e^{-\bar{\gamma}(t'-t_0)} - 1] - \frac{2\gamma\Omega}{(\bar{\gamma}^2+\Omega^2)^2}\sin\Omega(t'-t_0)e^{-\bar{\gamma}(t'-t_0)} - \frac{K}{4\bar{\gamma}}\frac{1}{\bar{\gamma}^2+\Omega^2}(e^{-2\bar{\gamma}(t'-t_0)} - 1) + \frac{K}{2\bar{\gamma}}\frac{1}{\bar{\gamma}^2+\Omega^2}[e^{-\bar{\gamma}(t'-t_0)}\cos\Omega(t'-t_0) - 1]\right\}.$$

As $t,t' \to \infty$, the equal time variance is

$$\langle\delta\phi(t)\delta\phi(t)\rangle = \frac{K^2}{2N}\left[\frac{\gamma}{\bar{\gamma}^2+\Omega^2}(t-t_0) - \frac{\gamma}{\bar{\gamma}}\frac{\bar{\gamma}^2-\Omega^2}{(\bar{\gamma}^2+\Omega^2)^2} - \frac{K}{4\bar{\gamma}}\frac{1}{\bar{\gamma}^2+\Omega^2}\right] \tag{30}$$

which means that at large times the equal time variance obeys $\langle\delta\phi(t)\delta\phi(t)\rangle \sim D(t-t_0)$ with a coefficient

$$D = \frac{K^2}{2N}\frac{\gamma}{\left(\gamma - \frac{K}{2}\right)^2 + \Omega^2}, \tag{31}$$

$\delta\Omega(t \to \infty)$ from Eq. (26) and D both appear in the spectrum shift at order $1/N$ for the marginal modes of the oscillator density in a linear stability analysis of the incoherent state; full details are in Ref. [7].

IV. COMPARISON TO NUMERICAL SIMULATIONS

Figure 5 compares the analytic computation of $\delta\Omega(t)$ with numerically simulated ensemble averages of the time derivative of $\delta\phi(t)$, approximated by a finite difference with time step δt. In each case, the measured oscillator was given initial condition $\phi(t_0) = 0$ and driving frequency Ω. The unmeasured oscillators' initial phases were drawn from a uniform distribution. The driving frequencies were drawn from a Cauchy-Lorentz distribution of width γ. The simulation used a simple Euler step with step length $\delta t = 0.1$ and the ensemble averages were taken over 1 million samples. From the figure one can see that the $1/N$ approximation works very well for N near 1000 and reasonably well for $N = 100$. Significant deviations appear for $N = 10$. Figures 6 and 7 compares the covariance of $d\phi(t)/dt$ [i.e., $G(t,t')$] between time points t and t' over 10 000 samples with the analytic prediction. Again, one sees that the $1/N$ expansion works well for large N. Also, one sees that deviations for large N become larger as one approaches synchrony, $K = 2\gamma$. Figures 8 and 9 show the evolution of the variance of the time derivative.

V. DISCUSSION

We have constructed an equivalent stochastic differential equation for a single unit in a collection of coupled oscillators. The distribution for this stochastic process is directly computed via marginalization over the remaining oscillators in the network, which we term "unmeasured" to suggest a comparison with an experimental situation in which only some components of a network are accessible. The distribution of the unmeasured oscillators is induced via the deterministic dynamics on the initial distribution. In the present example we used the incoherent state of the Kuramoto model for the background. In principle, this initial distribution corresponds to the state of information the external observer has about the network. The

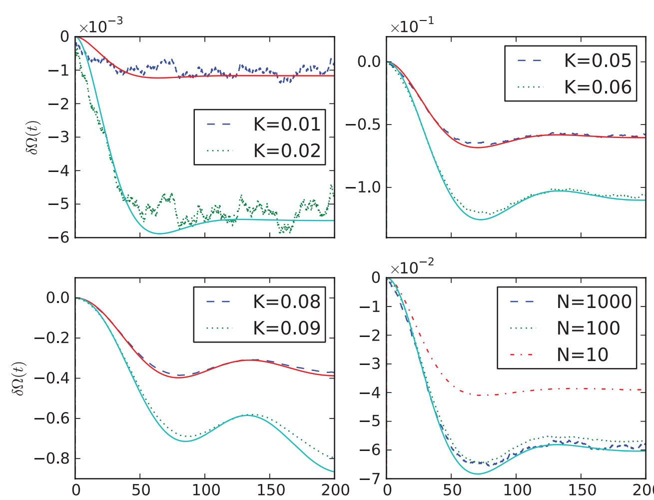

FIG. 5. (Color online) $\delta\Omega(t)N$ vs t. Comparison of analytic and simulation results for different values of K and $N = 1000$. Bottom right figure shows the comparison with various values of N; $K = 0.05$. Other parameters are $\gamma = 0.05$, $\Omega = 0.05$. Time step for simulation was $\delta t = 0.1$ and the ensemble average was taken over 1 million samples. Dashed lines are the values indicated in the legend; adjoining solid lines are the analytic prediction.

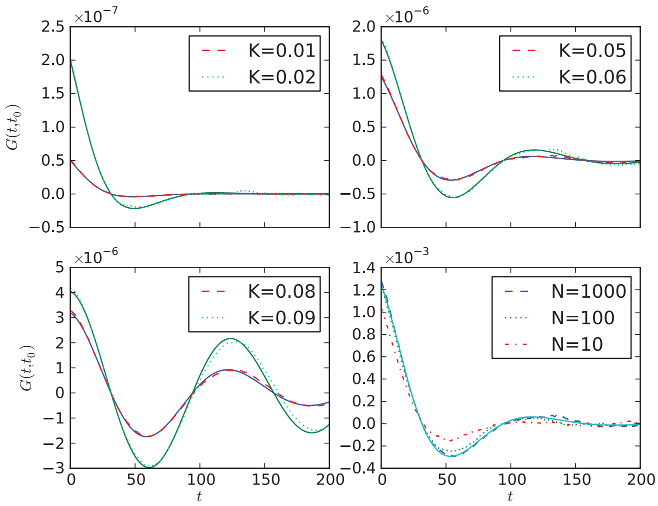

FIG. 6. (Color online) Covariance of $d\phi(t)/dt$ at t and $t_0 = 0$ s. Abscissa is t. Bottom right compares simulation and analytic prediction for different values of N at $K = 0.05$. All others compare analytic and simulation results for different values of K at $N = 1000$. Other parameters are $\gamma = 0.05$, $\Omega = 0.05$. Time step for simulation was $\delta t = 0.1$ and the ensemble average was taken over 10 000 samples.

equivalent stochastic equation describes the effect the initial uncertainty in the network state has on the evolution of the measured oscillator. The terms in the equation are necessarily dependent upon the characteristics of the system. Because of this, measurement of a single oscillator evolution can be used to estimate global properties of the network. This is potentially important for applications such as neuroscience, in which one typically measures only a handful of neurons which are themselves components of extremely large networks.

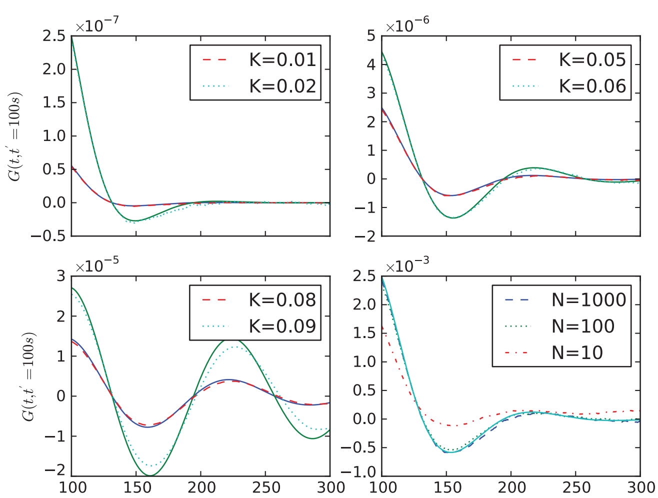

FIG. 7. (Color online) Covariance of $d\phi(t)/dt$ at t and $t' = 100$ s, with $t_0 = 0$ s. Abscissa is t. Bottom right compares simulation and analytic prediction for different values of N at $K = 0.05$. All others compare analytic and simulation results for different values of K at $N = 1000$. Other parameters are $\gamma = 0.05$, $\Omega = 0.05$. Time step for simulation was $\delta t = 0.1$ and the ensemble average was taken over 10 000 samples.

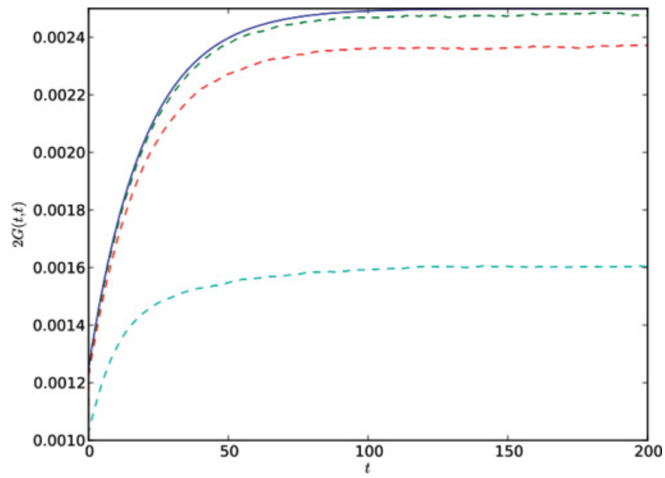

FIG. 8. (Color online) Variance of $\delta\phi(t + \delta t) - \delta\phi(t)$ for different values of N with a comparison of simulation and analytic prediction (solid curve). Each curve has been normalized by N. The curves from bottom to top correspond to $N = 10$, $N = 100$, and $N = 1000$, respectively. Other parameters are $\gamma = 0.05$, $\Omega = 0.05$, $K = 0.05$. Deviations correspond to $O(1/N^2)$ corrections.Time step for simulation was $\delta t = 0.1$ and the ensemble average was taken over 1 million samples.

Other authors, in particular [8–11], have used an averaging approach for coupled systems of harmonic oscillators in order to derive Brownian motion. They compute the distribution imposed by the Hamiltonian dynamics on an initial state distributed under the (Gaussian) canonical ensemble. Owing to the stationary and linear character of the dynamics, or, put another way, the exact integrability of the quadratic potential or Gaussian generating functional, the resulting distributions are Gaussian. Furthermore, [10] shows that a specific choice of interaction matrix yields a standard memoryless stochastic

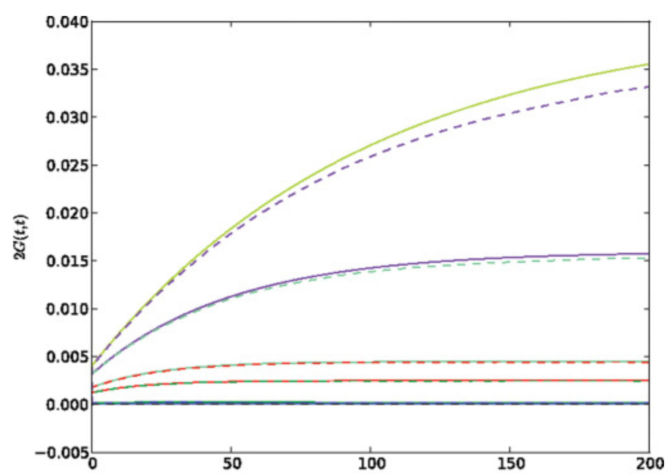

FIG. 9. (Color online) Variance of $\delta\phi(t + \delta t) - \delta\phi(t)$ for different values of K with a comparison of simulation (dashed lines) and analytic prediction (solid curves). Curves from bottom to top correspond to $K = 0.01$, $K = 0.02$, $K = 0.05$, $K = 0.06$, $K = 0.08$, and $K = 0.09$, respectively. Each curve has been normalized by N. Other parameters are $\gamma = 0.05$, $\Omega = 0.05$, $N = 1000$. Time step for simulation was $\delta t = 0.1$ and the ensemble average was taken over 1 million samples.

differential equation with δ correlated Gaussian noise, which holds in the limit of large N and for time scales that are long compared to a cutoff interaction time. These computations are facilitated by the quadratic nature of the Hamiltonian and the corresponding Gaussian character of the canonical distribution. Our method is a generalization of these approaches in that it does not rely on a Gaussian assumption of the bath nor does it require the system to be Hamiltonian. In exchange, our approach reduces the problem of deriving an effective stochastic equation to the problem of dealing with the BBGKY hierarchy. For example, should the hierarchy be truncatable, our approach will yield tractable results, regardless of the other underlying features of the model. In the present example the global coupling provides us with a loop expansion parameter which we can use to truncate the hierarchy, which thus becomes a finite size expansion. However, we stress that our model is not limited to global coupling. It could be applied to any coupled dynamical system in which the bulk dynamics have some ready means of approximation for the moments. In the Kuramoto model this is supplied by the $1/N$ loop expansion. In other models it could be due to the presence of other small parameters, such as the inverse of the distance to a critical point [19] or the inverse of the plasma parameter in plasma dynamics [5]. For gas dynamics, traditional approximations to the pair-potential approximations could possibly be utilized [4]. We aim to explore this in future work.

The construction also readily generalizes to any arbitrary function of the configuration variables. For example, an effective stochastic equation for the Kuramoto order parameter $Z(t)$ or for a fixed subset of oscillators can be derived in a similar manner via marginalization.

ACKNOWLEDGMENT

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APPENDIX

1. Generating functional for the initial configuration

Uncertainty in the initial configuration of $\phi(t_0)=\phi_0$ and $\eta(\theta,\omega,t_0)=\rho(\theta,\omega)$ corresponds to a distribution of possible configurations. In the case of the measured oscillator $\phi(t)$ this is a distribution over the variables ϕ_0 and Ω. For the population of unmeasured oscillators, the initial data obeys a distribution $\rho(\vec{\theta},\vec{\omega})$, which induces a distribution over the densities $\eta(\theta,\omega,t)$. The generating functional of the moments of the distribution in $\eta(\theta,\omega,t_0)$ is given by

$$Z[\tilde{\eta}(\theta,\omega,t_0)]=\left\langle \exp\left[N\int d\theta d\omega \tilde{\eta}(\theta,\omega,t_0)\eta(\theta,\omega,t_0)\right]\right\rangle$$

where moments of $\eta(\theta,\omega,t_0)$ are given by functional derivatives with respect to $\tilde{\eta}(\theta,\omega,t_0)$:

$$\left\langle \prod_i \eta(\theta_i,\omega_i,t_0)\right\rangle = \left[\prod_i \frac{1}{N}\frac{\delta}{\delta\tilde{\eta}(\theta_i,\omega_i,t_0)}\right] Z[\tilde{\eta}(\theta,\omega,t_0)]\Bigg|_{\tilde{\eta}=0}$$

In general this is given by

$$\begin{aligned}Z[\tilde{\eta}(\theta,\omega,t_0)] &= \left\langle \exp\left[N\int d\theta d\omega \tilde{\eta}(\theta,\omega,t_0)\eta(\theta,\omega,t_0)\right]\right\rangle \\ &= \int\prod_i d\theta_i d\omega_i \rho(\vec{\theta},\vec{\omega})\exp\left\{\int d\theta d\omega\tilde{\eta}(\theta,\omega,t_0)\right. \\ &\quad \left.\times\sum_i \delta[\theta-\theta_i(t)]\delta(\omega-\omega_i)\right\} \\ &= \int\prod_i d\theta_i d\omega_i \rho(\vec{\theta},\vec{\omega})\exp\left[\sum_i \tilde{\eta}(\theta_i,\omega_i,t_0)\right] \\ &= \exp W[\tilde{\eta}(\theta,\omega,t)],\end{aligned}$$

where W is the connected generating functional for $\rho(\vec{\theta},\vec{\omega})$.

In the text we assume that each oscillator's initial position $\theta_i(t_0)$ is independently drawn from a distribution $\rho_0(\theta)$ and each driving frequency is independently drawn from a distribution $g(\omega)$. This means that

$$\rho(\vec{\theta},\vec{\omega})=\prod_i \rho_0(\theta_i)g(\omega_i).$$

Thus

$$\begin{aligned}&Z[\tilde{\eta}(\theta,\omega,t_0)] \\ &\quad =\int\prod_i d\theta_i d\omega_i \rho_0(\theta_i)g(\omega_i)\exp\left[\sum_i\tilde{\eta}(\theta_i,\omega_i,t_0)\right] \\ &\quad =\left[\int d\theta d\omega\rho_0(\theta)g(\omega)e^{\tilde{\eta}(\theta(t_0),\omega,t_0)}\right]^N \\ &\quad =\left[1+\int d\theta d\omega\rho_0(\theta)g(\omega)\left(e^{\tilde{\eta}(\theta(t_0),\omega,t_0)}-1\right)\right]^N \\ &\quad =\exp\left\{N\ln\left[1+\int d\theta d\omega\rho_0(\theta)g(\omega)(e^{\tilde{\eta}(\theta(t_0),\omega,t_0)}-1)\right]\right\}.\end{aligned}$$

The variables ϕ_0 and Ω are considered known and so the generating functional is simply

$$Z[\tilde{\phi}(t_0)]=\exp[\tilde{\phi}(t_0)\phi_0].$$

(It is not necessary to consider Ω because it is fixed for all time and hence does not influence the future distribution.)

2. Time evolution of the generating functional

Before deriving the action for the Kuramoto model, we give an example for a simple dynamical system

$$\dot{x}=f(x) \tag{A1}$$

with some initial distribution in x_0, $P(x_0)$ and x may be a vector. Given the distribution at time t', the distribution at a later time t can be written as

$$P[x(t)]=\int dx(t')\delta[x(t)-X(t)]P[x(t')] \tag{A2}$$

where $X(t)$ is the solution of the dynamical system (A1) with fixed initial condition $X(t') = x(t')$. The generating function is the Laplace transform of the distribution:

$$
\begin{aligned}
Z[\tilde{x}(t)] &= \int dx(t) P[x(t)] e^{\tilde{x}(t)x(t)} \\
&= \int dx(t') e^{\tilde{x}(t)X(t)} P[x(t')] \\
&= \frac{1}{2\pi i} \int dx(t') d\tilde{x}(t') e^{\tilde{x}(t)X(t) - \tilde{x}(t')x(t')} Z[\tilde{x}(t')],
\end{aligned}
$$

where we have used (A2) and the inverse Laplace transform for $P[x(t')]$.

Consider $t = t' + \Delta t$. Then we have, using (A1),

$$
\begin{aligned}
Z[\tilde{x}(t)] &= \frac{1}{2\pi i} \int dx(t') d\tilde{x}(t') \\
&\quad \times e^{\tilde{x}(t'+\Delta t)\{x(t') + f[x(t')]\Delta t\} - \tilde{x}(t')x(t')} Z[\tilde{x}(t')] \\
&= \frac{1}{2\pi i} \int dx(t') d\tilde{x}(t') \\
&\quad \times e^{[\frac{\tilde{x}(t'+\Delta t) - \tilde{x}(t')}{\Delta t} x(t') + \tilde{x}(t'+\Delta t) f[x(t')]]\Delta t} Z[\tilde{x}(t')].
\end{aligned}
$$

Given a time interval $t \in [t_0, T)$, we can divide this interval into M subintervals of length Δt. We can use this to develop a path integral, by computing the generating functional at time $t = T$ via repeated application of the above formula:

$$
\begin{aligned}
Z[\tilde{x}(T)] &= \frac{1}{(2\pi i)^M} \int \prod_{j=0}^{M-1} dx(t_j) d\tilde{x}(t_j) \\
&\quad \times e^{\sum_{j=0}^{M-1} \{\frac{\tilde{x}(t_j+\Delta t) - \tilde{x}(t_j)}{\Delta t} x(t_j) + \tilde{x}(t_j+\Delta t) f[x(t_j)]\}\Delta t} Z[\tilde{x}(t_0)],
\end{aligned}
$$

where $t_j = t_0 + j\Delta t$. Because the initial distribution is normalized we have

$$
Z[0] = \int dx P(x) = 1. \tag{A3}
$$

The path integral thus defines a normalized measure when $\tilde{x}(T) = 0$. Taking the $M \to \infty$ limit gives us

$$
Z[0] = \int D\tilde{x}(t) Dx(t) e^{-S[\tilde{x}(t), x(t)]},
$$

where the measure is defined as

$$
D\tilde{x}(t) Dx(t) = \lim_{M\to\infty} \prod_{j=0}^{M-1} \frac{dx(t_j) d\tilde{x}(t_j)}{2\pi i}
$$

with the $\tilde{x}(t_i)$ integrations following a contour parallel to the imaginary axis and the $x(t_j)$ following a contour parallel to the real axis. The action $S[\tilde{x}(t), x(t)]$ is

$$
S[\tilde{x}(t), x(t)] = \int_{t_0}^{T} dt \tilde{x}(t) \left\{ \frac{d}{dt} x(t) - f[x(t)] \right\} - W[\tilde{x}(t_0)],
$$

where we have integrated by parts to put the derivative on $x(t)$ and expressed the initial generating functional in terms of the cumulant generating functional $W[\tilde{x}(t_0)] = \ln Z[\tilde{x}(t_0)]$. Expectation values can be computed by inserting factors of $x(t)$ and $\tilde{x}(t)$ at various times t.

3. The action for the Kuramoto system

Using the above construction and the equations of motion for the Kuramoto system (4), we have the action as

$$
S[\tilde{\eta}, \eta, \tilde{\phi}(t), \phi(t)] = S_{\text{bulk}} + S_{\text{int}} + S_{\phi},
$$

where

$$
\begin{aligned}
S_{\text{bulk}} &= N \int d\omega d\theta dt \tilde{\eta}(\theta,\omega,t) \Big[(\partial_t + \omega\partial_\theta)\eta(\theta,\omega,t) \\
&\quad + K\partial_\theta \int d\omega' d\theta' \sin(\theta' - \theta)\eta(\theta',\omega',t)\eta(\theta,\omega,t) \Big] \\
&\quad - N \ln\left[1 + \int d\theta d\omega \rho_0(\theta) g(\omega)(e^{\tilde{\eta}(\theta,\omega,t_0)} - 1) \right]
\end{aligned} \tag{A4}
$$

with

$$
S_{\text{int}} = K \int d\omega d\theta dt \, \tilde{\eta}(\theta,\omega,t)\partial_\theta\{\sin[\phi(t) - \theta]\eta(\theta,\omega,t)\} \tag{A5}
$$

and

$$
\begin{aligned}
S_\phi = \int dt \, \tilde{\phi}(t) \Big\{ &\dot{\phi}(t) - \Omega \\
&- K \int d\theta d\omega \sin[\theta - \phi(t)]\eta(\theta,\omega,t) \Big\}.
\end{aligned}
$$

To arrive at the action shown in the main text, we perform two transformations. The first is given by

$$
\begin{aligned}
\psi(\theta,\omega,t) &= \eta(\theta,\omega,t) e^{-\tilde{\eta}(\theta,\omega,t)}, \\
\tilde{\psi}(\theta,\omega,t) &= e^{\tilde{\eta}(\theta,\omega,t)} - 1,
\end{aligned}
$$

which we call the Doi-Peliti-Janssen transformation [7,18]. In particular, this implies

$$
\eta(\theta,\omega,t) = \tilde{\psi}(\theta,\omega,t)\psi(\theta,\omega,t) + \psi(\theta,\omega,t).
$$

The second transformation is to translate by the mean field solutions:

$$
\begin{aligned}
\varphi(\theta,\omega,t) &= \psi(\theta,\omega,t) - \rho_0(\theta,\omega,t), \\
\delta\phi(t) &= \phi(t) - \Omega(t - t_0),
\end{aligned}
$$

where $\rho_0(\theta,\omega,t)$ is the solution of the mean field equation for the unmeasured oscillators to zeroth order in $1/N$, that is,

$$
\begin{aligned}
&\partial_t \rho_0(\theta,\omega,t) - \omega\partial_\theta \rho_0(\theta,\omega,t) \\
&\quad -\partial_\theta \left\{ \left[K \int d\theta' d\omega' \sin(\theta' - \theta)\rho_0(\theta',\omega',t) \right] \rho_0(\theta,\omega,t) \right\} = 0
\end{aligned}
$$

with initial condition determined by the mean field contribution to $W[\tilde{\eta}(\theta,\omega,t_0)]$, that is, the linear term. In the text we use the incoherent state of the Kuramoto model, which implies

$$
\rho_0(\theta,\omega,t) = \frac{g(\omega)}{2\pi}. \tag{A6}
$$

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