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The Projective Envelope of a Cuspidal Representation of $\mathrm{GL}_n(\mathbb{F}_q)$

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The Projective Envelope of a Cuspidal Representation of $\mathrm{GL}_n(\mathbb{F}_q)$

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The Projective Envelope of a Cuspidal Representation of $\mathrm{GL}_n(\mathbb{F}_q)$

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Let ℓ be a prime and let q be a prime power not divisible by ℓ. Put $G = \mathrm{GL}_n(\mathbb{F}_q)$ and fix a representation $\bar{\pi}$ of G over a sufficiently large finite field, k, of characteristic ℓ, such that $\bar{\pi}$ is cuspidal but not supercuspidal. We compute the $\mathrm{W}(k)[G]$-endomorphism ring of the projective envelope of $\bar{\pi}$ under the assumption that $\ell > n$.

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Chapter 1

Introduction

Let p be a prime and suppose that F is a p-adic field. One of the primary goals of number theory is to understand the structure of the absolute Galois group, Gal_F, of the field F and an understanding of the representations of Gal_F would be very useful in this regard. Classically, much has been understood about the representations of Gal_F by considering representations of the linear groups $\mathrm{GL}_n(F)$ for varying n. In particular, one has the celebrated local Langlands correspondence, which we now summarize, following [8].

Attached to F is the normalized absolute value $|\cdot|_F$ (that is, the one that takes a uniformizer in F to the reciprocal of the order of the residue field). Also attached to F and closely related to Gal_F is the Weil group, W_F, of F (see for example [19]). Local class field theory (again see [19]) gives a canonical isomorphism

$$\mathrm{Art}_F : F^\times \to \mathrm{W}_F^{\mathrm{ab}}$$

(up to an appropriate normalization). Here $\mathrm{W}_F^{\mathrm{ab}}$ is the abelianization of W_F.

In that Gal_F is a profinite group, technical issues arise that prevent a meaningful study of $\mathbb{C}$-representations of Gal_F over $\mathbb{C}$ in that the topology on $\mathbb{C}$ does not allow enough continuous representations to be constructed. Thus, a proxy is needed for an n-dimensional representation of Gal_F and the notion of a so-called n-dimensional Frobenius semi-simple Weil-Deligne representation of W_F was developed to fill this role. Explicitly, the latter is pair (r, N), where r is a finite dimensional semi-simple representation of W_F that is trivial on an open subgroup and N is an element of $\mathrm{End}_{\mathbb{C}}(r)$ that satisfies

$$r(\sigma)Nr(\sigma)^{-1} = |\mathrm{Art}_F^{-1}(\sigma)|_F N.$$

The reason that this notion is an appropriate choice to serve for such a proxy will be made more clear later. We denote by $\mathrm{WDRep}^n_{\mathbb{C}}(\mathrm{W}_F)$ the collection of isomorphism classes of such representations. Likewise, denote by $\mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_n(F))$ the collection of isomorphism classes of irreducible admissible $\mathbb{C}$-representations of $\mathrm{GL}_n(F)$.

To formulate the Local Langlands Correspondence, one must fix a non-trivial additive character $\Psi : F \to \mathbb{C}^\times$, but, as we will note later, the correspondence ultimately does not depend on this choice. Once such a character is fixed, we may, given $[\pi_1] \in \mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_{n_1}(F))$ and $[\pi_2] \in \mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_{n_2}(F))$, construct an L-factor $L(\pi_1 \times \pi_2, s)$ and an epsilon factor $\epsilon(\pi_1 \times \pi_2, s, \Psi)$ (for this construction see [11]). Similarly if $[(r, N)] \in \mathrm{WDRep}^n_{\mathbb{C}}(\mathrm{W}_k)$, one has an L factor $L((r, N), s)$ and an epsilon factor $\epsilon((r, N), s, \Psi)$ (see Chapter VII.2 of [8] and [21]). As our notation suggests, only the ϵ-factor construction depends on our choice of Ψ.

The local Langlands correspondence is a correspondence

$$\mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_n(F)) \to \mathrm{WDRep}^n_{\mathbb{C}}(\mathrm{W}_F)$$

that behaves well with respect to these and other notions. Explicitly, it is the following result:

Theorem 1.1. *There is a unique set of bijections*

$$\mathrm{rec}_F : \mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_n(F)) \to \mathrm{WDRep}_{\mathbb{C}}^n(\mathrm{W}_F)$$

satisfying the following properties:

1. *The bijections generalize local class field theory, that is, if* $[\pi] \in \mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_1(F))$ *then* $\mathrm{rec}_F(\pi) = \pi \circ \mathrm{Art}_F^{-1}$.

2. *The bijections behave well with respect to L-factors and epsilon factors, that is, if* $[\pi_1] \in \mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_{n_1}(F))$ *and* $[\pi_2] \in \mathrm{Irr}_{\mathbb{C}}(\mathrm{GL}_{n_2}(F))$ *then*

$$L(\pi_1 \times \pi_2, s) = L(\mathrm{rec}_F(\pi_1) \otimes \mathrm{rec}_F(\pi_2), s)$$

and

$$\epsilon(\pi_1 \times \pi_2, s, \Psi) = \epsilon(\mathrm{rec}_F(\pi_1) \otimes \mathrm{rec}_F(\pi_2), s, \Psi).$$

3. *If* $[\pi] \in \mathrm{Irr}(\mathrm{GL}_n(F))$ *and* $[\chi] \in \mathrm{Irr}(\mathrm{GL}_1(F))$ *then*

$$\mathrm{rec}_F(\pi \otimes (\chi \circ \det)) = \mathrm{rec}_F(\pi) \otimes \mathrm{rec}_F(\chi).$$

4. *If* $[\pi] \in \mathrm{Irr}(\mathrm{GL}_n(F))$ *and* π *has central character* χ *then*

$$\det \mathrm{rec}_F(\pi) = \mathrm{rec}_F(\chi).$$

5. *If* $[\pi] \in \mathrm{Irr}(\mathrm{GL}_n(F))$ *then* $\mathrm{rec}_F(\pi^\vee) = \mathrm{rec}_F(\pi)^\vee$ *(where* $\vee$ *denotes contragredient).*

Moreover, the bijection rec_F *that satisfies these properties does not depend on the choice of* Ψ*.*

Proof. The local Langlands correspondence was proven by Harris and Taylor (see [8]) and also by Henniart (see [10]). □

An important observation for our purposes regarding the local Langlands correspondence is the fact that none of the notions or results depend on the topology of $\mathbb{C}$ (indeed, the notion of a Frobenius semi-simple Weil-Deligne representation was more or less created to avoid an interaction with this topology). As a result, if ℓ is a prime and $\bar{\mathbb{Q}}_\ell$ is a fixed algebraic closure of $\mathbb{Q}_\ell$, we may fix an algebraic isomorphism $\iota : \mathbb{C} \to \bar{\mathbb{Q}}_\ell$ and formulate a local Langlands correspondence over $\bar{\mathbb{Q}}_\ell$ simply by applying ι to all the notions.

Thus, if $\mathrm{Irr}_\ell(\mathrm{GL}_n(F))$ denotes the admissible $\mathrm{GL}_n(F)$-representations over $\bar{\mathbb{Q}}_\ell$ and if $\mathrm{WDRep}_\ell^n(\mathrm{W}_F)$ denotes the set of n-dimensional Frobenius semi-simple Weil-Deligne representation over $\bar{\mathbb{Q}}_\ell$ (defined just as it was for $\mathbb{C}$), we obtain a bijection

$$\mathrm{Irr}_\ell(\mathrm{GL}_n(F)) \to \mathrm{WDRep}_\ell^n(\mathrm{W}_F).$$

Moreover, since the L and ϵ-factors are also algebraic objects, the properties characterizing the bijection do not depend on ι and so the bijection does not either.

Moreover, the local Langlands correspondence is, in some sense, better behaved over $\bar{\mathbb{Q}}_\ell$ than it is over $\mathbb{C}$. Indeed, from 32.6.3 of [4] we see that there is a canonical equivalence of categories

$$\mathrm{WDRep}_\ell^n(\mathrm{W}_F) \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}^n(\mathrm{W}_F),$$

where $\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}^n(\mathrm{W}_F)$ is the category of n-dimensional continuous representations of W_F over $\bar{\mathbb{Q}}_\ell$. That this equivalence exists is the motivation for considering Weil-Deligne representations in the first place. While no such correspondence occurs over $\mathbb{C}$, the notion of Weil-Deligne representation may still be considered. Composing this equivalence of categories with the local Langlands correspondence for $\bar{\mathbb{Q}}_\ell$, we obtain a canonical bijection

$$\mathrm{Irr}_\ell(\mathrm{GL}_n(F)) \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\mathrm{W}_F),$$

which we might call the ℓ-adic local Langlands correspondence. Thus in some sense, it is more natural to formulate the local Langlands correspondence over $\bar{\mathbb{Q}}_\ell$ than it is to do so over $\mathbb{C}$.

Certainly, part of the importance of the local Langlands correspondence is that notions on one side give rise to notions on the other side. In particular, an important notion of any type of representation theory is that of Jordan-Hölder constituents and the related notion of semi-simplification. When one translates this notion of semi-simplification on the Weil representation side to the $\mathrm{GL}_n(F)$ side, one obtains a notion called supercuspidal support.

More precisely, suppose that π is a representation of $\mathrm{GL}_n(F)$ over $\bar{\mathbb{Q}}_\ell$. Then π corresponds under local Langlands to an n-dimensional Weil representation ρ. The representation ρ then has a semisimplification,

$$\rho^{\mathrm{ss}} = \rho_1 \oplus \cdots \oplus \rho_r,$$

where we denote the dimension of ρ_i by n_i. Then we have $n_1+\cdots+n_r = n$ and from ρ_i, we obtain a $\mathrm{GL}_{n_i}(F)$-representation π_i. The collection (π_i), typically denoted by a formal sum $\sum \pi_i$, is the *supercuspidal support* of π. In particular, we see that the representation π is its own supercupsidal support, in which case we say that π is *supercuspidal*, if and only if it corresponds to an irreducible Weil representation.

We should remark that these notions may be constructed without an appeal to the local Langlands correspondence and actually apply to more general reductive groups. We will suppress these more classical definitions here but refer the reader to Sections II.2.5. and II.2.6 of [22]. In addition, we note that we will discuss the analogous notions for the finite reductive group $\mathrm{GL}_n(\mathbb{F}_q)$ carefully in Chapter 3.

From the traditional perspective, one shows that the notion of supercuspidal support on the $\mathrm{GL}_n(F)$ side of the correspondence translates to semi-simplification on the Weil side and this aspect of the correspondence is often called the semi-simple (ℓ-adic) local Langlands correspondence. It is important partially because it translates well to the case of modular representations. Indeed, if ℓ is a prime distinct from p, and if k is the algebraic closure of a finite field of characteristic ℓ, one may construct a notion of supercuspidal support for $\mathrm{GL}_n(F)$-representation over k analogously as in characteristic zero. Vignéras has shown the semi-simple local Langlands correspondence then translates to representation over k:

Theorem 1.2. *There is a unique bijection between supercuspidal supports of*

$\mathrm{GL}_n(F)$*-representations over* k *and* n*-dimensional semi-simple Weil representations over* k *that is compatible with the semi-simple local Langlands correspondence and reduction modulo* ℓ*.*

Proof. See [24]. □

For obvious reasons, Vignéras's correspondence is often called the ℓ-modular semi-simple local Langlands correspondence.

When one passes to the ℓ-modular case, the situation regarding supercuspidality becomes a bit more complicated. Indeed, a supercuspidal representation of $\mathrm{GL}_n(F)$ should correspond to an irreducible Weil representation. On the other hand, it is possible for a reducible modular representation to have an irreducible lift to characteristic zero. Thus, we introduce the more general notion of cuspidality to reflect this possibility. Explicitly then, we say a modular representation of $\mathrm{GL}_n(F)$ is *cuspidal* if it has a lift to characteristic zero that is supercuspidal. Since supercupsidal representations of $\mathrm{GL}_n(F)$ correspond to irreducible Weil representations in both characteristics, one can also show that a cuspidal representation of $\mathrm{GL}_n(F)$ corresponds to a semi-simple modular Weil representation with an irreducible lift.

Once again cuspidality may be defined, in any characteristic, independently of the local Langlands correspondence (see Section II.2.2 of [22]). The fact that the traditional definition is equivalent to our can be found in Section III.5.10 of [22]. In that the notions of cuspidality and supercuspidality coincide in characteristic zero, we did not define cuspidality in characteristic zero.

One is led to consider whether these correspondences can be understood geometrically. To fix ideas then, let π be an irreducible representation of $\mathrm{GL}_n(F)$ over $\mathrm{W}(k)$ such that π is cuspidal but not supercuspidal. Assume that $\ell > n$. Denote by ρ the semi-simple Weil representation over k corresponding to π in the ℓ-modular local Langlands correspondence. Attached to ρ is the framed universal deformation ring, $R_\rho^\square$, which parameterizes lifts of ρ together with a choice of basis. One is led to consider whether a corresponding object can be found on the $\mathrm{GL}_n(F)$ side of local Langlands.

To this end, one considers the notion of the Bernstein center. The *Bernstein center* of any category $\mathcal{A}$ is the endomorphism ring of the identity functor on $\mathcal{A}$. In particular, it acts naturally on every object in $\mathcal{A}$. Classically, the Bernstein center has been important in the study of representations of $\mathrm{GL}_n(F)$. In particular, Bernstein and Deligne were able calculate the center of the category, $\mathrm{Rep}_{\mathbb{C}}(\mathrm{GL}_n(F))$, of smooth $\mathbb{C}$-representations of $\mathrm{GL}_n(F)$ (see [3]). Moreover, they give a decomposition of the category into a product of blocks (called Bernstein components) and a description of the center of each block, which they show to be a finitely generated $\mathbb{C}$-algebra. As usual, these results can be translated to the field $\bar{\mathbb{Q}}_\ell$.

In consideration of a geometric interpretation of the local Langlands correspondence, Helm has considered the situation for integral representation, that is for representations over k-Witt vectors, $\mathrm{W}(k)$. In particular, he obtains a block decomposition of the category $\mathrm{Rep}_{\mathrm{W}(k)}(\mathrm{GL}_n(F))$ which is analogous to but coarser than that obtained by Bernstein and Deligne (Theorem 10.8 of [9]).

Denote by $\mathcal{C}_\pi$ the block corrsponding to π. Then if A_π is the Bernstein center of $\mathcal{C}_\pi$, Helm also shows that A_π is finitely generated over $\mathrm{W}(k)$ (Theorem 12.1 of [9]).

Most important for our purposes, if Π is an irreducible representation in $\mathcal{C}_\pi$, Schur's lemma gives a map $f_\Pi : A_\pi \to k$ and Helm shows (Theorem 12.2 of [9]) that two irreducible representations induce the same map if and only if they have the same supercuspidal support. In this way, we get a bijection between maps $A_\pi \to k$ and possible supercuspidal supports of representation in $\mathcal{C}_\pi$. In particular, the supercuspidal support of π gives a maximal ideal, $\mathfrak{m}_\pi$, of A_π. In that the ring $R_\rho^\square$ is a complete ring, Helm ([9]) has conjectured that there is an isomorphism

$$(A_\pi)_{\mathfrak{m}_\pi} \to (R_\rho^\square)^{\mathrm{GL}_n}$$

(where the GL_n superscript denotes invariants under the change of basis action of GL_n and the $\mathfrak{m}_\pi$ denotes completion at $\mathfrak{m}_\pi$) which is compatible with the local Langlands correspondence. It is therefore advantageous to study the structure of the ring A_π.

To this end, attached to π is a so-called maximal distinguished cuspidal type. We omit the precise definition of this notion (see [23], IV.3.1B) but remark that a maximal distinguished cuspidal type is a special sort of pair (H, τ) where H is a compact open subgroup of $\mathrm{GL}_n(F)$ and τ is an $k[\mathrm{GL}_n(F)]$-module that is finitely generated as an k-module. The pair (H, τ) is constructed from an extension E/F of degree dividing n, say of ramification index e and residue degree f, and some other data. In addition, H contains a pro-p subgroup H^1 such that H/H^1 is isomorphic to $\mathrm{GL}_{\frac{n}{ef}}(\mathbb{F}_{q^f})$ and τ has the form $\kappa\otimes\sigma$ where σ is the inflation of a cuspidal representation of H/H^1 over k and the restriction of κ to H^1 is irreducible. Furthermore, there is an embedding $\mathrm{GL}_{\frac{n}{ef}}(E) \to \mathrm{GL}_n(F)$ such that $E^\times$ normalizes H and H^1 and τ extends to a representation, $\hat{\tau}$ of $E^\times H$ and for any such $\hat{\tau}$, we have $\pi = \text{c-Ind}_{E^\times K}^{\mathrm{GL}_n(F)}\hat{\tau}$ (here we are essentially applying IV.1.1-IV.1.3 of [23]).

We denote by $P_\sigma \to \sigma$ the projective envelope of σ in the category of $\mathrm{W}(k)[\mathrm{GL}_{\frac{n}{ef}}(\mathbb{F}_q^f)]$-modules. It can also be shown (Lemma 4.7 of [9]) that κ lifts to a representation $\tilde{\kappa}$ of H over $\mathrm{W}(k)$, the Witt vectors of k. Under these circumstances, it turns out (Lemma 4.8 of [9]) that $\tilde{\kappa}\otimes P_\sigma$ is a projective envelope of $\kappa\otimes\sigma$ in the category of $\mathrm{W}(k)[H]$-modules.

Furthermore, if we put $P_{\kappa,\tau} = \text{c-Ind}_H^G\tilde{\kappa}\otimes P_\sigma$, then we have $A_\pi = \mathrm{End}(P_{\kappa,\tau})$ (this fact follows from Corollary 11.11 of [9]). Finally, from Corollary 10.19 and Proposition 7.21 of [9], we have that

$$A_\pi = \mathrm{End}(P_{\kappa,\tau}) = \mathrm{End}(P_\sigma)[\theta_1,\ldots,\theta_{\frac{n}{def}}^{\pm 1}]/\left\langle \theta_1,\ldots,\theta_{\frac{n}{def}-1}\right\rangle \cdot I_0,$$

where I_0 is an ideal which we do not describe explicitly but which is constructed in [9].

In this thesis, we calculate the ring $\mathrm{End}(P_\sigma)$ in that it is of interest in understanding the structure of A_π and its relation to $R_\rho^\square$. We note that the representation σ comes attached with a notion of degree (see Definition 3.15). Explicitly, our result is the following (see Theorem 6.5).

Theorem. *Suppose that k is a finite field of characteristic ℓ. Let q be a power of a prime distinct from ℓ and let w be the order of q modulo ℓ. Suppose that σ is an irreducible cuspidal representation of $G = \mathrm{GL}_n(\mathbb{F}_q)$ over k of degree $d < n$. Let $c = \gcd(w, d)$. Then, under the assumptions that $2 \leq n < \ell$ and that k is large enough to contain the ℓ-regular $|G|$th roots of unity, the* $\mathrm{End}(P_\sigma)$ *is isomorphic to the ring of invariants of*

$$\mathrm{W}(k)[X]/(X^{\ell^r} - 1)$$

under the map $X \mapsto X^{q^c}$, where r is the ℓ-valuation of $q^w - 1$.

Moreover, we will formulate an explicit isomorphism in that we will find a generator of the ring of invariants of

$$\mathrm{W}(k)[X]/(X^{\ell^r} - 1)$$

and give its action on the direct summands of $P_\sigma \otimes_{\mathrm{W}(k)} L$, where L is a sufficiently large finite extension of the field of fractions of $\mathrm{W}(k)$ (again see Theorem 6.5).

Chapter 2

Invariants in a Cyclotomic Algebra

Let k be a finite field of characteristic $\ell > 2$ and let K be the field of fractions of the corresponding Witt vectors $\mathrm{W}(k)$. Suppose that q is an integer which is not divisible by ℓ. Let w be the order of q modulo ℓ and let $r > 0$ be the ℓ-valuation of $\Phi_w(q)$, where $\Phi_w(X)$ is the cyclotomic polynomial of order w. This chapter provides the technical commutative algebraic results we will need. More specifically, we will give an action of the subgroup of $(\mathbb{Z}/\ell^r\mathbb{Z})^\times$ generated by q on a certain $\mathrm{W}(k)$-algebra and compute the invariants of this algebra with respect to this action.

We begin by working over certain ℓ-power cyclotomic extensions of K, which will turn out to be the relevant quotients of our algebra by maximal ideals. Explicitly, for each $0 < i \leq r$, fix a primitive ℓ^ith root unity, ζ_i. Then we may identify $(\mathbb{Z}/\ell^i\mathbb{Z})^\times$ with the Galois group of $K(\zeta_i)/K$ in the usual way. Our first aim is to understand the fixed field, L_i, of the subgroup generated of $\mathrm{Gal}(K(\zeta_i)/K)$ generated by q.

To this end, put

$$T = \mathbb{Z}[X]/\left\langle X^{q^w-1} - 1\right\rangle$$

(T is not the algebra which will be the focus of this chapter). For $a \in \mathbb{Z}$, we have an endomorphism of T given by $\alpha(X) \mapsto \alpha(X^a)$. In this way, we get an action of $(\mathbb{Z}/(q^w-1)\mathbb{Z})^\times$ on T. In particular, the cyclic subgroup generated by q acts on T.

Lemma 2.1. *Consider an element of the form X^a in T. If the orbit of X^a has order strictly less than w, then a is divisible by ℓ^r.*

Proof. Suppose the orbit in question has order b. Then

$$X^{q^b a} - X^a = X^a(X^{a(q^b-1)} - 1)$$

is zero in T which is to say that it is divisible by X^{q^w-1}. We conclude that $q^w - 1$ divides $a(q^b - 1)$. Hence a is divisible by

$$\frac{q^w-1}{q^b-1}.$$

In particular, a is divisible by $\phi_w(q)$ and so by ℓ^r. □

Next, consider the element

$$N(X) = (X-1)(X^q-1)\cdots(X^{q^{w-1}}-1)$$

in T. We remark that $N(\zeta_i)$ is the norm of $\zeta_i - 1$ along the extension $K(\zeta_i)/L_i$. Since $\zeta_i - 1$ is a uniformizer for the ring of integers of $K(\zeta_i)$, we see that $N(\zeta_i)$ is a uniformizer for the ring of integers of L_i. Put

$$\omega_i = \zeta_i + \zeta_i^q + \cdots + \zeta_i^{q^w}.$$

Then we have the following:

Proposition 2.2. *Suppose that* $0 < i \leq r$. *Then* $L_i = K(\omega_i)$ *and the ring of integers of* L_i *is* $\mathrm{W}(k)[\omega_i]$. *Moreover,* $\omega_i - w$ *is a uniformizer for* $\mathrm{W}(k)[\omega_i]$.

Proof. Put $\zeta = \zeta_i$, $\omega = \omega_i$, and $L = L_i$. In addition let S be the collection of orbits of powers of X appearing in the expansion of $N(X)$. We claim that $N(X)$ may be written in the form

$$N(X) = \sum_{\mathcal{O} \in S} \left[(-1)^{\mathcal{O}} \sum_{X^a \in \mathcal{O}} X^a \right],$$

where for each $\mathcal{O} \in S$, $(-1)^{\mathcal{O}}$ has the value ± 1.

To show this claim, we first assume that $q \neq 2$. We see that all the powers of X appearing in the expansion of $N(X)$ are distinct (looking for example at q-adic expansions). Moreover, the degree of $N(X)$ is $1 + q + \cdots + q^{w-1}$, which is less than $q^w - 1$, and so all of the powers of X in appearing in the expansion of $N(X)$ are linearly independent over $\mathbb{Z}$ as elements of T. Since the polynomial is q-invariant in T we conclude the sign of any two monomials in an orbit is the same and the claim follows. In the case $q = 2$, the argument applies to all of the terms except for the first and the last. But those terms are each q-invariant themselves and so the claim about their orbits is trivial.

In particular, $N(1) = \sum_{\mathcal{O} \in S} (-1)^{\mathcal{O}} |\mathcal{O}|$, but $N(1)$ is clearly zero. Thus we have

$$N(\zeta) = \sum_{\mathcal{O} \in S} \left[(-1)^{\mathcal{O}} \left(\sum_{X^a \in \mathcal{O}} \zeta^a - |\mathcal{O}| \right) \right]$$

and so the latter sum is a uniformizer for L_i. Moreover, if $\mathcal{O} \in S$ does not have order w, Lemma 2.1 implies that for any $X^a \in \mathcal{O}$, ℓ^r divides a. Hence ζ^a

is 1 and so the term corresponding to such an orbit in the above sum is zero. Thus if S' is the collection of orbits of order w, we obtain

$$N(\zeta) = \sum_{\mathcal{O}\in S'} \left[(-1)^{\mathcal{O}} \left(\sum_{X^a \in \mathcal{O}} \zeta^a - w \right) \right].$$

Since ζ is an ℓth power root of unity, ζ reduces to 1 in the residue field of $K(\zeta_i)$ and so each term in the sum indexed by S' reduces to 0. Hence the sum over S' is a sum of elements of positive valuation which add to a uniformizer of L. We conclude that at least one of the terms of this sum is a uniformizer for L. That is, we have an a so that

$$\zeta^a + \zeta^{aq} + \cdots + \zeta^{aq^{w-1}} - w$$

is a uniformizer for L. We claim that a is prime to ℓ. Indeed, otherwise our sum is equal to

$$\xi + \xi^q + \cdots + \xi^{q^{w-1}},$$

where ξ is a primitive ℓ^tth root of unity for $t < i$. But this element is contained in the fixed field of q in $K[\xi]$, an extension of K of order

$$\ell^{t-1}(\ell - 1)/w.$$

Hence its degree is bounded by this last number and so it cannot generate L, which has degree $\ell^{s-1}(\ell-1)$ (a uniformizer always generates a totally ramified extension).

Thus a is prime to ℓ. But now we may apply the Galois automorphism $\zeta \mapsto \zeta^a$ to take $\omega - w$ to

$$\zeta^a + \zeta^{aq} + \cdots + \zeta^{aq^{w-1}} - w.$$

In particular, $\omega - w$ is uniformizer and so it generates L. Since ω and $\omega - w$ differ by an integer, ω also generates L. □

Corollary 2.3. *Over k, the polynomial*

$$g(X) = X + X^q + \cdots + X^{q^{w-1}} - w$$

is divisible by $(X-1)^w$, but not $(X-1)^{w+1}$.

Proof. Consider $g(X)$ as an element of $\mathrm{W}(k)[X]$. We will show by induction that it has the form $g(X) = (X-1)^w g_w(X) + \ell h_w(X)$. Certainly $f(1) = 0$ and so $f(X)$ is divisible by $X - 1$ even over $\mathrm{W}(k)$. Thus we may find a polynomial $g_1(X) \in \mathrm{W}(k)[X]$ with

$$g(X) = (X-1)g_1(X).$$

Suppose then that for some $i < w$, we have polynomials $g_i(X), h_i(X) \in \mathrm{W}(k)[X]$ with

$$g(X) = (X-1)^i g_i(X) + \ell h_i(X).$$

Put $\zeta = \zeta_r$ and consider the maps

$$\mathrm{W}(k)[X] \to \mathrm{W}(k)[\zeta] \to k$$

(the first map is $X \mapsto \zeta$ and the second is the reduction map). From Proposition 2.2, $g(X)$ maps to an element of valuation w in $\mathrm{W}(k)[\zeta]$ (taking the valuation on $\mathrm{W}(k)[\zeta]$ to be normalized).

On the other hand, $g(X)$ also maps to $(\zeta-1)^i g_i(\zeta) + \ell h_i(\zeta)$. The $K[\zeta]$-valuation of ℓ is $\phi(\ell)$, which is at least w and so the valuation of $g_i(\zeta)$ is at least $w-i$ and so at least 1. In particular, $g_i(\zeta)$ reduces to zero in k and so $g_i(1)=0$ in k. That is, we may write $g_i(X)=(X-1)g_{i+1}(X)+\ell s(X)$. Hence

$$g(X)=(X-1)^{i+1}g_{i+1}(X)+\ell[(X-1)^i s(X)+h_i(X)]$$

and we see inductively that $(X-1)^w$ divides $g(X)$ in k.

Conversely, suppose that we have

$$g(X)=(X-1)^i g_i(X)+\ell h_i(X)$$

with $i>w$. We again evaluate at ζ. But then the right-hand side has valuation at least

$$\min\{i,\ell^r(\ell-1)\}.$$

Under the assumption that we do not have $r=1$ and $w=\ell-1$, this integer is larger than w, which is the valuation of $g(\zeta)$. Thus we have a contradiction and are reduced to the case that $w=\ell-1$ and $r=1$.

Needless to say, it suffices in this case to show that, over k, $g^{(w+1)}(1)\neq 0$. We have

$$g^{(w+1)}(1)=q^{w-1}(q^{w-1}-1)\cdots(q^{w-1}-(w-1))+\cdots+q(q-1)\cdots(q-(w-1)).$$

Since $\ell=w+1$, w is even. Putting $w=w'/2$, we see that $q^{w'}$ is equivalent to -1 modulo ℓ. Furthermore, no other power of q between 0 and $w-1$ is equivalent to -1. In other words, all the other powers of q are equivalent to one of

$$\{1,2,\ldots,\ell-2\}=\{1,\ldots,w-1\}.$$

Thus the only product in our expansion for $g^{(w+1)}(1)$ which survives is

$$q^{w'} \cdots (q^{w'} - (w-1))$$

and it is the product of all the elements of $(\mathbb{Z}/\ell\mathbb{Z})^{\times}$, which is nonzero in k. $\square$

Before we proceed with our computations, we will need to make several general commutative algebraic observations. Indeed, suppose that R is a commutative ring and that $\{I_0, \ldots, I_s\}$ is a sequence of ideals in R. Then we have the canonical injection

$$R/(I_1 \cap \cdots \cap I_s) \to R/I_1 \times \cdots \times R/I_s$$

and we will consider the image. Considering first the simplest nontrivial case, suppose that $s = 2$. Let $(x_1, x_2) \in R/I_1 \times R/I_2$ and assume that $x_1 - x_2 \in I_1 + I_2$. Then we have $z_1 \in I_1$ and $z_2 \in I_2$ with $x_1 - x_2 = z_2 - z_1$ which is to say that $x_1 + z_1 = x_2 + z_2$. But then $x_1 + z_1 \in R$ maps to (x_1, x_2) and so (x_1, x_2) is in the image in question.

Proceeding inductively, suppose that

$$(x_1, x_2, x_3) \in R/I_1 \times R/I_2 \times R/I_3.$$

Then, if $x_1 - x_2 \in I_1 + I_2$, we may apply the previous paragraph to conclude that there is an element $y_2 \in R/(I_1 \cap I_2)$ which maps to (x_1, x_2). Supposing that $y_2 - x_3 \in I_1 \cap I_2 + I_3$, we see (again by the previous paragraph) that there is a $y_3 \in R$ which maps to (y_1, x_3) in $R \to R/(I_1 \cap I_2) \times R/I_3$. Thus y_3 maps to (x_1, x_2, x_3) in $R \to R/I_1 \times R/I_2 \times R/I_3$.

In general, if $(x_1, \ldots, x_j) \in R/I_1 \times \cdots \times R/I_s$ and if there is a $y_j \in R$ which maps to $(x_1, \ldots, x_j)$ then to find a y_{j+1} which maps to $(x_1, \ldots, x_{j+1})$, we need only know that

$$y_j - x_{j+1} \in (I_1 \cap \cdots \cap I_j) + I_{j+1}.$$

Thus we have conditions that guarantee that $(x_1, \ldots, x_s)$ is in the image of R and any element in the image of R certainly satisfies the conditions named.

The importance of these conditions for our purposes is the following lemma:

Lemma 2.4. *Suppose that $\phi : S \to R$ is an inclusion of commutative rings. Suppose in addition that $\{I_0, \ldots, I_s\}$ is a sequence of ideals of R whose intersection is zero and which satisfy*

$$\phi^{-1}(I_1 \cap \cdots \cap I_i + I_{i+1}) = \phi^{-1}(I_1 \cap \cdots \cap I_i) + \phi^{-1}(I_{i+1})$$

for all i. Then an element $a \in R$ lies in the image of ϕ if and only if for each i, it lies in the image of $S \to R \to R/I_i$.

Proof. The idea of this proof is that we have the commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\phi} & R \\ \downarrow & & \downarrow \\ S/\phi^{-1}(I_1) \times \cdots \times S/\phi^{-1}(I_r) & \longrightarrow & R/I_1 \times \cdots \times R/I_r \end{array}$$

where all the maps are injective. By assumption we may write the image of a in $R/I_1 \times \cdots \times R/I_r$ as

$$(\phi(s_1), \ldots, \phi(s_r))$$

with $s_i \in S$. Since this sequence comes from an element of R, we also know that the element $\phi(s_1 - s_2)$ lies in $I_1 + I_2$. Thus $s_1 - s_2$ lies in $\phi^{-1}(I_1 + I_2)$ and so also lies in $\phi^{-1}(I_1) + \phi^{-1}(I_2)$. By the discussion before the statement of the lemma, we may find a $t_2 \in S$ such that $t_2 \mapsto (s_1, s_2)$ in

$$S \to S/\phi^{-1}(I_1) \times S/\phi^{-1}(I_2).$$

Inductively, suppose that there is a $t_i \in S$ with $t_i \mapsto (s_1, \ldots, s_i)$. Then $\phi(t_i)$ maps to $(\phi(s_1), \ldots, \phi(s_i))$ so that $a - \phi(t_i) \in I_1 \cap \cdots \cap I_{i+1}$. Thus $\phi(t_i) - \phi(s_{i+1}) \in I_1 \cap I_{i+1} + I_{s+1}$ and so $t_i - s_{i+1} \in \phi^{-1}(I_1 \cap I_{i+1}) + \phi^{-1}(I_{s+1})$. By induction we conclude that there is a $t \in S$ which maps to $(s_1, \ldots, s_r)$. Thus $\phi(t)$ maps to $(\phi(s_1), \ldots, \phi(s_r))$. Since the intersection of the I_i is zero, we have $\phi(t) = a$. □

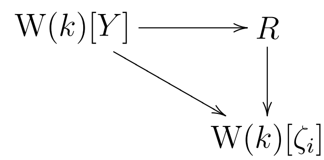
We will now apply Lemma 2.4 to the case of interest. For the remainder of the chapter, we put

$$R = \mathrm{W}(k)[X]/(X^{\ell^r} - 1) \text{ and } f(X) = X + X^q + \cdots + X^{q^{w-1}}.$$

We also denote the minimal polynomial of ω_i over $\mathrm{W}(k)$ by $m_i(X)$ and we put $m(X) = \prod_{i=0}^{r} m_i(X)$.

Lemma 2.5. *Consider the map* $\mathrm{W}(k)[Y] \to R$ *given by* $Y \mapsto f(X)$. *The pre-image of the ideal of* R *generated by* $\phi_{\ell^i}(X)$ *is* $m_i(Y)$.

Proof. We have the commutative diagram:



where X maps to ζ_i and Y maps to ω_i. The kernel of the map $R \to W(k)[\zeta_i]$ is $\Phi_{\ell^i}(X)$. Thus a polynomial in $W(k)[Y]$ which maps in R to the ideal generated by $\Phi_{\ell^i}(X)$ must map to zero in $W(k)[\zeta_i]$. In other words, it must be divisible by $m_i(Y)$. □

Since the intersection of the ideals generated by the $\Phi_{\ell^i}(X)$ is zero in R, we conclude that the kernel of $W(k)[Y] \to R$ is the ideal generated by $m(Y)$. Thus, putting

$$S = W(k)[Y]/(m(Y)),$$

we get an injective map $S \to R$ given by $Y \mapsto f(X)$.

As above, $a \in \mathbb{Z}$ gives an endomorphism, $g(X) \mapsto g(X^a)$, on R and we get an action of $(\mathbb{Z}/\ell^r\mathbb{Z})^\times$ on R. In particular, the subgroup (of order w) generated by q acts on R. Since ℓ^r divides $q^w - 1$, we see that $f(X)$ is fixed under the action of q and so the image of S is fixed under the action of q. We will show the converse of this result, beginning with a lemma that applies all of our technical work above.

Proposition 2.6. *The conditions of Lemma 2.4 apply to the injection $S \to R$ and the ideals $I_i = (\Phi_{\ell^i}(X))$.*

Proof. We have already noted that the intersection of the ideals is zero and so we need only show that

$$\phi^{-1}(X^{\ell^i}-1)+\phi^{-1}(\Phi_{\ell^{i+1}}(X))=\phi^{-1}(X^{\ell^i}-1,\Phi^{\ell^{i+1}}(X)).$$

The forward containment holds in general and so it suffices to show the reverse containment.

We begin by showing that the first ideal contains ℓ (which will imply that the other one does as well). By Lemma 2.5, we have

$$\phi^{-1}(X^{\ell^i}-1)=(\prod_{j=0}^{i} m_j(Y)).$$

Accordingly, put $\mu_i(Y)=\prod_{j=0}^{i} m_j(Y)$. Likewise, $\phi^{-1}(\Phi_{\ell^{i+1}}(X))=m_{i+1}(Y)$. Thus we want to show that ℓ is in the ideal $(m_{i+1}(Y),\mu_i(Y))$. But the latter is the kernel of the map

$$S\to \mathrm{W}(k)[\omega_{i+1}]/\left\langle \mu_i(\omega_{i+1})\right\rangle.$$

Thus it suffices to show that the valuation of ℓ in $\mathrm{W}(k)[\omega_{i+1}]$ is greater than that of $\mu_i(\omega_{i+1})$.

All the roots of $\mu_i(Y)$ are Galois conjugates of various ω_j all of which reduce to w modulo ℓ. Hence, modulo ℓ, $\mu_i(Y)$ is a power of $Y-w$ and its degree is

$$1+\frac{\ell^i-1}{w}=\left\lceil\frac{\ell}{w}\right\rceil.$$

Thus we may write $\mu_i(Y)=(Y-w)^{\lceil\frac{\ell}{w}\rceil}+\ell\nu(Y)$ with $\nu(Y)\in \mathrm{W}(k)[Y]$. Evaluating at ω_{i+1} gives

$$\mu_i(\omega_{i+1})=(\omega_{i+1}-w)^{\lceil\frac{\ell}{w}\rceil}+\ell\nu(\omega_i).$$

But, viewing the valuation on $\mathrm{W}(k)[\zeta_{i+1}]$ as normalized, we know from Proposition 2.2 that the valuation of $\omega_{i+1} - w$ is w. Thus the valuation of the first term in the sum above is $\ell^i - 1 + w$. On the other hand, the valuation of latter term is at least that of ℓ, which is $\ell^i(\ell - 1) \geq 2\ell^i > \ell^i - 1 + w$. We conclude that the valuation of $\mu_i(\omega_{i+1})$ is that of the first term and that it is less than that of ℓ.

Now modulo ℓ, $m_{i+1}(Y)$ and $\mu_i(Y)$ are both powers of $(Y - w)$ and the former has higher degree than the latter. We conclude that

$$\phi^{-1}(X^{\ell^i} - 1) + \phi^{-1}(\Phi_{\ell^{i+1}}(X)) = (\ell, (Y - w)^{\lceil \frac{\ell^i}{w} \rceil})$$

Likewise, since the pullback of $(X^{\ell^i} - 1, \Phi^{\ell^{i+1}}(X))$ contains ℓ, the ideal itself must also contain ℓ. From this fact, we conclude that

$$(X^{\ell^i} - 1, \Phi^{\ell^{i+1}}(X)) = (\ell, (X - 1)^{\ell^i}).$$

Thus we want to show that

$$\phi^{-1}(\ell, (X - 1)^{\ell^i}) = (\ell, (Y - w)^{\lceil \frac{\ell^i}{w} \rceil}).$$

Since both ideals contain ℓ, it suffices to consider the map

$$k[Y]/(Y - w)^{\lceil \frac{\ell^r}{w} \rceil} \to k[X]/(X - 1)^{\ell^r}$$

and show that the pullback of the ideal generated by $((X - 1)^{\ell^i})$ is the ideal generated by

$$(Y - w)^{\lceil \frac{\ell^i}{w} \rceil}$$

(as this map is the reduction modulo ℓ of the map $S \to R$). By Corollary 2.3, $(Y - w)$ maps to polynomial of the form $(X-1)^w s(X)$, with s prime to $X-1$. In particular, $(Y - w)^{\lceil \frac{\ell}{w} \rceil}$ maps to a polynomial divisible by $(X - 1)$ at least ℓ^i times. In particular it lies in the $(X - 1)^{\ell^i}$. On the other hand, if $j < \lceil \frac{\ell^i}{w} \rceil$, then $j \leq \frac{\ell^i - 1}{w}$ and so the image of $(Y - w)^j$ is divisible by $(X - 1)$ at most $\ell^i - 1$ times. Thus we have shown the claim. □

We have therefore arrived at our key result.

Theorem 2.7. *The subring of R invariant under $X \mapsto X^q$ is equal to S.*

Proof. Keeping the notation of the Proposition 2.6 suppose that $a \in R$ is q invariant. Then the image of a in R/I_i is q invariant for each i which, by Proposition 2.2, is to say that the image is a polynomial in ω_i. In other words, the image of a in R/I_i is contained in the image of S. We conclude by Lemma 2.4 and Proposition 2.6 implies that a itself is contained in S. □

Chapter 3

Modular Blocks of Finite General Linear Groups

Once again, k is a field of characteristic ℓ and K is the field of fractions of $\mathrm{W}(k)$. We let q be a power of a prime distinct from ℓ and we let w be the order of q modulo ℓ. Fix an algebraic closure of $\mathbb{F}_q$ so that we may consider $\mathbb{F}_{q^d}$ for any d. We fix $n \in \mathbb{N}$ and assume $\ell > n$. We also will assume that k is sufficiently large, in a sense made explicit momentarily.

The aim of this chapter is to characterize the characteristic zero and ℓ-modular irreducible generic representations in the block coming from an irreducible cuspidal (but not supercuspidal) representation of $\mathrm{GL}_n(\mathbb{F}_q)$, where $n \geq 2$. For the reader who is unfamiliar with this terminology, we begin by reviewing these notions. First, we discuss the elementary representation theory of finite groups over finite fields (a good general reference is [16]).

A representation of a group over a finite field is often called a *modular representation* and so we are discussing the elementary modular representation theory of a finite group. To this end, suppose that G is a finite group and assume that L is a finite extension of K that is large enough to contain all the roots of unity of order dividing that of G so that L admits all the characteristic zero irreducible representations of G (i.e., L admits one irreducible representation for every conjugacy class of G; see [18] for details). We assume that k is large enough so that L may be chosen to have residue field k.

We denote by $\mathcal{O}$ the ring of integers of L. We remark that, given a representation of G over L, we may obtain a representation of G over k. Indeed, if π is a representation of G over L, one may always choose a G-stable $\mathcal{O}$-lattice, Λ, in π (for example, choose an L-basis, B, of π and put $\Lambda = \mathcal{O}[G].B$). The $k[G]$-module $\Lambda \otimes_{\mathcal{O}} k$ is then a representation of G over k.

Defintion 3.1. We call the $k[G]$-module $\Lambda \otimes_{\mathcal{O}} k$ an *ℓ-modular reduction of* π.

Remark. Unfortunately, the notion of ℓ-modular reduction is not really well-defined. Indeed, one can choose two different lattices in π and arrive at non-isomorphic $k[G]$-modules. On the other hand, two different ℓ-modular reductions of π will also have the same Jordan Hölder constituents, as can be seen from the theory of Brauer characters (a generalization of classical character theory; see [18] for a good reference). In other words, two different ℓ-modular reductions of π are equal up to semi-simplification.

Central idempotents of the group ring play an important role in modular representation theory just as they do in classical representation theory. Indeed, from classical representation theory, we have a correspondence between the irreducible representations of G over L and the primitive central idempotents in the L-algebra $L[G]$. Explicitly, if π is such an irreducible representation, π corresponds to the idempotent

$$e_\pi = \frac{1}{|G|} \sum_{g \in G} \mathrm{tr}|_\pi(g^{-1}) \cdot g.$$

In the case that ℓ divides the order of G, e_π need not lie in the ring $\mathcal{O}[G]$ (which we view as embedded in $L[G]$). Nevertheless, an easy argument shows that we may find a unique minimal sum of primitive central idempotents of $L[G]$ that contains e_π and lies in $\mathcal{O}[G]$. In this way, the primitive central idempotents of $L[G]$ are canonically partitioned into classes. Correspondingly, the irreducible representations of G over L are partitioned into classes.

Defintion 3.2. A class in the partition of the irreducible representations is called a *characteristic zero block* (or just a *block*).

We remark that the blocks thus constructed depend on the choice of ℓ. One can see, however, that in the obvious sense, they do not depend on the choice of a sufficiently large L (and one may pass to the algebraic closure of K if one desires; we have not taken this perspective because for us it will be convenient to work with a finite residue field).

The direct product decomposition of $\mathcal{O}[G]$ corresponding to these blocks (i.e., to the central idempotents used to construct them) decomposes any $\mathcal{O}[G]$-module into a direct product. In particular, if we have an irreducible $L[G]$-module, π, we find a unique idempotent in $\mathcal{O}[G]$ which does not annihilate π (namely the one that contains the idempotent in $L[G]$ which corresponds to π) and we see that this idempotent acts as the identity on π. More importantly, similar remarks apply for an irreducible $k[G]$-module and so the collection of irreducible $k[G]$-modules is also partitioned into blocks corresponding to the idempotents of $\mathcal{O}[G]$ (or equivalently of $k[G]$ once one knows a bit about the nature of the reduction map $\mathcal{O}[G] \to k[G]$ on idempotents). We call these blocks the *ℓ-modular blocks.*

Thus, given an irreducible representation, π, of G over either L or k, we may speak of the characteristic zero block or the ℓ-modular block of π. Indeed, we find the minimal idempotent in $\mathcal{O}[G]$ which does not annihilate π and consider the corresponding class of either representations over L or over k. We remark also that it is a common practice to conflate the characteristic zero blocks with the ℓ-modular blocks and refer to both notions as 'blocks.' We will avoid this practice and reserve the term block for characteristic zero block.

Understanding the blocks and the ℓ-modular blocks of G is crucial in understanding its modular representation theory. We give an important tool for this purpose. Suppose that π_1 and π_2 are irreducible representations of G over L. Then we say that π_1 and π_2 are *linked* if ℓ-modular reductions of π_1 and π_2 have a Jordan-Hölder constituent in common (a notion which does not depend on the choices of modular reduction).

Proposition 3.3. *Suppose that π and ρ are irreducible $L[G]$-modules. Then π and ρ are in the same block if and only if there is a sequence*

$$\pi = \pi_0, \pi_1, \ldots, \pi_s = \rho$$

such that π_i and π_{i-1} are linked for all i.

Proof. This is a well-known result. See for example Theorem 3.6.20 of Section 3.6 of [16]. □

For the remainder of the chapter, we will work exclusively in the case $G = \mathrm{GL}_n(\mathbb{F}_q)$. Our next aim is to formulate the definition of a generic irreducible representation (a notion which essentially applies only to this case and the case of a linear group over a p-adic field). To this end, the *mirabolic subgroup*, H, of G defined to be the collection of matrices whose bottom row is the vector $[0, \ldots, 0, 1]$.

Over either k or L, H has an extremely important representation, known as the "mirabolic" representation (and so named because its existence is viewed as a miracle), which we now construct (summarizing the results and constructions found at the beginning of III.1 of [22]). Choose a nontrivial character $\psi : \mathbb{F}_q \to F^\times$, where $F = L$ or $F = k$. Then if U is the subgroup of G consisting of strictly upper-triangular matrices (that is, upper-triangular matrices with ones of the diagonal), we get a character, ψ_U, of U by putting

$$\psi_U(u_{i,j}) = \psi(\sum_{i=1}^{n-1} u_{i,i+1}).$$

Defintion 3.4. The *mirabolic representation* is defined to be the representation $\mathrm{Ind}_U^H \psi_U$.

In fact, the isomorphism class of this representation is independent of the choice of nontrivial character ψ. We see from construction that the mirabolic representation over k is a modular reduction of that over L and it turns out that both are irreducible.

Defintion 3.5. A representation π of G (over L or k) is called *generic* if $\mathrm{Res}_H^G \pi$ contains the mirabolic representation as a Jordan-Hölder constituent.

Finally, we discuss the notion of cuspidality. In Chapter 1, we gave some indications regarding this notion in relation to the linear group $\mathrm{GL}_n(F)$, where F is a p-adic field, but only by appealing to the local Langlands correspondence. We will now construct the analogous notion in the case of the finite linear group G without reference to Galois (or Weil) representations. We should also remark that, from the proper perspective, G is an example of a reductive group (in fact it is probably the quintessential example of a reductive group, at least of a finite reductive group). We will leave the term reductive group undefined (its formal definition is not necessary for our work), but we remark that the representation theory of reductive groups is very well-developed (see for example [2], [22], or [5]) and much of the terminology, including cuspidality and parabolic induction (which we discuss momentarily) is applicable to general reductive groups. Nevertheless, in that the general definitions and constructions can be a bit more involved and abstract, our practice will be to give the definitions only as they apply to G.

The first important notion attached to the representation theory of G as a reductive group is a process for building representations of G from representations of smaller linear groups (that is, general linear groups over

$\mathbb{F}_q$ of smaller dimension). We describe this process now. Let $\lambda \neq (n)$ be a partition of n. A matrix in G is called λ-block upper-diagonal if it has the form:

$$\begin{pmatrix} B_1 & * & * & \cdots & * \\ 0 & B_2 & * & \cdots & * \\ 0 & 0 & B_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_n \end{pmatrix}$$

where B_i is an invertible $\lambda_i \times \lambda_i$ block. The collection of λ-block upper-diagonal matrices is of course a subgroup of G, which we will denote by P_λ.

A *parabolic subgroup*, P, of G is a subgroup which is, up to conjugation, equal to P_λ for some such λ. In that no harm is done to this theory by allowing a conjugation, one can typically assume without loss of generality that $P = P_\lambda$ and we will adopt this practice. The corresponding collection, $M = M_\lambda$, of λ-block diagonal matrices (i.e. those matrices in P whose off-diagonal blocks are zero) is a subgroup of P, called the *Levi subgroup of* P. The Levi subgroup has a normal complement, $U = U_\lambda$, that consists of those matrices in P whose diagonal blocks are all equal to the identity matrix. One thus has the semi-direct product decomposition $P = M \ltimes U$, which is called the *Levi decomposition* of P.

Next, suppose that for each i, we have a representation π_i of $\mathrm{GL}_{\lambda_i}(\mathbb{F}_q)$ (over any fixed ring of coefficients). Then since the Levi subgroup M_λ is canonically isomorphic to the direct product of the $\mathrm{GL}_{\lambda_i}(\mathbb{F}_q)$, we may view the representation

$$\pi = \pi_1 \otimes \cdots \otimes \pi_s$$

as a representation of M_λ. Likewise, via inflation, we may view π as a representation of P. Finally, inducting to G gives us the representation $\mathrm{Ind}_P^G \pi$.

Defintion 3.6. The representation $\mathrm{Ind}_P^G \pi$ is called the *parabolic induction* of the π_i.

The representations of G (over either L or k) are of two kinds: those that arise as Jordan-Hölder constituents of a parabolic induction and those that do not. We call an irreducible representation of the latter kind *supercuspidal*. We see that, in some sense, the supercuspidal representations are the ones that are not built from smaller general linear groups. Likewise, we say that an irreducible representation is *cuspidal* if it does not occur as a quotient of a parabolic induction. Since characteristic zero representations of G are always semi-simple, these two notions coincide in characteristic zero (or in any characteristic not dividing the order of G).

The notion of supercuspidality leads to the notion of supercuspidal support.

Defintion 3.7. If λ is a partition of n and if for each i, π_i is a supercupsidal representation of $\mathrm{GL}_{\lambda_i}(\mathbb{F}_q)$ (over either L or k), we say that a representation π of G has *supercuspidal support* $\sum_i \pi_i$ if it is a Jordan-Hölder constituent of the parabolic induction coming from the π_i. Likewise, if the π_i are cuspidal, we say that π has *cuspidal support* $\sum_i \pi_i$ if it is a quotient of the corresponding parabolic induction.

Again we see that these notions coincide in characteristic zero.

Proposition 3.8. *The cuspidal and supercuspidal supports of a representation are unique (up to isomorphism and reordering of course).*

Proof. This is the content of the corollary in Section III.2.5 of [22]. □

We have now established the general framework in which we will work and begin the process of characterizing the irreducible generic representations in the block of a cuspidal representation. We begin with a simple but important elementary number theoretic observation.

Lemma 3.9. *The prime ℓ divides at most one integer of the form $\Phi_m(q)$ where $m < n$.*

Proof. Suppose that $m < n$ and that ℓ divides $\Phi_m(q)$. We claim that q is a primitive mth root of unity modulo ℓ. Indeed, ℓ certainly divides $q^m - 1$ (as $\phi_m(q)$ divides $q^m - 1$). On the other hand, if $q^d = 1$ for some d dividing m (and not equal to m), we see that q is a root, modulo ℓ, of the integer polynomials $X^d - 1$ and $\phi_m(X)$. Since these polynomials are relatively prime over $\mathbb{Z}$ and they both divide $X^m - 1$, we conclude that q is a double root of $X^m - 1$ modulo ℓ. This polynomial, however, is separable over $\mathbb{F}_\ell$ since ℓ does not divide m (as $\ell > m$).

But now suppose that q also divides $\Phi_{m'}(q)$ for some $m' < n$. Then, as above, q is also a primitive m'th root of unity modulo ℓ. Writing the greatest common divisor, d, of m and m' as $d = am + bm'$, we see that $q^d = 1$. We conclude that $d = m$ and so m divides m'. Likewise, m' divides m and so $m = m'$. □

Recall that we have denoted the order of q modulo ℓ by w. We see then that ℓ divides $\Phi_w(q)$ and that it does not divide any $\Phi_{w'}(q)$ for any other w' with $w' < n$.

We also recall that in any finite abelian group, an element is called ℓ-regular if its order is prime to ℓ and ℓ-power torsion if its order is a power of ℓ. Every element, s, of the group may be written uniquely as the product of an ℓ-regular element and an ℓ-power torsion element. We call the former the ℓ-regular part of s and the latter the ℓ-power torsion part of s. Both the ℓ-regular and ℓ-power torsion parts of s are powers of s.

Lemma 3.10. *Suppose that $\sigma \in \mathbb{F}_{q^n}^\times$ is an element of degree n over $\mathbb{F}_q$ and that its ℓ-regular part, σ', has degree $d < n$. Then n is the least common multiple of w and d. Furthermore, $\sigma'\tau$ has degree n over $\mathbb{F}_q$ for any nontrivial ℓ-power torsion element, τ, in $\mathbb{F}_{q^n}^\times$.*

Proof. Because σ differs from its ℓ-regular part, ℓ must divide $q^n - 1$ (as $q^n - 1$ is the order of $\mathbb{F}_{q^n}^\times$). In particular, w divides n. Let $r > 0$ be the ℓ-valuation of $q^n - 1$. Since ℓ divides $\Phi_w(q)$, Lemma 3.9 implies that ℓ does not divide $q^{w'} - 1$ for any $w' < w$ and that ℓ^r divides $\Phi_w(q)$. Hence Frobenius has order w on any nontrvial ℓ-torsion element of $\mathbb{F}_{q^n}^\times$.

Next consider an element of the form $\sigma'\tau$ with τ a nontrivial ℓ-torsion element. Since τ and σ' are both powers of $\sigma'\tau$, we conclude that the field generated over $\mathbb{F}_q$ by $\sigma'\tau$ coincides with that generated by σ' and τ. The latter has degree $\mathrm{lcm}(w,d)$. Applying this reasoning to the case that τ is actually the ℓ-regular part of σ, we conclude that $n = \mathrm{lcm}(w,d)$. The second assertion follows. □

Before we move more specifically into the representation theory of G, we will need to establish some notations and conventions. In their influential paper, [6], Dipper and James construct certain combinatorial objects that they call indices and use them to classify the characteristic zero and ℓ-modular representations of G (and reflect the relationships among them). We will now review their terminology so that we may apply it for our purposes.

To start, we fix an arbitrary total ordering on the $\mathbb{F}_q$-Galois orbits in $\mathbb{F}_{q^{n!}}^\times$. For $n' < n$, an (n',∞)-*index* is an array of the form

$$\left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ s_1 \cdots s_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right)$$

with $d_i, v_i \in \mathbb{N}$, $s_i \in \mathbb{F}_{q^{n!}}^\times$, and $\lambda^{(i)}$ a partition of v_i subject to the conditions that s_i has degree d_i over $\mathbb{F}_q$ and $\sum_i d_i v_i = n'$. We also call the (n',∞)-index *primary* if $N = 1$. We remark that Dipper and James do not use this latter terminology, but we will find it convenient to have a word for this notion. We have chosen the word "primary" to correspond with Green's notion of "primary characters" (see [7]) so that, in the characteristic zero case, primary indices will correspond to primary characters (see our discussion of Green's work in Chapter 4 for more details).

Given a permutation in S_N, the index obtained from I by permuting the s_i, d_i, k_i, and λ^i in the obvious way is considered to be equal to I. Likewise, two indices are considered equivalent if the corresponding elements of $\mathbb{F}_{q^{n!}}$ are $\mathbb{F}_q$-Galois conjugate and all the other entries are equal. Again we have deviated slightly from the conventions in [6]. In that paper, only a chosen representative of each Galois orbit is allowed to appear in indices. We will find it more convenient to allow any element of $\mathbb{F}_{q^{n!}}^\times$ and use the equivalence relation named, these two formulations being clearly equivalent.

The (n', ∞)-index, I, is called a *head* if the Galois orbits of the s_i are strictly increasing (under our fixed total order). We see immediately that the (equivalence classes of) head (n', ∞)-indices parameterize the conjugacy classes of $\mathrm{GL}_{n'}(\mathbb{F}_q)$. Indeed, if $s \in \mathbb{F}_{q^{n}!}^\times$, we may consider the minimal polynomial, f, of s over $\mathbb{F}_q$. If d is the degree of f then, given any partition λ, we may form an $d|\lambda| \times d|\lambda|$ matrix over $\mathbb{F}_q$ by making an arrangement of Jordan-type blocks of the companion matrix of f as specified by λ. In general,

$$\left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ s_1 \cdots s_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right),$$

corresponds to a conjugacy class represented by a block matrix with blocks formed in this way for each s_i.

An (n', ℓ)-index is defined similarly to an (n', ∞)-index, except that (in the notation above) we require s_i to be ℓ-regular but only require it to be the ℓ-regular part of some element of $\mathbb{F}_{q^{n!}}^\times$ of degree d_i (so that s_i need not itself have degree d_i). For an (n', ℓ)-index to be a head, however, we do require that s_i has exact degree d_i (and that the corresponding classes appear in strictly increasing order). Again we see that the head (n', ∞)-indices parameterize the ℓ-regular conjugacy classes of $\mathrm{GL}_{n'}(\mathbb{F}_q)$. The term primary is applied analogously for (n', ∞)-indices to the manner in which it is applied for (n', ∞)-indices

We also have a notion of reducing indices. Explicitly, the *ℓ-modular reduction* of the (n', ∞)-index

$$I = \left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma_1 \cdots \sigma_N & \lambda^{(1)} \ldots \lambda^{(N)} \end{array} \right),$$

is the (n', ℓ)-index

$$\bar{I} = \left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma'_1 \cdots \sigma'_N & \lambda^{(1)} \ldots \lambda^{(N)} \end{array} \right),$$

where σ'_i is the ℓ-regular part of σ_i. Since an element of $\mathbb{F}_{q^{n!}}^{\times}$ may differ in degree over $\mathbb{F}_q$ from its ℓ-regular part, we see that the reduction of a head (n', ∞)-index need not be a head (n', ℓ)-index.

Finally, we define the combinatorial notion of a foot index. We recall that if λ is any partition, $m_i(\lambda)$ denotes the number of parts of λ equal to i. If e is any integer, λ is said to be *e-regular* if e does not divide $m_i(\lambda)$ for any i (excluding those i for which $m_i(\lambda) = 0$). Furthermore, if $d \in \mathbb{N}$, we define $e(d)$ to be the least e so that ℓ divides

$$1 + q^d + q^{2d} + \cdots + q^{(e-1)d}.$$

The (n', ℓ)-index

$$\left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma_1 \cdots \sigma_N & \lambda^{(1)} \ldots \lambda^{(N)} \end{array} \right)$$

is called a *foot* if $\lambda^{(i)}$ is $e(d_i)$-regular for each i. By convention, any (n', ∞)-index is considered a foot. Likewise, the index is called *special* if s_i being Galois conjugate to s_j implies either $d_i \neq d_j$ or $i = j$.

The important combinatorial fact about these notions is the existence of an algorithm constructed in [6] on pp. 268-271. This algorithm takes any (n', ℓ)-index to a corresponding foot (n', ℓ)-index and we will describe it now. First of all, if I is the primary index

$$\left(\begin{array}{c|c} d & k \\ s & \lambda \end{array} \right),$$

then if I is not a foot, λ is $e(d)$-singular so that there is a unique pair of partitions $\lambda^{(0)}$ and $\lambda^{(1)}$ of weight $k_0 \geq 0$ and $k_1 \geq 1$ respectively with

$$\lambda' = (\lambda^{(0)})' + e(d)(\lambda^{(1)})',$$

where $\lambda^{(0)}$ is $e(d)$-regular and where we have denoted conjugation of partitions with a prime (and addition of partitions is performed pointwise). If $k_0 \neq 0$, we put

$$I' = \left(\begin{array}{c|c} d\; de(d) & k_0\, k_1 \\ s \;\; s & \lambda^{(0)}\, \lambda^{(1)} \end{array} \right)$$

and if $k_0 = 0$, we put

$$I' = \left(\begin{array}{c|c} de(d) & k_1 \\ s & \lambda^{(1)} \end{array} \right).$$

We then write $I \to I'$ for the first step in the algorithm.

In general, if

$$\left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma_1 \cdots \sigma_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right)$$

is an (n', ℓ)-index which is not a foot, we choose an i such that $\lambda^{(i)}$ is $e(d_i)$-singular and apply the algorithm given above to s_i, d_i, k_i, and $\lambda^{(i)}$ in the obvious way (either by changing them or by inserting more columns into I, depending on what the algorithm demands). In general, (see p.271 of [6]), one arrives at a (n', ℓ)-foot index which is uniquely determined by I and so we may speak of the foot of an (n', ℓ)-index. The algorithm induces a bijection between head (n', ℓ)-indices and special feet (n', ℓ)-indices (Theorem 2.13 of [6]). If I is an (n', ℓ)-index, we will write I^* for the associated foot (n', ℓ)-index.

Moving forward with representations, we fix a choice of generator for $\mathbb{F}_{q^{n!}}^{\times}$ and an embedding $\mathbb{F}_{q^{n!}}^{\times} \to L^{\times}$. With respect to these choices Dipper and James construct (in [6] and [12]) representations $M_L(I)$, $S_L(I)$, and $D_L(I)$ of $\mathrm{GL}_{n'}(\mathbb{F}_q)$ over L for any (n', ∞)-index I. Likewise, if I is an (n', ℓ)-index, they construct representations $M_k(I)$, $S_k(I)$, and $D_k(I)$ of $\mathrm{GL}_{n'}(\mathbb{F}_q)$ over k.

We will not look to define or construct these representations, but we will invoke many of their properties (as given in [6] and [12]). We will also consider some of their characters in Chapter 4. Loosely speaking, the S representations allow us to parameterize the characteristic zero representations of G. Likewise, the D representations parameterize the ℓ-modular representations of G (see Theorem 3.12). We make these notions precise now.

Theorem 3.11. *If I is a head (n', ∞)-index, $S_L(I) = D_L(I)$ is irreducible. The map S_L induces a bijection between equivalence classes of head (n', ∞)-indices and characteristic zero irreducible representations of* $\mathrm{GL}_{n'}(\mathbb{F}_q)$.

Proof. This result follows from [6] Theorem 4.6 and 4.7 and the fact that head (n, ∞)-indices are in bijection with the conjugacy classes of G. $\square$

Theorem 3.12. *If I is a head (n', ℓ)-index, $D_k(I)$ is irreducible. The map D_k induces a bijection between equivalence classes of head (n', ℓ)-indices and characteristic ℓ irreducible representations of $\mathrm{GL}_{n'}(\mathbb{F}_q)$. If I and I' are two indices and $I \to I'$, then $D_k(I) \cong D_k(I')$. Thus D_k also induces a bijection between special feet (n', ℓ)-indices and characteristic ℓ irreducible representations of $\mathrm{GL}_{n'}(\mathbb{F}_q)$.*

Proof. This result is Theorems 5.1 and 6.3 of [6]. $\square$

The M representations are perhaps more of an artifact of the Dipper and James's construction, but for our purposes, we will see that they are also associated to the notion of supercuspidal support. Indeed, if I is the (n', ∞)-index

$$\left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma_1 \cdots \sigma_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right),$$

then the representation, from Dipper and James's construction, $M_L(I)$ is equal to the parabolic induction of the representations

$$M_L \left(\begin{array}{c|c} d_i & v_i \\ \sigma_i & \lambda^{(i)} \end{array} \right).$$

Corresponding statements also apply to $D_L(I)$ and $S_L(I)$ and over the field k. In fact, Dipper and James construct these representations for primary indices first and then define them for arbitrary indices using parabolic induction in this way (see p.277-278 of [6]). As a result, it suffices in many contexts to study the representations attached to primary indices. The key set of properties of these representations is the next result.

Proposition 3.13. *Let*

$$I = \left(\begin{array}{c|c} d & v \\ \sigma & \lambda \end{array} \right)$$

be a primary (dv, ℓ)-index. Then $S_k(I)$ has a unique maximal $k[\mathrm{GL}_{dv}(\mathbb{F}_q)]$-submodule and the quotient by this submodule is $D_k(I)$. Moreover, every Jordan-Hölder constituent of $S_k(I)$ has the form $S_k(J)$ where

$$J = \left(\begin{array}{c|c} d & v \\ \sigma & \mu \end{array} \right)$$

and μ is greater than λ in dominance order.

Proof. This is 4.1.iv and 4.1.v of [6]. □

One also has a good characterization of cuspidality and supercuspidality in terms of indices. Indeed, from the discussion on p. 275 of [6] (or from Proposition 3.16 below) the irreducible cuspidal (or equivalently supercuspidal) representations of $\mathrm{GL}_{n'}(\mathbb{F}_q)$ in characteristic zero are those of the form

$$S_L \left(\begin{array}{c|c} n' & 1 \\ \sigma & (1) \end{array} \right).$$

Likewise, in characteristic ℓ, we have the following characterization:

Proposition 3.14. *Suppose that $\bar{\pi}$ is an irreducible cuspidal representation of $\mathrm{GL}_{n'}(\mathbb{F}_q)$ over k. Then $\bar{\pi}$ has the form*

$$D_k \left(\begin{array}{c|c} n' & 1 \\ \sigma' & (1) \end{array} \right).$$

Moreover $\bar{\pi}$ is supercuspidal if and only if σ' has degree n' over $\mathbb{F}_q$.

Proof. From part (3) of the theorem in Section III.2.2 of [22], we know that that irreducible cuspidal representation $\bar{\pi}$ is an ℓ-modular reduction of a characteristic zero cuspidal representation. Applying the characterization of irreducible cuspidal representations in characteristic zero, together with Proposition 3.13, we conclude that $\bar{\pi}$ has the form claimed. The second assertion is part (a) of the corollary in [22] Section III.2.5. □

This characterization of cuspidal representations motivates the following definition:

Defintion 3.15. In the notation of Proposition 3.14, we will say that the *degree* of the ℓ-modular irreducible cuspidal representation $\bar{\pi}$ is the degree of σ' over $\mathbb{F}_q$.

We remark that the cuspidal representation $\bar{\pi}$ is supercuspidal if and only if it has degree n'.

Because of Theorems 3.11 and 3.12, one may speak of the special foot index or the head index of an irreducible representation of $\mathrm{GL}_{n'}(\mathbb{F}_q)$ over either L or k (noting that a head (n', ∞)-index is the same as a special foot (n', ∞)-index, up to equivalence). One may obtain significant information about modular reduction and cuspidal and supercuspidal support from indices.

Proposition 3.16. *Suppose that $F = L$ or $F = k$ and put*

$$I = \left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma_1 \cdots \sigma_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right).$$

If I is a head index, the supercuspidal support of $D_F(I)$ is

$$\sum_i v_i D_F \left(\begin{array}{c|c} d_i & 1 \\ \sigma_i & (1) \end{array} \right).$$

If I is a special foot index than the sum given is the cuspidal support of $D_F(I)$.

Proof. Thus is part (b) of the corollary of Section III.2.5 of [22]. □

The fact that, for $F = L$, a head index is the same as a special foot index is compatible with Proposition 3.16 because a supercuspidal representation is the same as a cuspidal representation in characteristic zero.

Proposition 3.17. *Suppose that I is a head (n', ∞)-index with modular reduction $\bar{I}$. Then $S_k(\bar{I})$ is a modular reduction of $S_L(I)$.*

Proof. This fact follows from the discussion at the beginning of Section 6 of [6]. □

These observations will allow us, given an irreducible characteristic zero representation, to read the supercuspidal support of the Jordan-Hölder constituents off the the corresponding index. We begin with a lemma.

Lemma 3.18. *Let*

$$\bar{\pi} = D_k \left(\begin{array}{c|c} n' & 1 \\ \sigma' & (1) \end{array} \right)$$

be an irreducible ℓ-modular cuspidal representation. Let d be the degree of $\bar{\pi}$ and put $v = n'/d$. Then the supercuspidal support of $\bar{\pi}$ is

$$v D_k \left(\begin{array}{c|c} d & 1 \\ \sigma' & (1) \end{array} \right).$$

Proof. Essentially this lemma is an application of Proposition 3.16, but we need to understand how the algorithm of Dipper and James applies. By assumption, we may find a generator, σ, for $\mathbb{F}_{q^{n'}}/\mathbb{F}_q$ whose ℓ-regular part is σ'. Since ℓ does not divide n' (as $n' \leq n < \ell$) and since σ is an element of degree n' over $\mathbb{F}_q$ whose ℓ-regular part has degree $d < n'$, Lemma 2.3 of [6] implies that $e(d)d = n$. Thus we have

$$\left(\begin{array}{c|c} d & v \\ \sigma' & (1^v) \end{array} \right) \to \left(\begin{array}{c|c} n & 1 \\ \sigma' & (1) \end{array} \right)$$

in Dipper and James's algorithm. Since the former index is a head, Proposition 3.16 implies that the supercuspidal support of $\bar{\pi}$ is as claimed. □

We remark that from Definition 4.2 of [12],

$$M_F \left(\begin{array}{c|c} d & v \\ \sigma & (1^v) \end{array} \right)$$

is the parabolic induction of v copies of the cuspidal representation

$$S_F \left(\begin{array}{c|c} d & 1 \\ \sigma & (1) \end{array} \right) = M_F \left(\begin{array}{c|c} d & 1 \\ \sigma & (1) \end{array} \right).$$

Proposition 3.19. *Let I be the head (n, ∞)-index*

$$\left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma_1 \cdots \sigma_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right)$$

and put $\rho = S_L(I)$. For each i, let σ_i' be the ℓ-regular part of σ_i. Assume that the degree of σ_i' over $\mathbb{F}_q$ is d_i' and put $v_i' = d_i/d_i'$. Then the supercuspidal support of any Jordan-Hölder constituent of an ℓ-modular reduction of ρ is

$$\sum_i v_i v_i' D_k \left(\begin{array}{c|c} d_i' & 1 \\ \sigma_i' & (1) \end{array} \right).$$

Proof. By Proposition 3.16, the supercuspidal support of ρ is $\sum v_i S_L(I_i)$ where

$$I_i = \left(\begin{array}{c|c} d_i & 1 \\ \sigma_i & (1) \end{array} \right),$$

which is to say that ρ is contained in the representation $M_L(J)$, where

$$J = \left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma_1 \cdots \sigma_N & (1^{v_1}) \cdots (1^{v_N}) \end{array} \right).$$

On the other hand, if $\bar{I}_i$ is the ℓ-modular reduction of I_i, Proposition 3.17 implies that $M_k(\bar{I}_i)$ is an ℓ-modular reduction of $M_L(I_i)$. Hence, since parabolic induction commutes with ℓ-modular reduction, $M_k(\bar{J})$ is an ℓ-modular reduction of $M_L(J)$ if $\bar{J}$ is the ℓ-modular reduction of J. We conclude that any Jordan-Hölder constituent of an ℓ-modular reduction of ρ is also contained in $M_k(\bar{J})$ as a Jordan-Hölder constituent.

Furthermore, Lemma 3.18 implies that the ℓ-modular supercuspidal support of $S_k(\bar{I}_i)$ is

$$v_i' D_k \left(\begin{array}{c|c} d_i' & 1 \\ \sigma_i' & (1) \end{array} \right)$$

so that (since parabolic induction is exact) the parabolic induction of the $S_k(\bar{I}_i)$, namely $M_k(\bar{J})$, is contained in

$$M_k \left(\begin{array}{c|c} d_1' \cdots d_N' & v_1 v_1' \cdots v_N v_N' \\ \sigma_1' \cdots \sigma_N' & (1^{v_1 v_1'}) \cdots (1^{v_N v_N'}) \end{array} \right).$$

Hence any Jordan-Hölder constituent of an ℓ-modular reduction of ρ is contained in this last representation, which is to say that it has the supercuspidal support claimed. $\square$

For the remainder of the chapter, we fix the irreducible ℓ-modular cuspidal representation

$$\bar{\pi} = D_k \left(\begin{array}{c|c} n & 1 \\ \sigma' & (1) \end{array} \right)$$

of G. We assume that σ' has degree d over $\mathbb{F}_q$ and that $v = n/d$. We also assume that $d \neq n$ so that $\bar{\pi}$ is not supercuspidal. Again, our goal is to compute the irreducible generic characteristic zero and modular representations that lie in the block corresponding to $\bar{\pi}$. We will achieve this goal by first considering ℓ-modular supercuspidal support.

Lemma 3.20. *Suppose that $\bar{\rho}$ is a irreducible generic ℓ-modular representation of G which has the same ℓ-modular supercuspidal support as $\bar{\pi}$. Then $\bar{\rho} = \bar{\pi}$.*

Proof. By assumption, $\bar{\rho}$ is a Jordan-Hölder of

$$M_k \left(\begin{array}{c|c} d & v \\ \sigma' & (1^v) \end{array} \right).$$

But Remark 4 of Section III.2.4 of [22] implies that the only generic representation contained in this representation is

$$D_k \left(\begin{array}{c|c} d & v \\ \sigma' & (1^v) \end{array} \right) = \bar{\pi}.$$

Thus we have the claim. □

A *generalized Steinberg representation* of G is a representation of the form

$$S_L \left(\begin{array}{c|c} a & b \\ t & (1^b) \end{array} \right).$$

As with cuspidal representations, we say that a is the degree of the generalized Steinberg representation. We remark in the case $a = 1$ and $t = 1$, the generalized Steinberg representation is just the classical Steinberg representation. In the degenerate case that $b = 1$, a generalized Steinberg representation is a characteristic zero supercuspidal representation.

Proposition 3.21. *Suppose that ρ is an irreducible characteristic zero generic representation of G such that the Jordan-Hölder constituents of an ℓ-modular reduction of ρ have the same ℓ-modular supercuspidal support as $\bar{\pi}$. Then ρ is either a lift of $\bar{\pi}$ or it is the generalized Steinberg representation*

$$S_L \left(\begin{array}{c|c} d & v \\ \sigma' & (1^v) \end{array} \right).$$

Proof. We may apply Theorem 3.11 to find a head (n, ∞)-index, I, such that $\rho = S_L(I)$. Put

$$\bar{I} = \left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ \sigma'_1 \cdots \sigma'_n & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right).$$

Then Proposition 3.19 implies that $\sigma'_i = \sigma'$ for all i (by our assumption regarding ℓ-modular supercuspidal support).

Thus the only elements of $\mathbb{F}_{q^{n!}}^{\times}$ appearing in I must have ℓ-regular part σ'. From Lemma 3.10, σ' itself is the only element with ℓ-regular part σ' that has degree over $\mathbb{F}_q$ strictly less than n. We conclude that only one element of $\mathbb{F}_{q^{n!}}^{\times}$ may appear in I and that this element has ℓ-regular part σ'. If the element has the form $\sigma'\tau$ for τ a nontrivial ℓ-torsion element, we see immediately that ρ is a cuspidal lift of $\bar{\pi}$ (again applying Lemma 3.10).

Thus it suffices to assume that I has the form

$$\left(\begin{array}{c|c} d & v \\ \sigma' & \lambda \end{array} \right).$$

Since $S_L(I)$ contains a generic factor, Lemma 3.20 implies that $\bar{\pi}$ is a Jordan-Hölder factor of $S_k(\bar{I})$. On the other hand, Proposition 3.13 implies that the factors of $S_k(\bar{I})$ all have the form

$$D_k \left(\begin{array}{c|c} d & v \\ \sigma' & \mu \end{array} \right)$$

with μ greater than λ in dominance order. Thus for some such μ, the foot of

$$\left(\begin{array}{c|c} d & v \\ \sigma' & \mu \end{array} \right) \text{ must equal } \left(\begin{array}{c|c} n & 1 \\ \sigma' & (1) \end{array} \right),$$

the foot corresponding to $\bar{\pi}$.

But since $v = e(d)$, μ is $e(d)$-singular if and only if $\mu = (1^v)$. Thus the only way the foot of

$$\left(\begin{array}{c|c} d & v \\ \sigma' & \mu \end{array} \right)$$

is the foot index associated to π is if $\mu = (1^v)$ (as otherwise this index is its own foot). We conclude that $\lambda = (1^v)$. Hence we have

$$\rho = S_L \left(\begin{array}{c|c} d & v \\ \sigma' & (1^v) \end{array} \right)$$

. $\square$

It remains to connect the notion of ℓ-modular supercuspidal support with the notion of ℓ-modular blocks.

Proposition 3.22. *If two irreducible ℓ-modular representations lie in the same ℓ-modular block then they have the same ℓ-modular supercuspidal support. If two irreducible characteristic zero representations lie in the same ℓ-modular block, then all of the Jordan-Hölder constituents of corresponding modular reductions have the same ℓ-modular supercuspidal support.*

Proof. By Proposition 3.3, it suffices to show that the Jordan-Hölder constituents of reductions of linked representations have the same ℓ-modular supercuspidal support. This fact, however, is more or less trivial given Proposition 3.19. Indeed, it follows from that proposition that all of the Jordan-Hölder constituents of a reduction of a fixed irreducible representation have the same ℓ-modular supercuspidal support.

The first claim follows from the second. Indeed, if two modular representations lie in the same block, then they must be Jordan-Hölder constituents of reductions of two characteristic zero representations that lie in the same block. Hence they must have the same ℓ-modular supercuspidal support. $\square$

We let B denote the collection of irreducible characteristic zero representations in the block coming from $\bar{\pi}$. Likewise we let $\bar{B}$ be the collection of irreducible modular representations in the corresponding block.

Corollary 3.23. *The representation $\bar{\pi}$ is the unique generic representation in $\bar{B}$. The generic irreducible representations of B are those named in Proposition 3.21.*

Proof. The claim regarding $\bar{\pi}$ follows immediately from Proposition 3.22 and Lemma 3.20. Likewise that the characteristic zero representation named are the only generic representations in V follows from Propositions 3.21 and 3.22. Certainly the characteristic zero supercuspidal representations in question lie in B as they reduce to $\bar{\pi}$. Likewise a modular reduction of the generalized Steinberg representation is

$$S_k \left(\begin{array}{c|c} d & \sigma' \\ 1 & (1^v) \end{array} \right),$$

which by Proposition 3.13 contains $\bar{\pi}$. □

Likewise we may characterize the cuspidal supports of the irreducible modular representations in the block coming from $\bar{\pi}$. Let

$$\bar{\pi}_0 = D_k \left(\begin{array}{c|c} d & 1 \\ \sigma' & (1) \end{array} \right)$$

Proposition 3.24. *The representation $\bar{\pi}$ is its own cuspidal support and every other representation in $\bar{B}$ has cuspidal support $v\bar{\pi}_0$.*

Proof. Since every representation in $\bar{B}$ is a Jordan-Hölder constituent of a modular reduction of an irreducible characteristic zero representation in B, we conclude that every representation in $\bar{B}$ has the same supercuspidal support as $\bar{\pi}$. Hence, Proposition 3.16 (together with Theorem 3.12) implies that every irreducible representation of $\bar{B}$ has the form

$$D_k \left(\begin{array}{c|c} d & v \\ \sigma' & \lambda \end{array} \right).$$

Again if $\lambda = (1^v)$, this representation is $\bar{\pi}$. Otherwise, the index given is a foot and so the cuspidal support is as claimed (again by Proposition 3.16). □

Chapter 4

Characters of Finite General Linear Groups

In this chapter, G is again the general linear group $\mathrm{GL}_n(F_q)$ and L is any field of characteristic zero that contains the $|G|$th roots of unity so that it is large enough to contain all of the representations of G. In his landmark paper, [7], Green gives an algorithm for computing all the characters of G over L. We will describe the portions of this algorithm which will be necessary for our purposes.

Essentially, we need the characters of the representations named in Proposition 3.21 and (neglecting the definition of certain integral polynomials) we will give the algorithm for computing the characters of all representations of the form

$$S_L\left(\begin{array}{c|c} d & v \\ s & \lambda \end{array} \right),$$

which Green calls *primary characters.* We remark that Green's notation differs from that of Dipper and James and he denotes the representation just named by

$$(-1)^{n-v} I_d^{\kappa}[\lambda],$$

for $\kappa \in \mathbb{N}$ relating to s in a sense we will explain below.

To make our characters concrete, we fix a generator ϵ of $\mathbb{F}_{q^{n!}}^\times$ and a multiplicative embedding $\theta : \mathbb{F}_{q^{n!}}^\times \to L^\times$. In other words, we choose a generator ϵ of $\mathbb{F}_{q^{n!}}^\times$ and a primitive $(q^{n!}-1)$th root of unity, $\theta(\epsilon)$, of L. For $d < n$, we also put

$$\epsilon_d = \epsilon^{(q^{n!}-1)/(q^d-1)},$$

so that ϵ_d is a generator for $\mathbb{F}_{q^d}^\times$.

We begin by summarizing Green's notion of a uniform function as given in [7] (these functions have since come to be called Green functions). First of all, we must attach some terminology to partitions of n. Fix a partition, $\rho = (1^{r_1}2^{r_2}\cdots)$, of n. As is the common practice, we denote by z_ρ the cardinality of the centralizer of a cycle in S_n of type ρ so that

$$z_\rho = (1^{r_1})(r_1!)(2^{r_2})(r_2!)\cdots . \tag{4.1}$$

Likewise, we attach to ρ a collection, X^ρ, of indeterminates, called the *ρ-variables*. They are variables $x_{d,i}^\rho$ for $d \in \mathbb{N}$ and $1 \le i \le r_d$, so that we have one variable for each part of ρ. If the partition ρ is clear from context, we will typically write $x_{d,i}$ rather than $x_{d,i}^\rho$. We say that the *degree* of $x_{d,i}$ is d and write $d = \deg(x_{d,i})$. Finally, if λ is another partition of n, we have a certain integral polynomial, $Q_\rho^\lambda(T) \in \mathbb{Z}[T]$, called a *Green polynomial* whose definition we will not specify (Definition 4.2 of [7]; see also [15] and Section III.7.8 of [13]).

Defintion 4.1 (Definition 4.5 of [7])**.** A *ρ-substitution* (or just a substitution if ρ is clear from context) is a map $\alpha : X^\rho \to \mathbb{F}_{q^{n!}}^\times$ such that for each $x \in X^\rho$, the degree of $\alpha(x)$ over $\mathbb{F}_q$ divides $\deg(x)$.

As with indices, we identify two substitutions α and β if, for all x, $\alpha(x)$ is Galois conjugate to $\beta(x)$ over $\mathbb{F}_q$. We remark that Green viewed the codomain of his substitutions as the collection of monic irreducible polynomials over $\mathbb{F}_q$ aside from the polynomial T. In keeping with Dipper and James's notation, we will find our equivalent perspective more convenient. We will also find it convenient to call any map $X^\rho \to \mathbb{Z}$ an *integral ρ-substitution* (maps of these type are essential in Green's work, but he does not use this terminology; he typically uses the word 'row' to describe this situation).

Two substitutions are considered equivalent if there is a degree-preserving permutation of X^ρ that takes one to the other (see Definition 4.6 of [7]). Green referred to an equivalence class of substitutions as a 'mode of substitution,' but we will not use this terminology. Instead we will just use the phrase 'equivalence class.'

A substitution α gives rise to a partition-valued function on $\mathbb{F}_{q^{n!}}^\times$. Indeed, if $s \in \mathbb{F}_{q^{n!}}$ has degree d over $\mathbb{F}_q$, we may define, for each i, $r_i(\alpha, s)$ to be the number of $x \in X^\rho$ such that $\alpha(x) = s$ and $d(x) = id$. We then define a partition, $\rho(\alpha, s)$, by

$$\rho(\alpha, s) = (1^{r_1(\alpha,s)} 2^{r_2(\alpha,s)} \cdots).$$

One can verify that two substitutions are equivalent if and only if the two induced partition-valued functions coincide (Lemma 4.7 of [7]).

Now if I is an (n,∞)-index, I also corresponds to a partition valued function, ν_I, on $\mathbb{F}_{q^{n!}}^\times$. Indeed, putting

$$I = \left(\begin{array}{c|c} d_1 \cdots d_N & v_1 \cdots v_N \\ s_1 \cdots s_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right),$$

ν_I is the function that takes s_i to $\lambda^{(i)}$ and any other element of $\mathbb{F}_{q^{n!}}^\times$ to the zero partition. A *ρ-substitution* into I is a ρ-substitution, α, that satisfies $|\rho(\alpha, t)| = |\nu_I(t)|$ for all $t \in \mathbb{F}_{q^{n!}}$ (see Definition 4.10 of [7]). We remark that Green views his substitution as into conjugacy classes as his paper predates Dipper and James's notion of indices, but we have found this perspective more convenient. We see immediately that whether α qualifies as a substitution into a given I depends only on the equivalence class of α. At this point, we also will find it convenient to denote the collection of indices by $\mathcal{I}$.

Defintion 4.2 (Definitions 4.9 and 4.12 of [7]). A *ρ-function*, U_ρ, is an L-valued function on the collection of equivalence classes of ρ-substitutions. If $\{U_\rho\}$ is a collection of ρ-functions for each partition, ρ, of n, the corresponding *uniform function*, $U : \mathcal{I} \to L$, is defined by

$$U(I) = \sum_\rho \sum_m Q(m, I) U_\rho(m). \tag{4.2}$$

Here the sums are indexed by partitions, ρ, of n and equivalence classes, m, of ρ-substitution into I and

$$Q(m, I) = \prod_{s \in \mathbb{F}_{q^n}^\times} \frac{1}{z_{\rho(m,s)}} Q_{\rho(m,s)}^{\nu_I(s)}(q^{\deg(s)})$$

($\deg(s)$ being the degree of s over $\mathbb{F}_q$).

In practice, it is often only necessary to specify the value of a ρ-function on certain equivalence classes of substitutions. In the notation of Definition 4.2, the function U_ρ is called the *ρ-part* of U. If U has only one nonzero part, U_ρ, we say that U is *principal of type ρ* (Definition 4.13 of [7]). By construction, $U(I)$ does not change if we replace any of the s_i occurring in I by a Galois conjugate. As a result, a uniform function, via passing to equivalence classes of head indices, induces a class function on G. In fact, a major result of [7] is that all of the characters of G arise as uniform functions, but we will give the construction only for those characters that we need in our calculations.

We now construct the first example of uniform function. Fix a partition,

$$\rho = (1^{r_1} 2^{r_2} \cdots),$$

and an integral ρ-substitution, h. Define a ρ-function by

$$B_\rho(h) = \prod_d \left[\sum_{\sigma \in S_{r_d}} S_{d,\sigma(1)}(h_{d,1}) S_{d,\sigma(2)}(h_{d,2}) \cdots S_{d,\sigma(r_d)}(h_{r,r_d}) \right],$$

where d ranges over the integers contained in ρ, $h_{d,i} = h(x_{d,i})$, and $S_{d,i}(\kappa)$ is the ρ-function which takes a ρ-substitution α to

$$\theta^\kappa(s) + \theta^{q\kappa}(s) + \cdots + \theta^{q^{d-1}\kappa}(s),$$

where $s = \alpha(x_{d,i})$ (since the degree of s over $\mathbb{F}_q$ divides d, we see that this expression indeed only depends on the Galois class of s). Then we define $B^\rho(h) : \mathcal{I} \to L$ to be the principal uniform function of ρ-part $B_\rho(h)$.

Next we want to define uniform functions $I_d^\kappa[\lambda] : \mathcal{I} \to L$, any time $d|\lambda| = n$. First of all, let d be an integer dividing n and $\mu = (1^{m_1} 2^{m_2} \cdots)$, a partition of $v = n/d$. Then we write $d \cdot \mu$ for the partition $(d^{m_1}(2d)^{m_2}\cdots)$, so that $d \cdot \mu$ is a partition of n. Likewise, we write $\kappa \cdot \frac{\mu}{d}$ to be the $(d \cdot \mu)$-integral substitution

$$x_{de,i} \mapsto \kappa(1 + q^d + \cdots + q^{(e-1)d})$$

(here $e \in \mathbb{N}$ and $1 \leq i \leq m_e$). Finally, we let

$$I_d^\kappa[\lambda] = \sum_{|\mu|=v} \frac{1}{z_\mu} \chi_\mu^\lambda B^{d\cdot\mu}\left(\kappa\frac{\mu}{d}\right), \tag{4.3}$$

where χ_μ^λ is character of the representation of S_n corresponding to λ evaluated at the class of cycle type μ.

The key result regarding these characters is the following:

Lemma 4.3. *Suppose that $s = \epsilon_d^\kappa$ has degree d over $\mathbb{F}_q$. Then the class function $(-1)^{n-v} I_d^\kappa[\lambda]$ is an irreducible character of G. Explicitly, it is the character of the representation*

$$S_L\left(\begin{array}{c|c} d & v \\ s & \lambda \end{array}\right).$$

Proof. The first assertion is implied by Lemmas 7.5 and 7.6 of [7]. The second is stated on p.273 of [6]. One of course must assume that the same embedding $\mathbb{F}_{q^{n!}}^\times \to L$ is used in both cases. □

A generalized Steinberg character of degree d is thus a character of the form $(-1)^{n-v} I_d^\kappa[(1^v)]$, where κ is such that $s = \epsilon_d^\kappa$ has degree d over $\mathbb{F}_q$. We note that

$$\chi_\mu^{(1^v)}$$

is the sign of the class of S_v of cycle type corresponding to μ under the sign representation. We see also that taking $d = n$ corresponds to a cuspidal character of G as the sign representation and the trivial representation coincide on $S_1 = \{1\}$. On the other hand, taking $s = 1$ and $\kappa = q - 1$ corresponds to the classical Steinberg character of G.

To illustrate the algorithm outlined in this chapter, we will now compute the term in (4.3) corresponding to the partition $\mu = (v)$. This computation will also be important for us later and it comprises the entire calculation of the character in the supercuspidal case $v = 1$. Hence it will subsume a calculation of the supercuspidal characters of G, which are well-known.

Firstly, a trivial, but incredibly useful observation is the following:

Lemma 4.4. *Suppose that I is an index of the form*

$$\left(\begin{array}{c|c} a_1 \cdots a_N & b_1 \cdots b_N \\ t_1 \cdots t_N & \lambda_1 \cdots \lambda_N \end{array} \right)$$

with $N > 1$. Then the term corresponding to the partition $\rho = (v)$ in (4.3) is zero on I.

Proof. Since $\mu = (v)$, we have $\rho = d \cdot \mu = (n)$. In particular, there is only a single ρ-variable $X_{n,1}$ and I admits no ρ-substitution. □

Thus to compute the term of (4.3) corresponding to $\mu = (v)$, it suffices to compute the case given in the following lemma:

Lemma 4.5. *Let I denote the index*

$$\left(\begin{array}{c|c} a & b \\ t & \lambda \end{array} \right).$$

Then the value on I of the term of (4.3) corresponding to $\mu = (v)$ is

$$(-1)^{n-1}\frac{1}{v}(1-q^a)\cdots(1-q^{(x-1)a})[\theta^{\kappa'}(t)+\cdots+\theta^{q^{a-1}\kappa'}(t)],$$

where $\kappa' = \kappa(1+q^d+\cdots+q^{(v-1)d})$ and x is the length of λ (that is, the number of nonzero parts).

Proof. Put $\rho = d \cdot \mu = (n)$. By definition, $\kappa \cdot \frac{\mu}{d}$ maps the single ρ-variable, $x_{n,1}$, to κ'. There is a unique ρ-substitution, α, of $x_{n,1}$ into I and it is given by $x_{n,1} \mapsto t$. On α, $B_\rho(\kappa\frac{\mu}{d})$ takes the value

$$\theta^{\kappa'}(t)+\cdots+\theta^{q^{n-1}\kappa'}(t) = b[\theta^{\kappa'}(t)+\cdots+\theta^{q^{a-1}\kappa'}(t)],$$

since $t^{q^a} = t$. Furthermore, by the expression for $Q^{\lambda}_{(b)}(q)$ given on p.445 of [7], we see that

$$Q(\alpha, I) = \frac{1}{b}(1-q^a)\cdots(1-q^{(x-1)a}).$$

Finally, $z_\mu = v$ and $\chi^{(1^v)}(v) = (-1)^{v-1}$ and we conclude that the term is as claimed. □

Chapter 5

The Central Action of the Group Algebra

For this chapter, k is again a finite field of characteristic ℓ and K is the field of fractions of $\mathrm{W}(k)$. We put $G = \mathrm{GL}_n(\mathbb{F}_q)$, where q is a power of a prime distinct from ℓ and denote the order of G by g. L is a finite extension of K which is large enough to admit all of the irreducible representations of G and k is assumed to be large enough so that L may be chosen to have residue field k. As in Chapter 3, we fix an irreducible cuspidal representation, $\bar{\pi}$, of G over k which is not supercuspidal. We denote by X the collection of irreducible generic representations of G over L in the block coming from $\bar{\pi}$.

If π is a representation in X then any element of $Z(\mathrm{W}(k)[G])$ acts on π by a scalar in L and so π induces a map $\delta_\pi : Z(\mathrm{W}(k)[G]) \to L$. Combining all of these maps, we obtain the map $\delta : Z(\mathrm{W}(k)[G]) \to R$, where

$$R = \prod_{\pi \in X} L.$$

The aim of this chapter is to compute the image of δ under certain conditions that we will outline now.

First of all, we assume that $n \geq 2$ and that $\ell > n$. We also assume that the ℓ-regular part of $\mathbb{F}_{q^n}^\times$ embeds into $\mathrm{W}(k)$. That is, we assume that k

contains the ℓ-regular $|G|$th roots of unity. In particular, the ℓ-regular part of $\mathbb{F}_{q^n}^\times$ embeds into $\mathrm{W}(k)$ (since $|G|$ is divisible by $q^n - 1$).

As in Chapters 3 and 4, we fix ϵ, ϵ_j, and $\theta : \mathbb{F}_{q^n}^\times \to L$. With respect to these choices, we may put

$$\bar{\pi} = D_k \left(\begin{array}{c|c} n & 1 \\ \sigma & (1) \end{array} \right)$$

for some $\sigma \in \mathbb{F}_{q^n}^\times$. We denote by d the degree of σ over $\mathbb{F}_q$ (so that $\bar{\pi}$ itself has degree d in our terminology) and put $v = n/d$.

We denote the order of q modulo ℓ by w. We will put $r = \mathrm{ord}_\ell(q^n - 1)$ and define m by $q^n - 1 = m\ell^r$ so that m is prime to ℓ. Since the index giving $\bar{\pi}$ is by assumption an allowable (n, ℓ)-index, there must be an element of $\mathbb{F}_{q^n}$ of degree n over $\mathbb{F}_q$ whose ℓ-regular part is σ. Lemma 3.10 then implies that n is the least common multiple of d and w. We have seen in Corollary 3.23 that the representations in X are those of the form

$$S_L \left(\begin{array}{c|c} n & 1 \\ \sigma\tau & (1) \end{array} \right) \text{ or } S_L \left(\begin{array}{c|c} d & v \\ \sigma & (1^v) \end{array} \right),$$

where τ is a nontrivial ℓ-power torsion element of $\mathbb{F}_{q^n}$.

Finally, we choose a $\kappa \in \mathbb{Z}$ with $\sigma = \epsilon_d^\kappa$ and put

$$\kappa' = \kappa \frac{q^n - 1}{q^d - 1} = \kappa(1 + q^d + q^{2d} + \cdots + q^{(v-1)d})$$

so that

$$\sigma = \epsilon_n^{\kappa(q^n-1)/(q^d-1)} = \epsilon_n^{\kappa'}.$$

The ℓ-torsion elements of $\mathbb{F}_{q^n}^\times$ are those of the form ϵ_n^{im} with $i \in \mathbb{Z}$. Hence the elements, $\sigma\tau$, of $\mathbb{F}_{q^n}$ under consideration are those of the form

$$\epsilon_n^{\kappa'+im}$$

for $i \in \mathbb{Z}$. In the notation of [7], then, the irreducible generic characteristic zero representations in the block of $\bar{\pi}$ have the form $I_d^\kappa[(1^v)]$ and $I_n^{\kappa'+im}[(1)]$ where $i \in \mathbb{Z}$ is not divisible by ℓ^r.

But of course these representations are not all distinct up to isomorphism. Indeed, by Theorem 4.7 of [6] (or the work in [7]), two representations of this sort are equivalent if and only if the given elements of $\mathbb{F}_{q^n}^\times$ are Galois conjugates over $\mathbb{F}_q$. Since acting by the qth power Frobenius automorphism, Fr, on an element of $\mathbb{F}_{q^n}^\times$ is the same as acting its ℓ-power torsion and ℓ-regular parts independently, in order to take an element of the form σ or $\sigma\tau$ to another element of this form, we must act by a multiple of Fr^d. Under this action, σ has an orbit of size one and $\sigma\tau$ has an orbit of size $q^{w/\gcd(d,w)}$. Since there are $\ell^r - 1$ choices for τ, we have exactly $1 + \gcd(d,w)(\ell^r - 1)/w$ inequivalent representations under consideration.

Translating to the notation of [7], we are considering, modulo $q^n - 1$, the multiplicative action of q^d on $\{\kappa' + im.\}$ Accordingly, let $\mathcal{S} \subset \{0, 1, \ldots, \ell^r - 1\}$ be a collection of representatives for the q^d-orbits on $\{im\}$ where i runs from 0 to $\ell^r - 1$ (modulo $q^n - 1$ of course). Then the inequivalent generic representations in our block are $\{\pi_i\}_{i \in \mathcal{S}}$ where π_0 is the generalized Steinberg representation with character $(-1)^{n-v} I_d^\kappa[(1^v)]$ and, for $i \neq 0$, π_i is the supercuspidal representation with character $(-1)^{n-1} I_n^{\kappa'+im}[1]$.

Thus we can view the ring $R = \prod_{\pi \in X} L$ as the ring $R = \prod_{i \in \mathcal{S}} L$. Notationally, we will often write elements of R as ordered pairs (x, y_i) where x is the zeroth (so generalized Steinberg) coordinate and y_i is the ith coordinate for $i \neq 0$ (and the value of y_i may of course depend on i). We will also write δ_i for δ_{π_i}.

The collection $Z(\mathrm{W}(k)[G])$ is of course generated as a $\mathrm{W}(k)$-module by elements of the form $\beta_C = \sum_{t \in C} t$, where C is a conjugacy class of G. Given the character of π_i, it is relatively straightforward to compute the image β_C under δ_i. Indeed, we see that $\mathrm{Tr}|_{\pi_i}(\beta_C) = \delta_i(\beta_C) \dim \pi_i$. On the other hand, linearity of the trace tells us that

$$\mathrm{Tr}|_{\pi_i}(\beta_C) = \sum_{t \in C} \mathrm{Tr}|_{\pi_i}(t) = |C| \mathrm{Tr}|_{\pi_i}(C).$$

We conclude that

$$\delta_i(\beta_C) = \frac{|C|}{\dim \pi_i} \mathrm{Tr}|_{\pi_i}(C). \tag{5.1}$$

As a result, it will be important for us to have control on the cardinality of conjugacy classes in G. These orders can be computed explicitly from the corresponding indices and the relevant portion for us is the next lemma.

Lemma 5.1. *Let C be the conjugacy class in G given by the index*

$$I = \left(\begin{array}{c|c} a_1 \cdots a_N & b_1 \cdots b_N \\ t_1 \cdots t_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right).$$

Then the cardinality of the centralizer of an element in C is a power of q multiplied by the expression

$$\prod_{i=1}^{N}\prod_{j\geq 1}(q^{a_i}-1)(q^{2a_i}-1)\cdots(q^{m_j(\lambda^{(i)})a_i}-1),$$

(where again $m_j(\lambda^{(i)})$ is the number of parts of $\lambda^{(i)}$ equal to j).

Proof. This formula follows from a discussion on pages 409 - 410 of [7]. Note there is a typographical error in the centered equation at the bottom of p. 409: the sum in the formula should be a product. Green's explanation in the footnote on that page makes it clear what the correct formula should be. We have also translated Green's expression (which is written in terms of the conjugate) to the expression given using Formula I.1.4 of [13]. □

For a partition λ and an $a \in \mathbb{N}$, we define $c_a(\lambda)$ to be the integer

$$\sum_j \#\{i : |m_i(\lambda)| \geq ja\}.$$

In other words $c_a(\lambda)$ basically counts the integers that appear in λ more than a times: It counts those that occur more than a times once, those that occur more than $2a$ times twice, etc.

Corollary 5.2. *Let C be the conjugacy class in G given by the index*

$$I = \left(\begin{array}{c|c} a_1\cdots a_N & b_1\cdots b_N \\ t_1\cdots t_N & \lambda^{(1)}\cdots\lambda^{(N)} \end{array}\right)$$

and let c be the cardinality of the centralizer of an element in C. Then we have

$$\mathrm{ord}_\ell(c) = \sum_{i=1}^{N} r_i c_{w_i}(\lambda^{(i)}), \tag{5.2}$$

where $w_i = w/\gcd(a_i, w)$ and $r_i = \mathrm{ord}_\ell(\Phi_{w'}(q^{a_i}))$.

Proof. Since w is the order of q modulo ℓ, w_i is the order of q^{a_i}. Thus ℓ divides $\Phi_{w_i}(a_i)$ and Lemma 3.9 implies that ℓ does not divide $\Phi_y(q^{a_i})$ for any other $y \leq b_i$. Thus our formula follows immediately from the one in Lemma 5.1. $\square$

We have seen in Lemma 4.4 that the supercuspidal characters vanish on a large number of conjugacy classes. Our first step is to consider the contribution to the image of δ coming from all of these classes. We begin with a few lemmas.

Lemma 5.3. *Suppose that $a, b \in \mathbb{N}$ with $ab \leq n$ and that λ and ρ are partitions of b. Let $w' = w/\gcd(a, w)$ and assume that w' does not divide any part of ρ. Then the ℓ-valuation of the integer $Q_\rho^\lambda(q^a)$ is at least $r_a c_{w'}(\lambda)$, where a is the ℓ-valuation of $\Phi_{w'}(q^a)$.*

Proof. The argument is essentially the one given in [14] at the bottom of p. 451. Explicitly, Definition III.2.12. of [13] defines a polynomial $b_\lambda(t)$ by

$$b_\lambda(t) = \prod_{i \geq 1} (1-t)(1-t^2)\cdots(1-t^{m_i(\lambda)}).$$

We see immediately that $\Phi_{w'}(t)^{c_{w'}(\lambda)}$ divides $b_\lambda(t)$. Likewise $Q_\lambda(x;t)$ is defined to be $b_\lambda(t)P_\lambda(x;t)$ for a certain integral polynomial $P_\lambda(x;t)$ (Definition III.2.11 of [13]) so that $Q_\lambda(x;t)$ is divisible by $\Phi_{w'}(t)^{c_{w'}(\lambda)}$.

On the other hand, from Formula III.7.5 of [13], we have

$$Q_\lambda(x;t) = \sum_\rho z_\rho(t)^{-1} X_\rho^\lambda(t) \mathscr{P}_\rho(x),$$

where

$$z_\rho(t) = z_\rho \prod (1 - t^{\rho_i})^{-1},$$

$X_\rho^\lambda(t)$ is an integral polynomial we will not specify (see Definition III.7.1 of [13]), and $\mathscr{P}_\rho(x)$ is the so-called "power-sum product" corresponding to ρ (see Section I.2 of [13]). If ρ is any partition which has no part divisible by w', we see that the $\mathrm{W}(k)$-polynomial $z_\rho(t)$ is not divisible by $\Phi_{w'}(t)$. Since the power-sum product polynomials are $\mathbb{Q}$-linearly independent, we may evaluate repeatedly at a primitive w'th root of unity over $\mathbb{Q}$ and conclude that $X_\rho^\lambda(t)$ is divisible by $\Phi_{w'}(t)$ at least $c_{w'}(\lambda)$ times.

Furthermore, Formula III.7.8 of [13] implies that $Q_\rho^\lambda(q^a)$ is a power of q times $X_p^\lambda(q^{-a})$. Since q is a unit in $\mathrm{W}(k)$, we conclude that $Q_\rho^\lambda(q^a)$ is divisible over $\mathrm{W}(k)$ by $[\Phi_{w'}(q^a)]^{c_{w'}(\lambda)}$. Hence $Q_\rho^\lambda(q^a)$ is divisible by ℓ at least $r_a c_{w'}(\lambda)$ times. □

Suppose that C is a conjugacy class in G of the type under consideration (that is of the type given in Lemma 4.4). The purpose of Lemma 5.3 is that it will allow us to control $\mathrm{ord}_\ell(c)$, where again c is the order of a centralizer of an element in C. We will also establish the $\mathrm{W}(k)$-rationality of the character values on C. We will do so using Lemma 5.4 which will also serve us later.

Lemma 5.4. *Consider the index*

$$I = \left(\begin{array}{c|c} a_1 \cdots a_N & b_1 \cdots b_N \\ t_1 \cdots t_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right).$$

Let $\mu \neq (v)$ be a partition of v, put $\rho = d \cdot \mu$ and suppose that α is a ρ-substitution into I. Then the value of $B_\rho(\kappa \cdot \frac{\mu}{d})$ on α is an element of $\mathrm{W}(k)$.

Proof. The key observation here is that w does not divide any a_i. Indeed, the existence of the ρ-substitution into I implies that we have a ρ-variable x so that a_i divides the degree of x. In particular, if w divides a_i, then w divides the degree of x. On the other hand, each part of $\rho = d \cdot \mu$ is divisible by d and so the degree of x is divisible by d and w. Hence it is divisible by n and we get $\rho = (n)$ contrary to our assumption that $\mu \neq v$.

By definition, the value of $B_\rho(\kappa \cdot \frac{\mu}{s})$ on α is in the $\mathrm{W}(k)$-algebra generated by sums of the form

$$\theta^j(t_i) + \theta^{jq}(t_i) + \cdots + \theta^{jq^{x-1}}(t_i).$$

Since t_i has degree a_i, the order of t_i divides $q^{a_i} - 1$. Since w does not divide a_i, we conclude by Lemma 3.9 that t_i is ℓ-regular. Hence the value of any character on t_i is an element of $\mathrm{W}(k)$. This fact implies the claim. $\square$

If C is a conjugacy class in G, the value of the generalized Steinberg character on C is expressed in (4.3) as a sum over partitions of v. In that, according to (5.1), $\delta_0(\beta_C)$ depends linearly on this character value, we may consider the contribution to $\delta_0(\beta_C)$ of each of the partitions of v separately and this perspective will be very useful to us.

Proposition 5.5. *Let C be a conjugacy class in G. Suppose that μ is a partition of v with $\mu \neq (v)$. Then the contribution from μ to $\delta_0(\beta_C)$ is contained in $\ell^r \mathrm{W}(k)$.*

Proof. As usual, represent C by an index

$$I = \left(\begin{array}{c|c} a_1 \cdots a_N & b_1 \cdots b_N \\ t_1 \cdots t_N & \lambda^{(1)} \cdots \lambda^{(N)} \end{array} \right).$$

Let c be the order of the centralizer of C and fix a partition μ of v as required. Put $\rho = d \cdot \mu$. Certainly it suffices to assume that μ actually contributes to the value of the generalized Steinberg character on C and so we may fix a ρ-substitution, α, into I. Lemma 5.4 then implies that the value of $B_\rho(\kappa \frac{\mu}{d})$ on α is an element of $\mathrm{W}(k)$.

Furthermore, consider the partition $\rho(\alpha, t_i)$. We remark that no part of this partition is divisible by w', where $w' = w/\gcd(a_i, w)$. Indeed, assuming to the contrary, we may find a ρ-variable x with $\deg(x)/a_i$ divisible by w'. Thus $\deg(x)$ is divisible by w. Since all the parts of ρ are divisible by d, we conclude that $\deg(x) = n$, contrary to the fact that $\rho \neq (n)$. Thus Lemma 5.3 implies that

$$\mathrm{ord}_\ell(Q^{\nu_I(t_i)}_{\rho(\alpha,t_i)}(q^{a_i}))$$

is at least the ith term in (5.2).

Hence, since (4.1) implies that $z_{\rho(\alpha,t_i)}$ is always a unit in $\mathrm{W}(k)$ (as $\rho(\alpha, t_i)$ has weight less than n), we see from Corollary 5.2 that $\mathrm{ord}_\ell(Q(\alpha, I)) \geq \mathrm{ord}_\ell(c)$. Summing over all equivalence classes of substitutions as in (4.2), we conclude that the value of the term of (4.3) corresponding μ on I is an element of $\ell^{\mathrm{ord}_\ell(c)}\mathrm{W}(k)$. Summing over all μ, we conclude that the value of the generalized Steinberg character on C is an element of $\ell^{\mathrm{ord}_\ell(c)}\mathrm{W}(k)$.

To complete the proof, we point out that the dimension of the generalized Steinberg representation is

$$q^{n(v-1)/2}\frac{\prod_{j=1}^{n}(q^j-1)}{\prod_{j=1}^{v}(q^{dj}-1)}$$

(by, for example, a formula given on p. 3346 of [20]). The ℓ-valuation of the numerator is $\mathrm{ord}_\ell(g)$, g being the order of G. The denominator is divisible by q^n-1 and it is easy to see that it has ℓ-valuation r. Thus $\mathrm{ord}_\ell(\dim\pi_0) = \mathrm{ord}_\ell(g)-r$. Applying all of our observations to the various components of (5.1), we have the claim. □

As a result, we may handle the conjugacy classes of G mentioned in Lemma 4.4. We let S be the $\mathrm{W}(k)$-subalgebra of R generated by the element $(\ell^r,0)$.

Corollary 5.6. *Let C be the conjugacy class represented by the index*

$$I=\left(\begin{array}{c|c} a_1\cdots a_N & b_1\cdots b_N \\ s_1\cdots s_N & \lambda^{(1)}\cdots\lambda^{(N)}\end{array}\right)$$

and assume that $N>1$. Then $\delta(\beta_C)\in S$.

Proof. Since there is no (n)-substitution into I, a supercupsidal character will vanish on C. By Proposition 5.5, this fact also implies that the action of the generalized Steinberg representation on C is contained in $\ell^r\mathrm{W}(k)$. □

Conversely we have the following:

Proposition 5.7. *The element $(\ell^r,0)\in R$ is contained in the image of δ.*

Proof. Consider the class C represented by the index

$$I = \left(\begin{array}{c|c} 1 & n \\ 1 & (n) \end{array} \right).$$

We will show that the generalized Steinberg character vanishes on this class. We remark that for any partition, $\rho = (1^{r_1} \cdots)$, of n, we have a ρ-substitution, α, given by $x \mapsto 1$ for all $x \in X^\rho$. Moreover, we see that for each i, $r_i(\alpha, 1) = r_i$ (since r_i is by definition the number of ρ-variables of degree i). In particular, $\rho(\alpha, 1) = \rho$ and so α is a ρ-substitution into I. It is certainly the unique ρ-substitution into I (up to equivalence).

Now fix a partition μ of v and put $\rho = d \cdot \mu$. Then if α is the substitution just discussed, the value of $S_{j,i}(\kappa \frac{\mu}{d})$ on α is trivial j. Hence the value of $B_\rho(h)$ on α is

$$\prod_j j^{r_j} r_j! = z_\rho.$$

Furthermore, a remark on p. 445 of [7], shows that $Q_\rho^{(n)}(q) = 1$. Hence $Q(\alpha, I) = z_\rho^{-1}$ and we see that the value of $B^\rho(\kappa \frac{\mu}{d})$ on C is 1.

Thus, applying (4.3), the value of the generalized Steinberg character on C is

$$\sum_{|\mu|=v} \frac{1}{z_\mu} \chi_\mu^{1^v}.$$

But this sum is the inner product of the sign character on S_v with the trivial character and so is zero. Thus we have shown the claim. We remark that it follows trivially from Lemma 4.5 that the value of the character of a supercuspidal representation is 1 on C.

Lemma 5.1 implies that the order of the centralizer of C is a power of q times $(q-1)$. Thus, since the dimension of a supercuspidal representation and the order of G differ by a factor of $q^n - 1$ and a power of q, we conclude that

$$\delta(\beta_C) = (0, (q^n - 1)/(q-1)q^y)$$

for some y. Since the ℓ-valuation of $q^n - 1$ is r and that of $q - 1$ is zero (otherwise, we would have $d = n$), we may multiply by a unit in $\mathrm{W}(k)$ to conclude that $(0, \ell^r)$ lies in the image of δ. Subtracting $(0, \ell^r)$ from the element $(\ell^r, \ell^r) \in \mathrm{W}(k)$, we get the claim. □

We next handle classes represented by indices of the form

$$I = \left(\begin{array}{c|c} a & b \\ t & \lambda \end{array} \right)$$

where a is not divisible by w. We begin with a routine calculation.

Lemma 5.8. *We have*

$$\frac{(q^d - 1)(q^{2d} - 1) \cdots (q^{(v-1)d} - 1)}{q^{n(v-1)/2}} \equiv v \mod \ell^r.$$

Proof. First suppose that ζ is a primitive vth root of unity over $\mathbb{Q}$ and consider the polynomial

$$\sum_{i=0}^{v-1} X^i = \frac{X^v - 1}{X - 1} = \prod_{i=1}^{v-1} (X - \zeta^i) = (-1)^{v-1} \prod_{i=1}^{v-1} (\zeta^i - X).$$

Evaluating at 1 shows that

$$(\zeta - 1)(\zeta^2 - 1) \cdots (\zeta^{v-1} - 1) = (-1)^{v-1} v.$$

In other words, the polynomial

$$(X-1)(X^2-1)\cdots(X^{v-1}-1)-(-1)^{v-1}v \tag{5.3}$$

is divisible by $\Phi_v(X)$.

But now we remark that q^d is a primitive vth root of unity modulo ℓ. Indeed, the order of q^d modulo ℓ is $w/\gcd(d,w)$. On the other hand, n is the least common multiple of w and d so that $n = dw/\gcd(d,w)$. Thus we have the claim. Furthermore, we know that ℓ^r divides $(q^d)^v - 1$ and so Lemma 3.9 implies that ℓ^r divides $\Phi_v(q^d)$. Since the polynomial given in (5.3) is divisible by $\Phi_v(X)$, we may evaluate at q to conclude that

$$(q^d-1)(q^{2d}-1)\cdots(q^{(v-1)d}-1) \equiv (-1)^{v-1}v \mod \ell^r.$$

Likewise consider $q^{n(v-1)/2}$. First suppose that v is odd. Then $(v-1)/2$ is an integer and so $q^{n(v-1)/2} = (q^n)^{(v-1)/2}$. Since q^n is equivalent to 1 modulo ℓ^r, we see that $q^{n(v-1)/2}$ is equivalent to 1 modulo ℓ^r. For the case that v is even, put $v = 2v'$. Since v divides n, we know that n is also even. Thus we have $q^{n(v-1)/2} = q^{v'n}q^{-n/2}$. The argument we have just given shows that $q^{v'n}$ is equivalent to 1 modulo ℓ^r. We claim that $q^{n/2}$ is congruent modulo ℓ^r to -1.

Indeed, since $q^{n/2}$ is a square root of unity modulo ℓ and it is not 1 modulo ℓ (n being the order of q), we see that $q^{n/2}$ is congruent to -1 modulo ℓ. Hence, we may find an $a' \in \mathbb{Z}$ prime to ℓ and an $s \geq 1$ so that $q^{n/2} = -1 + a'\ell^s$. Squaring both sides of this equation and using the fact that

q^n is congruent to 1 modulo ℓ^r, we conclude that ℓ^r divides $-2a'\ell^s + (a')^2\ell^{2s}$. Since $s < 2s$, we see that

$$\mathrm{ord}_\ell(-2a'\ell^s + (a')^2\ell^{2s}) = s$$

and conclude that $s \geq r$ as desired. Combining the two cases, we get that $q^{n(v-1)/2}$ is congruent modulo ℓ^r to $(-1)^{v-1}$ and the claim follows. $\square$

Using Lemma 5.8, we may prove the following:

Proposition 5.9. *Suppose that C is represented by the index*

$$I = \left(\begin{array}{c|c} a & b \\ t & \lambda \end{array} \right).$$

Let h_C be the value on I of the term corresponding to $\mu = (v)$ in (4.3) for the generalized Steinberg character and let c be the part of the order of the centralizer of C which is prime to q. Then there exists an $x \in \mathrm{W}(k)$ and a unit, u, in $\mathrm{W}(k)$ such that

$$\beta'_C = u'(\beta_C - x(\ell^r, 0)),$$

where

$$\beta'_C = \left(\frac{v(q^n-1)}{c} h_C, \frac{q^n-1}{c} \mathrm{Tr}|_{\pi_i}(C) \right).$$

Proof. Proposition 5.5 implies that, aside from the contribution from (v), the action of C on the generalized Steinberg representation is an element of $\ell^r\mathrm{W}(k)$. Thus we may subtract a $\mathrm{W}(k)$-multiple of $(\ell^r, 0)$ from β_C and consider only the contribution of (v) to the first coordinate. In other words, the element under consideration is

$$\left(\frac{q^y(q^d-1)(q^{2d}-1)\cdots(q^n-1)}{q^{n(v-1)/2}c} h_C, \frac{q^y(q^n-1)}{c}\mathrm{Tr}|_{\pi_0}(C) \right),$$

where y is the power of q dividing $|C|$ (using the formulas for the dimensions of π_i, $i \geq 0$).

From Lemma 4.5, we see that h_C is an element of $\mathrm{W}(k)$ and is divisible by

$$Q^{\lambda}_{(b)}(q^a) = (q^a-1)(q^{2a}-1)\cdots(q^{(x-1)a}-1),$$

where x is the length of λ (since $\theta^{\kappa'}$ takes values in the (q^d-1)th roots of unity, its values lie in $\mathrm{W}(k)$). We remark that $c_a(\lambda)$ is bounded by the length of λ. In particular, Corollary 5.2 implies that the minimal ℓ-valuation of $Q^{\lambda}_{(b)}(q^a)/c$ is $-r$ (as the maximal ℓ-valuation of $q^{ax}-1$ is r). Since ℓ^r divides q^n-1, we see that the zeroth coordinate above is an element of $\mathrm{W}(k)$.

Lemma 5.8 shows that it is equivalent modulo ℓ^r to

$$\frac{vq^y(q^n-1)}{c}h_C.$$

Hence, by subtracting another $\mathrm{W}(k)$-multiple of $(\ell^r, 0)$, we may replace the zeroth coordinate with this expression. Dividing by the unit q^y, we obtain β'_C. □

Corollary 5.10. *Suppose that C is represented by the index*

$$I = \left(\begin{array}{c|c} a & b \\ t & \lambda \end{array} \right)$$

and suppose that a is not divisible by w. Then $\beta_C \in S$.

Proof. By Proposition 5.9, it suffices to show that

$$\left(\frac{v(q^n-1)}{c}h_C, \frac{q^n-1}{c}\mathrm{tr}|_{\pi_i}(C)\right)$$

is an element of S. Since t is a (q^a-1)th root of unity and since w does not divide a, the order of t is prime to ℓ. In particular, θ^m, a character of order ℓ^r, annihilates t.

Hence Lemma 4.5 shows that $vh_C = \mathrm{tr}_{\pi_i}(C)$ for any $i \neq 0$. Since we have already remarked in the proof of Proposition 5.9 that the zeroth coordinate of the element in consideration is an element of $\mathrm{W}(k)$, we conclude that the element in consideration is also an element of $\mathrm{W}(k)$ (that is, of the image of $\mathrm{W}(k)$ in R). □

To summarize our work so far, Corollaries 5.6 and 5.10, together with Proposition 5.7, show that the image of δ is equal to the $\mathrm{W}(k)$-subalgebra of R generated by $(\ell^r, 0)$ and the elements β'_C, where C is represented by an index of the form

$$I = \left(\begin{array}{c|c} a & b \\ t & \lambda \end{array}\right)$$

with a divisible by w.

In considering these remaining classes, we remark again that if ρ has the form $d \cdot \mu$, I admits a ρ-substitution only in the case that $\mu = (v)$. Indeed, if, under some substitution, the ρ-variable x maps to t, we must have $a|\deg(x)$ so that $w'|\deg(x)$. Since d must also divide $\deg(x)$, we have $n = \deg(x)$, giving the claim. Hence, Lemma 4.5 gives us the value on C of all the characters under consideration.

Let c be the cardinality of the centralizer of an element in C. First consider the special case that $\lambda = (b)$. In this case, Lemma 5.1 shows that c is actually $q^a - 1$ multiplied by a power of q. In particular, $\text{ord}_\ell(c) = r$. Hence $(q^n - 1)/c$ is a unit in $\mathrm{W}(k)$ and we may divide β'_C by this unit and multiply by n/a to obtain the element

$$\beta_t = (\theta^{\kappa' + im}(t) + \cdots + \theta^{q^{n-1}(\kappa' + im)}(t)).$$

In the case that $\lambda \neq (b)$, we saw in the course of the proof of Proposition 5.9 that

$$\text{ord}_\ell((q^n - 1)Q^\lambda_{(b)}(q^a)) \geq \text{ord}_\ell(c).$$

As a result, the corresponding β'_C is just a $\mathrm{W}(k)$-multiple of β_t. Hence the image in question is generated by $(\ell^r, 0)$ and the β_t.

In fact, it is these elements β_t which are really the heart of the matter, and we will now use them to obtain a certain element in the image. To make our calculations more explicit, decompose ϵ_n as $\epsilon_n = \epsilon_0 \epsilon_\ell$ where ϵ_ℓ is the ℓ-torsion part of ϵ_n and ϵ_0 is the ℓ-regular part. Put $\zeta_\ell = \theta^m(\epsilon_\ell)$ so that ζ_ℓ is a primitive ℓ^rth root of unity in L. Likewise, put $\zeta_0 = \theta^{\kappa'}(\epsilon_0)$ so that ζ_0 is a root of unity whose order under on the action of exponentiation by q is d (because $\epsilon_n^{\kappa'} = \epsilon_d^\kappa$ is a field generator for $\mathbb{F}_{q^d}/\mathbb{F}_q$ by assumption and since ℓ^r does not divide $q^d - 1$).

Before we proceed, we need to make some observations about the set, Z, of ℓ^rth roots of unity. Certainly the subgroup of $(\mathbb{Z}/\ell^r\mathbb{Z})^\times$ generated by q acts on Z by exponentiation. To simplify notation, we also put $c = \gcd(w, d)$.

Lemma 5.11. *The orbits in Z of the subgroup generated by q^c are the same as those of the subgroup generated by q^d. The set $(\zeta_\ell^i)_{i \in \mathcal{S}}$ is a collection of representatives for these orbits.*

Proof. Find a and b with $c = aw + bd$. Then if ζ is an ℓ^rth root of unity, we have

$$\zeta^{q^c} = \zeta^{q^{aw+bd}} = (\zeta^{q^{aw}})^{q^{bd}} = \zeta^{q^{bd}},$$

since w certainly divides aw (so that $q^w - 1$ divides $q^{aw} - 1$ and $\zeta^{q^{aw}-1} = 1$). Conversely, since c divides d, acting by q^d is the same as acting by a $q^{d'c}$ for some d'. Hence we have the first assertion.

Likewise, by definition, I is a collection of representatives for the multiplicative action of q^d on $\{im\}$, modulo $q^n - 1$. As ℓ is prime to q and m, I is then also a collection of representatives for the multiplicative q^d action on $\mathbb{Z}/\ell^r\mathbb{Z}$. Since the choice of primitive ℓ^rth root of unity, ζ_ℓ, gives an isomorphism from $\mathbb{Z}/\ell^r\mathbb{Z} \to Z$, the second assertion follows. □

Now for $j \in \mathbb{Z}$, put $\beta_j = \beta_t$, where $t = \epsilon_0^j \epsilon_\ell$ (noting that this element is a valid choice for t since its ℓ-part, ϵ_ℓ, has degree w over $\mathbb{F}_q$; thus it will have degree divisible by w). Then since $\zeta_0^{qd} = \zeta_0$, we have

$$\begin{aligned}\beta_j &= (\zeta_0^j \zeta_\ell^i + \zeta_0^{jq} \zeta_\ell^{iq} + \cdots + \zeta_0^{jq^{n-1}} \zeta_\ell^{iq^{n-1}}) \\ &= (\zeta_0^j [\zeta_\ell^i + \zeta_\ell^{iq^d} + \cdots + \zeta_\ell^{iq^{n-d}}] + \zeta^{jq} [\zeta_\ell^{iq} + \zeta_\ell^{iq^{d+1}} + \cdots + \zeta_\ell^{iq^{n-d+1}}] \\ &\quad + \cdots + \zeta^{jq^{d-1}} [\zeta_\ell^{iq^{d-1}} + \zeta_\ell^{iq^{2d-1}} + \cdots + \zeta_\ell^{iq^{n-1}}]).\end{aligned}$$

Furthermore, suppose that we have a, b, and y with $b = a + cy$. Then from Lemma 5.11, the q^d-orbit of $\zeta_\ell^{iq^b}$ is the same as that of $\zeta_\ell^{iq^a}$ (since they are by assumption in the same q^c-orbit) and the sum over this orbit is

$$\zeta_\ell^{iq^a} + \zeta_\ell^{iq^{a+c}} + \zeta_\ell^{iq^{a+2c}} + \cdots + \zeta_\ell^{iq^{a+w-c}}.$$

As a result, we may factor the expression we have given for β_j to conclude that

$$\begin{aligned}\beta_j = ([\zeta_0^j + \zeta_0^{jq^c} + \cdots + \zeta_0^{jq^{d-c}}][\zeta_\ell^i + \zeta_\ell^{iq^c} + \cdots + \zeta_\ell^{iq^{w-c}}] \\ + \cdots + [\zeta_0^{jq^{c-1}} + \zeta_0^{jq^{2c-1}} + \cdots + \zeta_0^{jq^{d-1}}][\zeta_\ell^{iq^{c-1}} + \zeta_\ell^{iq^{2c-1}} + \cdots + \zeta_\ell^{iq^{w-1}}]).\end{aligned}$$

To simplify our notation, we will denote by $T_\nu(x)$ the (d/c)-tuple

$$(x^{q^\nu}, x^{q^{\nu+c}}, x^{q^{\nu+d-c}})$$

and, for $0 \le \nu \le c-1$, we will put

$$\Gamma_\nu = (\zeta_\ell^{iq^\nu} + \zeta_\ell^{iq^{c+\nu}} + \cdots + \zeta_\ell^{iq^{w-c+\nu}}).$$

Then we have shown that $\beta_j = \sum_\nu \mathscr{P}_j(T_\nu(\zeta_0))\Gamma_\nu$, where again $\mathscr{P}_j$ is the jth power sum on d/c variables, a symmetric polynomial. We note that some authors take the convention that $\mathscr{P}_0 = 1$, but here we are using the convention $\mathscr{P}_0 = (d/c)$ so that we get the correct equation $\beta_0 = (d/c)\sum_\nu \Gamma_\nu$. The element we seek is Γ_0.

Lemma 5.12. *Let M be the* W(k)*-submodule generated by the β_j. Then $\Gamma_0 \in M$. In particular, Γ_0 is contained in the image of $Z($*W$(k))$.

Proof. One verifies trivially that if ζ is a $(q^d - 1)$th root of unity then

$$\mathscr{P}_i(T_\nu(\zeta))\mathscr{P}_j(T_\nu(\zeta)) = \sum_{x=0}^{d/c-1} \mathscr{P}_{i+jq^{xc}}(T_\nu(\zeta)).$$

Since M is by definition the W(k)-module consisting of W(k)-linear combinations of the β_j, we conclude that if M contains two W(k)-linear combinations of the Γ_ν, it contains the linear combination resulting from the pairwise product of their coefficients.

On the other hand, for $0 < x \leq c - 1$, the collection

$$S_0 = \{\zeta_0, \zeta_0^{q^c}, \ldots, \zeta_0^{q^{d-c}}\}$$

is different modulo ℓ from the collection

$$S_x = \{\zeta_0^{q^x}, \zeta_0^{q^{c+x}}, \ldots, \zeta_0^{q^{d-c+x}}\}.$$

Indeed, otherwise, injectivity of the reduction map on $(q^d - 1)$th roots of unity would give a y with $\zeta_0^{q^{yc}} = \zeta_0^{q^x}$ so that $\zeta_0^{q^x(q^{yc-x}-1)} = 1$, allowing us to conclude that d and so c divides $yc - x$ (since q^x is prime to the order of ζ_0) and contradict the fact that c does not divide x. Hence the polynomial which has S_0 as roots will be different modulo ℓ from the one that has S_x as roots. Thus there is some elementary symmetric function, e, such that

$$e(T_0(\zeta_0)) \not\equiv e(T_x(\zeta_0)) \mod \ell.$$

By Formula I.2.14' of [13], we may write the function e as a W(k)-linear combination of products of the $\mathscr{P}_i$ (since the number of variables we are using is bounded by n). In particular, applying the argument of the first paragraph, we see that the element $\sum_\nu e(T_\nu(\zeta_0))\Gamma_\nu$ lies in M. Likewise, the element $\sum_\nu \Gamma_\nu$ is an element of M and so the W(k)-multiple $\sum_\nu e(T_x(\zeta_0))\Gamma_\nu$ is as well. Subtracting these elements, we obtain an expression in M whose coefficient on Γ_0 is a unit in W(k) and whose coefficient on Γ_x is zero. Taking the coefficient-wise product over the x, we obtain $\Gamma_0 \in M$. $\square$

From now on, we put $\Gamma = \Gamma_0$. In that we will eventually show that Γ generates the image of δ, it will be useful us to record at this point how Γ acts on our representations a bit more intrinsically. Explicitly, this action is given by the following:

Proposition 5.13. *Suppose that $\tau \in \mathbb{F}_{q^n}^\times$ is a nontrivial ℓ-torsion element. Then the action of Γ on the supercuspidal representation*

$$S_L\left(\begin{array}{c|c} n & 1 \\ \sigma\tau & (1) \end{array}\right)$$

is given by

$$\theta(\tau) + \theta(\tau^{q^c}) + \cdots + \theta(\tau^{q^{w-c}}).$$

Moreover, Γ acts on the generalized Steinberg representation

$$S_L\left(\begin{array}{c|c} d & v \\ \sigma & (1^v) \end{array}\right)$$

by w/c.

Proof. Put $\tau = \epsilon_n^{im}$ so that $\sigma\tau = \epsilon_n^{\kappa'+im}$. Then by definition Γ acts on the supercuspidal representation named by

$$\theta(\epsilon_\ell^{mi}) + \theta(\epsilon_\ell^{miq^c}) + \cdots + \theta(\epsilon_\ell^{miq^{w-c}}).$$

But then since ϵ_ℓ is the ℓ-torsion part of ϵ_n, we have $\epsilon_n^m = \epsilon_\ell^m$ so that $\tau = \epsilon_\ell^{im}$ and the first claim follows. The second claim is trivial from the definition of Γ. □

The action of Γ on the supercuspidal representations depends on the choice of θ, but this dependence is merely an artifact of the fact that the characters of representations of G depend on this choice.

We next give a converse to the Lemma 5.12.

Lemma 5.14. *The β_t are contained in the* $\mathrm{W}(k)$*-subalgebra generated by* Γ*.*

Proof. Putting $t = \epsilon_n^j$, we have $t = \epsilon_0^j \epsilon_\ell^j$ so that

$$\begin{aligned} \beta_t &= (\zeta_0^j \zeta_\ell^{ij} + \zeta_0^{jq} \zeta_\ell^{ijq} + \cdots + \zeta_0^{jq^{n-1}} \zeta_\ell^{ijq^{n-1}}) \\ &= ([\zeta_0^j + \zeta_0^{jq^c} + \cdots + \zeta_0^{jq^{d-c}}][\zeta_\ell^{ij} + \zeta_\ell^{ijq^c} + \cdots + \zeta_\ell^{ijq^{w-c}}] + \cdots) \end{aligned}$$

Correspondingly, consider the polynomial

$$f_t(X) = ([\zeta_0^j + \zeta_0^{jq^c} + \cdots + \zeta_0^{jq^{d-c}}][X^j + X^{jq^c} + \cdots + X^{jq^{w-c}}] + \cdots)$$

in $W(k)[X]$. Modulo $(X^{\ell^r} - 1)$, this polynomial is fixed by the action of q^c and so, since w/c is the order of q^c modulo ℓ (and r is the valuation of $\Phi_{w/c}(q^c)$), Theorem 2.7 shows that we may find a polynomial $g_t(X) \in \mathrm{W}(k)[X]$ so that

$$f_t(X) = g_t(X + X^{q^c} + \cdots + X^{q^{w-c}}).$$

Evaluating at ζ_ℓ^i (for each i), we conclude that $\beta_t = g_t(\Gamma)$ and we have the claim. □

Thus we have characterized the image in question as being the subalgebra generated by the elements Γ and $(\ell^r, 0)$. To conclude our calculation of this image, we will now show that the $(\ell^r, 0)$ is not a necessary generator. To this end, put

$$f(X) = \prod_{i \in \mathcal{S}} [X - (\zeta_\ell^i + \zeta_\ell^{iq^c} + \cdots + \zeta_\ell^{iq^{w-c}})].$$

Proposition 5.15. *The element* $(\ell^r, 0)$ *is contained in the* $\mathrm{W}(k)$*-subalgebra generated by* Γ.

Proof. Since Lemma 5.11 implies that $\mathcal{S}$ is a collection of representative for the orbits of q^c in Z, $f(\Gamma)$ has the form $(a, 0)$ with

$$a = \prod_{i \in \mathcal{S}} [w/c - (\zeta^i + \zeta^{iq^c} + \cdots + \zeta^{iq^{w-c}})],$$

where every nontrivial ℓ^rth root of unity is represented in exactly one factor. On the other hand, by Proposition 2.2, if $i \in \mathcal{S}$ is such that ζ_ℓ^i is a primitive ℓ^sth root of unity, the $\mathrm{W}(k)$-valuation of the factor coming from i is $(w/c)/\phi(\ell^s)$. Since there are $\phi(\ell^s)/(w/c)$ such factors, we get a contribution of 1 from the factors coming from primitive ℓ^sth roots. Thus the valuation of a is r and, up to multiplication by a unit in $\mathrm{W}(k)$, $f(\Gamma)$ is equal to $(\ell^r, 0)$. □

We remark that the minimal polynomial of Γ is $g(X) = (X - w/c)f(X)$. We may thus show the following key result:

Theorem 5.16. *The image of* $\delta : Z(\mathrm{W}(k)[G]) \to R$ *is isomorphic as a* $\mathrm{W}(k)$*-algebra to*

$$\mathrm{W}(k)[Y]/(g(Y)).$$

That is, it is isomorphic to the invariants of

$$\mathrm{W}(k)[X]/(X^{\ell^r} - 1)$$

under the action of q^c*. Moreover, this isomorphism may be chosen in such a way that the element in* $Z(\mathrm{W}(k)[G])$ *corresponding to* Y *maps to*

$$\theta(\tau) + \theta(\tau^{q^c}) + \cdots + \theta(\tau^{q^{w-c}})$$

at the coordinate of R *corresponding to the supercuspidal representation*

$$S_L \left(\begin{array}{c|c} n & 1 \\ \sigma\tau & (1) \end{array} \right)$$

and to w/c *on that corresponding to the generalized Steinberg representation*

$$S_L \left(\begin{array}{c|c} d & v \\ \sigma & (1^v) \end{array} \right).$$

Proof. Since Γ generates the image of δ and has minimal polynomial g, the theorem follows from Theorem 2.7 and Proposition 5.13. □

Chapter 6

Projective Envelopes of Cuspidal Representations

We keep the notation and hypotheses of Chapter 5. In this chapter, we will compute the endomorphism ring of the projective envelope of the cuspidal representation $\bar{\pi}$. We work over the category of $\mathrm{W}(k)[G]$-modules or more precisely over the category, $\mathcal{C}$, of $\mathrm{W}(k)[G]$-modules in the same block as $\bar{\pi}$.

We begin by giving a nice characterization of the projective envelope of $\bar{\pi}$. Let U be the subgroup of G consisting of strictly upper-triangular matrices and let $\psi : U \to \mathrm{W}(k)$ be a fixed generic character (as in Chapter 3, choose a nontrivial homomorphism $\psi' : \mathbb{F}_q \to \mathrm{W}(k)^\times$ and define the image under ψ of an element in U to be the value of the sum of the values of the off-diagonal entries under ψ'). We recall that a G-representation is generic if and only if it admits a nonzero U-map from ψ (i.e., if it contains a copy of the mirabolic representation). Moreover, an irreducible generic representation contains a unique vector (up to scalars) on which U acts by ψ.

We define $P_{\bar{\pi}}$ to be the representation $e_{\bar{\pi}}\mathrm{Ind}_U^G\psi$, where $e_{\bar{\pi}}$ is the idempotent in $\mathrm{W}(k)[G]$ coming from $\bar{\pi}$ (see the beginning of Chapter 3).

Proposition 6.1. *$P_{\bar{\pi}}$ is a projective envelope for $\bar{\pi}$.*

Proof. Since the order of U is a power of q, it is prime to ℓ and so ψ is a projective representation. As a result, the G-representation $\mathrm{Ind}_U^G\psi$ is also projective. Moreover, the remarks preceding the proposition imply that we have a nonzero U-homomorphism $\psi \to \bar{\pi}$. Frobenius reciprocity thus gives a nonzero G-homomorphism $\mathrm{Ind}_U^G\psi \to \bar{\pi}$. Since $\bar{\pi}$ is irreducible, this map is surjective. Finally, $e_{\bar{\pi}}$ acts as the identity on $\bar{\pi}$ and so we get a surjective map $P_{\bar{\pi}} \to \bar{\pi}$. $P_{\bar{\pi}}$ is projective as a direct summand of the projective module $\mathrm{Ind}_U^G\psi$.

On the other hand, suppose that $\bar{\pi}'$ is another simple $\mathrm{W}(k)[G]$-module admitting a surjective map from $P_{\bar{\pi}}$. Since $\bar{\pi}'$ is a simple representation of G, it is annihilated by ℓ and may be viewed as a representation over k. Likewise, since $e_{\bar{\pi}}$ acts as the identity on $P_{\bar{\pi}}$, $e_{\bar{\pi}}$ does not annihilate $\bar{\pi}'$. Thus $\bar{\pi}'$ may be viewed as a representation in the same block as $\bar{\pi}$. Composing with the projection map $\mathrm{Ind}_U^G\psi \to P_{\bar{\pi}}$, we also get a nonzero map $\mathrm{Ind}_U^G\psi \to \bar{\pi}'$. Frobenius reciprocity then gives a nonzero map $\psi \to \bar{\pi}'$ and we conclude that $\bar{\pi}'$ is generic. Thus Corollary 3.23 implies that $\bar{\pi}' = \bar{\pi}$.

Suppose now for a contradiction that $P_{\bar{\pi}}$ is not a projective envelope of $\bar{\pi}$. Then since the projective module $P_{\bar{\pi}}$ maps surjectively onto $\bar{\pi}$, the projective envelope must be a direct summand of $P_{\bar{\pi}}$ (see for example Lemma 2.3 of [1]). Thus we may find nonzero $\mathrm{W}(k)[G]$-modules M and N with $P_{\bar{\pi}} = M \oplus N$. In particular, if M and N are both nonzero, we have surjective maps $P_{\bar{\pi}} \to M, N$. Since M and N must each admit a simple quotient, we conclude (from the previous argument) that they each admit simple quotients isomorphic to $\bar{\pi}$. In particular, the $\mathrm{W}(k)$-rank of $\mathrm{Hom}_G(P_{\bar{\pi}}, \bar{\pi})$ is at least two, a contradiction, by Frobenius reciprocity, to the remarks preceding the proposition. $\square$

Thus the aim of the rest of the chapter is to calculate the endomorphism ring of $P_{\bar{\pi}}$. An important observation to this end is the following:

Proposition 6.2. *The representation $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} L$ is the direct sum of a single copy of the generalized Steinberg representation,*

$$\mathrm{St}_L = S_L \left(\begin{array}{c|c} d & \sigma \\ 1 & (1^v) \end{array} \right),$$

and a single copy of each of the supercuspidal representations that lift $\bar{\pi}$ to L.

Proof. A representation over L admits a nonzero map from $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} L$ if and only if it is generic and in the block coming from $\bar{\pi}$ (in which case such maps form an L-vector space of dimension 1). Therefore, since G is a finite group and L is a sufficiently large field of characteristic zero, Corollary 3.23 implies that the statement in the proposition $\square$

In particular, the ring $\mathrm{End}_{L[G]}(P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} L)$ is commutative (isomorphic to a direct sum of copies of R, one for each of the representations named in Proposition 6.2). But since $P_{\bar{\pi}}$ is projective over $\mathrm{W}(k)$, we have an injection

$$\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}}) \hookrightarrow \mathrm{End}_{L[G]}(P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} L).$$

Thus $\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}})$ is commutative as well.

To relate the calculations we performed in Chapter 5 to $\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\overline{\pi}})$, we consider the Bernstein center of the category $\mathcal{C}$. Recall from Chapter 1 that the *Bernstein center* of an abelian category $\mathcal{A}$ is the endomorphism ring of the identity functor $\mathrm{id} : \mathcal{A} \to \mathcal{A}$. That is, it is the ring of natural transformation of this identity functor. In other words, an element of the Bernstein center is a choice of endomorphism on each object of $\mathcal{A}$ which commutes with any morphism of $\mathcal{A}$ in the obvious sense.

We remark that for the category of modules over a ring, R, the Bernstein center is just the center of R. Indeed, an element of the center of R certainly gives an appropriate choice of endomorphisms (namely multiplication by the element). On the other hand, R is itself an object in the category and one can recover the element chosen from the center of R from its action on R (and it is not hard to see from the required commutativity that the action on R determines all the choices). In particular, the Bernstein center of our category is $e_{\overline{\pi}} Z(\mathrm{W}(k)[G])$.

More generally, we have the following:

Proposition 6.3 ([17], Theorem 1.1)**.** *Suppose that Q is a faithfully projective object in the abelian category $\mathcal{A}$. Then $M \mapsto \mathrm{Hom}(P, M)$ is an equivalence of categories from $\mathcal{A}$ to the category of right $\mathrm{End}_{\mathcal{A}}(Q)$-modules.*

In particular, we have an isomorphism from the Bernstein center of $\mathcal{A}$ to $Z(\mathrm{End}_{\mathcal{A}}(Q))$. The map takes an element of the Bernstein center to its action on Q.

In order to use this last result to our advantage, we construct a faithfully projective object in the category $\mathcal{C}$. We let $\bar{\pi}_0$ be the representation

$$D_k \left(\begin{array}{c|c} d & 1 \\ \sigma & (1) \end{array} \right)$$

of $\mathrm{GL}_d(\mathbb{F}_q)$. Then if P is the subgroup of G corresponding to v copies of $\mathrm{GL}_d(\mathbb{F}_q)$, Proposition 3.24 implies that $i_P^G \bar{\pi}_0^{\otimes v}$ maps surjectively onto any irreducible representation in the block of $\bar{\pi}$ other than $\bar{\pi}$ itself (since any such representation has the corresponding cuspidal support). Hence, if we let π_0 be a $\mathrm{W}(k)$-lattice in the unique lift of $\bar{\pi}_0$ to K (unique since ℓ does not divide the order of $\mathrm{GL}_d(\mathbb{F}_q)$), we see that $P_0' = i_P^G \pi_0^{\otimes v}$ maps surjectively onto all the representations named. Moreover, the representation π_0 is projective (again since ℓ again does not divide the order of $\mathrm{GL}_d(\mathbb{F}_q)$). Since parabolic induction takes projectives to projectives, we conclude that P_0' is projective as well.

Putting $P_0 = e_{\bar{\pi}} P_0'$, we conclude that $Q_{\bar{\pi}} = P_0 \oplus P_{\bar{\pi}}$, a projective $\mathrm{W}(k)$-module, surjects onto all the simple objects in $\mathcal{C}$. Thus $Q_{\bar{\pi}}$ is a faithfully projective object in $\mathcal{C}$.

Since $Q_{\bar{\pi}}$ is constructed as a direct product, its $\mathrm{W}(k)[G]$-endomorphism ring can be represented as the collection of block matrices

$$\begin{pmatrix} \mathrm{End}(P_0) & \mathrm{Hom}(P_{\bar{\pi}}, P_0) \\ \mathrm{Hom}(P_0, P_{\bar{\pi}}) & \mathrm{End}(P_{\bar{\pi}}) \end{pmatrix}.$$

In particular we get an embedding

$$Z(\mathrm{End}(Q_{\bar{\pi}})) \hookrightarrow Z(\mathrm{End}(P_0)) \times Z(\mathrm{End}(P_{\bar{\pi}})).$$

The relevant point to all of this is the following proposition:

Proposition 6.4. *The map $Z(\mathrm{End}_{\mathrm{W}(k)[G]}(Q_{\bar{\pi}})) \to Z(\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}}))$ that sends an endomorphism to its action on $P_{\bar{\pi}}$ is surjective.*

Proof. The key point is that $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} L$ and $P_0 \otimes_{\mathrm{W}(k)} L$ each contain a unique copy of the same generalized Steinberg representation,

$$\mathrm{St}_L = S_L \left(\begin{array}{c|c} d & 1 \\ \sigma & (1^v) \end{array} \right),$$

and that this representation is the only irreducible representation they have in common. To see this fact, recall that we have already characterized the irreducible representations contained in $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} L$ in Proposition 6.2. Likewise, $P_0 \otimes L$ is a parabolic induction so that of the representations named in Proposition 6.2, only St_L can admit a nonzero map to or from $P_0 \otimes L$. Furthermore, $P_0 \otimes L$ is (by Definition 4.2 of [12]) equal to the representation

$$M_L \left(\begin{array}{c|c} d & 1 \\ \sigma & (1^v) \end{array} \right).$$

We may then apply 4.2.i of [6] to conclude that it contains a single copy of St_L, as claimed.

In particular, $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K$ and $P_0 \otimes_{\mathrm{W}(k)} K$ have only one irreducible $K[G]$-representation, St_K, in common, each representation contains a unique copy of St_K, and St_K satisfies

$$\mathrm{St}_L = \mathrm{St}_K \otimes_K L.$$

Let f be the canonical surjection $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K \to \mathrm{St}_K$. Since $P_{\bar{\pi}}$ is projective, it is torsion free and so the map $P_{\bar{\pi}} \to P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K$ is injective. We denote by

M the image of $P_{\bar{\pi}}$ in St_K so that we have the commutative diagram

$$\begin{array}{ccc} P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K & \xrightarrow{f} & \mathrm{St}_K \\ \uparrow & & \uparrow \\ P_{\bar{\pi}} & \longrightarrow & M \end{array}$$

and so that M is a G-stable $\mathrm{W}(k)$-lattice in St_K.

We claim that $g \in \mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}})$ acts on M by a scalar in $\mathrm{W}(k)$. Indeed, g induces (via extension of scalars) a map on $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K$ and so, since there is a unique copy of St_K contained in $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K$, induces a map on St_K via restriction. Since St_K is absolutely irreducible, this map must be multiplication by a scalar in K. Moreover, g preserves M since

$$g(M) = g(f(P_{\bar{\pi}})) = f(g(P_{\bar{\pi}})) \subset f(P_{\bar{\pi}}) = M.$$

Thus g is a scalar in K which preserves the $\mathrm{W}(k)$-lattice M. We conclude that g is a scalar in $\mathrm{W}(k)$ as claimed.

We thus obtain a map $s : \mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}}) \to \mathrm{W}(k)$ which takes a map to the scalar by which it acts on M (and so also on St_K in the appropriate sense). Certainly then, for $g \in Z(\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}}))$, we may view $s(g)$ as an element of $\mathrm{End}_{\mathrm{W}(k)[G]}(P_0)$ and consider the element

$$\Phi(g) = \begin{pmatrix} s(g) & 0 \\ 0 & g \end{pmatrix} \in \mathrm{End}_{\mathrm{W}(k)[G]}(Q_{\bar{\pi}}).$$

We claim that $\Phi(g)$ lies in the center of $\mathrm{End}_{\mathrm{W}(k)}(Q_{\bar{\pi}})$. Indeed, for $\psi \in \mathrm{Hom}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}}, P_0)$, to say that $\Phi(g)$ commutes with

$$\begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix}$$

is to say that for $x \in P_{\bar{\pi}}$, we have

$$\psi(g(x)) = s(g)\psi(x)$$

in P_0. But now since P_0 and $P_{\bar{\pi}}$ embed into $P_0 \otimes_{\mathrm{W}(k)} K$ and $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K$, respectively, it suffices to check this equality over $P_0 \otimes_{\mathrm{W}(k)} K$ for an element of $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K$. Furthermore, $P_0 \otimes_{\mathrm{W}(k)} K$ and $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} K$ have only the factor St_K in common and so $\psi = \psi \circ f$. Since g commutes with f and since g acts by $s(g)$ on St_k, we conclude that $\psi(g(x)) = \psi(s(g)x)$ and the equality sought holds trivially.

Likewise, for $\psi \in \mathrm{Hom}_{\mathrm{W}(k)[G]}(P_0, P_{\bar{\pi}})$ the commutativity of $\Phi(g)$ and

$$\begin{pmatrix} 0 & 0 \\ \psi & 0 \end{pmatrix}$$

is trivial. Finally, $\Phi(g)$ commutes with a diagonal endomorphism because g is in the center of $\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}})$. Certainly any element of $\mathrm{End}_{\mathrm{W}(k)[G]}(Q_{\bar{\pi}})$ is a sum of elements of the form we have considered and so we conclude that $\Phi(g)$ lies in the center of $\mathrm{End}_{\mathrm{W}(k)[G]}(Q_{\bar{\pi}})$ and maps to g. $\square$

Since we have already given an isomorphism

$$e_{\bar{\pi}} Z(\mathrm{W}(k)[G]) \to \mathrm{End}_{\mathrm{W}(k)[G]}(Q_{\bar{\pi}}),$$

we conclude that we have a surjective map $e_{\bar{\pi}} Z(\mathrm{W}(k)[G]) \twoheadrightarrow Z(\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}}))$ (that takes an element to the corresponding multiplication map; recall from the remark following Proposition 6.2 that $\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}})$ is commutative).

Thus we have maps

$$e_{\bar{\pi}} Z(\mathrm{W}(k)[G]) \twoheadrightarrow \mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}}) \hookrightarrow \mathrm{End}_{L[G]}(P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} L),$$

but, from Proposition 6.2, the last object isomorphic to the ring R considered in Chapter 5 and under this isomorphism, the map given here is the map considered in Section 5 (certainly the map $Z(\mathrm{W}(k)) \to R$ factors through $e_{\bar{\pi}} Z(\mathrm{W}(k))$). Thus we see that the image we calculated is isomorphic to $\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}})$, the ring in question. To summarize, we have shown the following:

Theorem 6.5. *Suppose that k is a finite field of characteristic ℓ. Let q be a power of a prime distinct from ℓ and let w be the order of q modulo ℓ. Suppose that $\bar{\pi}$ is an irreducible cuspidal representation of $G = \mathrm{GL}_n(\mathbb{F}_q)$ over k of degree $d < n$. Put $c = \gcd(w, d)$ and*

$$\bar{\pi} = D_k \left(\begin{array}{c|c} n & 1 \\ \sigma & (1) \end{array} \right)$$

and let $P_{\bar{\pi}}$ be the projective envelope of $\bar{\pi}$ in the category of $\mathrm{W}(k)[G]$-modules. Then, under the assumptions that $2 \leq n < \ell$ and that k is large enough to contain the ℓ-regular $|G|$th roots of unity, $\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}})$ is isomorphic to

$$\mathrm{W}(k)[Y] / \prod_{\zeta \in \mathcal{S}} [Y - (\zeta^i + \zeta^{iq^c} + \cdots + \zeta^{iq^{w-c}})],$$

where $\mathcal{S}$ is a collection of representatives for the action of q^c on the ℓ^rth roots of unity in $\mathrm{W}(k)$ and $r = \mathrm{ord}_\ell(q^w - 1)$. That is, $\mathrm{End}_{\mathrm{W}(k)[G]}(P_{\bar{\pi}})$ is isomorphic to the invariants of

$$\mathrm{W}(k)[X]/(X^{\ell^r} - 1)$$

under the action $X \mapsto X^{q^c}$.

Moreover, if L *is a finite extension of the field of fractions of* $\mathrm{W}(k)$, *large enough to contain the* $|G|$*th roots of unity and* θ *is a fixed embedding of* $\mathbb{F}_{q^{n!}}^{\times} \to L^{\times}$, *this isomorphism may be chosen in such a way that the generator* Y *acts on the direct summand of* $P_{\bar{\pi}} \otimes_{\mathrm{W}(k)} L$ *corresponding to the supercuspidal representation*

$$S_L \left(\begin{array}{c|c} n & 1 \\ \sigma\tau & (1) \end{array} \right)$$

by

$$\theta(\tau) + \theta(\tau^{q^c}) + \cdots + \theta(\tau^{q^{w-c}})$$

and on that corresponding to the generalized Steinberg representation

$$S_L \left(\begin{array}{c|c} d & v \\ \sigma & (1^v) \end{array} \right)$$

by w/c.

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