Global Coordinate Systems: Continuously Moving Finite-Dimensional Unit Norm Tight Frames on Smooth Manifolds

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Abstract

Continuously moving bases for tangent spaces of manifolds are important in the study of differential geometry and mathematical physics. However, globally continuous bases do not exist for the tangent spaces of all manifolds, for instance the Möbius strip and the 2-dimensional sphere. Some applications, particularly those in signal processing, call for a more general coordinate system known as a tight frame. Finite-dimensional unit-norm tight frames (FUNTFs) are a natural generalization of an orthonormal basis which satisfy a useful reconstruction formula for all vectors in their span. Thus, we are motivated to study the existence of FUNTFs for the tangent spaces of manifolds. We investigate questions about the existence of FUNTFs on manifolds, the minimum number of vectors needed for a FUNTF, and potential applications. In particular, we study the Möbius strip, its higher dimensional generalization of vector bundles on the circle, and n-spheres.

Acknowledgements

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Chapter 1

General Description

Often people need a digital representation of some analog data. This requires making a discrete approximation of the analog signal. An example of an analog signal is a sound wave. To manipulate the sound on a computer it must be in a digital format, so we must discretize the analog signal. Linear techniques tend to be useful, so we might choose to represent the signal with a discretization in a vector space. A *Hilbert space*, $\mathcal{H}$, is a vector space with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathcal{H}$ is complete when it is paired with the norm $\|x\| = \langle x, x \rangle^{1/2}$. Through the clever use of mathematics, analysts optimize the representation of the signal and improve the quality of the digital approximation.

If a basis is used to represent some digital signal, each component is independent of the others, so the loss or corruption of one component in the digital signal would be undesirable. That is, no information in the signal is repeated when a basis is used. Thus, we might decide to use some redundant coordinate system so that information is repeated and not totally lost when a single component gets corrupted. This effectively spreads out the error over more components.

A *frame* for an n-dimensional Hilbert space, $\mathcal{H}$, is a sequence of at least n vectors,

$\{x_1, x_2, \ldots, x_k\}$ such that for all vectors $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_{i=1}^{k} |\langle x, x_i \rangle|^2 \leq B\|x\|^2. \tag{1.1}$$

One characterization of frames for finite dimensional vector spaces that is typically easier to imagine is any finite set of at least n vectors such that every vector in the space can be represented as a linear combination of the frame vectors.

Some of the most useful frames are able to reconstruct everything in their span using the same reconstruction formula used by bases. These are called tight frames and they satisfy Equation 1.1 with $A = B$. If all the vectors in a tight frame have norm one, we call it a finite unit-norm tight frame (FUNTF). In signal processing applications, FUNTFs have been shown to minimize the mean-squared error in signals subject to random noise [3].

For many years frames were primarily studied for fixed Hilbert spaces, but recent applications have shown the value in developing a theory of smoothly changing frames for smoothly changing spaces. For example, frames have been used to describe symmetries that appear in quasicrystals [4]. Additionally, frames made up of exponentially-decaying Wannier functions over certain vector bundles on the torus were recently the key to solving a question in solid-state physics [11].

A manifold is essentially a topological space that locally looks like $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Note that our usual notion of a surface living in three dimensions is a two-dimensional manifold. For example, the surface of the Earth is a sphere, yet a person standing at any point on the Earth is likely to describe the area around them using a coordinate system such that each point is represented by some (x, y) such as latitude and longitude. If we look at the example of the surface a sphere as a two-dimensional manifold, we can imagine the tangent plane at each point as being a two-dimensional Euclidean space we associate with that point. Any two nearby points have tangent

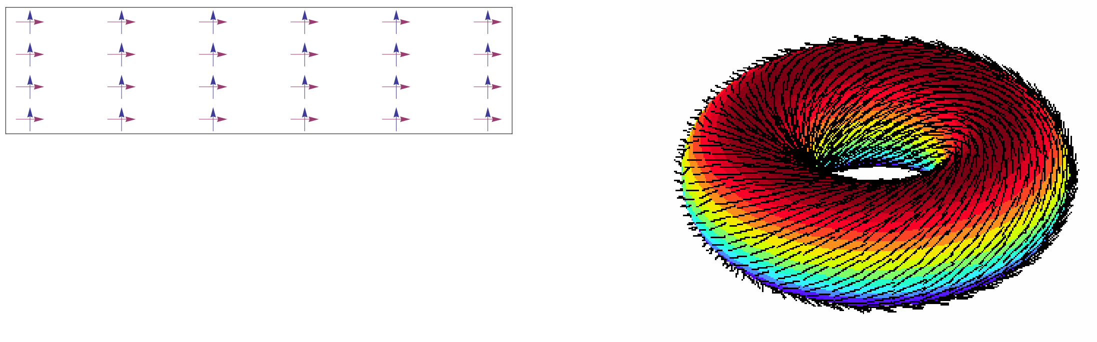

Figure 1.1: (Left) At each point on a rectangle we can form an orthogonal basis to represent vectors in the region around that point. This rectangle can then have its top and bottom glued together to form a tube, and that tube's ends can be attached to form the torus. (Right) At each point on the sphere we could imagine some vector. We can use a basis at each point to talk about the vector at that point. Image from http://en.wikipedia.org/wiki/File:Hairy_doughnut.png .

planes that are close to each other, so we say that these tangent planes vary continuously. The longitude and latitude coordinate system is useful locally, but fails to work at the North and South Poles and hence is not a global coordinate system. In fact, there is no global coordinate system consisting of only two vectors that moves continuously through its tangent space.

Because not all manifolds have a globally continuous basis, we are motivated to look at a more general class of coordinate systems, frames. We study frames more generally for smoothly varying finite-dimensional spaces, or *vector bundles*, by choosing some underlying smooth n-dimensional manifold and identifying each point with some smoothly changing m-dimensional Euclidean space, where m is any positive integer. The m-dimensional Euclidean space may or may not be the tangent space, but the tangent space is one example that is easy to visualize.

Because FUNTFs have been shown to be useful in fixed vector spaces and frames on manifolds have applications as well, we are motivated to combine these two ideas and study the more specific case of FUNTFs for smoothly varying finite-dimensional spaces. General Parseval frames on manifolds have been studied in [7], and we look more specifically at FUNTFs.

In this work, we investigate a number of general questions. As we show later, some manifolds like the Möbius strip have no continuous basis, but do have a FUNTF, while others, like the two-dimensional sphere have neither. What kinds of manifolds and vector bundles yield a FUNTF that changes continuously over the manifold? If a frame is being used to analyze some function, one thing that could make the analysis easier is to have as few frame vectors representing the space as possible. What is the minimum number k so that a given manifold and its associated vector spaces to have a continuously moving FUNTF of k vectors? We will address these questions for certain classes of manifolds and vector spaces. We also wonder what potential applications there are of this work to science and engineering.

We show that for any $k \geq 4$ there is a continuous FUNTF of k vectors for the tangent space of the Möbius strip. We investigate, as well, generalizations of these results to higher dimensions. We also look at the case of the tangent space of the circle, the sphere, and the analogous higher-dimensional generalizations of the sphere and circle. We show that there is a continuously moving FUNTF for the tangent spaces of the n-sphere if and only if n is odd. We then investigate methods of decreasing the upper bound on the number of vectors needed for such a FUNTF.

Chapter 2

Introduction, Context, Related Applications

Recall that a *Hilbert space*, $\mathcal{H}$, is a vector space with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathcal{H}$ with the norm $\|x\| = \langle x, x \rangle^{1/2}$ is a complete metric space. A *frame* for a Hilbert space $\mathcal{H}$ is a sequence of vectors $\{x_i\} \subset \mathcal{H}$ for which there exist constants $0 < A \leq B < \infty$ such that, for every $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_i |\langle x, x_i \rangle|^2 \leq B\|x\|^2. \tag{2.1}$$

The constants A and B are called the *frame bounds*. In this work, we specifically focus on frames for $\mathbb{R}^n$, so we can actually say more. Let $k \geq n$ and let $\{v_1, v_2, \ldots v_k\}$ be a finite sequence of vectors in $\mathbb{R}^n$. We show in Theorem 3.0.1 that a sequence of vectors is a frame if and only if $\{v_1, v_2, \ldots, v_k\}$ is a spanning set for $\mathbb{R}^n$. We say that the sequence $\{v_1, v_2, \ldots, v_k\}$ has a *dilation to a basis for* $\mathbb{R}^k$ if there exist vectors $\{w_1, w_2, \ldots, w_k\}$ in $\mathbb{R}^{k-n}$ such that $\{v_1 \oplus w_1, \ \ldots \ , v_k \oplus w_k \}$ is a basis for $\mathbb{R}^k$. In terms of matrices, this means that

$$\left\{ \begin{bmatrix} v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ w_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_k \\ w_k \end{bmatrix} \right\} \tag{2.2}$$

is a basis for $\mathbb{R}^k$.

A finite sequence of vectors $\{v_1, v_2, \ldots, v_k\} \in \mathbb{R}^n$ is called a *tight frame* if it satisfies equation 2.1 with $A = B$. If $A = B = 1$, it is called a *Parseval frame*. When $\{v_1, v_2, \ldots, v_k\}$ is a tight frame, this leads to the formula,

$$\|x\|^2 = \frac{1}{A} \sum_{i=1}^{k} |\langle x, v_i \rangle|^2. \tag{2.3}$$

for all $x \in \mathbb{R}^n$. Another way of looking at tight frames, shown in Theorem 3.0.2, is that they are frames for $\mathbb{R}^n$ that have a dilation to a scaled orthonormal basis for $\mathbb{R}^k$. This gives an easy way to create tight frames. An orthonormal basis for $\mathbb{R}^k$ is easy to create from any basis for $\mathbb{R}^k$ via the Gram-Schmidt process, then a tight frame for $\mathbb{R}^n$ can be created by chopping off the last $k - n$ components of each of the vectors in the orthonormal basis then multiplying by a constant.

In this research we are specifically interested in finite-length tight frames with frame vectors that are all length one, also called *finite unit-norm tight frames* (FUNTFs). The most useful frames to use as wavelet dictionaries for signal processing are FUNTFs [3], as these frames are the optimal coordinate system for diluting errors in signals by spreading the errors out over many coordinates. They still retain many of the useful properties of orthonormal bases.

One example of a FUNTF for $\mathbb{R}^2$ is the set of frame vectors

$$\{(0, 1), (-\sqrt{3}/2, -1/2), (\sqrt{3}/2, -1/2)\}$$

shown in Figure 2.1 along with its extension to a scaled orthonormal basis for $\mathbb{R}^3$. This is known as the Mercedes-Benz frame due, not surprisingly, to its resemblance

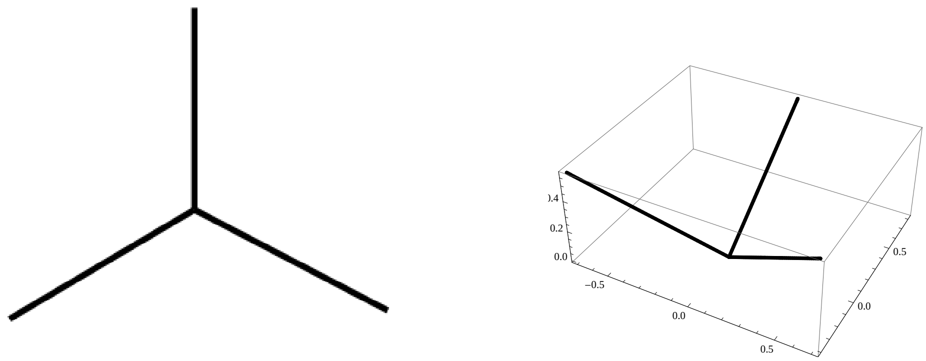

Figure 2.1: (Left) The Mercedes-Benz frame spans two-dimensional space with three vectors. (Right) Parseval frames have an extension to an orthonormal basis. The Mercedes-Benz frame has an extension to an orthonormal basis for $\mathbb{R}^3$.

to the company's logo. To see that this is a FUNTF, note that all three vectors are of length one. To test whether the definition of a FUNTF holds for this set of vectors, we do the following calculation on a general vector, $(x, y) \in \mathbb{R}^2$

$$\begin{aligned}
\sum_{i=1}^{k} |\langle (x,y), v_i \rangle|^2 &= y^2 + \left(-\frac{x\sqrt{3}}{2} - \frac{y}{2}\right)^2 + \left(\frac{x\sqrt{3}}{2} - \frac{y}{2}\right)^2 \\
&= y^2 + \frac{3x^2}{4} + \frac{xy\sqrt{3}}{2} + \frac{y^2}{4} + \frac{3x^2}{4} - \frac{xy\sqrt{3}}{2} + \frac{y^2}{4} \\
&= \frac{3x^2}{2} + \frac{3y^2}{2} \\
&= \frac{3}{2}\|x\|^2
\end{aligned}$$

In this case we see that $A = B = 3/2$. To keep all of these types of frames straight, see Figure 2.2, which includes example illustrations.

Until recently, frames have primarily been studied in fixed Hilbert spaces. However, in some contexts it is useful to imagine multiple vector spaces, each spanned by their own frame. One extension of this idea is to think of a smooth manifold with a tangent space at each point spanned by some (hopefully) continuously moving frame. Often mathematicians will use a basis at each point because they can rep-

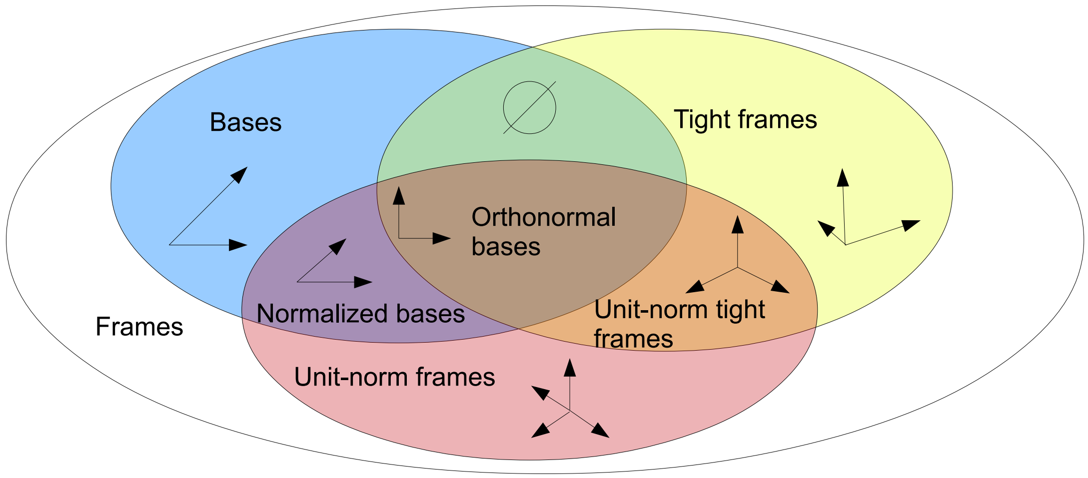

Figure 2.2: There are a variety of properties that frames can have. FUNTFs are scaled versions of Parseval frames, so they are more general versions of orthonormal bases that require that all vectors have the same length and that the frame coefficients, A and B, are equal.

resent each vector at that point uniquely. Frames on manifolds have recently been used to describe the symmetries of quasicrystals [4]. Another more recent application has been in solid state physics. Certain vector bundles on the torus which have no continuously moving basis were shown to have a continuously moving frame of exponentially-decaying Wannier functions [11].

Typically when there is no basis moving continuously over a manifold, someone who wants to do global analysis on the manifold will chop it up into coordinate patches, do their analysis on each of those patches, then glue the patches back together. Although methods for putting these patches together, such as partitions of unity, have been studied extensively, we seek to find globally continuous coordinate systems with the hope that their use could make some analyses easier or more elegant.

Some of the questions motivating this research are:

- Which smooth manifolds have FUNTFs in their tangent space? Are there certain characteristics that imply the existence of a FUNTF?

- On manifolds which do have FUNTFs, how many vectors are needed to create a FUNTF?

- The tangent space is a special case of a vector bundle. Could we say something more general about FUNTFs on vector bundles?

To address these questions we are primarily looking at certain classes of manifolds, particularly higher-dimensional spheres, the Möbius strip, and more general vector bundles on the circle.

Chapter 3

Background and Preliminaries

There are a number of already known results that are useful to keep in mind when studying frames. Although these results are known (see [9] for a reference), we include original proofs here. The following theorem gives a particularly useful characterization of frames.

Theorem 3.0.1. *A finite sequence of vectors in $\mathbb{R}^n$ is an $\mathbb{R}^n$ -frame if and only if it is a spanning set for $\mathbb{R}^n$.*

Proof. If $\{v_1, v_2, \ldots, v_k\}$ is a frame for $\mathbb{R}^n$ then by the definition of a frame, for all $v \in \mathbb{R}^n$, $A\|v\|^2 \leq \sum_{i=1}^{k} |\langle v, v_i\rangle|^2 \leq B\|v\|^2$ for some positive A, B. Assume that $\{v_1, v_2, \ldots, v_k\}$ were not a spanning set for $\mathbb{R}^n$. Then there would exist $v \in \mathbb{R}^n$ such that $\langle v, v_i\rangle = 0$ for all i. That is, this vector is orthogonal to all vectors in the frame. Then $\sum_i |\langle v, v_i\rangle|^2 = \sum_i 0 = 0$, so $A = B = 0$. Since A and B are not positive, the definition of a frame is violated, and we get a contradiction. Therefore, all frames are spanning sets.

If $\{v_1, v_2, \ldots, v_k\}$ is a spanning set for $\mathbb{R}^n$, we will show that it is also a frame. Assume that for all $B \in \mathbb{R}^+$ there exists $v \in \mathbb{R}^n$ such that
$\sum_{i=1}^{k} |\langle v, v_i\rangle|^2 > B\|v\|^2$.
By the Cauchy-Schwarz inequality, the largest that $|\langle v, v_i\rangle|^2$ could be is $\|v\|\|v_i\|$, so

$$\|v\|^2 \sum_{i=1}^{k} \|v_i\|^2 > B\|v\|^2$$
$$\sum_{i=1}^{k} \|v_i\|^2 > B$$

which is not possible for all B since $\{v_1, v_2, \ldots v_k\}$ is a finite set of vectors. Therefore, it must be the case that there exists B such that $\sum_{i=1}^{k} |\langle v, v_i\rangle|^2 \leq B\|v\|^2$. Now I must show that the other side of the inequality that defines a frame also holds. Assume that for all $A \in \mathbb{R}^+$ there exists v such that

$$\sum_{i=1}^{k} |\langle v, v_i\rangle|^2 < A\|v\|^2$$
$$\sum_{i=1}^{k} \frac{|\langle v, v_i\rangle|^2}{\|v\|^2} < A$$
$$\sum_{i=1}^{k} \left|\left\langle \frac{v}{\|v\|}, v_i\right\rangle\right|^2 < A$$

That is, the inner products of unit vectors with the spanning set vectors get arbitrarily small. However, this implies that the unit vectors in $\mathbb{R}^n$ get arbitrarily close to being orthogonal to all vectors in the spanning set. Then, for each $n \in \mathbb{N}$ there must exist some unit vector, x_n, such that $\sum_{i=1}^{k} |\langle x_n, v_i\rangle|^2 < \frac{1}{n}$. Because S^{n-1}, the set of unit vectors in $\mathbb{R}^n$, is compact, there must exist some convergent subsequence which converges to some particular unit vector. Call the vector that this sequence converges to x. However, we see that x must be orthogonal to all the spanning set vectors, which contradicts the definition of a spanning set. Thus, there must exist $A > 0$ such that $\sum_{i=1}^{k} |\langle v, v_i\rangle|^2 \geq A\|v\|^2$ for all $v \in \mathbb{R}^n$. Both sides of the inequality defining a frame are satisfied, so all spanning sets must be frames. □

To get a better idea of how to visualize Parseval frames, we can view them as projections of orthonormal bases.

Theorem 3.0.2. *A set of vectors $\{v_1, v_2, \ldots, v_k\}$ is a Parseval frame for $\mathbb{R}^n$ if and only if it has a dilation to an orthonormal basis of $\mathbb{R}^k$.*

Proof. Let $\left\{ \begin{bmatrix} v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ w_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_k \\ w_k \end{bmatrix} \right\}$ be an orthonormal basis for $\mathbb{R}^k$ such that for each $i \in \{1, \ldots, k\}$ $v_i \in \mathbb{R}^n$ and $w_i \in \mathbb{R}^{k-n}$. Say that $x \in \mathbb{R}^n$. Let o be a vector of $k-n$ zeros.

$$\begin{aligned}
\| x \|^2 &= \left\| \begin{pmatrix} x \\ o \end{pmatrix} \right\|^2 && \text{since } o \text{ is all zeros} \\
&= \sum_{i=1}^{k} \left| \left\langle \begin{pmatrix} x \\ o \end{pmatrix}, \begin{pmatrix} v_i \\ w_i \end{pmatrix} \right\rangle \right|^2 && \text{since } \begin{pmatrix} v_i \\ w_i \end{pmatrix} \text{ are orthonormal basis elements} \\
&= \sum_{i=1}^{k} |\langle x, v_i \rangle|^2 && \text{since } o \text{ is all zeros}
\end{aligned}$$

Therefore, $\{v_1, \ldots, v_k\}$ satisfies Equation 2.3, so any orthonormal basis projected into a lower dimension via having its last $k-n$ components deleted gives a Parseval frame.

Now we will show that Parseval frames have an extension to an orthonormal basis. If $\{v_1, v_2, \ldots, v_k\}$ is a Parseval frame for $\mathbb{R}^n$ then it must satisfy

$$\begin{aligned}
\|x\|^2 &= \sum_{i=1}^{k} |\langle x, v_i \rangle|^2 \\
x^T x &= \sum_{i=1}^{k} |x^T v_i|^2 \\
x^T x &= \sum_{i=1}^{k} x^T v_i x^T v_i \\
x^T x &= \sum_{i=1}^{k} x^T v_i v_i^T x && \text{since inner products are commutative} \\
x^T x &= x^T \sum_{i=1}^{k} v_i v_i^T x \\
x^T x &= x^T \sum_{i=1}^{k} (v_i v_i^T) x
\end{aligned} \tag{3.1}$$

I now claim that the matrix $\sum_{i=1}^{k}(v_i v_i^T)$ is self-adjoint, meaning that for all $x, y \in \mathbb{R}^n$,

$\langle \sum_{i=1}^{k}(v_i v_i^T)x, y\rangle = \langle x, \sum_{i=1}^{k}(v_i v_i^T)y\rangle$. Let $x, y \in \mathbb{R}^n$,

$$\begin{aligned} \langle \sum_{i=1}^{k}(v_i v_i^T)x, y\rangle &= \langle \sum_{i=1}^{k} v_i \langle v_i, x\rangle, y\rangle \\ &= \sum_{i=1}^{k}\langle v_i, y\rangle\langle v_i, x\rangle \\ &= \sum_{i=1}^{k}\langle x, v_i\rangle\langle v_i, y\rangle \\ &= \langle x, \sum_{i=1}^{k}(v_i v_i^T)y\rangle. \end{aligned}$$

Now I will show that if there is some matrix, A, such that $\langle x, x\rangle = \langle x, Ax\rangle$ and $\langle Ax, y\rangle = \langle x, Ay\rangle$, that matrix must be the identity matrix. Because $\langle x, x\rangle = \|x\|^2$ we know it takes only positive values. Since $\langle x, Ax\rangle = \langle x, x\rangle$ we can say that it is then a positive definite matrix. Because A is self-adjoint, we know that it has some eigenvectors that are all orthonormal to each other. In particular, this means that for any eigenvector, x_i, we can say that $\langle x_i, x_i\rangle = \langle x_i, Ax_i\rangle = \lambda_i\langle x_i, x_i\rangle$. Thus, all eigenvalues are one.

Say that $x \in \mathbb{R}^n$. Then it can be expressed in the form $x = \sum_{i=1}^{n}\langle x, x_i\rangle x_i$, so this means that

$$\begin{aligned} Ax &= A\sum_{i=1}^{n}\langle x, x_i\rangle x_i \\ &= \sum_{i=1}^{n}\langle x, x_i\rangle Ax_i \\ &= \sum_{i=1}^{n}\langle x, x_i\rangle x_i \\ &= x \end{aligned}$$

so we see that $A = \sum_{i=1}^{k} v_i v_i^T$ is the identity matrix, which means that the rows of the matrix $[v_1 v_2 \ldots v_k]$ are orthonormal. Now we can create $k - n$ more orthonormal rows. Because the rows of the new $k \times k$ matrix are orthonormal, so are the columns. Thus, the columns of this matrix form an orthonormal basis for $\mathbb{R}^k$, so there is an extension of the Parseval frame to an orthonormal basis. $\square$

One way to understand why a Parseval frame is a natural generalization of an orthonormal basis is stated in the following theorem.

Theorem 3.0.3. *The set of vectors* $\{v_1, v_2, \ldots, v_k\}$ *is a Parseval frame if and only if it satisfies*

$$x = \sum_{i=1}^{k} \langle x, v_i \rangle v_i$$

for all $x \in span\{v_1, v_2, \ldots, v_k\}$.

This equation is known as Parseval's identity, and is what the name "Parseval frames" refers to. Since all orthonormal bases are Parseval frames, they all must satisfy this formula.

Proof. If $\{v_1, v_2, \ldots, v_k\}$ is a Parseval frame for $\mathbb{R}^n$ then it has an extension to an orthonormal basis for $\mathbb{R}^k$. Call the extension $\left\{ \begin{bmatrix} v_1 \\ w_1 \end{bmatrix}, \begin{bmatrix} v_2 \\ w_2 \end{bmatrix}, \ldots, \begin{bmatrix} v_k \\ w_k \end{bmatrix} \right\}$. Like all orthonormal bases, this satisfies the formula $u = \sum_{i=1}^{k} \left\langle u, \begin{pmatrix} v_i \\ w_i \end{pmatrix} \right\rangle \begin{pmatrix} v_i \\ w_i \end{pmatrix}$ for all $u \in \mathbb{R}^k$. Let x be any vector in $\mathbb{R}^n$ and let o be a vector of $k - n$ zeros, then

$$\begin{aligned}
\begin{pmatrix} x \\ o \end{pmatrix} &= \sum_{i=1}^{k} \left\langle \begin{pmatrix} x \\ o \end{pmatrix}, \begin{pmatrix} v_i \\ w_i \end{pmatrix} \right\rangle \begin{pmatrix} v_i \\ w_i \end{pmatrix} \\
\begin{pmatrix} x \\ o \end{pmatrix} &= \sum_{i=1}^{k} \langle x, v_i \rangle \begin{pmatrix} v_i \\ w_i \end{pmatrix} \\
x &= \sum_{i=1}^{k} \langle x, v_i \rangle v_i \quad \text{looking at first } n \text{ components}
\end{aligned}$$

Now lets look at the other direction of the proof. If $\{v_1, v_2, \ldots, v_k\}$ satisfies $x = \sum_{i=1}^{k} \langle x, v_i \rangle v_i$ for all $x \in span\{v_1, v_2, \ldots, v_k\}$, then we can multiply both sides of this equation by x^T on the left:

$$\begin{aligned} x^T x &= x^T \sum_{i=1}^{k} \langle x, v_i \rangle v_i \\ \|x\|^2 &= \sum_{i=1}^{k} x^T v_i \langle x, v_i \rangle \\ \|x\|^2 &= \sum_{i=1}^{k} \|\langle x, v_i \rangle\|^2 \end{aligned}$$

Therefore, if $\{v_1, v_2, \ldots, v_k\}$ satisfies $x = \sum_{i=1}^{k} \langle x, v_i \rangle v_i$ then it is a Parseval frame. □

One useful tool that we will use later to construct FUNTFs is using the union of two FUNTFs with fewer vectors to create a new FUNTF with more vectors.

Proposition 3.0.4. *If F and G are FUNTFs spanning $\mathbb{R}^n$, then $F \cup G$ is a FUNTF spanning $\mathbb{R}^n$.*

Proof. Let $F = \{f_i\}_{i=1}^{j}$ and $G = \{g_i\}_{i=1}^{k}$ be two FUNTFs spanning $\mathbb{R}^n$. It follows that for any $x \in \mathbb{R}^n$, $A_f \|x\|^2 = \sum_{i=1}^{j} |\langle x, f_i \rangle|^2$ and $A_g \|x\|^2 = \sum_{i=1}^{k} |\langle x, g_i \rangle|^2$. Thus we can add these two equations to see that

$$(A_f + A_g)\|x\|^2 = \textstyle\sum_{i=1}^{j} |\langle x, f_i \rangle|^2 + \sum_{i=1}^{k} |\langle x, g_i \rangle|^2$$

so $F \cup G$ is also a FUNTF for $\mathbb{R}^n$. □

Chapter 4

FUNTFs on Vector Bundles on the Circle

4.1 FUNTFs on the Möbius Strip

The Möbius strip is an example of a manifold which lacks a continuously moving orthonormal basis for its tangent space. To create a Möbius strip, take a strip of paper, fold it over, and attach its two ends. By tracing along the strip you will find that this only has one face, and is a non-orientable manifold. This means that if you look at a normal vector at a point and draw any continuous closed loop along the strip, when you get back to the same point, the normal vector will have flipped directions. If one were to attempt to draw an orthonormal basis over the Möbius strip, it would become apparent that it is not possible because the only way that an orthonormal basis can be continuously changed in two-dimensions, while still meeting the requirements for an orthonormal basis is through rotation. However, the Möbius strip flips the orientation of bases, something that is not possible with rotation alone, as seen in Figure 4.1. It might also be tempting to try making a FUNTF of three vectors, but as shown in Lemma 4.1.2, this is not possible. However, there is a light

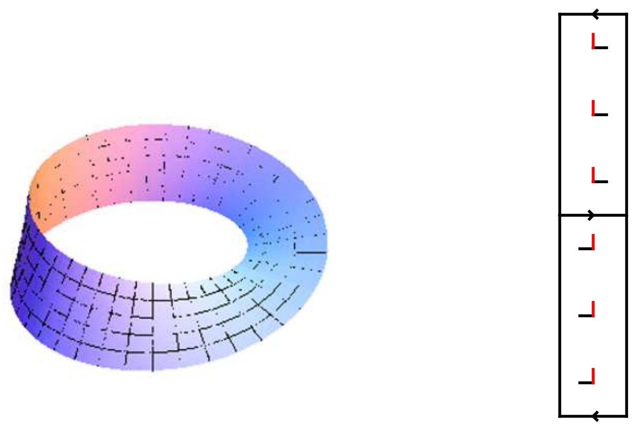

Figure 4.1: (Left) The Möbius strip can be made as a strip of paper which has been twisted and then had its ends glued together. (Right) A basis on the Möbius strip must experience an orientation change to preserve continuity, but this is impossible when we are only able to rotate the basis. Here is a failed attempt at making a basis for the Möbius strip.

at the end of the tunnel.

Theorem 4.1.1. *For any integer $n \geq 4$, there exists a FUNTF of n vectors that moves continuously over the Möbius strip.*

It's already been discussed why no FUNTF of two vectors exists for the Möbius strip. To prove this theorem, we must first prove a couple of lemmas about FUNTFs on the Möbius strip.

Lemma 4.1.2. *There is no FUNTF of three vectors for the Möbius strip.*

Proof. First, we show that up to rotation and reflection there are only two possible FUNTFs for the Möbius strip. Then we show that there is no way to make a continuous FUNTF over the whole Möbius strip using these FUNTFs.

All FUNTFs of three vectors that span $\mathbb{R}^2$ are projections of a scaled orthonormal basis for $\mathbb{R}^3$ onto a plane. FUNTFs must also have all vectors that are the same

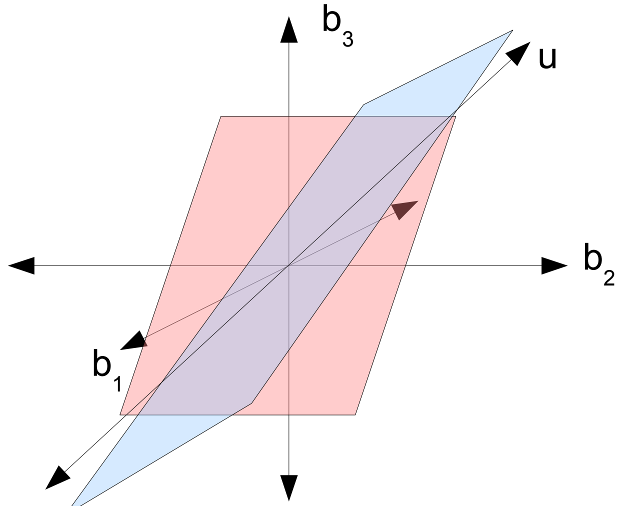

Figure 4.2: The pink plane defines the set of all vectors such that $\langle b_1, u\rangle = -\langle b_3, u\rangle$ and the blue plane defines the set of all vectors such that $\langle b_2, u\rangle = \langle b_3, u\rangle$. The intersection of these two planes defines a particular u that would be normal to a plane onto which b_1, b_2 and b_3 can be projected to create a FUNTF for $\mathbb{R}^2$.

length, so the projections of the basis vectors onto a plane that creates a FUNTF must be vectors that are the same length. Say that u is the unit vector normal to the plane that the basis is being projected onto. Then we know that for each orthonormal basis vector, b_i, the length of the projection, v_i, must be equal

$$\|v_1\|^2 = \|v_2\|^2 = \|v_3\|^2$$
$$\|b_1\|^2 - (\langle b_1, u\rangle)^2 = \|b_2\|^2 - (\langle b_2, u\rangle)^2 = \|b_3\|^2 - (\langle b_3, u\rangle)^2$$
$$(\langle b_1, u\rangle)^2 = (\langle b_2, u\rangle)^2 = (\langle b_3, u\rangle)^2$$
$$|\langle b_1, u\rangle| = |\langle b_2, u\rangle| = |\langle b_3, u\rangle|$$

For each pair of b_i, b_j, where $i, j \in \{1, 2, 3\}$, we can define a plane such that all vectors in the plane satisfy $\langle b_i, u\rangle = \langle b_j, u\rangle$ and a plane such that $langle b_i, u\rangle = -\langle b_j, u\rangle$. Then we can take any two pairs of b_i, b_j and choose either plane for each pair, then find the intersection of these two planes. The line created by the intersection of these two planes is a u that satisfies the equation above, as shown in Figure 4.1.

By going through these combinations, we find that there are only 8 possible u that satisfy the equation above. Two of these u gives a plane that projects the orthonormal basis to the Mercedes-Benz frame, and the other six project the orthonormal basis into a FUNTF that looks like the Mercedes-Benz frame except that one of the vectors in the frame has been multiplied by negative one. Thus, there are only two FUNTFs of three vectors spanning $\mathbb{R}^2$.

Up to rotation, there are only two FUNTFs of three vectors spanning $\mathbb{R}^2$, meaning that there is no continuous path of FUNTFs between these two, so any FUNTF on the Möbius strip must start and end with the same FUNTF. The only way these FUNTFs can change continuously over the surface is rotation. However, it is not possible to get the orientation flip that is required for global continuity on the Möbius strip through rotation alone, so there is no continuous FUNTF on the Möbius strip of three vectors.

□

Lemma 4.1.3. *Given two FUNTFs such that one is a pair of orthogonal vectors in $\mathbb{R}^2$ and the other is either another pair of orthogonal vectors or a Mercedes-Benz frame in $\mathbb{R}^2$, it is possible to use one FUNTF to flip the orientation of the other FUNTF.*

Proof. This can be proven visually. On the Möbius strip shown in Figure 4.3, there is a way to continuously move one frame from one picture to the next which ensures that at all times there are two FUNTFs. Note that between images the vectors associated with each FUNTF change. In Figure 4.3 is an example of two pairs of orthogonal vectors such that one flips the orientation of the other and then vice-versa. This figure also shows how the inner product of a vector that points straight up along the Möbius strip has a continuous inner product in each of the four frame components. The orthogonal basis and Mercedes-Benz frame work basically the same way. One frame flips the orientation of the other frame, then the second frame flips the orientation of the first frame.

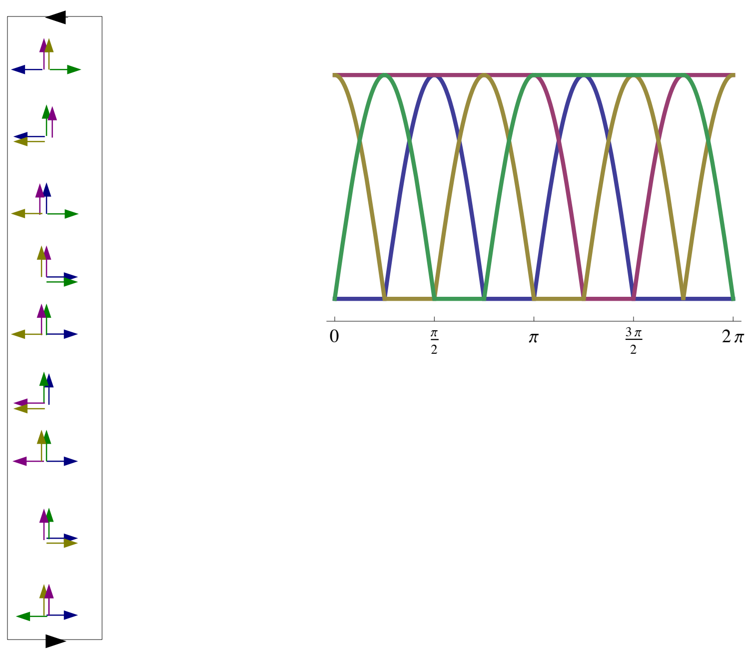

Figure 4.3: (Left) Between each image, imagine that pairs of vectors can move continuously. This shows a FUNTF for the Möbius strip that is continuous. (Right) If a vector pointing straight along the strip at each point were to be reconstructed, its inner product with each of the four FUNTF vectors would be continuous and piecewise as shown above.

□

Proof of Theorem 4.1.1 . Proposition 3.0.4 establishes that if we have FUNTFs with a certain number of vectors, we can combine those frames and scale their vectors to make a new FUNTF with even more vectors. The second lemma establishes the small building blocks of 2+2 vectors and 2+3 vectors. This means that for any k, there exist $a, b \in \mathbb{Z}$ such that $k = 2a + 3b$, and that there is a FUNTF that is continuous over the Möbius strip that is made of of k vectors that is made up of the scaled union of a pairs of orthonormal vectors and b Mercedes-Benz frames. □

4.2 Extension to the Klein Bottle

The Klein bottle is a non-orientable manifold which can be made by gluing the edges of the Möbius strip to itself. As shown in Figure 4.4 if a vector was drawn on the Klein bottle such that it moved continuously over the surface, it would require an orientation flip from the top of the diagram to the bottom, just like the Möbius strip. Left to right it can stay constant and it will move continuously over the surface. Thus, the FUNTF on the Möbius strip which we provided can be copied onto the Klein bottle along these vertical strips, and it creates a FUNTF on the Klein bottle.

4.3 Vector Bundles on the Circle and Properties

On the Möbius strip it is easy to imagine that there is some circle going along the center of the strip, and at each point on the circle there is a straight line perpendicular to the circle that twists as it goes along the circle. If it did not twist, then the manifold created over this circle would be a cylinder. We can then imagine what it might mean to generalize these two manifolds to higher dimensions. We hope to generalize our results on the Möbius strip to higher dimensions to find a new partial result for the

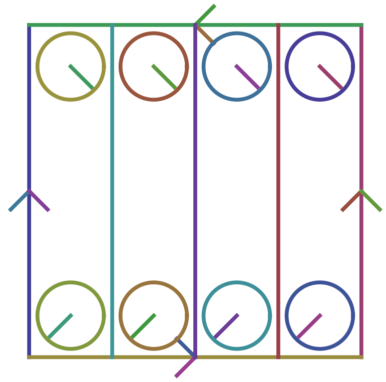

Figure 4.4: The Klein bottle glues the left and right sides of a square together, then the top and bottom once one has flipped its orientation. It intersects itself when constructed in only three dimensions. If a vector were drawn on the Klein bottle such that it moved continuously over the surface, it would require an orientation flip from the top of the diagram to the bottom, just like the Möbius strip.

long-open frame theory problem of identifying when two FUNTFs are path-connected [8]. As seen in Figure 4.5 at each point on the circle, there could be some vector space, $\mathbb{R}^n$, and we would hope that these vector spaces would have some smooth mapping between them. This idea can be formalized to the idea of a *vector bundle* with an underlying manifold that is the circle.

Definition 4.3.1. *Let E and M be manifolds and let $n \in \mathbb{N}$. Following the definitions from [10], a rank-n vector bundle is a map $\pi : E \to M$ together with a real vector space structure on $\pi^{-1}(m)$ for each $m \in M$, such that the following local triviality condition is satisfied: There is a cover of M by open sets U_α for each of which there exists a homeomorphism $h_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ taking $\pi^{-1}(m)$ to $\{m\} \times \mathbb{R}^n$ by a vector space isomorphism for each $m \in U_\alpha$. Such an h_α is called a local trivialization of the vector bundle. The space M is called the base space, E is the total space, and the vector spaces $\pi^{-1}(m)$ are the fibers.*

Let $\pi_1 : E_1 \to M_1$ and $\pi_2 : E_2 \to M_2$ be two vector bundles. Note that $\pi_i^{-1}(p_i), p_i \in M_i$ is a vector space for $i = 1, 2$. These two vector bundles are equivalent if there exists $\psi : E_1 \to E_2$ such that ψ is a smooth bijection with smooth inverse,

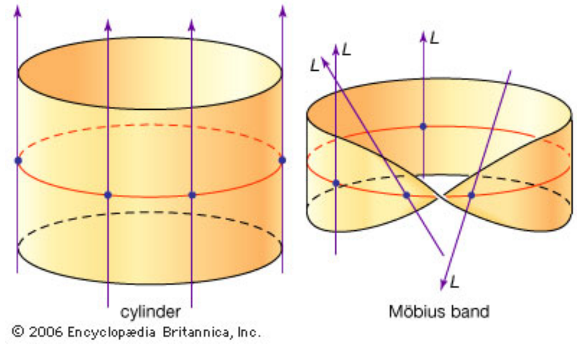

Figure 4.5: The two cases of vector bundles on the circle are the generalizations of the cylinder and the Möbius strip to higher dimensions (from Encyclopaedia Britannica, http://www.britannica.com/EBchecked/media/2221/Vector-bundles-As-the-circle-is-followed-clockwise-around-the).

and $\psi(av_1 + v_2) = a\psi(v_1) + \psi(v_2)$ for all $v_1, v_2 \in \pi_1^{-1}(\{p \in M_1\})$. Imagine that we have a circle and at each point we have a line (a one dimensional vector space) that is perpendicular to the circle so that once a line is associated with each point it looks like a cylinder. We will call this the *trivial bundle*, V_s. Now imagine that we have a nontrivial bundle because the lines twist and rotate 180 degrees over the circle. This will be called V_f, and is the same as the Möbius strip.

One way to distinguish between the trivial bundle and the Möbius strip is by looking at a *section* on each bundle. A section is a continuous function, $f : M \to E$, such that for each $m \in M$, $f(m) \in \pi^{-1}(m)$. That is, each point in the base space is sent to a point in the vector space at that base space. If we do this continuously over each of the manifolds, we find that it is possible to have a section on the trivial bundle that does not have the origin of any of those lines anywhere (that is, the section never has to cross the circle). Yet on the Möbius strip the flip can be thought of as forcing a function to go from positive to negative. Thus, there must be some point on any section on the Möbius strip where the section intersects the circle.

Theorem 4.3.2. *All rank-k vector bundles over the circle are equivalent to* $V_f \oplus \left(\oplus_{i=1}^{k-1} V_s\right)$ *or* $\oplus_{i=1}^{k} V_s$.

Proof. This can be proven through the use of induction, so first we will start with our base case. For $k = 1$ there are two vector bundles, V_s and V_f, so the theorem proposed is satisfied in this case. V_s is the rank-1 bundle on the circle that has a nowhere-zero section, while V_f is the line bundle such that no section is nowhere-zero.

Inductive Hypothesis: Let $k \geq 1$. Assume that all rank-k vector bundles over the circle are equivalent to $V_f \oplus \left(\oplus_{i=1}^{k-1} V_s\right)$ or $\oplus_{i=1}^{k} V_s$.

Inductive Step: Let $\pi : E \to S^1$ be a rank-$(k+1)$ vector bundle over the circle. This vector bundle must have some nowhere-zero section. The only vector bundle where this fails is on a rank one vector bundle (i.e. the Möbius strip). Call this nowhere-zero section V. We can then represent E as the direct sum of V and U, where U is a rank-k vector bundle over the circle. By the inductive hypothesis, U is either of the form $V_f \oplus \left(\oplus_{i=1}^{k-1} V_s\right)$ or $\oplus_{i=1}^{k} V_s$. Let $\psi_V : V \to U$ be the smooth, invertible bijection such that for all $v_1, v_2 \in \pi^{-1}(\{p \in S^1\})$, $\psi_V(av_1 + v_2) = a\psi_V(v_1) + \psi_V(v_2)$. Let $P_V : E \to V$ be a projection from E onto V and $P_U : E \to U$ be the projection from E onto U. These are calculated using the vectors spanning each space, $\{b_i\}$ and the equation $P(v) = \sum_i \langle b_i, v \rangle b_i$. Define $\psi : E \to V \oplus U$ such that for all $v \in E$, $\psi(v) = (\psi_V(P_V(v)), P_U(v))$. Because ψ_V, P_V, and P_U are all smooth, invertible bijections, so is ψ. It is important to ensure that linearity is preserved. Let $v_1, v_2 \in E$. Then

$$\begin{aligned}
\psi(av_1 + v_2) &= (\psi_V(P_V(av_1 + v_2)), P_U(av_1 + v_2)) \\
&= (a\psi_V(P_V(v_1)) + \psi_V(P_V(v_2)), aP_U(v_1) + P_U(v_2)) \\
&= a(\psi_V(P_V(v_1)), P_U(v_1)) + (\psi_V(P_V(v_2)), P_U(v_2)) \\
&= a\psi(v_1) + \psi(v_2)
\end{aligned}$$

Because such a ψ exists between E and $V \oplus U$ for an arbitrary rank-$k+1$ vector bundle, induction shows that the theorem is true. □

Because there are only two types of vector bundles on the circle, we see that one is the generalization of the cylinder to higher dimensions and the other type is the generalization of the Möbius strip to higher dimensions. The cylinder's generalization has a continuously moving basis that is easy to create, but as seen in the Möbius strip example, its generalization, $V_f \oplus \left(\oplus_{i=1}^{k-1} V_s\right)$, requires a FUNTF for a continuous spanning set since vectors on this type of vector bundle undergo an orientation change.

Chapter 5

FUNTFs on n-Spheres

Classifying the number of independent vector fields on spheres was a major area of study in differential topology. Thanks to this work we now know that all of the odd-dimensional spheres are the only ones which have nowhere-zero tangent vector fields and the only spheres which have a continuous basis are S^1, S^3, and S^7 [1]. The n-sphere, S^n, is the set of all $x = (x_1, x_2, \ldots, x_{n+1})$ such that $\|x\| = 1$. At any x on the n-sphere is an n-dimensional tangent space. This tangent space is the set of all vectors v such that $\langle v, x\rangle = 0$. Because we know so much about tangent vector fields on spheres and because their symmetry makes it easy to talk about any general x on the sphere, we decided to investigate FUNTFs on n-spheres.

We will show that there is a natural construction of FUNTFs for n-spheres, and that the number of vectors needed for a FUNTF can be reduced. We know that the existence of a FUNTF requires the existence of a nowhere zero vector field, but we do not know for all manifolds whether the existence of a nowhere zero vector field in the tangent space implies the existence of a FUNTF. On spheres, we know that if and only if n is odd, the n-sphere has a nowhere zero vector field in its tangent space. If a single vector cannot move continuously in the tangent space, there is no way that an entire frame could. Thus, even dimensional n-spheres can not have a FUNTF. The proof below shows that all odd n-spheres have a FUNTF. Thus, an n-sphere has a FUNTF if and only if n is odd.

5.1 Construction of a FUNTF for odd n-spheres

Say that you are at a point on the n-sphere, $(x_1, \ldots, x_{n+1})$, and you want to find a vector in the tangent space (i.e., one that is perpendicular to this vector). One thing you could do is to swap pairs of components and give one of the elements of each pair a negative sign. For example, on the 3-sphere, you can see that

$(x_1, x_2, x_3, x_4) \cdot (-x_2, x_1, x_4, -x_3) = -x_1x_2 + x_1x_2 + x_3x_4 - x_3x_4 = 0.$

We will show that by taking the set of all such possible swaps and negative signs, this creates a FUNTF. One way to approach this proof is to imagine a set of operators acting on $(x_1, \ldots, x_{n+1})$ to give a set of these vectors. The operator that gives the example above is:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ x_4 \\ -x_3 \end{pmatrix}$$

The set of all possible tangent vectors that can be created by pairwise flipping coordinates and adding negative signs is:

$$(-x_2, x_1, -x_3, x_4), (-x_2, x_1, x_3, -x_4), (x_2, -x_1, -x_3, x_4), (x_2, -x_1, x_3, -x_4),$$
$$(-x_3, -x_4, x_1, x_2), (-x_3, x_4, x_1, -x_2), (x_3, -x_4, -x_1, x_2), (x_3, x_4, -x_1, -x_2),$$
$$(-x_4, -x_3, x_2, x_1), (-x_4, x_3, -x_2, x_1), (x_4, -x_3, x_2, -x_1), (x_4, x_3, -x_2, -x_1)$$

Let A be a collection made up of operators $T((\epsilon_i)_{i=1}^{n+1}, (n_i)_{i=1}^{n+1}) \in A$, where each $\epsilon_i = \pm 1$ denotes whether the element that T moves to the i^{th} component of the vector will be negative or positive and n_i denotes that x_{n_i} will be put in the i^{th} spot. This means that $T((\epsilon_i)_{i=1}^{n+1}, (n_i)_{i=1}^{n+1})(x_1, \ldots, x_{n+1})$ will have $\epsilon_i x_{n_i}$ in the i^{th} spot for each $i = 1, \ldots, n+1$. Also, the i^{th} row of T will be all zeros except for an ϵ_i in the n_i^{th} column.

Definition 5.1.1. *Let A be a collection of operators of the form, $T((\epsilon_i)_{i=1}^{n+1}, (n_i)_{i=1}^{n+1})$. A is balanced if it satisfies the following two properties:*

- *(1) For all $i, j \in \{1, \ldots, n+1\}$ such that $i \neq j$ the number of operators, $T((\epsilon_l)_{l=1}^{n+1}, (n_l)_{l=1}^{n+1}) \in A$ such that $\epsilon_i \epsilon_j = -1$ equals the number of operators such that $\epsilon_i \epsilon_j = 1$.*

- *(2) For all $i, j, k = 1, \ldots, n+1$ the number of operators, $T((\epsilon_l)_{l=1}^{n+1}, (n_l)_{l=1}^{n+1}) \in A$ such that $n_i = k$ equals the number of operators such that $n_j = k$.*

Let A_T be the set containing all $T((\epsilon_i)_{i=1}^{n+1}, (n_i)_{i=1}^{n+1})$ such that for each $i \in \{1, \ldots, n+1\}$, $n_{n_i} = i$ and either $\epsilon_i = +1$ and $\epsilon_{n_i} = -1$ or vice-versa. In other words, A_T is the set of all $T \in \{-1, 0, +1\}^{(n+1)\times(n+1)}$ such that $T = -T^T$, and each row and column of T has one nonzero entry. Therefore, A_T is the set of all T such that when T acts on a vector, it swaps the components in pairs and multiplies one component in each pair by -1. Let A be the set containing $2\frac{(n-1)!}{((n-1)/2)!}$ copies of the identity and the subset A_T.

There are $\frac{(n+1)!}{((n+1)/2)!2^{(n+1)/2}}$ ways to swap $n+1$ components pairwise, and $2^{(n+1)/2}$ ways to assign one negative sign to each pair. Therefore there are $\frac{(n+1)!}{((n+1)/2)!}$ different $T \in A_T$.

Let $x \in \mathbb{R}^{n+1}$ such that $\| x \|_2 = 1$. That is, $x \in S^n$. We will prove that A is constructed such that for $i = 1, \ldots, n+1$, $e_i = \alpha \left(\sum_{T \in A_T} \langle e_i, Tx \rangle Tx + 2\frac{(n-1)!}{((n-1)/2)!} \langle e_i, Ix \rangle Ix \right)$, where α is some scalar that will control the ammount that vectors in the frames are scaled by. Since all vectors in $\mathbb{R}^{n+1}$ are linear combinations of $\{e_i\}_{i=1}^{n+1}$, this will imply that Ax is a FUNTF for $\mathbb{R}^{n+1}$. All vectors in the tangent space of S^n at x have zero component normal to the sphere and all Tx are tangent to the sphere when $T \neq I$, so this will lead to a natural choice of a FUNTF for TS^n.

Lemma 5.1.2. *A_T is balanced.*

Proof:

To prove this lemma we must first show that A_T satisfies (1). This is a consequence of the fact that for an arbitrary j if you have $T((\epsilon_i)_{i=1}^{n+1}, (n_i)_{i=1}^{n+1}) \in A_T$ where $\epsilon_j = \pm 1$ and $\epsilon_{n_j} = \mp 1$, then there must also be another $T((\epsilon_i)_{i=1}^{n+1}, (n_i)_{i=1}^{n+1}) \in A_T$ with an identical set of $(n_i)_{i=1}^{n+1}$ and all ϵ_i the same except that $\epsilon_j = \mp 1$ and $\epsilon_j = \pm 1$. No other matrix in A_T would be related to this second matrix in that same way, so we see that the set of matrices can be paired up in this way (giving a 1-1 correspondence for any given i, j pair).

Now we will show that A_T satisfies (2). That is, all pairs of components are paired the same number of times. If we wanted to know how many T caused any pair of components to be swapped, we would see that this set has the same cardinality as the similarly constructed set for $n-1$, except that for each T in that set there are two ways for negative signs to be assigned to the pair being looked at. Therefore each component is swapped with each other component $\frac{(n-1)!}{((n-1)/2)!2^{(n-1)/2}} * 2^{(n+1)/2} = 2\frac{(n-1)!}{((n-1)/2)!}$ times. □

Theorem 5.1.3. *Let A_T be a balanced set of operators. Then $\{Tv : v \in S^n\}_{T \in A}$ is a continuously moving FUNTF for the tangent space of S^n.*

Using the fact that A_T is balanced, we will show that Ax creates a scaled Parseval frame for $\mathbb{R}^{n+1}$. To do this we will show that each of the basis vectors $(1, 0, 0, \ldots)$,

$(0, 1, 0, \ldots)$, etc... for $\mathbb{R}^{n+1}$ can be created using the reconstruction formula with the vectors in Ax. Since each vector in $\mathbb{R}^{n+1}$ is a linear combination of these vectors, the reconstruction formula will work for any vector in the span of Ax. Then we can remove the copies of the normal vector so that we're left with a set of vectors that lie in the tangent space of the sphere at x so we have $A_T x$. Any vector in the tangent space spanned by these vectors would be orthogonal to the normal vectors, so the reconstruction formula for $A_T x$ would work out the same for these vectors as the reconstruction formula for Ax. Therefore, all vectors in the tangent space can be reconstructed by $A_T x$ if all the orthonormal basis vectors for $\mathbb{R}^{n+1}$ can be reconstructed by Ax. We start with the way that all T (including the copies of the identity) act on x through the frame operator, then prove that this actually is a non-zero scalar multiple of x.

$$\sum_{T \in A} \langle e_i, Tx \rangle Tx = \sum_{T \in A} \epsilon_i x_{n_i} Tx$$

If you look at a specific entry of this reconstruction, say the j^{th} one:

$$\left[\sum_{T \in A} \langle e_i, Tx \rangle Tx \right]_j = \sum_{T \in A} \epsilon_i x_{n_i} \epsilon_j x_{n_j}$$

To reconstruct every e_i, this must equal to 0 when $j \neq i$ and 1 when it does. What this implies:

Case 1 ($j \neq i$):

For all $j = 1, \ldots, m$ but $j \neq i$, there are as many T such that $\epsilon_i \epsilon_j = +1$ as there are T such that $\epsilon_i \epsilon_j = -1$. Therefore these terms ($+x_i x_j$ and $-x_i x_j$) cancel out and we are left with $e_j = 0$.

Case 2 ($j = i$):

Since we want A that reconstructs e_i for $i = 1, \ldots, m$ this implies that for all of these i, $\alpha \sum_{T\in A}(\epsilon_i x_{n_i})^2 = 1$. Since $\epsilon_i = \pm 1$, the sum reduces to $\alpha \sum_{T\in A} x_{n_i}^2 = 1$. Remember that since x is on a sphere, the sum of each component squared is 1. Since any given pair of components is paired the same number of times, each component is paired with itself that same number of times. In our case, we know this is $2\frac{(n-1)!}{(n-1)/2)!}$ times. Therefore, e_i is reconstructed as $2\frac{(n-1)!}{(n-1)/2)!}e_i$, so all the vectors can be scaled by $(2\frac{(n-1)!}{(n-1)/2)!})^{-1}$ to give the exact reconstruction.

5.2 Reducing the Number of Vectors Needed

We have a FUNTF for S^n whenever n is odd, but it would be desirable to use fewer vectors. One thing that could be done is to look at the subsets of A. If there were a subset that paired each component with each other component exactly once, but still used the same combinations of $\pm$ signs as in A this would also yield a FUNTF. In fact it is possible to create such a set for any odd n.

We have already shown that a FUNTF of $\frac{(n+1)!}{2((n+1)/2)!}$ vectors exists for all n-spheres when n is odd (that is, when S^n has a nowhere-zero vector field), but it can also be shown for all odd n that a FUNTF of $n2^{(n-1)/2}$ vectors exists. These are created by pairing each component with every other component exactly once then having all possible combinations of positive and negative signs that still give vectors in unique directions. In fact, there is at least one subset of vectors that can be easily chosen that create a FUNTF.

The way that the components are paired up to make vectors can be displayed as a graph made up of subgraphs such that each subgraph represents a set of components being paired up with each edge representing a pair so that there are $\frac{n+1}{2}$ edges per subgraph. Note that $2^{\frac{n}{2}-1}$ vectors come from each pairing due to plus/minus signs. If every possible pair of components is used in exactly one pairing then the union of the resulting subgraphs is a complete graph with $n+1$ vertices. This is shown in

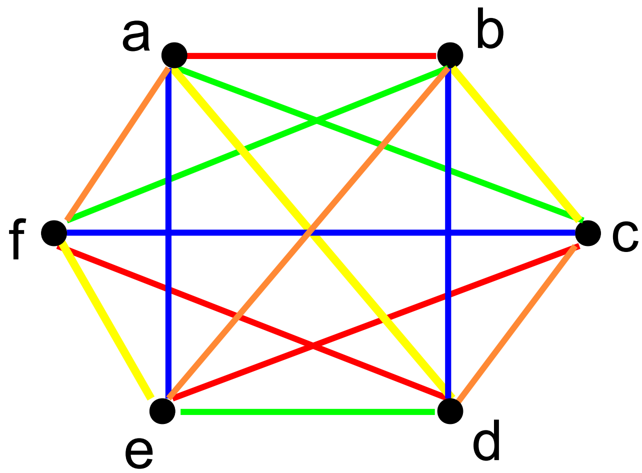

Figure 5.1: Each subgraph of three edges is represented in a different color and represents one way of pairing components where each edge represents two components being swapped. For instance, the red edges would represent all possible combinations of putting one negative sign in each pair of components in (b, a, e, f, c, d) so that it will be orthogonal to (a, b, c, d, e, f). This graph ensures that each component is only paired with each other component once.

Figure 5.1. It is common to represent graphs as matrices where a 1 in the (i, j) entry means that the i^{th} vertex in the graph is connected to the j^{th} vertex, and a 0 in the (i, j) entry means that the i^{th} vertex in the graph is not connected to the j^{th} vertex. Notice that the graphs being used to represent the component-wise pairings are not directed graphs (since a swapping places with b is the same as b swapping places with a) so the matrices representing these subgraphs are symmetric matrices. If we were to write out one subgraph for each pairing in the example from the complete graph in Figure 5.1, we would get the following subgraphs, each representing a pairing where every component is paired with exactly one other component:

	a	b	c	d	e	f
a	0	1	0	0	0	0
b	1	0	0	0	0	0
c	0	0	0	0	1	0
d	0	0	0	0	0	1
e	0	0	1	0	0	0
f	0	0	0	1	0	0

,

a	b	c	d	e	f
0	0	1	0	0	0
0	0	0	0	0	1
1	0	0	0	0	0
0	0	0	0	1	0
0	0	0	1	0	0
0	1	0	0	0	0

,

a	b	c	d	e	f
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	0	0	0
1	0	0	0	0	0
0	0	0	0	0	1
0	0	0	0	1	0

,

a	b	c	d	e	f
0	0	0	0	1	0
0	0	0	1	0	0
0	0	0	0	0	1
0	1	0	0	0	0
1	0	0	0	0	0
0	0	1	0	0	0

,

a	b	c	d	e	f
0	0	0	0	0	1
0	0	0	0	1	0
0	0	0	1	0	0
0	0	1	0	0	0
0	1	0	0	0	0
1	0	0	0	0	0

The union of these subgraphs results in a complete graph (left), which can have each of its subgraphs numbered to make generalization to higher dimensions easier (right):

	a	b	c	d	e	f
a	0	1	1	1	1	1
b	1	0	1	1	1	1
c	1	1	0	1	1	1
d	1	1	1	0	1	1
e	1	1	1	1	0	1
f	1	1	1	1	1	0

$\rightarrow$

	a	b	c	d	e	f
a	0	1	2	3	4	5
b	1	0	3	4	5	2
c	2	3	0	5	1	4
d	3	4	5	0	2	1
e	4	5	1	2	0	3
f	5	2	4	1	3	0

By numbering the subgraphs in their union, we can see a way to reframe this problem yet again to make its solution easier to find. In this context we can pose the following simple question: if n is an odd number, does there exist a symmetric $(n+1)\times(n+1)$ matrix with zero diagonal such that every row and every column contains the numbers 1 to n exactly once? This puzzle, reminiscent of a Sudoku, is simple enough that given some time to play around, a person with no mathematics background could solve it for a given n. By using a construction inspired by Hankel matrices, we can show that for all odd n the answer is yes, such a matrix does exist.

Given an $(n+1)\times(n+1)$ empty matrix, fill in the diagonal with all zeros, then for the 1^{st} through n^{th} antidiagonals contain 1 through n respectively, as shown on the left below for $n=5$. Then fill in the following antidiagonals starting again with 1, except don't fill in the $n+1^{th}$ row and column yet, as shown in the center below. Finally, fill in the remaining number in each row and column, shown on the right below. Each row and column can be described explicitly based on this procedure. The 0^{th} row and column are just the numbers 0 through n in consecutive order. The n^{th} row and column contain $(n, 2*1 \mod n, 2*2 \mod n, \ldots, 2*(n-1) \mod n, 2*n \mod n)$. Since n is odd, 2 times all the numbers from 0 to $n-1$ mod n gives n unique numbers, each of the integers from 0 to $n-1$. If $0<i<n$, the i^{th} row is $(i, i+1 \mod n, \ldots, i+i-1 \mod n, 0, i+i+1 \mod n, \ldots i+n-1 \mod n, i+i \mod n)$.

	a	b	c	d	e	f
a	0	1	2	3	4	5
b	1	0	3	4	5	
c	2	3	0	5		
d	3	4	5	0		
e	4	5			0	
f	5					0

	a	b	c	d	e	f
a	0	1	2	3	4	5
b	1	0	3	4	5	
c	2	3	0	5	1	
d	3	4	5	0	2	
e	4	5	1	2	0	
f	5					0

	a	b	c	d	e	f
a	0	1	2	3	4	5
b	1	0	3	4	5	2
c	2	3	0	5	1	4
d	3	4	5	0	2	1
e	4	5	1	2	0	3
f	5	2	4	1	3	0

Here are some larger examples:

0	1	2	3	4	5	6	7
1	0	3	4	5	6	7	2
2	3	0	5	6	7	1	4
3	4	5	0	7	1	2	6
4	5	6	7	0	2	3	1
5	6	7	1	2	0	4	3
6	7	1	2	3	4	0	5
7	2	4	6	1	3	5	0

0	1	2	3	4	5	6	7	8	9
1	0	3	4	5	6	7	8	9	2
2	3	0	5	6	7	8	9	1	4
3	4	5	0	7	8	9	1	2	6
4	5	6	7	0	9	1	2	3	8
5	6	7	8	9	0	2	3	4	1
6	7	8	9	1	2	0	4	5	3
7	8	9	1	2	3	4	0	6	5
8	9	1	2	3	4	5	6	0	7
9	2	4	6	8	1	3	5	7	0

Chapter 6

Conclusions and Suggestions for Future Research

By focusing on specific types of manifolds, we have been able to investigate each of our primary questions. To respond to the question of which manifolds have FUNTFs in their tangent spaces, we showed that the Möbius strip and odd n-spheres do have FUNTFs. On n-spheres the existence of a nowhere-zero vector field in the tangent space exactly corresponds to the existence of a FUNTF, but this has not been shown in general for all manifolds.

On the Möbius strip we provided a complete solution to the number of vectors needed to create a FUNTF, and on the spheres we got an initial upper bound on the minimum number of vectors needed, then were able to lower that. In the process of lowering this upper bound, we found that connecting the problem to graph theory and representing that graph in a matrix was a useful tool.

In response to generalizing the tangent space to vector bundles, we looked at vector bundles on the circle as a generalization of the Möbius strip and found that we were really only dealing with two cases.

6.1 Potential Applications of FUNTFs on Manifolds

FUNTFs on manifolds may have applications in both pure mathematics and in data analysis for the sciences and engineering. Often mathematicians may wish to analyze some function on a manifold that lacks a globally continuous basis. A number of techniques have been developed to deal with these problems so that we can essentially cut the manifold into patches, each of which have some continuous coordinate system. We then deal with the function on each patch separately, and glue those patches back together. Although these techniques are useful, the use of a single continuous coordinate system as opposed to multiple patches, would allow for alternate analyses of functions on manifolds that may be simpler.

One potential application of FUNTFs on manifolds is in data analysis. Say that someone is conducting an experiment and knows that their parameter space lies on some manifold. They may also know something about the dimension of the outputs that they will measure. Using this information, they can represent all of their data using a FUNTF over that vector bundle, then interpolate that data to look for patterns. Because the FUNTF is geometry-specific, they may be able to see patterns which would not show up if the data were naively represented as just a set of coordinates embedded in some Euclidean space. On smooth manifolds which lack a continuous global basis but do have a FUNTF, this allows for the continuity of that manifold to carry through to the representation of the points on that manifold. Thus, the data analysis will stay true to the experimental setup.

6.2 The Frame Force

Although we have achieved some results for the special cases of vector bundles on the circle and n-spheres, we would like to be able to say something about broader classes of manifolds, so our original questions remain open. Through the use of more advanced differential topology techniques, it may be possible to understand FUNTFs on manifolds in more generality. One technique known as the *frame force* shows some promise for future investigations.

Any manifold with a nowhere-zero tangent vector field has a unit-norm frame in its tangent space, so we are interested in finding a method of turning these unit-norm frames into FUNTFs. Introduced in [2], the frame force is a function of any two vectors that shares some properties of physical forces and is minimized by orthogonal sets of vectors or FUNTFs. The hope is that if we study the frame force's action on any frame as a dynamical system, that we will find a way to turn many unit-norm frames into FUNTFs. Specifically, if $\mathcal{H}$ is a Hilbert space, we can define the frame force for $u, v \in \mathcal{H}$, some vector space, as

$$FF(u,v) = \langle u, v\rangle(u - v). \tag{6.1}$$

If u and v are orthogonal or parallel to each other, the force is zero. Also note that we can think of the net force on a vector that interacts with multiple vectors. Say that u, v, and w are all vectors in the same vector space. Then

$$FF_{net}(u) = FF(u,v) + FF(u,w) = \langle u, v\rangle(u - v) + \langle u, w\rangle(u - w). \tag{6.2}$$

The frame force is discussed in [2] via the *frame potential.* Much like a physical potential,

$$\bigtriangledown_u P(u,v) = -FF(u,v). \tag{6.3}$$

The reason the frame force seems like it would be so useful is because of the following theorem by [2]:

Theorem 6.2.1. *A local minimizer for the frame potential is either an orthonormal set or a finite normalized tight frame for S, where S is a vector space.*

In [2] the frame force is treated primarily as a function of a static system. However, it seems natural to imagine that if the frame force on a vector were nonzero, it would move into a more energetically favorable position to lower the value of the frame potential, and thus bring the forces into equilibrium. We first considered the frame force like a force that was proportional to the angular acceleration of the vector about the origin. However, our goal is to start with a fixed frame, have the frame force act on it, and have the frame stop moving once the forces are in equilibrium. Treating the force like an angular acceleration could allow the vectors to continue moving after the forces have been equilibrated. We thought it would be simpler, but perhaps still useful, to define the angular velocity as being proportional to the frame force. Thus when the force goes to zero, the angular velocity goes to zero.

One particular example we looked at was the frame force acting on a unit-norm frame spanning $\mathbb{R}^2$. The ends of the vectors all lay on the same circle with a fixed radius, so their positions can be described by their angles. Some arithmetic shows that

$$\frac{d\theta_i}{dt} = \sum_{j=1}^{k} \frac{1}{2} \sin(2(\theta_i - \theta_j)). \tag{6.4}$$

It would be interesting to find a simple way to generalize this to higher dimensions.

As shown in Figure 6.1 the frame force pushes a general unit-norm frame into a FUNTF usually. However, some frames are not pushed into FUNTFs, especially those that have too many vectors spanning some lower dimensional subspace.

We wonder what kinds of frames actually get pushed into FUNTFs rather than orthonormal sets of vectors. What exactly does it mean to have too many vectors

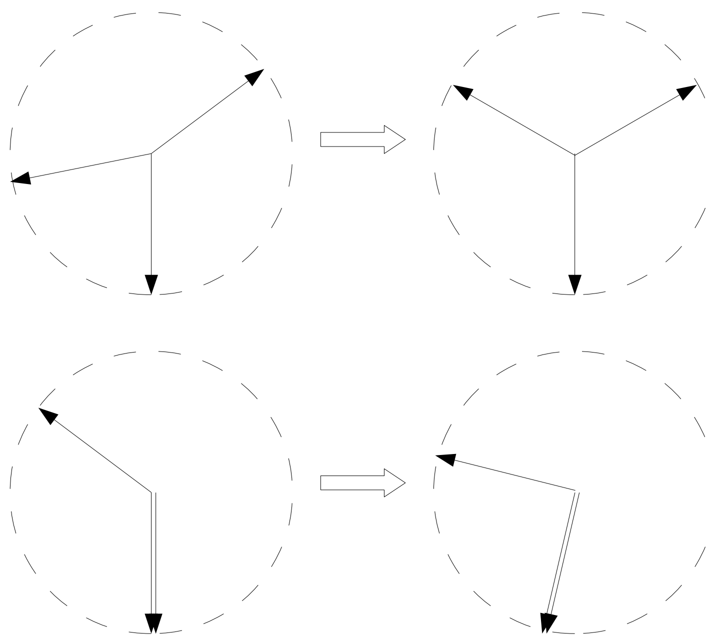

Figure 6.1: (Top Left) A typical unit-norm frame before the frame force acts on it. (Top Right) The typical unit-norm frame is pushed into a FUNTF after the frame force acts on it. (Bottom Left) A unit-norm frame which has multiple vectors spanning the same one-dimensional subspace. (Bottom Right) Some unit-norm frames act differently, and after the frame force acts on them they are pushed into an orthogonal set of vectors that is not a FUNTF.

spanning a lower dimensional subspace? We must first be able to answer these questions for a single vector space. The next step is to extend this to vector bundles on manifolds. In particular we need to know whether two nearby frames that look fairly similar necessarily end up as similar frames after the frame force has acted on them. Once we understand this better, we may be able to take advantage of the frame force as a tool for turning general frames into FUNTFs, at least in some cases.

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