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Problems in non linear PDE: Equilibrium configurations in periodic media and non local diffusion

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Problems in non linear PDE: Equilibrium configurations in periodic media and non local diffusion

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Problems in non linear PDE: Equilibrium configurations in periodic media and non local diffusion

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We study three different problems in non linear PDE. The first problem relates to finding equilibrium configurations in periodic media, more precisely, given an Area-Dirichlet functional $\mathcal{J}$, which is periodic under integer translations and given three planes in $\mathbb{R}^d$, we proof there exists at least one minimizer such that it's positive part, negative part and zero set remain at a uniform bounded distance of each plane. The second and third problem are related to non local diffusion, in the elliptic non symmetric case and parabolic case. In both cases we are interested in proving interior regularity for solutions of the aforementioned equations.

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Chapter 1

Introduction

In this work we study three problems in non linear PDE. First we consider functionals $\mathcal{J}$ of the form

$$\mathcal{J}(v) = \int \nabla u(x)^t A(x) \nabla u(x) dx + \int_\Omega F(x, \nu) d|m_E| + \int_\Omega h(x) \mathbf{1}_E dx, \qquad (1.1)$$

where E is a set of finite perimeter that contains the set $\{u \leq 0\}$ and the level surface $\{u = 0\}$, and $d|m_E|$ denotes the boundary measure. Here E plays the role of the zero set of u, which a priori may not be a set of bounded perimeter. Because of this technicality, we need to substitute $\{u = 0\}$ by a set of finite perimeter E.

The goal is to prove that if A, F, and h are periodic under integer translation, then, given three planes of slopes λ, μ and ω in $\mathbb{R}^d$ one can find a minimizing pair (u, E) of $\mathcal{J}$ such that u^+, u^- and ∂E remain at a uniform bounded distance of the planes of slope λ, μ and ω. These type of minimizers are called plane-like minimizers.

Plane-like minimizers have been studied in different settings and cases. The case $A = 0$ was studied by L. Caffarelli and R. de la Llave in [9], where they showed the existence of a plane-like minimizer and also showed that the minimizer was smooth depending on the properties of the metric. Moreover, they were able to conclude that the measure of the singular set of E, $\partial E \setminus \partial^* E$ had zero $s - 8 + \delta$ Haussdorff measure. In the case $A = 0$ these problems include as a particular case the problem of finding hypersurfaces of codimension one of minimal area.

The class of operators $\mathcal{J}$ is also connected with the Allen-Cahn or double well problem. In fact, for a Ginzburg-Landau like operator, E. Valdinoci proved in [20] the existence and regularity of plane-like minimizers and also was able to recover the results of [9] as a limiting case. More precisely he consider functionals of the form

$$\int \Big(a_{i,j}(x) \partial_i v \partial_j v + W(x, v) \Big) dx$$

where W is roughly a double well potential. As stated before, he was able to recover the result form L. Caffarelli and R. de la Llave in [9], which is to be expected, as one can recover minimal surfaces as limits when $\varepsilon \to 0$ of the zero level surfaces of minimizers of the energy functional

$$\int \varepsilon |\nabla v|^2 + \frac{1}{\varepsilon} W(v),$$

where W is a double well potential.

We also would like to make a connection with the Dirichlet problem studied in [8] I. Athanasopoulos et al. Given a smooth domain D and a boundary data g, they studied the problem, of finding a minimizer of

$$\int |\nabla v|^2 + Area\{v = 0\},$$

with boundary condition g. They also had to replace $Area\{v = 0\}$ by a suitable set of finite perimeter in order to have a good framework to work with. The tools used in [8] will be extremely useful when studying the regularity of the u across the free boundary. This paper combined with [9] are the motivation for studying plane-like minimizer for our functional $\mathcal{J}$.

Finally, let us go back to equation (1.1). We note, as in [8], that the problem has a natural scaling for u, which is Hölder $1/2$. This comes from the scaling $1/\zeta^{1/2}u(\zeta x)$. This fact will be used extensively in proving Hölder regularity across the free boundary ∂E.

Also note, even though we won't use it, that there is (formally) an equilibrium condition for the surface $\{u = 0\}$. In the case of [8], i.e, $A = Id$, $F = 1$ and $h = 0$ the condition is given by $\kappa = (u_\nu^+)^2 - (u_\nu^-)^2$, where κ denotes the mean curvature. In the general case, we will get a weighted curvature, a weighted normal derivative of u and an extra term given by the volume part h. In [8] this equation plays an important role when analyzing analyticity for $\partial^* E$.

Chapters 3 and 4 describe the results obtained in a joint work with Hector Chang on the regularity of solutions to fully non linear, non local equations. In general we are interested in studying integro differential equations that arise when studying discontinuous stochastic processes. By the Lèvy-Khintchine formula, the generator of an n-dimensional Lèvy process is given by

$$\begin{aligned} Lu(x) = &\sum_{i,j} a_{i,j}u_{i,j} + \sum_i b_i u_i \\ &+ \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x)\cdot y\chi_{B_1}(y))d\mu_x(y), \end{aligned}$$

where μ is a positive measure such that $\int |y|^2/(|y|^2+1)d\mu(y) < \infty$. The first and second term corresponds to the diffusion and drift part, and the third one correspond to the jump. The effect of first term is already well understood as it regularizes the solution.

The type of equations that we will study in Chapter 3 come from processes with only the jump part,

$$Lu(x) = \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x) \cdot y\chi_{B_1}(y))d\mu_x(y). \tag{1.2}$$

More general than the linear operator are the fully non linear ones, which are also important in stochastic control as seen in [18]. For example, a convex type of equation takes the form,

$$Iu(x) = \sup_{\alpha} L_\alpha u(x). \tag{1.3}$$

Equation (1.3) can be seen as a one player game, for which he can choose different strategies at each step to maximize the expected value of some function at the first exit point of the domain. A natural extension for (1.3), when there are two players competing is

$$Iu(x) = \inf_{\beta} \sup_{\alpha} L_{\alpha\beta} u(x).$$

We are mainly interested in studying interior regularity for solutions of

$$Iu(x) = f(x), \quad \text{in } \Omega, \tag{1.4}$$

for f continuous, Ω a given domain and I a fully non linear operator of fractional order to be defined in the next section. In [16] the regularity for this type of problem was already established by using analytic techniques. However those estimates blow up as the order of the equation goes to the classical one, so it was expected that better estimates could be possible. Those results are more elaborated and presented in [8], [9] and [10] in the case that the kernels are symmetric. We remove this symmetry hypothesis of the kernel and are able to obtain C^α regularity and $C^{1,\alpha}$ regularity for translation invariant equations.

In Chapter 4 we are interested in studying regularity for solutions of

$$u_t - Iu = f(x,t), \tag{1.5}$$

The type measures we are considering are of the form $d\mu = K(x,t;y)dy$ for some kernel positive K even in the y variable, i.e. $K(x,t;y) = K(x,t;-y)$. This assumption allows us to rewrite the operator without the principal value in the following way,

$$Lu(x,t) = \int_{\mathbb{R}^n} \delta(u,x,t;y)K(x,t;y)dy,$$

where $2\delta(u,x,t;y) = u(x+y,t) + u(x-y,t) - 2u(x,t)$ is the second order difference in space of u at (x,t). We note that we the setting is as in the previous case, but now the kernel is symmetric and may depend on time.

In the case of a family of symmetric kernels $K_{\alpha,\beta}(y) = a_{\alpha,\beta}(y)|y|^{-(n+1)}$, L. Silvestre studied in [17] the regularity of the solution of the Hamilton-Jacobi equation

$$u_t - \sup_{\alpha} \inf_{\beta} \left(c_{\alpha,\beta} + b_{\alpha,\beta} \cdot \nabla u + \int \delta(u,x,y) \frac{a_{\alpha,\beta}(y)}{|y|^{n+1}} dy \right) = 0,$$

where $c_{\alpha,\beta}$ is a family of constants, $b_{\alpha,\beta}$ is a bounded family of vectors and $\lambda \leq a_{\alpha,\beta} \leq \Lambda$. He was able to prove that the solution of the equation was classic using a non-variational approach to proof a diminish of oscillation lemma. A modification of the proof in [17] allows to get the regularity for for the same type of equations we study here, but the estimates would blow up as $\sigma \to 2$.

The variational problem was studied recently by L. Caffarelli, C. Chan and A. Vasseur in [6] by using De Giorgi's technique. Also, M. Felsinger and M. Kassmann in [12], obtained a Harnack inequality where the constants remain uniform as the order of the equation goes to the classical one by using Moser's technique.

The focus of this paper is to study regularity of solutions of (1.5), that remain uniform as $\sigma \to 2$. This will provide a natural extension to part of the theory already developed by L. Wang in [21].

Chapter 2

Plane like minimizers in periodic media for an Area-Dirichlet integral

2.1 Preliminaries

We will study first the case when ω is a rational vector and we will prove uniform bounds that will allow us to pass to the limit of minimizers with irrational slope. The whole setting of the problem deals with this case. Also we note that λ and μ depend on ω since the planes $\lambda \cdot x$ and $\mu \cdot x$ have to intersect the plane $\omega \cdot x$ for x such that $\omega \cdot x = 0$. With this, λ and μ are a one parameter family that depends on ω. Later on, for simplicity we will assume $\omega = e_d$, $\lambda = \lambda e_d$, $\mu = \mu e_d$ with $\lambda, \mu \in \mathbb{R}$.

Given $\omega \in \mathbb{Q}^d$ and an interval $M = [M^-, M^+]$ denote by $\Gamma^{\omega,M}$ the slab

$$\Gamma^{\omega,M} = \Gamma^{\omega,[M_-,M^+]} = \{x \in \mathbb{R}^d | x \cdot \omega/|\omega| \in M\}.$$

We consider the class $\mathcal{A}$ of Cacciopoli sets defined by

$$\mathcal{A}_M = \{E|\, \Gamma^{\omega,(-\infty,0]} \subset E \subset \Gamma^{\omega,(-\infty,M]}, \tau_k E = E,\ \forall k \in \mathbb{Z}^d,\ \omega \cdot k = 0\},$$

where τ_k denotes the translation operator.

Note that, as in [1], we can assume that the topological boundary of E coincides with the essential boundary $\partial_M E$ defined by

$$\partial_M E = \{x \in \mathbb{R}^d :\ 0 < |B_r(x) \cap E| < |B_r(x)|,\ \text{for each r} > 0\},$$

without changing the perimeter.

Now, since the only way ω enters is through the planes with normal ω and the module $\{k \in \mathbb{Z}^d, k\cdot\omega = 0\}$, we can assume without loss of generality that $\omega \in \mathbb{Z}^d$, and also assume that

$$\omega_d = \max |\omega_i| > 0.$$

Moreover, after rotation of the space we can further assume that $\omega = e_d$.

Given $\omega \in \mathbb{Z}^d$, consider $\approx$ the equivalence relation defined by

$$x \approx y \iff x = y + k,\ k \in \mathbb{Z}^d,\ when\ \omega \cdot k = 0.$$

Note that the quotient $(\Gamma^{\omega,(-\infty,M]} - \Gamma^{\omega,(-\infty,0]})/\approx$ is just $\mathbb{T}^{d-1} \times (0, M]$ and with this identification we have $\partial E \subset \mathbb{T}^{d-1}\times[0, M]$ and hence it is contained in a compact set.

We also denote S^M_ω by

$$S^M_\omega = \Gamma^{\omega,[0,M]}/\approx .$$

Given any domain Ω we define $\mathcal{J}_\Omega$ as

$$\mathcal{J}_\Omega(u, E) = \int_\Omega \nabla u(x)^t A(x)\nabla u(x)dx + \int_\Omega F(x, \nu)d|m_E| + \int_\Omega h(x)\mathbf{1}_E dx,$$

where ν denotes the unit normal to the set E and $d|m_E|$ denotes the boundary measure. For the moment $A(x)$ is a $d\times d$ positive definite matrix, F and h are continuous functions. The precise hypothesis on A, F and h are given in the next section.

Definition 2.1.1. *Given slopes ω, λ, μ, we will say a pair (u, E) is an admissible pair for $\mathcal{J}_\Omega$ if:*

- $u \in H^1_{loc}(\Omega)$, $E \in \mathcal{A}_M$.
- $u|_{E^c} \geq 0, a.e.$, $u|_E \leq 0, a.e.$.
- $\lambda \cdot x - M \leq u(x) \leq \lambda \cdot x$ *for* $x \in E^c$, $\mu \cdot x - M \leq u(x) \leq \mu \cdot x$ *for* $x \in E$

We point out again that E contains the set where u is negative and sometimes we may call $E = E^-$, $E^c = E^+$.

Given the plane-like behaviour of the admissible pairs at infinity, the functional is clearly unbounded when considered in $\mathbb{R}^d$. We give an alternative definition of minimizer for $\mathcal{J}$, which is the natural extension as the one proposed by L. Caffarelli and R. de la Llave in [7].

Definition 2.1.2. *Given $\omega \in \mathbb{Z}^d$, $\lambda, \mu \in \mathbb{R}^d$ we say that (u, E) is a Class A minimizer if for all compact sets B, and all admissible pairs (v, G) such that $u = v$ in $\mathbb{R}^d \setminus B$ and $G - (\mathbb{R}^d \setminus B) = E - (\mathbb{R}^d \setminus B)$ we have*

$$\mathcal{J}_B(u, E) \leq \mathcal{J}_B(v, G).$$

Such pair (v, G) is also called a compact perturbation of (u, E).

Remark 2.1.1. *We point out that in the case $\omega \in \mathbb{Q}^d$ we have the identification of the space, for which ∂E is contained in a compact set. In this case the perturbations only have to satisfy $u = v$ in $\mathbb{R}^d \setminus B$.*

We will use also the following notation.

Definition 2.1.3. *Let A be a positive definite matrix. We say a function u is A harmonic in Ω if u solves*

$$div(A\nabla u) = 0 \ \ in\ \Omega$$

in the weak sense.

2.2 Assumptions and Statement of results

In this section we state the main results of this work and we specify the hypothesis needed.

We assume the following conditions

(H1) A is C^1 and h is a continuous function in x, F is continuous in x, ν.

(H2) $A(x+e) = A(x)$, $h(x+e) = h(x)$ and $F(x+e,\nu) = F(x,\nu)$ for all $e \in \mathbb{Z}^d$.

(H3) $F(x, a\nu) = aF(x,\nu)$ for all $a \in \mathbb{R}^+$ and F is convex in ν.

(H4) There is $\Lambda \geq 1$ such that

$$0 < \Lambda^{-1} \leq F(x,\nu) \leq \Lambda$$

$$0 < \Lambda^{-1} \leq A(x) \leq \Lambda$$

for all $x \in \mathbb{R}^d$, $|\nu| = 1$.

Remark 2.2.1. *We could just ask Hölder regularity for A, but for the sake of making the presentation clearer we ask the coefficients to be C^1. We also point out that now an A harmonic function is smooth.*

Now, given λ, μ there has to be a relation between ω, μ, λ and the average of h. This condition will play an important role in the proof of the Birkhoff property.

(H5)

$$\int_{\mathbb{T}^d} h(x)dx = \left(\int_{\mathbb{T}^d} \nabla p_\mu(x)^t A(x) \nabla p_\mu(x) dx - \int_{\mathbb{T}^d} \nabla p_\lambda(x)^t A(x) \nabla p_\lambda(x) dx \right),$$

where p_λ, p_μ satisfy

$$div(A(x)\nabla p_{\lambda,\mu}(x)) = 0, \ in\, \mathbb{R}^d,$$

and

$$\lambda \cdot x - M \leq p_\lambda(x) \leq \lambda \cdot x, \ \forall x \in \mathbb{R}^d,$$

$$\mu \cdot x - M \leq p_\mu(x) \leq \mu \cdot x, \ \forall x \in \mathbb{R}^d.$$

The existence and study of the functions $p_{\lambda,\mu}$ can be found in [2] and [13].

(H6)

$$||h||_{L^\infty(\mathbb{R}^d)} \leq \varepsilon,$$

where ε depends only on the ellipticity constants of A and F, the dimension d and the isoperimetric inequality constant.

Remark 2.2.2. *Note that condition* $(H5)$ *and* $(H6)$ *implies that* λ *and* μ *cannot be too far apart, i.e.,* $|\lambda - \mu| \leq c(\varepsilon)$. *Heuristically, if* λ *and* μ *are too far apart, then the equilibrium condition for the surface won't be satisfied. In other words, there will be a region for which the curvature of* ∂E *will be positive, hence it won't be a plane-like surface.*

Now we are able to state the main result

Theorem 2.2.3. *Assume $\mathcal{J}$ satisfy hypothesis $(H1)$ to $(H6)$. Then for every $\omega \in \mathbb{R}^d - \{0\}$, $\lambda, \mu \in \mathbb{R}^d$ satisfying $(H5)$ and $(H6)$, we can find a class A minimizer (u_ω, E_ω) such that*

(A) For some $M = M$ independent of ω we have:

$$\partial E_\omega \subset \left\{ x \in \mathbb{R}^d | \, |x \cdot \omega| \leq M|\omega| \right\}.$$

(B) (u_ω, E_ω) is a class A minimizer, i.e., for all compact sets B, we have that all (v, F) such that $u = v$ in B and $L - (\mathbb{R}^d \setminus B) = E - (\mathbb{R}^d \setminus B)$ satisfy

$$\mathcal{J}_B(u_\omega, E_\omega) \leq \mathcal{J}_B(v, F).$$

(C) $(u_\omega, \partial E_\omega)$ is quasiperiodic, i.e. if $\omega \in \mathbb{Q}^d$, then $(u_\omega, \partial E_\omega)$ is periodic (with respect to the identification induced by ω), if $\omega \in \mathbb{R}^d \setminus \mathbb{Q}^d$ then u_ω can be approximated uniformly on compact sets and ∂E_ω in the local BV sense by sets that are periodic under integer translations.

We also are able to obtain regularity for u across the free boundary ∂E. The result is stated below.

Theorem 2.2.4. *Let $\mathcal{J}$ be as in Theorem 2.2.3 and let (u, E) be a class A minimizer for $\mathcal{J}$. If u is bounded, then u is $C^{1/2}$ in $\mathbb{R}^d$.*

Remark 2.2.5. *We can push the regularity up to Lipschitz by using the same ideas as in [1]. We will not present a proof of the result, since it is just an adaptation of the techniques used in [1].*

2.3 Existence of minimizers

In this section we will take the identification $\omega = e_d$, $\lambda = \lambda e_d$, $\mu = \mu e_d$ for some scalars λ and μ.

Given a functional of the form

$$\mathcal{J}_\Omega(u,E) = \int_\Omega \nabla u(x)^t A(x) \nabla u(x) dx + \int_\Omega F(x,\nu) d|m_E| + \int_\Omega h(x) \mathbf{1}_E dx,$$

we define the effective energy of the functional as

$$\tilde{\mathcal{J}}_\Omega(u,E) = \mathcal{J}_\Omega(u,E) - J_\lambda - J_\mu - \int_\Omega h(x) \mathbf{1}_{\mathbb{T}^{d-1}\times[M,\infty]},$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded set and J_λ, μ are given by

$$\begin{aligned} J_{\lambda,\Omega} &= \int_\Omega \nabla p_\lambda(x)^t A(x) \nabla p_\lambda(x) \mathbf{1}_{\mathbb{T}^{d-1}\times[M,\infty]}, \\ J_{\mu,\Omega} &= \int_\Omega \nabla p_\mu(x)^t A(x) \nabla p_\mu(x) \mathbf{1}_{\mathbb{T}^{d-1}\times[-\infty,0]} dx. \end{aligned}$$

In the case Ω is unbounded, the functional is understood in the principal value sense.

We note first that $\tilde{\mathcal{J}}_\Omega$ is bounded by below for all $\Omega \subseteq \mathbb{R}^d$, when evaluated in admissible pairs. This comes from the fact that we are removing the plane-like behaviour of the solution near infinity. More precisely, given an admissible pair (u,E), consider u_h^-, u_h^+ the solutions of

$$\begin{aligned} div(A(x)\nabla u_h^-(x)) &= 0, \text{ in } \mathbb{T}^{d-1}\times[-\infty,0], \\ u_h^- &= u, \text{ in } \mathbb{T}^{d-1}\times\{0\}, \end{aligned}$$

satisfying

$$\mu\cdot x - M \le u_h^-(x) \le \mu\cdot x, \ \forall x\in\mathbb{T}^{d-1}\times[-\infty,0].$$

Similarly define

$$\begin{aligned} div(A(x)\nabla u_h^+(x)) =& 0, \text{ in } \mathbb{T}^{d-1}\times[M,\infty], \\ u_h^+ =& u, \text{ in } \mathbb{T}^{d-1}\times\{M\}, \end{aligned}$$

satisfying

$$\lambda\cdot x - M \le u_h^+(x) \le \lambda\cdot x,\ \forall x\in\mathbb{T}^{d-1}\times[M,\infty].$$

Define now u_h to be u in $\mathbb{T}^{d-1}\times[0,M]$ and $u_h^\pm$ in $\mathbb{T}^{d-1}\times[-\infty,0]$ ($\mathbb{T}^{d-1}\times[M,\infty]$ resp.). Then, since $u_h^\pm$ minimize the energy integral outside of $\mathbb{T}^{d-1}\times[0,M]$ we have $\tilde{\mathcal{J}}_\Omega(u,E)\ge\tilde{\mathcal{J}}_\Omega(u_h,E)$. Now, we have

$$\tilde{\mathcal{J}}_\Omega(u_h,E)\ge -M\|h\|_{\mathbb{T}^d}+I_1+I_2,$$

where

$$I_1=\int_{\Omega\cap\mathbb{T}^{d-1}\times[M,\infty]}(\nabla u_h^+(x)^tA(x)\nabla u_h^+(x)-\nabla p_\lambda(x)^tA(x)\nabla p_\lambda(x))dx,$$

and

$$I_2=\int_{\Omega\cap\mathbb{T}^{d-1}\times[-\infty,0]}(\nabla u_h^-(x)^tA(x)\nabla u_h^-(x)-\nabla p_\mu(x)^tA(x)\nabla p_\mu(x))dx.$$

Let us bound I_1 (the bounds for I_2 are exactly the same). We have that $u_h^+-p_\lambda$ is an A-harmonic bounded function in $\mathbb{T}^{d-1}\times[M,\infty]$, hence it decays exponentially to a constant, see for example [14] and [15]. Now using the Cacciopoli inequality we are able to conclude that $\nabla(u_h^+-p_\lambda)^tA\nabla(u_h^++p_\lambda)$ is integrable in the cylinder, and so $I_1\ge -C$. Hence the functional is bounded by below.

Now note also that there is an admissible pair (u,E) for which the energy $\tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}}(u,E)$ is finite (note that the integrals are computed in the principal value sense). To construct such a pair just take the extensions u_h^- with boundary data $\mu \cdot x - M/2$ and u_h^+ with boundary data $\lambda \cdot x - M/2$. Construct u_h as follows

$$u_h(x) = \begin{cases} u_h^+ & x \in \mathbb{T}^{d-1} \times [M, \infty] \\ \eta & x \in \mathbb{T}^{d-1} \times [0, M] \\ u_h^- & x \in \mathbb{T}^{d-1} \times [-\infty, 0], \end{cases}$$

where η is any monotone C^∞ function such that $\eta = \lambda \cdot x - M/2$ in $\mathbb{T}^{d-1} \times \{M\}$ and $\eta = \mu \cdot x - M/2$ in $\mathbb{T}^{d-1} \times \{0\}$. Finally take E as the zero set of η, which has finite perimeter. Thanks to the hypothesis on the average of h it is clear than that the energy is finite.

Now using the fact that the functional is lower semicontinuous, it is possible to construct an absolute minimizer for $\tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}} = \tilde{\mathcal{J}}$. We point out that the Dirichlet integral is lower semicontinuous and in [7] it is proven that the weighted area term of the energy is also lower semicontinuous.

Lemma 2.3.1. *There exist an absolute minimizer for $\tilde{\mathcal{J}}$.*

Proof:

Since $\tilde{\mathcal{J}}$ is bounded by below, consider a minimizing sequence (u_n, E_n) of admissible pairs. Extract a converging subsequence and use the lower semicontinuity of the operator to conclude.□

Remark 2.3.2. *We could have prove existence by constructing a minimizers in balls of size B_k, $k \in \mathbb{N}$, satisfying boundary conditions the plane data. A diagonal trick would allow us then to find class A minimizers.*

Now, we want to relate the absolute minimizer for $\tilde{\mathcal{J}}$ with $\mathcal{J}$. This is done in the next lemma.

Lemma 2.3.3. *Let (u, E) be an admissible pair. Then (u, E) is an absolute minimizer for $\tilde{\mathcal{J}}$ if and only if (u, E) is a Class A minimizer for $\mathcal{J}$.*

Proof:

Assume first (u, E) is an absolute minimizer for $\tilde{\mathcal{J}}$. Now let $K \subset \mathbb{T}^{d-1} \times \mathbb{R}$ be a compact set and (v, G) a compact perturbation, we want to prove that

$$\mathcal{J}_K(v, G) \geq \mathcal{J}_K(u, E),$$

which is equivalent to

$$\tilde{\mathcal{J}}_K(v, G) \geq \tilde{\mathcal{J}}_K(u, E).$$

The last equation comes from the fact that we are subtracting a finite amount to each side of the inequality.

Now, note that the quantity

$$\tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}\setminus K}(u, E) = \tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}\setminus K}(v, G)$$

since (v, G) is a compact perturbation. Moreover this quantity is finite, so we can add it up on both sides of the inequality to conclude that

$$\tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}}(v, G) \geq \tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}}(u, E),$$

which concludes the argument.

Suppose now that (u, E) is a class A minimizer for $\mathcal{J}$ and assume also that there is (v, G) such that

$$\begin{aligned}\tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}}(u, E) >& \tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}}(v, G),\\ =& \tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}}(v, G) + \varepsilon.\end{aligned}$$

Let L be large enough such that for $K_{2L} = \mathbb{T}^{d-1} \times [-2L, 2L]$

$$\begin{aligned}\left|\tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}}(u, E) - \tilde{\mathcal{J}}_{K_{2L}}(u, E)\right| \leq& \frac{\varepsilon}{10},\\ \left|\tilde{\mathcal{J}}_{\mathbb{T}^{d-1}\times\mathbb{R}}(v, G) - \tilde{\mathcal{J}}_{K_{2L}}(v, G)\right| \leq& \frac{\varepsilon}{10},\end{aligned}$$

and note that $\mathcal{J}_{K_{2L}}(u, E) \geq \mathcal{J}_{K_{2L}}(v, G) + \frac{8\varepsilon}{10}$. Construct now the following extension

$$e = \eta v + (1 - \eta)u,$$

where $\eta = \eta(x_d)$ is a cut off function such that $\eta(\pm x_d) = 1$ for $x \in [-L, L]$ and $\eta(\pm 2L) = 0$. Now we can compare the energies

$$\mathcal{J}_{K_{2L}}(e, G) = \mathcal{J}_{K_{2L}}(v, G) + \int_{K_{2L}\setminus K_L} \nabla(v - e)^t A \nabla(u - e),$$

and since $v - e = (1 - \eta)(v - u)$ we get that

$$\int_{K_{2L}\setminus K_L} \nabla(v - e)^t A \nabla(u - e) \leq \frac{C}{L}.$$

Hence for L large enough we conclude

$$\mathcal{J}_{K_{2L}}(u, E) \geq \mathcal{J}_{K_{2L}}(e, G) + \frac{\varepsilon}{2},$$

which is a contradiction.□

Remark 2.3.4. *The previous lemma tells us that the minimizers for $\mathcal{J}$ and $\tilde{\mathcal{J}}$ are the same, which was to be expected since the operators only differ by a constant. From this point we will not distinguish between class A minimizer for $\mathcal{J}$ or absolute minimizers for $\tilde{\mathcal{J}}$.*

Finally we state some local properties of the minimizer (u, E).

Lemma 2.3.5. *Let (u, E) be a minimizer of $\mathcal{J}$. Then for every ball $B = B_r(x)$ such that it is strictly contained either in E or E^c, $B \cap \partial E = \emptyset$ we have*

$$div(A(x)u(x)) = 0, \tag{2.1}$$

in the weak sense.

Proof:

Just note that if $B_r \subset E$, then the solution u_r of (2.1) with boundary data u, has less integral energy than u. Hence, since u is a minimizer, it has to minimize the Dirichlet integral over that ball.□

2.4 Regularity across the free boundary

In this section we study the regularity of u across ∂E and the uniform density of ∂E. Since we want to study local properties, we will consider as the reference domain the unit ball $B_1 = B_1(0)$ and we will assume that $0 \in \partial E \cap B_1$. The first goal of this section is to prove the following theorem.

Theorem 2.4.1. *Let (u, E) be a minimizer in B_1. If u is bounded, then u is $C^{1/2}$ in $B_{1/2}$,*

$$\|u\|_{C^{1/2}(B_{1/2})} \leq C \|u\|_{L^\infty(B_1)},$$

and, for every $x \in \partial E \cap B_1$,

$$|B_r(x) \cap E, E^c| \geq Cr^d, \ \forall r \leq r_0,$$

where $C = C(\Lambda_1, \Lambda_2, d)$, $r_0 = r_0(\Lambda_1, \Lambda_2, d, \|h\|_\infty)$.

The proof of Theorem 2.4.1 is done in several intermediate steps, as in the case of [1].

We note that Section 2 of [1] can be re-done in our case with some minor obvious modifications. It is useful to introduce the following functional $\mathcal{I}_r^\pm$ defined as

$$\mathcal{I}_r^\pm = \mathcal{I}_r(u^\pm) = \int_{B_r} \frac{|\nabla u^\pm|^2}{|x|^{n-2}}.$$

Note also that it satisfies the following scaling property

$$\mathcal{I}_r(u) = r\mathcal{I}_1(u_r)$$

where

$$u_r(y) = \frac{1}{r^{1/2}} u(ry).$$

To help clarify some of the arguments, we will use the following notation $E^- = E$ and $E^+ = E^c$.

Lemma 2.4.2. *There exists positive universal constants δ_0 and c_0, such that if $\mathcal{I}_r \leq \delta_0 r$, then*

$$|E^{\pm} \cap B_{r/2}| \geq c_0 r^d.$$

Proof:

We will prove just the bounds for E^+. The case for E^- is done in the same way. Also, by rescaling, we may assume $r = 1$.

Consider now the following perturbation of the free boundary

$$E_*^- = E^- \setminus \bar{B}_{1-s}, \quad E_*^+ = E^+ \cup B_{1-s},$$

where $0 < s < 1$. Let u_*^+ be an extension of u^+ in E_*^+, that is u_*^+ solves the following equation

$$\begin{aligned} div(A\nabla u_*^+) &= 0, \text{ in } E_*^+, \\ u_*^+ &= u^+, \text{ on } \partial E_*^+. \end{aligned}$$

Now define $M_s = \sup u^-$ in $B_{1-s/4}$ and let G be the solution of the following equation

$$\begin{aligned} div(A\nabla G) &= 0, \text{ in } R_s = B_{1-s/4} \setminus B_{1-s}, \\ G &= 1, \text{ on } \partial B_{1-s/4}, \\ G &= 0, \text{ on } \partial B_{1-s}. \end{aligned}$$

Finally define the perturbation u_* as

$$u_*(x) = \begin{cases} -u^- & x \in E_*^- \setminus R_s, \\ -\min\{u^-, M_s G\} & x \in E_*^- \cap R_s, \\ u_*^+ & x \in E_*^+, \end{cases}$$

so that the pair (u_*, E_*^-) is admissible. Since (u, E) is a minimizer we have

$$\mathcal{J}_{B_1}(u_*, E_*^-) \geq \mathcal{J}_{B_1}(u, E^-).$$

Now, note that $E^+ \subseteq E_*^+$, hence we get

$$\int_{B_1} \nabla u_*^{+t} A \nabla u_*^+ \leq \int_{B_1} \nabla u^{+t} A \nabla u^+.$$

We also have

$$\int_{B_1} F(x,\nu) d|m_{E_*^-}| = \int_{B_1 \setminus \bar{B}_{1-s}} F(x,\nu) d|m_{E_*^-}| + \int_{\partial B_{1-s} \cap E^-} F(x,\nu) dH^{d-1}$$

and

$$\int_{B_1} F(x,\nu) d|m_{E^-}| \geq \int_{B_1 \setminus \bar{B}_{1-s}} F(x,\nu) d|m_{E_*^-}| + \int_{B_{1-s}} F(x,\nu) d|m_{E^-}|,$$

where H^{d-1} denotes de $d-1$ Haussdorff measure.

Finally, since $|\nabla G| \leq cs^{-1}$, we have

$$\int_{B_1} \nabla u_*^{-t} A \nabla u_*^- \leq \int_{B_1} \nabla u^{-t} A \nabla u^- + cM_s{}^2 s^{-2},$$

and since $||h||_\infty \leq \varepsilon$ we conclude

$$P(E^-, B_{1-s}) \leq C(H^{d-1}(\partial B_{1-s} \cap E^-) + M_s{}^2 s^{-2} + \varepsilon |B_{1-s} \cap E^-|). \tag{2.1}$$

Now we will use a De Giorgi iterative scheme to prove the desired estimate density.

Let $r_0 = 1/2$, $c \leq 1/10$ and define

$$r_{m+1} = R_m - c2^{-m}.$$

Our goal is to prove the following recursive formulas

$$\mathcal{I}_{m+1} \leq C^m \mathcal{I}_m V_m, \quad V_{m+1} + \mathcal{I}_{m+1} \leq C^m (V_m + \mathcal{I}_m)^{d/(d-1)}, \tag{2.2}$$

where $\mathcal{I}_m = \mathcal{I}^-_{r_m}$ and $V_m = |E^- \cap (B_{r_m} \setminus \bar{B}_{r_{m+1}})|$.

Following the steps of [1] we note that if $r'_m = r_{m+1} + (c/2)2^{-m}$ then we have

$$\sup_{B_r}(u^-)^2 \leq \frac{C}{(r_m - r)^n} \mathcal{I}_m, \text{ for } r < r_m, \tag{2.3}$$

and

$$\mathcal{I}_{m+1} \leq c2^{2nm} \int_{B_{r'_m} \setminus B_{r_{m+1}}} (u^-)^2 \leq c2^{2nm} \sup_{B_{r'_m}} (u^-)^2 V_m. \tag{2.4}$$

Note that these inequalities come from the fact the u^+ and u^- are A sub harmonic everywhere.

Combining (2.3) and (2.4) we obtain

$$\mathcal{I}_{m+1} \leq C2^{4nm} \mathcal{I}_m V_m \leq C_0^m \mathcal{I}_m V_m.$$

To estimate V_m, we use the isoperimetric inequality

$$V_m^{(d-1)/d} \leq \gamma_d (A_{m+1} + P_{m+1}),$$

where γ_d is the isoperimetric constant and

$$\begin{aligned} A_{m+1} =& H^{d-1}(E^- \cap \partial(B_{r_{m+1}} \setminus \bar{B}_{r_{m+2}})), \\ P_{m+1} =& P(E^- \cap (B_{r_{m+1}} \setminus \bar{B}_{r_{m+2}})). \end{aligned}$$

Now, if r is any intermediate radius $r_m < r < r''_m = r_m + 1/4c^{-m}$, then we get that $A_{m+1} \leq A_r$, where $A_r = H^{d-1}(E^- \cap \partial B_r)$. We also observe that $P_{m+1} \leq P_r$, where $P_r = P(E^- \cap B_r)$.

Note that for $\beta \geq 1$ and $a, b > 0$ small, we have $(a+b)^\beta \leq c_\beta(a^\beta + b^\beta) \leq c_\beta(a+b)$, hence, if we are in a small ball, we have

$$|E^- \cap B_r| \leq C_d(H^{d-1}(E^- \cap \partial B_r) + P(E^- \cap B_r)).$$

Hence, from the estimate (2.1) and using that ε is small, we deduce

$$P(E^- \cap B_r) \leq C(H^{d-1}(\partial B_r \cap E^-) + C_0^m \mathcal{I}_m). \tag{2.5}$$

Using (2.5) we get

$$V_{m+1}^{(d-1)/d} \leq C(A_r + \mathcal{I}_r),$$

which can be integrated to obtain

$$V_{m+1} \leq C^m (V_m + \mathcal{I}_m)^{d/(d-1)}.$$

We are now in shape to conclude the desired result. If $V_0 + \mathcal{I}_0$ is smaller than δ, then we also have

$$(V_{m+1} + \mathcal{I}_{m+1}) \leq C^m (V_m + \mathcal{I}_m)^{d/(d-1)}. \tag{2.6}$$

and from (2.6) we conclude then $\mathcal{I}_m \to 0$, which is a contradiction, since $r_m \to r_\infty > 0$. Hence, if $\mathcal{I}_0$ is small enough, let say smaller, than $\delta/2$, then we must have that

$$|E^- \cap B_{1/2}| > V_0 \geq \delta/2. \ \square$$

The following lemmas are derived in the same way as the ones found in [1], replacing F and A by their ellipticity bounds, hence we omit their proofs.

Lemma 2.4.3. *Assume $\mathcal{I}_1^+\mathcal{I}_1 \leq K$, then*

$$\mathcal{I}_{1/8}^+ \leq C_1, \quad \mathcal{I}_{1/8}^- \leq C_2.$$

Lemma 2.4.4. *Assume $\mathcal{I}_1^\pm \leq K^\pm$, then*

$$|B_{1/2} \cap E^\pm| \geq C(d, K^\pm) > 0.$$

We are now able to conclude Theorem 2.4.1.

First note that by the monotonicity formula (see [1]) we have the following estimate

$$\mathcal{I}_r^+\mathcal{I}_r^- \leq Cr^4\|u\|_{L^\infty(B_1)}^4.$$

Let $v_r(y) = r^{-1/2}u(ry)$, and note that after rescaling we have

$$\mathcal{I}_1(v_r^+)\mathcal{I}_1(v_r^-) \leq Cr^2\,\|u\|_{L^\infty(B_1)}^4\,.$$

Using Lemma 2.4.3 we deduce $\mathcal{I}_{1/2}(v_r^\pm) \leq c\,\|u\|_{L^\infty(B_1)}^2$. Also note that $u^\pm$ satisfies

$$\sup_{B_{1/2}}(u^\pm)^2 \leq C\mathcal{I}_{1/2}^\pm,$$

which combined with previous inequality gives us

$$(u^\pm)^2 \leq \mathcal{I}_r(u^\pm) \leq cr\|u\|_{L^\infty}^2.$$

The last inequality with Lemma 2.4.4 finishes the proof.□

Finally we state the Lipschitz regularity of u across the free boundary. The proof is just a minor modification of the one found in [1] and we omit it. We point out though, that we do not use the Lipschitz regularity in the upcoming sections.

Theorem 2.4.5. *Let (u, E) be a minimizer. If u is bounded, then u is Lipschitz in $B_{1/2}$.*

2.5 Density estimates

So far we have established that if (u, E) is a minimizer of our functional, then E and E^c have positive density, i.e.

$$|E \cap B_r| \geq cr^d.$$

In this section we establish the following estimate.

Lemma 2.5.1. *Let (u, E) be a minimizer of $\mathcal{J}$. Then, there exist a constant $C = C(R)$ such that for every $x \in \partial E$, we have*

$$|m_E|(B_r(x)) = |m_E|[\partial E \cap B_r(x)] \leq C(R). \tag{2.1}$$

Moreover there exist a universal σ such that for every $x \in \partial E$, $r \leq r_0$, we have

$$\int_{B_r} F(x, nu) d|m_E| + \int_{B_r} h(x) \mathbf{1}_E dx \geq \sigma r^{d-1}. \tag{2.2}$$

Remark 2.5.2. *We note that the restriction of the radius $r \leq r_0$ is natural, since for large values of r the volume term controls the area terms. There is a natural trade off between the size of the ball and the size of h, i.e., if we want r_0 in Lemma 2.5.1 to be large enough, we have to ask $||h||_{C^0}$ to be small.*

Proof of Lemma 2.5.1:

The proof goes along the lines of the Hölder regularity.

Recall the estimate (2.1), but for a general R,

$$P(E, B_{1-s}) \leq C(H^{d-1}(\partial B_{1-s} \cap E^-) + {M_s}^2 s^{-2} + \varepsilon|B_{1-s} \cap E^-|)$$

where $M_s = \sup u^-$ in $B_{1-s/4}$, and $0 < s < 1$. The proof was done for the case $R = 1$, but for a general R it reads

$$P(E, B_{R-s}) \leq C(H^{d-1}(\partial B_{R-s} \cap E^-) + {M_{s,R}}^2 s^{-2} + \varepsilon|B_{R-s} \cap E^-|),$$

and here $M_{R,s} = \sup u^-$ in $B_{R-s/4}$. Take $R > 1$, $s = 1$ and use the regularity across the free boundary of u to conclude

$$\begin{aligned} P(E, B_{R-1}) &\leq c(R-1)^{d-1} + C(R)^{2\alpha} + \varepsilon(R-1)^d \\ &\leq C(R). \end{aligned}$$

For the second inequality we note that for $r \leq r_0$

$$\begin{aligned} \int_{B_r} F(x, nu) d|m_E| + \int_{B_r} h(x)\mathbf{1}dx \geq& \lambda Per(E \cap B_r) - ||h||_{C^0}|E \cap B_r| \\ \geq& Cr^{d-1} - ||h||_{C^0} r^d \\ \geq& \sigma r^{d-1}, \end{aligned}$$

which ends the proof.□

As a direct consequence we have a uniform bound in the $H^1(B_R)$ norm of u.

Corollary 2.5.3. *Let (u, E) be a class A minimizer for $\mathcal{J}$ and let $R > 0$. Then*

$$||u||_{H^1(B_R)} \leq C(R),$$

where $C(R)$ is a constant only depending on R and the properties of the functional.

Proof:

After an identification of the space we can consider $\mathbb{R}^d = \mathbb{T}^{d-1} \times \mathbb{R}$. We will just prove it for slabs. Let $(v, \Gamma^{\omega,[-\infty,0]})$ be a compact perturbation of the form

$$v = \begin{cases} \eta_1 u + (1-\eta_1)p_\lambda & x \in \mathbb{T}^{d-1} \times [R, R+1], \\ p_\lambda & x \in \mathbb{T}^{d-1} \times [1, R], \\ p_\lambda \eta_2 & x \in \mathbb{T}^{d-1} \times [0,1], \\ p_\mu \eta_3 & x \in \mathbb{T}^{d-1} \times [-1, 0], \\ p_\mu & x \in \mathbb{T}^{d-1} \times [-R, -1], \\ u\eta_4 + (1-\eta_4)p_\mu & x \in \mathbb{T}^{d-1} \times [-R, -R-1], \end{cases}$$

where η_i are smooth cut-off functions.

Since (u, E) is a minimizer, we have for $K_{R+1} = \mathbb{T}^{d-1} \times [-R-1, R+1]$

$$\mathcal{J}_{K_{R+1}}(u, E) \leq \mathcal{J}_{K_{R+1}}(v, G),$$

Disregarding the area term in the left hand side of the inequality ($F > 0$), we get

$$\int_{K_{R+1}} \nabla u^t A \nabla u dx \leq \int_{K_{R+1}} \nabla v^t A \nabla v dx + (\|h\|_\infty + \Lambda)(R+1)^d.$$

Decomposing v and using the ellipticity of A, we conclude then

$$\|\nabla u\|_{L^2(B_R)} \leq C(R+1)^d,$$

where C depends on Λ, λ, μ and d.

2.6 The infimal minimizer and Birkhoff Property

Following the steps of [7] we focus our attention to a special type of minimizer. First we recall the following lemma from L. Caffarelli and R. de la Llave (see Section 6 from [7]).

Proposition 2.6.1 (L. Caffarelli, R. de la Llave). *Let E and L be sets of finite perimeter. Then*

$$\mathcal{L}(E) + \mathcal{L}(L) \geq \mathcal{L}(E \cap L) + \mathcal{L}(E \cup L),$$

where

$$\mathcal{L}(E) = \int F(x, \nu) d|m_E|.$$

Now we can prove the following lemma.

Lemma 2.6.2. *Let (u, E) be a minimizer in the class $\mathcal{A}_{S^-,S^+}$ and let (v, L) be a minimizer in the class $\mathcal{A}_{T^-,T+}$ with $T^- \subseteq S^-$ and $T^+ \subseteq S^+$. Then we have*

- $(\max(u, v), E \cup L)$ *is a minimizer in the class $\mathcal{A}_{S^-,S^+}$.*
- $(\min(u, v), E \cap L)$ *is a minimizer in the class $\mathcal{A}_{T-,T^+}$.*

Proof:

First note that $(\max(u, v), E \cup L)$ is an admissible pair in the class $\mathcal{A}_{S^-,S^+}$. We also have that $(\min(u, v), E \cap L)$ is an admissible pair in the class $\mathcal{A}_{T^-,T^+}$. To avoid problems at infinity we will look at the effective energy. Since (u, E) and (v, L) are minimizers in their respective classes we have the following inequalities,

$$\begin{aligned}\tilde{\mathcal{J}}(u, E) \leq& \tilde{\mathcal{J}}(\max(u, v), E \cup L),\\ \tilde{\mathcal{J}}(v, L) \leq& \tilde{\mathcal{J}}(\min(u, v), E \cap L).\end{aligned}$$

On the other hand, thanks to Proposition 2.6.1, we have the following inequality

$$\tilde{\mathcal{J}}(u, E) + \tilde{\mathcal{J}}(v, L) \geq \tilde{\mathcal{J}}(\max(u, v), E \cup L) + \tilde{\mathcal{J}}(\min(u, v), E \cap L),$$

hence we conclude

$$\tilde{\mathcal{J}}(u, E) = \tilde{\mathcal{J}}(\max(u, v), E \cup L),$$

and

$$\tilde{\mathcal{J}}(v, L) = \tilde{\mathcal{J}}(\min(u, v), E \cap L),$$

which ends the proof. □

Now we are able to define the minimal minimizer.

Lemma 2.6.3. *Consider the class $\mathcal{A}_{S^-,S^+}$ and define (u, E)*

$$u = \min v, \quad E = \cap L,$$

where (v, L) is a minimizer. Then (u, E) is also a minimizer in the class $\mathcal{A}_{S^-,S^+}$.

Proof:

As in [7], one can prove using Lemma 2.6.2 that any finite intersection is also a minimizer. Given the a priori bounds for u and the fact that one can uniformly bound the perimeter of the intersection, it is easy to find a convergent subsequence. Using the fact that the operator is lower semicontinuous one conclude the desired result.□

Remark 2.6.4. *The infimal minimizer can be characterized as the minimizer which is smaller (resp. contained) than all the minimizers.*

We will now take advantage of the periodicity assumptions on the operator in order to prove the following proposition.

Proposition 2.6.5. *Let $k \in \mathbb{Z}^d$ and (u_*, E_*) be the infimal minimizer in the class $\mathcal{A}_M$, then*

(a) If $k \cdot \omega \leq 0$, we have that $\tau_k E_* \subset E_*$,

(b) If $k \cdot \omega \geq 0$, we have that $\tau_k E_* \supset E_*$.

Proof:

We will just prove the case $k \cdot \omega \leq 0$. The proof for the case $k \cdot \omega \leq 0$ is identical.

First note that $\tau_k \Gamma^{w,M} \subseteq \Gamma^{w,M}$ and that $(\tau_k u_*, \tau_k E_*)$ should be the infimal minimizer in the class $\mathcal{A}_{\tau_k M}$. In fact, given any compact $K_L = \mathbb{T}^{d-1} \times [-L, L]$ its easy to see that for $k \cdot \omega \geq 0$

$$\mathcal{J}_{K_L}(\tau_k u_*, \tau_k E_*) = \mathcal{J}_{K_L}(u_*, E_*) + R_L,$$

where

$$\begin{aligned} R_L = & -k \cdot \omega \int_{\mathbb{T}^d} h(x)dx + \int_{\mathbb{T}^{d-1} \times [L, L+k\cdot\omega]} \nabla u_*{}^t A \nabla u_* dx \\ & - \int_{\mathbb{T}^{d-1} \times [-L+k\cdot\omega, -L]} \nabla u_*{}^t A \nabla u_* dx \end{aligned}$$

Note now that for u_*^+ we have

$$\nabla u_*^{+t} A \nabla u_*^+ = \nabla p_\lambda{}^t A \nabla p_\lambda + R_L^{'},$$

where

$$|R_L^{'}| \leq \Lambda \left(|\nabla (u_*^+ - p_\lambda)|^2 + |\nabla (u_*^+ - p_\lambda)||\nabla (u_*^+ + p_\lambda)| \right).$$

As in the proof of existence, we know that

$$\int_{\mathbb{T}^{d-1}\times[m,\infty]} |\nabla(u_*^+ - p_\lambda)|^2 < \infty,$$

hence as $L \to \infty$ we have

$$\int_{\mathbb{T}^{d-1}\times[L,L+k\cdot\omega]} |\nabla(u_*^+ - p_\lambda)|^2 dx \to 0.$$

Moreover, since $|\nabla(u_*^+ + p_\lambda)|$ is bounded uniformly, we conclude using Hölder inequality, that as $L \to \infty$

$$\int_{\mathbb{T}^{d-1}\times[L,L+k\cdot\omega]} |\nabla(u_*^+ - p_\lambda)||\nabla(u_*^+ + p_\lambda)|dx \to 0.$$

Applying the same treatment to the negative part and combining this with hypothesis $(H5)$ which controlled the average of h, we conclude then that

$$\mathcal{J}_{K_L}(\tau_k u_*, \tau_k E_*) = \mathcal{J}_{K_L}(u_*, E_*) + \varepsilon(L),$$

where $\varepsilon(L) \to 0$ as $L \to \infty$. So $(\tau_k u_*, \tau_k E_*)$ is in fact the minimizer in the translated class. Now it is also easy to check that is the infimal minimizer.

Finally apply Lemma 2.6.2 in order to conclude $\tau_k E_* \cup E_*$ is also a minimizer. In particular, since $\tau_k E_*$ is the infimal minimizer we conclude $\tau_k E_* \subseteq E_*$.□

2.7 Proof of Theorem 1 in the case $\omega \in \mathbb{Q}^d$

In this section we prove Theorem 1 in case $\omega \in \mathbb{Q}^d$. The next section deals with the irrational case.

Proposition 2.7.1. *If* $\mathbb{R}^d - E_*$ *contains a cube of side* 2*,* $Q_2(x)$*, or a ball of radius* $\sqrt{d}$*, then it contains a whole semi space, which includes the center of the cube*

Proof:

Note that we are assuming $Q_2(x) \subset \mathbb{R}^d - E_*$, then by proposition 2.6.5 we conclude

$$\bigcup_{k \in \mathbb{Z}^d,\ \omega \cdot k \geq 0} \tau_k Q_2(x) \subset \mathbb{R}^d - E_*.$$

Finally note that the left hand side is the intersections of a semi space with a lattice of size 1. Hence if we place a cube of size 2 or a ball of radius $\sqrt{d}$ in each of the centers in the semi lattice, we cover a semi space.□

Proposition 2.7.2. *When* $M_+ > \sqrt{d}$*,* $\Gamma^{\omega,[0,\sqrt{d}]}$ *intersects* $\mathbb{R}^d - E_*$*.*

Proof:

We proceed by contradiction. If the proposition is not true, then the pair $(\tau_{-d} u_*, \tau_{-d} E_*)$ is an admissible pair and also a minimizer. This statement contradicts the fact that (u_*, E_*) is the infimal minimizer.□

Now we are to show that ∂E_* is contained in a strip of fixed width $\Gamma^{\omega,[0,M^*]}$, when M^* is large enough.

Lemma 2.7.3. *There exist* M^* *that only depends on the properties of the functional such that, for all* $a > 0$ *we have*

$$E_*^{\omega,M^*+a} = E_*^{\omega,M^*}.$$

Proof:

First note that is enough to show that

$$E_* = E_*^{\omega,M} \subset \{|x| \mid x \cdot \omega/|\omega| \leq M^*\},$$

for all $M \geq M^*$. Also note that if for some M there is a cube $Q_2(x)$ completely contained in $\mathbb{R}^d - E_*$, then we can apply Proposition 2.7.1 to conclude $E_* \subset \{|x| \mid x \cdot \omega/|\omega| \leq \hat{M}\}$ for some $\hat{M} \leq M$. Hence we can always assume that ∂E intersects every cube of side 2 in $\Gamma^{\omega,[0,M]}$.
We will assume also that $\mu \leq \lambda$. The other case follows in the same way.

Note that by Proposition 2.7.2 we know that there is a point $x_0 \in \mathbb{R}^d - E_*$ such that $d(x_0, \Pi^{\omega,0}) \leq \sqrt{d}$. Hence by the Birkhoff property (Proposition 2.6.5), we conclude

$$x_k = \tau_k(x_0) \in \mathbb{R}^d - E_*,$$

for any $k \in \mathbb{Z}^d$ such that $k \cdot \omega \geq 0$.

Assume now that we also have the following property. For any $k \in \Gamma^{\omega,[0,M^*]}$, $Q_2(x_k) \cap E_*$ is non empty. Now by using the density estimates (2.2) we can conclude

$$\int_{Q_4(x_k)} F(x,\nu) d|m_E| + \int_{Q_4(x_k)} h(x) \mathbf{1}_E dx \geq c_0.$$

We note that there are TM^* of those cubes, where T is the number of cubes in the periodic $d-1$ dimensional base. Also they overlap at most 4^d times, so we can get the following estimate in the strip $K_{2M^*} = \mathbb{T}^{d-1} \times [-2M^*, 2M^*]$

$$\int_{K_{2M^*}} F(x,\nu) d|m_E| + \int_{K_{2M^*}} h(x) \mathbf{1}_E dx \geq 4c_0 4^{-d} TM^*. \tag{2.1}$$

On the other hand, we know that the pair (u_*, E_*) is a minimizer of our functional $\mathcal{J}$, hence we have the following inequality

$$\mathcal{J}_{K_{2M^*}}(v, \Gamma^{\omega,[-\infty,0]}) \geq \mathcal{J}_{K_{2M^*}}(u_*, E_*) \tag{2.2}$$

where the v is given by

$$v = \begin{cases} \eta_1 u + (1-\eta_1)p_\lambda & x \in \mathbb{T}^{d-1} \times [2M^* - 1, 2M^*], \\ p_\lambda & x \in \mathbb{T}^{d-1} \times [1, 2M^* - 1], \\ p_\lambda \eta_2 & x \in \mathbb{T}^{d-1} \times [0, 1], \\ p_\mu \eta_3 & x \in \mathbb{T}^{d-1} \times [-1, 0], \\ p_\mu & x \in \mathbb{T}^{d-1} \times [-2M^* + 1, -1], \\ u\eta_4 + (1-\eta_4)p_\mu & x \in \mathbb{T}^{d-1} \times [-2M^* + 1, -2M^*], \end{cases}$$

and η_i are smooth cut-off functions. Basically, what we are doing, is connecting the 0 function to $p_{\lambda,\mu}$ and the connecting it with u, using different cut off functions. This is the natural choice of competitor, since it is just the plane like solution, modified to satisfy the boundary conditions.

Let us denote $\Gamma^{\omega,[-\infty,0]} = \Gamma_0$, and observe that we have the following estimate on the area term

$$\int_{K_{2M^*}} F(x, \nu) d|\Gamma_0| \leq C_1,$$

and also the estimate on the Dirichlet integral

$$\int_{K_{2M^*}} \left| \nabla u^t A \nabla u - \nabla v^t A \nabla v \right| dx \leq C_2$$

where C_1, C_2 does not depend on M^*. Using the previous estimates and equation (2.1), we can rewrite inequality (2.2) to get

$$4c_0 4^{-d} T M^* \leq C + \int_{K_{2M^*}} h(x) \mathbf{1}_{\Gamma_0} dx, \tag{2.3}$$

with $C = C_1 + C_2$. Bounding h by its supremum norm, we conclude

$$4c_0 4^{-d} T M^* \leq C + 4\|h\|_{C^0} M^*.$$

Since $\|h\|_\infty$ is assumed to be small ($\|h\|_\infty \leq C_0 4^{-d} T/4$ is enough), we are able to conclude that for large M^* there is at least one $Q_2(x_k)$ contained in $\mathbb{R}^d - E_*$. Now we can apply proposition 2.7.1 to conclude the desired for $M \in [M^*, 2M^*]$.□

We proved that for M large enough, the upper constraint of the minimizer (u_*, E_*) is irrelevant. We need now to prove that the lower constraint is irrelevant to conclude that the minimizer is unconstrained.

Proposition 2.7.4. *The pair (u_*, E_*) is an unconstrained minimizer.*

Remark 2.7.5. *Note that we just need to prove that the set E_* is unconstrained. The fact that u_* remains unconstrained comes from the maximum principle.*

Proof of Proposition 2.7.4:

Note first that Lemma 2.7.3 states that E_* has to be contained in a semiplane $\Gamma^{\omega,[0,M^*]}$, where M^* only depends on the isoperimetric constants, the dimension and the properties of the functional. Now note that, for any $m \in \mathbb{N}$ the pair $(\tau_{me_d} u_*, \tau_{me_d} E_*)$ is a minimizer, where $\tau_{me_d} E_*$ is to be considered in the class of periodic sets that contain the lower constraint $\Gamma^{\omega,[-\infty,0]}$. Note also that by the hypothesis over the average of h, the contribution of the Dirichlet integral and volume term is 0.

The boundary of set has to be contained in $\Gamma^{\omega,[m\omega_d,M^*+m\omega_d]}$, hence this shows that we cannot make the functional decrease by making perturbations of E_* of size $m\omega_d/|\omega|$. Since m is arbitrary we conclude that the minimizer is unconstrained.□

We need to check now that the minimizer is a class A minimizer when considered as set/function of $\mathbb{R}^d$.

Proposition 2.7.6. *The pair (u_*, E_*) is a class A minimizer.*

Remark 2.7.7. *The following proof is the same as the one found in [7], but it is included for the convenience of the reader.*

Proof:

We need to prove now that (u_*, E_*) defined in $\mathbb{R}^d$ is a minimizer subject to compact perturbations. Given any compact perturbation, we can consider consider E_* as periodic with a period larger than the diameter of the perturbation and as a minimizer among the class of sets contained in a slab larger than the size of the perturbation. By uniqueness of the infimal minimizer, (u_*, E_*) will still be the infimal minimizer in this class, hence the value of the functional will be smaller when restricted to the larger period.□

This concludes the proof of Theorem 1 in the case $\omega \in \mathbb{Q}^d$.

2.8 Proof of Theorem 1 for ω irrational

Note that in the rational case, the width of the strip does not depend on ω, hence the case $\omega\mathbb{R}^d - \mathbb{Q}^d$ will follow as a limiting case. In fact, given any frequency

$\omega \in \mathbb{R}^d$, we can construct a sequence $\omega_n \in \mathbb{Q}^d$ such that $\omega_n \to \omega$. Denote by (u_*^n, E_*^n) the infimal minimizer associated to ω_n. Now, note that given any ball B we have by the density estimates that

$$Per(E_*^n \cap B) \leq C$$

and by the a priori estimates we also have that

$$||u_*^n||_{C^\alpha(B)} \leq C.$$

Also thanks to Corollary 2.5.3, we have a uniform bound of the H^1 norm along the sequence. Hence given any ball $B_R(0)$, we can extract a subsequence, that we still call (u_*^n, E_*^n) such that u_*^n converges to u_* uniformly in B_R and $E_*^n \cap B_R$ converges in L^1 to E_*. Moreover, we can extract a subsequence such that $u_*^n \to u$ weakly in $H^1(B_R)$. By applying a diagonal trick we can obtain that $u_*^n \to u_*$ uniformly in compact sets, $E_*^n \to E_*$ in L^1_{loc} and $u_*^n \to u$ weakly in H^1_{loc} .
Note that

$$\partial E_* \subset \{x \in \mathbb{R}^d |\, 0 \leq x \cdot (\omega/|\omega|) \leq L\},$$

and that u_* remains between its respective planes too. Hence the only thing left to check is that the pair (u_*, E_*) is a class A minimizer.
Given any pair (v, G) and any ball $B_r(0)$ such that they agree outside $B_r(0)$, we have

$$\mathcal{J}_{B_r(0)}(u_*^n, E_*^n) \leq \mathcal{J}_{B_r(0)}(v, G).$$

By the lower semicontinuity of the functional we then conclude that

$$\mathcal{J}_{B_r(0)}(u_*, E_*) \leq \mathcal{J}_{B_r(0)}(v, G),$$

which finishes the proof.□

Chapter 3

Regularity for solutions of non local equations, non symmetric equations

3.1 Preliminaries and Viscosity Solutions

In this work we restrict ourselves to measures $d\mu_x = K(x,y)dy$. From equation (1.2) we formally can write

$$Lu(x) = \int \delta_e(u,x;y)K_e(x;y)dy + \int \delta_o(u,x;y)K_o(x;y)dy + b(x)\cdot\nabla u(x), \quad (3.1)$$

where

$$\delta_e(u,x;y) = u(x+y) + u(x-y) - 2u(x),$$
$$\delta_o(u,x;y) = u(x+y) - u(x-y).$$

$K_{e,o}$ are the even and odd part of K with respect to y and b is a vector valued function given by

$$b(x) = \int_{B_1} K_o(x;y)y\,dy.$$

Notice that if the total kernel K is even the last two terms in (3.1) disappear. This was convenient in [8] as these bring additional difficulties with the scaling.

The second term can be considered as a drift term, in the sense that has a "direction", by K_o being odd. If the singularity of K_o at the origin is of order $n+\tau$, with $\tau \to 1^-$, then this integral becomes a gradient term. For this reason, one can consider studying the regularizing effect of the first two terms. The linear operators we are interested are always of the form,

$$Lu(x) = P.V. \int_{\mathbb{R}^n} (u(x+y) - u(x))K(x;y)dy, \tag{3.2}$$
$$:= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon} (u(x+y) - u(x))K(x;y)dy.$$

3.1.1 Integrability conditions.

We want here to make sense of the decomposition (3.1). All we need for that is that the kernels are not too singular whenever $u \in C^{1,1}(x_0)$. The following definition is the same as in [8].

Definition 3.1.1. *We say that a function u is $C^{1,1}$ at the point x_0 and write $u \in C^{1,1}(x_0)$ if and only if there exists a vector $v \in \mathbb{R}^n$ and a number $M > 0$ such that*

$$|u(x+y) - u(x) - v \cdot y| < M|y|^2 \textit{ for } |y| \textit{ small enough.}$$

This implies in particular that $|\delta_e(u, x_0; y)| = O(|y|^2)$ and $|\delta_o(u, x_0; y)| = O(|y|)$ as $|y|$ goes to zero.

With this notion at hand we ask for the kernel K, when decomposed in its symmetric and skew symmetric parts, $K = K_e + K_o$ respectively, to satisfy the following integrability conditions,

$$\int \frac{|y|^2}{|y|^2+1}|K_e(y)|dy < \infty, \tag{3.3}$$
$$\int \frac{|y|}{|y|+1}|K_o(y)|dy < \infty. \tag{3.4}$$

These conditions allow us to write rigorously

$$Lu(x) = \int \delta_e(u, x; y)K_e(y)dy + \int \delta_o(u, x; y)K_o(y)dy,$$

for $u \in C^{1,1}(x) \cap L^\infty(\mathbb{R}^n)$.

We say that a family $\mathcal{L}$ of linear operators satisfy the integrability conditions uniformly when the upper bounds in (3.3) and (3.4) can be taken independent of $L \in \mathcal{L}$.

3.1.2 Non linear, non local operators.

Before defining what will be for us a fully non linear non local operator we present some examples to keep in mind. They are constructed from the linear operators in (3.2).

$$\text{(Inf-sup type) } Iu(x) = \inf_\beta \sup_\alpha L_{\alpha,\beta} u(x), \tag{3.5}$$

$$\text{(Maximal) } \mathcal{M}^+_{\mathcal{L}} u(x) = \sup_{L \in \mathcal{L}} Lu(x), \tag{3.6}$$

$$\text{(Minimal) } \mathcal{M}^-_{\mathcal{L}} u(x) = \inf_{L \in \mathcal{L}} Lu(x). \tag{3.7}$$

Definition 3.1.2. *We say that I is a non local fully non linear operator if it satisfies the following:*

(i) If u is any bounded $C^{1,1}(x)$ function then $Iu(x)$ is well defined.

(ii) If $u \in C^{1,1}(\Omega)$ for some open set $\Omega \subseteq \mathbb{R}^n$, then Iu is a continuous function in Ω.

Our examples satisfy immediately (i) in the definition above. In order to have the continuity stated in (ii) we need to check a uniform integrability condition in the kernels.

Lemma 3.1.1. *Let I be of the form* (3.5) *where $\mathcal{L} = \{L_{\alpha,\beta}\}$ satisfy the integrability conditions* (3.3) *and* (3.4) *uniformly. Then $Iu \in C(\Omega)$ for every $u \in C^{1,1}(\Omega)$.*

Proof. We need to prove that $L_{\alpha,\beta}u$ are equicontinuous over compact sets of Ω in order to conclude by Arzela Ascoli's Theorem. Fix $\delta > 0$ and lets work over the points $x \in \Omega$ that are at least δ away from $\mathbb{R}^n \setminus \Omega$.

Let $L_{\alpha,\beta}$ has associated the kernels $K_{\alpha,\beta}(y) = (K_{\alpha,\beta})_e(y) + (K_{\alpha,\beta})_o(y)$, decomposed in its symmetric and skew symmetric parts. Because $u \in C^{1,1}(\Omega)$ we can write for $x \in \Omega$,

$$\begin{aligned} L_{\alpha,\beta}u(x) &= \int \delta_e(u,x;y)(K_{\alpha,\beta})_e(y)dy + \int \delta_o(u,x;y)(K_{\alpha,\beta})_o(y)dy, \\ &= \int_{B_r} \delta_e(u,x;y)(K_{\alpha,\beta})_e(y)dy + \int_{\mathbb{R}^n\setminus B_r} \delta_e(u,x;y)(K_{\alpha,\beta})_e(y)dy, \\ &+ \int_{B_r} \delta_o(u,x;y)(K_{\alpha,\beta})_o(y)dy + \int_{\mathbb{R}^n\setminus B_r} \delta_o(u,x;y)(K_{\alpha,\beta})_o(y)dy. \end{aligned}$$

The first and third integrals can be smaller than any $\varepsilon > 0$ if r is small enough. Use that u is $C^{1,1}$ to get that $|\delta_e(u,x;y)| \leq C|y|^2$ and $|\delta_o(u,x;y)| \leq C|y|$ if $r < \delta$ and for some constant C independent of x. By the integrability condition and the absolute continuity of the integral we get that, for even smaller radius r, the aforementioned terms are smaller than ε, independently of x and $L_{\alpha,\beta}$.

Now if we fix a radius r, we get that the second and fourth terms are equicontinuous in x. For this we just need to apply Lemma 4.1 in [8].

As a consequence of the previous two paragraphs, we obtain that the difference $|L_{\alpha,\beta}u(x) - L_{\alpha,\beta}u(x')|$ is arbitrarily small when $|x - x'|$ is sufficiently small, independently of x,x' (both at least δ away from $\mathbb{R}^n \setminus \Omega$) and $L_{\alpha,\beta}$. □

3.1.3 Extremal operators comparable to the fractional laplacian.

An important family, that will be used for the study of regularity, is given by $\mathcal{L}_0 = \mathcal{L}_0(\sigma, \tau, \lambda, \Lambda, b)$ with all the linear operators L such that the kernels $K_{e,o}$ are comparable to those the σ fractional Laplacian and some derivation of order τ.

$$(2-\sigma)\frac{\lambda}{|y|^{n+\sigma}} \leq K_e \leq (2-\sigma)\frac{\Lambda}{|y|^{n+\sigma}}, \tag{3.8}$$

$$|K_o| \leq (1-\tau)\frac{b}{|y|^{n+\tau}}. \tag{3.9}$$

In order to satisfy the integrability conditions all we need is $\sigma \in (0,2)$ and $\tau \in (0,1)$.

In this family the operators (3.6), (3.7) take the explicit form

$$\mathcal{M}^+_{\mathcal{L}_0} v(x) = \mathcal{M}^+_{\sigma} v(x) + b(1-\tau)\int_{\mathbb{R}^n} \frac{|\delta_o(v,x,y)|}{|y|^{n+\tau}}, \tag{3.10}$$

$$\mathcal{M}^-_{\mathcal{L}_0} v(x) = \mathcal{M}^-_{\sigma} v(x) - b(1-\tau)\int_{\mathbb{R}^n} \frac{|\delta_o(v,x,y)|}{|y|^{n+\tau}}, \tag{3.11}$$

where $\mathcal{M}^{\pm}_{\sigma}$ are the extremal operators found in [8], i.e.

$$\mathcal{M}^+_{\sigma} v(x) = (2-\sigma)\int_{\mathbb{R}^n} \frac{\Lambda\delta_e^+(v,x;y) - \lambda\delta_e^-(v,x;y)}{|y|^{n+\sigma}},$$

$$\mathcal{M}^-_{\sigma} v(x) = (2-\sigma)\int_{\mathbb{R}^n} \frac{\lambda\delta_e^+(v,x;y) - \Lambda\delta_e^-(v,x;y)}{|y|^{n+\sigma}}.$$

$\delta^{\pm}_{e,o}$ denote the positive and negative parts of $\delta_{e,o}$ respectively, ($\delta_{e,o} = \delta^+_{e,o} - \delta^-_{e,o}$).

For ease of notation we also introduce what we call the maximal τ derivative $|D_\tau|$, given by

$$|D_\tau|v(x) = (1-\tau)\int_{\mathbb{R}^n} \frac{|\delta_o(v,x;y)|}{|y|^{n+\tau}}dy,$$

so that we can rewrite the operators as

$$\mathcal{M}^{\pm}_{\mathcal{L}_0}v(x) = \mathcal{M}^{\pm}_{\sigma}v(x) \pm b|D_\tau|v(x).$$

The factors $(2-\sigma)$ and $(1-\tau)$ become important as $\sigma \to 2$, and $\tau \to 1$, as they will allow us to recover second order differential equations with gradient terms as limits of integro differential equations.

Notice that this family admits kernels that could be positive and negative. The natural assumption, due the positivity of the measure in the Lèvy-Khintchine formula, is to consider operators which are elliptic with respect to a family $\mathcal{L}$ with non negative kernels. Because of this reason we consider also the family $\tilde{\mathcal{L}}_0 \subseteq \mathcal{L}_0$, given by all possible operators L with total kernel $K = K_e + K_o \geq 0$ satisfying the conditions (3.8) and (3.9). We point out that given v smooth, we have the following natural inequalities,

$$\mathcal{M}^{+}_{\mathcal{L}_0}v(x) \geq \mathcal{M}^{+}_{\tilde{\mathcal{L}}_0}v(x) \geq \mathcal{M}^{-}_{\tilde{\mathcal{L}}_0}v(x) \geq \mathcal{M}^{-}_{\mathcal{L}_0}v(x).$$

This control will be useful, since we have explicit formulas for the maximal operators in the larger class $\mathcal{L}_0$.

3.1.4 Ellipticity.

The reason why we introduce extremal operators is because they are the ones that control *elliptic* non linear operators. Here is the definition of ellipticity for a general family $\mathcal{L}$ of linear operators.

Definition 3.1.3. *Let $\mathcal{L}$ be a class of linear integro differential operators satisfying* (3.3) *and* (3.4). *We say that a fully non linear operator I is elliptic with respect to the class $\mathcal{L}$ if*

$$\mathcal{M}^-_{\mathcal{L}}(u-v)(x) \le Iu(x) - Iv(x) \le \mathcal{M}^+_{\mathcal{L}}(u-v)(x). \tag{3.12}$$

3.1.5 Scaling.

A tool we will be using frequently is the scaling. Consider a smooth, bounded function u and a operator I, elliptic with respect to $\mathcal{L} \subseteq \mathcal{L}_0(\sigma, \tau, \lambda, \Lambda, b)$, such that

$$Iu = f \text{ in } \Omega.$$

If we rescale u by $u_{\alpha,\beta}(x) = \alpha u(\beta x)$ then the equation gets rescaled in the following way,

$$I_{\alpha,\beta} u_{\alpha,\beta} = f_{\alpha,\beta} \text{ in } \beta^{-1}\Omega,$$

where

$$(I_{\alpha,\beta} v)(x) = \alpha I(\alpha^{-1} v(\beta^{-1} \cdot))(\beta x),$$
$$f_{\alpha,\beta}(x) = \alpha f(\beta x).$$

In particular, if $I = L$ is linear with kernel K then the kernel $K_{\alpha,\beta}$ for $L_{\alpha,\beta}$ gets transformed according to the change of variables formula,

$$K_{\alpha,\beta}(x,y) = \beta^n K(\beta x, \beta y).$$

The extremal operators $\mathcal{M}_\sigma^\pm$ and $|D_\tau|$ scale with order σ and τ respectively, because by the change of variables formula,

$$\mathcal{M}_\sigma^\pm u_{\alpha,\beta}(x) = \alpha\beta^{-\sigma}(\mathcal{M}_\sigma^\pm u)(\beta x),$$
$$|D_\tau| u_{\alpha,\beta}(x) = \alpha\beta^{-\tau}(|D_\tau| u)(\beta x).$$

This implies that, going back to I non linear, the operator $I_{\alpha,\beta}$ belongs to some rescaled family of linear operator $\mathcal{L}_{\alpha,\beta} \subseteq \mathcal{L}_0(\sigma, \tau, \beta^{-\sigma}\lambda, \beta^{-\sigma}\Lambda, \beta^{-\tau}b)$.

At many points we will use that when $\sigma > \tau$ and β is small then the rescaled equation is dominated by the derivatives of order σ.

3.1.6 Viscosity solutions.

Viscosity solutions provide the right framework to study fully non linear equations, as seen in the local case in [5], and also in the non local case in [3].

Definition 3.1.4. *A bounded function $u : \mathbb{R}^n \to \mathbb{R}$, upper (lower) semicontinuous in $\bar{\Omega}$, is said to be a sub solution (super solution) to $Iu = f$, and we write $Iu \geq f$ ($Iu \leq f$), if every time φ is a second order polynomial touching u by above (below) at x in a neighborhood N, i.e.*

(i) $\varphi(x) = u(x)$,

(ii) $\varphi(y) > u(y)$ ($\varphi(y) < u(y)$) *for every* $x \in N \setminus \{x\}$,

then $Iv(x) \geq f(x)$ ($Iv(x) \leq f(x)$), *for v defined as*

$$v = \begin{cases} \varphi & \text{in } N, \\ u & \text{in } \mathbb{R}^n \setminus N. \end{cases}$$

Later on section 3.3 we will see that in many cases this definition is equivalent to one which includes many more test functions.

3.2 Statement of Results

In this section we state the main results obtained in this work. An important tool used to prove the following theorems is a point estimate, also known as L^ε Lemma. This comes from a partial ABP inequality similar to the one in [8] and a scaling argument which decreases the effect of the lower order term.

In order to prove our regularity results we will need to impose some assumptions on σ and τ. Given $\sigma_0, \tau_0, m, A_0 > 0$, considered as universal constants, we will assume that the following holds.

(H1) $2 > \sigma \geq \sigma_0 > 0, \min(1, \sigma) > \tau \geq \tau_0 > 0,$

(H2) $\sigma - \tau \geq m > 0,$

(H3) $\lambda A_0(2 - \sigma) \geq b(1 - \tau).$

Theorem 3.2.1. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that H1, H2 and H3 hold. Let u be a bounded function in $\mathbb{R}^n$ such that in B_1,*

$$\mathcal{M}^+_{\tilde{\mathcal{L}}_0} u \geq -C_0 \;\; \text{and} \;\; \mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leq C_0,$$

in the viscosity sense. Then there exists a universal exponent $\alpha > 0$ such that $u \in C^\alpha(B_{1/2})$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(\|u\|_\infty + C_0)$$

for some universal constant $C > 0$.

An immediate corollary is the following.

Corollary 3.2.2. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that H1, H2 and H3 hold. Let I be an elliptic operator of the inf-sup type as in* (3.5) *with all the linear operators in $\tilde{\mathcal{L}}_0$ and let $f \in C(\bar{B}_1)$. Let u be a bounded function in $\mathbb{R}^n$ such that in B_1,*

$$Iu = f,$$

in the viscosity sense. Then there exists a universal exponent $\alpha > 0$ such that $u \in C^\alpha(B_{1/2})$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leq C(\|u\|_\infty + \|f\|_\infty)$$

for some universal constant $C > 0$.

Coming back to Theorem 3.2.1, we would like to point out that our bounds remain uniform as $\sigma \to 2$ and $\tau \to 1$, which allows us to recover Hölder regularity for equations with bounded measurable coefficients including gradient terms. For fixed σ and τ these results were proven in [16] and [4] by using analytic techniques. These estimates are not uniform in σ and blow up as the order goes to the classical one.

The order α of our Hölder estimates deteriorates as $\tau \to \sigma$. In this critical case $\sigma = \tau$, both terms in the equation are of the same order and rescaling the equation does not have any effect on the τ derivative, hence our argument doesn't work. It is known from the previous work in [16] and [4] that the same result holds even when $\sigma = \tau$. By combining both results, we can get regularity uniformly in σ and τ, disregarding the separation between σ and τ (hypothesis H2).

To get higher regularity we will need to add an extra assumption to the kernels, which is a modulus of continuity of K_e and K_o in measure. More precisely, given $\rho_0 > 0$, we define the class $\mathcal{L}_1 = \mathcal{L}_1(\sigma, \tau, \lambda, \Lambda, b, \rho_0, C) \subseteq \tilde{\mathcal{L}}_0(\sigma, \tau, \lambda, \Lambda, b)$ such that it contains all the linear operators L with kernels $K = K_e + K_o \geq 0$ such that K_e and K_o satisfy (3.8) and (3.9) respectively and

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y-h)|}{|h|} dy \leq C \tag{3.13}$$

for every $|h| \leq \rho_0/2$.

A sufficient condition for (3.13) is for example that $|\nabla K(y)| \leq \Lambda/|y|^{n+1+\sigma}$.

In this smaller class we are able to get $C^{1,\alpha}$ by studying the incremental quotients of solutions and using the a priori C^α estimates given by Theorem 3.2.1. The proof follows the ideas of [5] and [8].

Theorem 3.2.3. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that H1, H2 and H3 holds. Let I be an elliptic operator of the inf-sup type as in* (3.5) *with all the linear operators in $\mathcal{L}_1$. There is $\rho_0 > 0$ small enough so that if u is a bounded function in $\mathbb{R}^n$ such that in B_1,*

$$Iu = 0,$$

in the viscosity sense. Then there is a universal $\alpha > 0$ such that $u \in C^{1,\alpha}(B_{1/2})$ and

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq C\|u\|_\infty$$

for some universal $C > 0$.

In the proofs of our regularity results the odd part doesn't have to be of a fixed order. We could ask for example

$$|K_o| \leq b \max\left(\frac{1-\tau_1}{|y|^{n+\tau_1}}, \frac{1-\tau_2}{|y|^{n+\tau_2}}\right)$$

with $0 < \tau_1 \leq \tau_2 < \min(1, \sigma)$. The reason is that the proofs will treat the lower order term as a perturbation term that can be made small enough after a dilation large enough. For the sake of keeping the exposition simpler we decided to restrict to the case of $\tau_1 = \tau_2 = \tau$.

3.3 Qualitative properties

This section is devoted to prove basic results that concern the definition of viscosity solution. First we take a look to the monotonicity properties which are inherited from assuming that the operator I is elliptic with respect to a family $\mathcal{L}$ with non negative kernels. Second we see how the set of test functions can be enlarged in the definition of viscosity solutions. We use these tools to prove the stability, comparison and maximum principle and existence of solutions for the Dirichlet problem.

3.3.1 Monotonicity.

Lemma 3.3.1 (Monotonicity)**.** *Let I be a elliptic with respect to a family $\mathcal{L}$ of linear operators with non negative kernels. Let u and v be two bounded functions in $C^{1,1}(x)$ such that $v \geq u$ and $v(x_0) = u(x_0)$, then*

$$Iv(x_0) \geq Iu(x_0).$$

Proof. By the ellipticity,

$$Iv(x_0) - Iu(x_0) \geq \mathcal{M}^-_{\mathcal{L}}(v-u)(x_0).$$

Let $w(y) = (v-u)(x_0+y)$ such that $w(y) \geq 0$ with equality for $y = 0$. Then for any $L \in \mathcal{L}$ with kernel $K \geq 0$ we have

$$Lw(0) = P.V. \int w(y)K(y) \geq 0.$$

By taking the infimum we get that $\mathcal{M}^-_{\mathcal{L}} w(x_0) \geq 0$ which concludes the proof. □

Lemma 3.3.2. *Let I be an elliptic operator with respect to a class $\mathcal{L}$ of non negative kernels. Let u, v be viscosity solutions of $Iu \leq f$, then $w = \min(u, v)$ is also a super solution.*

Proof. Let φ be a function touching w by below at x in N and assume without loss of generality that $w(x_0) = u(x_0)$. Then φ also touches u by below at x_0 in N and we use its equation. For

$$v = \begin{cases} \varphi & \text{in } N, \\ u & \text{in } \mathbb{R}^n \setminus N, \end{cases}$$

we have $Iv(x_0) \leq f(x_0)$.

Let $\tilde{v}$ be defined by

$$\tilde{v} = \begin{cases} \varphi & \text{in } N, \\ w & \text{in } \mathbb{R}^n \setminus N. \end{cases}$$

Then by the monotonicity Lemma 3.3.1 applied to v and $\tilde{v}$ at x_0 we get $I\tilde{v}(x_0) \leq Iv(x_0) \leq f(x_0)$ which concludes that $Iw \leq f$. □

3.3.2 A larger class of test functions.

Lemma 3.3.3. *Let I be elliptic with respect to a class $\mathcal{L}$ of non negative kernels satisfying* (3.3) *and* (3.4) *uniformly. Let $u : \mathbb{R}^n \to \mathbb{R}$ such that $Iu \geq f$ in the viscosity sense and φ touching u by above at x in a neighborhood N. Then $Iv(x) \geq f(x)$ for v defined as*

$$v = \begin{cases} \varphi & \text{in } N, \\ u & \text{in } \mathbb{R}^n \setminus N. \end{cases}$$

given that $\varphi \in C^{1,1}(x)$.

Proof. Fix p and q second order polynomials that touch φ, by below and above respectively, at x in $B_r(x) \subseteq N$. Let

$$w = \begin{cases} q & \text{in } B_r(x), \\ u & \text{in } \mathbb{R}^n \setminus B_r(x), \end{cases} \qquad v_r = \begin{cases} q & \text{in } B_r(x), \\ v & \text{in } \mathbb{R}^n \setminus B_r(x). \end{cases}$$

By the ellipticity

$$Iv(x) \geq Iv_r(x) + \mathcal{M}^-_{\mathcal{L}}(v - v_r)(x),$$

and thanks to the monotonicity Lemma 3.3.1 applied to $v_r \geq w$ we have $Iv_r(x) \geq Iw(x)$, so that

$$Iv(x) \geq Iw(x) + \mathcal{M}^-_{\mathcal{L}}(v - v_r)(x).$$

Note that $Iw(x) \geq f(x)$, so we only need to estimate the second term. Now, $v - v_r$ is supported in $B_r(x)$ and it is equal to $\varphi - q$ which is bounded by $-(q - p)$ and zero.

For $L \in \mathcal{L}$ with kernel K,

$$\begin{aligned} L(v - v_r)(x) &= \int_{B_r} \delta_e((v - v_r), x; y) K_e(y) dy + \int_{B_r} \delta_o((v - v_r), x; y) K_o(y) dy \\ &\geq -C \left\{ \int_{B_r} |y|^2 K_e(y) dy + \int_{B_r} |y| K_o(y) dy \right\} \\ &\geq -C\varepsilon, \end{aligned}$$

for $\varepsilon > 0$ arbitrarily small if $r = r(\varepsilon)$ is small enough (independent of $L \in \mathcal{L}$). By taking the infimum above among every $L \in \mathcal{L}$ we get that $Iv(x) \geq f(x) - C\varepsilon$ and we just need to take $\varepsilon \to 0$ to conclude. □

Next we have an even stronger result, that tells us that we can compute I classically every time we have a $\varphi \in C^{1,1}(x)$ touching by below.

Lemma 3.3.4. *Let I be an elliptic operator of the inf-sup (sup-inf) type as in* (3.5) *with all the linear operators in $\tilde{\mathcal{L}}_0$ satisfying H1. Let $u : \mathbb{R}^n \to \mathbb{R}$ such that $Iu \leq f$ in the viscosity sense and φ touching u by below at x in a neighborhood N. Then $Iu(x)$ is defined in the classical sense and we have $Iu(x) \leq f(x)$ given that $\varphi \in C^{1,1}(x)$.*

To prove Lemma 3.3.4 we need an interpolation result that will allow us to replace the τ derivative by the σ derivative and a residue term evaluated at the test function φ. This result is also useful when the function touching by below is the convex envelope as $\delta_e^-(\varphi) = 0$.

Lemma 3.3.5. *Let $u : \mathbb{R}^n \to \mathbb{R}$, $x \in \mathbb{R}^n$, $2 > \sigma > \tau > 0$ and $r_0 > 0$ such that the following integrals are finite,*

$$\int_{B_{r_0}} \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} dy \ \ \text{and} \ \ \int_{B_{r_0}} \frac{|\delta_o(u, x; y)|}{|y|^{n+\tau}} dy.$$

Let φ be a function defined in $B_{r_0}(x)$ and touching u by below at x. Then

$$\int_{B_{r_0}} \lambda(2-\sigma)\frac{\delta_e^+(u,x;y)}{|y|^{n+\sigma}} - b(1-\tau)\frac{|\delta_o(u,x;y)|}{|y|^{n+\tau}}dy \geq$$
$$\int_{B_{r_0}} \alpha\lambda(2-\sigma)\frac{\delta_e^+(u,x;y)}{|y|^{n+\sigma}} - b(1-\tau)\frac{\delta_e^-(\varphi,x;y)+|\delta_o(\varphi,x;y)|}{|y|^{n+\tau}}dy,$$

for $\alpha \in (0,1)$ given that

$$r_0 \leq \left(\frac{(1-\alpha)\lambda(2-\sigma)}{b(1-\tau)}\right)^{1/(\sigma-\tau)}. \tag{3.14}$$

Proof. Since φ touches u by below, we have that for every $y \in B_r$,

$$\begin{aligned}\delta_e^+(u-\varphi,x;y) &= (u-\varphi)(x+y)+(u-\varphi)(x-y),\\ &\geq |(u-\varphi)(x+y)-(u-\varphi)(x-y)|,\\ &= |\delta_o(u-\varphi,x;y)|,\end{aligned}$$

and also,

$$\begin{aligned}\delta_e^+(u,x;y) &\geq \delta_e^+(u-\varphi,x;y)-\delta_e^-(\varphi,x;y),\\ |\delta_o(u-\varphi,x;y)| &\geq |\delta_o(u,x;y)|-|\delta_o(\varphi,x;y)|,\end{aligned}$$

so that

$$\delta_e^+(u,x;y)-|\delta_o(u,x;y)| \geq -\delta_e^-(\varphi,x;y)-|\delta_o(\varphi,x;y)|.$$

Now we can replace $|\delta_o|$ by δ_e^+ in the integral,

$$\int_{B_{r_0}} \lambda(2-\sigma)\frac{\delta_e^+(u)}{|y|^{n+\sigma}} - b(1-\tau)\frac{|\delta_o(u)|}{|y|^{n+\tau}}dy \geq$$
$$\int_{B_{r_0}} \delta_e^+(u)\left\{\frac{\lambda(2-\sigma)}{|y|^{n+\sigma}} - \frac{b(1-\tau)}{|y|^{n+\tau}}\right\} - b(1-\tau)\frac{\delta_e^-(\varphi)+|\delta_o(\varphi)|}{|y|^{n+\tau}}dy.$$

By using that

$$r_0 \leq \left(\frac{(1-\alpha)\lambda(2-\sigma)}{b(1-\tau)} \right)^{1/(\sigma-\tau)},$$

and that $\sigma > \tau$ we can substitute the difference of the fractions by α times $|y|^{-(n+\sigma)}$,

$$\int_{B_{r_0}} \delta_e^+(u) \left\{ \frac{\lambda(2-\sigma)}{|y|^{n+\sigma}} - \frac{b(1-\tau)}{|y|^{n+\tau}} \right\} dy \geq \alpha\lambda(2-\sigma) \int_{B_{r_0}} \frac{\delta_e^+(u)}{|y|^{n+\sigma}}.$$

□

Proof of Lemma 3.3.4. We check first that Lu can be computed in the classical sense at x. Because u is bounded we only care about the convergence of the integrals around the origin.

Let φ be defined in $B_{r_0}(x)$ and for $r \leq r_0$

$$v_r(y) = \begin{cases} u \text{ in } B_r(x), \\ \varphi \text{ in } \mathbb{R}^n \setminus B_r. \end{cases}$$

The differences $\delta_e^-(v_r, x; y)$, parametrized by r, decrease to $\delta_e^-(u, x; y)$ as r goes to zero. Since

$$\int_{B_{r_0}} \frac{\delta_e^-(v_{r_0}, x; y)}{|y|^{n+\sigma}} dy < \infty,$$

we have by monotone convergence that

$$\int_{B_{r_0}} \frac{\delta_e^-(u, x; y)}{|y|^{n+\sigma}} dy < \infty.$$

By using $f(x) \geq \mathcal{M}^-_{\tilde{\mathcal{L}}_0} v_r(x)$, which implies $f(x) \geq \mathcal{M}^-_{\mathcal{L}_0} v_r(x)$, and the boundedness of u,

$$M \geq \int_{B_{r_0}} \lambda(2-\sigma) \frac{\delta_e^+(v_r, x; y)}{|y|^{n+\sigma}} - b(1-\tau) \frac{|\delta_o(v_r, x; y)|}{|y|^{n+\tau}} dy$$

for some M independent of r. Use now Lemma 3.3.5 to keep only the term with $\delta_e^+(v_r)$, this requires r_0 sufficiently small,

$$M + b(1-\tau)\int_{B_{r_0}} \frac{|\delta_o(\varphi, x; y)| + \delta_e^-(\varphi, x; y)}{|y|^{n+\tau}} dy \geq$$
$$\frac{\lambda(2-\sigma)}{2}\int_{B_{r_0}} \frac{\delta_e^+(v_r, x; y)}{|y|^{n+\sigma}} dy.$$

The left hand side above is finite and independent of r. By Fatou's Lemma,

$$\int_{B_{r_0}} \frac{\delta_e^+(u, x; y)}{|y|^{n+\sigma}} dy < \infty.$$

Now recall from the proof of Lemma 3.3.5 the identity

$$|\delta_o(v_r, x; y)| \leq |\delta_o(\varphi, x; y)| + \delta_e^+(v_r, x; y) + \delta_e^-(\varphi, x; y).$$

The last two terms are integrable against $|y|^{-(n+\sigma)}$ around the origin and therefore they are also integrable against $|y|^{-(n+\tau)}$ around the origin as well as the whole right hand side. Moreover the integral can be bounded by above independently of r. By Fatou's Lemma we then get that $|\delta_o(u, x; y)|$ is integrable against $|y|^{-(n+\tau)}$ in B_{r_0}.

We have shown that each term $\delta_e(u, x; y)/|y|^{n+\sigma}$ and $|\delta_o(u, x; y)|/|y|^{n+\tau}$ is integrable, then for every linear operator $L_{\alpha,\beta}u(x)$ is well defined. Therefore $Iu(x)$ can be computed by being an inf-sup combination of $L_{\alpha,\beta}$. To see that $Iu(x) \leq f(x)$ we use the ellipticity,

$$\begin{aligned} Iu(x) &\leq Iv_r(x) + \mathcal{M}^+_{\mathcal{L}}(u - v_r) \\ &\leq f(x) + \mathcal{M}^+_{\sigma}(u - v_r)(x) + b|D_\tau|(u - v_r)(x). \end{aligned}$$

Both integrals go to zero by absolute continuity. □

3.3.3 Stability.

We are interested in studying limit of sub or super solutions. To state the result we need first to recall the definition of Γ convergence.

Definition 3.3.1. *We say that a sequence of lower semicontinuous functions u_k Γ-converge to u in a set Ω if the two following conditions hold*

(i) For every sequence $x_k \to x$ in Ω, $\liminf_{k\to\infty} u_k(x_k) \geq u(x)$.

(ii) For every $x \in \Omega$, there is a sequence $x_k \to x$ in Ω such that

$$\limsup_{k\to\infty} u_k(x_k) = u(x).$$

Lemma 3.3.6. *Let I be an elliptic operator with respect to a class $\mathcal{L}$ with non negative kernels and satisfying the integrability conditions* (3.3) *and* (3.4) *uniformly. Let u_k be a sequence of functions that are uniformly bounded in $\mathbb{R}^n$ and lower semicontinuous in $\Omega \subseteq \mathbb{R}^n$ such that*

(i) $Iu_k \leq f_k$ in Ω

(ii) $u_k \to u$ in the Γ sense in Ω,

(iii) $u_k \to u$ a.e. in $\mathbb{R}^n$ and

(iv) $f_k \to f$ locally uniformly un Ω for some continuous function f.

Then $Iu \leq f$ in Ω.

Proof. Let φ be a test function touching u by below at x in N. Because $u_k - \varphi$ Γ-converges to $(u-\varphi)$ there exists a sequence $x_k \to x$ such that,

$$(u_k - \varphi)(x_k) = \inf_N (u_k - \varphi) = d_k.$$

Therefore $\varphi + d_k$ touches u_k at x_k in N, starting at some k sufficiently large.

Let

$$v_k = \begin{cases} \varphi + d_k & \text{in } B_r(x), \\ u_k & \text{in } \mathbb{R}^n \setminus B_r(x), \end{cases} \qquad v = \begin{cases} \varphi & \text{in } B_r(x), \\ u & \text{in } \mathbb{R}^n \setminus B_r(x). \end{cases}$$

By using the equation we know that $Iv_k(x_k) \leq f_k(x_k)$.

For $z \in B_{r/2}(x)$ we have by ellipticity,

$$\begin{aligned} |Iv_k(z) - Iv(z)| &\leq \max(|\mathcal{M}^+_{\mathcal{L}}(v_k - v)|, |\mathcal{M}^+_{\mathcal{L}}(v_k - v)|), \\ &\leq \sup_{L \in \mathcal{L}} |L(v_k - v)(z)|. \end{aligned}$$

For a given $L \in \mathcal{L}$ with kernel K we have,

$$|L(v_k - v)(z)| \leq \int_{\mathbb{R}^n \setminus B_{r/2}} |(v_k - v)(x+y)| K(y) dy.$$

The integrand goes to zero a.e. when $k \to \infty$ and it is dominated by

$$(\|v_k\|_\infty + \|v\|_\infty) K(y) \chi_{\mathbb{R}^n \setminus B_{r/2}} \in L^1.$$

Then by dominated convergence $|L(v_k - v)(z)| \to 0$ as $k \to \infty$ uniformly in $z \in B_{r/2}(x)$ and $L \in \mathcal{L}$. This implies that $|Iv_k(z) - Iv(z)|$ also goes to zero uniformly in $z \in B_{r/2}(x)$.

Finally, using that Iv is continuous in $B_{r/2}(x)$,

$$|Iv_k(x_k) - Iv(x)| \leq |Iv_k(x_k) - Iv(x_k)| + |Iv(x_k) - Iv(x)| \to 0.$$

Finally we have

$$\begin{aligned} Iv(x) &\leq Iv_k(x_k) + |Iv_k(x_k) - Iv(x)|, \\ &\leq f(x_k) + |Iv_k(x_k) - Iv(x)|, \\ &\leq f(x) + |Iv_k(x_k) - Iv(x)| + |f_k(x_k) - f(x)|. \end{aligned}$$

Take $k \to \infty$ and use also that $f_k \to f$ locally uniformly to conclude. □

3.3.4 Comparison and maximum principle for viscosity solutions.

Lemma 3.3.7 says that the difference of two viscosity solutions is the solution of an equation in the same ellipticity class. Theorem 3.3.10 is the comparison principle which implies in particular the maximum principle for sub solution. Instead of having to prove an ABP type result, as it is used in chapter 5 of [5], we take advantage of Lemma 3.3.4 in order to evaluate the operators in the classical sense whenever is needed.

Lemma 3.3.7. *Let I be an elliptic operator of the inf-sup type as in* (3.5) *with all the linear operators in $\tilde{\mathcal{L}}_0$ satisfying H1. Let u and v two bounded functions such that $Iu \geq f$ and $Iv \leq g$ in the viscosity sense in Ω. Then $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u - v) \geq f - g$ in the viscosity sense in Ω.*

The proof is straightforward when either u or v is smooth because of the non negativity of the kernels. In the general case we proceed by regularizing the functions by their inf or sup convolutions.

Definition 3.3.2. *Given an lower (upper) semi continuous function u and a parameter $\varepsilon > 0$ the inf (sup) convolution u_ε (u^ε) is given by*

$$u_\varepsilon(x) = \inf_y u(x+y) + \frac{|y|^2}{\varepsilon} \quad \left(u^\varepsilon(x) = \sup_y u(x+y) - \frac{|y|^2}{\varepsilon} \right).$$

The proof of the following property con be found for instance in the beginning of chapter 5 in [5].

Lemma 3.3.8. *If u is bounded and lower semicontinuous in $\mathbb{R}^n$ then u_ε Γ-converges to u.*

Lemma 3.3.9. *If f is a continuous function and I is elliptic with respect to a class $\mathcal{L}$ with non negative kernels. Then if $Iu \leq f$ in the viscosity sense, $Iu_\varepsilon \leq f - d_\varepsilon$ also in the viscosity sense, where $d_\varepsilon \to 0$ as $\varepsilon \to 0$ depending only on the modulus of continuity ρ of f.*

Proof. Let φ be a test function that touches u_ε by below at x in N.

For ε sufficiently small, there is some $(x+h) \in N$ such that $u_\varepsilon(y) \leq u(y+h) + |h|^2/\varepsilon$ with equality at $y = x$. (See the beginning of chapter 5 in [5]).

Then $\varphi - |h|^2/\varepsilon$ touches u at $x+h$ in N and $Iv(x+h) \leq f(x+h) \leq f(x) + \rho(|h|)$ for

$$v = \begin{cases} \varphi - \frac{|h|^2}{\varepsilon} & \text{in } B_{r/2}(x+h), \\ u & \text{in } \mathbb{R}^n \setminus B_{r/2}(x+h). \end{cases}$$

By ellipticity the value of Iv does not change by adding a constant, $I(v + |h|^2/\varepsilon)(x) \leq f(x) + \rho(|h|)$. Then by the monotonicity Lemma 3.3.1 we also have that $Iw(x) \leq f(x) + \rho(|h|)$ for

$$w = \begin{cases} \varphi & \text{in } N, \\ u_\varepsilon & \text{in } \mathbb{R}^n \setminus N, \end{cases}$$

because as we already noticed $u_\varepsilon(y) \leq u(y+h)+|h|^2/\varepsilon$. This concludes the proof. □

Proof of Lemma 3.3.7. Assume first that u is upper semicontinuous in $\mathbb{R}^n$ and v is lower semicontinuous in $\mathbb{R}^n$.

Thanks to Lemma 3.3.9, we have that $Iu^\varepsilon \geq f - d_\varepsilon$ and $Iv_\varepsilon \leq f + d_\varepsilon$ with $-u^\varepsilon \to -u$ and $v_\varepsilon \to v$ in the Γ-sense and $d_\varepsilon \to 0$. By the stability of viscosity solutions, Lemma 3.3.6, we just need to prove that $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u^\varepsilon - v_\varepsilon) \geq f - g - 2d_\varepsilon$ in Ω in the viscosity sense.

Let φ be a test function touching $u^\varepsilon - v_\varepsilon$ from above at a point x. For any $\varepsilon > 0$, u^ε, $-v_\varepsilon$ and $u^\varepsilon - v_\varepsilon$ are semiconvex functions, hence there is a paraboloid for each of them touching then from below at x. If $\varphi \in C^{1,1}(x)$ then both u^ε and v_ε are also $C^{1,1}(x)$. By Lemma 3.3.4 we can evaluate $Iu^\varepsilon(x)$, $Iv_\varepsilon(x)$ and $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u^\varepsilon - v_\varepsilon)(x)$ in the classical sense and they satisfy

$$\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u^\varepsilon - v_\varepsilon)(x) \geq Iu^\varepsilon(x) - Iv_\varepsilon(x) \geq f(x) - g(x) - 2d_\varepsilon.$$

Since φ touches $u^\varepsilon - v_\varepsilon$ from above at x

$$\mathcal{M}^+_{\tilde{\mathcal{L}}_0}\varphi(x) \geq f(x) - g(x) - 2d_\varepsilon.$$

This says that $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(u^\varepsilon - v_\varepsilon) \geq f - g - 2d_\varepsilon$ in the viscosity sense and completes the proof under the semicontinuity assumptions in $\mathbb{R}^n$.

Now we will not assume the lower and upper semincontinuity outside of $\bar{\Omega}$. There are sequences u_k and v_k, upper and lower semicontinuous respectively such that

(i) $u_k = u$ and $v_k = v$ in $\bar{\Omega}$ for every k,

(ii) $u_k \to u$ and $v_k \to v$ a.e. in $\mathbb{R}^n \setminus \bar{\Omega}$,

(iii) $Iu_k \geq f_k$ and $Iv_k \leq g_k$, with $f_k \to f$, $g_k \to g$ locally uniformly in Ω.

By having such sequences we just have to apply the first part of this proof and the stability, Lemma 3.3.6, to conclude the proof.

We can construct the sequences satisfying the first two items above by doing a standard mollification of u and v away from Ω and then filling the gap in a semi-continuous way. The function $u_k - u$ vanishes in Ω, hence $\mathcal{M}^-_{\tilde{\mathcal{L}}_0}(u_k - u)$ is defined in the classical sense in Ω and

$$\begin{aligned}\mathcal{M}^-_{\tilde{\mathcal{L}}_0}(u_k - u)(x) &\geq -\int_{\mathbb{R}^n \setminus B_{dist(x,\partial\Omega)}} |u_k(x+y) - u(x)| K(y)dy \\ &= h_k(x),\end{aligned}$$

where $K = \Lambda\frac{2-\sigma}{|y|^{n+\sigma}} + b\frac{1-\tau}{|y|^{n+\tau}}$.

The functions $h_k(x)$ are continuous in Ω and by dominated convergence $h_k \to 0$ locally uniformly in Ω as $k \to 0$. Let $\varphi \in C^{1,1}(x)$ touching u_k from above at x in N and v_k defined by

$$v_k = \begin{cases} \varphi & \text{in } N, \\ u_k & \text{in } \mathbb{R}^n \setminus N. \end{cases}$$

The functions $v_k + u - u_k$ are also in $C^{1,1}(x)$ and touch u by above at x. By Lemma 3.3.3 we have that $I(v_k + u - u_k)(x) \geq f(x)$. By ellipticity

$$
\begin{aligned}
Iv_k(x) &\geq I(v_k + u - u_k)(x) + \mathcal{M}^{-}_{\tilde{\mathcal{L}}_0}(u - u_k)(x), \\
&\geq f(x) + h_k(x).
\end{aligned}
$$

So we have that (iii) above is also satisfied. □

Theorem 3.3.10 (Comparison Principle)**.** *Let I be an elliptic operator of the inf-sup type as in* (3.5) *with all the linear operators in $\tilde{\mathcal{L}}_0$ satisfying H1. Let Ω be a bounded open set and u, v two bounded functions such that*

(i) $Iu \geq f$ and $Iv \leq f$ in Ω in the viscosity sense for some $f \in C(\Omega)$,

(ii) $u \leq v$ in $\mathbb{R}^n \setminus \Omega$.

Then $u \leq v$ in Ω.

Here, as in [8], the proof is also based on using a barrier function as

$$\varphi(x) = \min(1, |x|^2/4).$$

Lemma 3.3.11. *Let $s \in (0, 1)$ and $\varphi_s = \varphi(sx)$ for φ defined above. There exists $\delta > 0$ and some s small enough such that,*

$$\mathcal{M}^{-}_{\mathcal{L}_0}\varphi_s \geq \delta \text{ in } B_1.$$

Proof. First take $s = 1$ to get that

$$\mathcal{M}^{-}_{\sigma}\varphi \geq \delta_1 \text{ in } B_1.$$

This inequality comes from the fact that $\delta_e(\varphi, x; y) \geq \delta_2$ for $x \in B_1$ and every y. In fact, if $x \pm y$ are both in B_2 or both outside B_2 it is immediate. If only $x + y$ is in B_2 then we use that, $\varphi(x) \leq 1/4$ for $x \in B_1$,

$$\delta_e(\varphi, x; y) = 1 + \varphi(x + y) - 2\varphi(x) \geq 1/2.$$

On the other hand, since φ is smooth we have that $|D_\tau|\varphi \leq \delta_3$ in B_1, for some finite $\delta_3 > 0$. Now recall scaling properties from Section 3.1. We have that

$$\begin{aligned}
\mathcal{M}^-_\sigma \varphi_s(x) &= s^{-\sigma}(\mathcal{M}^-_\sigma \varphi)(sx) \geq s^{-\sigma}\delta_1, \\
|D_\tau|\varphi_s(x) &= s^{-\tau}(|D_\tau|\varphi)(sx) \leq s^{-\tau}\delta_3.
\end{aligned}$$

Which implies

$$\begin{aligned}
\mathcal{M}^-_{\mathcal{L}_0}\varphi_s &\geq s^{-\sigma}(\delta_1 - s^{\tau-\sigma}\delta_3), \\
&\geq \delta.
\end{aligned}$$

For s small enough. □

Proof of Theorem 3.3.10. By Lemma 3.3.7 we know that for $w = u - v$, $\mathcal{M}^+_{\tilde{\mathcal{L}}_0} w \geq 0$ in the viscosity sense in Ω. We will proof from here that $\sup_\Omega w \leq \sup_{\mathbb{R}^n \setminus \Omega} w := M$.

Let $\Omega \subseteq B_R$ ($R \geq 1$) and take $\psi(x) = \varphi_s(x/R)$ for φ_s as in the previous lemma. Since $\tilde{\mathcal{L}}_0 \subseteq \mathcal{L}_0$,

$$\begin{aligned}
\mathcal{M}^-_{\tilde{\mathcal{L}}_0}\psi &\geq \mathcal{M}^-_{\mathcal{L}_0}\psi, \\
&\geq R^{-\sigma}\left(\mathcal{M}^-_{\mathcal{L}_0}\varphi_s\right)(x/R), \\
&\geq R^{-\sigma}\delta,
\end{aligned}$$

in Ω. Fix $\varepsilon > 0$ and consider

$$\psi_\varepsilon(x) = M + \varepsilon(1 - \psi(x)),$$

which satisfies $\mathcal{M}^+_{\tilde{\mathcal{L}}_0} \psi_\varepsilon \leq -\varepsilon R^{-\sigma}\delta < 0$ in Ω.

If $\inf(\psi_\varepsilon - w) < 0$ then there is some translation $\psi_\varepsilon + d$ such that $\psi_\varepsilon + d$ touches w by above in $x \in \Omega$. This can not happen because of Lemma 3.3.3 which says that in that case $\mathcal{M}^+_{\tilde{\mathcal{L}}_0}(\psi_\varepsilon + d)(x) = \mathcal{M}^+_{\tilde{\mathcal{L}}_0}\psi_\varepsilon(x) \geq 0$. Therefore $\psi_\varepsilon \geq w$ and by letting $\varepsilon \to 0$ we get to the conclusion of the Theorem. □

3.3.5 Existence of solutions for the Dirichlet problem.

Theorem 3.3.12. *Let I be an elliptic operator of the inf-sup type as in* (3.5) *with all the linear operators in $\tilde{\mathcal{L}}_0$ satisfying H1. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set satisfying the exterior ball condition. Let $g : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ be a function which is globally bounded and continuous on $\partial\Omega$. Then there exist a viscosity solution $u \in C(\bar{\Omega})$ of*

$$Iu(x) = 0, \ \textit{in } \Omega,$$
$$u = g, \ \textit{in } \mathbb{R}^n \setminus \Omega.$$

The proof is based on the Perron's method. The first two lemmas account to the construction of a solution and the third one regards with achieving the boundary data.

Lemma 3.3.13. *Let I be an elliptic operator with respect to a class $\mathcal{L}$ with non negative kernels and satisfying the integrability conditions* (3.3) *and* (3.4) *uniformly.*

Let S a set of viscosity solutions of $Iv \geq 0$ in Ω. Then $\bar{u}$, the upper semicontinuous envelope in Ω of the function u defined by

$$u(x) = \sup_{v \in S} v(x),$$

also satisfies $I\bar{u} \geq 0$ in Ω in the viscosity sense.

Proof. Let φ be a test function touching $\bar{u}$ by above at $x \in \Omega$ in a neighborhood N.

The fact that $\bar{u}$ is defined as the upper semicontinuous envelope of $u(y) = sup_{v \in S} v(y)$ implies that for our given x there exist a sequence $\{(x_k, v_k)\} \subseteq (N \cap \Omega) \times S$ such that

(i) $(x_k, v(x_k)) \to (x, \bar{u}(x))$ as $k \to \infty$,

(ii) For any $y_k \to x$ we have that $\liminf_{k \to \infty} v_k(y_k) \geq \bar{u}(x)$.

These are the two sufficient conditions to prove the stability Lemma 3.3.6. The same proof applies here to show that $I\varphi(x) \geq 0$ and then conclude that $Iu \geq 0$ in the viscosity sense. □

Lemma 3.3.14. *Let I be an elliptic operator with respect to a class $\mathcal{L}$ with non negative kernels and satisfying the integrability conditions* (3.3) *and* (3.4) *uniformly. Let u be a viscosity subsolution of $Iu \geq 0$ in Ω such that $\underline{u}$, its lower semicontinuous envelope, it is not a viscosity supersolution of $Iu \leq 0$. Then there is function U such that*

(i) U is a viscosity subsolution of $IU \geq 0$ in Ω.

(ii) $U = u$ *in* $\mathbb{R}^n \setminus \Omega$.

(iii) $\sup_{x\in\Omega}(U - u)(x) > 0$.

Proof. Let φ be a test function touching $\underline{u}$ by below in $x_0 \in B_{r_0}(x_0) \subseteq \Omega$ such that $Iv(x_0) > 0$ for,

$$v = \begin{cases} \varphi & \text{in } B_{r_0}(x_0), \\ \underline{u} & \text{in } \mathbb{R}^n \setminus B_{r_0}(x_0). \end{cases}$$

By continuity we also have that $Iv > \delta > 0$ in $B_{r_1}(x_0) \subseteq B_{r_0/2}(x_0)$.

Let $\varepsilon_1, \varepsilon_2 > 0$ to be fixed and

$$\psi(y) = \varphi(y) - \varepsilon_1|y - x_0|^2 + \varepsilon_2.$$

We have that $\psi \leq u$ in $\mathbb{R}^n \setminus B_{r_1}(x_0) \subseteq \mathbb{R}^n \setminus \Omega$ if $\varepsilon_2 < r_1^2$. We want to choose ε_1 and ε_2 such that $U = \min(\psi, u)$ satisfies $IU \geq 0$.

Let η be a test function touching U by above at $x_1 \in \Omega$ in a neighborhood N. If $U(x_1) = u(x_1)$ then the inequality follows from the Lemma 3.3.3. If $U(x_1) = \psi(x_1) > u(x_1)$ then necessarily $x_1 \in B_{r_1}(x_0)$. By the lower semicontinuity, $\psi > u$ in some open neighborhood around x_1 and contained in $B_{r_1}(x_0)$. Because ψ is smooth, $IU(x_0)$ is classically defined and we just have to check that it is non negative.

Let

$$w = \begin{cases} \psi & \text{in } B_{r_0}(x_0), \\ \underline{u} & \text{in } \mathbb{R}^n \setminus B_{r_0}(x_0). \end{cases}$$

By monotonicity and ellipticity,

$$
\begin{aligned}
IU(x_1) &\geq Iw(x_1), \\
&\geq Iv(x_1) + \mathcal{M}^-_{\mathcal{L}}(w - v)(x_1), \\
&\geq \delta + \inf_{L \in \mathcal{L}} L(w - v)(x_1).
\end{aligned}
$$

Notice that $w - v = \varepsilon_2 - \varepsilon_1 |y - x_0|^2$ in $B_{r_0}(x_0)$ and it is zero outside. Recall that $x_1 \in B_{r_0/2}(x_0)$, so for any $L \in \mathcal{L}$ with kernel K,

$$
\begin{aligned}
L(w - v)(x_1) &= \int \delta(w - v, x_1; y) K(y) dy, \\
&\geq -\varepsilon_1 \int_{B_{2r_0}} \delta(|\cdot - x_0|^2, x_1; y) K(y) dy, \\
&\quad - \min\left(\varepsilon_1 r_0^2, \varepsilon_2\right) \int_{\mathbb{R}^n \setminus B_{r_0/2}} K(y) dy.
\end{aligned}
$$

The second term in the inequality appears since

$$
\delta(w - v, x_1, y) \geq - \min\left(\varepsilon_1 r_0^2, \varepsilon_2\right)
$$

for any x_1 in $B_{r_0/2}(x_0)$ and $|y| \geq B_{r_0/2}$. Therefore,

$$
L(w - v)(x_1) \geq -C\left(\varepsilon_1 + \min(\varepsilon_1 r_0^2, \varepsilon_2)\right).
$$

Then we choose $\varepsilon_2 = r_1^2/2$ and ε_1 sufficiently small to make $L(w - v)(x) \geq -\delta/2$ uniformly in $L \in \mathcal{L}$ and $x \in B_{r_1}(x_0)$. This finally implies that $IU \geq 0$ and concludes the prove of the lemma. □

Lemma 3.3.15. *Let $\varphi(x) = \min(1, C(|x| - 1)_+^{\alpha})$, where C and α has been chosen as in [9]. Then for any pair σ, τ satisfying H1 we have*

$$
\mathcal{M}^+_{\tilde{\mathcal{L}}_0} \varphi(x) \leq 0, \quad x \in \mathbb{R}^n \setminus B_1.
$$

Moreover,

$$\mathcal{M}^+_{\tilde{\mathcal{L}}_0}\varphi(x) \leq -\delta < 0, \quad x \in B_2 \setminus B_1.$$

Proof. Let r_0 and α the radius and exponent from Lemma 3.1 in [9]. We know that $\mathcal{M}^+_\sigma v(x_0) = -d$ $(d > 0)$ and $|D_\tau|v(x_0) = e < \infty$ for every $x_0 \in \partial B_{1+r_0}$.

Let $s \in (0, 1)$ and rescale v by

$$v_s(x) = s^{-\alpha} v(sx) = (|x| - s^{-1})^\alpha_+.$$

Recall the scaling remarks on section 3.1. For $L \in \tilde{\mathcal{L}}_0(\sigma, \tau, \lambda, \Lambda, b)$ with kernel K we consider $L_s \in \tilde{\mathcal{L}}_0(\sigma, \tau, s^{-\sigma}\lambda, s^{-\sigma}\Lambda, s^{-\tau}b)$ with kernel $K_s(y) = s^n K(sy)$ such that

$$L_s v_s(x) = s^{-\alpha}(Lv)(sx).$$

Let x such that $(1 + r_0)x = (1 + sr_0)x_0$ and translate v_s such that it remains below v but touches it in a whole ray passing through x and x_0. We still denote the translation v_s. By the scaling,

$$Lv(x) = s^\alpha (L_s v_s)(x_0) \leq (L_s v_s)(x_0)$$

By the monotonicity lemma 3.3.1 applied to v and v_s at x_0,

$$L_s v_s(x_0) \leq L_s v(x_0).$$

Since $L_s \in \tilde{\mathcal{L}}_0(\sigma, \tau, s^{-\sigma}\lambda, s^{-\sigma}\Lambda, s^{-\tau}b) \subseteq \mathcal{L}_0(\sigma, \tau, s^{-\sigma}\lambda, s^{-\sigma}\Lambda, s^{-\tau}b)$,

$$\begin{aligned} L_s v(x_0) &\leq s^{-\sigma}\mathcal{M}^+_\sigma v(x_0) + s^{-\tau} b|D_t|v(x_0), \\ &\leq s^{-\sigma}\left\{-d + s^{\sigma-\tau} be\right\}, \\ &\leq -d/2, \end{aligned}$$

if s is small enough. By transitivity $Lv(x) \leq -d/2$ for any x with $|x| = 1 + sr_0$ with $s \in (0, s_0)$. Finally we just multiply v by a constant C big enough such that $Cv(s_0 r_0) \geq 1$ and use Lemma 3.3.2 to conclude the lemma for the truncation of v. □

Proof of Theorem 3.3.12. Let S be the set of all viscosity subsolutions of $Iv \geq 0$ with boundary data smaller than g,

$$S = \{v \in USC(\Omega) \cap L^\infty(\mathbb{R}^n) : Iv \geq 0 \text{ in viscosity in } \Omega$$
$$\text{and } v \leq g \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

The set S is non empty because the constant function $u = -\|g\|_\infty$ satisfies $Iu = 0$ given that I is of the inf-sup type.

The first Lemma assures us that $\bar{u}$, defined as the upper semicontinuous envelope in Ω of $u(x) = \sup_{v \in S} v(x)$, is a viscosity sub solution of $I\bar{u} \geq 0$. Then $\bar{u} \in S$ and $\bar{u} = u$ is a sub solution too. By the second lemma the lower semicontinuous envelope $\underline{u}$, is a super solution. If not that would contradict the fact that u is the biggest subsolution. We conclude, by the comparison principle, that $\underline{u} \geq u$ and therefore both have to be equal and u is a viscosity solution of $Iu = 0$ in Ω.

The next step is to prove that we actually attain the boundary values in a continuous way. We have to show that for any $x \in \mathbb{R}^n \setminus \Omega$ and any $\varepsilon > 0$ we can find continuous barriers v and w such that,

(i) $Iw \leq 0$ and $Iv \geq 0$ in Ω in the viscosity sense,

(ii) $w \geq g$ and $v \leq g$ in $\mathbb{R}^n \setminus \Omega$,

(iii) $w(x) \leq g(x) + \varepsilon$ and $v(x) \geq g(x) - \varepsilon$.

We just prove it for w.

If x belongs to the interior of $\mathbb{R}^n \setminus \Omega$ then a function w which is equal to $\|g\|_\infty$ for every $y \neq x$ and equal to $g(x)$ for $y = x$ is in $USC(\Omega)$ and is a super solution. If $x \in \partial\Omega$ then there is a ball $B_{r_0}(x + r_0\eta)$ such that $\bar{B}_{r_0}(x + r_0\eta) \cap \partial\Omega = \{x\}$, where η is a unitary vector and r_0 less than one. Let

$$w(y) = 2\|g\|_\infty \varphi\left(\frac{y - (x + r\eta)}{r}\right) + g(x) + \varepsilon$$

with φ from Lemma 3.3.15 and some $r < r_0$. By the construction of φ we already have that (i) and (iii) are satisfied.

To check (ii) let $\delta > 0$ such that $|g(y) - g(x)| \leq \varepsilon$ whenever $|x - y| \leq \delta$. Take r such that $B_{2r}(x + r\eta) \subseteq B_\delta(x)$. If $y \in B_\delta(x) \cap (\mathbb{R}^n \setminus \Omega)$ then $w(y) = g(x) + \varepsilon \geq g(y)$. If $y \in \mathbb{R}^n \setminus B_\delta(x) \subseteq \mathbb{R}^n \setminus B_{2r}(x + r\eta)$ then $\varphi \geq 1$ and $w(y) \geq \|g\|_\infty \geq g(y)$. □

3.4 Partial ABP Estimates

The classical ABP theorem states that for a super solution, positive in ∂B_3, the supremum of u^- is controlled by the L^n norm of the right hand side, integrated only over the contact set for the convex envelope. These estimates are useful to get lower bounds in the measure of the contact set which are then needed to get point estimates.

We denote by Γ the convex envelope supported in B_3. For a lower semicontinuous function $u \geq 0$ in $\mathbb{R}^n \setminus B_1$,

$$\Gamma(x) = \sup\{v(x) : v : B_3 \to \mathbb{R} \text{ is convex and } v \leq u^-\}.$$

We get the same definition if v is only affine. Every time we refer to $\nabla\Gamma(x)$ we are actually referring to a sub differential of Γ at x which always exists.

In the next lemma we see that we can almost put a paraboloid above Γ, with the opening controlled by $f(x)$, the supremum of u outside B_1 and the τ derivative of Γ at x.

Lemma 3.4.1. *Let $u \geq 0$ in $\mathbb{R}^n \setminus B_1$ be a globally bounded viscosity solution of,*

$$\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leq f \text{ in } B_1,$$

and $x \in \{u = \Gamma\}$. Assume H1 holds ($2 > \sigma > \sigma_0$ and $\min(1, \sigma) > \tau > \tau_0$) and $2b \leq \lambda(2-\sigma)/(1-\tau)$. Let $\rho_0 = 1/(128\sqrt{n})$, $r_k = \rho_0 2^{-1/(2-\sigma)-k}$ and $R_k = B_{r_k} \setminus B_{r_{k+1}}$. Then there is a constant C_0 such that for any $M > 0$ there is a k such that

$$|\left\{y \in R_k : u(y+x) > u(x) + y \cdot \nabla\Gamma(x) + Mr_k^2\right\}| \leq C_0 \frac{F(x)}{M} |R_k|,$$

where

$$F(x) = f(x) + (1-\tau)b \int_{B_2} \frac{|\delta_0(\Gamma, x; y)|}{|y|^{n+\tau}} dy + \frac{1-\tau}{\tau} b \|u^+\|_{L^\infty(\mathbb{R}^n \setminus B_1)}.$$

Proof. Notice that $\delta_e^-(u, x; y) = 0$. If $x \pm y \in B_3$ then we use that there is a plane touching u by below in B_3. If $x + y \notin B_3$ then $x - y \notin B_1$ and the boundary value gives that $u(x \pm y) \geq 0$ and then $\delta_e(u, x; y) \geq 0$ because $u(x) \leq 0$.

By Lemma 3.3.4 the following quantities can be computed and satisfy,

$$\int_{B_{r_0}} (2-\sigma)\lambda\frac{\delta_e^+(u,x;y)}{|y|^{n+\sigma}} - (1-\tau)b\frac{|\delta_o(u,x;y)|}{|y|^{n+\tau}}dy \leq$$
$$f(x) + (1-\tau)b\int_{\mathbb{R}^n\setminus B_{r_0}} \frac{|\delta_o(u,x;y)|}{|y|^{n+\tau}}dy \leq$$
$$f(x) + C(n)b\frac{1-\tau}{\tau}\|u^+\|_{L^\infty(\mathbb{R}^n\setminus B_1)}.$$

We want to use Lemma 3.3.5 with Γ as the test function. The assumption $2b \leq \lambda(2-\sigma)/(1-\tau)$ guarantees (3.14) with $\alpha = 1/2$,

$$\int_{B_{r_0}} (2-\sigma)\lambda\frac{\delta_e^+(u,x;y)}{|y|^{n+\sigma}} - (1-\tau)b\frac{|\delta_o(u,x;y)|}{|y|^{n+\tau}}dy \geq$$
$$\int_{B_{r_0}} \frac{(2-\sigma)\lambda}{2}\frac{\delta_e^+(u,x;y)}{|y|^{n+\sigma}}dy - (1-\tau)b\int_{B_{r_0}} \frac{|\delta_0(\Gamma,x;y)|}{|y|^{n+\tau}}dy.$$

Adding what we have so far

$$(2-\sigma)\int_{B_{r_0}} \frac{\delta_e^+(u,x;y)}{|y|^{n+\sigma}}dy \leq CF(x) \tag{3.15}$$

The rest of the proof goes as in [8]. Fix M and assume that none of the dyadic rings satisfies the conclusion of the lemma for C_0 still to be fixed. For every $y \in R_k$ where

$$u(y+x) > u(x) + y\cdot\nabla\Gamma(x) + Mr_k^2,$$

we have that,

$$\begin{aligned}\delta_e(u,x;y) &= u(x+y) + u(x-y) - 2u(x),\\ &> y\cdot\nabla\Gamma(x) + Mr_k^2 + u(x-y) - u(x),\\ &\geq Mr_k^2.\end{aligned}$$

because by the convexity of Γ,

$$-y \cdot \nabla\Gamma(x) + u(x) = -y \cdot \nabla\Gamma(x) + \Gamma(x) \leq \Gamma(x-y) \leq u(x-y).$$

Adding all the contributions into the estimate (3.15) to get,

$$\begin{aligned}(2-\sigma)\int_{B_{r_0}} \frac{\delta_e^+(u,x;y)}{|y|^{n+\sigma}} &\geq (2-\sigma)C_0 M \frac{F(x)}{M} \sum_{k=0}^{\infty} r_k^{2-\sigma}, \\ &\geq C_0 F(x) r_0^{2-\sigma} \frac{2-\sigma}{1-2^{-(2-\sigma)}}.\end{aligned}$$

Now it just a matter to take C_0 large enough to get a contradiction. Notice that the quotient $(2-\sigma)/(1-2^{-(2-\sigma)})$ is uniformly bounded by above and away from zero when σ varies in $(0,2)$. □

The following is just a modification of the previous lemma. The aim is to replace the second term in $F(x)$ by $\|u\|_\infty$.

Remark 3.4.2. *By the intermediate value theorem, for each $x \in B_1$ and $y \in B_2$, $|\delta_0(\Gamma, x; y)|$ is equal to $2|\nabla\Gamma(x')||y|$ for x' an intermediate point in the segment between $x+y$ and $x-y$. So that*

$$\int_{B_2} \frac{|\delta_0(\Gamma, x, y)|}{|y|^{n+\tau}} dy \leq \frac{C(n)}{1-\tau} \|\nabla\Gamma\|_\infty.$$

By the geometry of the convex envelope $\|\nabla\Gamma\|_\infty \leq \|u^-\|_\infty/2$. We can also consider that $\tau > \tau_0 > 0$ for τ_0 universal, so that $F(x)$ can be simplified to

$$F(x) = f(x) + b\|u\|_\infty.$$

Notice that we haven't absorb the constant b into the universal constants of the estimate. The importance of this choice will be seen in the results of the next sections.

Corollary 3.4.3. *Under the assumptions of Lemma 3.4.1 there exists a small fraction $\varepsilon_0 > 0$ and a constant $M_0 > 0$ such that for some radius r and $R = B_r \setminus B_{r/2}$:*

(i) $|\left\{y \in R : u(y + x) > u(x) + y \cdot \nabla\Gamma(x) + M_0 F(x) r^2\right\}| \leq \varepsilon_0 |R|$.

(ii) $\Gamma(y + x) \leq u(x) + y \cdot \nabla\Gamma(x) + M_0 F(x) r^2$ *for* $y \in B_{r/2}$.

(iii) $|\nabla\Gamma(B_{r/4}(x))| \leq C|B_{r/4}|F(x)^n$

Proof. Let A be the following set

$$A = (B_1 \setminus B_{1/2}) \cap \{x_1 > 1/2\}.$$

Take

$$\varepsilon_0 = \frac{|A|}{2|B_1 \setminus B_{1/2}|} \text{ and } M_0 = \frac{C_0}{\varepsilon_0},$$

with C_0 from Lemma 3.4.1. Apply Lemma 3.4.1 with $M = F(x)M_0$ to get a radius $r(= r_k)$ such that

$$|\left\{y \in R : u(y + x) > u(x) + y \cdot \nabla\Gamma(x) + M_0 F(x) r^2\right\}| \leq \varepsilon_0 |R|. \tag{3.16}$$

By convexity we can assume without loss of generality that Γ attains its maximum N on $B_{r/2}(x)$ at the point $(r/2)e_1 + x$ and

$$\Gamma(y + x) \geq \Gamma((r/2)e_1 + x),$$

for every $y \in R$ with $y \cdot e_1 \geq r/2$. Therefore,

$$\begin{aligned} 2\varepsilon_0 |R| &\leq |\{y \in R : \Gamma(y + x) \geq N\}|, \\ &\leq |\{y \in R : u(y + x) \geq N\}|. \end{aligned}$$

Then N has to be smaller or equal than $u(x)+y\cdot\nabla\Gamma(x)+M_0F(x)r^2$ because otherwise we get a contradiction with (3.16). This implies (ii).

Finally, by Γ being trapped between two planes in $B_{r/2}$, separated by a distance $M_0F(x)r^2$, we get by the geometry of convex functions a control in the oscillation of $\nabla\Gamma$ in $B_{r/4}$. Namely $\nabla\Gamma(B_{r/4})$ is contained in the ball of radius $4M_0F(x)r$ with center at $\nabla\Gamma(x)$. This concludes the proof. □

Now we are able to state and prove an ABP type estimate.

Theorem 3.4.4. *Let $u \geq 0$ in $\mathbb{R}^n \setminus B_1$ be a globally bounded viscosity solution of,*

$$\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leq f \text{ in } B_1.$$

Assume H1 holds and $2b \leq \lambda(2-\sigma)/(1-\tau)$. There is a disjoint family of cubes Q_j with diameters $d_j \leq \rho_0 2^{-1/(2-\sigma)}$ ($\rho_0 = 1/(32\sqrt{n})$) which covers the contact set $\{\Gamma = u\}$ such that the following holds

(i) $\{u = \Gamma\} \cap \bar{Q}_j \neq \emptyset$ *for any Q_j.*

(ii) $\left|\left\{y \in 8\sqrt{n}Q_j : u(y) < \Gamma(y) + C\left(\max_{x\in Q_j\cap\{\Gamma=u\}} F(x)\right) d_j^2\right\}\right| \geq \mu|Q_j|.$

(iii) $|\nabla\Gamma(\bar{Q}_j)| \leq C\left(\max_{x\in Q_j\cap\{\Gamma=u\}} F(x)\right)^n |Q_j|.$

where μ ($= (1-\varepsilon_0)$ from Lemma 3.4.3) and C above are universal (independent of σ and τ) and

$$F(x) = f(x) + (1-\tau)b\int_{B_2} \frac{|\delta_0(\Gamma, x; y)|}{|y|^{n+\tau}}dy + \frac{1-\tau}{\tau}b\|u^+\|_{L^\infty(\mathbb{R}^n\setminus B_1)}.$$

Proof. Lets proceed as in [8] and cover B_1 with a tiling of cubes of diameter $\rho_0 2^{-1/(2-\sigma)}$. We discard all those that do not intersect the contact set $\{u = \Gamma\}$. Whenever a cube does not satisfy (ii) and (iii), we split it into 2^n congruent cubes of half diameter and discard those whose closure does not intersect $\{u = \Gamma\}$. We want to prove that eventually this procedure finishes.

Let's assume that the covering process does not stop. We end up getting a sequence of nested cubes intersecting at a point $x_0 \in \{u = \Gamma\}$. We will prove that there is a cube in the family that did not split, reaching then a contradiction.

Due to Corollary 3.4.3 there is a radius $0 < r < \rho_0 2^{-1/(2-\sigma)}$ such that for $R = B_r \setminus B_{r/2}$,

$$|\left\{y \in R : u(y + x) > u(x) + y \cdot \nabla\Gamma(x) + CF(x_0)r^2\right\}| \leq \varepsilon_0|R|,$$

and

$$|\nabla\Gamma(B_{r/4}(x_0))| \leq CF(x_0)^n|B_{r/4}|.$$

There is a cube Q_j with diameter $r/8 \leq d_j < r/4$ such that, $B_{r/4}(x_0) \supset \bar{Q}_j$ and $B_r(x_0) \subset 32\sqrt{n}Q_j$.

Using the fact that the diameter of the cube and the radius are comparable and that, by the convexity of Γ, $\Gamma(y) \geq u(x_0) + (y - x_0) \cdot \nabla\Gamma(x_0)$, we get

$$\begin{aligned}
&\left|\left\{y \in 32\sqrt{n}Q_j : u(y) \leq \Gamma(y) + C\left(\max_{Q_j\cap\{\Gamma=u\}} F\right)d_j^2\right\}\right| \geq \\
&\left|\left\{y \in 32\sqrt{n}Q_j : u(y) \leq u(x_0) + (y - x_0) \cdot \nabla\Gamma(x_0) + C\left(\max_{Q_j\cap\{\Gamma=u\}} F\right)d_j^2\right\}\right| \\
&\geq (1 - \varepsilon_0)|R| \geq \mu|Q_j|.
\end{aligned}$$

This is (ii) in the statement of the Theorem. Since $\bar{Q}_j$ is contained in B_r we conclude also that (iii) holds and Q_j did not split. □

As τ and σ go to one and two respectively in a controlled way (recall the hypothesis $2b \leq \lambda(2-\sigma)/(1-\tau)$), this theorem recovers a sufficient step to complete the proof of the classical ABP estimate. However, to prove regularity for u it will be sufficient to use a weaker version where $F(x) = f(x) + b\|u\|_\infty$ (see Remark 3.4.2).

We also point out that the condition on b is not too restrictive. By the scaling discussion on Section 3.1 we can always consider a dilation of u to make the assumption valid.

3.5 Point Estimate

The point estimate for non linear operators works in someway like the mean value theorem for super harmonic functions. If a non negative super harmonic function is bigger or equal than 1 in half of the points in B_1 (in measure) then it gets automatically separated from zero at $B_{1/4}$ a fixed quantity. This is the key step to prove a decay of oscillation and then Hölder regularity for the solutions of our equations. For non local operators, point estimates were already given in [16]. Those estimates are easier to obtain than in the local case because the definition of the non local operators already involve some sort of averaging. However, the estimates in [16] blow up when the order of the equation go to the classical one. Our goal here is to see that the same estimates still hold with constant that remain uniform when $\sigma \to 2$ and $\tau \to 1$ in a controlled way.

From this point on we will always assume that, for $\sigma_0, \tau_0, m, A_0 > 0$ given, the set of hypothesis H1, H2 and H3 holds.

(H1) $2 > \sigma \geq \sigma_0 > 0$, $\min(1, \sigma) > \tau \geq \tau_0 > 0$,

(H2) $\sigma - \tau \geq m > 0$,

(H3) $\lambda A_0(2 - \sigma) \geq b(1 - \tau)$.

We recall the special function constructed in [8].

Lemma 3.5.1. *Let $2 > \sigma_0 > 0$, there is a function Φ such that,*

(i) Φ *is continuous in* $\mathbb{R}^n$,

(ii) $\Phi(x) = 0$ *for x outside* $B_{2\sqrt{n}}$,

(iii) $\Phi(x) < -2$ *for x in Q_3, and*

(iv) $\mathcal{M}^+_\sigma \Phi \leq \psi(x)$ *in $\mathbb{R}^n$ for some non negative function $\psi(x)$ supported in* $\bar{B}_{1/4}$

for every $\sigma > \sigma_0$.

The following lemma provides the first and also the inductive step towards an inductive proof of the point estimate.

Lemma 3.5.2. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume H1, H2 and H3. There exists constants $\mu \in (0, 1)$, $\varepsilon_0 > 0$ and $M > 1$, such that if*

(i) $u \geq 0$ *in* $\mathbb{R}^n$,

(ii) $\inf_{Q_{3\kappa}} u \leq 1$,

(iii) $\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leq 1$ *in* $Q_{4\sqrt{n}\kappa}$,

then

$$|\{u \leq M\} \cap Q_\kappa| > \mu |Q_\kappa|,$$

for

$$\kappa = \frac{\varepsilon_0}{(1 + \|u\|_\infty)^{1/(\sigma-\tau)}}.$$

Proof. Consider $\tilde{u}(x) = u(\kappa x)$ and note that by the scaling of the equation $\tilde{u}$ satisfies (i), (ii) in the cube of side 3 and

$$\mathcal{M}_{\tilde{\mathcal{L}}_0(\tilde{b})} u\tilde{u} \leq \kappa^\sigma \text{ in } Q_{4\sqrt{n}},$$

where $\tilde{\mathcal{L}}_0(\tilde{b}) = \tilde{\mathcal{L}}_0(\sigma, \tau, \lambda\,\Lambda, \tilde{b})$, for $\tilde{b} = \kappa^{\sigma-\tau} b$. We will prove that the lemma holds for $\tilde{u}$ in Q_1, which implies the desired result. The proof follows as in [8] but we point out that the ABP type results that we have are different.

First thing we require from ε_0^m is to be small enough, with respect to A_0, such that the condition of smallness on $\tilde{b} \leq \varepsilon_0^m b$ from Lemma 3.4.1 and Theorem 3.4.4 holds. Namely, $2\varepsilon_0^m \leq A_0^{-1}$.

Consider $v = \tilde{u} + \Phi$, where Φ is the special function given in [8]. We have that v satisfies in $Q_{4\sqrt{n}}$ (since $\tilde{\mathcal{L}}_0(\tilde{b}) \subseteq \mathcal{L}_0(\tilde{b})$)

$$\begin{aligned} \mathcal{M}^-_\sigma v - \tilde{b}|D_\tau| v &\leq \kappa^\sigma + \mathcal{M}^+_\sigma \Phi + \kappa^{\sigma-\tau} b |D_\tau| \Phi, \\ &\leq \kappa^\sigma + \psi + \kappa^{\sigma-\tau} bC, \end{aligned}$$

for a universal constant C.

Let Γ be the concave envelope of v supported in the ball $B_{6\sqrt{n}}$. Let Q_j be the cubes from a rescaled version of our partial ABP estimate. We have

$$\begin{aligned}\max_{B_{2\sqrt{n}}} v^- &\leq C|\nabla\Gamma(B_{2\sqrt{n}})|^{1/n} \leq C\left(\sum_j |\nabla\Gamma(\bar{Q}_j)|\right)^{1/n},\\ &\leq C\left(\sum_j \left(\max_{Q_j}\psi + \kappa^{\sigma} + \kappa^{\sigma-\tau} b\,(1+\|v\|_\infty)\right)^n\right)|Q_j|^{1/n}.\end{aligned}$$

We can make the terms $\kappa^{\sigma} + \kappa^{\sigma-\tau} b\,(1+\|v\|_\infty)$ small enough by choosing ε_0 small enough,

$$\kappa^{\sigma} + \kappa^{\sigma-\tau} b\,(1+\|v\|_\infty) \leq C(\varepsilon_0^{\sigma} + \varepsilon_0^{\sigma-\tau}) \leq C\varepsilon_0^{m}.$$

Using that $\Phi \leq -2$ in Q_3

$$1 \leq C\varepsilon_0^{m} + C\left(\sum_j \left(\max_{Q_j}\psi^+\right)^n |Q_j|\right)^{1/n},$$

which implies then, for ε_0 small enough, the following inequality,

$$\frac{1}{2} \leq C\left(\sum_j \left(\max_{Q_j}\psi^+\right)^n |Q_j|\right)^{1/n}.$$

Since ψ is supported in $\bar{B}_{1/4}$ and is bounded, we get

$$\sum_{Q_j\cap\bar{B}_{1/4}\neq\emptyset} |Q_j| \geq c, \tag{3.17}$$

where c is universal. Now, the diameters of all cubes Q_j are bounded by $\rho_0 2^{-1/(2-\sigma)}$, which is smaller than $\rho_0 = 1/(128\sqrt{n})$. So, every time we have that Q_j intersects $B_{1/4}$ the cube $32\sqrt{n}Q_j$ will be contained in $B_{1/2}$.

Since ε_0 is universal, the partial ABP estimates translate into

$$|\left\{x \in 32\sqrt{n}Q_j : \ v(x) \leq \Gamma(x) + Cd_j^2\right\}| \geq c|Q_j|, \tag{3.18}$$

for C universal and $Cd_j^2 < C\rho_0^2$. Let us consider now the cubes $32\sqrt{n}Q_j$ for every Q_j that intersects $B_{1/4}$. This provides an open cover of the union of the corresponding cubes $\bar{Q}_j$ and it is contained in $B_{1/2}$. Taking a subcover with finite overlapping and using (3.17) and (3.18) we get

$$|\left\{x \in B_{1/2} : \ v(x) \leq \Gamma(x) + C\rho_0^2\right\}| \geq c.$$

Hence, if we let $-M_0 = \min_{B_{1/2}} \Phi$ we get

$$|\left\{x \in B_{1/2} : \ \tilde{u}(x) \leq M_0 + C\rho_0^2\right\}| \geq c.$$

Finally let $M = M_0 + C\rho_0^2$ and $\mu = c$. Since $B_{1/2} \subset Q_1$,

$$|\{x \in Q_1 : \ \tilde{u}(x) \leq M\}| \geq c,$$

which concludes the result for $\tilde{u}$. □

Remark 3.5.3. *In the previous proof the scaling is necessary to have:*

(i) $\tilde{b}(1 + ||u||_\infty) \leq \varepsilon_0^m$,

(ii) Right hand side smaller than ε_0^m.

In future references we will use that if these identities hold, then the conclusion also holds without any further scaling.

Remark 3.5.4. *Consider for u and $\tilde{u}$ as before and $0 < r \le 1$, $v(x) = \tilde{u}(rx)$. Then v satisfies,*

$$\mathcal{M}_{\tilde{\mathcal{L}}_0(b(r\kappa)^{\sigma-\tau})} v \le (r\kappa)^{\sigma},$$

and in particular

$$\mathcal{M}^-_{\sigma} v(x) - b(r\kappa)^{\sigma-\tau} |D_\tau v| \le (r\kappa)^{\sigma}.$$

From the previous remark we check that $b(r\kappa)^{\sigma-\tau}(1 + \|v\|_\infty) \le \varepsilon_0^m$ and that the right hand side is also smaller or equal than ε_0^m if $r \le 1$.

In particular, the transformations required to prove the full L^ε lemma are of the form

$$v(y) = \frac{u(x_0 + 2^{-i} y)}{M^k}.$$

The previous lemma can still be applied and we can iterate by means of a Calderón Zygmund decomposition as in [5].

Lemma 3.5.5. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and $u \ge 0$, $\tilde{u}$ as in Lemma 3.5.2. Then we have*

$$\left| \left\{ \tilde{u} > M^k \right\} \cap Q_1 \right| \le (1 - \mu)^k,$$

for $k = 1, 2, ...$ where M and μ are as in Lemma 3.5.2. As a consequence we have the following inequality,

$$|\{\tilde{u} > t\} \cap Q_1| \le d t^{-\varepsilon}, \quad \forall t > 0,$$

where d and ε are positive universal constants.

By standard covering arguments one can pass from cubes to balls.

Corollary 3.5.6. *Let* $\sigma_0, \tau_0, m, A_0 > 0$ *and* $u \geq 0$*,* $\tilde{u}$ *as in Lemma 3.5.2 with u super solution of* $\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leq 1$ *in* $B_{2\kappa}$ *and* $u(0) \leq 1$*. Then we have*

$$|\{\tilde{u} > t\} \cap B_1| \leq ct^{-\varepsilon}, \quad \forall t > 0,$$

where c and ε *are positive universal constants.*

By using Remark 3.5.4 one more time we can prove a rescaled version of the Corollary 3.5.6.

Corollary 3.5.7. *Let* $\sigma_0, \tau_0, m, A_0 > 0$*,* $u \geq 0$*,* $\tilde{u}$ *as as in Lemma 3.5.2 with u super solution of* $\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leq C_0$ *in* $B_{2\kappa r}$*, for* $r \leq 1$*. Then we have*

$$|\{\tilde{u} > t\} \cap B_r| \leq Cr^n(u(0) + C_0 r^{\sigma})^{\varepsilon} t^{-\varepsilon}, \quad \forall t > 0, \tag{3.19}$$

where C and ε *are positive universal constants.*

3.6 Hölder Regularity

The first lemma in this section is the decay of oscillation, that follows from the point estimate already proved, applied at every scale. It is well known that when the oscillation of a function decays geometrically in geometrically decaying balls it implies a Hölder modulus of continuity at the center of such ball. By applying it at every point of a ball strictly contained in the domain we get Hölder regularity.

We still assume the general hypothesis, H1, H2 and H3, of the previous section.

Lemma 3.6.1. *Let* $\sigma_0, \tau_0, m, A_0 > 0$ *and assume H1, H2 and H3. Let u be a function such that:*

(i) $\frac{-1}{2} \leq u \leq \frac{1}{2}$ *in* $\mathbb{R}^n$*,*

(ii) $\mathcal{M}^+_{\tilde{\mathcal{L}}_0} u \geq -1$ *and* $\mathcal{M}^-_{\tilde{\mathcal{L}}_0} u \leq 1$ *in* B_κ*,*

in the viscosity sense. Let $\tilde{u}(x) = u(\kappa x)$ *for*

$$\kappa = \frac{\varepsilon_1}{(1 + \|u\|_\infty)^{1/(\sigma-\tau)}}.$$

Then there are universals $\alpha, C > 0$ *such that*

$$|\tilde{u}(x) - \tilde{u}(0)| \leq C|x|^\alpha.$$

Our proof relies in noticing that a dilation powerful enough puts us in the same hypothesis as in the proof of [8]. The detail is that the rescaling considered in such proof consists of a dilation of the domain, which as we already saw are good for our situation, times some constants that grow geometrically which compete against the smallness condition on the coefficient b. We want to check that by making α small enough we can control the effect of this second multiplication.

Proof. Let $\varepsilon_1^m \leq \varepsilon_0^m/2$ to start such that the estimates from the previous section are valid with the same constants.

We will show that there exists sequences m_k and M_k such that $m_k \leq \tilde{u} \leq M_k$ in $B_{4^{-k}}$ and

$$M_k - m_k = 4^{-\alpha k}$$

so that result holds for $C = 4^\alpha$.

For $k = 0$ we choose $m_0 = -1/2$ and $M_0 = 1/2$ and by (i) we have $m_0 \leq \tilde{u} \leq M_0$ in $\mathbb{R}^n$. We construct the sequence by induction. Assume then that we have the sequences up to some k and then we want to find m_{k+1} and M_{k+1}. In the ball $B_{4^{-(k+1)}}$, either $\tilde{u} \geq (M_k + m_k)/2$ in at least half the points (in measure) or we have the other inequality. Let's assume, without loss of generality, that

$$\left|\left\{\tilde{u} \geq \frac{M_k + m_k}{2}\right\} \cap B_{4^{-(k+1)}}\right| \geq \frac{|B_{4^{-(k+1)}}|}{2}.$$

Consider now

$$v(x) = \frac{\tilde{u}(4^{-k}x) - m_k}{(M_k - m_k)/2},$$

so that $v \geq 0$ in B_1 and $|\{v \geq 1\} \cap B_{1/4}| \geq |B_{1/4}|/2$.

From the inductive hypothesis, we have that for any index j between 1 and k,

$$\begin{aligned} v \geq \frac{m_{k-j} - m_k}{(M_k - m_k)/2} &\geq \frac{m_{k-j} - M_{k-j} + M_k - m_k}{(M_k - m_k)/2} \\ &\geq 2(1 - 4^{\alpha j}), \end{aligned}$$

in B_{4^j}. Therefore $v(x) \geq -2(|4x|^\alpha - 1)$ outside B_1. Let $w = v^+$, it satisfies

$$\begin{aligned} &\mathcal{M}^-_{\tilde{\mathcal{L}}_0(4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau}b)} w \leq \\ &4^{-k\alpha}\kappa^\sigma + \mathcal{M}^+_\sigma v^- + 4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau} b|D_\tau|v^- \leq \\ &\varepsilon_1^m + \mathcal{M}^+_\sigma v^- + \varepsilon_1^m b|D_\tau|v^-. \end{aligned} \tag{3.20}$$

We still have $|\{w \geq 1\} \cap B_{1/4}| \geq |B_{1/4}|/2$. Use the other bound $v^- \leq 2(|4x|^\alpha - 1)$ outside B_1, also proved by induction, and $v^- = 0$ in B_1 to get that the right hand side can be made smaller than ε_0^m in $B_{3/4}$ by choosing a small exponent α.

We recall the conditions in the Remark 3.5.3. So far we have shown the second one which is satisfied with a right hand side ε_0^m. For the first condition note that

$$4^{-k(\sigma-\tau)}\kappa^{\sigma-\tau}b(1+\|w\|_\infty) \le 4^{-k(\sigma-\tau)}\varepsilon_0^m(1+4^{\alpha k}),$$
$$\le 4^{-k(m-\alpha)}\varepsilon_0^m.$$

so we have to choose $\alpha \le m$.

Now, given any $x \in B_{1/4}$ we can apply Corollary 3.5.7 in $B_{1/2}(x)$ to get

$$C(w(x)+\varepsilon_1)^\varepsilon \ge |\{w > 1\} \cap B_{1/2}(x)| \ge \frac{|B_{1/4}|}{2},$$

hence, since ε_1 can be made even smaller, we conclude $w \ge \theta > 0$ in $B_{1/4}$ for some $\theta > 0$. If we let $M_{k+1} = M_k$ and $m_{k+1} = m_k + \theta(M_k - m_k)/2$ we have the inductive step

$$m_{k+1} \le \tilde{u} \le M_{k+1}, \quad \text{in } B_{4^{-(k+1)}}.$$

Moreover $M_{k+1} - m_{k+1} = (1-\theta/2)4^{-\alpha k}$, so choosing α and θ such that $1-\theta/2 = 4^{-\alpha}$ we conclude $M_{k+1} - m_{k+1} = 4^{-\alpha(k+1)}$. □

As a result we get that $\tilde{u}$ is C^α. Here is the proof of Theorem 3.2.1.

Proof. Let u be as in Theorem 3.2.1 and consider now

$$v(x) = \frac{u(x)}{2(\|u\|_\infty + C_0)}.$$

Since the equation is homogeneous of degree 1 we are now under the hypothesis of the previous lemma. Hence we conclude that the dilation $\tilde{v}$ of v satisfies

$$|\tilde{v}(x) - \tilde{v}(0)| \le C|x|^\alpha,$$

where C is a universal constant. Coming back to v, this translates to

$$
\begin{aligned}
|v(x) - v(0)| &\leq C|x|^{\alpha}\kappa^{-\alpha} \\
&= \frac{C}{\varepsilon_0^{\alpha}}(\|v\|_{\infty} + 1)^{\alpha/(\sigma-\tau)}|x|^{\alpha},
\end{aligned}
$$

since $\|v\|_{\infty} \leq 1/2$, $\sigma - \tau \geq m$ and ε_0 is universal we get

$$|v(x) - v(0)| \leq C|x|^{\alpha},$$

for a different universal C. In terms of u we recover the estimate

$$|u(x) - u(0)| \leq C(\|u\|_{\infty} + C_0)|x|^{\alpha}.$$

Hence u is C^{α} at 0 and it's C^{α} seminorm is controlled as desired. This concludes the proof. □

3.7 $C^{1,\alpha}$ Regularity

For translation invariant equations, $C^{1,\alpha}$ regularity comes by proving C^{α} regularity for the incremental quotients of a given solution. This procedure allows to improve the regularity from C^{α} to $C^{2\alpha}$ and so forth all the way up to $C^{0,1}$ and then to $C^{1,\alpha}$, see [5]. We need to use the comparison principle to see that these incremental quotients satisfy a uniformly elliptic equation with bounded measurable coefficients and zero right hand side, for which we already have C^{α} estimates. The difficulty in this case is that we need, in each step, these incremental quotients to be uniformly bounded in $\mathbb{R}^n$. The previous regularity only guaranties this on $B_{r-\delta}$, given that the equation is satisfied in B_r.

Recall the class $\mathcal{L}_1 = \mathcal{L}_1(\sigma, \tau, \lambda, \Lambda, b, \rho_0) \subseteq \tilde{\mathcal{L}}_0(\sigma, \tau, \lambda, \Lambda, b)$ of all possible linear operators L with non negative kernels K such that they satisfy (3.8) and (3.9), and the following integrability assumption for some radius ρ_0,

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y-h)|}{|h|} dy \leq C \text{ every time } |h| < \frac{\rho_0}{2}.$$

Theorem 3.7.1. *Let $\sigma_0, \tau_0, m, A_0 > 0$ and assume that H1, H2 and H3 holds. There is $\rho_0 > 0$ small enough so that if I is an elliptic operator of the inf-sup type as in* (3.5) *with all the linear operators in $\mathcal{L}_1$ and u a bounded viscosity solution of $Iu = 0$ in B_1, then there is a universal $\alpha > 0$ such that $u \in C^{1,\alpha}(B_{1/2})$ and*

$$\|u\|_{C^{1,\alpha}(B_{1/4})} \leq C\|u\|_\infty$$

for some universal $C > 0$.

Proof. Let $\bar{\alpha}$ the Hölder exponent obtained by Theorem 3.2.1 and assume that it is not the reciprocal of an integer by making it smaller if necessary. Let $\delta = 1/(4[1/\bar{\alpha}])$. We want to see that, for $k = 0, 1, \ldots, [1/\bar{\alpha}] - 1$, the estimate

$$\|u\|_{C^{0,k\bar{\alpha}}(B_{3/4-k\delta})} \leq C(k)\|u\|_\infty, \tag{3.21}$$

implies the next estimate,

$$\|u\|_{C^{0,(k+1)\bar{\alpha}}(B_{3/4-(k+1)\delta})} \leq C(k+1)\|u\|_\infty. \tag{3.22}$$

Fix a unit vector $e \in \mathbb{R}^n$ and η a smooth cut-off function supported in $B_{(3/4-k\delta)-\delta/4}$ and equal to one in $B_{(3/4-k\delta)-\delta/2}$. For given $h \in (-\delta/8, \delta/8)$ we define the following incremental quotients

$$w_h(x) = \frac{u(x+he) - u(x)}{|h|^{\bar{\alpha}k}},$$
$$w_1^h(x) = \frac{(\eta u)(x+he) - (\eta u)(x)}{|h|^{\bar{\alpha}k}},$$
$$w_2^h(x) = \frac{((1-\eta)u)(x+he) - ((1-\eta)u)(x)}{|h|^{\bar{\alpha}k}}.$$

When $x \in B_{(3/4-k\delta)-\delta/8}$, $|w_1^h(x)|$ is bounded above by $C(k,\eta)\|u\|_\infty$. By interpolation and (3.21),

$$\begin{aligned} |w_1^h(x)| &\leq \|\eta u\|_{C^{0,k\bar{\alpha}}(B_{3/4-k\delta})}, \\ &\leq \|u\|_{C^{0,k\bar{\alpha}}(B_{3/4-k\delta})} + \|u\|_\infty \|\eta\|_{C^{0,k\bar{\alpha}}(B_{3/4-k\delta})}, \\ &\leq C(k,\eta)\|u\|_\infty. \end{aligned}$$

If $x \in \mathbb{R}^n \setminus B_{(3/4-k\delta)-\delta/8}$ then $w_1^h(x)$ just cancels.

By using that the equation is translation invariant we have that u and $u(\cdot + he)$ satisfy equations in the same ellipticity family, *with positive kernels*. Then w_h also satisfy an equation in the same ellipticity family by Lemma 3.3.7. The function w_1^h satisfy a similar equation as w_h, the difference is on the right hand side introduced by the cut-off,

$$\mathcal{M}^+_{\mathcal{L}_1} w_1^h \geq -\mathcal{M}^+_{\mathcal{L}_1} w_2^h \text{ and } \mathcal{M}^-_{\mathcal{L}_1} w_1^h \leq -\mathcal{M}^-_{\mathcal{L}_1} w_2^h.$$

For $x \in B_{(3/4-k\delta)-3\delta/4}$ the terms $|\mathcal{M}^\pm_{\mathcal{L}_1} w_2^h|$ are controlled by $\|u\|_\infty$ by using that

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y-h)|}{|h|} dy \leq C \text{ every time } |h| < \frac{\rho_0}{2}.$$

with $\rho_0 = \delta/8$. Indeed, for $L \in \mathcal{L}_1$ with kernel K and $x \in B_{(3/4-k\delta)-3\delta/4}$ and $|y| \leq \delta/8$, $w_2^h(x+y) = 0$ and

$$\begin{aligned}
|Lw_2^h(x)| &= \left| \int w_2^h(x+y)K(y)dy \right|, \\
&= \left| \int_{\mathbb{R}^n \setminus B_{\delta/8}} \frac{(1-\eta)u(x+y+h) - (1-\eta)u(x+y)}{|h|^{\bar{\alpha}k}} K(y)dy \right|, \\
&= \left| \int_{\mathbb{R}^n \setminus B_{\delta/8}} (1-\eta)u(x+y)|h|^{1-\bar{\alpha}k} \frac{K(y) - K(y-h)}{|h|} dy \right|, \\
&\leq C\|u\|_\infty.
\end{aligned}$$

We get then the equations for w_h^1 in $B_{(3/4-k\delta)-3\delta/4}$

$$\mathcal{M}^+_{\mathcal{L}_1} w_1^h \geq C\|u\|_\infty \ \text{ and } \ \mathcal{M}^-_{\mathcal{L}_1} w_1^h \leq -C\|u\|_\infty.$$

By applying Theorem 3.2.1 to w_1^h from $B_{(3/4-k\delta)-3\delta/4}$ to $B_{3/4-(k+1)\delta}$ we conclude that for a constant $C(k+1)$ independent of h,

$$\|w_h^1\|_{C^{0,\bar{\alpha}}(B_{3/4-(k+1)\delta})} \leq C(k+1)\|u\|_\infty.$$

This implies the estimate (3.22) by using Lemma 5.6 in [5].

From $k = [1/\bar{\alpha}] - 1$ to $k + 1 = [1/\bar{\alpha}]$ we get that u is Lipschitz in $B_{3/4}$ with the estimate

$$\|u\|_{C^{0,1}(B_{3/4})} \leq C\|u\|_\infty.$$

By applying the previous step one more time to the Lipschitz quotient we conclude the theorem. □

Chapter 4

Regularity for solutions of non local parabolic equations

4.1 Definitions and Preliminaries

4.1.1 Non local operators

We now study the parabolic case. Most of the definitions provided in Chapter 3 still apply in this Chapter, though we need to ask for some uniformity in time. To be precise about the formulas we presented in the previous section, we need to ask and integrability condition to K around the origin,

$$\int_{B_1} |y|^2 K(x,t;y)dy < \infty. \tag{4.1}$$

It allows us to write, as before, rigorously

$$Lu(x,t) = \int \delta(u,x,t;y)K(x,t;y)dy$$

not only when u, with compact support, is in C_0^∞, but also when $u(\cdot,t) \in C^{1,1}(x)$.

Definition 4.1.1. *$u(\cdot,t) \in C^{1,1}(x)$, if there exists a vector $v \in \mathbb{R}^n$ and a number $M > 0$ such that*

$$|u(y+x,t) - u(x,t) - v \cdot y| < M|y|^2 \textit{ for } |y| \textit{ small enough.}$$

Notice that this definition implies $|\delta(u, x, t; y)| = O(|y|^2)$ as $|y|$ is close to zero. This is why we can get rid of the principal value in the integral.

We say that a family $\mathcal{L}$ of linear operators satisfy the integrability condition uniformly in $\Omega \times [-T, 0]$ when the upper bounds in (4.1) can be taken independent of $L \in \mathcal{L}$ and $(x, t) \in \Omega \times [-T, 0]$.

We will consider as in [9] absolute continuous weights ω which measure the contributions of the tails to the non local operators. This allows to compute Lu even when u does not have compact support.

Definition 4.1.2. *The space of function $L^1(\omega)$ consist of all $u : \mathbb{R}^n \to \mathbb{R}$ such that*

$$||u||_{L^1(\omega)} := \int |u(y)|\omega(y)dy < \infty.$$

Definition 4.1.3 (Non local operators)**.** *We say that I is a non local, fully non linear operator with respect to ω, if for every $u(\cdot, t) \in C^{1,1}(x) \cap L^1(\omega)$, $Iu(x, t)$ is a well defined real number.*

For $\sigma \in (0, 2)$ fixed, linear operators with kernels of the form $K \sim |y|^{-(n+\sigma)}$ or combinations of those (by taking supremums and infimums) are contained by this definition. In such cases we say that the operator have order σ and use $\omega = 1/(1 + |y|^{n+\sigma})$.

We say that I is translation invariant in space if

$$I\tau_{(x-y,0)}u(y, s) = Iu(x, t),$$

where τ is the shift operator,

$$\tau_{(x,t)}u(y, s) = u(y + x, s + t).$$

4.1.2 Continuous operators

For time dependent problems, the natural topology to use in $\mathbb{R}^n \times \mathbb{R}$ is the so called parabolic topology. It is generated by neighborhoods of the form $B_r(x) \times (t, t-\tau]$, for a given point $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. For instance, in this topology a function f is continuous if and only if $f(y,s) \to f(x,t)$ as $(y,s) \to (x,t^-)$.

Whenever we want to see that a linear operator L has "continuous coefficients" we fix a smooth test function u in an open set $O \subseteq \mathbb{R}^n \times \mathbb{R}$ and check if Lu evaluated at O is continuous. In the non local case, we need to use not only functions which are smooth in O but also that the contributions from their tails vary in a sufficiently smooth way. This is the motivation to introduce the following space.

Definition 4.1.4. *Let $C(a,b;L^1(\omega))$ be the space of function $u : (a,b] \to \mathbb{R}$ such that*

1. *for every $t \in (a,b]$, $u(\cdot,t) \in L^1(\omega)$,*
2. *for every $t_2 \in (a,b]$, $\|u(\cdot,t_1) - u(\cdot,t_2)\|_{L^1(\omega)} \to 0$ as $t_1 \to t_2^-$.*

It comes additionally with the norm,

$$\|u\|_{C(a,b;L^1(\omega))} = \sup_{t \in (a,b]} \|u(\cdot,t)\|_{L^1(\omega)}.$$

The space of functions against which we test the continuity of I are given by parabolic second order polynomials and functions in $C(a,b;L^1(\omega))$.

Definition 4.1.5 (Test functions)**.** *The space $S = S(\Omega \times (-T,0])$ of test functions is the set of all pairs $(v, B_r(x) \times (t-\tau,t])$ such that $v \in C(t-\tau,t;L^1(\omega))$, $B_r(x) \times (t-\tau,t] \subseteq$*

$\Omega \times (-T, 0]$ *and* v *restricted to* $B_r(x) \times (t-\tau, t]$ *is a quadratic parabolic polynomial, i.e.*

$$v(x,t) = \sum_{i,j=1}^{n} a_{i,j} x_i x_j + \sum_{i=1}^{n} b_i x_i + ct + d.$$

Definition 4.1.6 (Continuous operators)**.** *We say that a non local operator* I*, with respect to* ω*, depends continuously on the position in* $\Omega \times (-T, 0]$ *if for every* $(v, B_r(x) \times (t-\tau, t]) \in S$*, we have that* Iu *is a continuous function in* $B_r(x) \times (t-\tau, t]$ *(with respect to the parabolic topology).*

We can understand a little bit better how the space $C(a, b; L^1(\omega))$ appears as a requirement for the continuity of the operator in time. Without this condition, even the fractional laplacian would not be a continuous operator in $B_1 \times (-1, 0]$ with respect to any positive ω. Take for example u equal to zero in $B_1 \times (-1, 0]$ and let vary u freely outside $B_1 \times (-1, 0]$.

4.1.3 Ellipticity.

In the classical stationary case ellipticity means that, for a solution u of a homogeneous problem, the positive eigenvalues of its Hessian control the negative ones and vice versa. Geometrically, the positive and negative curvatures of the graph of u control each other. A way to define this precisely is by imposing the following condition on F,

$$\mathcal{M}^-(D^2(u-v)) \leq F(D^2u) - F(D^2v) \leq \mathcal{M}^+(D^2(u-v)),$$

where we are using the notation from [5].

When we have in mind equations of order $\sigma \in (0,2)$, we may want to use $\mathcal{L}_0$, the family of all linear operators L which are comparable to the fractional laplacian of order σ (to be defined), in order to define the ellipticity of I as $\mathcal{M}^-_{\mathcal{L}_0}(u-v) \le Iu - Iv \le \mathcal{M}^+_{\mathcal{L}_0}(u-v)$. This family is however too big and in order to get further regularity we need to impose further assumptions on the linear operators. By this reason we give a definition of ellipticity which is more general.

Definition 4.1.7. *Let $\mathcal{L}$ be a class of linear integro differential operators. We say that a fully non linear operator I is elliptic with respect to the class $\mathcal{L}$ if*

$$\mathcal{M}^-_{\mathcal{L}}(u-v) \le Iu - Iv \le \mathcal{M}^+_{\mathcal{L}}(u-v). \tag{4.2}$$

Going back to the definition of $\mathcal{L}_0 = \mathcal{L}_0(\Lambda,\sigma)$ $(\Lambda \ge 1)$. The precise condition for L to be in $\mathcal{L}_0$ with kernel K is the following one,

$$(2-\sigma)\frac{\Lambda^{-1}}{|y|^{n+\sigma}} \le K(y) \le (2-\sigma)\frac{\Lambda}{|y|^{n+\sigma}}. \tag{4.3}$$

In this family the extremal operators take the explicit form

$$\begin{aligned}
\mathcal{M}^+_{\mathcal{L}_0} v(x,t) :&= \sup_{L\in\mathcal{L}_0} (Lv)(x,t)\\
&= (2-\sigma)\int_{\mathbb{R}^n} \frac{\Lambda\delta^+(v,x,t;y) - \Lambda^{-1}\delta^-(v,x,t;y)}{|y|^{n+\sigma}}dy,\\
\mathcal{M}^-_{\mathcal{L}_0} v(x,t) :&= \inf_{L\in\mathcal{L}_0} (Lv)(x,t)\\
&= (2-\sigma)\int_{\mathbb{R}^n} \frac{\Lambda^{-1}\delta^+(v,x,t;y) - \Lambda\delta^-(v,x,t;y)}{|y|^{n+\sigma}}dy.
\end{aligned}$$

Where $\delta^\pm$ denote the positive and negative parts of δ $(\delta = \delta^+ - \delta^-)$.

The factors $(2-\sigma)$ become important as $\sigma \to 2^-$ as they will allow us to recover second order differential operators.

Hölder regularity for the spatial gradient of u requires ellipticity with respect to a smaller class. Given $\rho_0 > 0$ we define the family $\mathcal{L}_1 = \mathcal{L}_1(\sigma, \Lambda, \rho_0) \subseteq \mathcal{L}_0(\sigma, \Lambda)$ by the operators $L \in \mathcal{L}_1$ with kernel K, such that

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y-h)|}{|h|} dy \leq \Lambda, \tag{4.4}$$

for every $|h| \leq \rho_0/2$. It is sufficient that $|DK| \leq \Lambda/(1 + |y|^{n+\sigma}))$ for (3.13) to hold. An important property of this stronger condition is that the parabolic equations associated with them remain invariant under scaling. We will not bother to give this family a specific notation because we will not use it in this work.

4.1.4 Viscosity solutions.

For viscosity solutions of an equation $u_t - Iu = f$ we always assume a minimum requirement of continuity for u. Here we denote the space of upper semi-continuous functions in $\bar{\Omega} \times [-T, 0]$, always with respect to the parabolic topology, by $USC(\bar{\Omega} \times [-T, 0])$. Similarly, $LSC(\bar{\Omega} \times [-T, 0])$ denotes the space of lower semicontinuous functions in $\bar{\Omega} \times [-T, 0]$.

With respect to the time derivative, it is natural for the parabolic topology to consider only the values of u towards the past. In this sense

$$u_{t^-}(x,t) = \lim_{h \to 0^+} \frac{u(x,t) - u(x,t-h)}{h}.$$

Definition 4.1.8. *A function $u \in USC(\bar{\Omega} \times [-T, 0])$ ($u \in LSC(\bar{\Omega} \times [-T, 0])$), is said to be a sub solution (super solution) to $u_t - Iu = f$, and we write $u_t - Iu \leq f$*

($u_t - Iu \geq f$), if every time $(v, B_r(x) \times (t-\tau, t]) \in S$ touches u by above (below) at (x,t), i.e.

(i) $v(x,t) = u(x,t)$,

(ii) $v(y,s) > u(y,s)$ ($\varphi(y,s) < u(y,s)$) *for every* $(y,s) \in B_r(x) \times (t-\tau, t] \setminus \{(x,t)\}$,

then $v_{t^-}(x,t) - Iv(x,t) \leq f(x,t)$ ($v_{t^-}(x,t) - Iv(x,t) \geq f(x,t)$).

An equivalent definition holds if instead of using parabolic second order polynomials as test functions we use test functions φ with less regularity around the contact point. This is important when we want to prove the maximum principle by means of and inf and sup convolutions. We omit it here and just assume that the maximum principle for viscosity solutions holds. The ideas for the proof of this result are standard and can be found in [5] or in the appendix of [17] for the non local case with $\sigma = 1$.

The following example illustrates the importance of having test functions in $C(a,b;L^1(\omega))$. Consider $u(x,t) = \chi_{E\times\{0\}}$ where $E \subset\subset \mathbb{R}^n \setminus \bar{B}_1$. In the domain $B_1 \times (-1,0)$ the function u satisfies $u_{t^-} + (-\Delta)^{\sigma/2}u = 0$ in the classical sense. When $t = 0$ the equation is not satisfied any more, $u_{t^-}(x,0)$ is still zero in B_1 but $(-\Delta)^{\sigma/2}u(x,0)$ becomes strictly positive in B_1. If we consider now the same equation in the viscosity sense, u is a solution even when $t = 0$. The restriction for the test functions to be in $C(-\tau,0;L^1(\omega))$ (in the case the contact occurs at $t = 0$) implies that such test function will not be able to see that u has a jump at $t = 0$.

4.1.5 Qualitative properties

Most of the qualitative behavior of solutions of fully non linear non local operators, as considered by us, have been already proven in [8] or in the appendix of [17]. Here we state some of the results already known, that we will need to use later on. The first lemma was proven in [8] and says that for a super solution regularity by below implies that the operator can be evaluated in the classical way. The following three results are the expected maximum and comparison principles which can be proven as in [17]. The last theorem regards to the existence of viscosity solutions of the Dirichlet problem by Perron's method, here we show that by using and appropriated barrier the solution achieves the boundary values in a continuous way.

Lemma 4.1.1. *Let I be a elliptic operator with respect to $\mathcal{L}_0$ and f a continuous function. If we have a super solution, $u_t - Iu = f$ in $\Omega \times (-T, 0]$ and φ is a C^2 function that touches u from below at a point (x_0, t_0), then $Iu(x_0, t_0)$ is defined in the classical sense and $\varphi_{t^-}(x_0, t_0) - Iu(x_0) \geq f(x_0, t_0)$.*

Theorem 4.1.2 (Equation for the difference of solutions)**.** *Let I be a continuous elliptic operator with respect to $\mathcal{L}_0$ and f and g continuous functions. Given u and v such that $u_t - Iu \leq f$ and $v_t - Iv \geq g$ hold in $\Omega \times (-T, 0]$ in the viscosity sense, then $(u - v)_t - \mathcal{M}^+_{\mathcal{L}}(u - v) \leq f - g$ also holds in $\Omega \times (-T, 0]$ in the viscosity sense.*

Theorem 4.1.3 (Maximum principle)**.** *Let u be a viscosity super solution of*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq 0 \text{ in } \Omega \times (-T, 0].$$

Then

$$\inf_{\bar{\Omega}\times[-T,0]} u = \inf_{((\mathbb{R}^n\setminus\Omega)\times(-T,0])\cup(\mathbb{R}^n\times\{-T\})} u.$$

Corollary 4.1.4 (Comparison principle)**.** *Let I be a continuous elliptic operator with respect to $\mathcal{L}_0$, u be a viscosity sub solution and v be a viscosity super solution of*

$$w_t - Iw = f \text{ in } \Omega \times (-T, 0].$$

Then $u \le v$ in $((\mathbb{R}^n \setminus \Omega) \times (-T, 0]) \cup (\mathbb{R}^n \times \{-T\})$ implies $u \le v$ in $\Omega \times (-T, 0]$.

Existence and uniqueness of a solution in the viscosity sense follows from the comparison principle by using Perron's method. The additional ingredient we need is a barrier that guarantees that the boundary and initial values are attained in a continuous way.

Lemma 4.1.5. *Let $\sigma \in (0, 2)$. There exists a non negative function $\psi : \mathbb{R}^n \times (-\infty, 0] \to \mathbb{R}$ such that:*

1. $\psi = 0$ *in* $B_1 \times \{0\}$
2. $\psi_t - \mathcal{M}^+_{\mathcal{L}_0(\sigma)}\psi \ge 0$ *in* $(\mathbb{R}^n \setminus B_1) \times (-\infty, 0]$,
3. $\psi \ge 1$ *in* $(\mathbb{R}^n \times (-\infty, 0]) \setminus (B_2 \times [-\kappa, 0])$,

for some κ universal.

Proof. Let $\varphi = \varphi(x)$ be the one from corollary 3.2 in [8] and

$$\kappa^{-1} = \inf_{B_2 \setminus B_1} |\mathcal{M}^+ \varphi| \wedge 1.$$

Then $\psi = (\varphi - \kappa^{-1} t) \wedge 1$ satisfy all the requirements. □

Theorem 4.1.6 (Existence)**.** *Let $\sigma \in (0,2)$, Ω be a smooth domain, I a continuous elliptic operator with respect to $\mathcal{L}_0$ and f and g bounded, continuous functions. The Dirichlet problem,*

$$u_t - Iu = f \text{ in } \Omega \times (-T, 0],$$
$$u = g \text{ in } ((\mathbb{R}^n \setminus \Omega) \times (-T, 0]) \cup (\mathbb{R}^n \times \{-T\}),$$

has a unique viscosity solution u.

Remark 4.1.7. *The boundary data g only needs to be continuous at the points in $((\mathbb{R}^n \setminus \Omega) \times (-T, 0]) \cup (\mathbb{R}^n \times \{-T\})$ with respect to the parabolic topology. With respect to smoothness of the domain, we only require that Ω satisfies the exterior ball condition.*

Proof. Let u be the solution obtained by Perron's method,

$$\begin{aligned} u(x,t) = \inf\{v(x,t) :& v_t - Iv \geq f \text{ in } \Omega \times (-T, 0] \\ & v \geq g \text{ in } ((\mathbb{R}^n \setminus \Omega) \times (-T, 0]) \cup (\mathbb{R}^n \times \{-T\})\}. \end{aligned}$$

It can be shown that $u \in C(\bar{\Omega} \times [-T, 0])$, solves $u_t - Iu = f$ in $\Omega \times (-T, 0]$ in the viscosity sense and $u \geq g$ in $((\mathbb{R}^n \setminus \Omega) \times (-T, 0]) \cup (\mathbb{R}^n \times \{-T\})$, see [11]. We will show now that u attains the initial and boundary values by comparison with appropriated barriers.

Let's see the case of initial values first. Let $b : \mathbb{R}^n \to [0,1]$ a smooth bump function such that $\text{supp}(1-b) = B_1$ and $b(0) = 0$. The function $\psi(y,s) = b(y) + \|\mathcal{M}^+_{\mathcal{L}_0} b\|_\infty s$ satisfies $\psi_t - \mathcal{M}^+_{\mathcal{L}_0}\psi \geq 0$. Let $(x,-T) \in \bar{\Omega}\times\{-T\}$ and $\varepsilon > 0$ fixed. By the continuity of g, there exists $\delta > 0$ such that $|g(x,-T) - g(y,s)| \leq \varepsilon$, given that $|x-y| + |T+s| \leq 2\delta$. Consider the following barrier,

$$\beta(y,s) = g(x,-T) + \varepsilon + 2\|g\|_\infty \left\{\psi\left(\frac{y-x}{\delta}, \frac{s+T}{\delta^\sigma}\right) + \frac{s+T}{\delta}\right\}.$$

β it is constructed such that $\beta_t - \mathcal{M}^+_{\mathcal{L}_0}\beta \geq 0$. Let's see that $\beta \geq g$ in $((\mathbb{R}^n \setminus \Omega) \times (-T,0]) \cup (\Omega \times \{-T\})$ and therefore $g(x,-T) \leq u(x,-T) \leq \beta(x,-T) = g(x,-T) + \varepsilon$ according to the definition of u. If $|x-y|+|T+s| \leq 2\delta$, then $\beta(y,s) \geq g(x,-T)+\varepsilon \geq g(y,s)$. If $|x-y| + |T+s| \geq 2\delta$ then either $|x-y| \geq \delta$ and then $b((y-x)/\delta) = 1$ or $(T+s) \geq \delta$ and then $(T+s)/\delta \geq 1$, in any case

$$\beta(y,s) \geq -\|g\|_\infty + 2\|g\|_\infty \left\{\psi\left(\frac{y-x}{\delta}, \frac{T+s}{\delta^\sigma}\right) + \frac{T+s}{\delta}\right\} \geq g(y,s).$$

After having that $u(x,-T) \in [g(x,-T), g(x,-T)+\varepsilon]$ we use that ε is arbitrary to conclude that $u(x,-T) = g(x,-T)$.

Let's consider now the case of boundary values. Let $(x,t) \in \partial\Omega \times (-T,0]$ and $\varepsilon > 0$ fixed. Let $\delta > 0$ such that $|g(x,t) - g(y,s)| \leq \varepsilon$, given that $|y-x| \leq \delta$ and $s \in [t-\kappa\delta, t]$, κ is the one from Lemma 4.1.5. By making δ even smaller we can also assume that the ball $B_{\delta/4}(x - (\delta/4)n)$ touches Ω by outside with n its normal vector. Let ψ the function from Lemma 4.1.5. The barrier,

$$\beta(y,s) = g(x,t) + \varepsilon + 2\|g\|_\infty \psi\left(\frac{y-(x-(\delta/4)n))4}{\delta}, s-t\right)$$

is constructed such that $\beta_t - \mathcal{M}^+_{\mathcal{L}_0}\beta \geq 0$. It also remain above g in $((\mathbb{R}^n \setminus \Omega) \times (-T,t]) \cup (\Omega \times \{-T\})$. If $|y-x| \leq \delta$ and $s \in [t-\kappa\delta, t]$ then $\beta(y,s) \geq g(x,t) + \varepsilon \geq g(y,s)$. If (y,s) is outside the cylinder $\bar{B}_\delta(x) \times [t-\kappa\delta, t]$, then it is also outside the cylinder $\bar{B}_{\delta/2}(x-(\delta/4)n) \times [t-\kappa\delta, t]$, then $\psi(\frac{y-(x-(\delta/4)n))4}{\delta}, s-t) \geq 1$ and $\beta(y,s) \geq -\|g\|_\infty + 2\|g\|_\infty \geq g(y,s)$. Then we conclude as before that $u(x,t) = g(x,t)$. □

4.2 Partial ABP Estimate

The classic ABP theorem says the following. If u satisfies $u_t - \mathcal{M}^- u \geq -f$ in $B_1 \times (-1,0]$ with $u \geq 0$ in $\partial B_1 \times (-1,0] \cup B_1 \times \{-1\}$ then,

$$\inf_{B_1 \times (-1,0]} u^- \leq c\left(\iint_{\{u=\Gamma\}} (f^+)^{n+1} dxdt\right)^{\frac{1}{n+1}},$$

where the domain of integration $\{u = \Gamma\}$ is the contact set of u with its parabolic convex envelope Γ.

In the non local case there is no hope to obtain a similar result by integrating only over $\{u = \Gamma\}$. In fact, consider the function $u(x,t) = (|x|^\sigma - 1)\chi_{B_2}(x)$. The contact set in this case has zero measure, however there is a constant $C \geq 0$ such that $u_t + (-\Delta)^\sigma u \geq -C$ holds in $B_1 \times (-1,0]$.

To sort out this difficulty, we consider the set where u is between Γ and $\Gamma + M$, for some positive and universal M. The theorem we prove in this section is the following one.

Theorem 4.2.1. *Let $f \in C([-1,0])$ positive and depending only on the time variable, $\rho_0 > 0$ such that $1/2 + 9\sqrt{n}2^{-1/(2-\sigma)}\rho_0 < 2$ and let u satisfying*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -f\chi_{B_{1/2}} \text{ in } B_2 \times (-2,0],$$
$$u \geq 0 \text{ in } ((\mathbb{R}^n \setminus B_1) \times [-2,0]) \cup (\mathbb{R}^n \times [-2,-1]\}),$$
$$\sup_{B_1 \times (-1,0]} u^- = 1.$$

Then,

$$c \leq \int_{-1}^{0} f(t)^{n+1} |\{u(\cdot,t) < \Gamma(\cdot,t) + C4^{-1/(2-\sigma)} f(t)\} \cap B_{1/2+9\sqrt{n}2^{-1/(2-\sigma)}\rho_0}| dt$$

for some constants c and C depending only on n, Λ, σ_0 and ρ_0.

Remark 4.2.2. *The domain of the equation $B_2 \times (-2,0]$ can be reduced to $B_{1+\varepsilon} \times (-1,0]$ and just indicates that we are going to need some room in the following proofs.*

Remark 4.2.3. *The radius ρ_0 will be a fixed universal constant in future sections. For this reason it may be noticed that we call also universal constants to some quantities that may also depend on ρ_0.*

The following is a more general version of the same theorem.

Theorem 4.2.4. *Let $\Omega \subseteq \mathbb{R}^n$ a bounded domain and $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0 \subset\subset \Omega$. Let $f \in C([-1,0])$ positive and depending only on the time variable, $d > 0$ and let u such that*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -f\chi_{\Omega_2} \text{ in } \Omega \times (-d^\sigma, 0],$$
$$u \geq 0 \text{ in } ((\mathbb{R}^n \setminus \Omega_0) \times (-d^\sigma, 0]) \cup (\mathbb{R}^n \times \{-d^\sigma\}).$$

Then

$$\sup_{\Omega_0\times(-d^\sigma,0]} u^- \le C\left(\int_{-d^\sigma}^{0} f(t)^{n+1}|\{u < \Gamma + C4^{-1/(2-\sigma)}f\}\cap\Omega_1|dt\right)^{1/(n+1)}$$

for some constant C, depending only on n, Λ, σ_0, Ω_2, Ω_1, Ω_0 *and* Ω.

The idea of the proof is to cover $\{u = \Gamma\}$ with a disjoint sequence of rectangles $K_j = Q_j \times I_j$, where Q_j is a cube in space and I_j is an interval in time such that:

1. The measure of the union of the covering is at least a fix constant.
2. In a dilation $\tilde{K}_j$ of K_j, u is most of the time between Γ and $\Gamma + M$.

4.2.1 Preliminaries

Here we will fix the notation that we will carry on for the next results.

From u we construct the following auxiliary functions which allow us to get important information for u. We enumerate them in the following list:

1. Let $(x_0, t_0) \in B_1 \times (-1, 0]$ such that $\sup_{B_1\times(-1,0]} u^- = |u(x_0, t_0)| = 1$.
2. Let $\bar{u}(x, t) = \inf_{s\in[-1,t]} u(x, s)$.
3. Let $\Gamma(\cdot, t)$ be the convex envelope of $\bar{u}(\cdot, t)$ supported in B_3. Notice that Γ is convex in space and non increasing in time. This is what we also call a parabolic convex function.

4. Let $\partial\Gamma(x,t) \subseteq \mathbb{R}^n$ be the set of spatial subdifferential of $\Gamma(\cdot,t)$ at (x,t). Notice that for every $t \in [-1,0]$, $\partial\Gamma(B_3,t) = \partial\Gamma(B_1,t)$, therefore every subdifferential p of Γ can be assume to be the slope of some supporting plane, namely $x \to p\cdot(x-x_0)+h$, to the graph of $\bar{u}(\cdot,t)$.

5. Let $h(\cdot,t) : \partial\Gamma(B_1,t) \to \mathbb{R}$ be the Legendre transform of $\Gamma(\cdot,t)$ centered at x_0,

$$h(p,t) = \sup\{h : p\cdot(x-x_0)+h \leq \Gamma(x,t) \text{ for all } x \in B_3\},$$

   or equivalently,

$$\begin{aligned} h(p,t) = \sup\{h :& p\cdot(x-x_0)+h \leq \bar{u}(x,t) \text{ for all } x \in B_1, \\ & p\cdot(x-x_0)+h \leq 0 \text{ for all } x \in B_3\}. \end{aligned}$$

6. Let $\Phi = (\partial\Gamma(x,t), h(\partial\Gamma(x,t),t))$.

This first lemma give some of the basic properties of the previously defined functions.

Lemma 4.2.5. *Let u, f, x_0, Γ, h and Φ as defined above and consider also $\Delta t > 0$. Then the following properties hold:*

1. *The domain of $h(\cdot,t)$ is non decreasing in time. i.e. $\partial\Gamma(B_1,t) \subseteq \partial\Gamma(B_1,t+\Delta t)$.*

2. *h is non increasing in time.*

3. *The function h restricted to $\{\Gamma = u\}$ is Lipschitz in time. Specifically, for $(x_1,t_1) \in \{\Gamma = u\}$ and $p_1 \in \partial\Gamma(x_1,t_1)$*

$$\Delta h := h(p_1,t_1+\Delta t) - h(p_1,t_1) \geq -2\|f\|_{L^\infty([t_1,t_1+\Delta t])}\Delta t.$$

Proof. The first two properties are consequences of the monotonicity of Γ. If at time t, the plane $x \to p \cdot (x - x_0) + h$ is a supporting plane for the graph of $\Gamma(\cdot, t)$ then at time $t + \Delta t$ it crosses or touches the graph of $\Gamma(\cdot, t + \Delta t) \leq \Gamma(\cdot, t)$, therefore by lowering h we can find a supporting plane for $\Gamma(\cdot, t + \Delta t)$ with the same slope p.

In order to see the next property notice that for every $p \in \Gamma(\cdot, t)$ and $x \in B_3$,

$$p \cdot (x - x_0) + h(p, t + \Delta t) \leq \Gamma(x, t + \Delta t) \leq \Gamma(x, t).$$

This makes $h(p, t + \Delta t)$ an admissible candidate in the definition of $h(p, t)$, and therefore $h(p, t) \geq h(p, t + \Delta t)$.

For the second part, notice first that $p_1 \in \partial\Gamma(B_1, t_1 + \Delta t)$ because of the first property, therefore Δh is well defined. We also have proved that $\Delta h \leq 0$ in the second property, so assume that $\Delta h < 0$ and consider the following test function,

$$v(x, s) = \left(p_1 \cdot (x - x_0) + h(p_1, t_1) + \frac{\Delta h}{2\Delta t}(s - t_1)\right)\chi_{B_1}(x).$$

The infimum of $u - v$ in $\mathbb{R}^n \times [-1, t_1 + \Delta t]$ is strictly negative and attained at some point $(x_2, t_2) \in B_1 \times (t_1, t_1 + \Delta t]$. Indeed, the plane $x \to p_1 \cdot (x - x_0) + h(p_1, t_1)$ crosses the graph of u in $B_1 \times (t_1, t_1 + \Delta t]$, otherwise Δh would not be strictly negative.

We have that $v_t(x_2, t_2) = \Delta h/2\Delta t$ and $\delta(v(\cdot, t_2), x_2; y) \geq 0$. This is immediate if $x_2 \pm y$ lie both inside B_1 or both lie outside B_1. If $x_2 + y \in B_1$ we use that $u^-(\cdot, t_2) \leq 1$ in order to see that $|v(x_2 + y, t_2)|$ is at most $2|v(x_2, t_2)|$. Because $u^- \leq 1$, the plane $x \to p_1 \cdot (x - x_0) + h(p_1, t_1)$ is above -1 at some point in B_1, lets recall also that the same plane is below zero in B_3. It tells us that its slope is at most $1/2$ and then $v(x_2, t_2) - v(x_2 + y, t_2) = -p_1 \cdot y \leq |p_1||y| \leq 1 \leq -v(x_2, t_2)$. Then, if $x_2 - y$ is outside B_1, $\delta(v(\cdot, t_1), x_1; y) = (v(x_1 + y, t_1) - 2v(x_1, t_1)) + v(x_1 - y, t_1) \geq 0$.

We conclude by the monotonicity of $d/dt - \mathcal{M}^-_{\mathcal{L}_0}$,

$$\begin{aligned} -\|f\|_{L^\infty([t_1,t_1+\Delta t])} &\leq (u_t - \mathcal{M}^-_{\mathcal{L}_0} u)(x_2,t_2), \\ &\leq (v_t - \mathcal{M}^-_{\mathcal{L}_0} v)(x_2,t_2), \\ &\leq \frac{\Delta h}{2\Delta t}. \end{aligned}$$

□

4.2.2 Configurations of the covering pieces

In the following lemmas we will study how the solution u detach from Γ around the contact set. In order to keep the statements as simple as possible we will describe here some recurrent geometric configurations and fix the notation for them. For u satisfying the hypothesis of Theorem 4.2.1, $k \in \mathbb{N}$, $\Delta t, \rho_0 \in (0,1)$, $(x_1,t_1) \in B_1 \times (-1,0]$ we consider:

1. $R_i = B_{r_i}(x_1) \setminus B_{r_{i+1}}(x_1)$ for $r_i = 2^{-i}2^{-1/(2-\sigma)}\rho_0$,

2. $S_i = R_i \times [t_1 - \Delta t, t_1 - \Delta t/2]$.

Eventually (x_1,t_1) will be fixed to be in the contact set, k will also be fixed of the order of $1/(2-\sigma)$.

Lemma 4.2.6. *Let u satisfy the hypothesis of Theorem 4.2.1 and for $k \in \mathbb{N}$, $\Delta t \in (0,1)$, $(x_1,t_1) \in B_1 \times (-1,0]$ let r_i and S_i as defined above. Given $(p_1,h_1) \in \mathbb{R}^{n+1}$, $M > 0$ and $\mu \in (0,1)$ such that*

1. $\Gamma(x,t) \geq p_1 \cdot (x - x_0) + h_1$ for $(x,t) \in B_3 \times (t_1 - \Delta t, t_1]$,

2. *For every* $i = 0, 1, \dots, k-1$,

$$\frac{|\{u - (p_1 \cdot (x - x_0) + h_1) \geq \|f^+\|_{L^\infty([t_1 - \Delta t, t_1])} M r_i^2\} \cap S_i|}{|S_i|} \geq \mu, \qquad (4.5)$$

then

$$u - (p_1 \cdot (x - x_0) + h_1) \geq \|f^+\|_{L^\infty([t_1-\Delta t, t_1])} \Delta t \text{ in } B_{r_{k+1}}(x_1) \times [t_1 - \Delta t/2, t_1]$$

if $\Delta t \in (0, r_k^2)$ *and* $M\mu(r_0^{2-\sigma} - r_k^{2-\sigma}) \geq K$, *for some constant* K *independent of* σ.

Corollary 4.2.7. *Let* u *satisfy the hypothesis of Theorem 4.2.1 and for* $k \in \mathbb{N}$, $\Delta t \in (0, 1)$, $(x_1, t_1) \in B_1 \times (-1, 0]$ *let* r_i *and* S_i *as defined above. For every* $\sigma < 2$ *there is some* $k \sim 1/(2-\sigma)$ *such that if,*

1. $\Delta t \in (0, r_k^2)$,
2. $(x_1, t_1) \in \{\Gamma = u\}$,
3. $p_1 \in \partial\Gamma(x_1, t_1)$,

then there is some sufficiently small radius $r \in (0, r_0)$, *so that for* $S = (B_r(x_1) \setminus B_{r/2}(x_1)) \times (t_1 - \Delta t, t_1 - \Delta t/2]$ *the following holds for every* $M > 0$,

$$\frac{|\{u - (p_1 \cdot (x - x_0) + h(p_1, t_1)) \geq \|f^+\|_{L^\infty([t_1-\Delta t, t_1])} M r^2\} \cap S|}{|S|} \leq \frac{2K}{r_0^2} M^{-1}.$$

Proof of Lemma 4.2.6. Let $\beta = \beta_0(x - x_1/r_k) \in [0, 1]$ be a smooth bump function such that *supp* $\beta_0 = B_{3/4}$ and $\beta_0 = 1$ in $B_{1/2}$. Moreover we can choose β_0 such that $\mathcal{M}^-_{\mathcal{L}_0}\beta_0 \geq 0$ if $\beta_0(x) \leq \beta_1$ for some positive constant β_1. We want to use a test function in $B_1 \times [t_1 - \Delta t, t_1]$ of the form

$$\begin{aligned} v(x,t) &= P(x) + m(t)\beta(x) - \|f^+\|_{L^\infty([t_1-\Delta t, t_1])}(t - (t_1 - \Delta t)), \\ P(x) &= (p_1 \cdot (x - x_0) + h_1)\chi_{B_1}(x), \end{aligned}$$

such that $m(t_1 - \Delta t) = 0$ and $m \geq 2\|f^+\|_{L^\infty([t_1-\Delta t,t_1])}\Delta t$ for $t \in [t_1 - \Delta t/2, t_1]$.

Assume by contradiction that,

$$\inf_{B_{r_{k+1}(x_1)} \times [t_1 - \Delta t/2, t_1]} (u - P) < \|f^+\|_{L^\infty([t_1-\Delta t,t_1])}\Delta t.$$

Then for some $(x_2, t_2) \in B_{3r_k/4}(x_1) \times [t_1 - \Delta t, t_1]$,

$$\inf_{\mathbb{R}^n \times [t_1 - \Delta t/2, t_1]} (u - v) = (u - v)(x_2, t_2) < 0.$$

Notice that $B_{3r_k/4}(x_1) \times [t_1 - \Delta t, t_1]$ is contained in the domain of the equation $B_2 \times (-2, 0]$ if $\Delta t < 1$ and $r_0 < 1$. We use Lemma 4.1.1 in order to do the following computations on u at the contact point (x_2, t_2). We also use that $\delta(u - v, x_2, t_2) = \delta^+(u - v, x_2, t_2)$ and $\delta(P, x_2, t_2; y) \geq 0$ for every $y \in \mathbb{R}^n$, as in the proof of Lemma 4.2.5,

$$\begin{aligned} &- m'(t_2)\beta(x_2) + m(t_2)r_k^{-\sigma}\mathcal{M}^-_{\mathcal{L}_0}\beta_0(x_2 - x_0), \\ &\leq (u_t - \mathcal{M}^-_{\mathcal{L}_0}u)(x_2, t_2) - (v_t - \mathcal{M}^-_{\mathcal{L}_0}v)(x_2, t_2), \\ &\leq (u - v)_t(x_2, t_2) - \mathcal{M}^-_{\mathcal{L}_0}(u - v)(x_2, t_2), \\ &\leq -\Lambda^{-1}(2 - \sigma)\int_{\bigcup_{i=0}^{k-1} R_i \times \{t_2\}} \frac{\delta^+(u - v, x_2, t_2; y)}{|y|^{n+\sigma}} dy, \\ &= -\Lambda^{-1}(2 - \sigma)\int_{\bigcup_{i=0}^{k-1} R_i \times \{t_2\}} \frac{\delta^+(u - m\beta, x_2, t_2; y)}{|y|^{n+\sigma}} dy. \end{aligned}$$

If $x_2 + y \in R_i$, then $|y| \sim r_i$, because $x_2 \in B_{3r_k/4}(x_1)$. Also

$$\delta^+(u - m\beta) \geq \|f^+\|_{L^\infty([t_1-\Delta t, t_1])} M r_i^2$$

, every time $x_2 + y \in G_i(t_2)$, where

$$G_i(t) = \{u - (p \cdot (x - x_0) + h_1) \geq \|f^+\|_{L^\infty([t_1-\Delta t, t_1])} M r_i^2\} \cap (R_i \times \{t\}).$$

Therefore we obtain, for some constant C_0 depending only on Λ^{-1} and the dimension,

$$-m'\beta + m r_k^{-\sigma} \mathcal{M}^-_{\mathcal{L}_0}\beta_0 \leq -C_0(2-\sigma)\|f^+\|_{L^\infty([t_1-\Delta t, t_1])} M \sum_{i=0}^{k-1} \frac{|G_i(t_0)|}{r_i^n} r_i^{2-\sigma}. \tag{4.6}$$

As in [16], if m satisfies:

$$m'(t) = c_1(2-\sigma)\|f^+\|_{L^\infty([t_1-\Delta t, t_1])} M \sum_{i=0}^{k-1} \frac{|G_i(t)|}{r_i^n} r_i^{2-\sigma} - C_2 r_k^{-\sigma} m(t),$$
$$m(t_1 - \Delta t) = 0,$$

with constants c_1 sufficiently small and C_2 sufficiently large then we get a contradiction. Indeed, if $\beta_0(x_2 - x_1/r_k) \leq \beta_1$ then $\mathcal{M}^-_{\mathcal{L}_0}\beta_0 \geq 0$ and we see that $0 < c_1 \leq C_0$ implies a contradiction by substituting m' in (4.6). Otherwise, if $\beta_0(x_2 - x_1/r_k) > \beta_1$, then we also get a contradiction in (4.6) if $C_2 \geq \|\mathcal{M}^-_{\mathcal{L}_0}\beta_0\|_\infty/\beta_1$.

We finally need to check that $m \geq 2\|f^+\|_{L^\infty([t_1-\Delta t, t_1])}\Delta t$ in $[t_1 - \Delta t/2, t_1]$ from the hypothesis of the lemma. We have an explicit formula for m,

$$m(t) = c_1(2-\sigma)\|f^+\|_{L^\infty([t_1-\Delta t, t_1])} M \sum_{i=0}^{k-1} r_i^{2-\sigma} \int_{t_1-\Delta t}^{t} \frac{|G_i(s)|}{r_i^n} e^{-C_2 r_k^{-\sigma}(t-s)} ds.$$

Using the hypothesis (4.5) of the lemma, for $t \geq t_1 - \Delta t/2$, we get

$$m \geq C(2-\sigma)\|f^+\|_{L^\infty([t_1-\Delta t, t_1])} M\mu\Delta t \frac{r_0^{2-\sigma} - r_k^{2-\sigma}}{1-2^{\sigma-2}} e^{-C_2 r_k^{-\sigma}\Delta t}.$$

The quotient $(2-\sigma)/(1-2^{\sigma-2})$ is bounded away from zero by a universal constant when $\sigma \in [0,2]$. Also $e^{-C_2 r_k^{-\sigma}\Delta t} \geq e^{-C_2}$ if $\Delta t \leq r_k^\sigma$. Finally $m \geq 2\|f^+\|_{L^\infty([t_1-\Delta t, t_1])}\Delta t$ is satisfied if $M\mu(r_0^{2-\sigma} - r_k^{2-\sigma}) \geq K$ for some K independent of σ. □

The following is a geometric lemma that can be applied to any parabolic convex function. As a reminder, we say that $\Gamma(x,t)$ is a parabolic convex function if it is convex in the variable $x \in \mathbb{R}^n$ and non increasing in the variable $t \in \mathbb{R}$.

Lemma 4.2.8. *Let* $\Gamma : B_3 \times [-\Delta t, 0] \to \mathbb{R}$ *parabolic convex function such that*

$$\frac{|\{\Gamma \geq M\} \cap (B_r \setminus B_{r/2}) \times [-\Delta t, -\Delta t/2]|}{|(B_r \setminus B_{r/2}) \times [-\Delta t, -\Delta t/2]|} \leq \varepsilon_0.$$

Then $\Gamma \leq M$ *in* $B_{r/2} \times [-\Delta t/2, 0]$ *if* ε_0 *is sufficiently small, depending only on n.*

Proof. By the convexity of Γ we can assume that its maximum N over $B_{r/2} \times [-\Delta t/2, 0]$, is attained at $(r/2e_1, -\Delta t/2)$. Therefore $\Gamma \geq N$ in $A = \{(x,t) \in (B_r \setminus B_{r/2}) \times [-\Delta t, -\Delta t/2] : x \cdot e_1 > r/2\}$. Therefore, if ε_0 is smaller than $|A|/|(B_r \setminus B_{r/2}) \times [-\Delta t, -\Delta t/2]|$, we obtain that N is necessarily smaller or equal than M in $B_{r/2} \times [-\Delta t/2, 0]$. □

By applying Lemma 4.2.5 and the previous lemma to $\Gamma(x,t) - (p_1 \cdot (x - x_0) + h(p_1, t_1))$ with all the hypothesis and conclusions of Corollary 4.2.7 we obtain the following result.

Corollary 4.2.9 (Flatness of Γ)**.** *Let u satisfy the hypothesis of Theorem 4.2.1 and (x_1, t_1), p_1 and r as in Corollary 4.2.7. There exist a universal constant $M > 0$ such that for every $(x, t) \in F = B_{r/2}(x_1) \times [t_1 - \Delta t/2, \min\{t_1 + \Delta t/2, 0\}]$*

$$-2\|f^+\|_{L^\infty([t_1,\min\{t_1+\Delta t,0\}])} r^2 \leq \Gamma(x,t) - (p_1 \cdot (x - x_0) + h(p_1, t_1)),$$
$$\leq M\|f^+\|_{L^\infty([t_1-\Delta t,t_1])} r^2.$$

The previous Corollary seems still insufficient to control

$$\frac{|\Phi(B_{r/4}(x_1) \times [t_1 - \Delta t/2, t_1 + \Delta t/2])|}{|B_{r/4}(x_1) \times [t_1 - \Delta t/2, t_1 + \Delta t/2]|}.$$

The flatness property takes care of the n-dimensional size of $\partial\Gamma(B_{r/4}(x_1) \times \{t\})$ by using the geometry of the convex function $\Gamma(\cdot, t)$ ($t \in [t_1 - \Delta t/2, t_1 + \Delta t/2]$). Note also that the image of $\Phi(\cdot, t) = (\partial\Gamma(\cdot, t), h(\partial\Gamma(\cdot, t), t))$ is the graph of $h(\cdot, t)$, for which we use again the properties in Lemma 4.2.5.

Corollary 4.2.10. *Let u satisfy the hypothesis of Theorem 4.2.1 and (x_1, t_1), p_1 and r as in Corollary 4.2.7. There exist a universal constant $C > 0$ such that for $K = B_{r/4}(x_1) \times [t_1 - \Delta t/2, \min\{t_1 + \Delta t/2, 0\}]$ we have*

$$\frac{|\Phi(K)|}{|K|} \leq C\|f^+\|^{n+1}_{L^\infty([t_1-\Delta t,\min\{t_1+\Delta t,0\}])}.$$

Proof. We do the proof for $t_1 + \Delta t \leq 0$ in order to avoid the difficulties that would arise if K goes beyond $t = 0$. In this case the proof does not differ to much from the one we present but makes the proof more technical.

As a consequence of Corollary 4.2.9 we have that $\partial\Gamma(B_{r/4}(x_1)\times\{t_1+\Delta t/2\}) \subseteq B_{Cr\|f^+\|_{L^\infty([t_1-\Delta t,t_1])}}(p_1)$ and then

$$\frac{|\partial\Gamma(B_{r/4}(x_1)\times\{t_1+\Delta t/2\}|}{|B_{r/4}(x_1)|} \leq C\|f^+\|^n_{L^\infty([t_1-\Delta t,t_1])}.$$

By Lemma 4.2.5,

$$\Phi(B_{r/4}(x_1)\times[t_1-\Delta t/2, t_1+\Delta t/2]) \subseteq \mathit{Cylinder},$$

where

$$\begin{aligned} \mathit{Cylinder} = \{(p,h) : & p \in \partial\Gamma(B_{r/4}(x_1)\times\{t_1+\Delta t/2\}), \\ & h \in [h(p,t), h(p,t) + 2\Delta t\|f^+\|_{L^\infty([t_1,t_1+\Delta t])}]\}. \end{aligned}$$

Finally the measure of *Cylinder* is controlled by its base times the height. □

4.2.3 Covering of the contact set

We state now a weak version of the ABP estimate. The result consists in finding a covering of the contact set where the solution does not separate too much from the convex envelope in a given fraction of the union of the covering.

Lemma 4.2.11. *Let u satisfy the hypothesis of Theorem 4.2.1. There exists a finite family of disjoint rectangles $\{K_j = Q_j \times I_j\}$, where $Q_j \subseteq \mathbb{R}^n$ is an open cube with diameter $d_j \leq r_0/4$ and $I_j = (-(l_j+1)\Delta t/2, -l_j\Delta t/2)$ with l_j non negative integer, such that:*

1. $K_j \cap \{u = \Gamma\} \neq \emptyset$,

2. $\bigcup_j \bar{K}_j \supseteq \{u = \Gamma\}$,

3. Γ *is between two planes in* $Q_j \times I_j$*, which are separated by a distance of size* $C\|f\|_{L^\infty(I_j)} d_j^2$,

4. $|\Phi(K_j)| \leq C\|f\|^{n+1}_{L^\infty(I_j)} |K_j|$,

5. $|\{u < \Gamma + C\|f\|_{L^\infty(I_j)} d_j^2\} \cap \tilde{K}_j| \geq (1 - \varepsilon_0)|\tilde{K}_j|$*, where* $\tilde{K}_j = 16\sqrt{n} Q_j \times [-(l_j + 3)\Delta t/2, -l_j \Delta t/2]$.

Proof. Fix a slice $B_1 \times I_l$ and cover it by a tiling of the form $\{Q \times I_l\}$ where Q have diameter $r_0/4$. Discard all of those rectangles that do not intersect $\{u = \Gamma\}$. Whenever $Q \times I_l$ does not satisfy (3), (4) or (5), we split Q into 2^n cubes Q' of half diameter and discard all of the rectangles $Q' \times I_l$ whose closure does not intersect $\{u = \Gamma\}$. We need to prove that eventually all rectangles satisfy (3), (4) and (5) and therefore the process finishes after a finite number of steps. In fact we will show that it will finish before $k \sim 1/(2 - \sigma)$ iterations.

As before, in order to avoid technical difficulties, we assume that $l \geq 1$. Let $Q_1 \times I_l \supseteq Q_2 \times I_l \supseteq \ldots \supseteq Q_k \times I_l \ni (x_1, t_1)$ such that $(x_1, t_1) \in \{u = \Gamma\}$ and let's see that at least one of those rectangles satisfy all the properties (3), (4) and (5). From Lemmas 4.2.9 and 4.2.10, there is some radius $r \in [r_k, r_0]$ and some subdifferential $p_1 \in \partial\Gamma(x_1, t_1)$ such that the following are true:

1. $|\Gamma - (p_1 \cdot (x - x_0) + h(p_1, t_1))| \leq C\|f\|_{L^\infty(I_l)} r^2$ in $F = B_{r/4} \times [t_1 - \Delta t/2, t_1 + \Delta t/2]$,

2. $|\Phi(K)| \leq C\|f\|^{n+1}_{L^\infty(I_l)} |K|$ for $K = B_{r/4} \times [t_1 - \Delta t/2, t_1 + \Delta t/2]$,

3. $|\{u - (p_1 \cdot (x - x_0) + h(p_1, t_1)) \geq \|f\|_{L^\infty(I_l)} M r^2\} \cap S| \leq \varepsilon_0 |S|$ for $S = B_r \times [t_1 - \Delta t, t_1 - \Delta t/2]$.

There is one of the rectangles $Q_j \times I_l$, with $\mathrm{diam}(Q_j) = d$, such that $r/8 \leq d < r/4$. Therefore $Q_j \times I_l \subseteq K(x_1, t_1)$ and conditions (3) and (4) from the lemma are verified. To check (5), notice that $S \subseteq 16\sqrt{n} Q_j \times [-(l+3)\Delta t/2, -l\Delta t/2] = \tilde{K}_j$, and that the volumes of S, K_j and $\tilde{K}_j$ are comparable, hence

$$
\begin{aligned}
&|\{u < \Gamma + \|f\|_{L^\infty(I_l)} M d_j^2\} \cap \tilde{K}_j|, \\
&\geq |\{u < p_1 \cdot (y - x_1) + h(p_1, t_1) + \|f\|_{L^\infty(I_l)} M d_j^2\} \cap S|, \\
&\geq (1 - \varepsilon_0)|S|.
\end{aligned}
$$

This is how μ is chosen and this concludes the proof. □

4.2.4 Proof of Theorem 4.2.1

We have as in [19] that

$$
\Phi(\{\Gamma = u\}) \supseteq Cone = \{(p, h) : h \in [-1, 0], |h| > 4|p|\}.
$$

The inclusion follows because for every $(p, h) \in Cone$ the plane $x \to p \cdot (x - x_0) + h$ can be brought from $t = -1$ towards the future until it hits the graph of u (and also the graphs of $\bar{u}$ and Γ) for the first time.

Therefore for some universal constants,

$$
C \leq |\Phi(\cup_j K_j)| \leq \sum_j |\Phi(K_j)| \leq C \sum_j \|f\|_{L^\infty(I_j)}^{n+1} |K_j|
$$

We group now the previous sum in each interval $J_l = (-(l+1)\Delta t/2, -l\Delta t/2)$,

$$C \leq \sum_l \|f\|_{L^\infty(J_l)}^{n+1} \sum_{I_j=J_l} |K_j| \leq \sum_l \|f\|_{L^\infty(J_l)}^{n+1} \left| \bigcup_{I_j=J_l} K_j \right|$$

By Besicovitch, we can take a sub set of $\{\tilde{K}_j\}_{I_j=J_l}$ (denoted by the same) with the finite intersection property and still covering $\cup_{I_j=J_l} K_j$ such that

$$\begin{aligned}\sum_{I_j=J_l} |\tilde{K}_j| &\leq C \sum_{I_j=J_l} |\{u < \Gamma + C4^{-1/(2-\sigma)}\|f\|_{L^\infty(J_l)}\} \cap \tilde{K}_j|,\\ &\leq C \left|\{u < \Gamma + C4^{-1/(2-\sigma)}\|f\|_{L^\infty(J_l)}\} \cap \left(\cup_{I_j=J_l}\tilde{K}_j\right)\right|.\end{aligned}$$

Notice that the contact set $\{u = \Gamma\}$ can only occur where $f\chi_{B_{\rho_0}}$ is positive. This implies that $\left(\cup_{I_j=J_l}\tilde{K}_j\right) \subseteq B_{9\sqrt{n}\rho_0}$ and then we have the following Riemann sum which is now independent of the covering,

$$c \leq \sum_l \|f\|_{L^\infty(J_l)}^{n+1} \left|\{u < \Gamma + C4^{-1/(2-\sigma)}\|f\|_{L^\infty(J_l)}\} \cap B_{9\sqrt{n}\rho_0} \times \tilde{J}_l\right|$$

where $\tilde{J}_l = (-(l+3)\Delta t/2, -l\Delta t/2)$. Now we just have to send Δt to zero to conclude the theorem.

In [8] the partial ABP involves a Riemann sum of $|f|^n$ which gets refined in the limit, when σ goes to 2, and allows to recover the classic ABP. The estimate presented here is weaker, assuming that the right hand side of the equation is $f(x,t)$ we notice that in Lemma 4.2.5 we need to take a global L^∞ norm in space and not just around the contact point. All the other proofs work fine in this sense. Our proofs can recover the following consequence of the classical ABP as σ goes to two,

$$\sup_{B_1\times(-1,0]} u^- \leq C\left(\int_{\{\Gamma=u\}} \sup_{y\in B_1}(f^+)^{n+1} dt\right)^{1/n+1}.$$

4.3 Point Estimate

We are interested now in proving a point estimate that will allow us to control the oscillation of the solution. Our goal is the following theorem. (Recall that $f \in C[-1,0]$ and non negative).

Theorem 4.3.1 (Point Estimate)**.** *Let $\sigma_0 \in (0,2)$ and $\sigma \in (\sigma_0, 2)$. Suppose u satisfies*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0(\sigma)} u \geq -f(t) \text{ in } B_1 \times (-1,0],$$
$$u \geq 0 \text{ in } \mathbb{R}^n \times [-1,0].$$

Then, for every $s \geq 0$,

$$\frac{|\{u > s\} \cap B_{1/2} \times [-1,-1/2]|}{|B_{1/2} \times [-1,-1/2]|} \leq C \left(\inf_{B_{1/2}\times[-1/2,0]} u + \|f^+\|_{L^\infty([-1,0])} \right)^\varepsilon s^{-\varepsilon},$$

for some constants ε, C depending only on n, Λ and σ_0.

The proof of Theorem 4.3.1 is done by induction as in [21]. The idea is to get a control of the measure of the set where u is bigger than a universal constant and then being able to reproduce the estimate at every scale.

4.3.1 Initial configurations

Lemma 4.3.2 (Special Function)**.** *Let $\sigma_0 \in (0,2)$, $\sigma \in (\sigma_0, 2)$. There is a function $p(x,t) \in C(\mathbb{R}^n \times [0,80])$ and a constant $C > 1$, such that for any $\sigma \in (\sigma_0, 2)$,*

$$p_{t^-} - \mathcal{M}^-_{\mathcal{L}_0(\sigma)} p \leq -1 + C\chi_{B_{1/4}\times(0,1]} \text{ in } B_{4\sqrt{n}} \times (0,80],$$
$$p(x,t) \leq 0 \text{ in } ((\mathbb{R}^n \setminus B_{2\sqrt{n}}) \times (0,80]) \cup (\mathbb{R}^n \times \{0\}),$$
$$p(x,t) > 2 \text{ in } Q_3 \times [1,80].$$

The function p will also be $C^{1,1}$ in the space variable and C^1 in the time variable (with respect to the parabolic topology), so that the computation of the equation is done in the classical sense.

Proof. Consider

$$f(x) = \begin{cases} |x|^{-p} & \text{in } \mathbb{R}^n \setminus B_\delta, \\ q & \text{in } B_\delta, \end{cases}$$

where q is a quadratic polynomial chosen so that f is $C^{1,1}$ across ∂B_δ. From Section 9 in [8] we know that it satisfies $\mathcal{M}^-_{\mathcal{L}_0(\sigma)} f > 0$ in $\mathbb{R}^n \setminus B_{1/4}$ for some sufficiently large $p > 0$ and some sufficiently small $\delta \in (0, 1/4)$, independently of $\sigma \in (\sigma_0, 2)$. By multiplying f by a sufficiently large constant we can also assume that

$$\begin{aligned} \mathcal{M}^-_{\mathcal{L}_0(\sigma)} f &\geq 1 - C_0 \chi_{B_{1/4}} \text{ in } B_{4\sqrt{n}}, \\ |Df(x) \cdot x| &\leq C_1 \text{ in } \mathbb{R}^n, \\ f &< C_2 \text{ in } \mathbb{R}^n. \end{aligned}$$

Consider also a continuous function $m(t) \geq 0$ such that for some $\tau \in (0, 1)$ to be fixed,

$$\begin{aligned} m(t) &= t^{1/2} \text{ in } [0, \tau], \\ m(t) &= \tau^{1/2} e^{-\frac{C_1 + C_0}{C_2 \tau}(t-\tau)} \text{ in } (\tau, 80], \end{aligned}$$

Let $\tilde{p}(x,t) = m(t) f(y)$, for $y = t^{-2/\sigma_0} x$. Notice that $\tilde{p}(\cdot, 0) \equiv 0$ defines $\tilde{p}$ continuously up to time zero.

In the region $B_{4\sqrt{n}} \setminus B_{1/4} \times (0, \tau]$ we have that,

$$
\begin{aligned}
(\tilde{p}_t - \mathcal{M}^-_{\mathcal{L}_0}\tilde{p})(x,t) &= \frac{f(y)}{2t^{1/2}} - (2/\sigma_0)t^{-1/2}Df(C_2y)\cdot y - t^{1/2-2\sigma/\sigma_0}\mathcal{M}^-_{\mathcal{L}_0}f(y),\\
&< (C_2/2 + 2C_1/\sigma_0)t^{-1/2} - t^{-3/2},
\end{aligned}
$$

Therefore, by choosing $\tau \leq 1/(C_2/2 + 2C_1/\sigma_0)$ we make $p_t - \mathcal{M}^-_{\mathcal{L}_0}p < 0$ in $B_{4\sqrt{n}} \setminus B_{1/4} \times [0, \tau]$.

In the region $B_{4\sqrt{n}} \times (\tau, 80]$ we have that,

$$
\begin{aligned}
(\tilde{p}_t - \mathcal{M}^-_{\mathcal{L}_0}\tilde{p})(x,t) &= m'(t)f(y) - \frac{m(t)}{t}Df(C_2y)\cdot y - \frac{m(t)}{t^{\sigma/\sigma_0}}\mathcal{M}^-_{\mathcal{L}_0}f(y),\\
&< m'(t)C_2 + m(t)(C_1 + C_0)/\tau,
\end{aligned}
$$

which is zero by the construction of m in $(\tau, 80]$.

Finally, we define $p = A(\tilde{p} - B)^+$ with $B \geq 0$ chosen such that $\tilde{p} - B \leq 0$ in $((\mathbb{R}^n \setminus B_{2\sqrt{n}}) \times (0, 80]) \cup (\mathbb{R}^n \times \{0\})$ and $A \geq 1$ chosen such that $p > 2$ in $Q_3 \times [1, 80]$ and $p_t - \mathcal{M}^-_{\mathcal{L}_0(\sigma)}p \leq -1$ in $B_{4\sqrt{n}} \times (0, 80]$. □

We are in shape now to prove a first control of the distribution.

Lemma 4.3.3 (Base configuration)**.** *Let $\sigma_0 \in (0,2)$, $\sigma \in (\sigma_0, 2)$, and u a function such that*

$$
\begin{aligned}
u_t - \mathcal{M}^-_{\mathcal{L}_0(\sigma)}u &\geq -1 \text{ in } B_{4\sqrt{n}} \times (0, 80],\\
u &\geq 0 \text{ in } \mathbb{R}^n \times [0, 80],\\
\inf_{Q_3\times[1,80]} u &\leq 1,
\end{aligned}
$$

Then

$$|\{u > M_0\} \cap Q_1 \times [0,1]| \le \mu_0 |Q_1 \times [0,1]|,$$

for some universal constants $\mu_0 \in (0,1)$ *and* $M_0 > 1$.

Proof. Let $v = u - p$, where p was constructed in the previous lemma. The graph of v goes below -1 at some point in $Q_3 \times [1,80]$, stays non negative in $((\mathbb{R}^n \setminus Q_3) \times (1,80]) \cup (\mathbb{R}^n \times \{0\})$ and satisfies

$$v_t - \mathcal{M}^-_{\mathcal{L}_0} v \ge (u_t - \mathcal{M}^-_{\mathcal{L}_0} u) - (p_t - \mathcal{M}^-_{\mathcal{L}_0} p) \ge -C\chi_{B_{1/4}\times(0,1]}.$$

We apply now a Theorem 4.2.4 to v with $\Omega = B_{4\sqrt{n}}$, $\Omega_0 = Q_3$, $\Omega_2 = B_{1/4}$ and $\Omega_1 = Q_1$,

$$c \le \int_0^1 |\{v < \Gamma_v + C\} \cap B_{1/2}| dt \le |\{u < p + C\} \cap Q_1 \times [0,1]|.$$

We just choose $\mu_0 = 1 - c$ and $M_0 = \sup_{Q_1\times[0,1]} p + C$ to conclude. □

We will need the following corollaries of the previous lemma. The first one will be necessary for the particular dyadic decomposition we will introduce in the next section. The second one iterates m times Corollary 4.3.4.

Corollary 4.3.4 (Flexible configuration)**.** *Let* $\sigma_0 \in (0,2)$*,* $\sigma \in (\sigma_0, 2)$*,* $\tau \in [1,8]$ *and u a function such that*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0(\sigma)} u \ge -1 \text{ in } B_{4\sqrt{n}} \times (0, (3^\sigma + 1)\tau],$$

$$u \ge 0 \text{ in } \mathbb{R}^n \times [0, (3^\sigma + 1)\tau],$$

$$\inf_{Q_3\times[\tau,(3^\sigma+1)\tau]} u \le 1.$$

Then for $\mu_1 = \frac{7+\mu_0}{8}$,

$$|\{u > M_0\} \cap Q_1 \times [0,\tau]| \leq \mu_1 |Q_1 \times [0,\tau]|,$$

for $\mu_0 \in (0,1)$ *and* $M_0 > 1$ *as in Lemma 4.3.3.*

Proof. Notice that $Q_3 \times [\tau, (3^\sigma + 1)\tau] \subseteq Q_3 \times [1, 80]$ therefore if u goes below 1 in $Q_3 \times [\tau, (3^\sigma + 1)\tau]$ then $|\{u > M_0\} \cap Q_1 \times [0,1]| \leq \mu_0 |Q_1 \times [0,1]|$ and

$$\begin{aligned} |\{u > M_0\} \cap Q_1 \times [0,\tau]| &\leq \mu_0 |Q_1 \times [0,1]| + |\{u > M_0\} \cap Q_1 \times [1,\tau]|\,, \\ &\leq \frac{7+\mu_0}{8} |Q_1 \times [0,\tau]|. \end{aligned}$$

□

Corollary 4.3.5 (Iteration)**.** *Let* $\sigma_0 \in (0,2)$*,* $\sigma \in (\sigma_0, 2)$*,* $\tau \in [1,8]$*,* $k \geq 1$ *a natural number,* $d_i = \frac{3^{\sigma(i+1)}-1}{3^\sigma - 1}$ *and u a function such that*

$$\begin{aligned} u_t - \mathcal{M}^-_{\mathcal{L}_0(\sigma)} u &\geq -1 \text{ in } B_{2\sqrt{n}3^k} \times (0, d_k\tau]\,, \\ u &\geq 0 \text{ in } \mathbb{R}^n \times [0, d_k\tau]\,, \\ \inf_{\cup_{i=1}^{k-1} Q_{3^{\sigma i}} \times [d_i\tau, d_{i+1}\tau]} u &\leq 1. \end{aligned}$$

Then

$$\left|\left\{u > M_0^k\right\} \cap Q_1 \times [0,\tau]\right| \leq \mu_1 |Q_1 \times [0,\tau]|,$$

for $\mu_1 \in (0,1)$ *and* $M_0 > 1$ *as in Corollary 4.3.4.*

Proof. Just apply Corollary 4.3.4, rescaled, k times. □

4.3.2 A Calderón - Zygmund Lemma

The purpose of a Calderón - Zygmund type Lemma is to find a cover of a given set A with dyadic boxes that capture a fraction of A around a given $\mu_1 \in (0, 1)$.

The boxes are chosen from a dyadic decomposition of $Q_1 \times [0, 1]$ that almost preserve the scaling of the equation. When σ is either 1, $\log_2 3$ or 2 the decomposition can be made preserving the scaling of the equation by dividing by 2, 3 or 4 in time respectively. The following algorithm for general σ was communicated to us by Luis Caffarelli.

Initially we consider $Q_1 \times [0, 1]$, split Q_1 into 2^n congruent cubes, $[0, 1]$ into 4 congruent intervals and take the 2^{n+2} possible cartesian products to form the new dyadic boxes. In each step we consider one of the cubes $Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]$ and always divide $Q_r(x_0)$ in 2^n congruent cubes; with respect to the time interval we do the following:

1. If $\tau < 2$ then we do not subdivide $[t_0, t_0 + r^\sigma \tau]$.

2. If $2 \leq \tau < 4$ then we subdivide $[t_0, t_0 + r^\sigma \tau]$ in 2 congruent intervals.

3. If $4 \leq \tau$ then we subdivide $[t_0, t_0 + r^\sigma \tau]$ in 4 congruent intervals.

Finally we take the cartesian product to form the new generation of dyadic boxes from $Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]$.

This procedure verifies that if $\tau \in [1, 8]$, then the boxes that $Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]$ generates have dimensions $r/2$ (in space) and $(r/2)^\sigma \tau'$ (in time) for some $\tau' \in [1, 8]$. To prove it, just consider each of the cases.

Given two dyadic boxes K and $\tilde{K}$ we say that $\tilde{K}$ is a predecessor of K if K is one of the boxes obtained from the decomposition of $\tilde{K}$.

The following lemma follows as the one in chapter 4 of [5] with the difference that the Lebesgue decomposition theorem is applied to rectangles of dimensions ρ and ρ^σ instead of the standard cubes.

Lemma 4.3.6. *Let $A \subseteq Q_1 \times [0, 1]$ and $\mu_1 \in (0, 1)$, such that $|A| \leq \mu_1 |Q_1 \times [0, 1]|$. Then there exists a set of disjoint dyadic boxes $\{K_j\}$ such that:*

1. $|\cup_j K_j \setminus A| = 0$,
2. $|A \cap K_j| > \mu_1 |K_j|$,
3. $|A \cap \tilde{K}_j| \leq \mu_1 |\tilde{K}_j|$.

Proof. Starting with $Q_1 \times [0, 1]$, we subdivide the dyadic boxes (with the previous algorithm) that capture a fraction of A smaller or equal to μ_1 and select those boxes $\{K_j\}$ that capture a fraction bigger than μ_1. Initially $Q_1 \times [0, 1]$ captures a fraction of A smaller or equal to μ_1, therefore we know that $Q_1 \times [0, 1]$ is subdivided and $\tilde{K}_j \subseteq Q_1 \times [0, 1]$.

This process selects a family of disjoint boxes $\{K_j\}$ that satisfy 2 and 3.

To verify 1 we use the Lebesgue differentiation theorem. For each $(x, t) \in \cup_j K_j \setminus A$ there exist a family of dyadic boxes $\{K_i^{(x,t)} = Q_{r_i}(x_j) \times [t_i, t_i + r_i^\sigma \tau_i]\}_{i \geq 1}$ such that,

1. $(x, t) \in K_i^{(x,t)}$,

2. $r_i \to 0$ as $i \to \infty$ and $\tau_i \in [0, 8]$,

3. $|A \cap K_i^{(x,t)}| \leq \mu_1 |K_i^{(x,t)}|$.

From $K_i^{(x,t)} = Q_{r_i}(x_j) \times [t_i, t_i + r^\sigma \tau]$ we construct a box with a scale σ, $\bar{K}_i^{(x,t)} = Q_{\rho_i}(x_j) \times [t_i, t_i + \rho_i^\sigma] \supseteq K_i^{(x,t)}$ such that $\rho_i = r_i \tau_i^{1/\sigma}$. They satisfy instead,

1. $(x, t) \in \bar{K}_i^{(x,t)}$,

2. $\rho_i \to 0$ as $i \to \infty$,

3. $|A \cap \bar{K}_i^{(x,t)}| \leq \bar{\mu}_1 |\bar{K}_i^{(x,t)}|$ with $\bar{\mu}_1 = \frac{(8^{\sigma_0} - 1) + \mu_0}{8^{\sigma_0}} < 1$.

Then we can apply a modified version of the Lebesgue differentiation theorem to conclude that $| \cup_j K_j \setminus A| = 0$. See for instance Exercise 3 in Chapter 7 of [22]. □

This lemma however can not be applied in our situation directly. The results of the previous section say that if u goes below 1 in some region in the future then we can control the distribution in the past; but the predecessor $\tilde{K}_j$ might give no information of what happens with u in the future. For this reason we need to consider shifts in time of $\tilde{K}_j$. For a given cube $K = Q \times [t_0, t_0 + r]$ and a natural number $m \geq 1$ let $K^m = Q \times [t_0 + r, t_0 + (m + 1)r]$.

The following lemma is proven as in section 3 of [21].

Lemma 4.3.7. *Let $A \subseteq Q_1 \times [0, 1]$ and $\mu_1 \in (0, 1)$, such that $|A| \leq \mu_1 |Q_1 \times [0, 1]|$. Then there exists a set of disjoint dyadic boxes $\{K_j\}$ such that:*

1. $| \cup_j K_j \setminus A| = 0$,

2. $|A \cap K_j| > \mu_1 |K_j|$,

3. $|A| \leq \frac{(m+1)\mu_1}{m} | \cup_j (\tilde{K}_j)^m|$.

Proof. Select the same covering $\{K_j\}$ of A from Lemma 4.3.6. Consider a disjoint sub covering of $\{\tilde{K}_j\}$ which also covers A in measure (denoted by the same). Then

$$|A| \leq \sum_j |A \cap \tilde{K}_j| \leq \mu_1 | \cup_j \tilde{K}_j| \leq \mu_1 | \cup_j (\tilde{K}_j \cup (\tilde{K}_j)^m)|.$$

Now we show that

$$| \cup_j (\tilde{K}_j \cup (\tilde{K}_j)^m)| \leq \frac{m+1}{m} | \cup_j (\tilde{K}_j)^m|. \tag{4.7}$$

Let E be the interior of $\cup_j (\tilde{K}_j)^m$ and consider the open bounded sets of the real line $E_x = \{t \in \mathbb{R} : (x,t) \in E\}$. Take the decomposition of E_x into a countable set of disjoint open intervals $E_x = \cup_j I_x^j$ and for every $I_x^j = (a,b)$ take $TI_x^j = (a - \frac{1}{m+1}(b - a), b)$. Define also $TE_x = \cup_j TI_x^j$ and $TE = \cup_x TE_x$. By Fubini,

$$\begin{aligned} |TE| &= \int_{\mathbb{R}^n} |TE_x|, \\ &= \int_{\mathbb{R}^n} | \cup_j TE_x^j|, \\ &\leq \frac{m+1}{m} \int_{\mathbb{R}^n} |E_x^j|, \\ &\leq \int_{\mathbb{R}^n} \frac{m+1}{m} |E_x|, \\ &= \frac{m+1}{m} |E|, \\ &\leq \frac{m+1}{m} | \cup_j (\tilde{K}_j)^m|. \end{aligned}$$

On the other hand, the interior of $\cup_j(\tilde{K}_j \cup (\tilde{K}_j)^m)$ is contained in TE, by the definition of T and the definition of the stacks. But $\cup_j(\tilde{K}_j \cup (\tilde{K}_j)^m)$ and its interior have the same measure and therefore we obtain (4.7) and conclude the proof. □

By combining Corollary 4.3.5 with Lemma 4.3.7 we get the following result.

Lemma 4.3.8. *Let $\sigma_0 \in (0, 2)$, $\sigma \in (\sigma_0, 2)$, $m \geq m_0$ and u a function such that*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0(\sigma)} u \geq -1 \text{ in } B_{2\sqrt{n}3^m} \times (0, d_m],$$
$$u \geq 0 \text{ in } \mathbb{R}^n \times [0, d_m]$$

Then for any $i \geq 1$, the dyadic covering $\{K_j\}$ of $A = \left\{u > M_1^{i+1}\right\} \cap (Q_1 \times [0, 1])$ with respect to the fraction μ_1 satisfies

$$\cup_j(\tilde{K}_j)^m \subseteq \left\{u > M_1^i\right\} \cap (Q_1 \times [0, d_m]),$$

for m_0 universal, $M_1 = M_0^m$, and $\mu_1 \in (0, 1)$, $M_0 > 1$, d_m as in Corollary 4.3.5.

Proof. Notice first that $M_1 = M_0^m$ implies by Corollary 4.3.5 that

$$|A| \leq |\{u > M_0^m\} \cap Q_1 \times [0, 1]| \leq \mu_1 |Q_1 \times [0, 1]|,$$

which allows us to apply Lemma 4.3.7 to A.

By the construction of $(\tilde{K}_j)^m$ we know that $\cup_j(\tilde{K}_j)^m \subseteq Q_1 \times [0, d_m]$ for m large enough. So we can assume by contradiction that there exists some $K_j = Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]$ such that

1. $\inf_{(\tilde{K}_j)^m} u \leq 1$,

2. $|A \cap K_j| > \mu_1 |K_j|$.

The rescaling $(x,t) \to (r^{-1}(x-x_0), r^{-\sigma}(t-t_0))$ sends K_j to $Q_1 \times [0,\tau]$; the stack $(\tilde{K}_j)^m$ to a subset of $\cup_{i=1}^m Q_{3^\sigma i} \times [d_i\tau, d_{i+1}\tau]$ if m is large enough; and it sends the domain of the equation $B_{2\sqrt{n}3^m} \times (0, d_m\tau]$ to a even bigger domain.

The function $v(x,t) = \frac{u(rx+x_0, r^\sigma t+t_0)}{M_1^i}$ satisfies

$$v_t - \mathcal{M}^-_{\mathcal{L}_0} v \geq -\frac{r^\sigma}{M_1^i} \geq -1 \text{ in } B_{2\sqrt{n}3^m} \times (0, d_m\tau],$$

$$v \geq 0 \text{ in } \mathbb{R}^n \times (0, d_m\tau],$$

$$\inf_{\cup_{i=1}^{m-1} Q_{3^\sigma i} \times [d_i\tau, d_{i+1}\tau]} v \leq 1.$$

By Corollary 4.3.5 we get,

$$\frac{\left|A \cap K_j\right|}{|K_j|} \leq \frac{\left|\left\{v > M_0^m\right\} \cap Q_1 \times [0,1]\right|}{|Q_1 \times [0,1]|} \leq \mu_1.$$

But this contradicts the construction of K_j. □

4.3.3 Proof of Theorem 4.3.1

By combining the previous results we get the following discrete version of Theorem 4.3.1 which implies the proof of Theorem 4.3.1 by rescaling and a covering argument.

Lemma 4.3.9. *Let $\sigma_0 \in (0,2)$, $\sigma \in (\sigma_0, 2)$ and u a function such that*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0} u \geq -1 \textit{ in } B_{2\sqrt{n}3^m} \times (0, d_m],$$

$$u \geq 0 \textit{ in } \mathbb{R}^n \times [0, d_m],$$

$$\inf_{Q_1 \times [C_0 - 1, C_0]} u \leq 1,$$

then for any $k \in \mathbb{N}$

$$|\{u > M_2^k\} \cap Q_1 \times [0,1]| \leq \mu_3^k |Q_1 \times [0,1]|,$$

for some universal constants $M_2 > 1$, $\mu_3 \in (0,1)$, C_0, m; and d_m as in Corollary 4.3.5.

Proof. Let $C_0 = \frac{8(6\sqrt{n})^{\sigma_0}}{3^{\sigma_0} - 1} + 2$.

We proceed by induction. The case $k = 1$ is the result of Corollary 4.3.5 with $\tau = 1$ if $C_0 \leq d_m$, $M_2 \geq M_0^m$ and $\mu_3 \in (\mu_0, 1)$. Assume then it is true for some k and lets prove it for $k + 1$. It means that we are assuming that

$$|\{u > M_2^{k+1}\} \cap Q_1 \times [0,1]| > \mu_3^{k+1} \geq \mu_3 |\{u > M_2^k\} \cap Q_1 \times [0,1]|.$$

Let $\{K_j\}$ the dyadic disjoint covering of $A = \{u > M_2^{k+1}\} \cap Q_1 \times [0,1]$ with respect to the fraction μ_1. Lets fix m sufficiently large such that:

1. $\mu_2 := \frac{(m+1)\mu_1}{m} < 1$,

2. $d_m \geq C_0$,

3. $m \geq m_0$ with m_0 from Lemma 4.3.8.

Lemma 4.3.7 and 4.3.8 tell us that for every piece K_j of the covering,

$$|\{u > M_2^{k+1}\} \cap K_j| > \mu_1 |K_j|,$$
$$|\{u > M_2^{k+1}\} \cap Q_1 \times [0,1]| \leq \mu_2 | \cup_j (\tilde{K}_j)^m|,$$
$$\cup_j (\tilde{K}_j)^m \subseteq \{u > M_2^k\} \cap Q_1 \times [0, d_m].$$

Therefore,

$$\begin{aligned}
\mu_3^{k+1} &< |\{u > M_2^{k+1}\} \cap Q_1 \times [0,1]|,\\
&\leq \mu_2 | \cup_j (\tilde{K}_j)^m|,\\
&= \mu_2(| \cup_j (\tilde{K}_j)^m \cap Q_1 \times [0,1]| + | \cup_j (\tilde{K}_j)^m \cap Q_1 \times [1, d_m]|),\\
&\leq \mu_2(|\{u > M_2^i\} \cap Q_1 \times [0,1]| + | \cup_j (\tilde{K}_j)^m \cap Q_1 \times [1, d_m]|),\\
&\leq \mu_2(\mu_3^k + | \cup_j (\tilde{K}_j)^m \cap Q_1 \times [1, d_m]|).
\end{aligned}$$

By fixing $\mu_3 = (1 + \mu_2)/2 \in (0,1)$ we get for some constant c,

$$c\mu_3^k \leq | \cup_j (\tilde{K}_j)^m \cap Q_1 \times [1, d_m]|.$$

Therefore there is a point $(x,t) \in \cup_j (\tilde{K}_j)^m$ with $t \geq c\mu_3^k + 1$. Hence there is a dyadic box $Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]$ in the covering with

1. $4(m+1) r^\sigma \tau \geq c\mu_3^k$,
2. $|\{u > M_2^{k+1}\} \cap Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]| > \mu_1 |Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]|$,

The first equation implies $d_{Nk} r^\sigma \tau \geq C_0$ for N large enough, independent of k, σ and τ.

In order to apply Corollary 4.3.5 (rescaled) we need to check at least that

$$Q_1 \times [C_0 - 1, C_0] \subseteq \cup_{i=1}^{k-1} Q_{3^{\sigma i} r}(x_0) \times [t_0 + d_i r^\sigma \tau, t_0 + d_{i+1} r^\sigma \tau] \tag{4.8}$$

By having chosen $C_0 - 2 = \frac{8(6\sqrt{n})^{\sigma_0}}{3^{\sigma_0}-1}$ we get that for $(x_0, t_0) \in Q_1 \times [0, 1]$ and $(x, t) \in Q_1 \times [C_0 - 1, C_0]$, $t - t_0 \geq \frac{\tau(6\sqrt{n})^{\sigma_0}}{3^{\sigma_0}-1}$ and $|x - x_0| \leq \sqrt{n}$. Therefore

$$Q_1 \times [C_0 - 1, C_0] \subseteq \{\tau \leq t - t_0 \leq \tau|6(x - x_0)|^\sigma/(3^\sigma - 1)\}.$$

Using the previous relation and that $d_{Nk} r^\sigma \tau \geq C_0$ we verify (4.8). It allows us then to conclude that $|\{u > M_0^{Nk}\} \cap Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]| \leq \mu_1 |Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]|$ which would contradict

$$|\{u > M_2^{k+1}\} \cap Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]| > \mu_1 |Q_r(x_0) \times [t_0, t_0 + r^\sigma \tau]|$$

if $M_2 \geq M_0^N$. □

4.4 Regularity Results

The purpose of this section is to prove that solutions of (1.5) are regular.

Hölder regularity follows by proving a geometric decay of the oscillation of the solution. Hölder regularity for the spatial gradient holds if the equation is translation invariant and the operator is elliptic with respect to $\mathcal{L}_1$.

Hölder regularity for the time derivative does not hold even for the fractional heat equation in a bounded domain if the boundary data is not sufficiently nice. Consider u, the solution of $u_{t^-} + (-\Delta)^\sigma u = 0$ in $B_1 \times (-1, 0]$, with initial value $u(\cdot, -1) \equiv 0$. The boundary data outside B_1 is equal to $\underline{u} = (c(t + 1/2) + \chi_{B_3 \setminus B_2}(x))\chi_{[-1,2,0]}(t)$ where the constant $c > 0$ is chosen small enough so that $\underline{u}$ is a subsolution to the fractional heat equation in $B_1 \times (-1, 0]$. By the comparison principle, we have that $u \geq \underline{u}$ in $B_1 \times (-1, 0]$. Also $u \equiv 0$ in $B_1 \times [-1, -1/2]$ by uniqueness. This shows that the time derivative of u have a jump at $t = -1/2$.

Theorem 4.4.1 (Hölder regularity)**.** *Let $\sigma_0 \in (0, 2)$ and $\sigma \in (\sigma_0, 2)$. Let $u \in C(\Omega \times (-1, 0])$ such that it satisfies the following two inequalities in the viscosity sense with $C_0 \geq 0$,*

$$u_t - \mathcal{M}^-_{\mathcal{L}_0(\sigma)} u \geq -C_0 \text{ in } B_2 \times (-1, 0],$$
$$u_t - \mathcal{M}^+_{\mathcal{L}_0(\sigma)} u \leq C_0 \text{ in } B_2 \times (-1, 0].$$

Then there is some $\alpha \in (0, 1)$ and $C > 0$, depending only on n, Λ and σ_0, such that for every $(y, s), (x, t) \in B_{1/2} \times (-1/2, 0]$

$$\frac{|u(y, s) - u(x, t)|}{(|x - y| + |t - s|^{1/\sigma})^\alpha} \leq C\left(\|u\|_{L^\infty(\bar{B}_2 \times [-1,0])} + \|u\|_{C(-1,0;L^1(\omega))} + C_0\right)$$

Proof. Assume without loss of generality that $(x, t) = (0, 0)$ and $u \in [-1/2, 1/2]$ in $B_2 \times [-1, 0]$. A truncation of u also satisfies a similar equation with a controlled right hand side. Let $v = u\chi_{B_2}$, then

$$v_t - \mathcal{M}^-_{\mathcal{L}_0} v \geq -C_0 - C\|u\|_{L^1(\omega)} \text{ in } B_1 \times (-1, 0],$$
$$v_t - \mathcal{M}^+_{\mathcal{L}_0} v \leq C_0 + C\|u\|_{L^1(\omega)} \text{ in } B_1 \times (-1, 0].$$

Assume without loss of generality that $C_0 + C\|u\|_{C(-1,0;L^1(\omega))} \leq \varepsilon_0$ for some $\varepsilon_0 > 0$ small enough to be fixed. The idea of the proof is to construct an increasing sequence m_k and a decreasing sequence M_k, such that $M_k - m_k = 2(1/2)^{\alpha k}$ and v is trapped between m_k and M_k in $B_{(1/2)^k} \times [-(1/2)^{\sigma k}, 0]$.

Take initially $m_0 = -1/2$ and $M_0 = 1/2$. Assume we have constructed the sequences up to some index k. We want to find now how to construct m_{k+1} and M_{k+1}. Consider

$$w(x,t) = \frac{v((1/2)^k x, (1/2)^{\sigma k} t) - (m_k + M_k)/2}{(1/2)^{\alpha k}},$$

so that $w \in [-1, 1]$ in $B_1 \times [-1, 0]$ and satisfies

$$w_t - \mathcal{M}^-_{\mathcal{L}_0} w \geq -\varepsilon_0 2^{-(\sigma-\alpha)k} \geq -\varepsilon_0 \text{ in } B_1 \times (-1, 0],$$
$$w_t - \mathcal{M}^+_{\mathcal{L}_0} w \leq \varepsilon_0 2^{-(\sigma-\alpha)k} \leq \varepsilon_0 \text{ in } B_1 \times (-1, 0],$$

given that $\alpha \in (0, \sigma_0)$.

In $B_{1/2} \times [-1/2, 0]$, w is between -1 and 1 and either is above or below zero at least $1/2$ of the measure of $B_{1/2} \times [-1, -1/2]$. Assume, without loss of generality, that

$$\frac{|\{w > 0\} \cap B_{1/2} \times [-1, -1/2]|}{|B_{1/2} \times [-1, -1/2]|} \geq \frac{1}{2}.$$

Then we want to use Theorem 4.3.1 to show that $w + 1$ raises from 0 in $B_{1/2} \times [-1/2, 0]$. Still w can have negative values outside B_1 which will not allow to apply such theorem. However, $w + 1 \geq -2(|2x|^\alpha - 1)$ outside B_1, so that w^+ satisfies in $B_{3/4} \times (-1, 0]$,

$$(w+1)^+_t - \mathcal{M}^-_{\mathcal{L}_0}(w+1) \geq -\varepsilon_0 - \|2(|2x|^\alpha - 1)\chi_{\mathbb{R}^n \setminus B_1}\|_{L^1(\omega)},$$

By choosing α small enough we can get right hand sides of magnitude $2\varepsilon_0$.

By Theorem 4.3.1 we get that,

$$C\left(\inf_{B_{1/2}\times[-1/2,0]} w + 2\varepsilon_0\right) \geq \frac{|B_{1/2}\times[-1/2,0]|}{2},$$

Which for ε_0 small enough implies that $\inf_{B_{1/2}\times[-1/2,0]} w \geq \theta$ for some universal $\theta > 0$.

Then in this case we can choose $M_{k+1} = M_k$ and $m_{k+1} = m_k + \theta(M_k - m_k)/2$. Notice that, $M_{k+1} - m_{k+1} = (1-\theta/2)2^{-\alpha k}$ which can be made smaller than $2^{-\alpha k}$ for α even smaller. In the case that $\frac{|\{w>0\}\cap B_{1/2}\times[-1,-1/2]|}{|B_{1/2}\times[-1,-1/2]|} \geq \frac{1}{2}$ we arrive to the conclusion that $M_{k+1} = M_k - \theta(M_k - m_k)/2$ and $m_{k+1} = m_k$ satisfies all the inductive hypothesis which concludes the proof. □

Theorem 4.4.2 ($C^{1,\alpha}$ Regularity for translation invariant operators)**.** *Let $\sigma_0 \in (0,2)$, $\sigma \in (\sigma_0, 2)$ and $f \in C([-1,0])$. There is ρ_0 depending only on $n, \lambda, \Lambda, \sigma_0$, such that if I is an inf sup (or sup inf) type operator, translation invariant in space and elliptic with respect to $\mathcal{L}_1(\sigma,\Lambda,\rho_0)$ and $u \in C(\bar{B}_1 \times [-1,0])$ is a viscosity solution of the equation,*

$$u_t - Iu = f(t) \text{ in } B_2 \times (-1,0],$$

then u is $C^{1,\alpha}$ in space for some universal $\alpha \in (0,1)$. More precisely, there is a constant $C > 0$, depending only on n, Λ and σ, such that for every $(x,t),(y,s) \in B_{1/4} \times (-1,0]$

$$\frac{|u_x(x,t) - u_x(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^{\alpha}} \leq C\left(\|u\|_{L^\infty(\bar{B}_2\times[-1,0])} + \|u\|_{C(-1,0;L^1(\omega))} + \|f\|_{L^\infty((-1,0])}\right)$$

Proof. Assume without loss of generality that

$$\|u\|_{L^\infty(\bar{B}_2\times[-1,0])} + \|f\|_{L^\infty((-1,0])} \leq 1.$$

By considering the cutoff of u in B_2 we have, as in the previous proof, that we can assume also that $u \equiv 0$ in $\mathbb{R}^n \setminus B_2$.

Let $\bar{\alpha}$ the Hölder exponent obtained by Theorem 4.4.1, and assume that it is not the reciprocal of an integer by making it smaller if necessary. Let us define $\delta = 1/(4\lfloor 1/\bar{\alpha}\rfloor)$. Fix a unit vector $e \in \mathbb{R}^n$, a number $h \in (0, \delta/8)$ and, for $k = 1, 2, \ldots, \lfloor 1/\bar{\alpha}\rfloor$, let η^k be a smooth cut-off function supported in $B_{(3/4-k\delta)-\delta/4}$ and equal to one in $B_{(3/4-k\delta)-\delta/2}$. Define the following incremental quotients,

$$\begin{aligned}
w^{h,k}(x,t) &= \frac{u(x+he,t) - u(x,t)}{|h|^{\bar{\alpha}k}},\\
w_1^{h,k}(x,t) &= \frac{(\eta^k u)(x+he,t) - (\eta^k u)(x,t)}{|h|^{\bar{\alpha}k}},\\
w_2^{h,k}(x,t) &= \frac{((1-\eta^k)u)(x+he,t) - ((1-\eta^k)u)(x,t)}{|h|^{\bar{\alpha}k}}.
\end{aligned}$$

We will show that for every $k = 0, 1, \ldots, \lfloor 1/\bar{\alpha}\rfloor - 1$, if

$$\|w^{h,k}\|_{L^\infty(B_{3/4-k\delta}\times[-(3/4-k\delta),0])} \leq C(k) \tag{4.9}$$

then in $B_{3/4-(k+1)\delta} \times [-(3/4-(k+1)\delta), 0]$ the following estimate holds

$$\frac{|w^{h,k}(x,t) - w^{h,k}(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^{\alpha}} \leq C(k+1) \tag{4.10}$$

where $w^{h,0} = u$ and the constants $C(k)$ are independent of h.

When $(x,t) \in B_{(3/4-k\delta)-\delta/8} \times [-((3/4-k\delta)-\delta/8), 0]$, $|w_1^{h,k}|$ is bounded above by the product rule and the hypothesis (4.9),

$$|w_1^{h,k}(x,t)| \leq C(k) + \|\eta^k\|_{L^\infty(B_{(3/4-k\delta)-\delta/8}\times[-((3/4-k\delta)-\delta/8),0])}.$$

If $x \in (\mathbb{R}^n \setminus B_{(3/4-k\delta)-\delta/8}) \times [-((3/4-k\delta)-\delta/8), 0]$ then $w_1^{h,k}(x,t)$ just cancels.

By using that the equation is translation invariant we have that u and $u(\cdot + he, \cdot)$ satisfy equations in the same ellipticity family. By Theorem 4.1.2, $w^{h,k}$ satisfy the following inequalities in $B_1 \times (-1, 0]$ in the viscosity sense,

$$w_t^{h,k} - \mathcal{M}^-_{\mathcal{L}_1} w^{h,k} \geq 0,$$
$$w_t^{h,k} - \mathcal{M}^+_{\mathcal{L}_1} w^{h,k} \leq 0.$$

The function $w_1^{h,k}$ satisfy a similar equation as $w^{h,k}$ in $B_{(3/4-k\delta)-3\delta/4} \times (-1, 0]$, the difference is on the right hand side introduced by the cutoff,

$$(w_1^{h,k})_t - \mathcal{M}^-_{\mathcal{L}_1} w_1^{h,k} \geq -\mathcal{M}^-_{\mathcal{L}_1} w_2^{h,k},$$
$$(w_1^{h,k})_t - \mathcal{M}^+_{\mathcal{L}_1} w_1^{h,k} \leq -\mathcal{M}^+_{\mathcal{L}_1} w_2^{h,k}.$$

For $x \in B_{(3/4-k\delta)-3\delta/4}$ the terms $|\mathcal{M}^\pm_{\mathcal{L}_1} w_2^h|$ are controlled by $\|u\|_\infty = 1$ by using that

$$\int_{\mathbb{R}^n \setminus B_{\rho_0}} \frac{|K(y) - K(y-h)|}{|h|} dy \leq C \text{ every time } |h| < \frac{\rho_0}{2}.$$

with $\rho_0 = \delta/8$. Indeed, for $L \in \mathcal{L}_1$ with kernel K, $x \in B_{(3/4-k\delta)-3\delta/4}$, $|y| \leq \delta/8$ we have that $w_2^{h,k}(x+y) = 0$ and by the product rule

$$\begin{aligned}
|Lw_2^{h,k}(x)| &= \left| \int w_2^{h,k}(x+y,t) K(y) dy \right|, \\
&= \left| \int_{\mathbb{R}^n \setminus B_{\delta/8}} \frac{(1-\eta^k)u(x+y+h) - (1-\eta^k)u(x+y)}{|h|^{\bar{\alpha}k}} K(y) dy \right|, \\
&= \left| \int_{\mathbb{R}^n \setminus B_{\delta/8}} (1-\eta^k)u(x+y)|h|^{1-\bar{\alpha}k} \frac{K(y) - K(y-h)}{|h|} dy \right|, \\
&\leq C.
\end{aligned}$$

We get then the equations for $w_1^{h,k}$ in $B_{(3/4-k\delta)-3\delta/4} \times (-((3/4-k\delta)-3\delta/4), 0]$

$$(w_1^{h,k})_t - \mathcal{M}^-_{\mathcal{L}_1} w_1^{h,k} \geq -C,$$
$$(w_1^{h,k})_t - \mathcal{M}^+_{\mathcal{L}_1} w_1^{h,k} \leq C.$$

By applying Theorem 4.4.1 to $w_1^{h,k}$ from $B_{(3/4-k\delta)-3\delta/4} \times (-((3/4-k\delta)-3\delta/4), 0]$ to $B_{3/4-(k+1)\delta} \times (-(3/4-(k+1)\delta), 0]$ we conclude that for a constant $C(k+1)$ independent of h,

$$\frac{|w_1^{h,k}(x,t) - w_1^{h,k}(y,s)|}{(|x-y| + |t-s|^{1/\sigma})^\alpha} \leq C(k+1).$$

Which is equivalent to (4.10).

By Lemma 5.3 in [5] we get that (4.10) implies that $w^{h,k+1}$ is also bounded by a constant independent of h. Therefore we can apply this procedure up to obtaining that u_x is bounded in $B_{1/2} \times [-1/2, 0]$. Finally, by applying the previous argument one more time to the Lipschitz quotient we conclude the theorem. □

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