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Periods of Modular Forms and Central Values of L-functions

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Periods of Modular Forms and Central Values of L-functions

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Dedicated to Mr. M, the $(\text{Dr. Spears})^2$, and J.

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Periods of Modular Forms and Central Values of L-functions

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This thesis is comprised of three problems in number theory. The introduction is Chapter 1. The first problem is to partially generalize the main theorem of Gross, Kohnen and Zagier to higher weight modular forms. In Chapter 2, we present two conjectures which do this and some partial results towards their proofs as well as numerical examples. This work provides a new method to compute coefficients of weight $k + 1/2$ modular forms for $k > 1$ and to compute the square roots of central values of L-functions of weight $2k > 2$ modular forms. Chapter 3 presents four different interpretations of the main construction in Chapter 2. In particular we prove our conjectures are consistent with those of Beilinson and Bloch. The second problem in this thesis is to find an arithmetic formula for the central value of a certain Hecke L-series in the spirit of Waldspurger's results. This is done in Chapter 4 by using a correspondence between special points in Siegel space and maximal orders in quaternion algebras. The third problem is to find a lower bound

for the cardinality of the principal genus group of binary quadratic forms of a fixed discriminant. Chapter 5 is joint work with Jeffrey Stopple and gives two such bounds.

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Chapter 1

Introduction

This thesis consists of five chapters. Chapters 2 and 3 are dedicated to formulating and discussing a conjecture which partially generalizes the Gross-Kohnen-Zagier theorem to higher weight modular forms. For $f \in S_{2k}(N)$ satisfying certain conditions, we construct a map from the Heegner points of level N to a complex torus associated to f and denoted $\mathbb{C}/L_f$. We define higher weight analogues of Heegner divisors on $\mathbb{C}/L_f$. We conjecture they all lie on a line, and their positions are given by the coefficients of a certain Jacobi form corresponding to f. In weight 2, our map is the modular parametrization map (restricted to Heegner points), and our conjectures are implied by the Gross-Kohnen-Zagier theorem. In Chapter 2 we describe this map and the conjectures and present an algorithm which produces numerical evidence to support our conjectures for a variety of examples. In Chapter 3 we show that the map constructed above has four different characterizations. The first is as a type of cycle integral and is how the map is defined in Chapter 2. The second is in terms of the Eichler integral of f followed by applications of a certain 'weight raising' differential operator. This interpretation of the map gives the intuition behind why it would be invariant modulo the periods L_f. The third characterization gives the map as a higher weight Abel-Jacobi map from a certain weight k modular variety into its k-th intermediate Jacobian. This map gives an interpretation of our conjectures in terms of the conjectures of Beilinson and Bloch. The fourth characterization is a function on quadratic forms instead of on Heegner points. This allows us to prove a partial result towards our second conjecture using the Shimura lift.

In Chapter 4 we derive a formula for the central value of a certain Hecke L-series in terms of the coefficients of a certain half-integer weight modular form. This extends the work of Pacetti in [52]. We define split-CM points to be certain points of the moduli space $\mathbb{H}_2/Sp_4(\mathbb{Z})$ corresponding to products $E \times E'$ of elliptic curves with the same complex multiplication. We prove that the number of split-CM points in a given class of $\mathbb{H}_2/Sp_4(\mathbb{Z})$ is related to the coefficients of a weight $3/2$ modular form studied by Eichler. The main application of this result is a formula for the central value $L(\psi_{\mathcal{N}}, 1)$ of a certain Hecke L-series. The Hecke character $\psi_{\mathcal{N}}$ is a twist of the canonical Hecke character ψ for the elliptic $\mathbb{Q}$-curve A studied by Gross, and formulas for $L(\psi, 1)$ as well as generalizations were proven by Villegas and Zagier. The formulas for $L(\psi_{\mathcal{N}}, 1)$ are easily computable and numerical examples are given. Our intention is to lay the groundwork for the higher weight case $L(\psi_{\mathcal{N}}^k, k)$ involving higher rank quadratic forms.

Chapter 5 is joint work with Jeffrey Stopple. We apply Tatuzawa's version of Siegel's theorem to derive two lower bounds on the size of the principal genus of positive definite binary quadratic forms. Comparing the two bounds shows that there is always a range of discriminants where the first bound is better than the second, but that the second bound is better in the limit as the discriminant tends to infinity.

Chapter 2

Higher weight Heegner points

2.1 Introduction

For integers $N, k \geq 1$, let $S_{2k}(N)$ denote the cusp forms of weight $2k$ on the congruence group $\Gamma_0(N)$. Let $X_0(N)$ be the usual modular curve and $J_0(N)$ its Jacobian. By D we will always mean a negative fundamental discriminant which is a square modulo $4N$. For each D, one can construct a Heegner divisor y_D in $J_0(N)$ and defined over $\mathbb{Q}$. Suppose $f \in S_2(N)$ is any normalized newform whose sign in the functional equation of $L(f,s)$ is -1. Then the celebrated theorem of Gross, Kohnen, and Zagier [21, Theorem C] says that, as D varies, the f-eigencomponents of the Heegner divisors y_D all 'lie on a line[1]' in the quotient $J_0(N)_f$. Furthermore it says that their positions on this line are given by the coefficients of a certain Jacobi form. In particular when N is prime, the positions are the coefficients of a half-integer weight modular form in Shimura correspondence with f.

Now suppose $f \in S_{2k}(N)$ is a normalized newform of weight $2k$ and level N. In addition, assume the coefficients in its Fourier series are rational, and

$*$ *This contents of this chapter is to appear in [36].*

[1] We will say a subset X of an abelian group J lies on a line if $X \subseteq \mathbb{Z} \cdot x_0$ for some $x_0 \in J$.

the sign in the functional equation of $L(f,s)$ is -1. Let $\mathbb{H}_N/\Gamma_0(N) \subset X_0(N)$ denote the Heegner points of level N. In this chapter we construct a map

$$\alpha : \mathbb{H}_N/\Gamma_0(N) \to \mathbb{C}/L_f$$

where $\mathbb{C}/L_f$ is a complex torus defined by the periods of f. Let $h(D)$ denote the class number of the imaginary quadratic field of discriminant D. For each D and fixed choice of its square root $(\bmod 2N)$, we get precisely $h(D)$ distinct representatives $\tau_1, \ldots, \tau_{h(D)}$ of $\mathbb{H}_N/\Gamma_0(N)$. Define $(\mathcal{Y}_D)_f = \alpha(\tau_1) + \cdots + \alpha(\tau_{h(D)})$ and define $(y_D)_f = (\mathcal{Y}_D)_f + \overline{(\mathcal{Y}_D)_f}$ in $\mathbb{C}/L_f$. When $k = 1$, α is the usual modular parametrization map restricted to Heegner points, and $(y_D)_f$ is equal to the f-eigencomponent of y_D in $J_0(N)$ as described in the first paragraph. For $k \geq 1$ we formulate conjectures similar to Gross-Kohnen-Zagier. We predict the $(y_D)_f$ all lie in a line in $\mathbb{C}/L_f$, that is, there exists a point $y_f \in \mathbb{C}/L_f$ such that

$$(y_D)_f = m_D y_f,$$

up to torsion, with $m_D \in \mathbb{Z}$. Furthermore we predict the positions m_D on the line are coefficients of a certain Jacobi form corresponding to f. In the case when N is prime and k is odd, the m_D should be the coefficients of a weight $(k + 1/2)$ modular form in Shimura correspondence with f.

In Chapter 3 we prove that our map is equivalent to the Abel Jacobi map on Kuga-Sato varieties in the following sense. Let $Y = Y^k$ be the Kuga-Sato variety associated to weight $2k$ forms on $\Gamma_0(N)$. (See [96, p.117] for details.) This is a smooth projective variety over $\mathbb{Q}$ of dimension $2k - 1$.

Set $\mathcal{Z}^k(Y)_{\mathrm{hom}}$ to be the nullhomologous codimension k algebraic cycles, and $\mathrm{CH}^k(Y)_{\mathrm{hom}}$ the group of $\mathcal{Z}^k(Y)_{\mathrm{hom}}$ modulo rational equivalence. Let Φ^k be the usual k-th Abel-Jacobi map

$$\Phi^k : \mathrm{CH}^k(Y)_{\mathrm{hom}} \to J^k(Y)$$

where $J^k(Y)$ is the k-th intermediate Jacobian of Y. Given any normalized newform $f = \sum_{n \geq 1} a_n q^n \in S_{2k}(N)$ with rational coefficients, there exists an f-isotypical component $J_f^k(Y)$ of $J^k(Y)$, and thus an induced map

$$\begin{tikzcd} \mathrm{CH}^k(Y)_{\mathrm{hom}} \arrow[r, "\Phi^k"] \arrow[dr, dashed, "\Phi_f^k"'] & J^k(Y) \arrow[d] \\ & J_f^k(Y) \end{tikzcd}$$

We prove in Chapter 3 that the image of Φ_f^k on classes of CM cycles in $\mathrm{CH}^k(Y)_{\mathrm{hom}}$ is equal (up to a constant) to the image of our map α on Heegner points in $X_0(N)$. This implies that our conjectures are consistent with the conjectures of Beilinson and Bloch. In this setting they predict

$$\mathrm{rank}_{\mathbb{Z}}\, \mathrm{CH}^k(Y_F)_{\mathrm{hom}} = \mathrm{ord}_{s=k} L_F(H^{2k-1}(Y), s).$$

If we assume $\mathrm{ord}_{s=k} L(f,s) = 1$ then a refinement of their conjecture predicts the image of Φ_f^k on CM divisors in $Y_{\mathbb{Q}}$ should have rank at most 1 in $J_f^k(Y)$.

To prove the equivalence of α and Φ_f^2 in the case of weight 4, we used an explicit description of Φ_f^2 on CM cycles given by Schoen in [69]. In fact, in [70] Schoen uses this map to investigate a consequence of Beilinson-Bloch similar to the one described above. For a specific $Y = Y^4$ and f, he computes

Φ_f on certain CM divisors in Y defined over the quadratic number field $\mathbb{Q}(i)$. From this he finds numerical evidence that the images lie on a line and their positions are given by a certain weight $5/2$ form corresponding to f. For all weights, we prove this result using an elegant description of the Abel-Jacobi map from [3]. (See Chapter 3 Section 3.2).

The algorithm we describe in this chapter provides a new and effective way to compute (what are conjecturally) the coefficients of weight $k + 1/2$ modular forms, for any $k > 1$. Hence by the theorems of Waldspurger, the algorithm also gives a new method to compute the square roots of central values of L-functions of higher weight modular forms. The intention is to implement and optimize this procedure into SAGE [84].

The sections of this chapter are divided as follows. In Section 2.2 we describe our map and its lattice of periods. In Section 2.3 we give explicit statements of our conjectures. In Section 2.4 we describe the algorithm we created to numerically verify the conjectures in a variety of examples. In sections 4.7 and 2.6 we compute some examples and use them to verify our conjectures in two different ways.

2.2 Higher Weight Heegner Points

Let $\mathbb{H}$ denote the upper half-plane. Suppose f is a normalized newform in $S_{2k}(N)$ having a Fourier expansion of the form,

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n, \qquad q = \exp(2\pi i \tau), \quad \tau \in \mathbb{H},$$

with $a_n \in \mathbb{Q}$.

Recall the L-function of f is defined by the Dirichlet series,

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \qquad \mathrm{Re}(s) > k + 1/2,$$

and has an analytic continuation to all of $\mathbb{C}$. Moreover the function $\Lambda(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$ satisfies the functional equation,

$$\Lambda(f,s) = \varepsilon\Lambda(f, 2k-s),$$

where $\varepsilon = \pm 1$ is the sign of the functional equation of $L(f,s)$.

For each prime divisor p of N, let $q = p^\ell$, $\ell \in \mathbb{N}$ such that $\gcd(q, N/q) = 1$ and set $\omega_q = \left(\begin{smallmatrix} qx_0 & 1 \\ Ny_0 & q \end{smallmatrix}\right)$, for some $x_0, y_0 \in \mathbb{Z}$, with $qx_0 - (N/q)y_0 = 1$. Define $\Gamma_0^*(N)$ to be the group generated by $\Gamma_0(N)$ and each ω_q. Let S be a set of generators for $\Gamma_0^*(N)$. Define the period integrals of f for the set S by

$$\mathcal{P} = \left\{ (2\pi i)^k \int_{i\infty}^{\gamma(i\infty)} f(z) z^m dz \;:\; m \in \{0, \ldots, 2k-2\},\ \gamma \in S \right\} \subseteq \mathbb{C}.$$

These are sometimes referred to as Shimura integrals. It is straightforward to see that every integral of the form

$$(2\pi i)^k \int_{i\infty}^{\gamma(i\infty)} f(z) z^m dz, \qquad \gamma \in \Gamma_0^*(N),\ 0 \le m \le 2k-2$$

is an integral linear combination of elements in $\mathcal{P}$. (See [77, Section 8.2], for example.) In fact, the $\mathbb{Z}$-module generated by $\mathcal{P}$ forms a lattice:

Lemma 2.2.1. $L_f := \mathit{Span}_{\mathbb{Z}}(\mathcal{P})$ *is a lattice in* $\mathbb{C}$.

Proof. By theorems of Razar [56, Theorem 4] and Šokurov [82, Lemma 5.6], the set $\mathcal{P}$ is contained in some lattice. Hence L_f is of rank ≤ 2. For $k = 1$, L_f contains the lattice of period integrals for $\Gamma_0(N)$ defined by Eichler and Shimura so the result is known. To show the rank of L_f is 2 for $k > 1$, it suffices to show there exist nonzero complex numbers $u^+, u^- \in L_f$ with $u^+ \in \mathbb{R}$ and $u^- \in i\mathbb{R}$.

Suppose m is a prime not dividing N, and χ a primitive Dirichlet character modulo m. Define $(f \otimes \chi) := \sum_{n\geq 1} \chi(n) a_n q^n$, and $L(f \otimes \chi, s)$ to be its Dirichlet series. Let $\Lambda(f \otimes \chi, s) = (2\pi)^{-s}(Nm^2)^{s/2}\Gamma(s)L(f \otimes \chi, s)$. Then for $\mathrm{Re}(s) > k + 1/2$, we have

$$i^s(Nm^2)^{-s/2}\Lambda(f \otimes \chi, s) = \int_0^{i\infty} (f \otimes \chi)(z) z^s \frac{dz}{z}. \tag{2.1}$$

Let $g(\chi)$ denote the Gauss sum associated to χ. Then an expression for χ in terms of the additive characters is given by,

$$\chi(n) = m^{-1} g(\chi) \sum_{u \bmod m} \bar{\chi}(-u) e^{2\pi i n u/m}.$$

So

$$(f \otimes \chi)(\tau) = m^{-1} g(\chi) \sum_{u \bmod m} \bar{\chi}(-u) f(z + u/m).$$

Substituting this into (2.1) gives

$$i^s(Nm^2)^{-s/2}\Lambda(f \otimes \chi, s) = m^{-1} g(\chi) \sum_{u \bmod m} \bar{\chi}(-u) \int_0^{i\infty} f(z + u/m) z^s \frac{dz}{z},$$

and replacing z by $z - u/m$ and rearranging implies

$$i^{-s} g(\chi)^{-1} N^{-s/2} \Lambda(f \otimes \chi, s) = (-1)^{s-1} \sum_{u \bmod m} \bar{\chi}(-u) \int_{i\infty}^{u/m} f(z)(mz-u)^{s-1} dz.$$

Now let $s = 2k-1$ in the above equation, and multiply both sides by $(2\pi i)^k$. In addition suppose χ is a quadratic Dirichlet character modulo m. If $m \equiv 3 \bmod 4$, then $g(\chi) = i\sqrt{m}$, and if $m \equiv 1 \bmod 4$ then $g(\chi) = \sqrt{m}$. The special value $\Lambda(f \otimes \chi, 2k-1)$ is nonzero for $k > 1$ by the absolute convergence of the Euler product. Hence since $\Lambda(f \otimes \chi, 2k-1)$ is real-valued and nonzero, the right hand side of this equation is either purely real or purely imaginary depending on the choice of m. Then this proves the lemma since the right hand side is in L_f for any m.

$\square$

Let $D < 0$ be a fundamental discriminant, and assume D is a square modulo $4N$. Fix a residue class $r \bmod 2N$ satisfying $D \equiv r^2 \bmod 4N$. Then

$$\mathcal{Q}_N^D(r) := \left\{ \begin{array}{c} [A, B, C] : A > 0, B, C \in \mathbb{Z}, \\ D = B^2 - 4AC, A \equiv 0 \bmod N, B \equiv r \bmod 2N \end{array} \right\}.$$

corresponds to a subset of the positive definite binary quadratic forms of discriminant D. We define $\mathbb{H}_N^D(r)$ to be the roots in $\mathbb{H}$ of $\mathcal{Q}_N^D(r)$,

$$\mathbb{H}_N^D(r) := \left\{ \tau = \frac{-B + \sqrt{D}}{2A} : [A, B, C] \in \mathcal{Q}_N^D(r) \right\}.$$

The group $\Gamma_0(N)$ preserves $\mathbb{H}_N^D(r)$, and the $\Gamma_0(N)$-orbits in $\mathbb{H}_N^D(r)$ are in bijection with the classes of reduced binary quadratic forms of discriminant D.

We will call $\mathbb{H}_N^D(r)/\Gamma_0(N)$ the set of Heegner points of level N, discriminant D, and root r. Define $\mathbb{H}_N$ to be the union of $\mathbb{H}_N^D(r)$ over all D and r, and so $\mathbb{H}_N/\Gamma_0(N)$ are the Heegner points of level N.

For each $\tau = \frac{-B+\sqrt{D}}{2A} \in \mathbb{H}_N^D(r)$, set $Q_\tau(z) := Az^2 + Bz + C$ so that $[A, B, C] \in \mathcal{Q}_N^D(r)$. We now define a function $\alpha = \alpha_f : \mathbb{H}_N \to \mathbb{C}$ by

$$\alpha(\tau) := (2\pi i)^k \int_{i\infty}^{\tau} f(z) Q_\tau(z)^{k-1} dz.$$

Lemma 2.2.2. *The map α induces a well-defined map (which we will also denote by α),*

$$\alpha : \mathbb{H}_N/\Gamma_0(N) \to \mathbb{C}/L_f.$$

Proof. For any $\tau \in \mathbb{H}_N$ of discriminant D and $\gamma \in \Gamma_0(N)$, we will show

$$\alpha(\gamma\tau) - \alpha(\tau) = (2\pi i)^k \cdot \int_{i\infty}^{\gamma(i\infty)} f(z) Q_{\gamma\tau}(z)^{k-1} dz.$$

Since $Q_{\gamma\tau}(z)$ has integer coefficients, this implies that $\alpha(\gamma\tau) - \alpha(\tau)$ belongs to L_f for all $\gamma \in \Gamma_0(N)$.

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. Then

$$\begin{aligned}
&\alpha(\gamma\tau) - (2\pi i)^k \cdot \int_{i\infty}^{\gamma(i\infty)} f(z)\, Q_{\gamma\tau}(z)^{k-1} dz \\
&= (2\pi i)^k \cdot \int_{\gamma(i\infty)}^{\gamma\tau} f(z) Q_{\gamma\tau}(z)^{k-1} dz \\
&= (2\pi i)^k \cdot \int_{i\infty}^{\tau} f(\gamma z) Q_{\gamma\tau}(\gamma z)^{k-1} d(\gamma z) \\
&= \alpha(\tau),
\end{aligned}$$

where in the last equality we used

$$f(\gamma z) = (cz+d)^{2k} f(z), \qquad Q_{\gamma\tau}(z) = (-cz+a)^2 Q_\tau(\gamma^{-1}z)$$

and $d(\gamma z) = (cz+d)^{-2}dz$.

$\square$

2.3 Conjectures

Let $\{\tau_1, \ldots, \tau_{h(D)}\} \in \mathbb{H}_N^D(r)$ be any set of distinct class representatives of $\mathbb{H}_N^D(r)/\Gamma_0(N)$. Define

$$P_{D,r} := \sum_{i=1}^{h(D)} \tau_i \in \mathrm{Div}(X_0(N)),$$

where $\mathrm{Div}(X_0(N))$ denotes the group of divisors on $X_0(N)$. If $D = -3$ (resp. $D = -4$), scale $P_{D,r}$ by 1/3 (resp. 1/2). Extend α to P_D by linearity and define

$$(y_{D,r})_f = \alpha(P_{D,r}) + \overline{\alpha(P_{D,r})} \in \mathbb{C}/L_f.$$

Here, bar denotes complex conjugation in $\mathbb{C}$. We write $y_{D,r}$ or y_D for $(y_{D,r})_f$, and P_D for $P_{D,r}$ when the context of f, r is clear.

By the actions of complex conjugation and Atkin-Lehner on $\mathbb{H}_N$, we have:

Lemma 2.3.1.

$$\overline{\alpha(P_{D,r})} = -\varepsilon\alpha(P_{D,r}),$$

where ε is the sign of the functional equation of $L(f,s)$.

Proof. Using the method in the proof of Lemma 2.2.2 with γ replaced by ω_N, $Q_{\omega_N\tau}(z) = Nz^2 \cdot Q_\tau(z)$, and the action of f under ω_N (see (2.3)), we have

$$\alpha(\omega_N\tau) = (-1)^k\varepsilon \cdot \alpha(\tau) + \int_{i\infty}^{0} f(z)Q_{\omega_N\tau}(z)^{k-1}dz.$$

On the other hand complex conjugation acts on α by

$$\overline{\alpha(\tau)} = (-1)^{k+1}\alpha(-\bar{\tau})$$

(using (2.5), for example). Therefore

$$\overline{\alpha(\tau)} = -\varepsilon \cdot \alpha(\omega_N(-\bar{\tau})) \bmod L_f.$$

The action of complex conjugation followed by ω_N permutes $\mathbb{H}_N^D(r)/\Gamma_0(N)$ simply transitively [20, p.89], so the trace $P_{D,r}$ satisifes

$$\overline{\alpha(P_{D,r})} = -\varepsilon\alpha(P_{D,r})$$

as wanted.

□

Thus if $\varepsilon = +1$, then $y_{D,r}$ are in L_f for all D, r. This is, in some sense, the trivial case. Hence we restrict our attention to the case when $\varepsilon = -1$.

Conjectures 2.3.2 and 2.3.3 give a partial generalization of the Gross-Kohnen-Zagier theorem to higher weights.

Conjecture 2.3.2. *Let $f = \sum_{n\geq 1} a_n q^n \in S_{2k}(N)$ be a normalized newform with rational coefficients, and assume $\varepsilon = -1$ and $L'(f,k) \neq 0$. Then for all* *fundamental discriminants $D < 0$ and $r \bmod 2N$ with $D \equiv r^2 \bmod 4N$, there exist integers $m_{D,r}$ such that*

$$t\, y_{D,r} = t\, m_{D,r} y_f \qquad \text{in } \mathbb{C}/L_f,$$

where $y_f \in \mathbb{C}/L_f$ is non-torsion and $t \in \mathbb{Z}$ are both nonzero and independent of D and r.

Remark 2.3.1. Equivalently we could say $y_{D,r} = m_{D,r} y_f$ up to a t-torsion element in $\mathbb{C}/L_f$.

To state the second conjecture we will need to use Jacobi forms. (See [13] for background). Let $J_{2k,N}$ denote the set of all Jacobi forms of weight $2k$ and index N. Then such a $\phi \in J_{2k,N}$ is a function $\phi : \mathbb{H} \times \mathbb{C} \to \mathbb{C}$, which satisfies the transformation law

$$\phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^{2k} e^{2\pi i N \frac{cz^2}{c\tau+d}} \phi(\tau, z),$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n,r\in\mathbb{Z} \\ r^2 \le 4Nn}} c(n,r) q^n \zeta^r, \qquad q = e^{2\pi i \tau},\ \zeta = e^{2\pi i z}. \tag{2.2}$$

The coefficient $c(n, r)$ depends only on $r^2 - 4Nn$ and on the class $r \bmod 2N$.

Suppose $f \in S_{2k}(N)$ is a normalized newform with $\varepsilon = -1$. Then by [81], there exists a non-zero Jacobi cusp form $\phi_f \in J_{k+1,N}$ which is unique up to scalar multiple and has the same eigenvalues as f under the Hecke operators T_m for m, N coprime. We predict that the coefficients of ϕ_f are related to the $m_{D,r}$ from above in the following way:

Conjecture 2.3.3. *Let* $f = \sum_{n\geq 1} a_n q^n \in S_{2k}(N)$ *be a normalized newform with rational coefficients, and assume* $\varepsilon = -1$ *and* $L'(f,k) \neq 0$. *Assume Conjecture 2.3.2. Then*

$$m_{D,r} = c(n,r)$$

where $n = \frac{|D|+r^2}{4N}$ *and* $c(n,r)$ *is (up to a scalar multiple) the* (n,r)*-th coefficient of the Jacobi form* $\phi_f \in J_{k+1,N}$.

Remark 2.3.2. When $k = 2$, the points $(y_{D,r})_f$ and y_f are the same as those defined in [21], and both of our conjectures are implied by Theorem C of their paper. (Actually their theorem is only for D coprime to $2N$ but they say the result remains 'doubtless true' with this hypothesis removed. See [31] and [6] for more details.) Particular to weight 2 is the fact that $\mathbb{C}/L_f$ corresponds to an elliptic curve defined over $\mathbb{Q}$ and that y_D is a rational point on the elliptic curve $E_f \simeq \mathbb{C}/L_f$. In contrast, we should stress that for $2k > 2$, the elliptic curve $E \simeq \mathbb{C}/L_f$ is not expected to be defined over any number field. For instance, the j-invariants for our examples all appear to be transcendental over $\mathbb{Q}$.

Remark 2.3.3. For $N = 1$ or a prime, and k odd we can state Conjecture 2.3.3 in terms of modular forms of half-integer weight. Specifically, let $\phi \in J_{k+1}(N)$ be a Jacobi form with a Fourier expansion as in (2.2), and set

$$g(\tau) = \sum_{M=0}^{\infty} c(M) q^M, \qquad q = e^{2\pi i \tau}$$

where $c(M)$ is defined by,

$$c(M) := \begin{cases} c\left(\frac{M+r^2}{4N}, r\right) & \text{if } M \equiv -r^2 \bmod 4N \text{ for any } r \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

This function is well-defined because $c(n,r)$ depends only on $r^2 - 4nN$ when $N = 1$ or a prime, and k is odd. Then by [13, p.69], g is in $M_{k+1/2}(4N)$, the space of modular forms of weight $k + 1/2$ and level $4N$. In addition, if $f \in S_{2k}(N)$ is a normalized newform with $\varepsilon = -1$, then the form g defined by ϕ_f is in Shimura correspondence with f.

2.4 Algorithms

Let $f = \sum_{n \geq 1} a_n q^n \in S_{2k}(N)$ be a normalized newform with rational Fourier coefficients. The sign ε of the functional equation of $L(f,s)$ can be computed with the identity,

$$f\left(\frac{-1}{Nz}\right) = (-1)^k \varepsilon N^k z^{2k} f(z) \tag{2.3}$$

given by the action of the Fricke involution of level N on f. We will only consider f such that $\varepsilon = -1$ and $L'(f,k) \neq 0$.

The first step is to find a basis of our lattice L_f, which is the $\mathbb{Z}$-module generated by the periods $\mathcal{P}$ as described above. Suppose p_1, p_2, p_3 are three periods in $\mathcal{P}$. Since L_f has rank 2, these are linearly dependent over $\mathbb{Z}$, that is

$$a_1 p_1 + a_2 p_2 + a_3 p_3 = 0, \quad \text{for some} \quad a_i \in \mathbb{Z}.$$

We may assume $\gcd(a_1, a_2, a_3) = 1$. Let $d = \gcd(a_1, a_2)$, then there exist integers $x, y \in \mathbb{Z}$ such that $xa_1 + ya_2 = d$. Similarly $\gcd(d, a_3) = 1$ so there exist integers $u, v \in \mathbb{Z}$ such that $ud + va_3 = 1$. Define the matrix M by,

$$M = \begin{pmatrix} a_1 & a_2 & a_3 \\ -y & x & 0 \\ -va_1/d & -va_2/d & u \end{pmatrix}.$$

Observe $M \in GL_3(\mathbb{Z})$ and $M \cdot {}^T(p_1, p_2, p_3) = {}^T(0, -yp_1 + xp_2, -va_1p_1/d - va_2p_2/d + up_3)$. Hence $-yp_1 + xp_2$ and $-va_1p_1/d - va_2p_2/d + up_3$ are a basis for the $\mathbb{Z}$-module generated by p_1, p_2, p_3.

We would also like our basis elements to have small norm. Given a basis ω_1, ω_2 of a lattice, its norm form is a real bilinear quadratic form defined by the matrix

$$B = \begin{pmatrix} 2|\omega_1|^2 & 2\,\mathrm{Re}(\omega_1\bar{\omega}_2) \\ 2\,\mathrm{Re}(\omega_1\bar{\omega}_2) & 2|\omega_2|^2 \end{pmatrix}.$$

Thus it is equivalent to a reduced form of the same discriminant, that is, there exists $U \in SL_2(\mathbb{Z})$ such that

$${}^TUBU = \begin{pmatrix} 2\alpha & \beta \\ \beta & 2\gamma \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R},$$

with $|\beta| \le \alpha \le \gamma$ and $\beta \ge 0$ if either $|\beta| = \alpha$ or $\alpha = \gamma$. Hence $(\omega_1', \omega_2') := (\omega_1, \omega_2)U$ is a 'reduced' basis. For a basis of all of L_f we simply apply this process iteratively on the elements of $\mathcal{P}$.

In fact it is not hard to see that L_f is a real lattice, that is, $\bar{L_f} = L_f$. Thus given a basis ω_1, ω_2 of L_f, we may assume $\omega_1 \in i\mathbb{R}$, and therefore $\tau := \omega_2/\omega_1$ has real part equal to either 0 or $\pm 1/2$. This implies $\mathrm{Re}(L_f) = \mathrm{Re}(\omega_2)$ which will help simplify our computations.

To actually compute the elements in $\mathcal{P}$ we need to split the path from $(i\infty)$ to $\gamma(i\infty)$ of integration at some point $\tau \in \mathbb{H}$ which gives,

$$\int_{i\infty}^{\gamma(i\infty)} f(z)z^m dz = \int_{i\infty}^{\gamma(\tau)} f(z)z^m dz - \int_{i\infty}^{\tau} f(z)(az+b)^m(cz+d)^{2k-2-m}dz,$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. We choose τ and $\gamma(\tau)$ to be points at which f has good convergence (i.e. their imaginary parts are not too small)[2]. To compute integrals of the form,

$$\int_{i\infty}^{\tau} f(z)z^m dz,$$

we use repeated integration by parts to get the formula

$$\int_{i\infty}^{\tau} f(z)z^m dz = m!\,(-1)^m \sum_{j=-1}^{m-1} \frac{(-1)^{j+1}}{(j+1)!}\tau^{j+1} f_{m-j}(\tau), \tag{2.4}$$

where $f_\ell(\tau)$ is defined to be the ℓ-fold integral of f evaluated at $\tau \in \mathbb{H}$, that is,

$$f_\ell(\tau) = \frac{1}{(2\pi i)^\ell} \sum_{n\geq 1} \frac{a_n}{n^\ell} q^n, \qquad q = \exp(2\pi i\tau)$$

which is well-defined for any $0 \leq \ell \leq 2k-1$.

The next task is to compute $\alpha(\tau)$ for $\tau \in \mathbb{H}_N$. We could do this using (2.4), but it is computationally faster[3] to use the following identity for α. Recall the modular differential operator,

$$\partial_m := \frac{1}{2\pi i}\frac{d}{dz} - \frac{m}{4\pi y}, \qquad z = x + iy \in \mathbb{H},$$

[2]For example, following Cremona [10] we could take $\tau := \frac{-d+i}{c}$ so that $\gamma(\tau) = \frac{a+i}{c}$.

[3]To compute α using (2.4) would require non-linear polynomials of the power series in (2.4) whereas Lemma 2.4.1 below requires computing only the single power series in (2.5).

for any integer m. Define $\partial_m^\ell(f) := \partial_{m+2(\ell-1)} \circ \cdots \circ \partial_{m+2} \circ \partial_m(f)$ to be the composition of the ℓ operators $\partial_m, \partial_{m+2}, \ldots, \partial_{m+2(\ell-1)}$. Then a straightforward combinatorial argument yields the following identity, whose proof we will omit,

Lemma 2.4.1. *Let τ be a Heegner point of level N and discriminant D. Then*

$$\alpha(\tau) = \kappa_D \cdot \partial_{-2k+2}^{k-1} \circ f_{2k-1}(\tau),$$

where $\kappa_D = (k-1)!\,(2\pi i)^k(2\pi\sqrt{|D|})^{k-1}$ is a constant depending only on D and $2k$.

A closed formula for ∂_m^ℓ (see [58] for example) allows us to write α as

$$\alpha(\tau) = \kappa_D(2\pi i)\left(\frac{-y}{\pi}\right)^k \sum_{n\geq 1} p\left(k, \frac{1}{4\pi y n}\right) a_n q^n, \tag{2.5}$$

where $p(m, x)$, is the polynomial,

$$p(m,x) = \sum_{\ell=m}^{2m-1} \binom{m-1}{2m-1-\ell} \frac{(\ell-1)!}{(m-1)!} x^\ell, \qquad m \in \mathbb{Z},\ x \in \mathbb{R}.$$

We compute $\alpha(\tau)$ using (2.5). Also notice that Lemma 2.4.1 perhaps provides further insight into why the map $\mathbb{H}_N \to \mathbb{C}/L_f$ inducing α is invariant under $\Gamma_0(N)$. Loosely speaking, this is because integrating f $(2k-1)$-times lowers its weight by $2(2k-1)$ and ∂_{-2k+2}^{k-1} increases its weight by $2(k-1)$ to get something morally of weight 0.

Given a set of Heegner point representatives of level N, discriminant D, and root r, we can use the above to compute $y_{D,r}$. Verifying the first conjecture for each D, r then amounts to choosing a complex number y_f, and an integer t, both non-zero, and showing the linear dependence,

$$\operatorname{Re}(y_{D,r}) - m_{D,r} \operatorname{Re}(y_f) + n_{D,r} \operatorname{Re}(\omega_2)/t = 0 \tag{2.6}$$

for some integers $m_{D,r}, n_{D,r}$. The second conjecture consists of comparing the coefficients $m_{D,r}$ of y_f we get above with the Jacobi form coefficients of the form ϕ_f.

2.5 Examples

The Fourier coefficients of the forms in these examples were computed using SAGE [84]. The rest of the calculations were done in PARI/GP [53]. Code and data from this chapter can be found at

`http://www.math.utexas.edu/users/khopkins/comp.html`.

We will always take a set of generators for $\Gamma_0(N)$ which includes the translation matrix $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ but no other matrix whose $(2,1)$ entry is 0. The period integrals for T are always 0 since $i\infty$ is its fixed fixed point, hence we can exclude it from our computations of $\mathcal{P}$. In addition the $(2\pi)^k$ factor in the definitions of y_D and L_f is left off from the computations, since it is just a scaling factor.

For each example below, we list the number of digits of precision and the number M of terms of f we used. Below that is a set of generators we chose for $\Gamma_0^*(N)$ and the bases, ω_1, ω_2, we got for L_f from computing $\mathcal{P}$ and applying the lattice reduction algorithm explained in Section 2.4. We then provide a table listing the m_D which satisfy equation (2.6) up to at least the number of digits of precision specified for t, y_f of our choosing, and D less than some bound. Without getting into details, the precision we chose depended on the size of the M-th term of f and on the a priori knowledge of the size of the coefficients satisfying (2.6).

Example 2.5.1. $2k = 10$, $N = 3$. The space of cuspidal newforms of weight 10 and level 3 has dimension 2, but only one form has $\varepsilon = -1$. The first few terms of it are

$$f = q - 36q^2 - 81q^3 + 784q^4 - 1314q^5 + 2916q^6 - 4480q^7 - 9792q^8 + \cdots$$

Precision	60
Number of terms	100
$\Gamma_0^*(3)$	$\left\langle T, \left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right), \omega_3 = \left(\begin{smallmatrix} 0 & -1 \\ 3 & 0 \end{smallmatrix}\right) \right\rangle$
ω_1	$-i \cdot 0.00088850361439085\ldots$
ω_2	$0.00002189032158611\ldots$
y_f	$y_{-8}/2$
t	1

The m_D in Table 2.1 give, up to scalar multiple, the coefficients of the weight 11/2 level 12 modular form found in [13, p. 144]. Note we can use the theorems of Waldspurger to get information about the values $L(f, D, k)$ from this table. For example, $L(f, -56, 5) = 0$.

$\|D\|$	m_D	$\|D\|$	m_D
8	2	104	380
11	-5	107	-507
20	8	116	-40
23	8	119	-560
35	42	131	235
47	-48	143	-376
56	0	152	-364
59	-155	155	-64
68	160	164	-1440
71	40	167	1528
83	353	179	2635
95	280	191	-400

Table 2.1: $f \in S_{10}(3)$. List of D, m_D such that $y_D - m_D y_f \in L_f$ for $|D| < 200$.

Example 2.5.2. $2k = 18$, $N = 1$.

The weight 18 level 1 eigenform in $S_{18}(1)$ has the closed form

$$f(z) = \frac{-E_6^3(z) + E_4^3(z)E_6(z)}{1728},$$

where $E_{2k}(z)$ is the normalized weight $2k$ Eisenstein series.

Precision	200
Number of terms	100
$\Gamma_0^*(1) = SL_2(\mathbb{Z})$	$\left\langle T, S = \omega_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle$
ω_1	$i \cdot 0.001831876775870191761\ldots$
ω_2	$0.000000000519923858624\ldots$
y_f	y_{-3}
t	1

$\lvert D\rvert$	m_D	$\lvert D\rvert$	m_D
3	1	51	108102
4	-2	52	-93704
7	-16	55	-22000
8	36	56	80784
11	99	59	-281943
15	-240	67	659651
19	-253	68	193392
20	-1800	71	-84816
23	2736	79	-109088
24	-1464	83	-22455
31	-6816	84	-484368
35	27270	87	1050768
39	-6864	88	143176
40	39880	91	195910
43	-66013	95	-370800
47	44064		

Table 2.2: $f \in S_{18}(1)$. List of D, m_D such that $y_D - m_D y_f \in L_f$ for $|D| < 100$.

The m_D in Table 2.2 are identical to the coefficients of the weight 19/2 level 4 half-integer weight form in [13, p.141], which is in Shimura correspondence with f.

Example 2.5.3. $2k = 4$, $N = 13$

The dimension of the new cuspidal subspace is 3 in this case, but only one has integer coefficients in its q-expansion.

$$f = q - 5q^2 - 7q^3 + 17q^4 - 7q^5 + 35q^6 - 13q^7 - 45q^8 + 22q^9 + \cdots$$

Precision	28
Number of terms	250
$\Gamma_0^*(13)$	$\left\langle T, \left(\begin{smallmatrix} 8 & -5 \\ 13 & -8 \end{smallmatrix}\right), \left(\begin{smallmatrix} -3 & 1 \\ -13 & 4 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 & -2 \\ 13 & -5 \end{smallmatrix}\right), \left(\begin{smallmatrix} -9 & 7 \\ -13 & 10 \end{smallmatrix}\right), \omega_{13} = \left(\begin{smallmatrix} 0 & -1 \\ 13 & 0 \end{smallmatrix}\right) \right\rangle$
ω_1	$i \cdot 0.003124357726009878347400865279\ldots$
ω_2	$-0.04271662498543992056668379773\ldots$
	$-i \cdot 0.001562178863004939178984383052\ldots$
y_f	$6 \cdot y_{-3}$
t	6

Notice this is the first example of a nonsquare lattice. In fact $\omega_2/\omega_1 = -0.5000\cdots + i \cdot 13.67212999\ldots$ so $\mathrm{Re}(\omega_2/\omega_1) = 1/2$ as explained earlier. This is also the first example where the choice of r matters, since $k = 2$ is not odd. For each D, we chose r in the interval $0 < r < 13$. In addition this is our only example where $t > 1$.

A closed form expression for the weight 3 index 13 Jacobi form $\phi = \phi_f$ corresponding to f was provided to us by Nils Skoruppa,

$$\phi(\tau, z) = \vartheta_1^5 \vartheta_2^3 \vartheta_3 / \eta^3.$$

Here η is the usual Dedekind eta-function, $\eta = q^{1/24} \prod_{n \geq 1}(1 - q^n)$ with $q = e^{2\pi i \tau}$, and $\vartheta_a = \sum_{r \in \mathbb{Z}} \left(\frac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{ar}{2}}$ for $a = 1, 2, 3$, $\zeta = e^{2\pi i z}$. (This has a nice product expansion using Jacobi's triple product identity.)

We verify that the (n, r)-th coefficient $c(n, r)$ in the Fourier expansion of ϕ is identically equal to the $m_{D,r}$ in Table 2.3 for $|D| < 200$.

$\lvert D\rvert$	$m_{D,r}$	$\lvert D\rvert$	$m_{D,r}$
3	1	107	4
4	-1	116	-8
23	2	120	-13
35	-7	127	14
40	3	131	-3
43	-17	139	29
51	9	152	2
55	-6	155	22
56	1	159	-6
68	-5	168	-21
79	4	179	-17
87	-6	183	-2
88	10	191	-10
95	4	199	4
103	-8		

Table 2.3: $f \in S_4(13)$. List of D, $m_{D,r}$ such that $ty_{D,r} - m_{D,r}y_f \in L_f$ with $t = 6$, for $|D| < 200$.

2.6 More Examples

The coefficients of Jacobi forms are difficult to compute, in particular for the cases when N is composite or when k is even. We chose the previous examples in part because the Fourier coefficients for their Jacobi forms already existed, thanks to the work of Zagier, Eichler, and Skoruppa mentioned above. However, given any weight and level, we can still provide convincing evidence for our conjecture without knowing the exact coefficients of its Jacobi form. This is done using a refinement of Waldspurger [90] given in [21, p.527].

Specifically, let $f \in S_{2k}(N)$ be a normalized newform with $\varepsilon = -1$. Let

$\phi = \phi_f \in J_{k+1,N}$, with Fourier coefficients denoted by $c(n,r)$, be the Jacobi form corresponding to f as described in Section 2.3. For a fundamental discriminant D with $\gcd(D,N) = 1$ and square root r modulo $4N$, [21, Corollary 1] says

$$|D|^{k-1/2} L(f, D, k) \doteq |c(n,r)|^2;$$

here $L(f,D,s)$ is L-series of f twisted by D, and $n \in \mathbb{Z}$ satisfies $D = r^2 - 4Nn$. By $\doteq$ we mean equality up to a nonzero factor depending on $N, 2k, f$, and ϕ, but independent of D. (Gross-Kohnen-Zagier give this constant explicitly in their paper, but for us it is unnecessary.)

Thus given two such discriminants $D_i = r_i^2 - 4Nn_i$, $i = 1, 2$, we have

$$\frac{|D_1|^{k-1/2} L(f, D_1, k)}{|D_2|^{k-1/2} L(f, D_2, k)} = \frac{|c(n_1, r_1)|^2}{|c(n_2, r_2)|^2}.$$

Hence by computing central values of twisted L-functions of f, we can test if ratios of squares of our m_{D_i,r_i} are equal to those of $c(n_i, r_i)$.

For the examples below we have the same format as the previous examples along with a fixed choice of discriminant D_1 for which we verified explicitly,

$$\frac{|D_1|^{k-1/2} L(f, D_1, k)}{|D|^{k-1/2} L(f, D, k)} = \frac{m_{D_1,r}^2}{m_{D,r}^2}$$

for all D coprime to N less than a certain bound.

Example 2.6.1. $2k = 4$, $N = 21$.

The dimension of the new cuspidal subspace of $S_4(21)$ is 4. We chose

$$f = q - 3q^2 - 3q^3 + q^4 - 18q^5 + 9q^6 + 7q^7 + \cdots .$$

Precision	40
Number of terms	500
$\Gamma_0^*(21)$	$\left\langle T, \left(\begin{smallmatrix} -4 & 1 \\ -21 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 11 & -5 \\ 42 & -19 \end{smallmatrix}\right), \left(\begin{smallmatrix} 13 & -9 \\ 42 & -29 \end{smallmatrix}\right), \left(\begin{smallmatrix} 8 & -5 \\ 21 & -13 \end{smallmatrix}\right), \left(\begin{smallmatrix} 26 & -19 \\ 63 & -46 \end{smallmatrix}\right), \left(\begin{smallmatrix} -16 & 13 \\ -21 & 17 \end{smallmatrix}\right) \right\rangle$
ω_1	$i \cdot 0.012130626847574141\ldots$
ω_2	$-0.03257318919429172\ldots$
y_f	y_{-3}
t	1
D_1	-20

For a consistent choice of each r we chose the first positive residue modulo $2N$ which satisfies $D \equiv r^2 \bmod 4N$ for each D.

$\|D\|$	$m_{D,r}$	$\|D\|$	$m_{D,r}$
3	1	111	4
20	-1	119	0
24	-1	131	3
35	0	132	8
47	2	143	2
56	0	152	-7
59	1	159	0
68	-2	164	-2
83	5	167	4
84	0	168	0
87	-4	195	8
104	-3		

Table 2.4: $f \in S_4(21)$. List of D, $m_{D,r}$ such that $y_{D,r} - m_{D,r} y_f \in L_f$ for $|D| < 200$.

Example 2.6.2. $2k = 12$, $N = 4$.

The space of new cuspforms in $S_{12}(4)$ is spanned by one normalized newform whose Fourier series begins with,

$$f = q - 516q^3 - 10530q^5 + 49304q^7 + 89109q^9 - 309420q^11 + \cdots .$$

Precision	80
Number of terms	200
$\Gamma_0^*(4)$	$\left\langle T, \left(\begin{smallmatrix} 1 & -1 \\ 4 & -3 \end{smallmatrix}\right) \right\rangle$
ω_1	$i \cdot 0.000960627675025996\ldots$
ω_2	$-0.02998129737318938\ldots$
y_f	y_{-7}
t	1
D_1	-7

Similar to the last example, we chose the first positive residue modulo $2N$ which satisfies $D \equiv r^2 \bmod 4N$ for each D.

$\lvert D\rvert$	$m_{D,r}$	$\lvert D\rvert$	$m_{D,r}$
7	1	103	1649
15	5	111	-765
23	-3	119	-90
31	-50	127	2664
39	-35	143	-3729
47	186	151	-505
55	215	159	-2825
71	-315	167	3819
79	-10	183	2539
87	-497	191	1830
95	405	199	-5755

Table 2.5: $f \in S_{12}(4)$. List of D, $m_{D,r}$ such that $y_{D,r} - m_{D,r}y_f \in L_f$ for $|D| < 200$.

Chapter 3

Interpretations of the higher weight Heegner Map

This Chapter presents four different interpretations of the map

$$\alpha : \mathbb{H}_N/\Gamma_0(N) \longrightarrow \mathbb{C}/L_f,$$

which was defined on $\tau \in \mathbb{H}_N$ by

$$\tau \mapsto (2\pi i)^k \int_{i\infty}^{\tau} f(z) Q_\tau(z)^{k-1} dz.$$

The four interpretations of α can be summarized as follows:

1. The definition of α given above as a cycle integral.

2. The Eichler (i.e. $(2k-1)$-fold) integral, f_{2k-1}, of f followed by the modular differential operator ∂_m applied $(k-1)$ times (see Theorem 3.1.1)
$$\alpha(\tau) = \kappa_D \cdot \partial_{-2k+2}^{k-1} \circ f_{2k-1}(\tau),$$
where $\kappa_D := (k-1)!\,(2\pi i)^k (2\pi\sqrt{|D|})^{k-1}$.

3. The image of the Abel Jacobi map Φ_f^k on classes of CM cycles in the Chow group $\mathrm{CH}^k(Y^k)_{\mathrm{hom}}$ of the Kuga-Sato variety $Y := Y^k$ associated

to $S_{2k}(N)$, after restricting to the f-isotypical component $J_f^k(Y)$ of the intermediate Jacobian $J^k(Y)$,

$$\begin{tikzcd} \mathrm{CH}^k(Y)_{\mathrm{hom}} \arrow[r, "\Phi^k"] \arrow[dr, dashed, "\Phi_f^k"'] & J^k(Y) \arrow[d] \\ & J_f^k(Y) \end{tikzcd} \tag{3.1}$$

(See Theorem 3.2.2).

4. A function on binary quadratic forms Q of discriminant D,

$$\alpha(f)(Q) := (2\pi i)^k \int_{i\infty}^{\tau_Q} f(z)Q(z)^{k-1}dz, \tag{3.2}$$

where τ_Q is the root corresponding to Q in $\mathbb{H}$.

Characterization (2) is the map originally suggested by F. Rodriguez Villegas and perhaps provides insight into why the map α is invariant modulo L_f under the action of $\Gamma_0(N)$. It says that integrating f $(2k-1)$-times lowers its weight by $2(2k-1)$ and differentiating by ∂_{-2k+2}^{k-1} increases its weight by $2(k-1)$ to give something which is modular of weight 0 modulo the periods L_f.

The third characterization implies these conjectures are consistent with the conjectures of Beilinson and Bloch. As explained in Chapter 2, in this setting the Beilinson and Bloch conjectures [2, 5, 70] predict that

$$\mathrm{rank}_{\mathbb{Z}}\, \mathrm{CH}^k(Y_F)_{\mathrm{hom}} = \mathrm{ord}_{s=k} L_F(H^{2k-1}(Y), s).$$

If we assume $\mathrm{ord}_{s=k}L(f,s) = 1$, then a refinement of their conjecture predicts the image of Φ_f^k on cycles in the Chow group of $Y_{\mathbb{Q}}$ should have rank at most one in $J_f^k(Y)$. When $k = 1$, this is the Birch and Swinnerton-Dyer conjecture.

Note that in Chapter 2 and the descriptions given above, α and y_D are only defined for discriminants D which are fundamental. Using an idea similar to Zagier's in [94], the fourth characterization above allows us to extend the definition of y_D to all integers $D < 0$, and extend the conjectures accordingly. This is proven here only for level 1 but is not difficult to generalize to any prime level. Assuming the first conjecture and thus the existence of the integers m_D, let

$$g(z) := \sum_{|D|=1}^{\infty} m_D\, q^{|D|}, \qquad q = e^{2\pi i z}.$$

For a fixed fundamental discriminant $D_0 < 0$, let $g\big|\mathscr{S}_{k,N,D_0}(z)$ denote the D_0-th Shimura lift of g. As a partial result towards proving Conjecture 2, we prove the D_0-th Shimura lift of g is f (see Theorem 3.3.3).

3.1 Eichler integrals and Maass Differential Operators

For the first interpretation of our map α, we need to recall two differential operators which were introduced in Chapter 2. Define the classical differential operator by

$$D := \frac{1}{2\pi i}\frac{d}{dz}$$

and the modular differential operator by

$$\partial_m := D - \frac{m}{4\pi y}, \qquad z = x + iy \in \mathbb{H}$$

for any integer m. Define $\partial_m^\ell(f) := \partial_{m+2(\ell-1)} \circ \cdots \circ \partial_{m+2} \circ \partial_m(f)$ to be the composition of the ℓ operators $\partial_m, \partial_{m+2}, \ldots, \partial_{m+2(\ell-1)}$.

The operators D and ∂_m have an interesting relationship with one another. While D preserves holomorphicity but in general destroys modularity, the operator ∂_m preserves modularity (i.e. sends weight m to weight $m+2$) but in general destroys holomorphicity. However there is a certain instance in which their compositions agree:

$$\begin{tikzcd} \text{weight } 2-2k \arrow[rr, "D^{2k-1}"] \arrow[dr, "\partial_{2-2k}^{k-1}"'] & & \text{weight } 2k \\ & \text{weight } 0 \arrow[ur, "\partial_0^k"'] & \end{tikzcd}$$

This says $D^{2k-1} = \partial_{2-2k}^{2k-1}$ where we have written ∂_{2-2k}^{2k-1} as the composition $\partial_{2-2k}^{k-1} \circ \partial_0^k$. The top arrow in the diagram is known as Bol's identity. It says D^{2k-1} intertwines the weights of $2-2k$ and $2k$. The formal antiderivative of the top arrow is the $(2k-1)$-fold integral, also sometimes referred to as the Eichler integral. Given

$$f(z) := \sum_{n \geq 1} a_n q^n, \qquad q := \exp(2\pi i z),\ z \in \mathbb{H}$$

in $S_{2k}(N)$, and any positive integer M, we define the *M-fold integral* of f by

$$f_M(z) := \sum_{n \geq 1} \frac{a_n}{n^M} q^n. \tag{3.3}$$

In particular, the Eichler integral of f is $f_{2k-1}(z)$. Intuitively the diagram above says that the Eichler integral of f followed by the operator ∂_{2-2k}^{k-1} should give something that is morally of weight 0, modulo certain periods. This is indeed be the case and gives us a second description of our map α:

Theorem 3.1.1.

$$\alpha(\tau) = \kappa_D \cdot \partial_{-2k+2}^{k-1} \circ f_{2k-1}(\tau),$$

where $\kappa_D = (k-1)!\,(2\pi i)^k (2\pi\sqrt{|D|})^{k-1}$ *is a constant depending only on* D *and* $2k$.

Proof. It suffices to show

$$\int_{i\infty}^{\tau} f(z) Q_\tau(z)^{k-1} dz = (k-1)!(2\pi\sqrt{|D|})^{k-1} \partial_{-2k+2}^{k-1} \circ f_{2k-1}(\tau). \tag{3.4}$$

Denote by 'LHS' the left hand side of this equation, and 'RHS' the right hand side. For the LHS, we complete the square on $Q_\tau(z)$ to get

$$Q_\tau(z) := Az^2 + Bz + C = A\left(z + \frac{B}{2A}\right)^2 - \frac{D}{4A}$$

and then apply the binomial theorem to $Q_\tau(z)^{k-1}$. This gives

$$\text{LHS} = \sum_{m=0}^{k-1} \binom{k-1}{m} A^m \left(\frac{-D}{4A}\right)^{k-1-m} \int_{i\infty}^{\tau} f(z)\left(z + \frac{B}{2A}\right)^{2m} dz. \tag{3.5}$$

Repeated integration by parts yields the identity:

$$\int f(z)(z+R)^n dz = (-1)^n n! \sum_{j=0}^{n} \frac{(-1)^j}{j!} (z+R)^j f_{n-j+1}(z).$$

Substitute this and the power series formula for $f_{n-j+1}(z)$ from (3.3) into (3.5).

Turning now to the RHS, a closed formula for ∂_h^m with $h, m \in \mathbb{Z}, m > 0$ was given by Rodriguez Villegas and Zagier [63] and stated explicitly in Chapter 2. Combining this with (3.3) implies that

$$\text{RHS} = (k-1)!(2\pi\sqrt{|D|})^{k-1}(2\pi i)\left(\frac{-y}{\pi}\right)^k \sum_{n\geq 1} p\left(k, \frac{1}{4\pi y n}\right) a_n q^n, \tag{3.6}$$

where $p(m,x)$ is the polynomial

$$p(m,x) := \sum_{\ell=m}^{2m-1} \binom{m-1}{2m-1-\ell} \frac{(\ell-1)!}{(m-1)!} x^\ell, \qquad m \in \mathbb{Z},\ x \in \mathbb{R}.$$

Proving (3.4) is therefore equivalent to proving the n-th terms of (3.5) and (3.6) are equal. Thus it suffices to show

$$\left(\frac{-D}{4A}\right)^{k-1} \sum_{m=0}^{k-1} \binom{k-1}{m} A^m \left(\frac{-D}{4A}\right)^{-m} (2m)! \sum_{j=0}^{2m} \frac{(-1)^j}{j!} (iy)^j \left(\frac{1}{2\pi in}\right)^{2m-j+1} \tag{3.7}$$

$$= i \cdot (-2y)^k (\sqrt{|D|})^{k-1} \sum_{j=k}^{2k-1} \binom{k-1}{2k-1-j} (j-1)! \left(\frac{1}{4\pi yn}\right)^j.$$

Set $\tau = x + iy$. Observe

$$\left(\frac{-D}{4A}\right) = \frac{\sqrt{|D|}}{2} \cdot \frac{\sqrt{|D|}}{2A} = \frac{\sqrt{|D|}}{2} \cdot y,$$

and so in particular $A^m \left(\frac{-D}{4A}\right)^{-m} = y^{-2m}$. Substituting these two identities into (3.7) and dividing through by $i \cdot y^k \sqrt{|D|}^{k-1}$ implies that it suffices to show

$$2^{1-k} \sum_{m=0}^{k-1} \binom{k-1}{m} (2m)! (-1)^{m+1} \left(\frac{1}{2\pi yn}\right)^{2m+1} \sum_{j=0}^{2m} \frac{(2\pi yn)^j}{j!} \tag{3.8}$$

$$=(-2)^k \sum_{j=k}^{2k-1} \binom{k-1}{j-k} (j-1)! \left(\frac{1}{4\pi yn}\right)^j,$$

where we used the identity $\binom{n}{m} = \binom{n}{n-m}$ on the RHS.

Replacing j with $(2m+1-j)$ on the LHS gives that it suffices to show

$$2^{1-k} \sum_{m=0}^{k-1} \binom{k-1}{m} (2m)! (-1)^{m+1} \sum_{j=1}^{2m+1} \frac{1}{(2m+1-j)!(2\pi yn)^j} \tag{3.9}$$

$$=(-2)^k \sum_{j=k}^{2k-1} \binom{k-1}{j-k} (j-1)! \left(\frac{1}{4\pi yn}\right)^j.$$

To the LHS we now apply the identity:

$$\sum_{m=0}^{N}\sum_{j=1}^{2m+1} F(m,j) = \sum_{j=1}^{2N+1}\sum_{m=\lceil \frac{j-1}{2}\rceil}^{N} F(m,j)$$

for any function $F(m,j)$. Then (3.9) is equivalent to showing

$$\begin{aligned}&2^{1-k}\sum_{j=1}^{2k-1}\sum_{m=\lceil \frac{j-1}{2}\rceil}^{k-1}\binom{k-1}{m}(2m)!\frac{(-1)^{m+1}}{(2m+1-j)!(2\pi yn)^j} \\ =&(-2)^k\sum_{j=k}^{2k-1}\binom{k-1}{j-k}(j-1)!\left(\frac{1}{4\pi yn}\right)^j.\end{aligned} \tag{3.10}$$

It suffices to show both sides of (3.10) are equal term by term as a function of j:

$$\begin{aligned}&2^{1-k}\sum_{m=\lceil \frac{j-1}{2}\rceil}^{k-1}\binom{k-1}{m}(2m)!\frac{(-1)^{m+1}}{(2m+1-j)!(2\pi yn)^j} \\ =&(-2)^k\binom{k-1}{j-k}(j-1)!\left(\frac{1}{4\pi yn}\right)^j.\end{aligned} \tag{3.11}$$

Replace j by $j+1$ on both sides and divide through by $j!$. Collecting powers of (-1) and 2's, we have that it suffices to show

$$\sum_{m=\lceil \frac{j}{2}\rceil}^{k-1}\binom{k-1}{m}\binom{2m}{j}(-1)^{k-1-m} = 2^{2k-2-j}\binom{k-1}{j-(k-1)}. \tag{3.12}$$

Set $R := k-1$ so that (3.12) becomes

$$\sum_{m=\lceil \frac{j}{2}\rceil}^{R}\binom{R}{m}\binom{2m}{j}(-1)^{m-R} = 2^{2R-j}\binom{R}{j-R}. \tag{3.13}$$

To prove the combinatorial identity in (3.13), we will expand the polynomial

$$(1-(1+x)^2)^R$$

in two different ways. First, by applying the binomial theorem two times we have

$$
\begin{aligned}
(1-(1+x)^2)^R &= \sum_{m=0}^{R} \binom{R}{m} (-1)^m (1+x)^{2m} \\
&= \sum_{m=0}^{R} \binom{R}{m} (-1)^m \sum_{j=0}^{2m} \binom{2m}{j} x^j \\
&= \sum_{j=0}^{2R} \sum_{m=\lceil \frac{j}{2} \rceil}^{R} \binom{R}{m} \binom{2m}{j} (-1)^m x^j, \qquad (3.14)
\end{aligned}
$$

where in the last equality we used the identity

$$\sum_{m=0}^{N} \sum_{j=0}^{2m} F(m,j) = \sum_{j=0}^{2N} \sum_{m=\lceil \frac{j}{2} \rceil}^{N} F(m,j).$$

On the other hand,

$$1-(1+x)^2 = -x(2+x).$$

Therefore a second expansion formula is given by

$$
\begin{aligned}
(1-(1+x)^2)^R &= (-1)^R x^R (2+x)^R \\
&= (-1)^R x^R \sum_{j=0}^{R} \binom{R}{j} 2^{R-j} x^j,
\end{aligned}
$$

and replacing j with $j-R$ gives

$$
\begin{aligned}
&= (-1)^R x^R \sum_{j=R}^{2R} \binom{R}{j-R} 2^{2R-j} x^{j-R} \\
&= (-1)^R \sum_{j=R}^{2R} \binom{R}{j-R} 2^{2R-j} x^j.
\end{aligned}
$$

If $j < R$ then $\binom{R}{j-R} = 0$, so we can write the sum starting from $j = 0$:

$$= (-1)^R \sum_{j=0}^{2R} \binom{R}{j-R} 2^{2R-j} x^j. \tag{3.15}$$

Comparing the j-th coefficients of (3.14) and (3.15) implies equality in (3.13) which proves the theorem. $\square$

3.2 Abel-Jacobi Maps and the Beilinson-Bloch Conjectures

Let $f \in S_{2k}(\Gamma_0(N))$ be a normalized newform with rational Fourier coefficients. Recall that the map $\alpha : \mathbb{H}_N \longrightarrow \mathbb{C}$ was defined in Chapter 2 on any Heegner point τ in $\mathbb{H}_N \subset \mathbb{H}$ by

$$\alpha(\tau) := (2\pi i)^k \int_{i\infty}^{\tau} f(z) Q_\tau(z)^{k-1} dz. \tag{3.16}$$

It was shown that this induces a map from $\mathbb{H}_N/\Gamma_0(N) \subset X_0(N) \rightarrow \mathbb{C}/L_f$. The goal of this section is to prove that the image of α on Heegner points is equivalent to the image of a certain Abel-Jacobi map on "Heegner cycles", modulo certain periods. More specifically, the latter map will be an Abel-Jacobi map from the k-th Chow group of a Kuga-Sato variety into its k-th intermediate Jacobian. Here the intermediate Jacobian refers to the one studied by Griffiths [18, 19]. For detailed background on Chow groups, intermediate Jacobians, and Abel-Jacobi maps in the general setting, see [7, 17, 87, 88]. For descriptions of these objects for Kuga-Sato varieties, see [3, 49, 69, 96]. Much of what follows is based on the second section of the paper *Generalized Heegner cycles* *and p-adic Rankin L-series* by Bertolini, Darmon and Prasanna [3]. The initial version of our main result in the case of weight 4 was inspired by Schoen's work in the paper *Complex multiplication cycles on elliptic modular threefolds* [69].

Fix $r := k - 1$ throughout this chapter. Let Γ be the congruence subgroup

$$\Gamma := \Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ : \ a-1, d-1, c \equiv 0 \pmod{N} \right\}.$$

Set $C^0 := Y_1(N) := \mathbb{H}/\Gamma$ and $C := X_1(N) := Y_1(N) \cup \{\text{cusps}\}$. Note that $\Gamma \subseteq \Gamma_0(N)$ and $S_{2k}(\Gamma_0(N)) \subseteq S_{2k}(\Gamma)$. Therefore, although the results below refer to the modular curve C, they can be extended naturally to $X_0(N)$ by viewing C as a covering for $X_0(N)$.

To define the Kuga-Sato variety, let

$$W_{2r}^{\sharp} := \mathcal{E} \times_C \mathcal{E} \times_C \times \cdots \times_C \mathcal{E}$$

denote the $2r$-fold product of the universal generalized elliptic curve $\mathcal{E}$ with itself over C. The *weight k Kuga-Sato variety* W_{2r} is defined to be the canonical desingularization of $W_{2r}^{\sharp}$ as described in [11, Lemmas 5.4 and 5.5] and [71, 1.0.3], for example. This is a smooth projective variety over $\mathbb{Q}$ of (complex) dimension $2r+1$.

Recall the k-th Chow group $\mathrm{CH}^{r+1}(W_{2r})_0(\mathbb{C})$ is defined to be the group of null-homologous codimension k algebraic cyles on W_{2r} modulo rational equivalence. We wish to study the k-th Abel Jacobi map, which is a function from the Chow group $\mathrm{CH}^{r+1}(W_{2r})_0(\mathbb{C})$ into a complex torus:

$$\mathrm{AJ} : \mathrm{CH}^{r+1}(W_{2r})_0(\mathbb{C}) \longrightarrow J^{r+1}(W_{2r}) := \frac{\mathrm{Fil}^{2r+1} H_{\mathrm{dR}}^{2r+1}(W_{2r}/\mathbb{C})^\vee}{\mathrm{Im}\, H_{2r+1}(W_{2r}(\mathbb{C}), \mathbb{Z})};$$

here $^\vee$ denotes the dual of a complex vector space, and $\mathrm{Im}\, H_{2r+1}(W_{2r}(\mathbb{C}), \mathbb{Z})$ is viewed as a sublattice of $\mathrm{Fil}^{2r+1} H_{\mathrm{dR}}^{2r+1}(W_{2r}/\mathbb{C})^\vee$ by integration of closed differential $(2r+1)$-forms against singular integral homology classes of dimension $(2r+1)$. Given $\Delta \in \mathrm{CH}^{r+1}(W_{2r})_0(\mathbb{C})$, the linear functional $\mathrm{AJ}(\Delta)$ is defined by choosing a $(2r+1)$-chain Ω such that the boundary $\partial(\Omega)$ is equal to Δ and setting

$$\mathrm{AJ}(\Delta)(\alpha) := \int_\Omega \alpha, \qquad \text{for all } \alpha \in \mathrm{Fil}^{2r+1} H_{\mathrm{dR}}^{2r+1}(W_{2r}/\mathbb{C}).$$

We will be interested only in the "holomorphic piece" of $\mathrm{Fil}^{2r+1} H_{\mathrm{dR}}^{2r+1}(W_{2r}/\mathbb{C})$. This is captured by a projection operator ε defined in [3, Lemma 1.8 and Section 2.1]. Its image on the product of elliptic curves is given by [3, Lemma 1.8]:

$$\varepsilon H_{\mathrm{dR}}^{2r}(E^{2r}) = \mathrm{Sym}^{2r} H_{\mathrm{dR}}^1(E).$$

On the variety W_{2r} the projection gives the relative deRham cohomology sheaf [3, Lemma 2.1]:

$$\varepsilon H_{\mathrm{dR}}^{2r+1}(W_{2r}) = H_{\mathrm{dR}}^1(C, \mathcal{L}_{2r})$$

where $\mathcal{L}_{2r}$ is the sheaf corresponding to $\mathrm{Sym}^{2r} H_{\mathrm{dR}}^1(\mathcal{E})$ as described in [3, p.8]. Combined with the $(2r+1)$-th Hodge filtration, the image is the 0-th sheaf cohomology [3, Lemma 2.1]:

$$\mathrm{Fil}^{2r+1} \varepsilon H_{\mathrm{dR}}^{2r+1}(W_{2r}/\mathbb{C}) = H^0(C, \mathrm{Sym}^{2r} \Omega^1_{\mathcal{E}/C} \otimes \Omega^1_C);$$

here $\Omega^1_{\mathcal{E}/C}$ is the line bundle of relative differentials on $\mathcal{E}/C$ and Ω^1_C the sheaf of holomorphic differentials on C. Analogous to the isomorphism of the global sections of Ω^1_C with $S_2(\Gamma)$, there is a standard construction which to each $f \in S_{2r+2}(\Gamma)$ attaches a differential $(2r+1)$-form

$$\omega_f := f(\tau)(2\pi i dw)^{2r} \otimes (2\pi i d\tau) \in \mathrm{Sym}^{2r}\Omega^1_{\mathcal{E}/C} \otimes \Omega^1_C.$$

The map

$$\begin{aligned} S_{2r+2}(\Gamma) &\longrightarrow \mathrm{Fil}^{2r+1}\varepsilon H^{2r+1}_{\mathrm{dR}}(W_{2r}/\mathbb{C}) \\ f &\mapsto \omega_f \end{aligned}$$

gives an identification [3, Corollary 2.2].

We wish to only consider the piece of the Abel-Jacobi map which survives under the projection operator ε. Therefore from now on we replace the map AJ with the map (still denoted AJ, by abuse of notation)

$$\mathrm{AJ} : \varepsilon\mathrm{CH}^{r+1}(W_{2r})_0(\mathbb{C}) \longrightarrow \frac{S_{2r+2}(\Gamma)^\vee}{\varepsilon\,\mathrm{Im}\,H_{2r+1}(W_{2r}(\mathbb{C}), \mathbb{Z})}. \tag{3.17}$$

(By definition ε has coefficients in $\mathbb{Q}$ so $\varepsilon\,\mathrm{Im}\,H_{2r+1}(W_{2r}(\mathbb{C}), \mathbb{Z})$ is a lattice in $\mathrm{Fil}^{2r+1}\varepsilon H^{2r+1}_{\mathrm{dR}}(W_{2r}/\mathbb{C})^\vee$.)

The next task is to construct Heegner cycles. These are (complex) dimension r cycles defined on W_{2r} and indexed by elliptic curves with Γ-level structure and with complex multiplication. Moreover these cycles are supported above CM points of C and are defined over abelian extensions of imaginary quadratic fields. From now on, fix any point $\tau \in \mathbb{H}/\Gamma \cap K$ where

K is an imaginary quadratic field of discriminant $D < 0$, and let $E := E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ represent the elliptic curve defined by τ with Γ-level structure. Let $\Gamma_{a\tau} \subset E \times E$ denote the image of the graph of $\{(z, a\tau z) \in \mathbb{C}^2\}$. A Heegner cycle Δ_τ is defined by

$$\begin{aligned} \Upsilon &:= \Upsilon_E := (\Gamma_{a\tau})^r \subset (E \times E)^r, \\ \Delta_\tau &:= \Delta_E := \varepsilon(\Upsilon). \end{aligned} \tag{3.18}$$

We wish to evaluate $\mathrm{AJ}(\Delta_\tau)(\omega_f)$ modulo a certain lattice $\Lambda_r \subset S_{2r+2}(\Gamma)^\vee$. Namely, let Λ_r denote the classical period lattice attached to $S_{2r+2}(\Gamma)$ (see [3, p.23-24], [77, p.239], [75, Part II], [74] or [83, Section 10.2] for a description). This is a $\mathbb{Z}$-submodule of $S_{2r+2}(\Gamma)^\vee$ of rank $2g$. Actually, we will need to work with the $\mathbb{Z}$-module generated by Λ_r and the cusp integrals

$$\left\{ \int_\alpha^\beta f(z)P(z)dz \;:\; \alpha, \beta \in \mathbb{P}^1(\mathbb{Q}), P(z) \in \mathbb{Z}[x]^{\deg = r} \right\}. \tag{3.19}$$

By the classical theory of modular symbols,

Lemma 3.2.1. *The $\mathbb{Z}$-module generated by the cusp integrals in* (3.19) *and Λ_r is a lattice in $S_{2k}(\Gamma_1(N))^\vee$ and contains Λ_r with finite index.*

Proof. This result is also used in [3, Proposition 3.13]. A proof for weight 4 is given by Schoen in [69, Proposition 3.5] and generalizes naturally to higher weight. For more on the construction of this lattice, see [56, Theorem 4] and [82, Theorem 5.1]. ∎

The purpose of this section is to prove the following theorem:

Theorem 3.2.2. *Suppose $\tau \in \mathbb{H}_N$ and $f \in S_{2k}(\Gamma_0(N))$ is a normalized newform with rational Fourier coefficients. Then*

$$\alpha(\tau) = (2r)! \cdot AJ(\Delta_\tau)(\omega_f) \, (\mathrm{mod} \Lambda_r), \tag{3.20}$$

where α is the map defined in (3.16) *and AJ is the Abel-Jacobi map defined in* (3.17)*.*

Remark 3.2.1. We can interpret (3.20) as holding under the action of $\Gamma_0(N)$ on $\mathbb{H}_N$ in the following manner. Let $W_{2r}(\Gamma_0(N))$ (resp. $W_{2r}(\Gamma_1(N))$) denote the $2r$-fold Kuga-Sato variety over the modular curve $X_0(N)$ (resp. $X_1(N)$). Denote by Δ'_τ the Heegner cycle for τ on $W_{2r}(\Gamma_0(N))$. The 'forgetful morphism'

$$\phi : W_{2r}(\Gamma_1(N)) \longrightarrow W_{2r}(\Gamma_0(N))$$

satisfies

$$\phi(\Delta_\tau) = \Delta'_\tau.$$

On the other hand, ϕ also induces by functoriality a map

$$\phi^* : H^{2r+1}_{\mathrm{dR}}(W_{2r}(\Gamma_0(N)) \longrightarrow H^{2r+1}_{\mathrm{dR}}(W_{2r}(\Gamma_1(N))$$

and

$$\mathrm{AJ}(\phi\Delta_\tau)(\omega) = \mathrm{AJ}(\Delta_\tau)(\phi^*\omega).$$

Here the map on the left is the Abel-Jacobi map on $X_0(N)$ and the on the right is the Abel-Jacobi map on $X_1(N)$.

3.2.1 Plan of Proof

We will denote Δ_τ by just Δ when the context is clear. Before proving Theorem 3.2.2, we will sketch the general method which will be used. The first idea in the proof is to show, roughly, that

$$\int\limits_{\partial^{-1}(\Delta)} \omega_f = \int\limits_{\tilde{\Delta}} F_f \ (\mathrm{mod}\Lambda_r). \tag{3.21}$$

Here $F_f(\tau) \in H^0(\mathbb{H}, \mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(\mathcal{E}/\mathbb{H}))$ is a primitive of ω_f over $\mathbb{H}$ and $\tilde{\Delta}$ is a null-homologous cycle over $\mathbb{H}$ satisfying $\mathrm{pr}(\tilde{\Delta}) = \Delta$. A primitive of ω_f means with respect to the Gauss-Manin connection. Recall that for any smooth projective variety X of dimension m defined over $\mathbb{C}$, Poincaré duality gives a non-degenerate pairing

$$\big\langle\ ,\ \big\rangle : H^j_{\mathrm{dR}}(X) \times H^{2m-j}_{\mathrm{dR}}(X) \longrightarrow \mathbb{C}$$

defined by the formula

$$\big\langle \omega_1, \omega_2 \big\rangle := \left(\frac{1}{2\pi i}\right)^m \int_{X(\mathbb{C})} \omega_1 \wedge \omega_2.$$

In terms of the Poincaré pairing, (3.21) is equivalent to showing

$$\mathrm{AJ}(\Delta)(\omega_f) = \big\langle F_f(\tau), \mathrm{cl}(\Delta) \big\rangle \quad \mathrm{mod}\ \Lambda_r, \tag{3.22}$$

where cl is the usual cycle class map

$$\mathrm{cl} : \mathrm{CH}^r(E^{2r}) \longrightarrow H^{2r}_{\mathrm{dR}}(E^{2r})$$

on the associated fibers. To explicitly define F_f, let $W^0_{2r} := W_{2r} \times_C C^0$ be the complement in W_{2r} of the fibers above the cusps. Let $\tilde{W}_{2r}$ be the $2r$-fold fiber product of the universal elliptic curve $\mathcal{E}$ over the upper half-plane $\mathbb{H}$. It is isomorphic as an analytic variety to the quotient $\mathbb{Z}^{4r}\backslash(\mathbb{C}^{2r}\times\mathbb{H})$, where $\mathbb{Z}^{4r}$ acts on $\mathbb{C}^{2r}\times\mathbb{H}$ via

$$(m_1,n_1,\ldots,m_{2r},n_{2r})\cdot(w_1,\ldots,w_{2r},\tau):=(w_1+m_1+n_1\tau,\ldots,w_{2r}+m_{2r}+n_{2r}\tau,\tau).$$

Therefore

$$W_{2r}^0(\mathbb{C})=\tilde{W}_{2r}/\Gamma$$

where Γ acts on $\tilde{W}_{2r}$ by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot(w_1,\ldots,w_{2r},\tau):=\left(\frac{w_1}{c\tau+d},\ldots,\frac{w_{2r}}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right).$$

Denote by pr the natural covering maps $\mathrm{pr}:\tilde{W}_{2r}\longrightarrow W_{2r}^0(\mathbb{C})$, $\mathrm{pr}:\mathbb{H}\longrightarrow C^0(\mathbb{C})$, and let $\tilde{\pi}:\tilde{W}_{2r}\longrightarrow\mathbb{H}$, $\pi:W_{2r}^0(\mathbb{C})\longrightarrow C^0(\mathbb{C})$ be the natural fiberings. These maps fit into the diagram

$$\begin{tikzcd} \tilde{W}_{2r} \arrow[r, "\mathrm{pr}"] \arrow[d, "\tilde{\pi}"'] & W_{2r}^0(\mathbb{C}) \arrow[d, "\pi"] \\ \mathbb{H} \arrow[r, "\mathrm{pr}"'] & C^0(\mathbb{C}) \end{tikzcd}$$

For a point $\tau\in\mathbb{H}$, we think of the fiber of $\tilde{W}_{2r}$ over τ as the "lift" $E^{2r}\times\mathbb{H}$ over its image in W_{2r}, and over $\mathrm{pr}(\tau)$ we think of the fiber of $W_{2r}^0(\mathbb{C})$ over τ as the "open modular" piece $E^{2r}\times Y_1(N)$ over its image in W_{2r}.

The coherent sheaf $\mathcal{L}_{2r}$ is equipped with a canonical integrable connection (the Gauss-Manin connection)

$$\nabla:\mathcal{L}_{2r}\longrightarrow\mathcal{L}_{2r}\otimes\Omega_C^1.$$

(See [41], for example.) A primitive of ω_f is an element $F_f \in H^0(\mathbb{H}, \mathrm{pr}^*(\mathcal{L}_{2r}))$ satisfying

$$\nabla F_f = \mathrm{pr}^*(\omega_f).$$

Because $\mathbb{H}$ is contractible and ∇ integrable, a primitive always exists, and is well-defined up to elements in the space of global horizontal sections of $\mathrm{pr}^*(\mathcal{L}_{2r})$ over $\mathbb{H}$.

Equation (3.22) is essentially Proposition 3.3 of [3]. A precise statement of this proposition together with a sketch of their proof is given at the end of this section. Assume for now that (3.22) holds. The proof of Theorem 3.2.2 then follows in two steps. The first is to show that our map α is equal to the Poincaré pairing of F_f with a multiple of the (r,r)-form $dw^r d\bar{w}^r$ on E^{2r}. The second step is to compute the coefficient of $dw^r d\bar{w}^r$ in the class $\mathrm{cl}(\Delta)$ and show that this is the only nonvanishing component in $\langle F_f, \mathrm{cl}(\Delta)\rangle$. These are Lemmas 3.2.4 and 3.2.5 below. Comparing these two formulas and applying (3.22) will give the desired result. To make these steps precise we now recall some necessary background from [3].

We will make use of two different bases for $H^1_{\mathrm{dR}}(E)$. One, $\{\eta_1, \eta_\tau\}$, will give us horizontal sections for $\mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(E)$ and the other, $\{\omega, \eta\}$, will lead to a natural (compatible with the Hodge decomposition) basis of $\mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(E)$ for computing pairings with F_f. First recall how elements of $\mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(E^{2r})$ are constructed from elements of $H^1_{\mathrm{dR}}(E)$. The Künneth decomposition gives an inclusion

$$\mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(E) \subset H^1_{\mathrm{dR}}(E)^{\otimes 2r} \hookrightarrow H^{2r}_{\mathrm{dR}}(E^{2r}).$$

For any $0 \leq j \leq 2r$, and for $0 < i_1 < i_2 < \cdots < i_{2r} \leq 2r$ set

$$m_{i_1} \otimes m_{i_2} \otimes \cdots \otimes m_{i_{2r}} := dw_{i_1} \otimes \cdots \otimes dw_{i_j} \otimes d\bar{w}_{i_{j+1}} \otimes \cdots \otimes d\bar{w}_{i_{2r}}$$

in $H^1_{\mathrm{dR}}(E)^{\otimes^{2r}}$. Then the element $dw^j d\bar{w}^{2r-j} \in \mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(E)$ viewed as an element in $H^{2r}_{\mathrm{dR}}(E^{2r})$ is defined by

$$dw^j d\bar{w}^{2r-j} := \frac{1}{(2r)!} \cdot \sum_{\sigma \in S_{2r}} m_{\sigma^{-1}(i_1)} \wedge m_{\sigma^{-1}(i_2)} \wedge \cdots \wedge m_{\sigma^{-1}(i_{2r})}$$

where the sum is over all elements of the symmetric group S_{2r}.

Let $p_1, p_\tau \in H_1(\mathrm{pr}^*(E), \mathbb{Q})$ correspond to closed paths on E from 0 to 1 and from 0 to τ, respectively. Denote by $\eta_1, \eta_\tau \in H^1_{\mathrm{dR}}(E)$ the basis which is Poincaré dual to p_1, p_τ, so that

$$\left\langle \omega, \eta_1 \right\rangle = \int_{p_1} \omega, \quad \left\langle \omega, \eta_\tau \right\rangle = \int_{p_\tau} \omega, \qquad \text{for all } \omega \in H^1_{\mathrm{dR}}(E).$$

The elements $\eta_\tau^j \eta_1^{2r-j}, 0 \leq j \leq 2r$ form a basis for $\mathrm{Sym}^{2r} H^1_{dR}(E)$.

Let w denote the natural holomorphic coordinate on E. Then the (i,j)-th entry in Table 3.2.1 is the pairing $\left\langle \ , \ \right\rangle$ of the differential in the i-th row with the differential in the j-th column:

	dw	$d\bar{w}$	η_1	η_τ
dw	0	$\frac{-1}{2\pi i}(\tau - \bar{\tau})$	1	τ
$d\bar{w}$	$\frac{1}{2\pi i}(\tau - \bar{\tau})$	0	1	$\bar{\tau}$

Table 3.1: Differential relations

It follows directly from this that

$$2\pi i dw = \tau \eta_1 - \eta_\tau, \quad 2\pi i d\bar{w} = \bar{\tau} \eta_1 - \eta_\tau,$$

and that

$$\left\langle dw^{2r}, \eta_\tau^j \eta_1^{2r-j} \right\rangle = \tau^j. \tag{3.23}$$

A second basis of $H^1_{\mathrm{dR}}(E)$ is defined by setting

$$\omega := 2\pi i dw, \qquad \eta := \frac{d\bar{w}}{\bar{\tau} - \tau}.$$

The expressions $\omega^j \eta^{2r-j}, 0 \leq j \leq 2r$ given using the notation above by

$$\omega^j \eta^{2r-j} := \frac{1}{(2r)!} \cdot \frac{(2\pi i)^r}{(\bar{\tau} - \tau)^{r+1}} \cdot \sum_{\sigma \in S_{2r}} m_{\sigma^{-1}(i_1)} \wedge m_{\sigma^{-1}(i_2)} \wedge \cdots \wedge m_{\sigma^{-1}(i_{2r})}$$

form a second basis for $\mathrm{Sym}^{2r} H^1_{dR}(E)$.

From the above calculations one can see that this basis satisfies the duality relations

$$\left\langle \omega^j \eta^{2r-j}, \omega^{2r-j'} \eta^{j'} \right\rangle = \begin{cases} 0 & \text{if } j \neq j' , \\ (-1)^j \frac{j!(2r-j)!}{(2r)!} & \text{if } j = j'. \end{cases} \tag{3.24}$$

We will need the following Proposition 3.5 of [3]:

Proposition 3.2.3. *The section F_f of $pr^*(\mathcal{L}_{2r})$ over $\mathbb{H}$ satisfying*

$$\left\langle F_f(\tau), \omega^j \eta^{2r-j} \right\rangle = \frac{(-1)^j (2\pi i)^{j+1}}{(\tau - \bar{\tau})^{2r-j}} \int_{i\infty}^{\tau} f(z)(z-\tau)^j (z - \bar{\tau})^{2r-j} dz, \quad (0 \leq j \leq 2r)$$

is a primitive of ω_f.

Proof of Proposition 3.2.3. We recall the proof given in [3]. By definition of the Gauss-Manin connection (particularly the Leibniz rule) and the fact that the sections $\eta_\tau^j \eta_1^{2r-j}$ are horizonal,

$$d\left\langle F_f, \eta_\tau^j \eta_1^{2r-j} \right\rangle = \left\langle \nabla F_f, \eta_\tau^j \eta_1^{2r-j} \right\rangle = \left\langle \mathrm{pr}^* \omega_f, \eta_\tau^j \eta_1^{2r-j} \right\rangle. \tag{3.25}$$

Substituting $\mathrm{pr}^*\omega_f = (2\pi i)^{2r+1} f(\tau) dw^{2r} d\tau$ and applying (3.23), this last expression is equal to

$$\left\langle \mathrm{pr}^*\omega_f, \eta_\tau^j \eta_1^{2r-j} \right\rangle = (2\pi i)^{2r+1} \left\langle f(\tau) dw^{2r}, \eta_\tau^j \eta_1^{2r-j} \right\rangle d\tau = (2\pi i)^{2r+1} f(\tau) \tau^j d\tau. \tag{3.26}$$

Therefore $d\left\langle F_f, \eta_\tau^j \eta_1^{2r-j} \right\rangle = (2\pi i)^{2r+1} f(\tau)\tau^j d\tau$. Integrating both sides of this identity with respect to τ gives

$$\left\langle F_f, \eta_\tau^j \eta_1^{2r-j} \right\rangle = (2\pi i)^{2r+1} \int_{i\infty}^{\tau} f(z) z^j dz, \quad (0 \le j \le 2r). \tag{3.27}$$

Therefore any F_f satisfying the above identity is a global primitive of ω_f. In particular, the relation (3.27) shows that for all homogenous polynomials $P(x,y)$ of degree $2r$,

$$\left\langle F_f, P(\eta_\tau, \eta_1) \right\rangle = (2\pi i)^{2r+1} \int_{i\infty}^{\tau} f(z) P(z,1) dz.$$

From Table 3.2.1 we have

$$\omega^j \eta^{2r-j} = Q(\eta_\tau, \eta_1), \quad \text{with } Q(x,y) := \frac{(-1)^j}{(2\pi i(\tau - \bar{\tau}))^{2r-j}} (x - \tau y)^j (x - \bar{\tau} y)^{2r-j},$$

which gives

$$\left\langle F_f(\tau), \omega^j \eta^{2r-j} \right\rangle = \frac{(-1)^j (2\pi i)^{j+1}}{(\tau - \bar{\tau})^{2r-j}} \int_{i\infty}^{\tau} f(z)(z-\tau)^j (z-\bar{\tau})^{2r-j} dz, \quad (0 \le j \le 2r).$$

$\square$

The proof of Theorem 3.2.2 can now be stated.

3.2.2 Proof of Theorem 3.2.2

Assume the notation and hypotheses from the previous sections. The two necessary steps referenced earlier are given in the following two lemmas.

Lemma 3.2.4.

$$\left\langle F_f(\tau), \omega^r \eta^r \right\rangle = \frac{(-1)^r}{(\sqrt{D})^r} \alpha(\tau). \tag{3.28}$$

Lemma 3.2.5.

$$\left\langle F_f(\tau), cl(\Delta_\tau) \right\rangle = \frac{1}{(2r)!} (-1)^r (\sqrt{D})^r \left\langle F_f(\tau), \omega^r \eta^r \right\rangle. \tag{3.29}$$

Lemma 3.2.4 follows immediately from Proposition 3.2.3 by setting $j := r$. It remains to prove Lemma 3.2.5.

Proof of Lemma 3.2.5. Since $\Delta := \Delta_\tau$ is an r-dimensional complex submanifold of E^{2r}, any (p, q)-form ω satisfies

$$\int_\Delta \omega = 0 \qquad \text{if} \quad (p, q) \neq (r, r).$$

Therefore, letting F_f denote $F_f(\tau)$, and writing it as a linear combination

$$F_f = \sum_{j=0}^{2r} F_{j,2r-j} \cdot \omega^j \eta^{2r-j} \in \operatorname{Sym}^{2r} H^1_{\mathrm{dR}}(E)$$

in the basis $\omega^j \eta^{2r-j}$ of $\operatorname{Sym}^{2r} H^1_{\mathrm{dR}}(E)$, we get

$$\int_\Delta F_f = \int_\Delta F_{r,r} \cdot \omega^r \eta^r, \tag{3.30}$$

which by the definition of the cycle class map cl is

$$= F_{r,r} \cdot \int_{E^{2r}} \omega^r \eta^r \wedge \mathrm{cl}(\Delta). \tag{3.31}$$

Suppose $\mathrm{cl}(\Delta)$ is given in the basis $\omega^j\eta^{2r-j}$ by

$$\mathrm{cl}(\Delta) = \sum_{j=0}^{2r} \Delta_{j,2r-j} \cdot \omega^j\eta^{2r-j} \in \mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(E).$$

The duality relations for $\omega^j\eta^{2r-j}$ in (3.24) imply

$$\left\langle F_{r,r} \cdot \omega^r\eta^r,\ \Delta_{j,2r-j} \cdot \omega^j\eta^{2r-j} \right\rangle = 0 \qquad \text{unless } j = r.$$

Putting this together with (3.30), we find

$$\left\langle F_f(\tau), \mathrm{cl}(\Delta) \right\rangle = \left\langle F_f,\ \Delta_{r,r} \cdot \omega^r\eta^r \right\rangle.$$

Thus to prove this lemma, it suffices to show $\Delta_{r,r} = \frac{1}{(2r)!}(-1)^r(\sqrt{D})^r$. By Poincaré duality, the coefficient $\Delta_{r,r}$ is given by the integral

$$\Delta_{r,r} = \frac{1}{(2\pi i)^r} \int_\Delta \omega^r\eta^r.$$

Therefore Lemma 3.2.5 reduces to the following claim.

Claim.

$$\int_\Delta \omega^r\eta^r = \frac{1}{(2r)!}(-2\pi i)^r(\sqrt{D})^r.$$

First note that $\Delta = \varepsilon(\Gamma_{a\tau})^r \subset E^{2r}$ by definition. Since the component $\omega^r\eta^r$ is in the image of projection ε, we have

$$\int_\Delta \omega^r\eta^r = \int_{(\Gamma_{a\tau})^r} \omega^r\eta^r.$$

Let $[a, b, c]$ be the coefficients of the quadratic form of discriminant D such that $\tau = \frac{-b+\sqrt{D}}{2a}$. We prove the claim by induction on r. First suppose

$r = 1$. The elliptic curve given by E_τ is isomorphic to $\Gamma_{a\tau}$ via the map

$$g : E_\tau \longrightarrow \Gamma_{a\tau}$$
$$u \mapsto (u, a\tau\, u).$$

In $H^2_{\mathrm{dR}}(E \times E)$,

$$\omega\eta = \frac{1}{2} \cdot \frac{2\pi i}{\bar{\tau} - \tau}(dw_1 \wedge d\bar{w}_2 + d\bar{w}_2 \wedge dw_2)$$

and so

$$\begin{aligned} g^*(\omega\eta) &= \frac{1}{2} \cdot \frac{2\pi i\, a}{\bar{\tau} - \tau}(\bar{\tau} \cdot du \wedge d\bar{u} + \tau \cdot d\bar{u} \wedge du) \\ &= a\pi i\, du \wedge d\bar{u}. \end{aligned}$$

Therefore

$$\begin{aligned} \int\limits_{\Gamma_{a\tau}} \omega\eta &= \int\limits_{E_\tau \times E_\tau} g^*(\omega\eta) \\ &= a\pi i \int\limits_{E_\tau \times E_\tau} du \wedge d\bar{u} \\ &= (-2\pi i) \cdot ai\, \mathrm{Im}(\tau) \\ &= \frac{1}{2} \cdot (-2\pi i) \cdot \sqrt{D}. \end{aligned}$$

Now for $r > 1$, suppose the following identity holds:

$$\int_{(\Gamma_{a\tau})^r} \omega^r \eta^r = \frac{1}{(2r)!} \cdot (-2\pi i)^r (\sqrt{D})^r.$$

We will show the identity also holds for $r + 1$. For any integer $n \geq 1$, set

$$m_{i_1} \wedge m_{i_2} \wedge \cdots \wedge m_{i_{2n}} := \underbrace{dw_{i_1} \wedge \cdots \wedge dw_{i_n}}_{n\,\text{times}} \wedge \underbrace{d\bar{w}_{i_{n+1}} \wedge \cdots \wedge d\bar{w}_{i_{2n}}}_{n\,\text{times}}.$$

Set $c_r := \frac{1}{(2r+2)!} \cdot \frac{(2\pi i)^{r+1}}{(\bar{\tau}-\tau)^{r+1}}$ for the moment. Then observe

$$
\begin{aligned}
c_r^{-1} \cdot \omega^{r+1}\eta^{r+1} &= \sum_{\sigma \in S_{2r+2}} m_{\sigma^{-1}(i_1)} \wedge m_{\sigma^{-1}(i_2)} \wedge \cdots \wedge m_{\sigma^{-1}(i_{2r+2})} \\
&= \sum_{\sigma \in S_{2r}} \Big[m_{\sigma^{-1}(i_1)} \wedge \cdots \wedge m_{\sigma^{-1}(i_{2r})} \wedge dw_{i_{2r+1}} \wedge d\bar{w}_{i_{2r+2}} \\
&\qquad + m_{\sigma^{-1}(i_1)} \wedge \cdots \wedge m_{\sigma^{-1}(i_{2r})} \wedge d\bar{w}_{i_{2r+1}} \wedge dw_{i_{2r+2}} \Big] \\
&= \sum_{\sigma \in S_{2r}} m_{\sigma^{-1}(i_1)} \wedge \cdots \wedge m_{\sigma^{-1}(i_{2r})} \\
&\quad \wedge (dw_{i_{2r+1}} \wedge d\bar{w}_{i_{2r+2}} + d\bar{w}_{i_{2r+1}} \wedge dw_{i_{2r+2}}).
\end{aligned}
$$

The first equality is by definition of $\omega^{r+1}\eta^{r+1}$ and the second follows because permutations of the type

$$
\begin{aligned}
&\cdots \wedge dw_{i_{2r+1}} \wedge dw_{i_{2r+2}} \\
\text{or} \quad &\cdots \wedge d\bar{w}_{i_{2r+1}} \wedge d\bar{w}_{i_{2r+2}}
\end{aligned}
$$

equal 0.

By Fubini's Theorem combined with above,

$$
\begin{aligned}
\int_{(\Gamma_{a\tau})^{r+1}} \omega^{r+1}\eta^{r+1} &= c_r \cdot \int_{(\Gamma_{a\tau})^{r}} \sum_{\sigma \in S_{2r}} m_{\sigma^{-1}(i_1)} \wedge \cdots \wedge m_{\sigma^{-1}(i_{2r})} \times \\
&\qquad \int_{\Gamma_{a\tau}} (dw_{i_{2r+1}} \wedge d\bar{w}_{i_{2r+2}} + d\bar{w}_{i_{2r+1}} \wedge dw_{i_{2r+2}}) \\
&= \frac{1}{(r+1)(2r+1)} \cdot \int_{(\Gamma_{a\tau})^{r}} \omega^r \eta^r \cdot \int_{\Gamma_{a\tau}} \omega\eta \\
&= \frac{1}{(2r+2)!} \cdot (-2\pi i)^{r+1} (\sqrt{D})^{r+1}
\end{aligned}
$$

as was to be shown. This completes the proof of Lemma 3.2.5. $\square$

It now remains to explicitly state and prove the condition alluded to in (3.22). The results and proofs below can also be found in [3]. We have modified them slightly for the (simpler) setting of Heegner cycles versus the ‘generalized Heegner cycles’ which are studied in their paper. The following is meant to serve as an introductory survey to their results; for more precise statements see [3, p.14-25].

The first task is to show that the Heegner cycle Δ is homologically trivial by constructing a $(2r+1)$-chain whose boundary is Δ. This will also provide a more explicit description of AJ on Δ. The proof involves the construction of two $(2r)$-chains, $\Delta^\sharp$ and $\tilde{\Delta}$, in the lift $\tilde{W}_{2r}$ of W_{2r}. The general idea behind each one is to:

- First, construct a $(2r)$-chain $\Delta^\sharp$ on $\tilde{W}_{2r}$ supported over τ that maps to Δ under pr_*. In general, $\Delta^\sharp$ is not homologically trivial on W_{2r} but its feature is that it is in the kernel of pr_*, which will imply that Δ is homologically trivial.

- Second, construct a $(2r)$-chain $\tilde{\Delta}$ on $\tilde{W}_{2r}$ from the data of $\Delta^\sharp$ supported on τ and its translates $\gamma\tau$ under the action of Γ. The chain $\tilde{\Delta}$ *will* be homologically trivial and map to [0] as well under pr_*. An explicit $(2r+1)$-chain will be constructed whose boundary is $\tilde{\Delta}$ which will allow for the Abel-Jacobi map to be explicitly computed. The end goal will be to translate this to a statement for the simpler $\Delta^\sharp$ supported on just one point τ.

The explicit statement [3, Proposition 2.6] is:

Proposition 3.2.6. *Assume $r > 0$. Then there exists a topological cycle $\tilde{\Delta}$ on $\tilde{W}_{2r}$ satisfying*

1. *The pushforward $pr_*(\tilde{\Delta})$ satisfies*

$$pr_*(\tilde{\Delta}) = \tilde{\Delta} + \partial\xi,$$

where ξ is a topological $(2r+1)$-chain supported on $\pi^{-1}(pr(\tau))$.

2. *The cycle $\tilde{\Delta}$ is homologically trivial on $\tilde{W}_{2r}$.*

Proof. The following is an outline of the basic steps required to prove this proposition in the case of classical Heegner cycles. For details see [3].

Let $P := \mathrm{pr}(\tau)$. The map pr gives an isomorphism between the fibers $\tilde{\pi}(\tau)$ and $\pi^{-1}(P)$. Therefore by the existence of the Heegner cycle Δ, there exists cycles $\Upsilon^\sharp$ and $\Delta^\sharp$ on $\tilde{W}_{2r}$ supported on $\tilde{\pi}^{-1}(\tau)$ such that

$$\mathrm{pr}_*(\Upsilon^\sharp) = \Upsilon \quad \text{and} \quad \mathrm{pr}_*(\Delta^\sharp) = \Delta.$$

The cycle $\Delta^\sharp$ is not necessarily homologically trivial on $\tilde{W}_{2r}$. In fact since $\mathbb{H}$ is contractible, the inclusion

$$\iota_\tau : \tilde{\pi}^{-1}(\tau) \longrightarrow \tilde{W}_{2r}$$

induces an isomorphism

$$\iota_{\tau *} : H_{2r}(\tilde{\pi}^{-1}(\tau), \mathbb{Q}) \longrightarrow H_{2r}(\tilde{W}_{2r}, \mathbb{Q}),$$

which identifies the classes $[\Upsilon^\sharp]$, $[\Delta^\sharp]$ in $H_{2r}(\tilde{W}_{2r}, \mathbb{Q})$ with the classes of Υ, Δ in $H_{2r}(E^{2r}(\mathbb{C}), \mathbb{Q})$, respectively.

The pushforward map

$$\mathrm{pr}_* : H_{2r}(\tilde{W}_{2r}, \mathbb{Q}) \longrightarrow H_{2r}(W^0_{2r}, \mathbb{Q})$$

has kernel which contains the module $I_\Gamma H_{2r}(\tilde{W}_{2r}, \mathbb{Q})$, where I_Γ is the augmentation ideal[1] of the group ring $\mathbb{Q}[\Gamma]$. Furthermore

$$\varepsilon H_{2r}(\tilde{W}_{2r}, \mathbb{Q}) \subset I_\Gamma H_{2r}(\tilde{W}_{2r}, \mathbb{Q}).$$

Therefore since $[\Delta^\sharp] = \varepsilon[\Upsilon^\sharp]$ we get

$$[\Delta] = \mathrm{pr}_*([\Delta^\sharp]) = 0.$$

Hence Δ is homologically trivial. The existence of $\tilde{\Delta}$ satisfying (1) and (2) above follows. $\square$

To describe $\tilde{\Delta}$ more concretely, we write $[\Delta^\sharp]$ in the module $I_\Gamma H_{2r}(\tilde{W}_{2r}, \mathbb{Q})$ as

$$[\Delta^\sharp] = \sum_{j=1}^{t} (\gamma_j^{-1} - 1) Z_j,$$

for some $\gamma_1, \ldots, \gamma_t \in \Gamma$ and $Z_1, \ldots, Z_t \in H_{2r}(\tilde{W}_{2r}, \mathbb{Q})$. Define $\mathcal{Z}(\tau, Z)$ to be any topological $2r$-cycle supported over τ such that $\iota_{\tau *}([\mathcal{Z}(\tau, Z)]) = [Z]$. Then $\tilde{\Delta}$ is defined by

$$\tilde{\Delta} := \sum_{j=1}^{t} (\mathcal{Z}(\gamma_j \tau, Z) - \mathcal{Z}(\tau, Z)).$$

[1]The augmentation ideal I of a group ring $R[G]$ is defined to be the kernel of the ring homomorphism $\varepsilon : R[G] \to R$ which sends $\sum_i r_i g_i$ to $\sum_i r_i$. It is generated by the elements $\{g - 1 \,:\, g \in G\}$.

The properties (1) and (2) of the proposition can be checked explicitly. For example, the homological triviality of $\tilde{\Delta}$ follows from the fact that

$$\tilde{\Delta} = \partial \tilde{\Delta}^\sharp \qquad \text{with} \qquad \tilde{\Delta}^\sharp := \sum_{j=1}^{t} \mathcal{Z}(\tau \to \gamma_j \tau, Z_j),$$

where

$$\mathcal{Z}(\tau \to \gamma_j \tau, Z_j) := \mathrm{path}(\tau \to \gamma_j \tau) \times Z_j,$$

for any continuous path on $\mathbb{H}$ from τ to $\gamma_j \tau$.

The next goal is to use the descriptions of $\Delta^\sharp$ and $\tilde{\Delta}^\sharp$ above to compute the Abel-Jacobi map on Heegner cycles. This is [3, Proposition 3.3] modified for the classical Heegner cycles:

Proposition 3.2.7. *For all $f \in S_{2r+2}(\Gamma)$*

$$AJ(\Delta)(\omega_f) = \sum_{j=1}^{t} \left(\langle F_f(\gamma_j \tau), cl(Z_j) \rangle - \langle F_f(\tau), cl(Z_j) \rangle \right).$$

for any primitive F_f of ω_f.

Proof. The definition of the Abel-Jacobi map together with Proposition 3.2.6 imply that

$$\begin{aligned} \mathrm{AJ}(\Delta)(\omega_f) &= \int_{\mathrm{pr}_*(\tilde{\Delta}^\sharp)} \omega_f = \int_{\tilde{\Delta}^\sharp} \mathrm{pr}^* \omega_f \\ &= \sum_{j=1}^{t} \int_{\tau}^{\gamma_j \tau} \langle \mathrm{pr}^* \omega_f, \mathrm{cl}(Z_j) \rangle. \end{aligned}$$

Here we view $\mathrm{cl}(Z_j)$ as the horizontal section of $\mathrm{pr}^*(\mathcal{L}_{2r})$ whose value at τ is equal to $\mathrm{cl}(Z_j)$. (Since the vector bundle $\mathrm{pr}^*(\mathcal{L}_{2r})$ is trivial, it admits a basis of horizontal sections over $\mathbb{H}$ given by $\eta_1^j\eta_\tau^{2r-j}$). Therefore by the definition of the Gauss-Manin connection,

$$\big\langle \mathrm{pr}^*\omega_f, \mathrm{cl}(Z_j)\big\rangle = \big\langle \nabla F_f, \mathrm{cl}(Z_j)\big\rangle = d\big\langle F_f, \mathrm{cl}(Z_j)\big\rangle.$$

Hence by Stokes theorem,

$$\mathrm{AJ}(\Delta)(\omega_f) = \sum_{j=1}^{t}\big(\big\langle F_f(\gamma_j\tau), \mathrm{cl}(Z_j)\big\rangle - \big\langle F_f(\tau), \mathrm{cl}(Z_j)\big\rangle\big).$$

$\square$

We will now use the above to get a formula for the Abel-Jacobi map in terms of the simpler cycle $\Delta^\sharp$ supported over τ. For any $Z \in \mathrm{Sym}^{2r} H^1_{\mathrm{dR}}(E)$, [3, Lemma 3.7][2] states:

$$\big\langle F_f(\gamma\tau), Z\big\rangle - \big\langle \gamma F_f(\tau), Z\big\rangle = (2\pi i)^{2r+1}\int_{i\infty}^{\gamma(i\infty)} f(z)P_Z(z,1)dz, \qquad \forall\, \gamma \in \Gamma; \tag{3.32}$$

here $P_Z(x,y) \in \mathbb{C}[x,y]$ is defined to be the homogenous polynomial of degree $2r$ such that

$$Z = P_Z(\eta_\tau, \eta_1).$$

The proof follows from Proposition 3.2.3 and a straightforward computation using that f has weight $2r+2$ and P_Z is homogenous of degree $2r$.

Recall Λ_r is the period lattice attached to $S_{2r+2}(\Gamma)$. This is a $\mathbb{Z}$-submodule of $S_{2r+2}(\Gamma)^\vee$ of rank $2g$. In particular it contains the periods from

[2]From here on out we require F_f to be an *integral* primitive. See [3, p.24] for a definition.

(3.32) above. This implies

$$\left\langle F_f(\gamma\tau), Z\right\rangle = \left\langle F_f(\tau), \gamma^{-1}Z\right\rangle \mod \Lambda_r \tag{3.33}$$

for all $Z \in H^{2r+1}_{\mathrm{dR}}(\varepsilon\tilde{W}_{2r}, \mathbb{Z})$.

This brings us to the precise statement of (3.22) from [3, Proposition 3.11]:

Proposition 3.2.8.

$$AJ(\Delta)(\omega_f) = \left\langle F_f(\tau), cl(\Delta^\sharp)\right\rangle \bmod \Lambda_r.$$

Proof. By Proposition 3.2.7 and (3.33) above,

$$\begin{aligned}
\mathrm{AJ}(\Delta)(\omega_f) &= \sum_{j=1}^{t}\left(\left\langle F_f(\gamma_j\tau), \mathrm{cl}(Z_j)\right\rangle - \left\langle F_f(\tau), \mathrm{cl}(Z_j)\right\rangle\right) \\
&= \sum_{j=1}^{t}\left(\left\langle F_f(\tau), \gamma_j^{-1}\mathrm{cl}(Z_j)\right\rangle - \left\langle F_f(\tau), \mathrm{cl}(Z_j)\right\rangle\right) \mod \Lambda_r \\
&= \left\langle F_f(\tau), \sum_{j=1}(\gamma_j^{-1} - 1)\mathrm{cl}(Z_j)\right\rangle \\
&= \left\langle F_f(\tau), \mathrm{cl}(\Delta^\sharp)\right\rangle \bmod \Lambda_r.
\end{aligned}$$

□

Proposition 3.2.8 combined with Lemmas 3.2.4 and 3.2.5 complete the proof of Theorem 3.2.2.

3.3 The Shimura Lift

The results in this section are proven for the case of level $N = 1$. See the end of the section for a discussion on how to generalize to level N as well as some examples. Set $\Gamma := SL_2(\mathbb{Z})$. By D we will always mean a negative integer. For $D \equiv 0, 1 \bmod 4$ (not necessarily fundamental), define

$$\mathscr{Q}_D := \{[a, b, c] : D = b^2 - 4ac,\ a, b, c \in \mathbb{Z}\}$$

and

$$\mathscr{Q}_D^0 := \{[a, b, c] : D = b^2 - 4ac,\ a, b, c \in \mathbb{Z},\ \gcd(a, b, c) = 1\}.$$

The set $\mathscr{Q}_D$ corresponds to binary quadratic forms of discriminant D and $\mathscr{Q}_D^0$ to the subset of primitive forms. For $Q := [a, b, c] \in \mathscr{Q}_D$, this correspondence is given by

$$Q(x, y) := ax^2 + bxy + cy^2.$$

For any $M := \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Mat}_2(\mathbb{Z})$ with $\det M \neq 0$, set

$$Q \circ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (x, y) := Q(\alpha x + \beta y, \gamma x + \delta y).$$

Let $M^* := M^{-1}(\det M)$ denote the matrix adjoint of M and define

$$(M \cdot Q)(x, y) := Q \circ M^*(x, y).$$

This defines an action of Γ on $\mathscr{Q}_D$ which is compatible with the linear fractional action of Γ on the roots of $Q(\tau, 1) = 0$.

Denote by τ_Q the root in $\mathbb{H}$ of $Q(\tau, 1) = 0$. We define P_D^0 to be the formal sum of the roots τ_Q as Q runs over a set of representatives of primitive binary quadratic forms of discriminant D,

$$P_D^0 := \begin{cases} \sum_{Q \in \mathscr{Q}_D^0/\Gamma} \tau_Q & \text{if } D \equiv 0, 1 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if D is fundamental then $P_D^0 = P_D$ from Chapter 2 and that $(y_D)_f$ from Chapter 2 is by definition $\alpha(P_D^0) + \overline{\alpha(P_D^0)}$. For any D (not necessarily fundamental) we define $y_D = (y_D)_f$ to be the sum of

$$\sum_{\substack{d>0 \\ d^2 \mid |D|}} d^{k-1} \alpha(P_{\frac{D}{d^2}}) \tag{3.34}$$

and its complex conjugate.

We now extend the conjectures from Chapter 2 to all integers $D < 0$ in the case of level 1.

Conjecture 3.3.1. *Let $f = \sum_{n \geq 1} a_n q^n \in S_{2k}(1)$ be a normalized newform with rational coefficients, and assume $\varepsilon = -1$ and $L'(f, k) \neq 0$. Then for all integers $D < 0$, there exist integers $m(D)$ such that*

$$t\, y_D = m(D) y_f \qquad \text{in } \mathbb{C}/L_f,$$

where $y_f \in \mathbb{C}/L_f$ is non-torsion and $t \in \mathbb{Z}$ are both nonzero and independent of D.

Conjecture 3.3.2. *Let $f = \sum_{n \geq 1} a_n q^n \in S_{2k}(1)$ be a normalized newform with rational coefficients, and assume $\varepsilon = -1$ and $L'(f, k) \neq 0$. Assume Conjecture 3.3.1. Then*

$$g(z) := \sum_{D<0} m(D) q^{|D|}$$

is in $S_{k+1/2}(4)$ and is in Shimura correspondence with f.

In this section we will prove the second part of Conjecture 3.3.2, namely that if we assume Conjecture 3.3.1 and the existence of g above, then the D-th Shimura lift of g is f for a certain D.

Given $g = \sum c(n)q^n \in S_{k+1/2}(4)$ and $D < 0$ fundamental, recall the D-th Shimura lift of g is defined by

$$g\big|\mathscr{S}_{2k,D}(z) := \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D}{d}\right)d^{k-1}\cdot c\left(\frac{|D|n^2}{d^2}\right)\right)q^n, \tag{3.35}$$

(see [43], for example).

Theorem 3.3.3. *Assume Conjecture 3.3.1 above. Assume as well that there exists a fundamental discriminant $D_0 < 0$ with $h(D_0) = 1$ such that $y_f = y_{D_0}$. Then the function*

$$g(z) := \sum_{D<0} m(D)q^{|D|}$$

satisfies

$$g\big|\mathscr{S}_{2k,D_0}(z) = f(z).$$

In order to prove this theorem we need to reinterpret our map α as a function on binary quadratic forms rather than on Heegner points. The general theory of such functions is described as follows. For additional background see [21, p. 504–508].

3.3.1 Functions on classes of binary quadratic forms

For an integer $D < 0$, instead of a sum of points, we define P_D^0 to be the formal sum of all primitive binary quadratic forms of discriminant D modulo

Γ,

$$P_D^0 := \sum_{Q \in \mathscr{Q}_D^0/\Gamma} Q, \quad \text{if } D \equiv 0, 1 \bmod 4,$$

and 0 otherwise. Define P_D to be the formal sum

$$P_D := \sum_{\substack{d>0 \\ d^2 \big| |D|}} d \cdot P^0_{\frac{D}{d^2}}, \quad \text{if } D \equiv 0, 1 \bmod 4,$$

and 0 otherwise. Notice $P_D = P_D^0$ if D is fundamental.

The union of $\mathscr{Q}_D/\Gamma$ over all integers $D < 0$ is denoted

$$\bigcup_D \mathscr{Q}_D/\Gamma$$

where we set $\mathscr{Q}_D := \emptyset$ if $D \not\equiv 0, 1 \bmod 4$.

We will say a function $F : \bigcup_D \mathscr{Q}_D/\Gamma \longrightarrow \mathbb{C}$ is of *weight* 0. Moreover we will call it *homogeneous of degree* r for some integer $r > 0$ if $F(\ell \cdot Q) = \ell^r \cdot F(Q)$. (Here $\ell \cdot [a, b, c] := [\ell a, \ell b, \ell c]$). Extend such an F to P_D^0 and P_D by linearity. Then observe that

$$F(P_D) = \sum_{\substack{d>0 \\ d^2 \big| |D|}} d^r \cdot F\left(P^0_{\frac{D}{d^2}}\right).$$

Hecke operators can be constructed for these functions as follows. Let

$$M(n) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \ : \ \det M = n \right\}.$$

Since there are only finitely many lattices Λ' of index n in a given lattice $\Lambda \subset \mathbb{C}$, there are only finitely many right cosets $\Gamma \cdot \sigma$ in $M(n)$. Thus there is a right coset decomposition

$$M(n) = \bigcup_{i=1}^{\nu(n)} \Gamma \cdot \sigma_i$$

with $\nu(n) < \infty$. A set of representatives is given explicitly by

$$\Gamma \backslash M(n) = \left\{ \begin{pmatrix} n/m & t \\ 0 & m \end{pmatrix} \,:\, m|n,\ t = 0, \ldots, m-1 \right\}. \qquad (3.36)$$

Definition 3.3.1. For an integer $n \geq 0$, the *n-th Hecke operator of weight* 0 is defined on any function $F : \bigcup_D \mathscr{Q}_D/\Gamma \longrightarrow \mathbb{C}$ by

$$T_0(n)[F](Q) := \sum_{\sigma \in \Gamma \backslash M(n)} F(\sigma \cdot Q), \qquad Q \in \bigcup_D \mathscr{Q}_D/\Gamma. \qquad (3.37)$$

This definition is equivalent (up to a scalar multiple of n) with the definition of the classical Hecke operators acting on modular forms for weight 0 [42, p.244]. For $f \in M_{2k}(\Gamma)$, recall the n-th Hecke operator is defined by

$$T_{2k}(n)[f](z) := n^{2k-1} \sum_{\sigma \in \Gamma \backslash M(n)} m^{-2k} f(\sigma z), \qquad z \in \mathbb{H}$$

where σ runs over the representatives in (3.36).

An identity for the action of the Hecke operators on $F(P_D)$ is given in [21, p.507][3]. For p prime it says

$$T_0(p)[F](P_D) = F(P_{Dp^2}) + p^r \left(\frac{D}{p}\right) F(P_D) + p^{2r+1} F\left(P_{\frac{D}{p^2}}\right)$$

where the last term is omitted if $p^2 \nmid D$.

[3] In their notation, $F(P_D)$ is denoted by $\mathscr{L}_D(F)$.

By induction on the powers of primes dividing m, they deduce from this the formula

$$T_0(n)[F](P_D) = \sum_{d|n} \left(\frac{D}{d}\right) d^r F\left(P_{\frac{Dn^2}{d^2}}\right) \tag{3.38}$$

for any fundamental discriminant $D < 0$ and integer $m \geq 1$.

We can now redefine our map α as a function on quadratic forms by

$$\alpha[f](Q) := (2\pi i)^k \int_{i\infty}^{\tau_Q} f(z)Q(z)^{k-1}dz, \tag{3.39}$$

where $Q(z) := Q(z, 1)$ and $\tau_Q \in \mathbb{H}$ is the solution to $Q(\tau) = 0$. Note that

$$\alpha[f](\gamma \cdot Q) = \alpha[f](Q) \bmod L_f$$

by Chapter 2 and that $\alpha[f](\ell \cdot Q) = \ell^{k-1}\, \alpha[f](Q)$; hence α is of weight 0 and homogeneous of degree $k-1$, modulo L_f. Define

$$y_D := \alpha[f](P_D) + \overline{\alpha[f](P_D)}.$$

Note that the definitions of α and y_D are equivalent to those defined in the beginning of this section on Heegner points.

The following lemma says that the Hecke operators commute with the map α.

Lemma 3.3.4. *Let $f = \sum_{n\geq 1} a_n q^n \in S_{2k}(1)$ be a normalized newform with rational coefficients, and assume $\varepsilon = -1$ and $L'(f, k) \neq 0$. Then for any integer $n \geq 1$,*

$$T_0(n)[\alpha(f)(Q)] = \alpha[T_{2k}(n)(f)](Q) \bmod L_f.$$

Proof. By Definition 3.3.1 and (3.36) we have

$$\begin{aligned} T_0(n)[\alpha(f)(Q)] &= \sum_{\sigma \in \Gamma \backslash M(n)} \alpha[f](\sigma \cdot Q) \\ &= \sum_{m|n} \sum_{t=0}^{m-1} \int_{i\infty}^{(n\tau_Q+tm)/m^2} f(z) Q(\sigma^* z)^{k-1} \big(\frac{n}{m}\big)^{2k-2} dz. \end{aligned} \tag{3.40}$$

The above uses the general fact that

$$Q \circ M^*(z) = Q(M^* z)(-\gamma z + \alpha)^2 \text{ for } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}).$$

Now apply a change of variables with $z \mapsto \sigma(z)$. Hence $\sigma^*(z) \mapsto z$ since $\sigma \circ \sigma^*$ acts as the identity on z, and (3.40) is equal to

$$\begin{aligned} &= \sum_{m|n} \sum_{t=0}^{m-1} \int_{i\infty}^{\tau_Q} f(\sigma z) Q(z)^{k-1} \big(\frac{n}{m}\big)^{2k-2} \frac{n}{m^2} dz \\ &= \int_{i\infty}^{\tau_Q} \big[n^{2k-1} \sum_{m|n} m^{-2k} \sum_{t=0}^{m-1} f\big(\frac{nz+tm}{m^2}\big)\big] Q(z)^{k-1} dz \\ &= \int_{i\infty}^{\tau_Q} [T_{2k}(n)(f)(z)] Q(z)^{k-1} dz \\ &= \alpha[T_{2k}(n)(f)](Q) \end{aligned}$$

□

The proof of Theorem 3.3.3 will follow almost immediately.

Proof of Theorem 3.3.3. For $D_0 < 0$ fundamental with $h(D_0) = 1$, we have $T_0(n)[F](P_{D_0}) = T_0(n)[F](Q_0)$ where $\mathscr{Q}_D/\Gamma = \{[Q_0]\}$. Set

$$y_f := y_{D_0} = \alpha[f](Q_0) + \overline{\alpha[f](Q_0)}.$$

Hence formula (3.38) gives

$$T_0(n)[F](Q_0) = \sum_{d|n} \left(\frac{D_0}{d}\right) d^r \cdot F\left(P_{\frac{D_0 n^2}{d^2}}\right).$$

Set $F := \alpha(f) \bmod L_f$. By Lemma 3.3.4 we also have

$$T_0(n)[\alpha(f)(Q_0)] = \alpha[T_{2k}(n)(f)](Q_0) \bmod L_f.$$

Combining the above formulas gives

$$\alpha[T_{2k}(n)(f)](Q_0) = \sum_{d|n} \left(\frac{D_0}{d}\right) d^{k-1} \cdot \alpha[f]\left(P_{\frac{D_0 n^2}{d^2}}\right) \bmod L_f.$$

On the other hand since f is a newform we have

$$\alpha[T_{2k}(n)(f)] = a(n)\alpha[f]$$

where $a(n)$ is the n-th Fourier coefficient of f. Together these imply

$$a(n)y_f = \sum_{d|n} \left(\frac{D_0}{d}\right) d^{k-1} \cdot y_{\frac{D_0 n^2}{d^2}} \bmod L_f. \tag{3.41}$$

Set $g := \sum_{D<0} m(D)q^{|D|}$ with the $m(D)$ defined by Conjecture 3.3.1. By definition of the Shimura lift,

$$g\big|\mathscr{S}_{2k,D_0}(z) = \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D_0}{d}\right)d^{k-1} \cdot m\left(\frac{|D_0|n^2}{d^2}\right)\right)q^n.$$

But we also have $y_D = m(D)y_f \bmod L_f$ by Conjecture 1. Therefore

$$\left(g\big|\mathscr{S}_{2k,D_0}(z)\right) \cdot y_f = \sum_{n=1}^{\infty}\left(\sum_{d|n}\left(\frac{D_0}{d}\right)d^{k-1} y_{\frac{|D_0|n^2}{d^2}}\right)q^n \tag{3.42}$$

Combining (3.41) and (3.42) gives

$$\left(g\middle|\mathscr{S}_{2k,D_0}(z)\right) \cdot y_f = f \cdot y_f \bmod L_f.$$

But the fact that y_f is non-torsion in $\mathbb{C}/L_f$ by hypothesis implies $g\big|\mathscr{S}_{2k,D_0}(z) = f$ as wanted. $\square$

3.3.2 Examples and generalization to level N

Example 3.3.5 $(2k = 18, N = 1)$**.** We recompute the values $m(D)$ for the weight 18 level 1 newform f given in Chapter 2 for all integers $|D| \leq 40$, omitting the ones with $D \not\equiv 0, 1 \bmod 4$ since these are trivially equal to 0. In red are the extra $m(D)$ we get by extending to all integers $D < 0$ rather than just fundamental ones. These are identical to the coefficients of the weight 19/2 level 4 half-integer weight form in [13, p.141].

$\lvert D\rvert$	m_D	$\lvert D\rvert$	m_D
3	1	23	2736
4	-2	24	-1464
7	-16	27	-4284
8	36	28	12544
11	-99	31	-6816
12	-272	32	-19008
15	-240	35	27270
16	1056	36	-4554
19	-253	39	-6864
20	-1800	40	39880

Table 3.2: $f \in S_{18}(1)$. List of D, m_D such that $y_D - m_D y_f \in L_f$ for $|D| \leq 40$.

Regarding level $N > 1$, the only discrepancy is in defining P_D for the case where $N|D$. The details are yet to be carried out but appear to be straightforward.

Chapter 4

Split-CM points and central values of Hecke L-series

4.1 Introduction

Let $D < 0$, $|D|$ prime be the discriminant of an imaginary quadratic field K with ring of integers $\mathcal{O}_K$. Suppose N is a prime which splits in $\mathcal{O}_K$ and is divisible by an ideal $\mathcal{N}$ of norm N. We will define Hecke characters $\psi_{\mathcal{N}}$ of K of weight one and conductor $\mathcal{N}$ (see Section 4.3). These are twists of the canonical Hecke characters studied by Rohrlich [64–66] and Shimura [76, 78, 80]. Denote by $L(\psi_{\mathcal{N}}, s)$ the corresponding Hecke L-series.

Our main theorem (Theorem 4.3.6) is a formula in the spirit of Waldspurger's results [89, 90]. It says approximately that

$$L(\psi_{\mathcal{N}}, 1) = \sum_{[R]} \sum_{[\mathfrak{a}]} \Theta_{[\mathfrak{a},R],\mathcal{N}} \cdot h^{\varepsilon}_{[\mathfrak{a},R]}(-N). \tag{4.1}$$

Here the first sum is over all conjugacy classes of maximal orders R in the quaternion algebra ramified only at ∞ and $|D|$, and the second sum is over the elements $[\mathfrak{a}]$ of the ideal class group of $\mathcal{O}_K$. We will see that the $h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$ are integers related to coefficients of a certain weight 3/2 modular form, and that the $\Theta_{[\mathfrak{a},R],\mathcal{N}}$ are algebraic integers equal to the value of a symplectic theta function on 'split-CM' points (defined in Section 4.3) in the Siegel space

$\mathbb{H}_2/Sp_4(\mathbb{Z})$. We expect the formula (4.1) to be useful for computing the central value $L(\psi_{\mathcal{N}}, 1)$.

Let $A(|D|)$ denote a $\mathbb{Q}$-curve as defined in [23]. This is an elliptic curve defined over the Hilbert class field H of K with complex multiplication by $\mathcal{O}_K$ which is isogenous over H to its Galois conjugates. Its L-series is a product of the squares of L-series $L(\psi, s)$ over the $h(D)$ Hecke characters of conductor $(\sqrt{D})$. A formula for the central value $L(\psi, 1)$ expressed as a square of linear combinations of certain theta functions was proven by Villegas in [60]. Extensions of his result to higher weight Hecke characters were given by Villegas in [61] and jointly with Zagier in [63]. The Hecke character $\psi_{\mathcal{N}}$ is a twist of ψ by a quadratic Dirichlet character of conductor $(\sqrt{D})\mathcal{N}$. Therefore our result (4.1) gives a formula for the central value of the corresponding twist of $A(|D|)$.

Our main theorem can be stated in a particularly nice form when the class number of $\mathcal{O}_K$ is one. Then $[\mathfrak{a}] = [\mathcal{N}] = [\mathcal{O}_K]$ and so in particular $\Theta_{[\mathfrak{a},R],\mathcal{N}} = \Theta_{[R]}$ and $h^{\varepsilon}_{[\mathfrak{a},R]}(-N) = h^{\varepsilon}_{R}(-N)$ are independent of $[\mathfrak{a}]$ and $\mathcal{N}$. This suggests that formula (4.1) will lead to a generating series for $L(\psi_{\mathcal{N}}, 1)$ as N varies in terms of linear combinations (with scalars in $\{\Theta_{[R]}\}$) of half-integer weight modular forms.

We hope to extend these results to higher weight as follows. For certain $k \in \mathbb{Z}_{\geq 1}$ it is well-known that the central value $L(\psi^k_{\mathcal{N}}, k)$ can be written as a trace over the class group of $\mathcal{O}_K$ of a weight k Eisenstein series evaluated at Heegner points of level N and discriminant D. It is a general philosophy (see

[95], for example) that such traces relate to coefficients of a corresponding modular form of half-integer weight. By the Siegel-Weil formula[1] we can write the central value of $L(\psi_{\mathcal{N}}^k, s)$ in terms of a sum of theta-series[2]

$$L(\psi_{\mathcal{N}}^k, k) \doteq \sum_{[\mathfrak{a}]} \sum_{[Q]} \frac{1}{\omega_Q} \Theta_Q(\tau_{\mathfrak{a}}). \tag{4.2}$$

Here the sum is over $[\mathfrak{a}]$ in the class group of $\mathcal{O}_K$ and over classes of positive definite quadratic forms $Q : \mathbb{Z}^{2k} \longrightarrow \mathbb{Z}$ in $2k$ variables and in a given genus. The point $\tau_{\mathfrak{a}} \in \mathbb{H}$ is a Heegner point of level N and discriminant D. Analogous to the case of two variables, these quadratic forms correspond to higher rank Hermitian forms (see [51] and [26–30]). An approach to counting the number of distinct theta values in (4.2) would be to associate the Hermitian forms to isomorphism classes of rank k R-modules of $(-1, D)_{\mathbb{Q}}$, for maximal orders R of $(-1, D)_{\mathbb{Q}}$. This paper does this for the case $k = 1$. Our intention here is to lay the groundwork for the generalization to arbitrary weight k.

This paper is organized as follows. Basic notation is given in Section 4.2. Background and a statement of results are in Section 4.3. In Section 4.4, we analyze the endomorphisms of the principally polarized abelian varieties for the split-CM points, and show they form an explicit maximal order in the quaternion algebra $(-1, D)_{\mathbb{Q}}$. In Section 4.5 we identify these orders with explicit right orders in $(-1, D)_{\mathbb{Q}}$. In Section 4.6 we prove the main results (Theorems 4.3.2, 4.3.3 and 4.3.6) and provide numerical examples.

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[1] The precise statement of this formula is simplified here for the sake of exposition.
[2] Here ω_Q is the number of automorphisms of the form Q.

4.2 Notation

Given any imaginary quadratic field M of discriminant $d < 0$, we denote by $\mathcal{O}_M$ its ring of integers, $Cl(\mathcal{O}_M)$ its ideal class group, $h(d)$ its class number, and $Cl(d)$ the isomorphic class group of primitive positive definite binary quadratic forms of discriminant d. A nonzero integral ideal of $\mathcal{O}_M$ with no rational integral divisors besides ± 1 is said to be *primitive*. Any primitive ideal $\mathfrak{a}$ of $\mathcal{O}_M$ can be written uniquely as the $\mathbb{Z}$-module

$$\mathfrak{a} = a\mathbb{Z} + \frac{-b+\sqrt{d}}{2}\,\mathbb{Z} = [a, \frac{-b+\sqrt{d}}{2}]$$

with $a := \mathbf{N}\mathfrak{a}$ the norm of $\mathfrak{a}$, and b an integer defined modulo $2a$ which satisfies $b^2 \equiv d \bmod 4a$. Conversely any $a, b \in \mathbb{Z}$ which satisfy the conditions above determine a primitive ideal of $\mathcal{O}_M$. The coefficients of the corresponding primitive positive definite binary quadratic form are given by $[a, b, c := \frac{b^2-D}{4a}]$. The form $[a, -b, c]$ corresponds to the ideal $\bar{\mathfrak{a}}$. We will always assume our forms are primitive positive definite and the same for ideals. The point

$$\tau_{\mathfrak{a}} := \frac{-b+\sqrt{d}}{2a}$$

is in the upper half-plane $\mathbb{H}$ of $\mathbb{C}$ and is referred to in general as a *CM point*. A *Heegner point* of level N and discriminant D is a CM point $\tau_{\mathfrak{a}}$ where $\mathfrak{a}$ is given by a form $[a, b, c]$ of discriminant D such that $N|a$. The *root* of $\tau_{\mathfrak{a}}$ is defined to be the reduced representative $r \in (\mathbb{Z}/2N\mathbb{Z})^\times$ such that $b \equiv r \bmod 2N$.

Square brackets $[\cdot]$ around an object will denote its respective equivalence class. The units of a ring R are written as $R^\times$.

4.3 Statement of Results

We first recall some basic results for Siegel space and symplectic modular forms.

Assume K is an imaginary quadratic field of prime discriminant $D < -4$. Let L be an imaginary quadratic field of discriminant $-N < 0$ where N is a prime which splits in $\mathcal{O}_K$, and is divisible by an ideal $\mathcal{N}$ of norm N. Note $h(D)$ and $h(-N)$ are both odd since $|D|$ and N are prime. Let $\mu : \mathcal{O}_K/\mathcal{N} \longrightarrow \mathbb{Z}/N\mathbb{Z}$ be the natural isomorphism. Composing this with the Jacobi symbol $(\frac{\cdot}{N}) : \mathbb{Z}/N\mathbb{Z} \longrightarrow \{0, \pm 1\}$ defines a character

$$\chi : (\mathcal{O}_K/\mathcal{N})^\times \longrightarrow \{\pm 1\}.$$

This is an odd quadratic Dirichlet character of conductor $\mathcal{N}$. Let $I_\mathcal{N}$ denote the group of nonzero fractional ideals of K which are coprime to $\mathcal{N}$, and let $P_\mathcal{N} \subset I_\mathcal{N}$ be the subgroup of principal ideals. The map $\psi_\mathcal{N} : P_\mathcal{N} \longrightarrow K^\times$ defined by

$$\psi_\mathcal{N}((\alpha)) := \chi(\alpha)\alpha$$

is a homomorphism. There are exactly $h(D)$ extensions of $\psi_\mathcal{N}$ to a Hecke character $\psi_\mathcal{N} : I_\mathcal{N} \longrightarrow \mathbb{C}^\times$. This produces $h(D)$ primitive Hecke characters of weight one and conductor $\mathcal{N}$. (See [20, 52] and [64, p.225] for more details). Fix a choice of $\psi_\mathcal{N}$. We can extend $\psi_\mathcal{N}$ to a multiplicative function on all of $\mathcal{O}_K$ by setting $\psi_\mathcal{N}(\mathfrak{a}) := 0$ if $\mathfrak{a}$ is not coprime to $\mathcal{N}$.

To $\psi_{\mathcal{N}}$ we associate the Hecke L-function

$$L(\psi_{\mathcal{N}}, s) := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\psi_{\mathcal{N}}(\mathfrak{a})}{\mathbf{N}\mathfrak{a}^s}, \qquad \mathrm{Re}(s) > 3/2.$$

The L-function has analytic continuation to an entire function and satisfies a functional equation under the symmetry $s \mapsto 2 - s$.

We now recall a result due to Hecke which gives the central value $L(\psi_{\mathcal{N}}, 1)$ as a linear combination of certain theta series evaluated at CM points. For each primitive ideal $\mathcal{Q}$ of $\mathcal{O}_L$, the associated theta series is defined by

$$\Theta_{\mathcal{Q}}(\tau) := \sum_{\lambda \in \mathcal{Q}} q^{\mathbf{N}(\lambda)/\mathbf{N}(\mathcal{Q})}, \qquad q = e^{2\pi i \tau}, \ \tau \in \mathbb{H}.$$

It is a modular form on $\Gamma_0(N)$ of weight one and character $\mathrm{sgn}(\cdot)\left(\frac{-N}{|\cdot|}\right)$ (see [14, p.49], for example).

For each primitive ideal $\mathfrak{a}$ of $\mathcal{O}_K$ with norm prime to N, the product ideal $\mathfrak{a}\bar{\mathcal{N}}$ is of the form $[a_1 N, \frac{-b_1+\sqrt{D}}{2}]$ for some $a_1, b_1 \in \mathbb{Z}$. The point

$$\tau_{\mathfrak{a}\bar{\mathcal{N}}} := \frac{-b_1 + \sqrt{D}}{2a_1 N} \in \mathbb{H}$$

is a Heegner point of level N and discriminant D. We will write $\tau_{\mathfrak{a}}$ or just τ for $\tau_{\mathfrak{a}\bar{\mathcal{N}}}$ when the context is clear. Note that as $\mathfrak{a}$ runs over a distinct set of representatives of $Cl(\mathcal{O}_K)$, so does $\mathfrak{a}\bar{\mathcal{N}}$. (The fact that representatives of $Cl(\mathcal{O}_K)$ can be chosen with norm prime to N is in [9, Lemmas 2.3, 2.25], for example.) By $\mathfrak{a}$ we will always mean a primitive ideal with norm prime to $\mathcal{N}$ as above.

Hecke's formula [32] for the central value of $L(\psi_{\mathcal{N}}, s)$ states

$$L(\psi_{\mathcal{N}}, 1) = \frac{2\pi}{\omega_N \sqrt{N}} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \sum_{[\mathcal{Q}] \in Cl(\mathcal{O}_L)} \frac{\Theta_{\mathcal{Q}}(\tau_{\mathfrak{a}\bar{\mathcal{N}}})}{\psi_{\bar{\mathcal{N}}}(\bar{\mathfrak{a}})} \tag{4.3}$$

where ω_N is the number of units in $\mathcal{O}_L$.

The theta function for $\mathcal{Q}$ arises from a certain specialization of a symplectic theta function. Let $Sp_4(\mathbb{Z})$ denote the Siegel modular group of degree 2. Let Γ_θ be the subgroup of $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in Sp_4(\mathbb{Z})$ ($\alpha, \beta, \gamma, \delta \in \mathrm{Mat}_2(\mathbb{Z})$) such that both $\alpha^T\gamma$ and $\beta^T\delta$ have even diagonal entries. The group Γ_θ inherits the action of $Sp_4(\mathbb{Z})$ on the Siegel upper half plane $\mathbb{H}_2 := \left\{z \in \mathrm{Mat}_2(\mathbb{C}) \;:\; {}^T z = z, \mathrm{Im}(z) > 0\right\}$. Define the symplectic theta function by

$$\theta(z) := \sum_{\vec{x} \in \mathbb{Z}^2} \exp[\,\pi i\, {}^T\vec{x}\, z\, \vec{x}\,], \qquad z \in \mathbb{H}_2.$$

The function θ satisfies the functional equation

$$\theta(M \circ z) = \chi(M)[\det(\gamma z + \delta)]^{1/2}\theta(z), \qquad M \in \Gamma_\theta \tag{4.4}$$

where $\chi(M)$ is an eighth root of unity which depends on the chosen square root of $\det(\gamma z + \delta)$ but is otherwise independent of z. It is a symplectic modular form on Γ_θ of dimension $-1/2$ with multiplier system χ (see [14, p.43] or [46, p.189], for example)[3].

Given a primitive ideal $\mathcal{Q}$ of $\mathcal{O}_L$, let $Q := [a, b, c]$ represent the corresponding binary quadratic form of discriminant $-N$. The product of the

[3]The symplectic theta function is sometimes defined with extra parameters, $\theta(z, u, v)$ where $u, v \in \mathbb{C}^2$, in which case the theta function above is equal to $\theta(z, \vec{0}, \vec{0})$.

matrix of Q with any Heegner point $\tau_{\mathfrak{a}}$ is the Siegel point

$$Q\tau_{\mathfrak{a}} := \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \cdot \tau_{\mathfrak{a}} \in \mathbb{H}_2.$$

We will refer to points constructed in this way as *split-CM points* of level N and discriminant D. This yields the relation

$$\Theta_{\mathcal{Q}}(\tau_{\mathfrak{a}}) = \theta(Q\tau_{\mathfrak{a}}) \tag{4.5}$$

which can be substituted into formula (4.3) to get

$$L(\psi_{\mathcal{N}}, 1) = \frac{2\pi}{\omega_N \sqrt{N}} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \sum_{[Q] \in Cl(-N)} \frac{\theta(Q\tau_{\mathfrak{a}})}{\psi_{\bar{\mathcal{N}}}(\bar{\mathfrak{a}})}. \tag{4.6}$$

If $Q \sim Q'$ in $Cl(-N)$, then $Q\tau_{\mathfrak{a}} \sim Q'\tau_{\mathfrak{a}}$ in $\mathbb{H}_2/Sp_4(\mathbb{Z})$, and if $\mathfrak{a} \sim \mathfrak{a}'$ in $Cl(\mathcal{O}_K)$, then $Q\tau_{\mathfrak{a}} \sim Q\tau_{\mathfrak{a}'}$ in $\mathbb{H}_2/Sp_4(\mathbb{Z})$ (see Remark 4.6.1 and Lemma 4.6.11). In addition it is shown in [52, Lemma 53] that these equivalences of Siegel points sustain modulo Γ_θ. The function $\theta/\psi_{\bar{\mathcal{N}}}$ is invariant on such points:

Lemma 4.3.1. *Fix an ideal $\mathfrak{a} \subset \mathcal{O}_K$ and a prime ideal $\mathcal{N} \subset \mathcal{O}_K$ of norm N. Let Q be a binary quadratic form of discriminant $-N$. Then the value*

$$\frac{\theta(Q\tau_{\mathfrak{a}\bar{\mathcal{N}}})}{\psi_{\bar{\mathcal{N}}}(\bar{\mathfrak{a}})} \tag{4.7}$$

depends only on the class $[Q] \in Cl(\mathcal{O}_L)$ and the class $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$.

Proof. The value $\theta(Q\tau_{\mathfrak{a}\bar{\mathcal{N}}})$ is independent of the class representative of $[Q]$ because equivalent forms represent the same values. That (4.7) is independent of the representative of $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$ is a short calculation using the functional equation for θ in (4.4) and is done in [52, Proposition 22]. $\square$

Therefore the set of points $[Q]\tau_{[\mathfrak{a}]\bar{\mathcal{N}}}$ as $[Q]$ runs over $Cl(-N)$ and $[\mathfrak{a}]$ runs over $Cl(\mathcal{O}_K)$ are equivalent in $\mathbb{H}_2/\Gamma_\theta$ and are identified under $\theta/\psi_{\bar{\mathcal{N}}}$. We refer to $[Q]\tau_{[\mathfrak{a}]\bar{\mathcal{N}}}$ as a *split-CM orbit*. Thus to determine which values $\theta(Q\tau_{\mathfrak{a}})$ are equal in (4.6) it is necessary to determine which split-CM orbits $[Q]\tau_{[\mathfrak{a}]\bar{\mathcal{N}}}$ are equivalent modulo Γ_θ. Since $\mathbb{H}_2/Sp_4(\mathbb{Z})$ is a moduli space for the principally polarized abelian varieties of dimension two ([46] or [4, Chp. 8]), the classes of split-CM points are determined by the isomorphism classes of the corresponding varieties.

To describe these, we will recall some basic facts about quaternion algebras. Let $B := (-1, D)_{\mathbb{Q}}$ be the quaternion algebra over $\mathbb{Q}$ ramified at ∞ and $|D|$. Recall two maximal orders R, R' in B are *equivalent* if there exists $x \in (-1, D)_{\mathbb{Q}}^{\times}$ such that $R' = x^{-1}Rx$. Moreover, two optimal embeddings $\phi : \mathcal{O}_L \hookrightarrow R$ and $\phi' : \mathcal{O}_L \hookrightarrow R'$ are *equivalent* if there exists $x \in (-1, D)_{\mathbb{Q}}^{\times}$ and $r \in R'^{\times}$ such that $R' = x^{-1}Rx$ and $\phi' = (xr)^{-1}\phi(xr)$. Let $\mathcal{R}$ denote the set of conjugacy classes of maximal orders in B and let $\Phi_{\mathcal{R}}$ denote the set of classes of optimal embeddings of $\mathcal{O}_L$ into the maximal orders of B. Let $\mathcal{R}_N \subset \mathcal{R}$ denote the maximal order classes which admit an optimal embedding of $\mathcal{O}_L$. Given an optimal embedding $(\phi : \mathcal{O}_L \hookrightarrow R) \in \Phi_{\mathcal{R}}$, let $(\bar{\phi} : \mathcal{O}_L \hookrightarrow R) \in \Phi_{\mathcal{R}}$ denote its quaternionic conjugate, so that $\phi(\sqrt{-N}) = \bar{\phi}(-\sqrt{-N})$. The quotient $\Phi_{\mathcal{R}}/-$ will denote the set $\Phi_{\mathcal{R}}$ modulo this conjugation. Let $h_R(-N)$ denote the number of optimal embeddings of $\mathcal{O}_L$ into R modulo conjugation by $R^{\times}$. This number is an invariant of the choice of representative of $[R]$ in $\mathcal{R}$.

Our first theorem says that the classes of split-CM points in Siegel space correspond to classes of maximal orders in B.

Theorem 4.3.2. *Fix* $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$*,* $\mathcal{N} \subset \mathcal{O}_K$ *a prime ideal of norm* N*, and* $\tau := \tau_{\mathfrak{a}\bar{\mathcal{N}}}$*. There is a bijection*

$$\Upsilon_1 : \{Q\tau \ : \ [Q] \in Cl(-N)\} \, / Sp_4(\mathbb{Z}) \longrightarrow \mathcal{R}_N.$$

This map is independent of the choice of representative $\mathfrak{a}$ *of* $[\mathfrak{a}]$*.*

Let $\Upsilon_1^{-1}([R])$ for $[R] \in \mathcal{R}_N$ denote the pre-image class in $\mathbb{H}_2/Sp_4(\mathbb{Z})$ and set $\Upsilon_1^{-1}([R]) := \emptyset$ if $[R] \in \mathcal{R} \setminus \mathcal{R}_N$. Our second theorem gives the number of split-CM orbits in a given class.

Theorem 4.3.3. *Assume the hypotheses of Theorem 4.3.2. For any* $[R] \in \mathcal{R}$*,*

$$\# \left\{[Q]\tau \in \Upsilon_1^{-1}([R]) : [Q] \in Cl(-N)\right\} = h_R(-N)/2.$$

That is, the number of split-CM orbits in the class in $\mathbb{H}_2/Sp_4(\mathbb{Z})$ *corresponding to* $[R]$ *under Theorem 4.3.2 is* $h_R(-N)/2$*.*

For a maximal order R of B, define $S_R := \mathbb{Z} + 2R$ and $S_R^0 \subset S_R$ to be the suborder of trace zero elements. The suborder S_R^0 is a rank 3 $\mathbb{Z}$-submodule of R. Define g_R to be its theta series

$$\begin{aligned} g_R(\tau) &:= \frac{1}{2} \sum_{x \in S_R^0} q^{\mathbb{N}(x)} \\ &= \frac{1}{2} + \sum_{N>0} a_R(N) q^N, \end{aligned}$$

where $a_R(N)$ are defined by its q-expansion. It is well known that g_R is a weight 3/2 modular form on $\Gamma_0(4|D|)$. Applying [24, Proposition 12.9] to fundamental $-N$ gives

$$a_R(N) = \frac{\omega_R}{\omega_N} h_R(-N)$$

where ω_R is the cardinality of the set $R^\times / < \pm 1 >$.

This gives immediately the following Corollary to Theorem 4.3.3.

Corollary 4.3.4. *Assume the hypotheses of Theorem 4.3.3. For any* $[R] \in \mathcal{R}$,

$$\#\left\{[Q]\tau \in \Upsilon_1^{-1}([R]) : [Q] \in Cl(-N)\right\} = a_R(N) \cdot \frac{2\omega_N}{\omega_R}.$$

That is, the number of split-CM orbits in the class in $\mathbb{H}_2/Sp_4(\mathbb{Z})$ *corresponding to* $[R]$ *under Theorem 4.3.2 is proportional to the* N*-th Fourier coefficient of the weight* $3/2$ *modular form* g_R.

The application of Theorems 4.3.2 and 4.3.3 to a formula for $L(\psi_{\mathcal{N}}, 1)$ proceeds as follows. Define the following normalization of θ given by [52]:

$$\hat{\theta}(Q\tau_{\mathfrak{a}\bar{\mathcal{N}}}) := \frac{\theta(Q\tau_{\mathfrak{a}\bar{\mathcal{N}}})}{\eta(\bar{\mathcal{N}})\eta(\mathcal{O}_K)} \tag{4.8}$$

where $\eta(z) := e_{24}(z)\prod_{n=1}^{\infty}(1 - e^{2\pi i z})$ for $\mathrm{Im}(z) > 0$ is Dedekind's eta function and the evaluation of η on ideals is defined in Section 4.6. It is proven in [52, Proposition 23] (see also [25]) that the numbers in $\hat{\theta}(Q\tau_{\mathfrak{a}\bar{\mathcal{N}}})/\psi_{\bar{\mathcal{N}}}(\bar{\mathfrak{a}})$ are algebraic integers.

Define

$$\Theta_{[\mathfrak{a},Q],\mathcal{N}} := \frac{\hat{\theta}(Q\tau_{\mathfrak{a}\bar{\mathcal{N}}})}{\psi_{\bar{\mathcal{N}}}(\bar{\mathfrak{a}})}.$$

This is well-defined by Lemma 4.3.1. The following lemma says that the theta-values which correspond to a given class $[R] \in \mathcal{R}$ under Theorem 4.3.2 are all equal up to ± 1.

Lemma 4.3.5. *Fix $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$, $\mathcal{N} \subset \mathcal{O}_K$ a prime ideal of norm N, and $\tau := \tau_{\mathfrak{a}\bar{\mathcal{N}}}$. Let $[R] \in \mathcal{R}$. Then the values*

$$\left\{\Theta_{[\mathfrak{a},Q],\mathcal{N}} \,:\, [Q]\tau \in \Upsilon_1^{-1}([R])\right\} \tag{4.9}$$

differ by ± 1.

Assume Lemma 4.3.5 holds (see Section 4.5 for the proof). Given $[R] \in \mathcal{R}_N$ and any $[Q]\tau \in \Upsilon_1^{-1}([R])$, define $\Theta_{[\mathfrak{a},R],\mathcal{N}}$ to be either $\Theta_{[\mathfrak{a},Q],\mathcal{N}}$ or $-\Theta_{[\mathfrak{a},Q],\mathcal{N}}$ so that it satisfies $\mathrm{Re}(\Theta_{[\mathfrak{a},R],\mathcal{N}}) > 0$. Set $\Theta_{[\mathfrak{a},R],\mathcal{N}} := 0$ if $[R] \in \mathcal{R} \setminus \mathcal{R}_N$.

We record the mysterious ± 1 signs appearing in Lemma 4.3.5 by defining

$$\begin{aligned} \varepsilon_{[\mathfrak{a},R]} : \left\{[Q]\tau \in \Upsilon_1^{-1}([R])\right\} &\longrightarrow \{\pm 1\} \\ [Q]\tau &\mapsto \mathrm{sgn}\left(\mathrm{Re}\left(\Theta_{[\mathfrak{a},Q],\mathcal{N}}\right)\right). \end{aligned} \tag{4.10}$$

Note $\Theta_{[\mathfrak{a},Q],\mathcal{N}} = \pm\Theta_{[\mathfrak{a},R],\mathcal{N}}$ by construction. This definition assigns, albeit somewhat arbitrarily, a fixed choice of sign for the theta-values as $[Q]$ varies.

We then define a corresponding twisted variant of $h_R(-N)$ by

$$h^{\varepsilon}_{[\mathfrak{a},R]}(-N) := \sum_{[Q]\tau \in \Upsilon_1^{-1}([R])} \varepsilon_{[\mathfrak{a},R]}([Q]\tau). \tag{4.11}$$

The formula for $L(\psi_{\mathcal{N}}, 1)$ can now be stated as follows.

Theorem 4.3.6. *Let $\mathcal{N} \subset \mathcal{O}_K$ be a prime ideal of norm N. Then*

$$L(\psi_{\mathcal{N}}, 1) = \frac{\pi \cdot \eta(\bar{\mathcal{N}})\eta(\mathcal{O}_K)}{\omega_N \sqrt{N}} \sum_{[R] \in \mathcal{R}} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} \Theta_{[\mathfrak{a},R],\mathcal{N}} \cdot h^{\varepsilon}_{[\mathfrak{a},R]}(-N). \tag{4.12}$$

where $\Theta_{[\mathfrak{a},R],\mathcal{N}}$ is an algebraic integer and $h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$ is an integer with $|h^{\varepsilon}_{[\mathfrak{a},R]}(-N)| \leq h_R(-N)$.

Remark 4.3.1. The signs in Lemma 4.3.5 and hence the function $h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$ depend on the character χ which appears in the functional equation (4.28) for θ. In particular, the values of χ depend on the entries of the transformation matrices in Γ_θ which takes one Siegel point to an equivalent one. This value is complicated to compute or even define, and is discussed in detail in [1, 83] and [14, Appendix to Chp 1]. An arithmetic formula for these signs and for $h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$ is yet to be determined. But since the $h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$ are a weighted count of optimal embeddings, we expect that, like the $h_R(-N)$, they will be related to coefficients of a half-integer weight modular form. This will be treated in a subsequent paper.

Theorem 4.3.6 gives us an upper bound on $L(\psi_{\mathcal{N}}, 1)$ in terms of the computable modular form coefficients $h_R(-N)$.

Corollary 4.3.7. *Assume the hypotheses of Theorem 4.3.6. Then*

$$|L(\psi_{\mathcal{N}}, 1)| \leq \frac{\pi \cdot |\eta(\bar{\mathcal{N}})\eta(\mathcal{O}_K)|}{\omega_N \sqrt{N}} \sum_{[R] \in \mathcal{R}} \sum_{[\mathfrak{a}] \in Cl(\mathcal{O}_K)} |\Theta_{[\mathfrak{a},R],\mathcal{N}}| \cdot h_R(-N).$$

If $h(D) = 1$, then (4.12) has a particularly simple form:

Corollary 4.3.8. *Assume the hypotheses of Theorem 4.3.6 and suppose $h(D) = 1$. Then $\Theta_{[\mathfrak{a},R],\mathcal{N}} = \Theta_{[R]}$ and $h^{\varepsilon}_{[\mathfrak{a},R]}(-N) = h^{\varepsilon}_{[R]}(-N)$ are independent of $\mathfrak{a}$ and $\mathcal{N}$ and*

$$L(\psi_{\mathcal{N}}, 1) = \frac{\pi \cdot |\eta(\mathcal{O}_K)|^2}{\omega_N \sqrt{N}} \sum_{[R] \in \mathcal{R}} \Theta_{[R]} \cdot h^{\varepsilon}_{[R]}(-N).$$

We conclude this section with a comment regarding varying N. The set

$$\bigcup_N \left\{ [Q]\tau_{[\mathfrak{a}]\bar{\mathcal{N}}} \,:\, [Q] \in Cl(-N), \quad [\mathfrak{a}] \in Cl(\mathcal{O}_K), \quad \mathcal{N} \subset \mathcal{O}_K \text{ of norm } N \right\},$$

of split-CM orbits over all prime N with $D \equiv \square \bmod 4N$ partitions into a finite number of Siegel classes in $\mathbb{H}_2/Sp_4(\mathbb{Z})$. This has a natural explanation from our viewpoint. As a complex torus, $X_{Q\tau}$ is isomorphic to a product $E \times E'$ of two elliptic curves E, E' defined over $\bar{\mathbb{Q}}$ and with complex multiplication by $\mathcal{O}_K$. (This is the reason the $Q\tau$ are called 'split-CM'.) It is a general result of [47] that there are only finitely many principal polarizations on a given complex abelian variety up to isomorphism. There are also only finitely many isomorphism classes of elliptic curves with CM by $\mathcal{O}_K$. Together these imply that the number of classes of Siegel points $(X_{Q\tau}, H_{Q\tau})$ for all split-CM points $Q\tau$ of discriminant D must be finite. See [52, Theorem 58] as well for an alternative interpretation.

4.4 Endomorphisms of X_z preserving H_z

In this section we prove that the endomorphisms of the abelian varieties corresponding to split-CM points give maximal orders in the quaternion algebra $(-1,D)_{\mathbb{Q}} = (-1,D)_{\mathbb{Q}}$. Let V, V' be complex vector spaces of dimension 2 with lattices $L \subset V$, $L' \subset V'$. The analytic and rational representations are denoted by $\rho_a : \mathrm{Hom}(X,X') \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V')$ and $\rho_r : \mathrm{Hom}(X,X') \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(L,L')$, respectively. Recall the periods matrices $\Pi, \Pi' \in \mathrm{Mat}_{2\times 4}(\mathbb{C})$ of X, X' commute with ρ_a and ρ_r in the following diagram

$$\begin{tikzcd} \mathbb{Z}^{2g} \arrow[r, "\Pi"] \arrow[d, "\rho_r(f)"'] & \mathbb{C}^g \arrow[d, "\rho_a(f)"] \\ \mathbb{Z}^{2g'} \arrow[r, "\Pi'"'] & \mathbb{C}^{g'} \end{tikzcd} \tag{4.13}$$

(see [4], for example).

For any Siegel point $z \in \mathbb{H}_2$, let $\Pi_z := [z, \mathbf{1}_2] \in \mathrm{Mat}_{2\times 4}(\mathbb{C})$ be its period matrix, $L_z := \Pi_z \mathbb{Z}^4$ be its defining lattice, and $X_z := \mathbb{C}^2/L_z$ be its corresponding complex torus. The Hermitian form $\mathcal{H}_z : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ defined by $\mathcal{H}_z(u,v) := {}^T u\, \mathrm{Im}(z)^{-1} \bar{v}$ determines a principal polarization on X_z. As a point in the moduli space $\mathbb{H}_2/Sp_4(\mathbb{Z})$, z corresponds to the principally polarized abelian variety $(X_z, \mathcal{H}_z)$. Throughout Sections 4.4, 4.5 and 4.6, fix a representative $\mathfrak{a}$ of $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$, $\mathcal{N} \subset \mathcal{O}_K$ a prime ideal of norm N, $\tau := \tau_{\mathfrak{a}\bar{\mathcal{N}}} := \frac{-b_1+\sqrt{D}}{2a_1 N}$, and a split-CM point $z = Q\tau$ of level N and discriminant D where $Q := [a,b,c]$ is of discriminant $-N$. The endomorphisms of $(X_z, \mathcal{H}_z)$ will be our first main object of study.

We define $\mathscr{B}$ to be the $\mathbb{Q}$-algebra of endomorphisms of X_z which fix $\mathcal{H}_z$

$$\mathscr{B} := \left\{ \alpha \in \mathrm{End}_{\mathbb{Q}}(X_z) \;:\; \mathcal{H}_z(\alpha u, v) = \mathcal{H}_z(u, \alpha^{\iota} v) \quad \forall\, u, v \in \mathbb{C}^2 \right\};$$

here ι is the canonical involution inherited from $\mathrm{Mat}_2(K)$ as defined in [73].

In terms of matrices, let $H_z := \mathrm{Im}(z)^{-1}$ denote the matrix of $\mathcal{H}_z$ with respect to the standard basis of $\mathbb{C}^2$. Then viewing $\mathrm{End}_{\mathbb{Q}}(X_z) \subseteq \mathrm{Mat}_2(K)$, the set $\mathscr{B}$ is

$$\mathscr{B} = \left\{ M \in \mathrm{End}_{\mathbb{Q}}(X_z) \ : \ {}^T\bar{M} H_z = H_z M^\iota \right\}.$$

The bar denotes complex conjugation restricted to K. The map ι sends a matrix M to its adjoint, or equivalently sends M to $\mathrm{Tr}(M) \cdot \mathbf{1}_2 - M$.

We define $\mathscr{R}_z$ to be the $\mathbb{Z}$-submodule of endomorphisms which fix H_z

$$\mathscr{R}_z := \left\{ M \in \mathrm{End}(X_z) \ : \ {}^T\bar{M} H_z = H_z M^\iota \right\}. \tag{4.14}$$

The first observation is that $\mathscr{B}$ is isomorphic to a rational definite quaternion algebra.

Proposition 4.4.1. *$\mathscr{B}$ is isomorphic to $(-1, D)_{\mathbb{Q}}$ as $\mathbb{Q}$-algebras.*

Remark 4.4.1. In [73, Proposition 2.6], Shimura proves $\mathscr{B}$ is a quaternion algebra over $\mathbb{Q}$ in a much more general setting by showing $\mathscr{B} \otimes \bar{\mathbb{Q}}$ is isomorphic to $\mathrm{Mat}_2(\bar{\mathbb{Q}})$. Here we give an alternative proof which explicitly gives the primes ramified in $\mathscr{B}$.

Proof. We will need the following elementary lemma.

Lemma 4.4.2. *Suppose $Q_1, Q_2 \in Mat_2(\mathbb{Z})$ with determinant N. Set $H_i := \mathrm{Im}(Q_i\tau)^{-1}$ and*

$$\mathscr{R}_i := \left\{ M \in End(X_{Q_i\tau}) \ : \ {}^T\bar{M} H_i = H_i M^\iota \right\}, \qquad i = 1, 2.$$

Let $S = \mathbb{Z}$ *or* $\mathbb{Q}$ *and suppose there exists* $A \in GL_2(S)$ *such that* $Q_2 = (\det A)^{-1} A Q_1{}^T A$*. Then the map*

$$\begin{aligned} End_S(X_{Q_1\tau}) &\longrightarrow End_S(X_{Q_2\tau}) \\ M &\mapsto AMA^{-1} \end{aligned} \tag{4.15}$$

and the induced map

$$\mathscr{R}_1 \otimes_{\mathbb{Z}} S \longrightarrow \mathscr{R}_2 \otimes_{\mathbb{Z}} S$$

are S*-algebra isomorphisms.*

Proof of Lemma. Let $\Pi_i := [Q_i\tau, \mathbf{1}_2]$ be the period matrices for $Q_i\tau$, $i = 1, 2$. Suppose $M \in \mathrm{End}_S(X_{Q_1\tau})$. By (4.13), this is if and only if $M\Pi_i = \Pi_i P$ for some $P \in \mathrm{Mat}_4(S)$. Set

$$\tilde{A} := \begin{pmatrix} (\det A^{-1})^T A & 0 \\ 0 & A^{-1} \end{pmatrix} \in GL_4(S).$$

Using the identity $A\Pi_1\tilde{A} = \Pi_2$ gives

$$(AMA^{-1})\Pi_2 = \Pi_2(\tilde{A}^{-1}P\tilde{A}).$$

Clearly $\tilde{A}^{-1}P\tilde{A} \in \mathrm{Mat}_4(S)$, hence $AMA^{-1} \in \mathrm{End}_S(X_{Q_2\tau})$.

Furthermore the identity $H_1 = (\det A^{-1})^T A H_2 A$ implies ${}^T(\overline{AMA^{-1}})H_2 = H_2(AMA^{-1})^\iota$ by a straightforward calculation. $\square$

Define matrices

$$A := \frac{1}{2a}\begin{pmatrix} 1 & 0 \\ -b & 2a \end{pmatrix} \in GL_2(\mathbb{Q}) \quad \text{and} \quad Q' := \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix}. \tag{4.16}$$

By Lemma 4.4.2, $\mathscr{B}$ is isomorphic as a $\mathbb{Q}$-algebra to

$$\mathscr{B}' := \left\{ M \in \mathrm{End}_{\mathbb{Q}}(X_{Q'\tau}) \; : \; {}^T\bar{M}H' = H'M^{\iota} \right\}$$

where $H' := \mathrm{Im}(Q'\tau)^{-1}$.

We will compute $\mathscr{B}'$ explicitly. Let $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ for any $\tau \in \mathbb{H}$. Clearly $X_{Q'\tau} \cong E_\tau \times E_{N\tau}$ as complex tori. The endomorphisms of $X_{Q'\tau}$ are characterized as follows.

Lemma 4.4.3.

$$End(X_{Q'\tau}) = \begin{pmatrix} \mathcal{O}_K & \mathbb{Z} + \mathbb{Z}\omega/N \\ N\mathbb{Z} + \mathbb{Z}\bar{\omega} & \mathcal{O}_K \end{pmatrix}$$

where $\omega := a_1 N\tau$.

Assuming this for a moment, we have $\mathrm{End}_{\mathbb{Q}}(X_{Q'\tau}) = \mathrm{Mat}_2(K)$, and a quick calculation shows any $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{Mat}_2(K)$ satisfies ${}^T\bar{M}H = HM^{\iota}$ if and only if $\delta = \bar{\alpha}$ and $\gamma = -N\bar{\beta}$. Therefore

$$\mathscr{B}' = \left\{ \begin{pmatrix} \alpha & \beta \\ -N\bar{\beta} & \bar{\alpha} \end{pmatrix} \; : \; \alpha, \beta \in K \right\} \subset \mathrm{Mat}_2(K).$$

The elements

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{D} & 0 \\ 0 & -\sqrt{D} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sqrt{D} \\ N\sqrt{D} & 0 \end{pmatrix}$$

form a basis of $\mathscr{B}'$ and clearly give an isomorphism to $(D, -N)_{\mathbb{Q}}$. We claim $(-1, D)_{\mathbb{Q}} \cong (D, -N)_{\mathbb{Q}}$. This is a general fact: if p, q are primes with $p \equiv q \equiv 3 \bmod 4$ and $-p$ is a square modulo q, then $(-p, -q)_{\mathbb{Q}}$ is ramified at ∞ and p only, so $(-p, -q)_{\mathbb{Q}} \cong (-1, p)_{\mathbb{Q}}$. Hence $\mathscr{B} \cong \mathscr{B}' \cong (-1, D)_{\mathbb{Q}}$ as $\mathbb{Q}$-algebras.

It remains to prove Lemma 4.4.3.

Proof of Lemma 4.4.3. For any quadratic surds $\tau, \tau' \in K$,

$$\mathrm{Hom}(E_\tau, E_{\tau'}) = \{\alpha \in K : \alpha(\mathbb{Z} + \mathbb{Z}\tau) \subseteq \mathbb{Z} + \mathbb{Z}\tau'\}.$$

Since $X_{Q'\tau} \cong E_\tau \times E_{N\tau}$, we have

$$\mathrm{End}(X_{Q'\tau}) = \begin{pmatrix} \mathrm{End}(E_\tau) & \mathrm{Hom}(E_{N\tau}, E_\tau) \\ \mathrm{Hom}(E_\tau, E_{N\tau}) & \mathrm{End}(E_{N\tau}) \end{pmatrix}.$$

We compute. $\mathrm{End}(E_{N\tau}) = \mathcal{O}_K$ since $\mathbb{Z} + \mathbb{Z}a_1 N\tau = \mathcal{O}_K$ and $[1, N\tau]$ is a (proper) fractional $\mathcal{O}_K$-ideal. Similarly $\mathrm{End}(E_\tau) = \mathcal{O}_K$ since $\mathbb{Z} + \mathbb{Z}\tau$ is a fractional $\mathcal{O}_K$-ideal.

It is straightforward to check $\mathbb{Z} + \mathbb{Z}a_1\tau \subseteq \mathrm{Hom}(E_{N\tau}, E_\tau)$. On the other hand, $\mathrm{Hom}(E_{N\tau}, E_\tau) \subset \mathbb{Z} + \mathbb{Z}\tau$ by definition, and this is proper containment since otherwise $\mathbb{Z} + \mathbb{Z}N\tau$ would preserve $\mathbb{Z} + \mathbb{Z}\tau$ which is impossible since the former contains $\mathcal{O}_K$. Therefore $\mathrm{Hom}(E_{N\tau}, E_\tau) = \mathbb{Z} + \mathbb{Z}m\tau$ for some integer $m|a_1$ but a quick calculation shows $m = a_1$ else it divides a_1, b_1 and c_1 whose gcd is assumed to be 1.

It remains to show

$$\mathrm{Hom}(E_\tau, E_{N\tau}) = N\mathbb{Z} + \mathbb{Z}\bar{\omega}.$$

First observe the ideal (N) in $\mathcal{O}_K$ is contained in $\mathrm{Hom}(E_\tau, E_{N\tau})$ since

$$N(\mathbb{Z} + \mathbb{Z}a_1 N\tau)(\mathbb{Z} + \mathbb{Z}\tau) \subseteq N(\mathbb{Z} + \mathbb{Z}\tau) \subseteq \mathbb{Z} + N\mathbb{Z}\tau.$$

Furthermore (N) splits as $(N) = \mathcal{N} \cdot \bar{\mathcal{N}}$ where $\mathcal{N} = N\mathbb{Z} + \mathbb{Z}\omega$. Therefore

$$\mathcal{N} \cdot \bar{\mathcal{N}} \subseteq \mathrm{Hom}(E_\tau, E_{N\tau}) \subseteq \mathcal{O}_K,$$

where the last containment follows because $\mathbb{Z}+\mathbb{Z}\tau$ is a proper fractional $\mathcal{O}_K$-ideal which contains $\mathbb{Z}+\mathbb{Z}N\tau$. But since $\mathcal{O}_K$ is Noetherian, there exists a maximal order M such that

$$\mathcal{N}\cdot\bar{\mathcal{N}} \subseteq \mathrm{Hom}(E_\tau, E_{N\tau}) \subseteq M \subseteq \mathcal{O}_K.$$

Therefore either $\mathcal{N}$ or $\bar{\mathcal{N}}$ is in M. Whichever is contained in M is actually equal to M since they are both prime and hence maximal. But $\mathrm{Hom}(E_\tau, E_{N\tau})$ is not contained in $\mathcal{N}$. For example, $\bar{\omega} \in \mathrm{Hom}(E_\tau, E_{N\tau})$ but not in $\mathcal{N}$. Thus

$$\mathrm{Hom}(E_\tau, E_{N\tau}) \subseteq \bar{\mathcal{N}}.$$

Finally since the index $[\bar{\mathcal{N}} : (N)] = N$ is prime, either $\mathrm{Hom}(E_\tau, E_{N\tau})$ is equal to $\mathcal{N}$ or $\bar{\mathcal{N}}$, but we already showed the former is impossible, hence it is the latter. $\square$

This also completes the proof of Proposition 4.4.1. $\square$

Lemma 4.4.4. *$\mathscr{R}_z$ is isomorphic to an order in $(-1, D)_{\mathbb{Q}}$ as $\mathbb{Z}$-algebras, and admits an optimal embedding of $\mathcal{O}_L$.*

Proof. The first part is immediate.

The embedding is given in matrix form by QS where $S := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. It is straightforward to check that $(QS)^2 = -N$ and $\frac{1+QS}{2} \in \mathscr{R}_z$ using definition (4.14). An embedding is optimal if it does not extend to any larger order in the quotient field, but this is immediate since $\mathcal{O}_L$ is the maximal order in L. (See [73] for additional discussion of this order.) $\square$

The next step is to prove the order $\mathscr{R}_z$ is maximal.

Theorem 4.4.5. *$\mathscr{R}_z$ is a maximal order.*

Proof. It suffices to show the local order $(\mathscr{R}_z)_p$ is maximal for all primes p. We do this with the following two lemmas.

Lemma 4.4.6. *$(\mathscr{R}_z)_p$ is maximal for all primes $p \neq 2$.*

Proof of Lemma. Define $\mathscr{R}' := \mathscr{B}' \cap \mathrm{End}(Q'\tau)$ with Q' defined in (4.16). From Lemma 4.4.3 and the definition of $\mathscr{B}'$ above it is clear that $\mathscr{R}'$ is an order given explicitly by

$$\mathscr{R}' = \left\{ \begin{pmatrix} \alpha & \beta \\ -N\bar{\beta} & \bar{\alpha} \end{pmatrix} \,:\, \alpha \in \mathcal{O}_K, \beta \in \mathbb{Z} + \mathbb{Z}\omega/N \right\}. \tag{4.17}$$

Its discriminant is D^2, which can be computed using the basis

$$u_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u_2 := \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}, u_3 = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}, u_4 = \begin{pmatrix} 0 & \omega/N \\ -\bar{\omega} & 0 \end{pmatrix}. \tag{4.18}$$

Hence $\mathscr{R}'$ is maximal. For $p \not| a$, the matrix A from (4.16) is in $\mathrm{Mat}_2(\mathbb{Z}_p)$ and so gives an isomorphism $M \mapsto AMA^{-1}$ from $(\mathscr{R}_z)_p \to \mathscr{R}'_p$. Hence $(\mathscr{R}_z)_p$ is maximal for $p \not| a$.

There exists a form $\tilde{Q} = \left(\begin{smallmatrix} 2\tilde{a} & \tilde{b} \\ \tilde{b} & 2\tilde{c} \end{smallmatrix}\right)$ properly equivalent to Q with $\gcd(2a, \tilde{a}) = 1$ (see [9, p. 25,35], for example). Applying Lemma 4.4.2 to the pair Q and $\tilde{Q}$ gives $\mathscr{R}_z \cong \mathscr{R}_{\tilde{Q}\tau}$. Hence for $p|a$ we can apply the paragraph above to $\mathscr{R}_{\tilde{Q}\tau}$ to conclude $(\mathscr{R}_z)_p$ is maximal. $\square$

Lemma 4.4.7. *$(\mathscr{R}_z)_2$ is maximal.*

Proof of Lemma. Note $\gcd(2a,b)=1$ because N is prime and b is odd. Define $U := \left(\begin{smallmatrix} 1 & 0 \\ -2cx-by & 1 \end{smallmatrix}\right)$ and $V := \left(\begin{smallmatrix} y & -b \\ x & 2a \end{smallmatrix}\right)$ where $x, y \in \mathbb{Z}$ such that $2ay + bx = 1$. Then $UQV = Q'$ where Q' was defined in (4.16). Define $\hat{H} := {}^T U^{-1} H U^{-1}$, $\hat{\mathscr{B}} := \left\{ M \in \mathrm{End}_{\mathbb{Q}}(X_{Q'\tau}) \,:\, {}^T\bar{M}\hat{H} = \hat{H}M^{\iota} \right\}$, and $\hat{\mathscr{R}} := \hat{\mathscr{B}} \cap \mathrm{End}(X_{Q'\tau})$. The period matrix $\Pi' := [Q'\tau, \mathbf{1}_2]$ satisfies $\Pi' = U\Pi_z\tilde{V}$ where $\tilde{V} := \left(\begin{smallmatrix} V & 0 \\ 0 & U^{-1} \end{smallmatrix}\right) \in \mathrm{Mat}_4(\mathbb{Z})$. Hence the map $M \mapsto UMU^{-1}$ from $\mathscr{R}_z \to \hat{\mathscr{R}}$ is an isomorphism over $\mathbb{Z}$. Therefore $(\mathscr{R}_z)_p \cong \hat{\mathscr{R}}_p$ for all primes p. We will show $\hat{\mathscr{R}}_2$ is maximal.

By Lemma 4.4.2 and the isomorphism $\mathscr{B} \cong \mathscr{B}'$, a basis for $\mathscr{B}$ is given by the set $\{A^{-1}u_iA\}$ with A defined in (4.16) and u_i in (4.18). Hence by above the set $\{v_i := UA^{-1}u_iAU^{-1}\}$ gives a basis for $\hat{\mathscr{B}}$ over $\mathbb{Q}$. Replace v_i with $2av_i$ for $i = 2, 3$ and v_4 by $2aNv_4$. Then explicitly,

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 2a\omega & 0 \\ -Nx(b_1+2\omega) & 2a\bar{\omega} \end{pmatrix}$$
$$v_3 = \begin{pmatrix} 2aNx & 4a^2 \\ -N(Nx^2+1) & -2aNx \end{pmatrix} \quad v_4 = \begin{pmatrix} 2aNx\omega & 4a^2\omega \\ -N(Nx^2\omega+\bar{\omega}) & -2aNx\omega \end{pmatrix}.$$

By Lemma 4.4.3 we see $v_i \in \hat{\mathscr{R}}$, $i = 1, \ldots, 4$. To prove $\hat{\mathscr{R}}_2$ is maximal we will use the elements $\{v_i\}$ to construct a basis of $\hat{\mathscr{R}}_2$ whose discriminant is a unit modulo $(\mathbb{Z}_2)^2$.

Associate any matrix $M := (m_{ij} + n_{ij}\omega) \in \mathrm{Mat}_2(\mathbb{Q}(\omega))$ with $m_{ij}, n_{ij} \in \mathbb{Q}$ to the vector

$$\vec{v}_M := {}^T(m_{11}, n_{11}, m_{12}, n_{12}, m_{21}, n_{21}, m_{22}, n_{22}) \in \mathbb{Q}^8.$$

Denote the vector $\vec{v}_{v_i}$ by $\vec{v}_i$ for simplicity. Let $M_{\mathrm{bas}} \in \mathrm{Mat}_{8\times 4}(\mathbb{Z})$ be the matrix whose i-th column is $\vec{v}_i$ for $i = 1, \ldots, 4$. Given $M \in \mathrm{Mat}_2(K)$, $M \in \hat{\mathscr{B}}$ if and only if

$$\vec{v}_M = M_{\text{bas}} \cdot \vec{\alpha}_M \tag{4.19}$$

for some $\vec{\alpha}_M \in \mathbb{Q}^4$. Moreover $M := (m_{ij} + n_{ij}\omega)$ is in $\text{End}(Q'\tau)$ if and only if

$$m_{11}, n_{11}, m_{12}, Nn_{12}, \frac{m_{21} - b_1 n_{21}}{N}, n_{21}, m_{22}, n_{22} \in \mathbb{Z} \tag{4.20}$$

by (4.17). Let $M_{\text{end}} \in \text{Mat}_{8\times 8}(\mathbb{Q})$ be the matrix which describes the conditions in (4.20) so that $M \in \text{End}(Q'\tau)$ if and only if $M_{\text{end}} \cdot \vec{v}_M \in \mathbb{Z}^8$. Therefore the elements M of $\hat{\mathscr{R}}$ correspond precisely under (4.19) to $\vec{\alpha}_M \in \mathbb{Q}^4$ such that

$$M_{\text{end}} \cdot M_{\text{bas}} \cdot \vec{\alpha}_M \in \mathbb{Z}^8. \tag{4.21}$$

To show the discriminant of $\hat{\mathscr{R}}$ is 1 mod $(\mathbb{Z}_2)^2$ amounts to finding solutions $\vec{\beta} \in \mathbb{Z}^4$ such that $M_{\text{end}} \cdot M_{\text{bas}} \cdot \vec{\beta} \equiv 0 \bmod 4$. (Then $\vec{\alpha} := \vec{\beta}/4$ satisfies (4.21).) Three linearly independent solutions for $\vec{\alpha}$ are given by the vectors

$$\vec{\alpha}_5 := {}^T(0,0,1,0)/2, \quad \vec{\alpha}_6 := {}^T(0,1,0,1)/2, \quad \text{and } \vec{\alpha}_7 := {}^T(2,0,1,0)/4.$$

Therefore $\vec{v}_i := M_{\text{bas}} \cdot \vec{\alpha}_i$ gives an element in $\hat{\mathscr{R}}$ for $i = 5, 6, 7$. Consider the set $S := \{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$. Observe the relations

$$\vec{v}_5 = \vec{v}_3/2, \qquad \vec{v}_7 = (\vec{v}_1 + \vec{v}_5)/2, \qquad \text{and } \vec{v}_6 = (\vec{v}_2 + \vec{v}_4)/2.$$

These imply $\vec{v}_5$ generates $\vec{v}_3$, while $\vec{v}_1$ and $\vec{v}_7$ generate $\vec{v}_5$, and finally $\vec{v}_2$ and $\vec{v}_6$ generate $\vec{v}_4$. Accordingly, replace $\vec{v}_3$ and $\vec{v}_4$ in S with $\vec{v}_6$ and $\vec{v}_7$ so that $S = \{\vec{v}_1, \vec{v}_2, \vec{v}_6, \vec{v}_7\}$. Now S is a set of linearly independent vectors over $\mathbb{Z}$ and contained in $\hat{\mathscr{R}}$, hence a basis. A computation (using PARI/GP [53]) of the discriminant of $\hat{\mathscr{R}}$ with respect to this basis shows it is $D^2 \cdot N^2 \cdot a^6$. This is a unit modulo $(\mathbb{Z}_2)^2$ since we may assume a is odd. Hence $\hat{\mathscr{R}}_2$ is maximal.

$\square$

This concludes the proof that $\mathscr{R}_z$ is a maximal order. $\square$

The next step is to prove $\mathscr{R}_z$ is the right order of an explicit ideal in $(-1, D)_{\mathbb{Q}}$. We first recall a result of Pacetti which constructs Siegel points from certain ideals of $(-1, D)_{\mathbb{Q}}$.

4.5 Split-CM points and right orders in $(-1, D)_{\mathbb{Q}}$

In this section we identify $\mathscr{R}_z$ with an explicit right order in $(-1, D)_{\mathbb{Q}}$. Let $\mathcal{M}$ be a maximal order of $(-1, D)_{\mathbb{Q}}$ such that there exists $u \in \mathcal{M}$ with $u^2 = D$. (Such an order must exist by Eichler's mass formula). Two left $\mathcal{M}$-ideals I and I' are in the same *class* if there exists $b \in (-1, D)_{\mathbb{Q}}^{\times}$ such that $I = I'b$. The number n of left $\mathcal{M}$-ideal classes is finite and independent of the choice of maximal order $\mathcal{M}$. Let $\mathcal{I}$ be the set of n left $\mathcal{M}$-ideal classes, and recall $\mathcal{R}$ is the set of conjugacy classes of maximal orders in $(-1, D)_{\mathbb{Q}}$. (Equivalently, $\mathcal{R}$ is the set of conjugacy classes of right orders with respect to $\mathcal{M}$, taken without repetition.) The cardinality t of $\mathcal{R}$ is less than or equal to n and is called the *type number*.

Recall $(-1, D)_{\mathbb{Q}} \cong (D, -N)_{\mathbb{Q}}$ and let $1, u, v, uv$ be a basis for $(-1, D)_{\mathbb{Q}}$

where $u^2 = D$, $v^2 = -N$, and $uv = -vu$. Define the $\mathbb{Z}$-module

$$I_z := \left\langle \left(\frac{b_1 - u}{2a_1 N}\right) av, \left(\frac{b_1 - u}{2a_1 N}\right)\left(\frac{N + bv}{2}\right), \frac{b - v}{2}, -a \right\rangle_{\mathbb{Z}}. \tag{4.22}$$

It is proven in [52, p. 369-372] that I_z is a left ideal for a maximal order $\mathcal{M}_{\mathfrak{a},[\mathcal{N}]}$ which is independent of the class representative of $[\mathcal{N}]$ and of the form Q, and contains the element u. Let R_z denote the right order of I_z. It is maximal because $\mathcal{M}_{\mathfrak{a},[\mathcal{N}]}$ is maximal.

We will show that the right order R_z has a natural identification with the maximal order $\mathscr{R}_z$. To do this, we recall a result of [52] which associates ideals of $(-1, D)_{\mathbb{Q}}$ to Siegel points. Namely, let (I_R, R) be a pair consisting of a left $\mathcal{M}$-ideal I_R with maximal right order R. Define the 4-dimensional real vector space $V := (-1, D)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$, so that V/I_R is a real torus. The linear map

$$\begin{aligned} J : V &\to V \\ x &\mapsto \frac{u}{\sqrt{|D|}} \cdot x \end{aligned}$$

induces a complex structure on V. Hence the data $(V/I_R, J)$ determines a 2-dimensional complex torus. Define a map $\mathcal{E}_R : V \times V \to \mathbb{R}$ by

$$\mathcal{E}_R(x, y) := \mathrm{Tr}(u^{-1} x \bar{y}) / \mathbf{N}(I_R),$$

where $\mathbf{N}(I_R)$ is the norm of the ideal I_R and the 'bar' denotes conjugation in $(-1, D)_{\mathbb{Q}}$. It is straightforward to check that $\mathcal{E}_R$ is alternating, satisfies $\mathcal{E}_R(Jx, Jy) = \mathcal{E}_R(x, y)$ for all $x, y \in V$, is integral on I_R, and that the form $\mathcal{H}_R : V \times V \to \mathbb{C}$ defined by

$$\mathcal{H}_R(x, y) := \mathcal{E}_R(Jx, y) + i\mathcal{E}_R(x, y), \qquad x, y \in V \tag{4.23}$$

is positive definite (see [52] for details). Thus $\mathcal{E}_R$ is a Riemann form and so there exists a symplectic basis $\{x_1, x_2, y_1, y_2\}$ of I_R with respect to $\mathcal{E}_R$. The matrix E_R of $\mathcal{E}_R$ with respect to this basis has determinant

$$\det(E_R) = \mathbf{N}(I_R)^{-4}\mathbf{N}(u)^{-2}\mathrm{disc}(I_R),$$

where we have used the fact that $\mathrm{disc}(I_R) = (\det(u_i u_j))_{ij}$ for any basis $\{u_1, \ldots, u_4\}$ of I_R. But the fact that R is maximal implies $\mathrm{disc}(I_R) = D^2\,\mathbf{N}(I_R)^4$ [54], [52, Proposition 32], hence $\det(E_R) = 1$. This implies $\mathcal{E}_R$ is of type 1, its matrix is $E_R = \left(\begin{smallmatrix} 0 & \mathbf{1}_2 \\ -\mathbf{1}_2 & 0 \end{smallmatrix}\right)$, and $\mathcal{H}_R$ is a principal positive definite Hermitian form.

The conclusion is that the data (I_R, J, E_R) determines a Siegel point in $\mathbb{H}_2/Sp_4(\mathbb{Z})$. The action of a $\gamma \in Sp_4(\mathbb{Z})$ on (I_R, J, E_R) is given as a $\mathbb{Z}$-linear isomorphism $I_R \to \gamma(I_R)$, which sends $J \to \gamma^{-1} \circ J \circ \gamma$, and $\mathcal{E}_R \to \mathcal{E}_R \circ \gamma$.

Left $\mathcal{M}$-ideals with the same right order class determine equivalent Siegel points under this construction [52, p. 364]. In other words, there is a well-defined map

$$\mathcal{R} \longrightarrow \mathbb{H}_2/Sp_4(\mathbb{Z}).$$

This can be seen as follows. Let I and I' be two left $\mathcal{M}$-ideals with the same right order class $[R]$. Assume first that they are equivalent, that is, $I = I'b$ for some $b \in (-1, D)^\times_{\mathbb{Q}}$. Then multiplication on the right by b determines a $\mathbb{Z}$-linear isomorphism

$$\begin{aligned} \gamma : I &\longrightarrow I' \\ x &\mapsto x \cdot b. \end{aligned}$$

Furthermore

$$E(\gamma(x), \gamma(y)) = \frac{\mathrm{Tr}(u^{-1}x \cdot b(\overline{y \cdot b}))}{\mathbf{N}(I)} = E(x, y) \cdot \frac{\mathbf{N}(b)}{\mathbf{N}(I)} = E'(x, y),$$

and since J is a multiplication on the left, and b on the right, clearly $\gamma^{-1} \circ J \circ \gamma = J$. Therefore $(I, J, E) \sim (I', J, E')$ for $I \sim I'$. Now suppose I and I' are not equivalent. Then uI has the same left order and right order class as I but is not equivalent to I (see Lemmas 4.6.6 and 4.6.8 below). Since there are at most two classes of left $\mathcal{M}$-ideals with the same right order class, it must be that $uI \sim I' \sim uIu^{-1}$. It is straightforward to check that the map from I to uIu^{-1} via conjugation by u gives $(I, J, E) \sim (uIu^{-1}, J, E)$ and so by the above case, $(I, J, E) \sim (I', J, E')$.

The ideal I_z in (4.22) corresponds to the Siegel point z under this construction. This is left as an exercise in [52] but can be seen as follows. Let $\{x_1, x_2, y_1, y_2\}$ denote the basis, taken in order, of I_z given in (4.22). A straightforward calculation done by Pacetti shows $\{x_1, x_2, y_1, y_2\}$ is symplectic with respect to $\mathcal{E}$, and of principal type. Then $\{y_1, y_2\}$ is a basis for the complex vector space (V, J), and the period matrix for the complex torus $(V/I_z, J)$ is the coefficient matrix of the basis of $\{x_1, x_2, y_1, y_2\}$ in terms of $\{y_1, y_2\}$. It suffices to show this period matrix is $\Pi_z := [z, \mathbf{1}_2]$. Thus one needs to verify

$$x_1 = 2a\tilde{\tau}y_1 + b\tilde{\tau}y_2$$

$$x_2 = b\tilde{\tau}y_1 + 2c\tilde{\tau}y_2,$$

where $\tilde{\tau} := \frac{-b_1+\sqrt{|D|}J}{2a_1N}$ is given by the complex multiplication J. This is a simple calculation using the relations $D = b_1^2 - 4a_1c_1N$ and $-N = b^2 - 4ac$.

Note this construction determines an isomorphism $\sigma : I_z \longrightarrow L_z$ by

$$x_1 \mapsto \begin{bmatrix} 2a \\ b \end{bmatrix} \tau, \quad x_2 \mapsto \begin{bmatrix} b \\ 2c \end{bmatrix} \tau, \quad y_1 \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad y_2 \mapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which maps $J \mapsto i$. In particular $\mathcal{H}_{R_z}(x, y) = \mathcal{H}_z\big|_{L_z \times L_z}(\sigma(x), \sigma(y))$ for all $x, y \in I_z$.

The elements of R_z and $\mathscr{R}_z$ can now be related as follows. Any $b \in R_z$ preserves I_z (on the right) as well as the complex structure J and hence defines an endomorphism f_b of X_z. Likewise, any $M \in \mathscr{R}_z$ defines an endomorphism f_M of the torus X_z by definition. We claim these rings give the same endomorphisms of X_z:

Proposition 4.5.1. *As endomorphisms, R_z is identified with $\mathscr{R}_z$.*

Proof. Suppose $f_b \in \mathrm{End}(X_z)$ for some $b \in R_z$. To show f_b comes from $\mathscr{R}_z$, it suffices to show $\rho_r(f_b)$ preserves $H_z\big|_{L_z \times L_z}$. Equivalently by the map σ it suffices to show

$$\mathcal{H}_{R_z}(x \cdot b, y) = \mathcal{H}_{R_z}(x, y \cdot b^\iota).$$

But this is immediate since $\mathrm{Tr}(u^{-1}(xb)\bar{y}) = \mathrm{Tr}(u^{-1}x(\bar{\bar{b}}\bar{y}) = \mathrm{Tr}(u^{-1}x(\overline{y\bar{b}}))$ and $\bar{b} = b^\iota$ in $(-1, D)_{\mathbb{Q}}$. Therefore as endomorphisms R_z is contained in $\mathscr{R}_z$. Conversely any $f_M \in \mathrm{End}(X_z)$ for $M \in \mathscr{R}_z$ defines a linear map from I_z to itself which commutes with the complex structure J, hence corresponds to an element in R_z. □

Corollary 4.5.2. *$\mathscr{R}_z$ is isomorphic to the maximal right order R_z in the quaternion algebra $\mathscr{B}$, and this map sends $\frac{1+QS}{2} \mapsto \frac{1+v}{2}$.*

Proof. The first part follows immediately from the proposition. Regarding the embedding, the rational representation in $\mathrm{Mat}_4(\mathbb{Z})$ of the endomorphism $\frac{1+QS}{2} \in R_z$ is

$$\begin{pmatrix} \frac{b+1}{2} & c & 0 & 0 \\ -a & \frac{1-b}{2} & 0 & 0 \\ 0 & 0 & \frac{1-b}{2} & a \\ 0 & 0 & -c & \frac{b+1}{2} \end{pmatrix}.$$

Its action on the basis x_1, x_2, y_1, y_2 of I_z shows immediately that it is the linear transformation given by multiplication on the right by $\frac{1+v}{2}$. $\square$

4.6 Formula for the central value $L(\psi_{\mathcal{N}}, 1)$

In this section we prove Theorems 4.3.2, 4.3.3 and 4.3.6.

Proof of Theorems 4.3.2 and 4.3.3. Fix $[\mathfrak{a}] \in Cl(\mathcal{O}_K)$, $\mathcal{N} \subset \mathcal{O}_K$ a prime ideal of norm N, $\tau := \tau_{\mathfrak{a}\bar{\mathcal{N}}}$. Throughout the rest of this section, fix $z := Q\tau$ and $z' := Q'\tau$ where Q, Q' are binary quadratic forms of discriminant $-N$. Define

$$\Upsilon_1 : \{Q\tau \ : \ [Q] \in Cl(-N)\} / Sp_4(\mathbb{Z}) \longrightarrow \mathcal{R}_N \qquad (4.24)$$
$$[Q]\tau \mapsto [R_{Q\tau}]$$

Given an $R_{Q\tau}$, let $\phi_Q : \mathcal{O}_L \hookrightarrow R_{Q\tau}$ be the optimal embedding defined in Lemma 4.4.4 and Corollary 4.5.2. Define a second map

$$\Upsilon_2 : Cl(-N) \longrightarrow \Phi_{\mathcal{R}}/- \qquad (4.25)$$
$$[Q] \mapsto [\phi_Q : \mathcal{O}_L \hookrightarrow R_{Q\tau}].$$

We will start by showing that the maps Υ_1 and Υ_2 are well-defined. First note Υ_1 is injective: if $R_{Q\tau} \sim R_{Q'\tau}$ in B, then we saw in the last section that

Pacetti's map $\mathcal{R} \longrightarrow \mathbb{H}_2/Sp_4(\mathbb{Z})$ sends $R_{Q\tau} \mapsto Q\tau$. After proving the maps are well-defined, we will prove Υ_2 is a bijection and independent of the choice of representative $\mathfrak{a}$ of $[\mathfrak{a}]$. This will simultaneously prove Theorems 4.3.3 and 4.3.2.

Lemma 4.6.1. *If $z \sim z'$ in $\mathbb{H}_2/\Gamma_\theta$, then $R_z \sim R_{z'}$ in $\mathcal{R}$.*

Remark 4.6.1. Note that if $Q \sim Q'$ with $Q = {}^TAQ'A$ for some $A \in SL_2(\mathbb{Z})$, then $Q\tau \sim Q'\tau$ as Siegel points via the matrix $\left(\begin{smallmatrix} {}^TA & 0 \\ 0 & A^{-1} \end{smallmatrix}\right) \in \Gamma_\theta$.

Proof. Recall $z \sim z'$ in $\mathbb{H}_2/Sp_4(\mathbb{Z})$ if and only if the abelian varieties (X_z, H_z) and $(X_{z'}, H_{z'})$ are isomorphic. Write X, X', H, H' for $X_z, X_{z'}, H_z, H_{z'}$, respectively. Suppose $f : X \longrightarrow X'$ is an isomorphism of (X, H) with (X', H'), so that $H'(f(x), f(y)) = H(x, y)$ for all $x, y \in \mathbb{C}^2$. We claim the isomorphism

$$\begin{aligned} \mathrm{End}(X) &\longrightarrow \mathrm{End}(X') \\ \alpha &\mapsto f \circ \alpha \circ f^{-1} \end{aligned} \tag{4.26}$$

induces an isomorphism of $\mathscr{R}_z$ and $\mathscr{R}_{z'}$. This follows immediately from the calculation

$$\begin{aligned} H'(f \circ \alpha \circ f^{-1}(x), y) &= H(\alpha(f^{-1}(x)), f^{-1}(y)) \\ &= H(f^{-1}(x), \alpha^\iota(f^{-1}(y))) \qquad \text{(since } \alpha \in \mathscr{R}_z\text{)} \\ &= H'(x, f(\alpha^\iota(f^{-1}(y)))) \\ &= H'(x, (f \circ \alpha \circ f^{-1})^\iota). \end{aligned}$$

The last equality follows because, as a matrix, $\rho_a(f)^\iota = \rho_a(f)^{-1}\det(\rho_a(f))$ and so the determinants in $(f \circ \alpha \circ f^{-1})^\iota$ cancel out. Therefore $\mathscr{R}_{z'} = f \circ \mathscr{R}_z \circ f^{-1}$ and so by Proposition 4.5.1, $R_z \sim R_{z'}$ in $\mathscr{B}$. $\square$

Lemma 4.6.2. *If $Q \sim Q'$ in $Cl(-N)$, then the corresponding optimal embeddings $\frac{v+1}{2} \hookrightarrow R_z$ and $\frac{v+1}{2} \hookrightarrow R_{z'}$ are equivalent.*

Proof. Suppose $Q \sim Q'$ with $Q' = AQ^TA$ for some $A \in SL_2(\mathbb{Z})$. Then by Lemma 4.4.2, the map $\mathscr{R}_z \to \mathscr{R}_{z'}$ by $M \mapsto AMA^{-1}$ is a $\mathbb{Z}$-algebra isomorphism, and extends to a $\mathbb{Q}$-algebra isomorphism from $\mathscr{B} \to \mathscr{B}'$. In particular it sends $QS \mapsto A(QS)A^{-1} = Q'S$. By Corollary 4.5.2, this induces a $\mathbb{Z}$-algebra isomorphism of $R_z \to R_{z'}$ which sends v to v, and extends to a $\mathbb{Q}$-algebra automorphism of $(-1, D)_{\mathbb{Q}}$. Hence by the Skolem-Noether theorem, the map $R_z \to R_{z'}$ must be conjugation by some unit of $(-1, D)_{\mathbb{Q}}$. $\square$

We now turn to proving Υ_2 is a bijection. The following six lemmas will be needed to prove Υ_2 is injective. Let $\mathcal{Q}$ denote the ideal in L which corresponds to Q.

Lemma 4.6.3.

$$I_z \cong \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}}$$

as right $\mathcal{O}_L$-modules.

Proof of Lemma. Define $v_1 := x_1, v_2 := x_2, v_3 := y_1, v_4 := -y_2$ where x_i, y_j is the basis of I_z defined in Section 4.4. The $\{v_i\}$ also form a basis for I_z. The map $f : I_z \longrightarrow \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}}$ defined by

$$\begin{aligned} v_1 &\mapsto (a, 0) & \qquad v_2 &\mapsto \left(\frac{b-\sqrt{-N}}{2}, 0\right) \\ v_4 &\mapsto (0, a) & \qquad v_3 &\mapsto \left(0, \frac{b-\sqrt{-N}}{2}\right) \end{aligned}$$

and extended $\mathbb{Z}$-linearly is an isomorphism of $\mathbb{Z}$-modules. To show it is an $\mathcal{O}_L$-module isomorphism, it suffices to show

$$f\left(v_i\left(\frac{b+v}{2}\right)\right) = f(v_i)\left(\frac{b+\sqrt{-N}}{2}\right) \quad \text{for all } \ i = 1, 2, 3, 4.$$

For this, use the identities:

$$\begin{aligned} v_1\left(\frac{b+v}{2}\right) &= bv_1 - av_2 & \qquad v_3\left(\frac{b+v}{2}\right) &= cv_4 \\ v_2\left(\frac{b+v}{2}\right) &= cv_1 & \qquad v_4\left(\frac{b+v}{2}\right) &= -av_3 + bv_4. \end{aligned}$$

$\square$

Lemma 4.6.4. *Suppose $S := I_z x$ where $x \in (-1, D)_{\mathbb{Q}}^{\times}$ commutes with $\frac{v+1}{2}$. Then*

$$S \cong \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}},$$

as right $\mathcal{O}_L$-modules.

Proof of Lemma. By Lemma 4.6.3 and the hypotheses on x, the composition from $S \to \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}}$ given by $g(v_i x) := f(v_i)$ is an isomorphism of $\mathcal{O}_L$-modules. $\square$

Lemma 4.6.5. *Suppose $\bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}} \cong \bar{\mathcal{Q}}' \oplus \bar{\mathcal{Q}}'$ as right $\mathcal{O}_L$-modules, and $h(-N)$ is odd. Then*

$$Q \sim Q'$$

in $Cl(-N)$.

Proof of Lemma. By a classical theorem of Steinitz [45, Theorem 1.6], $\bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}} \cong \bar{\mathcal{Q}}' \oplus \bar{\mathcal{Q}}'$ as right $\mathcal{O}_L$-modules if and only if $[\bar{\mathcal{Q}}']^2 = [\bar{\mathcal{Q}}]^2$ as classes in the ideal class group of $\mathcal{O}_L$. This is if and only if $[\bar{\mathcal{Q}}'/\bar{\mathcal{Q}}]^2 = [\mathrm{id}]$ where id is the identity class. But since the class number $h(-N)$ is odd, this implies $[\mathcal{Q}] = [\mathcal{Q}']$ in $Cl(-N)$. □

The next three lemmas we need are general results for quaternion algebras. Assume for Lemmas 4.6.6, 4.6.7, and 4.6.8 below that B is a quaternion algebra ramified precisely at ∞ and a prime p. In addition, assume M and R are maximal orders and there exists $u \in M$ such that $u^2 = -p$.

Lemma 4.6.6.

$$uMu^{-1} = M.$$

Proof. This is clear locally at primes $q \neq p$ because $u^{-1} = -u/p$. This is also clear locally at p because there is a unique maximal order in the division algebra B_p (see [44, Theorem 6.4.1, p.208] or [86] for example). □

Lemma 4.6.7. *Suppose I, I' are left M-ideals with right order R. In addition assume R admits an embedding of a ring of integers $\mathcal{O}$ of some imaginary quadratic field. Set $J := I(I')^{-1}$. Then*

$$JI' \cong I'$$

as right $\mathcal{O}$-modules.

Proof. First note J is a bilateral M-ideal. Since $u \in M$, $uM = Mu$ by Lemma 4.6.6 and so is a principal M-ideal of norm p. Hence it is the unique integral bilateral M-ideal of norm p, and so every bilateral M-ideal is equal to $uM \cdot m$ for some $m \in \mathbb{Q}$ [12, Proposition 1, p. 92]. In particular, this implies the bilateral M-ideals are principal. Therefore $J = tM = Mt$ for some $t \in B^{\times}$, and the map,

$$f : I' \longrightarrow JI'$$

$$w \rightarrow tw$$

is a $\mathbb{Z}$-module isomorphism. Since the multiplication by t is on the left, f is an isomorphism of right $\mathcal{O}$ modules. $\square$

Lemma 4.6.8. *Suppose I is a left M-ideal with right order R. Then uI is also a left M-ideal with right order R. Furthermore, any left M-ideal with right order R is equivalent to I or uI (or both).*

Proof. The right order of uI is clearly R. The left order is $uMu^{-1} = M$ by Lemma 4.6.6.

Suppose J is any left M-ideal with right order R. The ideal $I^{-1}J$ is R-bilateral, hence

$$I^{-1}J = \mathcal{P}^i m, \qquad i = 0, 1, \ m \in \mathbb{Q}$$

where $\mathcal{P}$ is the unique bilateral R-ideal of norm p [12, Proposition 1, p. 92].

If $I^{-1}J$ is principal, then $I \sim J$. Otherwise $i = 1$. Then since the ideal $I^{-1}uI$ is R-bilateral of norm p, by uniqueness $I^{-1}uI = \mathcal{P}$ and so

$$I^{-1}J = I^{-1}uI \cdot m.$$

Multiplying through by I we see $J \sim uI$ as left M-ideals. □

Now the injectivity of Υ_2 can be proven.

Proposition 4.6.9. *Suppose* $(R_z, \frac{v\pm1}{2}) \sim (R_{z'}, \frac{v\pm1}{2})$. *Then* $Q \sim Q'$ *in* $Cl(-N)$.

Proof. The assumption $(R_z, \frac{v\pm1}{2}) \sim (R_{z'}, \frac{v\pm1}{2})$ implies there exists $x \in (-1, D)_{\mathbb{Q}}^{\times}$ such that

$$x^{-1}R_z x = R_{z'}$$

and $r \in R_{z'}^{\times}$ such that

$$(xr)^{-1}\left(\frac{v+1}{2}\right)xr = \frac{v+1}{2}.$$

The proof is broken up into two cases.

Case 1

Assume $I_z \sim I_{z'}$. Then $I_z x \sim I_{z'}$ and they both have right order $R_{z'}$. Set $J := I_z x I_{z'}^{-1}$. Then $JI_{z'} \cong I_{z'}$ as right $\mathcal{O}_L$-modules by Lemma 4.6.7. Combining with Lemma 4.6.3 applied to $I_{z'}$ implies

$$JI_{z'} \cong \bar{\mathcal{Q}}' \oplus \bar{\mathcal{Q}}'$$

as right $\mathcal{O}_L$-modules.

On the other hand, $JI_{z'} = I_z x$. Since r is a unit, $I_z x = I_z x r$, so replacing x by xr if necessary we may assume $r = 1$ and $x^{-1}\left(\frac{v+1}{2}\right)x = \frac{v+1}{2}$. Lemma 4.6.4 applied to $I_z x$ gives

$$JI_{z'} \cong \bar{\mathcal{Q}} \oplus \bar{\mathcal{Q}}$$

as right $\mathcal{O}_L$-modules. Hence $Q \sim Q'$ by Lemma 4.6.5.

Case 2

Assume $I_z \not\sim I_{z'}$. For each maximal order R, there can be at most two left M-ideal classes with right orders in the class $[R]$. Therefore since $I_{z'}$ has right order $R_{z'} \in [R_z]$, but $I_z \not\sim I_{z'}$, by Lemma 4.6.8 it must be that

$$uI_z \sim I_{z'};$$

note uI_z is a left $\mathcal{M}$-ideal by Lemma 4.6.6. Then $uI_z x \sim I_{z'}$ and they have the same right order. Let $J := uI_z x I_{z'}^{-1}$ and use the same argument from Case 1, noting that Lemmas 4.6.3 and 4.6.4 hold with I_z replaced by uI_z since the multiplication by u is on the left. This concludes the proof that Υ_2 is injective. $\square$

It remains to show that Υ_2 is a surjection. This follows from the fact:

Lemma 4.6.10.

$$h(-N) = \#\Phi_{\mathcal{R}}/- \,.$$

Proof of Lemma. For $[R] \in \mathcal{R}$, let $h_R(-N)$ denote the number of optimal embeddings of $\mathcal{O}_L$ into R, modulo conjugation by $R^\times$. Then

$$\begin{aligned} \#\Phi_{\mathcal{R}}/- &= \frac{1}{2}\sum_{[R]\in\mathcal{R}} h_R(-N) && \text{by definition,} \\ &= h(-N) && \text{by Eichler's mass formula [20, (1.12)].} \end{aligned}$$

□

The last task is to prove the maps Υ_1 and Υ_2 are independent of the choice of representative $\mathfrak{a}$ of $[\mathfrak{a}]$. In fact we will prove a slightly stronger result regarding the right orders:

Lemma 4.6.11. *If $\mathfrak{a} \sim \mathfrak{a}'$ in $Cl(\mathcal{O}_K)$ then $R_{Q\tau_{\mathfrak{a}\bar{\mathcal{N}}}} = R_{Q\tau_{\mathfrak{a}'\bar{\mathcal{N}}}}$.*

Proof. The hypothesis $\mathfrak{a} \sim \mathfrak{a}'$ implies $\mathfrak{a}\bar{\mathcal{N}} \sim \mathfrak{a}'\bar{\mathcal{N}}$. Suppose $\bar{\mathcal{N}}$ corresponds to a form $[N, b, c]$. Then we can choose bases so that the products $\mathfrak{a}\bar{\mathcal{N}}$, $\mathfrak{a}'\bar{\mathcal{N}}$ both correspond to forms with middle coefficient congruent to $b \bmod 2N$ (see [60, Lemma 2.3], for example). The CM-points $\tau_{\mathfrak{a}\bar{\mathcal{N}}}, \tau_{\mathfrak{a}'\bar{\mathcal{N}}}$ are Heegner points of level N and discriminant D by construction, and by the comment above they have the same 'root' $b \bmod 2N$ of $\sqrt{D \bmod 4N}$. Hence there exists $M := \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ such that

$$M(\tau_{\mathfrak{a}\bar{\mathcal{N}}}) = \tau_{\mathfrak{a}'\bar{\mathcal{N}}}.$$

Set

$$\tilde{M} := \begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}$$

where

$$\tilde{\alpha} := \alpha \cdot \mathbf{1}_2, \quad \tilde{\beta} := \beta \cdot Q, \quad \tilde{\gamma} := \gamma \cdot Q^{-1}, \quad \tilde{\delta} := \delta \cdot \mathbf{1}_2.$$

It is shown in [1, p.233], for example, that $\tilde{M} \in \Gamma_\theta \subseteq Sp_4(\mathbb{Z})$. Therefore the relation

$$\tilde{M}(Q\tau_{\mathfrak{a}\bar{\mathfrak{N}}}) = Q\tau_{\mathfrak{a}'\bar{\mathfrak{N}}}$$

implies $Q\tau_{\mathfrak{a}\bar{\mathfrak{N}}} \sim Q\tau_{\mathfrak{a}'\bar{\mathfrak{N}}}$ in $\mathbb{H}_2/\Gamma_\theta$. Let $\tau := \tau_{\mathfrak{a}\bar{\mathfrak{N}}}$ and $\tau' := \tau_{\mathfrak{a}'\bar{\mathfrak{N}}}$. An isomorphism $f_M : X_{Q\tau'} \longrightarrow X_{Q\tau}$ is given by

$${}^T(\tilde{\gamma}Q\tau + \tilde{\delta})[Q'\tau, \mathbf{1}_2] = [Q\tau, \mathbf{1}_2]\, {}^T\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{pmatrix}.$$

The analytic representation of this isomorphism, which we will also denote by f_M, is

$$f_M = {}^T(\tilde{\gamma}Q\tau + \tilde{\delta}) = (\gamma\tau + \delta) \cdot \mathbf{1}_2,$$

where recall $\gamma, \delta \in \mathbb{Z}$. Therefore the map

$$\begin{aligned} \mathrm{End}(X_{Q\tau'}) &\longrightarrow \mathrm{End}(X_{Q\tau}) \\ A &\mapsto f_M A f_M^{-1} = A \end{aligned}$$

is the identity map, hence $\mathrm{End}(X_{Q\tau'}) = \mathrm{End}(X_{Q\tau})$. Moreover the equivalence

$${}^T\bar{A}H_{Q\tau} = H_{Q\tau}A^\iota \qquad \Leftrightarrow \qquad {}^T\bar{A}Q^\iota = Q^\iota A^\iota$$

implies the relation on the left hand side is independent of τ. Hence $R_{Q\tau} = R_{Q\tau'}$. □

It follows immediately since $R_{Q\tau} = R_{Q\tau'}$ that the maps Υ_1 and Υ_2 are independent of the choice of representative $\mathfrak{a}$ of $[\mathfrak{a}]$.

This completes the proofs of Theorems 4.3.2 and 4.3.3. $\square$

Recall the definitions of: the normalized theta values $\Theta_{[\mathfrak{a},R],\mathcal{N}}$ in (4.8), the sign function $\varepsilon_{[\mathfrak{a},R]}$ on the embeddings in (4.10), and the twisted number of optimal embeddings $h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$ in (4.11). The η function in (4.8) is defined on an ideal $\mathfrak{a} = [a, \frac{-b+\sqrt{D}}{2}]$ of $\mathcal{O}_K$ by

$$\eta(\mathfrak{a}) := e_{48}(a(b+3)) \cdot \eta\Big(\frac{-b+\sqrt{D}}{2a}\Big) \tag{4.27}$$

where $e_n(x) := \exp(2\pi i x/n)$ for $n \in \mathbb{Z}$, $x \in \mathbb{C}$, and $\eta(z) := e_{24}(z)\prod_{n=1}^{\infty}(1 - e^{2\pi i z})$ for $\mathrm{Im}(z) > 0$ is Dedekind's eta function. Using Shimura's reciprocity law it can be shown that the value $\Theta_{[\mathfrak{a},R],\mathcal{N}}$ is an algebraic integer (see [52, Proposition 23, p. 355] and [25]).

We now prove Lemma 4.3.5.

Proof of Lemma 4.3.5. Theorem 31 of [52] says that if $Q\tau_{\mathfrak{a}\bar{\mathcal{N}}} \sim Q'\tau_{\mathfrak{a}\bar{\mathcal{N}}}$ in $\mathbb{H}_2/\Gamma_\theta$, then

$$\Theta_{[\mathfrak{a},Q],\mathcal{N}} = \pm\Theta_{[\mathfrak{a},Q'],\mathcal{N}}.$$

The lemma therefore follows immediately by this fact and Theorem 4.3.2. $\square$

We now prove Theorem 4.3.6.

Proof of Theorem 4.3.6. The remaining step in deriving formula (4.12) for $L(\psi_{\mathcal{N}}, 1)$ is to determine how θ behaves on equivalent split-CM points. The following is a special case of [52, Theorem 31] but we give a slightly simplified proof.

Lemma 4.6.12. *Let Q and Q' be binary quadratic forms of discriminant $-N$. If $Q\tau \sim Q'\tau$ in $\mathbb{H}_2/\Gamma_\theta$, then $\theta(Q\tau) = \pm\theta(Q'\tau)$.*

Proof of Lemma. Suppose $Q\tau \sim Q'\tau$ in $\mathbb{H}_2/\Gamma_\theta$. Then there exists $M := \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_\theta$ such that $M(Q\tau) = Q'\tau$. Recall the functional equation for θ is

$$\theta(M \circ z) = \chi(M)[\det(\gamma z + \delta)]^{1/2}\theta(z), \qquad M \in \Gamma_\theta \tag{4.28}$$

where $\chi(M)$ is a certain 8th root of unity.

Then

$$\frac{\theta(Q'\tau)}{\theta(Q\tau)} = \chi(M)[\det(\gamma Q\tau + \delta)]^{1/2}.$$

Applying Smith Normal Form, there exists $U, V \in SL_2(\mathbb{Z})$ such that $UQV = \left(\begin{smallmatrix} 1 & 0 \\ 0 & N \end{smallmatrix}\right)$, and $U', V' \in SL_2(\mathbb{Z})$ such that $U'Q'V' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & N \end{smallmatrix}\right)$. These give isomorphisms $f_U : X_{Q\tau} \to E_\tau \times E_{N\tau}$ and $f_{U'} : X_{Q'\tau} \to E_\tau \times E_{N\tau}$ respectively. From the relation $M(Q\tau) = Q'\tau$, we also get an isomorphism $f_M : X_{Q'\tau} \to X_{Q\tau}$ given by

$${}^T(\gamma Q\tau + \delta)[Q'\tau, \mathbf{1}_2] = [Q\tau, \mathbf{1}_2]\, {}^T\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Thus the composition

$$f_U \circ f_M \circ f_{U'}^{-1} : E_\tau \times E_{N\tau} \longrightarrow E_\tau \times E_{N\tau}$$

is an automorphism, and the determinant of its analytic representation is a unit and an algebraic integer. This last fact follows from linear algebra or can be deduced directly using Lemma 4.4.3. Since U and U' are both in $SL_2(\mathbb{Z})$, we get $\det(\gamma Q\tau + \delta) \in \mathcal{O}_K^\times$. Since $D < -4$ this implies $\det(\gamma Q\tau + \delta) = \pm 1$. Therefore $[\det(\gamma Q\tau + \delta)]^{1/2} = \pm\sqrt{\pm 1}$.

This proves $\frac{\theta(Q'\tau)}{\theta(Q\tau)} = \pm\sqrt{\pm 1} \cdot \chi(M)$. But by Theorem 17 of [52], the ratio of theta values on the left is an algebraic integer in the Hilbert class field of K. Hence $\pm\sqrt{\pm 1} \cdot \chi(M)$ is an 8th root of unity and an algebraic integer in the Hilbert class field of K, which does not contain i. Therefore

$$\pm\sqrt{\pm 1} \cdot \chi(M) = \pm 1.$$

$\square$

The theorem follows immediately from Lemma 4.6.12 and Theorems 4.3.2 and 4.3.3. $\square$

4.7 Examples

This section provides tables for two class number one examples. All calculations were done in gp/PARI [53]. Given D of class number one, for each admissable N we compute a form $[N, b_1, c_1]$ corresponding to $\mathcal{N}$. We set $\mathfrak{a}\mathcal{N} = \mathcal{N}$ since $Cl(\mathcal{O}_K)$ is trivial, and $\tau_{\mathfrak{a}\mathcal{N}} := \tau_{\mathcal{N}} := \frac{-b_1+\sqrt{D}}{2N}$ to be a Heegner point of level N and discriminant D. We choose $[1, \frac{-b_1+\sqrt{D}}{2}]$ for a basis of $\mathcal{O}_K$ so that following definition (4.27),

$$\eta(\mathcal{N})\eta(\mathcal{O}_K) := e_{48}^2(N(b_1+3)^2) \cdot \eta\Big(\frac{-b_1+\sqrt{D}}{2N}\Big) \cdot \eta\Big(\frac{-b+\sqrt{D}}{2}\Big).$$

From left to right, the columns of the table are N, the absolute values of the integers $\Theta_{[R]}$ for each $[R] \in \mathcal{R}$, the number, denoted $\#\Theta_{[R]}$, of classes $[Q] \in Cl(-N)$ with value $\pm\Theta_{[R]}$ (this equals $h_R(-N)$ by Theorem 4.3.3), and the values $h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$.

For $D = -7$, the type number is 1 and so $\#\Theta_{[R]} = \frac{1}{2}h_R(-N) = h(-N)$ gives the N-th coefficient of the weight $3/2$ level $4D$ form $\frac{1}{2}+\omega_R \sum_{N>0} H_D(N)q^N$ defined by the modified Hurwitz invariants $H_D(N)$ (see [20, p. 120] for their definition).

N	$\Theta_{[R]}$	$\#\Theta_{[R]}$	$h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$	N	$\Theta_{[R]}$	$\#\Theta_{[R]}$	$h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$
11	1	1	-1	107	1	3	-3
23	1	3	-1	127	1	5	1
43	1	1	1	151	1	7	-1
67	1	1	-1	163	1	1	1
71	1	7	-3	179	1	5	-3
79	1	5	-1	191	1	13	-5

Table 4.1: $D = -7$, $N \leq 200$, $t = 1$.

N	$\Theta_{[R]}$	$\#\Theta_{[R]}$	$h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$	N	$\Theta_{[R]}$	$\#\Theta_{[R]}$	$h^{\varepsilon}_{[\mathfrak{a},R]}(-N)$
23	0	2	2	103	0	3	3
	2	1	1		2	2	2
31	0	2	2	163	0	1	1
	2	1	-1		2	0	0
47	0	3	3	179	0	2	2
	2	2	2		2	3	1
59	0	2	2	191	0	8	8
	2	1	-1		2	5	1
67	0	0	0	199	0	5	5
	2	1	-1		2	4	4
71	0	4	4	223	0	4	4
	2	3	-3		2	3	3

Table 4.2: $D = -11$, $N \leq 250$, $t = 2$.

Chapter 5

Lower bounds for the principal genus of definite binary quadratic forms

5.1 Introduction

The following is joint work with Jeffrey Stopple†. Suppose $-D < 0$ is a fundamental discriminant. By genus theory we have an exact sequence for the class group $\mathcal{C}(-D)$ of positive definite binary quadratic forms:

$$\mathcal{P}(-D) \overset{\text{def.}}{=} \mathcal{C}(-D)^2 \hookrightarrow \mathcal{C}(-D) \twoheadrightarrow \mathcal{C}(-D)/\mathcal{C}(-D)^2 \simeq (\mathbb{Z}/2)^{g-1},$$

where D is divisible by g primary discriminants (i.e., D has g distinct prime factors). Let $p(-D)$ denote the cardinality of the principal genus $\mathcal{P}(-D)$. The genera of forms are the cosets of $\mathcal{C}(-D)$ modulo the principal genus, and thus $p(-D)$ is the number of classes of forms in each genus. The study of this invariant of the class group is as old as the study of the class number $h(-D)$ itself. Indeed, Gauss wrote in [16, Art. 303]

> . . . Further, the series of [discriminants] corresponding to the same given classification (i.e. the given number of both genera and classes) always seems to terminate with a finite number . .

† *To appear in [38].*

. However, *rigorous* proofs of these observations seem to be very difficult.

Theorems about $h(-D)$ have usually been closely followed with an analogous result for $p(-D)$. When Heilbronn [32] showed that $h(-D) \to \infty$ as $D \to \infty$, Chowla [8] showed that $p(-D) \to \infty$ as $D \to \infty$. An elegant proof of Chowla's theorem is given by Narkiewicz in [48, Prop 8.8 p. 458].

Similarly, the Heilbronn-Linfoot result [34] that $h(-D) > 1$ if $D > 163$, with at most one possible exception was matched by Weinberger's result [91] that $p(-D) > 1$ if $D > 5460$ with at most one possible exception. On the other hand, Oesterlé's [50] exposition of the Goldfeld-Gross-Zagier bound for $h(-D)$ already contains the observation that the result was not strong enough to give any information about $p(-D)$.

In [85] Tatuzawa proved a version of Siegel's theorem: for every ε there is an explicit constant $C(\varepsilon)$ so that

$$h(-D) > C(\varepsilon)D^{1/2-\varepsilon}$$

with at most one exceptional discriminant $-D$. This result has never been adapted to the study of the principal genus. It is easily done; the proofs are not difficult so it is worthwhile filling this gap in the literature. We present two versions. The first version contains a transcendental function (the Lambert W function discussed below). The second version gives, for each $n \geq 4$, a bound which involves only elementary functions. For each fixed n the second version is stronger on an interval $I = I(n)$ of D , but the first is stronger as $D \to \infty$.

The second version has the added advantage that it is easily computable. (N.B. The constants in Tatuzawa's result have been improved in [35] and [40]; these could be applied at the expense of slightly more complicated statements.)

5.1.1 Notation

We will always assume that $g \geq 2$, for if $g = 1$ then $-D = -4, -8$, or $-q$ with $q \equiv 3 \bmod 4$ a prime. In this last case $p(-q) = h(-q)$ and Tatuzawa's theorem [85] applies directly.

5.2 First version

Lemma 5.2.1. *If $g \geq 2$,*

$$\log(D) > g\log(g).$$

Proof. Factor D as $q_1, \ldots q_g$ where the q_i are (absolute values) of primary discriminants, i.e. 4, 8, or odd primes. Let p_i denote the ith prime number, so we have

$$\log(D) = \sum_{i=1}^{g} \log(q_i) \geq \sum_{i=1}^{g} \log(p_i) \stackrel{\text{def.}}{=} \theta(p_g). \tag{5.1}$$

By [67, (3.16) and (3.11)], we know that Chebyshev's function θ satisfies $\theta(x) > x(1 - 1/\log(x))$ if $x > 41$, and that

$$p_g > g(\log(g) + \log(\log(g)) - 3/2).$$

After substituting $x = p_g$ and a little calculation, this gives $\theta(p_g) > g\log(g)$ as long as $p_g > 41$, i.e. $g > 13$. For $g = 2, \ldots, 13$, one can easily verify the inequality directly. $\square$

Let $W(x)$ denote the Lambert W-function, that is, the inverse function of $f(w) = w\exp(w)$ (see [15], [55, p. 146 and p. 348, ex 209]). For $x \geq 0$ it is positive, increasing, and concave down. The Lambert W-function is also sometimes called the product log, and is implemented as `ProductLog` in *Mathematica*.

Theorem 5.2.2. *If* $0 < \varepsilon < 1/2$ *and* $D > \max(\exp(1/\varepsilon), \exp(11.2))$*, then with at most one exception*

$$p(-D) > \frac{1.31}{\pi}\varepsilon D^{1/2-\varepsilon-\log(2)/W(\log(D))}.$$

Proof. Tatuzawa's theorem [85], says that with at most one exception

$$\frac{\pi \cdot h(-D)}{\sqrt{D}} = L(1, \chi_{-D}) > .655\varepsilon D^{-\varepsilon}, \tag{5.2}$$

thus

$$p(-D) = \frac{2h(-D)}{2^g} > \frac{1.31\varepsilon \cdot D^{1/2-\varepsilon}}{\pi \cdot 2^g}.$$

The relation $\log(D) > g\log(g)$ is equivalent to

$$\log(D) > \exp(\log(g))\log(g),$$

Thus applying the increasing function W gives, by definition of W

$$W(\log(D)) > \log(g),$$

and applying the exponential gives

$$\exp(W(\log(D)) > g.$$

The left hand side above is equal to $\log(D)/W(\log(D))$ by the definition of W. Thus

$$-\log(D)/W(\log(D)) < -g,$$
$$D^{-\log(2)/W(\log(D))} = 2^{-\log(D)/W(\log(D))} < 2^{-g},$$

and the Theorem follows. $\square$

Remark 5.2.1. Our estimate arises from the bound $\log(D) > g\log(g)$, which is nearly optimal. That is, for every g, there exists a fundamental discriminant (although not necessarily negative) of the form

$$D_g \stackrel{\text{def.}}{=} \pm 3 \cdot 4 \cdot 5 \cdot 7 \ldots p_g,$$

and

$$\log|D_g| = \theta(p_g) + \log(2).$$

From the Prime Number Theorem we know $\theta(p_g) \sim p_g$, so

$$\log|D_g| \sim p_g + \log(2)$$

while [67, 3.13] shows $p_g < g(\log(g) + \log(\log(g))$ for $g \geq 6$.

5.3 Second version

Theorem 5.3.1. *Let $n \geq 4$ be any natural number. If $0 < \varepsilon < 1/2$ and $D > \max(\exp(1/\varepsilon), \exp(11.2))$, then with at most one exception*

$$p(-D) > \frac{1.31\varepsilon}{\pi} \cdot \frac{D^{1/2-\varepsilon-1/n}}{f(n)},$$

where

$$f(n) = \exp\left[(\pi(2^n) - 1/n)\log 2 - \theta(2^n)/n\right];$$

here π is the prime counting function and θ is the Chebyshev function.

Proof. First observe

$$f(n) = \frac{2^{\pi(2^n)}}{2^{1/n}\prod_{\text{primes } p<2^n} p^{1/n}}.$$

From Tatuzawa's Theorem (5.2), it suffices to show $2^g \le f(n)D^{1/n}$. Suppose first that D is not $\equiv 0 \pmod 8$.

Let $S = \{4, \text{ odd primes } < 2^n\}$, so $|S| = \pi(2^n)$. Factor D as $q_1 \cdots q_g$ where q_i are (absolute values) of coprime primary discriminants, that is, 4 or odd primes, and satisfy $q_i < q_j$ for $i < j$. Then, for some $0 \le m \le g$, we have $q_1, \ldots, q_m \in S$ and $q_{m+1}, \ldots, q_g \notin S$, and thus $2^n < q_i$ for $i = m+1, \ldots, g$. This implies

$$\begin{aligned} 2^{gn} &= \underbrace{2^n \cdots 2^n}_{m} \cdot \underbrace{2^n \cdots 2^n}_{g-m} \le 2^{mn}\ q_{m+1}q_{m+2}\cdots q_g \\ &= \frac{2^{mn}}{q_1 \cdots q_m} D \le \frac{2^{|S|\cdot n}}{\prod_{q\in S} q} \cdot D \end{aligned}$$

as we have included in the denominator the remaining elements of S (each of which is $\le 2^n$). The above is

$$= \frac{2^{\pi(2^n)\cdot n}}{2\prod_{\text{primes } p<2^n} p} \cdot D = f(n)^n \cdot D.$$

This proves the theorem when D is not $\equiv 0 \bmod 8$. In the remaining case, apply the above argument to $D' = D/2$; so

$$2^{gn} \le f(n)^n D' < f(n)^n D.$$

$\square$

Examples 5.3.2. If $0 < \varepsilon < 1/2$ and $D > \max(\exp(1/\varepsilon), \exp(11.2))$, then with at most one exception, Theorem 5.3.1 implies

$$p(-D) > 0.10199 \cdot \varepsilon \cdot D^{1/4-\varepsilon} \quad (n = 4)$$

$$p(-D) > 0.0426 \cdot \varepsilon \cdot D^{3/10-\varepsilon} \quad (n = 5)$$

$$p(-D) > 0.01249 \cdot \varepsilon \cdot D^{1/3-\varepsilon} \quad (n = 6)$$

$$p(-D) > 0.00188 \cdot \varepsilon \cdot D^{5/14-\varepsilon} \quad (n = 7)$$

5.4 Comparison of the two theorems

How do the two theorems compare? Canceling the terms which are the same in both, we seek inequalities relating

$$D^{-\log 2/W(\log D)} \quad \text{v.} \quad \frac{D^{-1/n}}{f(n)}.$$

Theorem 5.4.1. *For every n, there is a range of D where the bound from Theorem 5.3.1 is better than the bound from Theorem 5.2.2. However, for any fixed n the bound from Theorem 5.2.2 is eventually better as D increases.*

For fixed n, the first statement of Theorem 5.4.1 is equivalent to proving

$$D^{\log(2)/W(\log(D))-1/n} \geq f(n)$$

on a non-empty compact interval of the D axis. Taking logarithms, it suffices to show,

Lemma 5.4.2. *Let $n \geq 4$. Then*

$$x\left(\frac{\log 2}{W(x)} - \frac{1}{n}\right) \geq \log f(n)$$

on some non-empty compact interval of positive real numbers x.

Proof. Let $g(n, x) = x\left(\log 2/W(x) - 1/n\right)$. Then

$$\frac{\partial g}{\partial x} = \frac{\log 2}{W(x)+1} - \frac{1}{n} \qquad \text{and} \qquad \frac{\partial^2 g}{\partial x^2} = \frac{-\log 2 \cdot W(x)}{x(W(x)+1)^3}.$$

This shows g is concave down on the positive real numbers and has a maximum at

$$x = 2^n(n\log 2 - 1)/e.$$

Because of the concavity, all we need to do is show that $g(n, x) > \log f(n)$ at *some* x. The maximum point is slightly ugly so instead we let $x_0 = 2^n n \log 2/e$.

Using $W(x) \sim \log x - \log\log x$, a short calculation shows

$$g(n, x_0) \sim \frac{1}{e} \cdot \frac{2^n}{n}.$$

By [68, 5.7)], a lower bound on Chebyshev's function is

$$\theta(t) > t\left(1 - \frac{1}{40\log t}\right), \quad t > 678407.$$

(Since we will take $t = 2^n$ this requires $n > 19$ which is not much of a restriction.) By [67, (3.4)], an upper bound on the prime counting function is

$$\pi(t) < \frac{t}{\log t - 3/2}, \quad t > e^{3/2}.$$

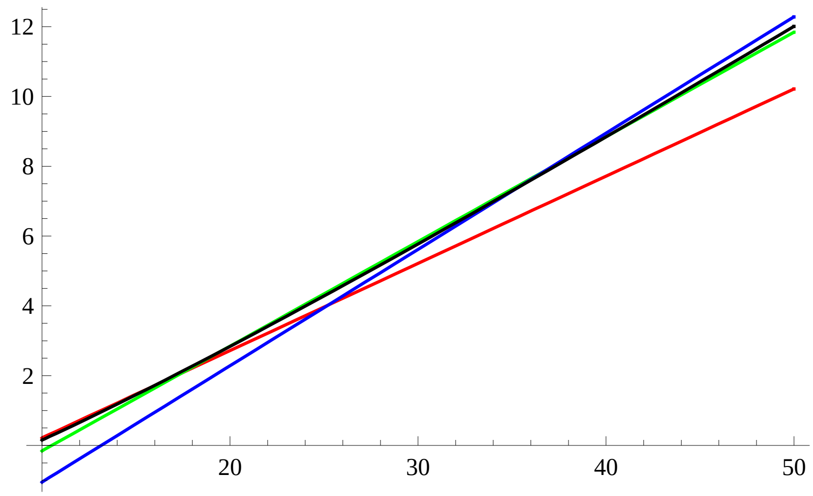

Figure 5.1: log-log plots of the bounds from Theorems 5.2.2 and 5.3.1

Hence $-\theta(2^n) < 2^n\left(1/(40n\log 2) - 1\right)$ and so

$$\begin{aligned}
\log f(n) &= \left(\pi(2^n) - \frac{1}{n}\right)\log 2 - \frac{\theta(2^n)}{n} \\
&< \left(\frac{2^n}{n\log 2 - 3/2} - \frac{1}{n}\right)\log 2 + \frac{2^n}{n}\left(\frac{1}{40n\log 2} - 1\right) \\
&\sim \frac{61}{40\log 2}\cdot\frac{2^n}{n^2}.
\end{aligned}$$

Comparing the two asymptotic bounds for g and $\log f$ respectively we see that

$$\frac{1}{e}\cdot\frac{2^n}{n} > \frac{61}{40\log 2}\cdot\frac{2^n}{n^2},$$

for $n \geq 6$; small n are treated by direct computation.[1] $\square$

[1]The details of the asymptotics have been omitted for conciseness.

Figure 5.1 shows a log-log plot of the two lower bounds, omitting the contribution of the constants which are the same in both and the terms involving ε. That is, Theorem 5.3.1 gives for each n a lower bound $b(D)$ of the form

$$b(D) = C(n)\varepsilon D^{1/2-1/n-\varepsilon}, \quad \text{so}$$
$$\log(b(D)) = (1/2 - 1/n - \varepsilon)\log(D) + \log(C(n)) + \log(\varepsilon).$$

Observe that for fixed n and ε, this is linear in $\log(D)$, with the slope an increasing function of the parameter n. What is plotted is actually $(1/2 - 1/n)\log(D) + \log(C(n))$ as a function of $\log(D)$, and analogously for Theorem 5.2.2. In red, green, and blue are plotted the lower bounds from Theorem 5.3.1 for $n = 4$, 5, and 6 respectively. In black is plotted the lower bound from Theorem 5.2.2.

Examples 5.4.3. The choice $\varepsilon = 1/\log(5.6\cdot 10^{10})$ in Theorem 5.2.2 shows that $p(-D) > 1$ for $D > 5.6 \cdot 10^{10}$ with at most one exception. (For comparison, Weinberger [91, Lemma 4] needed $D > 2 \cdot 10^{11}$ to get this lower bound.) And, $\varepsilon = 1/\log(3.5\cdot 10^{14})$ in Theorem 5.2.2 gives $p(-D) > 10$ for $D > 3.5\cdot 10^{14}$ with at most one exception. Finally, $n = 6$ and $\varepsilon = 1/\log(4.8 \cdot 10^{17})$ in Theorem 5.3.1 gives $p(-D) > 100$ for $D > 4.8 \cdot 10^{17}$ with at most one exception.

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Vita

Kimberly Michele Hopkins was born in Portland, Oregon on December 21, 1981. She is the daughter of Drs. Barbara and William Spears, a lattice theorist and a category theorist. She graduated from Oregon Episcopal School in the spring of 2000. She then enrolled at the University of California at Santa Barbara where she participated in the UC LEADS program under the supervision of Jeffrey Stopple, and received her B.S. in Mathematics in Spring 2004. In Fall of 2004, she began graduate studies at the University of Texas at Austin as a Harrington Fellow and Wendell T. Gordon Endowed Fellow.

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