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# Local Elliptic Boundary Value Problems for the Dirac Operator

Committee:

Dan Freed, Supervisor

Jacques Distler

Dan Knopf

Lorenzo Sadun

Karen Uhlenbeck

# Local Elliptic Boundary Value Problems for the Dirac Operator

by

**Matthew Gregory Scholl, B.S.**

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**Local Elliptic Boundary Value Problems for the Dirac Operator**

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Matthew Gregory Scholl, Ph.D.
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Supervisor: Dan Freed

Two classes of local elliptic boundary conditions for the Dirac operator are studied: one posed on a family of even-dimensional spin manifolds and one posed on a family of odd-dimensional spin manifolds. It is shown that for such families of elliptic boundary value problems an associated determinant line bundle may be constructed, much as in the standard setting of a family of manifolds without boundary. The determinant line of the first class (the even problem) is shown to be isomorphic to the determinant line bundle associated to a Dirac operator on the double of the family. The second class (the odd problem) is related to the determinant line of a Dirac operator on the boundary family: we show that the squares of these determinant lines are isomorphic.

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# Chapter 1

# Introduction

The study of determinant line bundles for families of Dirac operators began in 1985 with the work of Quillen [23]. Quillen constructed the determinant line bundle for the family of $\bar{\partial}$-operators acting on a vector bundle over a Riemann surface, parametrized by the space of holomorphic structures on the surface. Furthermore, he discovered a natural metric on this determinant line bundle. The following year, Bismut and Freed in [6] generalized Quillen's work to arbitrary families of twisted Dirac operators over compact closed manifolds and constructed a natural connection on the determinant line bundle compatible with the Quillen metric. In [7], they determined the curvature and holonomy of the Bismut-Freed connection, thereby proving conjectures of Witten in [28]. Another invention of Quillen features prominently in these works: the notion of a super vector bundle with superconnection, first introduced in [24].

A crucial prerequisite for the determinant line bundle construction of Bismut and Freed is that the fibers of the underlying family of manifolds must be closed. Naturally, one might suspect that analogous constructions could be made for a family of Dirac operators with suitable boundary conditions.

As Atiyah and Bott observed in [1], Dirac operators there is a topological obstruction to the existence of well-posed local boundary conditions for the Dirac operator. For this reason, Atiyah, Patodi, and Singer introduced a global boundary condition for the Dirac operator in [2]. Atiyah-Patodi-Singer boundary conditions always exist. In 1996, Piazza [21] constructed a determinant line bundle with metric and compatible connection for a family of compact even-dimensional spin manifolds with boundary using the $b$-calculus. The $b$-calculus is related to Atiyah-Patodi-Singer boundary conditions. Indeed, it was introduced by Melrose in [19] to provide a natural setting for proving a version of the Atiyah-Patodi-Singer index theorem for families. This was done by Melrose and Piazza in [20].

Although local boundary conditions for the Dirac operator do not always exist, for certain twisted Dirac operators they do. Recent developments in physics have provided an incentive to study families of Dirac operators with local boundary conditions beyond their intrinsic interest; [13] and [18] use local boundary conditions that are special cases of the two types of boundary conditions that we consider.

We have two main goals here. The first is to show that for a certain class of *local* boundary conditions it's possible to construct a determinant line bundle with Quillen metric and compatible Bismut-Freed connection. The second is to calculate the geometry of these determinant line bundles. We do this indirectly by constructing flat isometries to the determinant line bundles of related Dirac operators over families of closed manifolds. We also prove index theorems for families of such problems.

The two boundary value problems we consider are:

**The even problem.** Suppose $X$ is an even-dimensional compact spin manifold with boundary. In this situation, the spinor bundle $SX \to X$ decomposes into a direct sum $SX \cong S^+X \oplus S^-X$ of equal rank bundles. Clifford multiplication by the outward unit conormal $du$ gives an isomorphism $\mathrm{cl}(du): S^+X|_{\partial X} \xrightarrow{\sim} S^-X|_{\partial X}$ of these bundles over the boundary; let $\sigma := \mathrm{cl}(du)$. As part of the data of the problem, we also have a complex vector bundle $E$ with hermitian metric and compatible connection, a decomposition $E \cong E^+ \oplus E^-$, and at the boundary an isomorphism $\gamma: E^+|_{\partial X} \xrightarrow{\sim} E^-|_{\partial X}$. The associated Dirac operator $D: C^\infty(SX \otimes E) \to C^\infty(SX \otimes E)$ decomposes as the direct sum of two chiral Dirac operators: $D^+: C^\infty(SX \otimes E)^+ \to C^\infty(SX \otimes E)^-$, and its formal adjoint $D^-$. The boundary condition on $D^+$ is that the boundary values of a section $\psi$ lie in the graph of $\sigma \otimes \gamma$. That is, $\psi^- = (\sigma \otimes \gamma)\psi^+$, where $\psi|_{\partial X} = \psi^+ + \psi^-$ is the decomposition of $\psi|_{\partial X}$ into a section of $(S^+X \otimes E^+)|_{\partial X}$ and a section of $(S^-X \otimes E^-)|_{\partial X}$.

**The odd problem.** Suppose $X$ is an odd-dimensional compact spin manifold with boundary. In this situation, the spinors $SX$ decompose at the boundary as a direct sum $SX|_{\partial X} \cong S^+X \oplus S^-X$. The grading into chiral bundles at the boundary comes from the outward unit conormal; if we set $\sigma := i\,\mathrm{cl}(du)$, then $\sigma^2 = 1$ and $S^\pm X$ are the $\pm 1$ eigenspaces of $\sigma$. Additionally, suppose that we have a complex vector bundle $E$ and an endomorphism $\tau : E|_{\partial X} \to E|_{\partial X}$ grading $E|_{\partial X}$ into subbundles $E^{\pm}$, not necessarily of equal rank. The Dirac operator for $SX \otimes E$ is formally self-adjoint. We let $D^+$ be the operator $D$ with the boundary condition $\psi|_{\partial X} = (\sigma \otimes \tau)\psi|_{\partial X}$, and set $D^-$ to be $D$ with the adjoint boundary condition. These boundary conditions were first considered by Råde in [25].

In Chapter 2, we achieve our first main goal, to show that a determinant line bundle for a family of Dirac operators with geometric local boundary conditions has a well-defined determinant line bundle with Quillen metric and Bismut-Freed connection. In particular, this is true for the two classes of boundary value problems we study here. We begin this chapter with the Riemannian geometry of a family of manifolds with boundary. We introduce family connections, the $L_2$ metric and connection, and integration along the fiber. While this material is by now standard, the presence of the boundary does introduce some new difficulties. In particular, in order for the $L_2$ connection to have a parallel transport, and in order for the boundary family to have an induced family connection, the family connection must have a special property near the boundary: it must be *parallel to the boundary.* This means that the horizontal lift of a vector in the base to a point in the boundary of the family lies in the tangent space of the boundary. We use the $L_2$ connection of the boundary family to define our class of geometric local boundary conditions. We formally introduce the even and odd problems at this point and show that they are local elliptic boundary conditions additionally satisfying the geometric condition. We go on to define the determinant line bundle, its Quillen metric, and its Bismut-Freed connection. The definitions are exactly the same as in the closed case. Although there is essentially no difficulty in making these definitions, in order to show compatibility of the Bismut-Freed connection with the Quillen metric we need the boundary conditions to be of the special class of geometric local boundary conditions we define here.

In Chapter 3 we relate the determinant line bundle for the even problem to the determinant line bundle of a related family of twisted Dirac operators on the double. Our main tools are *cutting maps* which relate smooth sections over the double to smooth sections satisfying the boundary condition on the original family. Properly constructed, the cutting maps are unitary and flat with respect to the $L_2$ metric and connection. Thus we are able to provide a flat isometry of the associated determinant lines by directly relating the defining data.

The main result for the even problem is:

**Theorem 1.0.1.** *To a family of elliptic boundary value problems of this type, there is an associated determinant line bundle $\mathcal{L}(D^+, \mathfrak{B})$. The zeta-regularized metric and connection are well-defined and compatible. Let $\mathsf{D}_d$ denote the family of chiral Dirac operators coupled to $E_d$ for $M^d$. There is a natural flat isometry*

$$\mathcal{L}(D^+, \mathfrak{B}) \cong \mathcal{L}(\mathsf{D}_d) \tag{1.0.1}$$

*This isometry identifies the canonical sections* $\det(D^+, \mathfrak{B})$ *and* $\det(\mathsf{D}_d)$.

Since the determinant line bundle associated to a family of compact closed even-dimensional manifolds is well-understood, this theorem has as a corollary formulas for the curvature and holonomy of the determinant line bundle. Furthermore, since the isometry identifies the canonical sections, we obtain a formula for the covariant derivative of the canonical section.

We also give a families index theorem for the even problem. This is

**Theorem 1.0.2.** *Let $\mathsf{D}_d$ be the family of chiral Dirac operators coupled to $E_d$ for $M^d$. We have*

$$\operatorname{Ind}(D^+, \mathfrak{B}) = \operatorname{Ind}(\mathsf{D}_d) \tag{1.0.2}$$

Again, we use the cutting maps in an essential way to prove this result.

In Chapter 4, we turn to the odd problem. After describing the odd problem in detail, we describe some special cases where we are able to find an explicit flat isometry of the determinant line bundle with the determinant line bundle of the boundary family. These special cases are families of closed odd-dimensional manifolds, doubles, and cylinders. Unfortunately, we are unable to find such flat isometries in general. In the following section, we use the Bunke isometry to relate the *square* of the general problem to these special cases. The Bunke isometry is an $L_2$ isometry from spaces of sections over one family of manifolds with boundary to another family with the same boundary first used by Bunke in [9] and later by Dai and Freed in [10]. This map preserves boundary conditions and maps smooth sections to smooth sections. Unfortunately, the pullback of the Dirac operator by the Bunke isometry is not a differential operator or even a pseudodifferential operator. In fact, this pullback operator differs from the Dirac operator on the second family by a non-local endomorphism supported on the interior. Although this problem turns out to be fairly easy to handle, it does take us beyond the scope of classical theorems on the heat kernels of Laplace-type operators with local elliptic boundary conditions. We give in section A.4 a construction of heat operators for non-local perturbations of Laplace-type operators with local boundary conditions. We show that the Bunke isometry, properly constructed, induces a flat isometry of the determinant lines bundle of the Dirac operator and the pullback Dirac operator. Regarding the non-local endomorphism as a perturbation, we make a straight line homotopy of the perturbed Dirac operator to the unperturbed Dirac operator and treat this homotopy as a new family of operators over the homotopy space. Restricted to the two ends of the homotopy, the determinant line bundle of this homotopy family is isomorphic to the determinant line bundles of the perturbed and unperturbed Dirac operators. Parallel transport in the homotopy direction therefore gives an isometry of the two determinant line bundles. To see that this isometry is actually flat, we make a curvature calculation. We learned this technique from Piazza in [22]. Unfortunately, the construction of the Bunke isometry involves a choice and it is not clear at this point that the constructed flat isometry is canonical. In the end, we show that

**Theorem 1.0.3.** *Let* $D^+_{\partial M}(E^+): C^\infty(S^+_{\partial M} \otimes E^+) \to C^\infty(S^-_{\partial M} \otimes E^+)$ *be the chiral Dirac operator for the boundary family coupled to* $E$ *and* $D^+_{\partial M}(E^-)$ *the* *chiral Dirac operator coupled to $E^-$. There is a flat isometry*

$$\mathcal{L}(D^+, \mathfrak{B})^2 \cong \mathcal{L}(D^+_{\partial M}(E^+)) \otimes \mathcal{L}(D^+_{\partial M}(E^-))^{-1} \tag{1.0.3}$$

The families index theorem for the odd problem is:

**Theorem 1.0.4.** *We have*

$$\operatorname{Ind}(D^+) = \operatorname{Ind}(D^+_{\partial M}(E^+)) = -\operatorname{Ind}(D^+_{\partial M}(E^-)) \tag{1.0.4}$$

This generalizes Theorem B of [11], which treats the special case where $E$ is a trivial complex line bundle and at each boundary component of a fiber either $E^+$ or $E^-$ is $\{0\}$.

# Chapter 2

# Determinant line bundles

In this section we construct determinant line bundles for families of non-local perturbations of Dirac-type operators with local elliptic boundary conditions. We also define the Quillen metric and Bismut-Freed connection for the determinant line [6]. The presentation closely follows [5], where these constructions are done for families of Dirac-type operators on a manifold without boundary.

## 2.1 Riemannian fiber bundles

### 2.1.1 Basic definitions

Suppose $B$ is a compact manifold with boundary and $\pi : M \to B$ is a fiber bundle, with fibers diffeomorphic to $X$, a compact manifold with boundary. Thus the total space of $M$ may be a manifold with corners.

The vertical tangent bundle $T(M/B)$ is the kernel of $\pi_*$. Let $(\iota_{M/B})_*$ denote the inclusion of $T(M/B)$ in $TM$. Let $\Lambda(M/B) := \Lambda T^*(M/B)$ be the exterior algebra of the dual vector space, and let

$$\iota^*_{M/B} : \Lambda M \to \Lambda(M/B)$$

be the map dual to $(\iota_{M/B})_*$. Forms in the kernel of $\iota^*_{M/B}$ and all 0-forms are said to be horizontal. The horizontal forms are a subbundle of $\Lambda M$ naturally isomorphic to $\pi^{-1}\Lambda B$.

**Definition 2.1.1.** A family connection for $M$ is a projection $H : TM \to T(M/B)$ splitting the exact sequence

$$0 \to T(M/B) \xrightarrow{(\iota_{M/B})_*} TM \xrightarrow{\pi_*} TB \to 0$$

A form in $\Lambda M$ is said to be vertical if it is in the image of the dual map

$$H^* :\ \Lambda(M/B) \to \Lambda M$$

Thus 0-forms are both horizontal and vertical. In fact, we have

$$\pi^* \otimes H^* :\ \pi^{-1}\Lambda B \otimes \Lambda(M/B) \xrightarrow{\sim} \Lambda M$$

Locally, then, a homogeneous form in $\Lambda M$ can be uniquely identified with the tensor product of a form in $\Lambda B$ and a form in $\Lambda(M/B)$, and all homogeneous forms have a well-defined horizontal and vertical degree, the sum of which is their degree.

The family connection determines a bundle of horizontal vectors $T^H M$, called the horizontal tangent bundle. A vector in $T_{\pi(p)}B$ lifts to a horizontal vector at $p$ by inverting the isomorphism

$$(\pi_*)_p :\ T^H_p M \to T_{\pi(p)}B$$

If $\xi$ is a vector in the base, we denote its horizontal lift by $\xi^H$. More generally, smooth vector fields lift to smooth vector fields.

**Definition 2.1.2.** The boundary family of $M$ is

$$\partial^\pi M := \{p \in M \mid p \in \partial X_{\pi(p)}\}$$

Let $\iota_{\partial^\pi M}$ denote the inclusion of the boundary family, and let $\partial\pi := \pi \circ \iota_{\partial^\pi M}$; clearly $\partial\pi : \partial^\pi M \to B$ is a family of manifolds over $B$ with fibers diffeomorphic to $\partial X$.

**Definition 2.1.3.** The extrinsic conormal bundle is the kernel of $\iota^*_{\partial^\pi M} : T^*M \to T^*\partial^\pi M$. The intrinsic conormal bundle is the kernel of $\iota^*_{\partial^\pi M/B} : T^*(M/B) \to T^*(\partial^\pi M/B)$.

The extrinsic conormal bundle is clearly rank 1. Differentials of boundary defining functions orient the extrinsic conormal bundle: $f \in C^\infty(M)$ is a boundary defining function if $f \neq 0$ off $\partial^\pi M$, $f = 0$ and $df \neq 0$ on $\partial^\pi M$. We say $df$ is *inward* if $f > 0$ off $\partial^\pi M$, *outward* otherwise. We always take the outward orientation to be positive. A simple diagram chase shows that the intrinsic conormal bundle is the isomorphic image of the extrinsic conormal bundle under the restriction map. Thus the intrinsic bundle is rank 1 and oriented.

**Proposition 2.1.1.** *The following are equivalent:*

(i) *The horizontal lift of a vector field $\xi$ on $B$ has no normal component, i.e., $\omega(\xi^H) = 0$ for any section $\omega$ of the extrinsic conormal bundle.*

(ii) *The extrinsic conormal bundle is vertical.*

(iii) *The extrinsic conormal bundle is the image of the intrinsic conormal bundle under* $H^*$.

(iv) *The image of* $H$ *on* $T\partial^\pi M$ *is contained in* $T(\partial^\pi M/B)$.

(v) *There is a uniquely determined connection for the boundary family,* $H^\partial$, *defined by the condition:*

$$H(\iota_{\partial^\pi M})_* = (\iota_{\partial^\pi M/B})_* H^\partial$$

(vi) *Relative to a choice of splitting for* $(\iota_{\partial^\pi M/B})_*$, *the family connection* $H$ *on* $TM|\partial^\pi M$ *has the form*

$$H = (\iota_{\partial^\pi M/B})_* H^\partial P + X \otimes \nu$$

*where* $P : TM \to T\partial^\pi M$ *is the induced splitting for* $(\iota_{\partial^\pi M})_*$, $X$ *is a non-vanishing section of the kernel of* $P$ *and* $\nu$ *is a non-vanishing section of the conormal bundle such that* $\nu(X) = 1$.

We omit the lengthy but elementary proof.

**Definition 2.1.4.** When any of the conditions above are satisfied, we say that the family connection is parallel to the boundary.

*Remark* 2.1.1. We see from part (i) of Prop. 2.1.1 that when the family connection is parallel to the boundary, the horizontal lift of a vector field generates a flow that maps the boundary family to itself. Thus the family connection has a well-defined parallel transport. Part (v) shows that the Riemannian structure of $M$ induces a Riemannian structure on $\partial^\pi M$.

The intrinsic outward unit conormal $\nu_{M/B}$ is the section of the intrinsic conormal bundle of unit length oriented outward; $\frac{\partial}{\partial r}$ is its metric dual. The unit outward conormal $\nu$ is $H^* \nu_{M/B}$. By part (iii) of Prop. 2.1.1, $\nu$ is a section of the extrinsic conormal bundle. Let $P^{\partial^\pi M}$ denote the splitting of $(\iota_{\partial^\pi M})_*$ with kernel spanned by $\frac{\partial}{\partial r}$. By part (vi) of Prop. 2.1.1, we have

$$H = (\iota_{\partial^\pi M/B})_* H^\partial P^{\partial^\pi M} + \frac{\partial}{\partial r} \otimes \nu \tag{2.1.1}$$

along $\partial^\pi M$.

**Theorem 2.1.2.** *(Collar neighborhood theorem for families) There is a neighborhood $U$ of $\partial^\pi M$ and an embedding*

$$\phi : (-1, 0] \times \partial^\pi M \to U$$

*such that $\phi_1 := \mathrm{pr}_1 \circ \phi^{-1}$ is an outward boundary-defining function, $\phi|_{\{0\} \times \partial^\pi M}$ is the standard inclusion of $\partial^\pi M$ in $M$, and $\pi \circ \phi = \partial\pi \circ \mathrm{pr}_2$.*

*Proof.* This is a consequence of the collar neighborhood theorem, which guarantees the existence of a neighborhood $V$ of $\partial^\pi M$ and an embedding

$$\tilde{\phi} : (-1, 0] \times \partial^\pi M \to V$$

with all the properties asked of $\phi$ above, except that the factor $(-1, 0]$ is not necessarily vertical with respect to $\pi \circ \tilde{\phi}$. (This property is equivalent to the factorization $\pi \circ \phi = \partial\pi \circ \mathrm{pr}_2$.)

The tangent space of $(-1, 0] \times \partial^\pi M$ is naturally isomorphic to $\mathrm{pr}_1^{-1} T(-1, 0] \oplus \mathrm{pr}_2^{-1} T\partial^\pi M$. Along $\{0\} \times \partial^\pi M$, $\pi_* \tilde{\phi}_*$ is a surjection from $\mathrm{pr}_2^{-1} T\partial^\pi M$ to $TB$, because on this set $\tilde{\phi}$ is the standard inclusion of the boundary family, and $\pi$ is a submersion when restricted to the boundary family. This is an open condition, thus there is a neighborhood $\tilde{U}$ of $\{0\} \times \partial^\pi M$ where this condition holds. Let $U := \phi(\tilde{U})$, and let

$$\tilde{L} : (\pi \circ \tilde{\phi})^{-1} TB \to \mathrm{pr}_2^{-1} T\partial^\pi M$$

be a splitting over $\tilde{U}$. Let $u$ denote the standard coordinate on $(-1, 0]$, thought of as a function on $(-1, 0] \times \partial^\pi M$. By construction, $\tilde{X} := \frac{\partial}{\partial u} - \tilde{L}[\pi_* \tilde{\phi}_* \frac{\partial}{\partial u}]$ is vertical with respect to the projection $\pi \circ \tilde{\phi}$; it follows that $X := \phi_* \tilde{X}$ is vertical with respect to $\pi$. Furthermore, $X\tilde{\phi}_1 = \tilde{X}u = 1$. This implies that all flow lines of $X$ reach $\partial^\pi M$ in finite time (since $\tilde{\phi}_1$ is bounded above by 0 and attains this maximum exactly at $\partial^\pi M$.) Let $\phi(t, x)$ for $t \leq 0$, $x \in \partial^\pi M$ be the point in $U$ obtained by flowing $x$ by $-X$ for time $|t|$. Renormalizing $X$ if necessary, we may assume that all such flows exist for $-1 < t \leq 0$. □

**Definition 2.1.5.** A family boundary collaring is an embedding $\phi$ with all the properties given in the statement of Theorem 2.1.2.

**Corollary 2.1.3.** *A family connection parallel to the boundary exists for all smooth families of manifolds with boundary.*

*Proof.* Since family connections patch nicely, we need only make the construction near the boundary, i.e., if $U$ and $\phi$ are as in the theorem, it suffices to construct a family connection parallel to the boundary for $U$.

Choose a family connection $H^\partial$ for $\partial^\pi M$. Then $\tilde{H} := H^\partial \circ (\mathrm{pr}_2)_* + \frac{\partial}{\partial u} \otimes \nu$ is a family connection for $(-1,0] \times \partial^\pi M$ with respect to $\partial\pi \circ \mathrm{pr}_2 = \pi \circ \phi$. Then $H := \phi_* \tilde{H} \phi_*^{-1}$ is the desired family connection. $\square$

**Definition 2.1.6.** A metric $g^{M/B}$ is a product near the boundary if there is a family boundary collaring $\phi$ and a vertical metric $\tilde{g}$ for the family projection

$$\partial\pi \circ \mathrm{pr}_2 : \ (-1,0] \times \partial^\pi M \to B$$

such that $\tilde{g} = \phi^* g^{M/B}$ and $\tilde{g}$ is a product metric with respect to the splitting $\mathrm{pr}_1^{-1} T(-1,0] \oplus \mathrm{pr}_2^{-1} T\partial^\pi M$ of the tangent bundle.

**Definition 2.1.7.** The geometry of a family of manifolds with boundary is a product near the boundary if the metric is a product near the boundary and relative to the family boundary collaring the family connection is a product.

From this point on we shall only consider families whose geometry is a product near the boundary.

The volume form $\mathrm{vol}_{M/B}$ of $M/B$ is the top-degree section of $\Lambda(M/B)$ given locally as

$$\mathrm{vol}_{M/B}(p) = e^1 \wedge \cdots \wedge e^n$$

where $\{e^1, \ldots, e^n\}$ is an oriented orthonormal basis of $T_p^*(M/B)$. The divergence of the the volume form is the horizontal 1-form $k_{M/B}$ defined implicitly by:

$$d_M H^* \mathrm{vol}_{M/B} = k_{M/B} \wedge H^* \mathrm{vol}_{M/B} + \text{terms of lower vertical degree}$$

The definition makes it clear that $k_{M/B}$ depends only on local data for the metric on $M/B$ and the family connection.

We induce an orientation on $T(\partial^\pi M/B)$ using the outward normal first convention: a frame $\{e_1, \ldots, e_{n-1}\}$ for $T(\partial^\pi M/B)$ is oriented if $\{\frac{\partial}{\partial r}, e_1, \ldots e_{n-1}\}$ is oriented for $T(M/B)$. Define $\mathrm{vol}_{\partial^\pi M/B}$ and $k_{\partial^\pi M/B}$ as above using the induced orientation, metric, and family connection on $\partial^\pi M$.

**Proposition 2.1.4.** *Along $\partial^\pi M$ we have $k_{M/B} = (P^{\partial^\pi M})^* k_{\partial^\pi M/B}$.*

*Proof.* Since the metric is a product near the boundary, we see that along $\partial^\pi M$, $\mathrm{vol}_{M/B} = \nu_{M/B} \wedge (P^{\partial^\pi M})^* \mathrm{vol}_{\partial^\pi M/B}$, thus

$$H^* \mathrm{vol}_{M/B} = \nu \wedge H^*(P^{\partial^\pi M})^* \mathrm{vol}_{\partial^\pi M/B}$$

Using the form of the family connection near the boundary (Eq. 2.1.1), we see that $H^*(P^{\partial^\pi M})^* = (P^{\partial^\pi M})^*(H^\partial)^*$. Thus

$$H^* \mathrm{vol}_{M/B} = \nu \wedge (P^{\partial^\pi M})^*(H^\partial)^* \mathrm{vol}_{\partial^\pi M/B}$$

Let $\phi : (-1, 0] \times \partial^\pi M \to U$ be a family boundary collaring that induces the metric. We have $(\phi^{-1})^*\nu = du$, $(\phi^{-1})^*(P^{\partial^\pi M})^* = \mathrm{pr}_2^*$, and

$$(\phi^{-1})^* H^* \mathrm{vol}_{M/B} = du \wedge \mathrm{pr}_2^*(H^\partial)^* \mathrm{vol}_{\partial^\pi M/B}$$

Thus

$$(\phi^{-1})^* dH^* \mathrm{vol}_{M/B} = (-1) du \wedge \mathrm{pr}_2^* d(H^\partial)^* \mathrm{vol}_{\partial^\pi M/B}$$

Identifying parts of equal vertical degree, we have

$$(\phi^{-1})^* k_{M/B} \wedge H^* \operatorname{vol}_{M/B} = \operatorname{pr}_2^* k_{\partial^\pi M/B} \wedge du \wedge \operatorname{pr}_2^* (H^\partial)^* \operatorname{vol}_{\partial^\pi M/B}$$

Applying $\phi^*$ to both sides, we have the proposition. □

#### 2.1.1.1 The Levi-Civita connection

The vertical tangent bundle of a Riemannian family has a natural connection, called the Levi-Civita connection, defined as follows. Choose a Riemannian metric for $B$. This induces a metric on the horizontal tangent bundle via $\pi_*$, and then $TM$ can be given the direct sum metric. If $\nabla$ is the Levi-Civita connection for this metric,

$$\nabla^{M/B} := H \nabla H$$

is a connection on the vertical tangent bundle, called the Levi-Civita connection for $M/B$.

**Proposition 2.1.5.** *(Prop 10.2 of [5]) The connection $\nabla^{M/B}$ is independent of the choice of metric on $B$.*

The Levi-Civita connection induces a connection on the relative cotangent bundle $T^*(M/B)$ by duality:

$$\langle \nabla_X^{M/B} \omega \mid Y \rangle := X \langle \omega \mid Y \rangle - \langle \omega \mid \nabla_X^{M/B} Y \rangle$$

**Proposition 2.1.6.** *If the metric is a product near the boundary, the unit outward conormal section is parallel along $T^H M | \partial^\pi M$. That is,*

$$\nabla_{X^H}^{M/B} \nu_{M/B} = 0$$

*for any vector field $X$ on $B$.*

*Proof.* If $Z$ is a section of $T(M/B)$, it's easy to check that

$$\langle \nabla^{M/B}_{X^H} \nu_{M/B} \mid Z \rangle = \langle \nabla^{M/B}_{X^H} \frac{\partial}{\partial r}, Z \rangle$$

Applying the definition of the Levi-Civita connection, we have:

$$\begin{aligned} 2\langle \nabla^{M/B}_{X^H} \frac{\partial}{\partial r}, Z \rangle = \langle H[X^H, \frac{\partial}{\partial r}], Z \rangle - \langle [\frac{\partial}{\partial r}, Z], X^H \rangle + \langle H[Z, X^H], \frac{\partial}{\partial r} \rangle \\ + X^H \langle \frac{\partial}{\partial r}, Z \rangle + \frac{\partial}{\partial r} \langle Z, X^H \rangle - Z \langle X^H, \frac{\partial}{\partial r} \rangle \end{aligned}$$

If $Z = \frac{\partial}{\partial r}$, the right hand side is zero. If $Z$ is orthogonal to $\frac{\partial}{\partial r}$, then it is tangent to $\partial^\pi M$. Since the connection is parallel to the boundary, $X^H$ is also parallel to the boundary, thus $[Z, X^H]$ is tangent to the boundary, and $\langle H[Z, X^H], \frac{\partial}{\partial r} \rangle = 0$. Therefore we have $2\langle \nabla^{M/B}_{X^H} \frac{\partial}{\partial r}, Z \rangle = \langle H[X^H, \frac{\partial}{\partial r}], Z \rangle$.

We claim $[X^H, \frac{\partial}{\partial r}] = 0$. Let $\phi$ be a family boundary collaring that induces the metric near the boundary. Since the metric is induced by the product metric on $(-1, 0] \times \partial^\pi M$, we have $(\phi^{-1})_* \frac{\partial}{\partial r} = \frac{\partial}{\partial u}$. Thus

$$(\phi^{-1})_* [X^H, \frac{\partial}{\partial r}] = [(\phi^{-1})_* X^H, (\phi^{-1})_* \frac{\partial}{\partial r}] = [(\phi^{-1})_* X^H, \frac{\partial}{\partial u}] = 0$$

Since $\phi^{-1}$ is an isometry, we see $[X^H, \frac{\partial}{\partial r}] = 0$. □

### 2.1.2 Smooth families of vector bundles

By definition, a smooth family of vector bundles on $M$ is a smooth vector bundle $V \to M$. By $\pi_* V$ we mean the bundle of smooth sections along the fiber of $V$, that is, at $b \in B$, $(\pi_* V)_b$ is $C^\infty(X_b; V|X_b)$. [1] We give $\pi_* V$ a smooth structure by defining a section of $\pi_* V$ to be smooth if and only if it is given by a smooth section of $V$.

Now suppose $M/B$ has an orientation as well as a Riemannian structure, and $V$ has a metric and compatible connection. Then we can define a corresponding metric and compatible connection on $\pi_* V$, called the $L_2$-metric and connection. In order to define the $L_2$-metric and connection, we first summarize some facts about the fiber integral.

#### 2.1.2.1 The fiber integral

The fiber integral is a linear map from $k$-forms on $M$ to $(k-n)$-forms on $B$ defined by the condition

$$\int_B \left( \int_{M/B} \alpha \right) \wedge \beta = \int_M \alpha \wedge \pi^* \beta \tag{2.1.2}$$

for all differential forms $\alpha$ on $M$ and for forms $\beta$ on $B$ compactly supported on a coordinate chart. Since the fibers are compact this is well-defined. Note that it is *not* necessary to orient $B$ or $M$ in order for this definition to make sense. Since $M/B$ is oriented, a (local) choice of orientation for $B$ induces a (local) orientation for $M$ via the isomorphism $TM \cong T(M/B) \oplus T^H M$. Reversing the orientation of $T^H M$ reverses the orientation of $TM$, introducing a minus sign on both sides of (2.1.2). Thus the definition is invariant.

[1]Note: the same notation has a slightly different meaning in [5].

**Proposition 2.1.7.** *Suppose* $\pi : M \to B$ *is a family of compact manifolds with boundary, and* $\phi : B' \to B$ *is a smooth map. If* $M' = \phi^{-1}M$ *is the pullback family and* $\hat{\phi} : M' \to M$ *is the natural map lifting* $\phi$*, then*

$$\int_{M'/B'} \hat{\phi}^* \alpha = \phi^* \int_{M/B} \alpha$$

*where* $\hat{\phi}$ *induces the orientation on* $M'/B'$*.*

*Proof.* The first observation is that since every smooth map can be factored as an immersion followed by a submersion, we can reduce to the case where $\phi$ is an immersion or submersion.

The second observation is that by using a partition of unity on the base, the fiber integral of $\alpha$ can be assumed to have support on an arbitrary open set $U$ in $B$.

Combining these two observations, and using the local straightening theorem for immersions and submersions, we can assume that $B$ is the unit open ball in $\mathbb{R}^k$, $M$ is $X_0 \times B$, and $\phi : \mathbb{R}^j \to \mathbb{R}^k$ is the canonical immersion ($j \leq k$) or submersion ($j \geq k$). Then $B'$ is the unit open ball in $\mathbb{R}^j$, $M'$ is $X_0 \times B'$, and $\hat{\phi} = \mathrm{id}_{X_0} \times \phi$.

In this situation, integration over the fiber is equivalent to partial Lesbegue integration over the factor $X_0$, and the proposition is clear. □

There is a fibered version of Stokes' Theorem for families of manifolds with boundary:

**Theorem 2.1.8.** *For any differential form $\alpha$ on $M$,*

$$\int_{M/B} d_M\alpha - (-1)^n d_B \int_{M/B} \alpha = \int_{\partial^\pi M/B} \iota^*_{\partial^\pi M}\alpha$$

*where $n$ is the rank of the vertical tangent bundle.*

**Corollary 2.1.9.** *Suppose $H$ is parallel to the boundary and $\alpha$ is a section of $\Lambda^n(M/B)$. Then*

$$\int_{M/B} d_M H^*\alpha = (-1)^n d_B \int_{M/B} H^*\alpha$$

*Proof.* By definition, the family connection is parallel to the boundary if and only if the outward conormal bundle is vertical. Since $\alpha$ is a top form in $\Lambda(M/B)$, $H^*\alpha$ is a top vertical form, and thus necessarily in the ideal generated by the outward conormal bundle.

We claim $\iota^*_{\partial^\pi M} H^*\alpha = 0$. To see this, recall that the outward conormal bundle is the kernel of $\iota^*_{\partial^\pi M}$ on 1-forms; the full kernel therefore contains the ideal in $\Lambda M$ generated by the outward conormal bundle. The corollary then follows easily from Stokes' Theorem. □

More generally, we can define integration along the fiber for forms with values in a pullback bundle. If $V \to B$ is a vector bundle and $\alpha$ is a section of $\Lambda^n(M/B) \otimes \pi^{-1}V$, then along the fiber $X_b$, the form $\alpha$ takes values in the fixed vector space $V_b$.

We will need the following corollary of Stokes' Theorem, which applies to such forms.

**Corollary 2.1.10.** *Suppose $V \to B$ is a vector bundle with connection $\nabla^V$, and $\alpha$ is a section of $\Lambda^n(M/B) \otimes \pi^{-1}V$, where $n$ is the rank of the vertical tangent bundle. Then*

$$\nabla^V \int_{M/B} H^*\alpha = (-1)^n \int_{M/B} \nabla^{\pi^{-1}V} H^*\alpha$$

*Proof.* Using a partition of unity on $B$, we may assume that the support of the integral along the fiber is covered by a trivialization of $V$. Therefore without loss of generality we assume that $V$ is trivial and $\nabla^V = d_B + \omega$. Then $\pi^{-1}V$ is also trivial and $\nabla^{\pi^{-1}V} = d_M + \pi^*\omega$. We have

$$\begin{aligned}\nabla^V \int_{M/B} H^*\alpha &= (d_B + \omega) \int_{M/B} \alpha \\ &= (-1)^n \int_{M/B} (d_M + \pi^*\omega)\alpha \\ &= (-1)^n \int_{M/B} \nabla^{\pi^{-1}V} H^*\alpha\end{aligned}$$

Where we have used Coro. 2.1.9 to Stokes' Theorem and the definition of integration along the fiber. □

Using the family connection, $\mathrm{vol}_{M/B}$ can be pulled back to $T^*M$, and then integrated along the fiber against functions.

**Proposition 2.1.11.** *Suppose $f$ is a smooth function on $M$. Then*

$$\int_{X_b} \mathrm{vol}_{X_b}\, f \; = \left[\int_{M/B} H^* \mathrm{vol}_{M/B}\, f\right]_b \tag{2.1.3}$$

*Proof.* From the definition above, it's easy to see that $\mathrm{vol}_{X_b} = \hat{\iota}_b^* H^* \mathrm{vol}_{M/B}$, where $\hat{\iota}_b$ is the inclusion of the fiber $X_b$. Therefore this is a special case of Prop 2.1.7, where $\phi$ is the inclusion $\iota_b$ of $b$ in $B$. □

### 2.1.3 The $L_2$ metric and connection

The $L_2$-metric on two sections $f$ and $g$ of $V$ is

$$\langle f, g\rangle_{L_2} := \int_{M/B} H^* \operatorname{vol}_{M/B} \langle f, g\rangle \tag{2.1.4}$$

Note that the same formula makes sense if $f$ or $g$ is a section of $\Lambda(M) \otimes V$ if we extend the pairing as usual to be bilinear (hermetian if $V$ is complex) for $\Lambda(M)$. We use this extended definition of the $L_2$-metric.

*Remark* 2.1.2. According to Prop. 2.1.11, this is just the usual $L_2$ pairing applied at each fiber. Consequently, we see that this definition is actually independent of the family connection $H$. This can also be seen directly: if $H_1$ and $H_2$ are two connections for $M$, the form $(H_1 - H_2)^* \operatorname{vol}_{M/B}$ is a horizontal form (i.e., it vanishes on vertical vectors.) Hence the integral along the fiber vanishes.

Defining a connection on $\pi_* V$ compatible with the $L_2$ metric is slightly more involved. The obvious thing to do is use the "horizontal part" of the connection on $V$:

$$\nabla^{\pi_* V}_{\xi} s := \nabla^{V}_{\xi^H} s$$

However, this connection is not in general compatible with the $L_2$-metric on $\pi_* V$. The difficulty is that the volume form varies from fiber to fiber, and the connection must account for this change.

**Definition 2.1.8.** The $L_2$-connection for $\pi_* V$ is

$$\nabla^{L_2(V)} := \nabla^{\pi_* V} + \frac{1}{2} k_{M/B}$$

**Proposition 2.1.12.** *The $L_2$-connection is compatible with the $L_2$-metric:*

$$d_B\langle f, g\rangle_{L_2} = \langle \nabla^{L_2} f, g\rangle_{L_2} + \langle f, \nabla^{L_2} g\rangle_{L_2}$$

*Proof.* Applying Coro. 2.1.9, we have:

$$\begin{aligned} d_B\langle f, g\rangle_{L_2} &= (-1)^n \int_{M/B} d_M \left\{ H^* \operatorname{vol}_{M/B} \langle f, g\rangle \right\} \\ &= \int_{M/B} \left\{ (-1)^n d_M H^* \operatorname{vol}_{M/B} \right\} \langle f, g\rangle \\ &\qquad + \int_{M/B} H^* \operatorname{vol}_{M/B} \wedge \left\{ \langle \nabla^V f, g\rangle + \langle f, \nabla^V g\rangle \right\} \\ &= \langle \nabla^{L_2} f, g\rangle_{L_2} + \langle f, \nabla^{L_2} g\rangle_{L_2} \end{aligned}$$

The terms of vertical degree less than $n$ in $d_M H^* \operatorname{vol}_{M/B}$ contribute nothing to the fiber integral. In the last line of the calculation, we use the fact that if $\omega$ is any form on $M$,

$$H^* \operatorname{vol}_{M/B} \wedge \omega = H^* \operatorname{vol}_{M/B} \wedge (\omega - H^* \omega)$$

Specifically, we apply this with $\omega = \left\{ \langle \nabla^V f, g\rangle + \langle f, \nabla^V g\rangle \right\}$. □

Restricting to the boundary, $V|\partial^\pi M$ is a smooth family of vector bundles over $\partial^\pi M$ with metric and connection induced from $V$. Let $\partial\pi_* V$ denote the Frechet bundle of smooth sections of $V|\partial^\pi M$. Since $\partial^\pi M$ has an induced family connection, relative orientation, and metric, $\partial\pi_* V$ has an induced $L_2$ metric and connection. As one might expect, the restriction map $\pi_* V \to \partial\pi_* V$ relates the $L_2$ connection on $\pi_* V$ to the $L_2$ connection on $\partial\pi_* V$ when the metric is a product near the boundary.

**Proposition 2.1.13.** *Suppose the metric is a product near the boundary. Then for all vector fields $X$ on base, we have*

$$\nabla_X^{L_2}(s_{|\partial^\pi M}) = (\nabla_X^{L_2} s)_{|\partial^\pi M}$$

*Proof.* Since the family connection is parallel to the boundary, $X^H|\partial^\pi M = (\iota_{\partial^\pi M})_* X^{H^\partial}$. By definition of the induced connection on the boundary vector bundle, we have

$$\nabla_{X^{H^\partial}}^{V|\partial^\pi M}(s_{|\partial^\pi M}) = (\nabla_{X^H}^V s)_{|\partial^\pi M}$$

By Prop. 2.1.4, $k_{M/B} = (P^{\partial^\pi M})^* k_{\partial^\pi M/B}$, thus

$$\begin{aligned} k_{M/B} X^H &= [(P^{\partial^\pi M})^* k_{\partial^\pi M/B}](\iota_{\partial^\pi M})_* X^{H^\partial} \\ &= k_{\partial^\pi M/B}[P^{\partial^\pi M}(\iota_{\partial^\pi M})_*] X^{H^\partial} \\ &= k_{\partial^\pi M/B} X^{H^\partial} \end{aligned}$$

□

### 2.1.4 Smooth families of operators

We define smooth families of differential and smoothing operators for a family of vector bundles $\pi_* V$.

**Definition 2.1.9.** A smooth family of vertical differential operators on $\pi_* V$ is a differential operator on $V$ that commutes with the multiplicative action of $C^\infty(B)$ on $C^\infty(V)$. That is $Q$ is vertical if and only if

$$Q\pi^* f\phi = \pi^* f Q\phi$$

for all smooth functions $f$ on $B$.

Using an approximate identity on $B$ and the local triviality of $M$, we see that a family of vertical differential operators has a well-defined restriction to sections defined over a particular fiber, and this restriction is a differential operator along the fiber.

Let $M \times_\pi M$ be the fiber product $\{(m_1, m_2) \in M^2 \mid \pi(m_1) = \pi(m_2)\}$; $M \times_\pi M$ is naturally a family over $M$ in two different ways, corresponding to $\mathrm{pr}_1, \mathrm{pr}_2 : M^2 \to M$. In keeping with our previous notation, we define

$$T(M \times_\pi M/M_1) := \ker(\mathrm{pr}_1)_*$$

The fiber $\mathrm{pr}_1^{-1}(m_1)$ of $M \times_\pi M$ is naturally isomorphic to the fiber $X_{\pi(m_1)}$ of $M$ via $\mathrm{pr}_2$. Thus, we have

$$(\mathrm{pr}_2)_* : T(M \times_\pi M/M_1) \xrightarrow{\sim} \mathrm{pr}_2^{-1} T(M/B)$$

This isomorphism induces a metric on $T(M \times_\pi M/M_1)$. There is also an induced family connection, given by

$$H_1 := (\mathrm{pr}_2)_*^{-1} H (\mathrm{pr}_2)_*$$

If $V$ and $W$ are vector bundles over $M$, define

$$V \boxtimes W := \mathrm{pr}_1^{-1} V \otimes \mathrm{pr}_2^{-1} W$$

A smooth section $k$ of $W \boxtimes (V^* \otimes \Lambda^n M/B)$ defines a family of smoothing operators $K : \pi_* V \to \pi_* W$ by integration along the fiber: if $f$ is a smooth section of $\pi_* V$, we have

$$Kf(x) := \int_{\mathrm{pr}_1} \langle H_1^* k(x, y) \mid \mathrm{pr}_2^* f(y) \rangle$$

where we identify $\mathrm{pr}_2^{-1}\Lambda(M/B)$ with $\Lambda(M \times_\pi M/M_1)$ in the natural way.

A family of operators $K$ defined in this way is called a smooth family of smoothing operators; we call $k$ the kernel of the family. Note that although the $W \boxtimes V^*$-valued form $H_1^* k(x,y)$ depends on the choice of $H$, Prop. 2.1.7 implies that the value of $Kf$ depends only on the kernel (just as the $L_2$ metric is independent of the family connection.)

**Proposition 2.1.14.** *Suppose $K : \pi_* V \to \pi_* W$ is a smooth family of smoothing operators with kernel $k$. Then $\nabla^W \circ K - K \circ \nabla^V$ is a smooth family of smoothing operators with kernel $(-1)^n \nabla^{W \boxtimes V^*} H_1^* k(x,y)$.*

*Proof.* Let $f$ be a smooth section of $\pi_* V$. From Coro. 2.1.10

$$\begin{aligned}\nabla^W \int_{\mathrm{pr}_1} \langle H_1^* k(x,y) \mid \mathrm{pr}_2^* f(y)\rangle &= (-1)^n \int_{\mathrm{pr}_1} \nabla^{\mathrm{pr}_1^{-1} W} \langle H_1^* k(x,y) \mid \mathrm{pr}_2^* f(y)\rangle \\ &= (-1)^n \int_{\mathrm{pr}_1} \langle \nabla^{W \boxtimes V^*} H_1^* k(x,y) \mid \mathrm{pr}_2^* f(y)\rangle \\ &\quad + \int_{\mathrm{pr}_1} \langle H_1^* k(x,y) \mid \nabla^{\mathrm{pr}_2^{-1} V} \mathrm{pr}_2^* f(y)\rangle\end{aligned}$$

Note that $\nabla^{\mathrm{pr}_2^{-1} V} \mathrm{pr}_2^* f = \mathrm{pr}_2^* \nabla^V f$. □

**Corollary 2.1.15.** *The $L_2$ variation of $K$,*

$$\nabla^{L_2} K := \nabla^{L_2(W)} \circ K - K \circ \nabla^{L_2(V)}$$

*is a smooth family of smoothing operators with kernel*

$$(-1)^n \nabla^{W \boxtimes V^*} H_1^* k + \frac{1}{2} k \wedge [\mathrm{pr}_1^* k_{M/B} - \mathrm{pr}_2^* k_{M/B}]$$

*Proof.* In view of Prop. 2.1.14, we only need to calculate the kernel of the difference

$$\left(\nabla^{L_2(W)} \circ K - K \circ \nabla^{L_2(V)}\right) - \left(\nabla^{\pi_* W} \circ K - K \circ \nabla^{\pi_* V}\right)$$
$$= \frac{1}{2}\left[k_{M/B} \circ K - K \circ k_{M/B}\right]$$

A simple calculation shows that the kernel of $k_{M/B} \circ K$ is $k \wedge \mathrm{pr}_1^* k_{M/B}$ and the kernel of $K \circ k_{M/B}$ is $k \wedge \mathrm{pr}_2^* k_{M/B}$. □

#### 2.1.4.1 Trace and integration along the diagonal

The diagonal $M_\triangle$ is the subset of $M \times_\pi M$ where $\mathrm{pr}_1 = \mathrm{pr}_2$. $M_\triangle$ is naturally a Riemannian family over $B$, with family projection $\pi \circ \mathrm{pr}_1|_{M_\triangle} = \pi \circ \mathrm{pr}_2|_{M_\triangle}$ and as a Riemannian family it is naturally isomorphic to $M$ via $\mathrm{pr}_1$ (or $\mathrm{pr}_2$.) Let $\iota_\triangle$ denote the inclusion of $M$ into $M \times_\pi M$ as the diagonal.

If $K$ is a smooth family of smoothing operators on $\pi_* V$ with kernel $k$, the pullback of $k$ by $\iota_\triangle$ is a smooth section of $\pi_*\left(V \otimes V^* \otimes \Lambda M/B\right)$. The trace of $\iota_\triangle^* k$ is a smooth section of $\Lambda M/B$, so that we may take the integral along the fiber of $H^* \operatorname{tr} \iota_\triangle^* k$ to obtain a smooth function on $B$.

**Definition 2.1.10.** The integral trace of $K$ is

$$\operatorname{Tr} K := \int_{M/B} H^* \operatorname{tr} \iota_\triangle^* k$$

*Remark* 2.1.3. The notation is justified by Prop. 2.1.11, which shows that value of the integral over a particular point $b \in B$ is the usual integral over the diagonal of the kernel $k$ restricted to the fiber over $b$. It is well known that this gives the trace of the corresponding smoothing operator.

**Proposition 2.1.16.** *We have*

$$d_B \operatorname{Tr} K = \operatorname{Tr} \nabla^{L_2} K$$

*Proof.* Note that

$$\iota_\triangle^* \frac{1}{2} k \wedge [\mathrm{pr}_1^* k_{M/B} - \mathrm{pr}_2^* k_{M/B}] = 0$$

because $\iota_\triangle^* \mathrm{pr}_1^* = \iota_\triangle^* \mathrm{pr}_2^*$. Using this observation and Coro. 2.1.15, we have

$$\begin{aligned}
\operatorname{Tr} \nabla^{L_2} K &= (-1)^n \int_{M/B} \operatorname{tr} \left[\iota_\triangle^* \nabla^{V \boxtimes V^*} H_1^* k\right] \\
&= (-1)^n \int_{M/B} \operatorname{tr} \left[\nabla^{V \otimes V^*} H^* \iota_\triangle^* k\right] \\
&= (-1)^n \int_{M/B} d_M \operatorname{tr} \left[H^* \iota_\triangle^* k\right] \\
&= d_B \int_{M/B} \operatorname{tr} \left[H^* \iota_\triangle^* k\right] \\
&= d_B \operatorname{Tr} K
\end{aligned}$$

□

### 2.1.5 Parallel transport for $L_2$ connections

Let $\gamma : (-\epsilon, \epsilon) \to B$ be a non-degenerate path. The smoothness of the distribution $T^H M$ implies that the horizontal lift of the tangent vector field $\dot\gamma$ is smooth. Furthermore, $\dot\gamma^H$ is tangent to the boundary, because we have required the family connection to be parallel to the boundary. It follows that the flow generated by $\dot\gamma^H$ exists and is a family of diffeomorphisms $\Phi_\gamma(s) : X_{\gamma(0)} \to X_{\gamma(s)}$, $-\epsilon < s < \epsilon$. Covering this family of diffeomorphisms is parallel transport along $\dot\gamma^H$. This gives us a parallel transport for the $L_2$ connection on $\pi_* V$.

**Definition 2.1.11.** The parallel transport of $\nabla^{L_2(V)}$ over a path $\gamma$ is the family of bounded maps

$$\tau_\gamma(s):\ \pi_* V_{\gamma(0)} \to \pi_* V_{\gamma(s)}$$

given by parallel transport along $\dot{\gamma}^H$ with respect to the connection $\nabla^V + \frac{1}{2}k_{M/B}$.

It's clear from the definition that $\tau_\gamma(0)$ is the identity, and that the maps $\tau_\gamma$ extend to unitary maps of the $L_2$ sections. This map has some other nice properties as well.

**Proposition 2.1.17.** *Restriction to the boundary intertwines the $L_2$ parallel transport map of $\partial\pi_* V$ with the $L_2$ parallel transport map of $\pi_* V$.*

*Proof.* Given an initial point $p$ in $X_{\gamma(0)}$, $\tilde{\gamma}_p(s) := \Phi_\gamma(s)(p)$ is a horizontal lift of $\gamma$. If $p$ is in $\partial^\pi M$, $\tilde{\gamma}_p$ is in $\partial^\pi M$ since the family connection is parallel to the boundary. With this observation, the proposition is an immediate consequence of Prop. 2.1.13. □

We will make use of this proposition when we define geometric local boundary conditions and prove the transgression formula.

## 2.2 The determinant line bundle

### 2.2.1 Clifford modules and Dirac operators

Let $\mathrm{Cliff}(M/B)$ denote the bundle of Clifford algebras associated to $T^*(M/B)$, and let

$$\mathrm{cl}: \Lambda(M/B) \to \mathrm{Cliff}(M/B)$$

denote the homomorphism induced by the natural inclusion of $T^*(M/B)$ in both algebras.

**Definition 2.2.1.** A $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle $S \cong S^+ \oplus S^-$ with hermetian metric and compatible connection is a $\mathrm{Cliff}(M/B)$-module bundle with compatible connection if the Clifford action of a covector is odd and anti-self-adjoint, and

$$[\nabla^S, \mathrm{cl}(\omega)] = \mathrm{cl}(\nabla^{M/B}\omega)$$

Let $S \to M$, $S \cong S^+ \oplus S^-$, be a $\mathrm{Cliff}(M/B)$-module bundle with compatible connection. As in the last section, $\pi_* S$ denotes the bundle of smooth sections of the corresponding smooth family of vector bundles, $\langle \cdot, \cdot \rangle_{L_2(S)}$ denotes its $L_2$ metric, and $\nabla^{L_2(S)}$ its $L_2$ connection.

Let $D_S$ denote the family of Dirac operators of $S$, and let $D_S^{\pm} := D_S | S^{\pm}$. Relative to a local orthonormal basis for $T(M/B)$, $\{e_i\}$, with dual basis $\{e^i\}$,

$$D_S = \sum \mathrm{cl}(e^i) \nabla^S_{e_i}$$

Near the boundary we can use an adapted basis to split $D_S$ into a normal and tangential part: $D_S = \mathrm{cl}(\nu_{M/B})[\nabla^S_{\frac{\partial}{\partial r}} + A]$, where

$$A := -\sum \mathrm{cl}(\nu_{M/B}) \mathrm{cl}(e^i) \nabla^S_{e_i}$$

relative to a local orthonormal frame for $T(\partial^\pi M/B)$. $D_S$ is odd and formally self-adjoint, thus $(D_S^\pm)^* = D_S^\mp$. For the constructions of Chapter 4 we need to consider more general families of operators. Suppose $\mathsf{D}$ is a family of operators that differs from $D_S$ by a smooth family of formally self-adjoint odd non-local differential operators of order 0 supported on the interior; thus $\mathsf{D}$ is odd and formally self-adjoint. (We define non-local differential operators in section A.4.) By definition, $\mathsf{D} = D_S$ near the boundary, so that for this operator as well we have a well-defined tangential operator $A$. We define $\mathsf{D}^\pm := \mathsf{D}|S^\pm$. We have $(\mathsf{D}^\pm)^* = \mathsf{D}^\mp$. The *associated Laplacians* are $\Delta^\pm := \mathsf{D}^\mp\mathsf{D}^\pm$.

### 2.2.2 Geometric local boundary conditions

If the fibers of $M$ have empty boundary, the data given so far is enough to define a determinant line bundle $\mathcal{L}(\mathsf{D}^+) \to B$ with Quillen metric and Bismut-Freed connection, and this metric and connection will be compatible. In general we need to put boundary conditions on $\mathsf{D}^+$ in order to define a determinant line bundle. Furthermore, in order for the metric and connection to be compatible we will require the boundary conditions to be compatible with the geometry of the Riemannian family in a certain sense.

In order to define local elliptic boundary conditions for the Dirac operator we need to introduce some additional machinery. Let $\sigma(\mathsf{D}^+)$ denote the principal symbol of $\mathsf{D}^+$. This by definition is equal to the principal symbol $\sigma(D_S^+)$ of $D_S^+$. From the factorization of $\mathsf{D}^+$ near the boundary,

$\mathsf{D}^+ = \mathrm{cl}(\nu_{M/B})[\nabla^S_{\frac{\partial}{\partial r}} + A]$ we see that

$$\sigma(\mathsf{D}^+) = \mathrm{cl}(\nu_{M/B})[i\nu_{M/B} + \sigma(A)]$$

Note that the ellipticity of $\mathsf{D}^+$ implies $\sigma(A)$ has no imaginary eigenvalues off the zero section of $T^*(M/B)$. For $(y, \xi')$ in the sphere bundle $S^*(\partial^\pi M/B)$, consider the model operator

$$\sigma(\mathsf{D}^+)(0, y; -i\frac{\partial}{\partial t}, \xi') = \mathrm{cl}(\nu_{M/B})[\frac{\partial}{\partial t} + \sigma(A)(y, \xi')]$$

and let $L^\pm(y, \xi')$ denote the vector space of initial values of solutions to

$$\sigma(\mathsf{D}^+)(0, y; -i\frac{\partial}{\partial t}, \xi')u(t) = 0$$

with $\lim_{t\to\pm\infty} u(t) = 0$. In fact, $L^\pm$ are smooth subbundles of $p^{-1}S^+|\partial^\pi M$.

**Definition 2.2.2.** A smooth family of local elliptic boundary conditions for $\mathsf{D}^+$ is a smooth family of self-adjoint projections $\mathfrak{B}^+ : S^+|\partial^\pi M \to S^+|\partial^\pi M$ such that the symbol $b$ of $\mathfrak{B}^+$ satisfies

$$b|_{L^+} : L^+ \to \mathrm{Ran}(b)$$

is an isomorphism.

Integration by parts shows that the adjoint boundary condition for $(\mathsf{D}^+, \mathfrak{B}^+)$ is

$$\mathfrak{B}^- := -\mathrm{cl}(\nu_{M/B})(1 - \mathfrak{B}^+)\mathrm{cl}(\nu_{M/B})$$

This condition is clearly reflexive, i.e., the adjoint boundary condition to $\mathfrak{B}^-$ is $\mathfrak{B}^+$, and $\mathfrak{B}^+$ is given in terms of $\mathfrak{B}^-$ by exchanging $\mathfrak{B}^+$ and $\mathfrak{B}^-$ in the formula above.

**Definition 2.2.3.** A local elliptic boundary condition $\mathfrak{B}^{\pm}$ for $\mathsf{D}^{\pm}$ is a geometric local boundary condition if $[\nabla^{L_2(S|\partial^{\pi} M)}, \mathfrak{B}^{\pm}] = 0$ and the boundary operator $A$ is odd with respect to the decomposition induced by $\mathfrak{B}^{\pm}$ in the sense that

$$A\mathfrak{B}^{\pm} = (1 - \mathfrak{B}^{\pm})A$$

**Proposition 2.2.1.** *If the boundary condition $\mathfrak{B}^{+}$ is a geometric local boundary condition, then $\mathfrak{B}^{-}$ is also.*

*Proof.* An easy computation shows that $A\mathrm{cl}(\nu_{M/B}) = -\mathrm{cl}(\nu_{M/B})A$ and $1 - \mathfrak{B}^{-} = -\mathrm{cl}(\nu_{M/B})\mathfrak{B}^{+}\mathrm{cl}(\nu_{M/B})$. Thus $\mathfrak{B}^{-}$ is odd with respect to $A$.

Prop. 2.1.6 shows that $\nabla^{M/B}_{X^H}\nu_{M/B} = 0$. ∎

A local elliptic boundary condition $\mathfrak{B}^{+}$ for $\mathsf{D}^{+}$ induces the boundary condition

$$\mathfrak{B}^{+}\phi = 0, \quad \mathfrak{B}^{-}\mathsf{D}^{+}\phi = 0 \tag{2.2.1}$$

on $\Delta^{+}$. Similarly, the induced boundary condition for $\Delta^{-}$ is

$$\mathfrak{B}^{-}\phi = 0, \quad \mathfrak{B}^{+}\mathsf{D}^{-}\phi = 0 \tag{2.2.2}$$

We will denote the corresponding boundary value problems as $(\Delta^{\pm}, \mathfrak{B})$.

**Proposition 2.2.2.** *The boundary conditions for $\Delta^{\pm}$ induced by $(\mathsf{D}^{+}, \mathfrak{B}^{+})$ are are equivalent to*

$$\mathfrak{B}^{\pm}\phi = 0, \quad (1 - \mathfrak{B}^{\pm})\nabla^{S}_{\frac{\partial}{\partial r}}\phi = 0$$

*Proof.* Writing $\mathsf{D}^{\pm}$ near the boundary as $\mathrm{cl}(\nu_{M/B})[\nabla^S_{\frac{\partial}{\partial r}} + A]$, and we see that the first order part $\mathfrak{B}^-\mathsf{D}^+$ of the boundary condition for $\Delta^+$ is

$$\mathfrak{B}^-\mathsf{D}^+ = \mathrm{cl}(\nu_{M/B})(1 - \mathfrak{B}^+)[\nabla^S_{\frac{\partial}{\partial r}} + A]$$

Using the fact that $(1 - \mathfrak{B}^+)A = A\mathfrak{B}^+$, and the fact that $\mathrm{cl}(\nu_{M/B})$ is an isomorphism, we see this is equivalent to

$$(1 - \mathfrak{B}^+)\nabla^S_{\frac{\partial}{\partial r}} + A\mathfrak{B}^+$$

Thus the pair of boundary conditions $\mathfrak{B}^+\phi = 0, \mathfrak{B}^-\mathsf{D}^+\phi = 0$ is equivalent to

$$\mathfrak{B}^+\phi = 0, \quad (1 - \mathfrak{B}^+)\nabla^S_{\frac{\partial}{\partial r}}\phi = 0$$

□

**Proposition 2.2.3.** *The boundary conditions for* $\Delta^{\pm}$ *induced by* $(\mathsf{D}^+, \mathfrak{B}^+)$ *are* p-*elliptic boundary conditions.*

*Proof.* (We define $p$-elliptic boundary conditions in section A.3.) We see from the last proposition that the boundary condition is a Dirichlet boundary condition on a subbundle of half-rank, and a Neumann boundary condition on its complement. A boundary condition of this type is known to be $p$-elliptic [4] for a Laplace-type operator. □

**Lemma 2.2.4.** *Suppose* $\gamma : (-\epsilon, \epsilon) \to B$ *is a non-degenerate path and* $\phi$ *is a section of* $S^{\pm}|X_{\gamma(0)}$. *Then* $\phi$ *satisfies the induced boundary conditions on* $\Delta^{\pm}$ *if and only if* $\tau_{\gamma}(z)^{-1}\phi$ *does.*

*Proof.* We first check that at the fiber over $\gamma(0)$,

$$0 = \mathfrak{B}^+_{\gamma(0)} - \tau_\gamma(z)^{-1}\mathfrak{B}^+_{\gamma(z)}\tau_\gamma(z)$$

This is obvious at $z = 0$. Differentiating in $z$, we obtain

$$-\tau_\gamma(z)^{-1}[\nabla^{S|\partial^\pi M}_{\dot\gamma^H} + k_{M/B}(\dot\gamma^H), \mathfrak{B}^+]\tau_\gamma(z) = -\tau_\gamma(z)^{-1}[\nabla^{L_2(S|\partial^\pi M)}_{\dot\gamma^H}, \mathfrak{B}^+]\tau_\gamma(z)$$

The commutator vanishes by definition of a geometric local boundary condition. Thus the derivative is zero and the equation holds over the whole path. By the same method, we see the corresponding result for $1-\mathfrak{B}^+$ holds. Therefore by Prop. 2.2.2, to check the first-order part of the boundary condition it suffices to show that

$$\nabla^S_{\frac{\partial}{\partial r}}\phi = \tau_\gamma(z)^{-1}\nabla^S_{\frac{\partial}{\partial r}}\tau_\gamma(z)\phi$$

Proceeding as before, we see that equality holds over the whole curve if the commutator

$$[\nabla^S_{\dot\gamma^H} + k_{M/B}(\dot\gamma^H), \nabla^S_{\frac{\partial}{\partial r}}] = [\nabla^S_{\dot\gamma^H}, \nabla^S_{\frac{\partial}{\partial r}}] + \frac{\partial}{\partial r}k_{M/B}(\dot\gamma^H)$$

vanishes. The first term is zero because the connection on $S$ is the pullback connection near the boundary, and the family connection is parallel to the boundary over the neighborhood of a family boundary collaring. Using Prop. 2.1.4 we see that $k_{M/B}(\dot\gamma^H)$ is constant along $\frac{\partial}{\partial r}$. Therefore the second term vanishes as well. □

We come to the main theorem of this section:

**Theorem 2.2.5.** *(The transgression theorem.) The smooth family of boundary value problems* $(\Delta^+, \mathfrak{B})$ *has a corresponding smooth family of heat operators* $\exp(-t\Delta^+)$*, and we have*

$$d_B \operatorname{Tr}\left\{\exp -t\Delta^+\right\} = -t \operatorname{Tr}\left\{[\nabla^{L_2}, \Delta^+] \exp -t\Delta^+\right\}$$

*We also have the corresponding results for* $(\Delta^-, \mathfrak{B})$*.*

*Proof.* The existence of the smooth family of smoothing operators is given by Prop. A.5.2.

From Prop. 2.1.16 we have

$$X \operatorname{Tr}\left\{\exp -t\Delta^+\right\} = \operatorname{Tr}\left\{\nabla_X^{L_2} \exp -t\Delta^+\right\}$$

Using the $L_2$ parallel transport map for a path $\gamma$ in $B$ with $\dot{\gamma}(0) = X$, we have

$$\operatorname{Tr}\left\{\nabla_X^{L_2} \exp -t\Delta^+\right\} = \operatorname{Tr}\left\{\frac{\partial}{\partial z}|_{z=0} \tau_\gamma(z)^{-1} \exp(-t\Delta^+)\tau_\gamma(z)\right\}$$

Let $\Delta_z^+ := \tau_\gamma(z)^{-1} - t\Delta^+\tau_\gamma(z)$. By Lemma 2.2.4, $\tau_\gamma(z)^{-1}\exp(-t\Delta^+)\tau_\gamma(z)$ satisfies the boundary conditions of $(\Delta_0^+, \mathfrak{B})$. Clearly $\tau_\gamma(z)^{-1}\exp(-t\Delta^+)\tau_\gamma(z)$ satisfies the heat equation for $\Delta_z^+$. Uniqueness of solutions to the heat equation (Prop. A.4.11) therefore implies that

$$\tau_\gamma(z)^{-1} \exp(-t\Delta^+)\tau_\gamma(z) = \exp(-t\Delta_z^+)$$

with the boundary conditions of $(\Delta_0^+, \mathfrak{B})$. Thus

$$\operatorname{Tr}\left\{\nabla_X^{L_2} \exp -t\Delta^+\right\} = \operatorname{Tr}\left\{\frac{\partial}{\partial z}|_{z=0} \exp(-t\Delta_z^+)\right\}$$

Applying Duhamel's principle for one-parameter families with fixed boundary conditions (Theorem A.5.3), we have:

$$\begin{aligned}\operatorname{Tr}\left\{\nabla_X^{L_2}\exp -t\Delta^+\right\} &= -t\operatorname{Tr}\left\{\frac{\partial}{\partial z}|_{z=0}\Delta_z^+\exp(-t\Delta_0^+)\right\}\\ &= -t\operatorname{Tr}\left\{[\nabla_X^{L_2},\Delta^+]\exp -t\Delta^+\right\}\end{aligned}$$

The corresponding statements for $(\Delta^-, \mathfrak{B})$ follow by similar reasoning.

□

#### 2.2.2.1 Examples of geometric local boundary conditions

In Chapters 3 and 4 we will study two classes of geometric local boundary conditions, which we introduce here as examples. In both cases we suppose that $T(M/B)$ has a spin-structure, and that besides the associated spinor bundle $\mathcal{S}$ there is an auxiliary bundle $E$.

The first class we will study is posed on a family of even dimensional manifolds with boundary and we refer to it as *the even problem.* When the fibers are even-dimensional, $\mathcal{S}$ decomposes as an orthogonal direct sum $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$. The grading operator for this decomposition can be written as a constant multiple of the volume form (acting by Clifford multiplication.) Thus this decomposition is parallel with respect to the spin connection, by our assumption of compatibility with the Levi-Civita connection. Near the boundary, where we have a well-defined intrinsic outward unit conormal,

$$\sigma := \mathrm{cl}(\nu_{M/B}) : \mathcal{S}^+|U \to \mathcal{S}^-|U$$

is a canonical isomorphism. This isomorphism is unitary and by Prop. 2.1.6 it is parallel with respect to boundary vector fields. We suppose that $E$ has similar properties: $E$ decomposes as an orthogonal direct sum $E^+ \oplus E^-$, that this $\mathbb{Z}/2\mathbb{Z}$ grading is parallel, and furthermore there is a unitary isomorphism defined near the boundary

$$\tau : \ E^+|U \to E^-|U$$

parallel with respect to boundary vector fields. We define $S = \mathcal{S} \otimes E$, and grade $S$ by

$$S^\pm := \mathcal{S}^+ \otimes E^\pm \oplus \mathcal{S}^- \otimes E^\mp$$

The boundary condition on $\mathsf{D}^+$ is

$$\mathfrak{B}^+ := \frac{1}{2}\left(1 - \begin{bmatrix} 0 & (\sigma \otimes \tau)^* \\ (\sigma \otimes \tau) & 0 \end{bmatrix}\right)$$

Equivalently, the boundary values of sections satisfying $\mathfrak{B}^+$ lie in the graph of

$$\sigma \otimes \tau : \ \mathcal{S}^+ \otimes E^+ \to \mathcal{S}^- \otimes E^-$$

**Proposition 2.2.6.** *Even-type boundary conditions are local elliptic boundary conditions.*

*Proof.* We have $\sigma(A)(y, \xi') = i\mathrm{cl}(\nu_{M/B})\mathrm{cl}(\xi') \otimes 1$. Since

$$\sigma(A)(y, \xi')^2 = -\mathrm{cl}(\nu_{M/B})\mathrm{cl}(\xi')\mathrm{cl}(\nu_{M/B})\mathrm{cl}(\xi') \otimes 1 = 1$$

for $\xi'$ in the sphere bundle, we see that the eigenvalues of $\sigma(A)(y, \xi')$ are $\pm 1$. The map $\mathrm{cl}(\xi') \otimes \tau$ is a canonical isomorphism from the $+1$ eigenspace to the

$-1$ eigenspace. Thus $L^{\pm}$ are each half-rank. The solution with initial value $u_0$ in $S^+_{(0,y)}$ is

$$u(t) = e^{-t\sigma(A)(y,\xi')}u_0$$

We see that $u$ is in $L^{\pm}$ if and only if it is in the $\pm 1$ eigenspace of $\sigma(A)$. In particular, $u$ is in $L^+$ if and only if $\mathrm{cl}(\nu_{M/B})u = -i\mathrm{cl}(\xi')u$. Thus

$$\mathfrak{B}^+ \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u^+ - i(\mathrm{cl}(\xi') \otimes \tau^*)u^- \\ u^- + i(\mathrm{cl}(\xi') \otimes \tau)u^+ \end{pmatrix}$$

Thus $\mathfrak{B}^+ u = 0$ if and only if

$$u^+ = i(\mathrm{cl}(\xi') \otimes \tau^*)u^-$$

$$u^- = -i(\mathrm{cl}(\xi') \otimes \tau)u^+$$

Substituting the first equation into the second, we arrive at the condition $2u^- = 0$. This in turn implies $u^+ = 0$, since $(\mathrm{cl}(\xi') \otimes \tau^*)$ is an isomorphism. Thus $\mathfrak{B}^+$ is injective on $L^+$. Since $L^+$ is half-rank, as is the range of $\mathfrak{B}^+$, we see that $\mathfrak{B}^+$ is an isomorphism of $L^+$. □

**Proposition 2.2.7.** *Even-type boundary conditions are geometric local boundary conditions.*

*Proof.* We rewrite the boundary condition as

$$\mathfrak{B}^+ := \frac{1}{2}\left(1 - \left((\sigma \otimes \tau)^* P^- + (\sigma \otimes \tau)P^+\right)\right)$$

where for the purpose of the proof we let $P^+$ denote the orthogonal projection

$$P^+ : (\mathcal{S} \otimes E)^+ \to \mathcal{S}^+ \otimes E^+$$

and $P^- := 1 - P^+$.

There are two conditions to check. The first is $[\nabla^{L_2(S|\partial^\pi M)}, \mathfrak{B}^+] = 0$. Since $\mathfrak{B}^+$ is a local boundary condition, this is equivalent to

$$[\nabla^{S|\partial^\pi M}_{X^H}, \mathfrak{B}^+] + [k_{M/B}(X^H), \mathfrak{B}^+] = 0$$

The first commutator is zero because $P^+$, and $\sigma \otimes \tau$ are parallel along boundary vector fields. It follows from compatibility of the metric and connection that $(\sigma \otimes \tau)^*$ is likewise parallel. The second commutator is zero because $\mathfrak{B}^+$ is zero-order and therefore commutes with multiplication by a function.

The second condition is $A\mathfrak{B}^+ = (1 - \mathfrak{B}^+)A$. Equivalently,

$$-A\left((\sigma \otimes \tau)^* P^- + (\sigma \otimes \tau)P^+\right) = \left((\sigma \otimes \tau)^* P^- + (\sigma \otimes \tau)P^+\right) A$$

Recall that $A$ is a sum of terms like $\mathrm{cl}(\nu_{M/B})\mathrm{cl}(e^i)\nabla_{e_i}$. Thus $A$ anticommutes with $\sigma \otimes \tau$ and $(\sigma \otimes \tau)^*$ and commutes with $P^+$. □

The second class we will study is posed on a family of odd dimensional manifolds with boundary and we refer to it as *the odd problem*. Near the boundary of an odd-dimensional manifold, $\mathrm{cl}(\nu_{M/B})$ determines an orthogonal decomposition $\mathcal{S}|U \cong \mathcal{S}^+ \oplus \mathcal{S}^-$. The grading operator is $i\mathrm{cl}(\nu_{M/B})$. Applying Prop. 2.1.6, we see that this decomposition is parallel. We suppose the auxiliary bundle $E$ has a similar property: near the boundary there is a parallel grading $E|U \cong E^+ \oplus E^-$ given by $\gamma$. We set $S^+ = S^- = \mathcal{S} \otimes E$, $S = S^+ \oplus S^-$ and let $\mathsf{D}^+$ be the Dirac operator for $\mathcal{S} \otimes E$ regarded in the obvious way as an odd operator on $S$. The boundary condition is

$$\mathfrak{B}^+ := \frac{1}{2}\left(1 - i\mathrm{cl}(\nu_{M/B}) \otimes \gamma\right)$$

Equivalently, a section satisfies the boundary condition if its boundary values lie in the positively-graded subbundle of $S^+|\partial^\pi M$

$$(S^+|\partial^\pi M)^+ = \mathcal{S}^+ \otimes E^+ \oplus \mathcal{S}^- \otimes E^-$$

These boundary conditions were first considered by Råde in [25].

**Proposition 2.2.8.** *Odd-type boundary conditions are local elliptic boundary conditions.*

*Proof.* As in the proof of Prop. 2.2.6, $L^\pm$ are of half-rank and are the $\pm 1$ eigenspaces of $\sigma(A)(y,\xi') = i\mathrm{cl}(\nu_{M/B})\mathrm{cl}(\xi')$. Since $\mathfrak{B}^+$ also has half-rank, it suffices to show that it is injective on $L^+$. Suppose that $u$ is in the kernel of $\mathfrak{B}^+$, i.e., it lies in $\mathcal{S}^+ \otimes E^+ \oplus \mathcal{S}^- \otimes E^-$. Let $u = s^+ \otimes e^+ \oplus s^- \otimes e^-$. Recall that $i\mathrm{cl}(\nu_{M/B})$ defines the grading $\mathcal{S}^+ \oplus \mathcal{S}^-$. Thus

$$\sigma(A)(y,\xi')u = \mathrm{cl}(\xi')s^- \otimes e^- \oplus -\mathrm{cl}(\xi')s^+ \otimes e^+$$

This lies in $\mathcal{S}^+ \otimes E^- \oplus \mathcal{S}^- \otimes E^+$ and is therefore equal to $u$ only if $u = 0$. □

**Proposition 2.2.9.** *Odd-type boundary conditions are geometric local boundary conditions.*

*Proof.* The proof is similar to that of Prop. 2.2.7. We see that the condition $[\nabla^{L_2(S|\partial^\pi M)}, \mathfrak{B}^+] = 0$ is equivalent to

$$[\nabla^{S|\partial^\pi M}_{X^H}, \mathfrak{B}^+] + [k_{M/B}(X^H), \mathfrak{B}^+] = 0$$

The first commutator is zero because $\mathrm{cl}(\nu_{M/B}) \otimes \gamma$ is parallel along boundary vector fields. The second commutator is zero because $\mathfrak{B}^+$ is zero-order and therefore commutes with multiplication by a function.

The second condition $A\mathfrak{B}^+ = (1 - \mathfrak{B}^+)A$ is equivalent to

$$-A(\mathrm{cl}(\nu_{M/B}) \otimes \gamma) = (\mathrm{cl}(\nu_{M/B}) \otimes \gamma)A$$

This follows because $\mathrm{cl}(\nu_{M/B})\mathrm{cl}(e^i)$ anticommutes with $i\mathrm{cl}(\nu_{M/B}) \otimes \gamma$ when $e_i$ is a boundary vector field and $\mathrm{cl}(\nu_{M/B})$ and $\gamma$ are parallel along boundary vector fields. □

### 2.2.3 Construction of the determinant line bundle

For each $\alpha \in \mathbb{R}$, let $U_\alpha$ be the set of $b \in B$ such that $\alpha$ is not in the spectrum of $\Delta_b^+$ or $\Delta_b^-$. (The spectra are the same except perhaps the multiplicity of 0.) For $b$ in $U_\alpha$ let $P^\pm_{[0,\alpha)}(b) := P_{[0,\alpha)}(\Delta_b^\pm)$ be the orthogonal projection from the space of $L_2$ sections onto the direct sum of eigenspaces with eigenvalue in $[0, \alpha)$. By Cor. A.4.15 these are finite rank smoothing operators satisfying the boundary conditions. Let $P^\pm_{(\alpha,\infty)} := 1 - P^\pm_{[0,\alpha)}$. For $\alpha < \beta < \infty$ and $b$ in $U_\alpha \cap U_\beta$, let $P^\pm_{(\alpha,\beta)}(b) := P^\pm_{[0,\beta)} - P^\pm_{[0,\alpha)}$. Let $\mathcal{H}^\pm_{(\alpha,\beta),b}$ be the range of $P^\pm_{(\alpha,\beta)}(b)$, and let $\mathcal{H}_{(\alpha,\beta),b}$ be the superbundle $\mathcal{H}^+_{(\alpha,\beta),b} \oplus \mathcal{H}^-_{(\alpha,\beta),b}$. Let

$$\mathsf{D}^\pm_{(\alpha,\beta),b} := \mathsf{D}^\pm_b P^\pm_{(\alpha,\beta)}(b)$$

and let $\mathsf{D}_{(\alpha,\beta),b}$ be the odd operator on $\mathcal{H}_{(\alpha,\beta),b}$ with components $\mathsf{D}^\pm_{(\alpha,\beta),b}$.

The following proposition follows immediately from Lemma 9.9 of [5] and Theorem A.6.2.

**Proposition 2.2.10.**

1. *The projections $P^{\pm}_{(\alpha,\beta)}$ are a smooth family of smoothing operators.*

2. *The sets $U_\alpha$ are an open covering of $B$.*

3. *The vector bundles $\mathcal{H}^{\pm}_{(\alpha,\beta)}$ are smooth and finite-rank.*

If $V$ is a complex vector bundle of finite rank, $\det V$ is the line bundle with fiber at $p$, $(\det V)_p := \bigwedge^{\dim V_p} V_p$. Note that $\det(V \oplus W) \cong \det V \otimes \det W$, and that if $g$ is a section of $V$, there is a corresponding section $\det g$ of $\det V$. If $g$ is a homomorphism from $V$ to $W$, $\det g$ is in $\det W \otimes \det V^*$; $\det g$ is non-zero if and only if $g$ is an isomorphism.

Let $\mathcal{L}_{(\alpha,\beta)}(\mathsf{D}^+)$ be the line-bundle over $U_\alpha \cap U_\beta$

$$\mathcal{L}_{(\alpha,\beta)}(\mathsf{D}^+) := \det \mathcal{H}^-_{(\alpha,\beta)} \otimes (\det \mathcal{H}^+_{(\alpha,\beta)})^*$$

$\det \mathsf{D}_{(\alpha,\beta)}$ is a smooth section of $\mathcal{L}_{(\alpha,\beta)}(\mathsf{D}^+)$. If $\alpha < \beta$,

$$\mathcal{L}_{[0,\beta)}(\mathsf{D}^+) \cong \mathcal{L}_{[0,\alpha)}(\mathsf{D}^+) \otimes \mathcal{L}_{(\alpha,\beta)}(\mathsf{D}^+)$$

over $U_\alpha \cap U_\beta$. For all $0 < \alpha < \beta$, let $g_{(\alpha,\beta)}$ be the map

$$\begin{aligned} g_{(\alpha,\beta)} :\ \mathcal{L}_{[0,\alpha)}(\mathsf{D}^+) &\to \mathcal{L}_{[0,\beta)}(\mathsf{D}^+) \\ s &\mapsto s \otimes \det \mathsf{D}^+_{(\alpha,\beta)} \end{aligned}$$

Because $\mathsf{D}_{(\alpha,\beta)}$ is an isomorphism if $0 < \alpha < \beta$, the $g_{(\alpha,\beta)}$ are isomorphisms; the cocycle condition $g_{(\alpha,\gamma)} = g_{(\beta,\gamma)} \circ g_{(\alpha,\beta)}$ is easily verified when $\alpha < \beta < \gamma$.

We define $\mathcal{L}(\mathsf{D}^+) \to B$ to be the line-bundle constructed from the line-bundles $\mathcal{L}_{[0,\alpha)}$ using the maps $g_{(\alpha,\beta)}$ as transition functions.

Over $U_{[0,\alpha)} \cap U_{[0,\beta)}$, we have

$$\begin{aligned}\det \mathsf{D}^+_{[0,\beta)} &= \det \mathsf{D}^+_{[0,\alpha)} \otimes \det \mathsf{D}^+_{(\alpha,\beta)} \\ &= g_{(\alpha,\beta)}(\det \mathsf{D}^+_{[0,\alpha)})\end{aligned}$$

Thus the sections $\det \mathsf{D}^+_{[0,\alpha)}$ glue together into a section of the determinant line. We define $\det \mathsf{D}^+$ to be this section; $\det \mathsf{D}^+$ is called the canonical section of the determinant line bundle.

### 2.2.4 The metric and connection of the determinant line bundle

As Quillen discovered in [23], the determinant line bundle can be given a hermetian metric, constructed in a canonical way from geometric data on the family of manifolds. Bismut and Freed extended this construction to a larger category of manifolds and showed how to construct a compatible connection on the determinant line, again using geometric data; furthermore, they gave curvature and holonomy formulas for this connection. For this reason, we call the metric on the determinant line bundle the Quillen metric and the compatible connection the Bismut-Freed connection.

The $L_2$-metric and connection for $\pi_* S$ induce metrics and connections on the bundles $\mathcal{H}^{\pm}_{(\alpha,\beta)}$ by restriction. That is,

$$\langle \cdot, \cdot \rangle_{L_2,(\alpha,\beta)} := \langle P_{(\alpha,\beta)} \cdot, P_{(\alpha,\beta)} \cdot \rangle_{L_2(S)}$$

and

$$\nabla^{L_2,(\alpha,\beta)} := P_{(\alpha,\beta)} \nabla^{L_2(S)} P_{(\alpha,\beta)}$$

The compatibility of the metric and connection on $\pi_* S$ and the fact that the projection $P_{(\alpha,\beta)}$ is orthogonal in the $L_2$ metric imply that the $L_2$-connection on $\mathcal{H}^{\pm}_{(\alpha,\beta)}$ is compatible with its $L_2$ metric. The metric and connection on $\mathcal{H}^{\pm}_{(\alpha,\beta)}$ induce a metric and connection on its determinant line $\mathcal{L}_{(\alpha,\beta)}(\mathsf{D}^+)$. However, the transition functions $g_{(\alpha,\beta)}$ are *not* unit length or flat in the induced metric and connection, meaning that they do not patch together to give a metric or connection on $\mathcal{L}(\mathsf{D}^+)$.

The Quillen metric and Bismut-Freed connection are defined by modifying the $L_2$ metric and connection. We will define real-valued functions $z_\alpha$ and 1-forms $\mu_\alpha$ on each set $U_{[0,\alpha)}$. Then the Quillen metric will be

$$\langle \cdot, \cdot \rangle_{\zeta,[0,\alpha)} := e^{z_\alpha} \langle \cdot, \cdot \rangle_{[0,\alpha)}$$

and the Bismut-Freed connection will be

$$\nabla^{\zeta,[0,\alpha)} := \nabla^{[0,\alpha)} + \mu_\alpha$$

In order to have the metric and connection patch properly, the transition functions must be flat and unit length. This means that on overlaps $U_{[0,\alpha)} \cap U_{[0,\beta)}$, $\alpha < \beta$, the functions $z_\alpha$ and 1-forms $\mu_\alpha$ must satisfy

1. $\exp(z_\alpha - z_\beta) = \|\det \mathsf{D}^+_{(\alpha,\beta)}\|^2$
2. $(\mu_\alpha - \mu_\beta) \det \mathsf{D}^+_{(\alpha,\beta)} = \nabla \det \mathsf{D}^+_{(\alpha,\beta)}$

The metric and connection are compatible if and only if on each $U_{[0,\alpha)}$ we have

3. $\operatorname{Re}\mu_\alpha = \frac{1}{2}dz_\alpha$

We start by making conditions (1) and (2) more explicit.

**Proposition 2.2.11.**

$$\|\det \mathsf{D}^+_{(\alpha,\beta)}\|^2 = \det \Delta^+_{(\alpha,\beta)} \tag{2.2.3}$$

$$\nabla \det(\mathsf{D}^+_{(\alpha,\beta)}) = \operatorname{tr}(\nabla\mathsf{D}^+_{(\alpha,\beta)}\,(\mathsf{D}^+_{(\alpha,\beta)})^{-1})\det(\mathsf{D}^+_{(\alpha,\beta)}) \tag{2.2.4}$$

*Proof.* These are statements about finite-rank operators, and follow exactly as in [5]. □

Therefore conditions conditions (1) and (2) can be restated as:

1. $z_\alpha - z_\beta = \log\det\Delta^+_{(\alpha,\beta)}$
2. $\mu_\alpha - \mu_\beta = \operatorname{tr}(\nabla\mathsf{D}^+_{(\alpha,\beta)}\,(\mathsf{D}^+_{(\alpha,\beta)})^{-1})$

In view of these statements, it's natural to try to define $z_\alpha = \log\det\Delta^+_{(\alpha,\infty)}$ and $\mu_\alpha = \operatorname{tr}(\nabla\mathsf{D}^+_{(\alpha,\infty)}\,(\mathsf{D}^+_{(\alpha,\infty)})^{-1})$, however these expressions are not well-defined. In order to make sense of them, we need to introduce $\zeta$-regularized determinants and traces, defined using the Mellin transform.

The Mellin transform of a function $f:\ (0,\infty)\to\mathbb{C}$ is

$$\mathbf{M}\,[f]\,(s) := \frac{1}{\Gamma(s)}\int_0^\infty f(\tau)\,\tau^s\,\frac{d\tau}{\tau}$$

To avoid repeating hypotheses, we introduce a class of admissible functions for the Mellin transform.

**Definition 2.2.4.** A function $f : (0, \infty) \to \mathbb{C}$ is admissible for the Mellin transform if

1. $f$ decays exponentially at infinity, i.e., $|f(\tau)| \leq \exp(-\tau K)$ for some $K > 0$ and all $\tau$ large enough;

2. $f$ is continuous; and

3. $f$ has an asymptotic expansion as $t \to 0^+$ of the form

$$f(\tau) \sim \sum_{k \geq -m}^{0} f_k \tau^{k/2} + g \log \tau$$

for some integer $m$.

**Proposition 2.2.12.** *(Lemma 9.34 of [5]) Suppose that $f$ is admissible for the Mellin transform. Then the Mellin transform of $f$ is a meromorphic function with simple poles contained in the set $m/2 - \mathbb{N}/2$, and the Laurent series of $\mathbf{M}[f]$ around $0$ begins $-gs^{-1} + (f_0 - \gamma g) + \cdots$, where $\gamma$ is the Euler constant.*

**Corollary 2.2.13.** *If $f$ is admissible, then so are*

$$g(t) = tf(t), \text{ and } h(t) = \int_t^\infty f(\tau) d\tau$$

*Furthermore,*

$$\mathbf{M}[tf(t)](s) = s\mathbf{M}\left[\int_t^\infty f(\tau) d\tau\right](s)$$

*Proof.* Checking that $g$ and $h$ are admissible is elementary.

At real $s > m + 1$, integration by parts shows that the formula holds. Using the previous proposition, both sides are meromorphic. Analytic continuation shows that the formula holds throughout the region of analyticity. $\square$

**Definition 2.2.5.** The renormalized limit of $f$ at 0 is

$$\operatorname*{LIM}_{t\to 0^+} f(t) := \frac{\partial}{\partial s}|_{s=0} \{s\mathbf{M}[f](s)\}$$

We will need the following simple properties of the renormalized limit:

**Proposition 2.2.14.**

1. *When both limits exist,* $\lim_{t\to 0^+} f(t) = \mathrm{LIM}_{t\to 0^+} f(t)$.

2. $$\operatorname*{LIM}_{t\to 0^+} \overline{f(t)} = \overline{\operatorname*{LIM}_{t\to 0^+} f(t)}$$

*Proof.* The first property is a simple corollary of Prop 2.2.12. The second follows from the fact that $\mathbf{M}\left[\bar{f}\right](s) = \overline{\mathbf{M}[f](\bar{s})}$. □

The following theorem is the crucial analytic fact which makes regularization possible.

**Theorem 2.2.15.** *Suppose that $Q$ is a non-local differential operator with non-local support contained in the non-local support of $\Delta$. Then*

$$\mathrm{Tr}\left\{Q\exp(-t\Delta)P_{(\alpha,\infty)}\right\}$$

*is admissible for the Mellin transform if $\alpha > 0$; furthermore, the coefficient of the log term in the asymptotic expansion is* 0.

*Proof.* The exponential decay at infinity is immediate, since $\alpha > 0$. Continuity in $t$ is likewise obvious. The asymptotic expansion as $t \to 0^+$ is a consequence of Theorem A.4.5. □

The zeta-function of $\Delta^+$ from $\alpha > 0$ is the function on $U_{[0,\alpha)}$

$$\zeta(s; b, \Delta^+, \alpha) := \mathbf{M}\left[\mathrm{Tr}_+ \left\{P^+_{(\alpha,\infty)} \exp(-t\Delta^+)\right\}_b\right](s) \tag{2.2.5}$$

This is a well-defined meromorphic function of $s$ by Theorem 2.2.15 and Prop 2.2.12, and, since the log term in the asymptotic expansion vanishes, $\zeta(s; b, \Delta^+, \alpha)$ is meromorphic at $s = 0$. We summarize this as:

**Proposition 2.2.16.** *The function $\zeta(s; b, \Delta^+, \alpha)$ has a meromorphic extension to the whole complex plane, and is holomorphic at $s = 0$.*

The zeta-determinant of $\Delta^+$ from $\alpha > 0$ is

$$\det{}_\zeta(b, \Delta^+, \alpha) = \exp\left\{-\frac{\partial}{\partial s}\zeta(0; b, \Delta^+, \alpha)\right\} \tag{2.2.6}$$

The following proposition shows that the zeta-determinant is a regularized determinant.

**Proposition 2.2.17.** *For $b$ in the overlap set $U_{[0,\alpha)} \cap U_{[0,\beta)}$, $0 < \alpha < \beta$,*

$$\zeta(s; b, \Delta^+, \alpha) = \zeta(s; b, \Delta^+, \beta) + \sum_{(\alpha,\beta)} \lambda_i^{-s}$$

*where $\lambda_i$ is an enumeration of the (finitely-many) eigenvalues of $\Delta_b^+$ with eigenvalue in the range $(\alpha, \beta)$, repeated according to multiplicity. Thus*

$$\frac{\partial}{\partial s}\zeta(0; b, \Delta^+, \alpha) = \frac{\partial}{\partial s}\zeta(0; b, \Delta^+, \beta) - \sum_{(\alpha,\beta)} \log \lambda_i$$

*Proof.* This is immediate from the linearity of the Mellin transform. □

Therefore if we define $z_\alpha(b)$ to be

$$z_\alpha(b) := -\frac{\partial}{\partial s}\zeta(0; b, \Delta^+, \alpha) = \log\det_\zeta(b, \Delta^+, \alpha)$$

then $z_\alpha$ will satisfy

$$z_\alpha - z_\beta = \sum_{(\alpha,\beta)} \log\lambda_i = \log\det\Delta^+_{(\alpha,\beta)}$$

That is, the $z_\alpha$ will satisfy the metric gluing property (1).

Turning now to the connection 1-forms, define

$$\mu_\alpha := \operatorname*{LIM}_{t\to 0^+}\int_t^\infty \operatorname{Tr}\left\{P^-_{(\alpha,\infty)}\nabla\mathsf{D}^+\,\mathsf{D}^-\exp(-\tau\Delta^-)\right\}\,d\tau$$

If $\xi$ is a vector field on $B$, and $\xi^H$ its horizontal lift, $\nabla_{\xi^H}\mathsf{D}^+\,\mathsf{D}^-$ is a non-local differential operator of order at most 2. Theorem 2.2.15 therefore applies, and we can conclude that $P^-_{(\alpha,\infty)}\nabla\mathsf{D}^+\,\mathsf{D}^-\exp(-\tau\Delta^-)$ is smoothing, and its trace is admissible. It follows from Coro 2.2.13 that the integral of the trace is admissible. Thus $\mu_\alpha$ is well-defined.

**Proposition 2.2.18.**

$$\mu_\alpha - \mu_\beta = \operatorname{tr}(\nabla\mathsf{D}^+_{(\alpha,\beta)}\,(\mathsf{D}^+_{(\alpha,\beta)})^{-1})$$

*Proof.* We have

$$P^-_{(\alpha,\infty)}\left\{\nabla\mathsf{D}^+\,\mathsf{D}^-\exp(-\tau\Delta^-)\right\} - P^-_{(\beta,\infty)}\left\{\nabla\mathsf{D}^+\,\mathsf{D}^-\exp(-\tau\Delta^-)\right\}$$
$$= P^-_{(\alpha,\beta)}\left\{\nabla\mathsf{D}^+\,\mathsf{D}^-\exp(-\tau\Delta^-)\right\}$$

From the definition $\mathrm{LIM}_{t\to 0^+}$ is linear, as is integration, as is trace. Therefore

$$\mu_\alpha - \mu_\beta = \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \mathrm{Tr}\left\{ P^-_{(\alpha,\beta)} \nabla \mathsf{D}^+ \, \mathsf{D}^- \exp(-\tau\Delta^-) \right\}$$

Using the fact that $P^-_{(\alpha,\beta)}$ is a projection commuting with $\mathsf{D}$ and $\Delta^-$, and the fact that Tr vanishes on commutators of trace class operators, we have

$$\begin{aligned} &= \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \mathrm{Tr}\left\{ P^-_{(\alpha,\beta)} \nabla \mathsf{D}^+ \, P^+_{(\alpha,\beta)} \mathsf{D}^- P^-_{(\alpha,\beta)} \exp(-\tau\Delta^-) P^-_{(\alpha,\beta)} \right\} \\ &= \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \mathrm{tr}\left\{ \nabla \mathsf{D}^+_{(\alpha,\beta)} \, \mathsf{D}^-_{(\alpha,\beta)} \exp(-\tau\Delta^-_{(\alpha,\beta)}) \right\} \end{aligned}$$

Using the linearity of trace,

$$\begin{aligned} &= \operatorname*{LIM}_{t\to 0^+} \mathrm{tr}\left\{ \nabla \mathsf{D}^+_{(\alpha,\beta)} \, \mathsf{D}^-_{(\alpha,\beta)} \int_t^\infty \exp(-\tau\Delta^-_{(\alpha,\beta)}) \right\} \\ &= \operatorname*{LIM}_{t\to 0^+} \mathrm{tr}\left\{ \nabla \mathsf{D}^+_{(\alpha,\beta)} \, \mathsf{D}^-_{(\alpha,\beta)} (\Delta^-_{(\alpha,\beta)})^{-1} \exp(-t\Delta^-_{(\alpha,\beta)}) \right\} \end{aligned}$$

This gives the proposition, because the regularized limit agrees with the limit for functions that have a limit, and $\mathsf{D}^-_{(\alpha,\beta)}(\Delta^-_{(\alpha,\beta)})^{-1} = (\mathsf{D}^+_{(\alpha,\beta)})^{-1}$. □

We now compute the real and imaginary parts of $\mu_\alpha$.

**Proposition 2.2.19.**

$$\overline{\mu}_\alpha = \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \mathrm{Tr}\left\{ P^-_{(\alpha,\infty)} \mathsf{D}^+ \, \nabla \mathsf{D}^- \exp(-\tau\Delta^-) \right\} \, d\tau$$

*Proof.* Using part (2) of Prop 2.2.14, we see that $\overline{\mu}_\alpha$ can be computed as

$$\overline{\mu}_\alpha = \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \overline{\mathrm{Tr}\left\{ P^-_{(\alpha,\infty)} \nabla \mathsf{D}^+ \, \mathsf{D}^- \exp(-\tau\Delta^-) \right\}} \, d\tau$$

Using the fact that trace vanishes on commutators and the semigroup property of the heat kernel, we have

$$\overline{\mathrm{Tr}\left\{P^-_{(\alpha,\infty)}\nabla \mathsf{D}^+\,\mathsf{D}^-\exp(-\tau\Delta^-)\right\}}$$
$$=\overline{\mathrm{Tr}\left\{P^-_{(\alpha,\infty)}\left(e^{-\frac{\tau}{4}\Delta^-}\nabla\mathsf{D}^+e^{-\frac{\tau}{4}\Delta^+}\right)\left(e^{-\frac{\tau}{4}\Delta^+}\mathsf{D}^-e^{-\frac{\tau}{4}\Delta^-}\right)\right\}}$$

The point is that inside the trace we have a product of Hilbert-Schmidt operators.

$$=\mathrm{Tr}\left\{\left(e^{-\frac{\tau}{4}\Delta^+}\mathsf{D}^-e^{-\frac{\tau}{4}\Delta^-}\right)^*\left(e^{-\frac{\tau}{4}\Delta^-}\nabla\mathsf{D}^+e^{-\frac{\tau}{4}\Delta^+}\right)^*P^-_{(\alpha,\infty)}\right\}$$

It's not hard to show that formally (that is, applied to smooth sections)

$$\left(e^{-\frac{\tau}{4}\Delta^+}\mathsf{D}^-e^{-\frac{\tau}{4}\Delta^-}\right)^*=\left(e^{-\frac{\tau}{4}\Delta^-}\,\mathsf{D}^+e^{-\frac{\tau}{4}\Delta^+}\right)$$

and

$$\left(e^{-\frac{\tau}{4}\Delta^-}\nabla\mathsf{D}^+e^{-\frac{\tau}{4}\Delta^+}\right)^*=\left(e^{-\frac{\tau}{4}\Delta^+}\nabla\mathsf{D}^-e^{-\frac{\tau}{4}\Delta^-}\right)$$

Since these are bounded operators the formal adjoints are true $L_2$ adjoints. The proposition follows. □

**Proposition 2.2.20.** *We have*

$$\mathrm{Re}\,\mu_\alpha=\frac{1}{2}dz_\alpha$$

*That is, the metric functions $z_\alpha$ and connection 1-forms $\mu_\alpha$ satisfy the compatiblity condition (3)*

*Proof.* In view of the last proposition, we have to show that

$$d_Bz_\alpha=\operatorname*{LIM}_{t\to0^+}\int_t^\infty\mathrm{Tr}\left\{P^-_{(\alpha,\infty)}\nabla(\Delta^-)\exp(-\tau\Delta^-)\right\}\,d\tau$$

Recall that $z_\alpha(b) := \log\det_\zeta(b, \Delta^+, \alpha)$; it's not hard to show that

$$\log\det_\zeta(b, \Delta^+, \alpha) = \log\det_\zeta(b, \Delta^-, \alpha)$$

Thus,

$$\begin{aligned} d_B z_\alpha &= -d_B \frac{\partial}{\partial s}|_{s=0}\mathbf{M}\left[\mathrm{Tr}_-\left\{P^-_{(\alpha,\infty)}\exp(-t\Delta^-)\right\}_b\right] \\ &= -\frac{\partial}{\partial s}|_{s=0}\mathbf{M}\left[d_B\,\mathrm{Tr}_-\left\{P^-_{(\alpha,\infty)}\exp(-t\Delta^-)\right\}_b\right] \\ &= -\frac{\partial}{\partial s}|_{s=0}\mathbf{M}\left[\mathrm{Tr}_-\,\nabla\left\{P^-_{(\alpha,\infty)}\exp(-t\Delta^-)\right\}_b\right] \end{aligned}$$

We have

$$\begin{aligned} &\mathrm{Tr}_-\,\nabla\left\{P^-_{(\alpha,\infty)}\exp(-t\Delta^-)\right\}_b \\ &\qquad = \mathrm{Tr}_-\left\{\nabla P^-_{(\alpha,\infty)}\exp(-t\Delta^-)\right\}_b + \mathrm{Tr}_-\left\{P^-_{(\alpha,\infty)}\nabla\exp(-t\Delta^-)\right\}_b \end{aligned}$$

Since for any projection $P$, $\nabla P = (1-P)\nabla P\,P + P\nabla P\,(1-P)$, the first term is zero. The second term is computed using the transgression formula, Theorem 2.2.5:

$$\begin{aligned} &\mathrm{Tr}_-\left\{P^-_{(\alpha,\infty)}\nabla\exp(-t\Delta^-)\right\}_b \\ &\qquad = -t\,\mathrm{Tr}_-\left\{P^-_{(\alpha,\infty)}\nabla(\Delta^-)\exp(-t\Delta^-)\right\}_b \end{aligned}$$

Returning to the calculation of $d_B z_\alpha$,

$$\begin{aligned} d_B z_\alpha &= \frac{\partial}{\partial s}|_{s=0}\mathbf{M}\left[t\,\mathrm{Tr}_-\left\{P^-_{(\alpha,\infty)}\nabla(\Delta^-)\exp(-t\Delta^-)\right\}_b\right] \\ &= \frac{\partial}{\partial s}|_{s=0}s\mathbf{M}\left[\int_t^\infty \mathrm{Tr}_-\left\{P^-_{(\alpha,\infty)}\nabla(\Delta^-)\exp(-\tau\Delta^-)\right\}_b\,d\tau\right] \\ &= \operatorname*{LIM}_{t\to 0^+}\int_t^\infty \mathrm{Tr}_-\left\{P^-_{(\alpha,\infty)}\nabla(\Delta^-)\exp(-\tau\Delta^-)\right\}_b\,d\tau \end{aligned}$$

where we have used Coro 2.2.13. □

# Chapter 3

# The even problem

## 3.1 Introduction

In section 2.2.2.1 we introduced the even problem, a class of local elliptic boundary value problems for the Dirac operator, and showed that the even problem is a geometric local boundary value problem. It follows from the results of Chapter 2 that smooth families of boundary value problems of this type have an associated determinant line bundle with Quillen metric and compatible Bismut-Freed connection.

There is another determinant line bundle we can naturally construct from the defining data of the even problem: the determinant line of the double. The family $M$ has a natural spin-double $\pi_d : M^d \to B$. The fibers of $M^d$ are diffeomorphic to $X^d$, the double of $X$. The vertical tangent bundle $T(M^d/B)$ has a spin-structure $\mathcal{F}_{\mathrm{Spin}}(M^d/B)$ and a connection $H^d$ defined naturally from the family connection of $M$. We will define a vector bundle $E_d$ over $M^d$. Let $\mathcal{S}_d := \mathcal{S}(M^d/B)$, let $\mathsf{D}_d$ be the Dirac operator for $\mathcal{S}_d \otimes E_d$, and let $D_d^{\pm}$ be its graded components, $D_d^{\pm} := \mathsf{D}_d | C^{\infty}(\mathcal{S}_d^{\pm} \otimes E_d)$. As is well-known, the Dirac operator $D_d^{+}$ of this family has a well-defined determinant line bundle with Quillen metric and Bismut-Freed connection.

The main theorem for this chapter asserts that these line bundles are naturally geometrically isomorphic.

**Theorem 3.1.1.** *There is a natural geometric isomorphism of the determinant lines $\mathcal{L}(D_d^+)$ and $\mathcal{L}(D^+, \mathfrak{B})$. The isometry maps the canonical section of $\mathcal{L}(D_d^+)$ to the canonical section of $\mathcal{L}(D^+, \mathfrak{B})$.*

We also prove that the index of the Dirac operator equals the index of the Dirac operator on the spin-double. To prove these results we interpret the boundary condition for the even problem as a gluing condition for sections of a vector bundle over the spin-double of the family. More precisely, we define maps

$$K^{\pm} : C^{\infty}(M^d; \mathcal{S}_d^{\pm} \otimes E_d) \to \mathrm{Dom}(D^{\pm}, \mathfrak{B})$$

called the cutting maps, such that $K^{\mp} D_d^{\pm} = D^{\pm} K^{\pm}$. We show that the cutting maps are unitary and flat with respect to the $L_2$-metric and connection.

## 3.2 Defining data

We begin by recalling the geometric data that defines a smooth family of even-type boundary value problems. Let $\pi : M \to B$ be a Riemannian fiber bundle with each fiber diffeomorphic to $X$, a compact even dimensional manifold with boundary. As in Chapter 2, we assume that $M$ has a fixed family boundary collaring over a collar neighborhood $U$ of the boundary family, and that all the geometry is a product near the boundary. That is, over $U$ the metric on the vertical tangent bundle is a product and the family connection has the form $H = 1 \oplus H^{\partial}$, where $H^{\partial}$ is the connection on the boundary family, relative to the local decomposition of the tangent bundle into normal and boundary directions induced by the family collaring. (See Defn. 2.1.5, Coro. 2.1.3, and Defn. 2.1.6 for a detailed explanation.)

The vertical tangent bundle $T(M/B)$ has a spin-structure $\mathcal{F}_{\mathrm{Spin}}(M/B)$. Let $\mathcal{S} := \mathcal{S}(M/B)$ denote the associated spinor bundle. The spinors are naturally $\mathbb{Z}/2\mathbb{Z}$-graded because the vertical tangent bundle has even rank. Over the collar neighborhood $U$, where we have a well-defined intrinsic outward unit conormal,

$$\sigma := \mathrm{cl}(\nu_{M/B}) : \mathcal{S}^{+}|U \to \mathcal{S}^{-}|U$$

is a canonical unitary isomorphism. Let $E \cong E^{+} \oplus E^{-} \to M$ be a complex vector bundle with hermetian metric and compatible connection. Over the collar neighborhood there is a unitary and flat bundle isomorphism

$$\tau : E^{+}|U \to E^{-}|U$$

Let $\mathsf{D}$ denote the Dirac operator for $\mathcal{S} \otimes E$. The gradings on $\mathcal{S}$ and $E$ induce a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{S} \otimes E$, defined by

$$(\mathcal{S} \otimes E)^{\pm} := \mathcal{S}^{\pm} \otimes E^{+} \oplus \mathcal{S}^{\mp} \otimes E^{-}$$

There is a corresponding decomposition of $\mathsf{D}$ into chiral Dirac operators

$$D^{\pm} : C^{\infty}((\mathcal{S} \otimes E)^{\pm}) \to C^{\infty}((\mathcal{S} \otimes E)^{\mp})$$

### 3.2.1 The boundary condition

At each fiber $X_b := \pi^{-1}(b)$, this data defines a boundary value problem of even type. The boundary condition $\mathfrak{B}^+$ we impose on $D^+$ is that a section $\phi$ of $(\mathcal{S} \otimes E)^+|X_b$ is in $\mathrm{Dom}(D_b^+, \mathfrak{B}^+)$ if and only if

$$\phi|_{\partial X} = f^+ \oplus (\sigma \otimes \tau) f^+$$

for some section $f^+$ of $(\mathcal{S}^+ \otimes E^+)|\partial X_b$. Integration by parts shows that the adjoint boundary condition $\mathfrak{B}^-$ is that a section $\phi$ of $(\mathcal{S} \otimes E)^-|X_b$ is in $\mathrm{Dom}(D_b^-, \mathfrak{B}^-)$ if and only if

$$\phi|_{\partial X} = f^- \oplus (-\sigma \otimes \tau) f^-$$

for some section $f^-$ of $(\mathcal{S}^- \otimes E^+)|\partial X_b$. (We have used the unitarity of $\tau$ and $\sigma$ here.) We give $\Delta^\pm := D^\mp D^\pm$ the induced boundary conditions: $\phi$ is in $\mathrm{Dom}(\Delta^\pm, \mathfrak{B})$ if and only if

$$\phi \text{ is in } \mathrm{Dom}(D^\pm, \mathfrak{B}) \text{ and } D^\pm \phi \text{ is in } \mathrm{Dom}(D^\mp, \mathfrak{B})$$

## 3.3 Doubling and the cutting maps

### 3.3.1 The spin-double of a family

In this section we construct $M^d$, its family connection $H^d$ and spin-structure, and discuss the relations between these structures and their counterparts on $M$.

In Appendix C, we construct the spin-double of a spin manifold with boundary in detail, beginning with constructing the underlying smooth manifold and then showing how to double the metric and spin structures. In brief, this construction proceeds in two steps:

(i) The collar neighborhood $U \cong (-1, 0] \times \partial X$ has an obvious double

$$U^d \cong (-1, 1) \times \partial X$$

The manifold $X$ is extended by gluing it to $U^d$ using the embedding $U \to X$. The resulting manifold is denoted $X^e$.

(ii) Let $\bar{X}^e$ denote $X^e$ with the opposite orientation and spin-structure. The double $X^d$ is the quotient of $X^e \sqcup \bar{X}^e$ by the reflection

$$\begin{aligned} \rho :\ U^d &\to \bar{U}^d \\ (u, y) &\mapsto (-u, y) \end{aligned}$$

Since we always glue by smooth maps and open sets, there is a natural smooth structure for the double. Furthermore, because we glue $X^e$ to $\bar{X}^e$, $\rho$ preserves orientation. Thus the double has a natural orientation. We explain in the Appendix what the opposite of a spin-structure is, and show how to lift $\rho$ to a gluing of the spin-structures. There are two natural inclusions of $X^e$ into $M^d$, $\iota_l$ and $\iota_r$

$$\iota_l :\ X^e \to X^e \sqcup \bar{X}^e \to X^d$$

$$\iota_r :\ X^e \to \bar{X}^e \to X^e \sqcup \bar{X}^e \to X^d$$

By restriction, $\iota_l$ and $\iota_r$ are inclusions of $X$ into $X^d$.

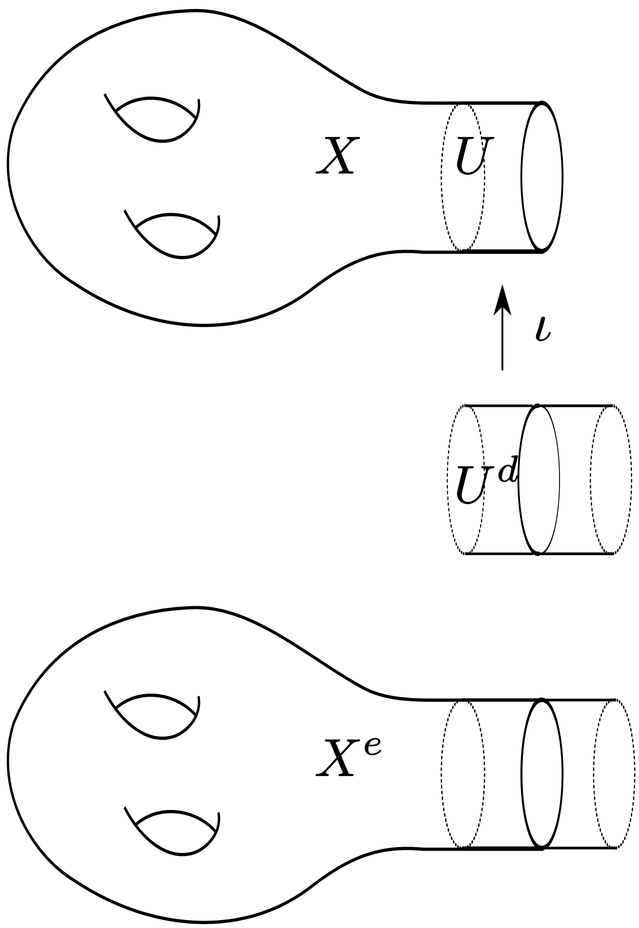


Figure 3.1: Extending $X$ with $U^d$

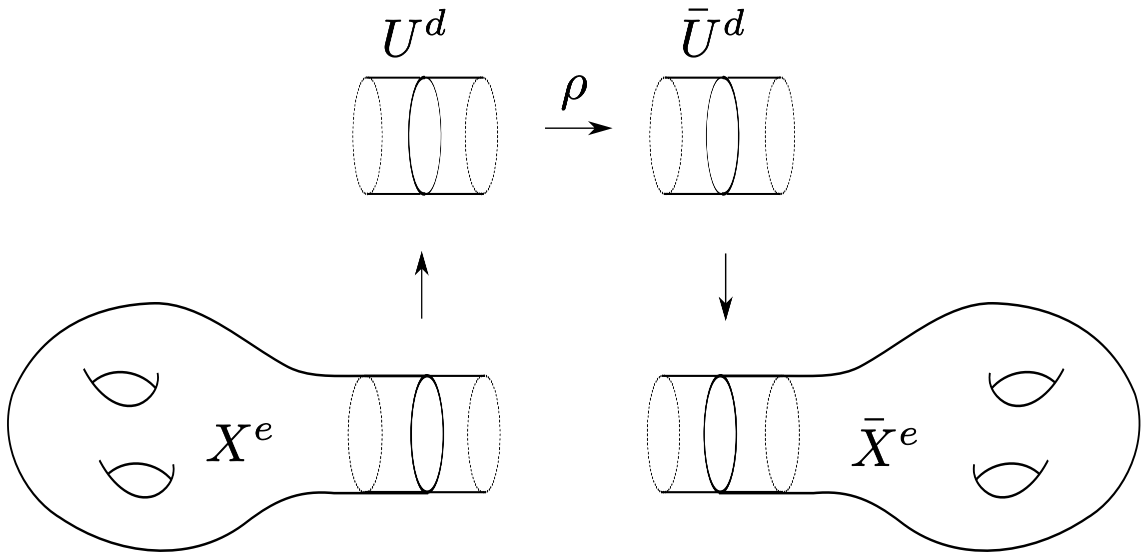


Figure 3.2: Doubling using the reflection map.

The family $M^d$ is constructed by applying this procedure fiber-by-fiber using the assumed family collaring. Since the construction of the double is natural, it works in families as well. The vertical spin-structure $\mathcal{F}_{\mathrm{Spin}}(M^d/B)$ agrees with the doubled spin-structure on each fiber.

There is one subtlety: in order to apply the construction at a particular fiber, we need to know that the collaring of the family induces a collaring of the fiber. Equivalently, we need to know that the induced splitting of $TM|U$ into normal and boundary directions makes the normal directions *vertical*. The definition of a family collaring (Defn. 2.1.5) guarantees this. A related issue is the compatibility of the family structure of the double with the family structure of $M$. The compatibility of the two family structures is expressed by requiring that

$$\pi_d \circ \iota_l = \pi, \qquad \pi_d \circ \iota_r = \pi,$$

which says that the two natural inclusions restrict to inclusions of the fibers. Away from the collar neighborhood where the gluing takes place, this requirement effectively defines $\pi_d$. On the intersection of the images of $\iota_l$ and $\iota_r$, it's equivalent to saying that $\iota_l^{-1} \circ \iota_r$ is vertical in the sense that it maps a fiber to itself. This map is induced by reflecting the natural double $(-1,1) \times \partial^\pi M$ of $(-1,0] \times \partial^\pi M$ across $\{0\} \times \partial^\pi M$. Again the key point is that the normal direction, the direction corresponding to the factor $(-1,0]$, is vertical for a family collaring.

We give the doubled family a connection by declaring that the differentials of $\iota_l$ and $\iota_r$ map horizontal vectors to horizontal vectors. At the collar there is a compatibility check to be made. More explicitly, the family connection is the map $H^d : TM^d \to T(M^d/B)$ defined by $H^d v = (\iota_{l*})H(\iota_{l*})^{-1}v$ for all vectors $v$ in the image of the differential $\iota_{l*}$ and $H^d v = (\iota_{r*})H(\iota_{r*})^{-1}v$ for all vectors $v$ in the image of the differential $\iota_{r*}$. The consistency of this definition is assured by the following

**Proposition 3.3.1.** *For all vectors $v$ in the image of both $\iota_{l*}$ and $\iota_{r*}$,*

$$(\iota_{l*})H(\iota_{l*})^{-1}v = (\iota_{r*})H(\iota_{r*})^{-1}v$$

*Proof.* By assumption, the family connection $H$ is a pullback from the boundary on the collar, i.e., $H = 1 \oplus H_{\partial M}$ with respect to the splitting induced by the collar neighborhood map. Let

$$v_l = (\iota_{l*})^{-1} v, \qquad v_r = (\iota_{r*})^{-1} v$$

$v_l$ and $v_r$ are tangent vectors in the collar neighborhood of the family's boundary. Using the collar neighborhood isometry $(r, \pi_{\partial M})$, we can decompose $v_l$ and $v_r$ into components normal to the boundary $r_*(v_l)$, $r_*(v_r)$, and parallel to the boundary $\pi_{\partial M*}(v_l)$, $\pi_{\partial M*}(v_l)$. We have $r_*(v_l) = -r_*(v_r)$ and $\pi_{\partial M*}(v_l) = \pi_{\partial M*}(v_l)$, because $((\iota_{l*})^{-1} \circ \iota_{r*}) v_l = \rho_* v$. That is, $v_l$ and $v_r$ are related by reflection across the tangent plane of the boundary. Using the special form assumed for $H$ near the boundary, $H = 1 \oplus H_{\partial M}$, we see that

$$H v_l = r_*(v_l) + H_{\partial M}(\pi_{\partial M*}(v_l)) = -r_*(v_r) + H_{\partial M}(\pi_{\partial M*}(v_r)) = \rho_* H v_r$$

Thus

$$\iota_{l*} H v_l = \iota_{l*} \rho_* H v_r = \iota_{r*} H v_r$$

□

**Proposition 3.3.2.** *The Riemannian geometry of the families $M$ and $M^d$ are compatible in the sense that:*

$$\iota_l^* k_{M^d/B} = k_{M/B} \qquad \iota_r^* k_{M^d/B} = k_{M/B}$$

*and for all vector fields $X$ on $B$ and sections $v$ of the vertical tangent bundle over $M$,*

$$(\iota_l)_*^{-1} \nabla_{X^{H^d}}^{M^d/B} (\iota_l)_* v = \nabla_{X^H}^{M/B} v \qquad (\iota_r)_*^{-1} \nabla_{X^{H^d}}^{M^d/B} (\iota_r)_* v = \nabla_{X^H}^{M/B} v$$

*Proof.* Recall that the divergence of the volume form $k_{M/B}$ can be defined implicitly by

$$dH^* \operatorname{vol}_{M/B} = k_{M/B} \wedge \operatorname{vol}_{M/B} + \text{terms of lower vertical degree}$$

The volume form is determined locally by the orientation and metric. The inclusions $\iota_l$ and $\iota_r$ are isometries; $\iota_l$ preserves orientation, and $\iota_r$ reverses it. The family connections are compatible with these inclusions by construction. Therefore we have

$$\iota_l^* d(H^d)^* \operatorname{vol}_{M^d/B} = dH^* \operatorname{vol}_{M/B}$$

and

$$\iota_r^* d(H^d)^* \operatorname{vol}_{M^d/B} = -dH^* \operatorname{vol}_{M/B}$$

Applying the pullback homomorphisms to both sides of the equation above, we have

$$dH^* \operatorname{vol}_{M/B} = (\iota_l^* k_{M/B}) \wedge \operatorname{vol}_{M/B}$$

and

$$-dH^* \operatorname{vol}_{M/B} = -(\iota_r^* k_{M/B}) \wedge \operatorname{vol}_{M/B}$$

modulo terms of lower vertical degree. The decomposition of the right hand side by vertical degree is preserved in the pullback because $\iota_l$ and $\iota_r$ are compatible with the family structure of $M^d$. We conclude that

$$\iota_l^* k_{M^d/B} = k_{M/B}, \qquad \iota_r^* k_{M^d/B} = k_{M/B}$$

The statements about the Levi-Civita connections are immediate from the naturality of the Levi-Civita connection and the definition of the family connection on the spin-double.

□

### 3.3.2 Cutting maps on doubled families

Suppose $V^+$ and $V^-$ are complex vector bundles over $M$ with hermetian metrics and compatible connections, and $\gamma_W$ is a flat isometry from $V^+|\partial M$ to $V^-|\partial M$. This data determines a complex vector bundle $W$ over the spin-double by a clutching construction, such that $V^+ = \iota_l^{-1}W$, $V^- = \iota_r^{-1}W$, and

$$\gamma_W = (\iota_r^{-1} \circ \iota_l) : \iota_l^{-1}W|\partial M \to \iota_r^{-1}W|\partial M$$

Since $\gamma_W$ is a flat isometry, $W$ naturally has a hermetian metric and compatible connection.

To each section $\phi$ of $W$, there corresponds a pair of sections $(\phi_l, \phi_r)$ of $V^+$, $V^-$ respectively, namely $\phi_l = \iota_l^*\phi$ and $\phi_r = \iota_r^*\phi$, with the property $\phi_r|\partial M = \gamma_W\, \phi_r|\partial M$. Define

$$K_W : \Gamma(W) \to \Gamma(V^+ \oplus V^-) \tag{3.3.1}$$

$$\phi \mapsto \phi_l \oplus \phi_r \tag{3.3.2}$$

Let $V := V^+ \oplus V^-$. Clearly $K_W$ maps smooth sections of $W$ to smooth sections of $V$. We will show that $K_W$ is $L_2$ unitary and flat.

**Proposition 3.3.3.** *$K_W$ is $L_2$ unitary. That is, for all smooth, compactly-supported sections $\phi$ of $W$,*

$$\|K_W\phi\|_k = \|\phi\|$$

*Proof.* Let $X^d$ be a fiber of $M^d$, and $X$ the corresponding fiber of $M$. $X^d$ is the oriented metric double of $X$, hence $\iota_l^* \operatorname{vol}_{X^d} = \operatorname{vol}_X$ and $\iota_r^* \operatorname{vol}_{X^d} = -\operatorname{vol}_X$. Let $\phi$ be a smooth section of $W$. The $L_2$-norm of $\phi$ is

$$\|\phi\|_0^2 = \int_{X^d} \operatorname{vol}_{X^d} |\phi|^2$$

$X^d = \iota_l(X) \cup \iota_r(X)$ and $\iota_l(X) \cap \iota_r(X) = \iota_l(\partial X)$ is measure zero, thus

$$\begin{aligned} \|\phi\|_0^2 &= \int_{\iota_l(X)} \operatorname{vol}_{X^d} |\phi|^2 + \int_{\iota_r(X)} \operatorname{vol}_{X^d} |\phi|^2 \\ &= \int_X \operatorname{vol}_X \iota_l^* |\phi|^2 + \int_X \operatorname{vol}_X \iota_r^* |\phi|^2 \end{aligned}$$

From the definition of $\phi_l$ and $\phi_r$, and the fact that the metrics on $V^\pm$ are induced from the metric on $W$,

$$\begin{aligned} \|\phi\|_0^2 &= \int_X \operatorname{vol}_X |\phi_l|^2 + \int_X \operatorname{vol}_X |\phi_r|^2 \\ &= \int_X \operatorname{vol}_X |\phi_l \oplus \phi_r|^2 = \|K_W \phi\|_0^2 \end{aligned}$$

□

**Proposition 3.3.4.** *For all smooth sections $\phi$ of $\pi_* W$ and vector fields $X$ on $B$, we have*

$$K_W \nabla_X^{L_2(W)} \phi = \nabla_X^{L_2(V)} K_W \phi$$

*That is, $K_W$ is $L_2$-flat.*

*Proof.* By definition of the $L_2$ connection on $W$, we have

$$K_W \nabla_X^{L_2(W)} \phi = K_W \left( \nabla_{X^{H^d}}^W \phi \right) + K_W \left( \frac{1}{2} k_{M^d/B}(X^{H^d}) \phi \right)$$

Using Prop 3.3.2, the first term is

$$\begin{aligned} K_W \nabla^W_{X^{H^d}} \phi &= \iota_l^* \nabla^W_{X^{H^d}} \phi \oplus \iota_r^* \nabla^W_{X^{H^d}} \phi \\ &= \nabla^{V+}_{X^H} \iota_l^* \phi \oplus \nabla^{V+}_{X^H} \iota_r^* \phi \\ &= \nabla^V_{X^H} K_W \phi \end{aligned}$$

and the second term is

$$\begin{aligned} K_W \frac{1}{2} k_{M^d/B}(X^{H^d})\phi &= \iota_l^* \frac{1}{2} k_{M^d/B}(X^{H^d})\phi \oplus \iota_r^* \frac{1}{2} k_{M^d/B}(X^{H^d})\phi \\ &= \frac{1}{2} k_{M/B}(X^H)[\iota_l^* \phi \oplus \iota_r^* \phi] \\ &= \frac{1}{2} k_{M/B}(X^H) K_W \phi \end{aligned}$$

Hence

$$\begin{aligned} K_W \nabla^{L_2(W)}_X \phi &= \nabla^V_{X^H} K_W \phi + \frac{1}{2} k_{M/B}(X^H) K_W \phi \\ &= \nabla^{L_2(V)}_X K_W \phi \end{aligned}$$

□

## 3.4 Application to the boundary value problem

In Appendix C, we show that there is a canonical isomorphism from the spinors to the spinors associated to the opposite spin-structure,

$$\mathfrak{R} : \ \mathcal{S}(M/B) \to \mathcal{S}(\overline{M}/B)$$

This isomorphism exchanges spinor chirality, and it is a flat isometry. Sections $\phi$ of $\mathcal{S}_d$ are in one-to-one correspondence with pairs $(\phi_l, \phi_r)$, where $\phi_l$ is a section of $\mathcal{S}(M/B)$, $\phi_r$ is a section of $\mathcal{S}(\overline{M}/B)$, and over $\partial M$, we have $\mathfrak{R}^{-1}\phi_r = \sigma\phi_l$.

We use $\tau$ as the clutching map for an (ungraded) complex vector bundle over $M^d$. Let $E_d \to M^d$ be the vector bundle defined by: $\iota_l^{-1}E_d = E^+$, $\iota_r^{-1}E_d = E^-$, and

$$\tau = \iota_r^{-1} \circ \iota_l : \; E^+|\partial X \to E_d|\iota_l(\partial X) \to E^-|\partial X$$

Since $\tau$ is unitary and flat, $E_d$ has a natural metric and compatible connection. Let $K_{E_d}$ be the cutting map for sections of $E_d$.

The cutting maps $K^\pm$

$$K^\pm : \; C^0(M^d; \mathcal{S}_d^\pm \otimes E_d) \to C^0(M; (\mathcal{S} \otimes E)^\pm)$$

are defined on homogeneous sections $\phi \otimes \psi$ by

$$K^+(\phi \otimes \psi) = (\phi_l \otimes \psi_l) \oplus (\mathfrak{R}^{-1}\phi_r \otimes \psi_r)$$
$$K^-(\phi \otimes \psi) = (\phi_l \otimes \psi_l) \oplus (-\mathfrak{R}^{-1}\phi_r \otimes \psi_r)$$

**Proposition 3.4.1.** *The image of $K^+$ is contained in* $\mathrm{Dom}(D^+, \mathfrak{B})$*, and the image of $K^-$ is contained in* $\mathrm{Dom}(D^-, \mathfrak{B})$*.*

*Proof.* We prove it for $K^+$; the proof for $K^-$ is similar.

Since both $K^+$ and the boundary condition are linear, it suffices to show the result for homogeneous sections $\phi \otimes \psi$, where $\phi$ is a section of $\mathcal{S}_d^+$ and $\psi$ is a section of $E_d$. Let $(\phi_l, \phi_r)$ and $(\psi_l, \psi_r)$ be the corresponding pairs of sections, as defined above. Then $\phi_l$ is a section of $\mathcal{S}^+$ and $\mathfrak{R}^{-1}\phi_r$ is a section of $\mathcal{S}^-$, and over $\partial M$, $\mathfrak{R}^{-1}\phi_r = \sigma\phi_l$ and $\psi_r = \tau\psi_l$. Thus $K^+(\phi \otimes \psi) = (\phi_l \otimes \psi_l) \oplus (\mathfrak{R}^{-1}\phi_r \otimes \psi_r)$ satisfies the boundary condition.

□

**Proposition 3.4.2.** *The cutting maps $K^\pm$ extend to $L_2$-unitary maps and are $L_2$-flat. That is, for all smooth sections $\phi$ of $\pi_*(\mathcal{S}_d \otimes E_d)^\pm$ and vector fields $X$ on $B$, we have*

$$K^\pm \nabla_X^{L_2} \phi = \nabla_X^{L_2} K^\pm \phi$$

*Proof.* These statements follows from Prop 3.3.3 and the fact that $\mathfrak{R}$ is a flat isometry. □

**Theorem 3.4.3.** *On smooth sections $\phi$ of $\mathcal{S}_d^\pm \otimes E_d$,*

$$K^- D_d^+ \phi = D^+ K^+ \phi \quad \textit{and} \quad K^+ D_d^- \phi = D^- K^- \phi \tag{3.4.1}$$

*Proof.* We did most of the work already in Props C.3.8 and C.3.11. We show that $K^- D_d^+ \phi = D^+ K^+ \phi$; the other statement has a similar proof.

Since $D^\pm$ and $K^\pm$ are vertical operators, we might as well work over a particular fiber $X := X_b$ and $X^d := X_b^d$. Let $(\phi_l, \phi_r)$ be the pair of sections corresponding to $\phi$. Using Prop C.3.11, we have

$$\begin{aligned} K^- D_{X^d}^+ \phi &= K^-(D_X^+ \phi_l, D_{\overline{X}}^- \phi_r) \\ &= D_X^+ \phi_l \oplus -\mathfrak{R}^{-1} D_{\overline{X}}^- \phi_r \end{aligned}$$

By Prop C.3.8, $-\mathfrak{R}^{-1}D^-_{\overline{X}} = D^+_X\mathfrak{R}^{-1}$. Thus

$$K^-D^+_{X^d}\phi = D^+_X\phi_l \oplus D^+_X\mathfrak{R}^{-1}\phi_r$$
$$= D^+_X(\phi_l \oplus \mathfrak{R}^{-1}\phi_r) = D^+_XK^+\phi$$

□

**Proposition 3.4.4.** *Suppose $\phi$ is a smooth section of $\mathcal{S}^\pm_d \otimes E_d$. Then $K^\pm\phi$ is in* $\mathrm{Dom}(\Delta^\pm, \mathfrak{B})$. *Furthermore,*

$$K^+\Delta^+_d\phi = \Delta^+K^+\phi \quad \textit{and} \quad K^-\Delta^-_d\phi = \Delta^-K^-\phi$$

*Proof.* Suppose $\phi$ is a smooth section of $\mathcal{S}^+_d \otimes E_d$; the case of sections of $\mathcal{S}^-_d \otimes E_d$ is similar. By definition, $K^+\phi$ is in $\mathrm{Dom}(\Delta^+, \mathfrak{B})$ if and only if $K^+\phi$ is in $\mathrm{Dom}(D^+, \mathfrak{B})$ and $D^+K^+\phi$ is in $\mathrm{Dom}(D^-, \mathfrak{B})$. By Prop 3.4.1, $K^+\phi$ is in $\mathrm{Dom}(D^+, \mathfrak{B})$. Using Theorem 3.4.3, $D^+K^+\phi = K^-D^+_d\phi$; using Prop 3.4.1 again, $K^-D^+_d\phi$ is in $\mathrm{Dom}(D^-, \mathfrak{B})$.

To see that $K^+\Delta^+_d = \Delta^+K^+$, use Theorem 3.4.3 twice:

$$K^+\Delta^+_d\phi = K^+D^-_dD^+_d\phi = D^-K^-D^+_d\phi = D^-D^+K^+\phi = \Delta^+K^+\phi$$

The other case is similar. □

**Theorem 3.4.5.** *We have*

$$\exp(-\tau\Delta^\pm) = K^\pm \exp(-\tau\Delta^\pm_d)(K^\pm)^{-1}$$

*Proof.* The smoothness of the kernel of $\exp(-\tau\Delta_d^{\pm})$ implies the smoothness of the kernel of $K^{\pm}\exp(-\tau\Delta_d^{\pm})(K^{\pm})^{-1}$. By Prop. A.4.11, it suffices to check that $K^{\pm}\exp(-\tau\Delta_d^{\pm})(K^{\pm})^{-1}$ satisfies the the heat equation for $(\Delta^{\pm}, \mathfrak{B})$ with the proper initial and boundary conditions. Prop. 3.4.4 implies that this operator satisfies the heat equation and has the proper boundary conditions. The continuity of $K^{\pm}$ implies that this operator satisfies the initial condition of the heat operator. □

### 3.4.1 The determinant line bundle isomorphism

If $\Delta$ is any of the four families of operators $\Delta_d^{\pm}$ or $\Delta_{\mathfrak{B}}^{\pm}$, let $P_{[0,\alpha)}(\Delta)$, $0 < \alpha < \infty$, be the family of spectral projections onto the span of eigenspaces of eigenvalue in $[0, \alpha)$, defined over $U_{[0,\alpha)}(\Delta)$. Let $P_{(\alpha,\beta)}(\Delta) := P_{[0,\beta)}(\Delta) - P_{[0,\alpha)}(\Delta)$, $P_{(\alpha,\infty)}(\Delta) := 1 - P_{[0,\alpha)}(\Delta)$. Let $\mathcal{H}_{(\alpha,\beta)}(\Delta)$ denote the vector bundle over $U_{(\alpha,\beta)}(\Delta)$ whose fiber at a point $b$ is the union of the eigenspaces of $\Delta_b$ with eigenvalue between $\alpha$ and $\beta$, i.e., the range of $P_{(\alpha,\beta)}(\Delta_b)$. We proved in Chapter 2 that the projections $P_{[0,\alpha)}(\Delta)$ are smooth families of smoothing operators, the sets $U_{[0,\alpha)}(\Delta)$ are open, and the vector bundles $\mathcal{H}_{[0,\alpha)}(\Delta)$ are smooth.

**Proposition 3.4.6.** *The spectral projections for $\Delta_d^{\pm}$ and $\Delta_{\mathfrak{B}}^{\pm}$ are conjugate via the cutting maps. In particular,*

$$P_{[0,\alpha)}(\Delta^{\pm}) = K^{\pm} P_{[0,\alpha)}(\Delta_d^{\pm})(K^{\pm})^{-1}$$

*Furthermore, we have $U_{[0,\alpha)}(\Delta_d^{\pm}) = U_{[0,\alpha)}(\Delta_{\mathfrak{B}}^{\pm})$.*

*Proof.* In section A.6 we defined the spectral projections $P_{[0,\alpha)}(\Delta)$ using a contour integral for the heat operator at $t = 1$. Therefore this proposition is an immediate corollary of Theorem 3.4.5. The equality of the open sets $U_{[0,\alpha)}(\Delta_d^{\pm}) = U_{[0,\alpha)}(\Delta_{\mathfrak{B}}^{\pm})$ also follows from this theorem, because these are the sets where $e^{-\alpha}$ is not an eigenvalue of the heat operators at $t = 1$. □

In view of the previous proposition, let

$$U_{[0,\alpha)} := U_{[0,\alpha)}(\Delta_d^{\pm}) = U_{[0,\alpha)}(\Delta_{\mathfrak{B}}^{\pm})$$

**Proposition 3.4.7.** *For* $0 < \alpha < \beta < \infty$*, the diagram*

$$\begin{array}{ccc} \mathcal{H}_{(\alpha,\beta)}(\Delta_d^+) & \xrightarrow{D_d^+} & \mathcal{H}_{(\alpha,\beta)}(\Delta_d^-) \\ {\scriptstyle K^+}\downarrow & & \downarrow{\scriptstyle K^-} \\ \mathcal{H}_{(\alpha,\beta)}(\Delta_{\mathfrak{B}}^+) & \xrightarrow[D^+]{} & \mathcal{H}_{(\alpha,\beta)}(\Delta_{\mathfrak{B}}^-) \end{array} \tag{3.4.2}$$

*commutes. The vertical maps are flat isometries with respect to the induced* $L_2$*-metric and connection.*

*Proof.* Commutivity follows from Theorem 3.4.3. Theorem 3.4.5 implies that the vertical arrows are smooth isomorphisms. The fact that these maps are flat isometries follows from Prop. 3.4.2. □

**Proposition 3.4.8.** *The zeta-regularized determinant functions* $\det_\zeta(\Delta_d^+, \alpha)$ *and* $\det_\zeta(\Delta_{\mathfrak{B}}^+, \alpha)$*,* $\alpha > 0$*, are equal at every point* $b$ *in* $B$*.*

*Proof.* Recall the definition of the zeta-function and zeta-determinant of $\Delta$ from $\alpha > 0$. The zeta-function of $\Delta$ from $\alpha > 0$ is the function on $U_{[0,\alpha)}$

$$\zeta(s; b, \Delta, \alpha) := \mathbf{M}\left[\operatorname{Tr}\left\{P_{(\alpha,\infty)}(\Delta)\exp(-t\Delta)\right\}_b\right](s)$$

The zeta-determinant is:

$$\det_\zeta(b,\Delta,\alpha) = \exp\left\{-\frac{\partial}{\partial s}\zeta(0;b,\Delta,\alpha)\right\}$$

From the definition, it's clearly enough to show that,

$$\operatorname{Tr}\left\{P_{(\alpha,\infty)}(\Delta_d^+)\exp(-t\Delta_d^+)\right\} = \operatorname{Tr}\left\{P_{(\alpha,\infty)}(\Delta_{\mathfrak{B}}^+)\exp(-t\Delta_{\mathfrak{B}}^+)\right\}$$

Theorem 3.4.5 implies that the operators in the traces are unitarily equivalent, hence have equal traces. The proposition follows. □

**Proposition 3.4.9.** *At all points $b$ in $U_{[0,\alpha)}$, the connection 1-forms for the determinant line bundle are equal: $\mu_\alpha = \mu_{d,\alpha}$.*

*Proof.* Recall from section 2.2.4 the definition of the connection 1-form for the determinant line bundle for a spectral cut at $\alpha > 0$:

$$\mu_{d,\alpha} := \operatorname*{LIM}_{t\to 0^+}\int_t^\infty \operatorname{Tr}\left\{P_{(\alpha,\infty)}(\Delta_d^-)\nabla D_d^+\, D_d^-\exp(-\tau\Delta_d^-)\right\}\, d\tau$$

The connection 1-form $\mu_\alpha$ for $(D^+,\mathfrak{B})$ is defined similarly, except that we use the heat operator and spectral projections for $\Delta_{\mathfrak{B}}^-$. To show that $\mu_\alpha = \mu_{d,\alpha}$, it suffices to show that

$$(K^-)^{-1}P^-_{(\alpha,\infty)}(\Delta_{\mathfrak{B}}^-)\nabla D_d^+\, D_d^-\exp(-\tau\Delta_{\mathfrak{B}}^-)K^- = P^-_{(\alpha,\infty)}(\Delta_d^-)\nabla D_d^+\, D_d^-\exp(-\tau\Delta_d^-)$$

because the trace is invariant under unitary conjugation. Using Theorem 3.4.5, we have

$$(K^-)^{-1}P^-_{(\alpha,\infty)}(\Delta_{\mathfrak{B}}^-)K^- = P^-_{(\alpha,\infty)}(\Delta_d^-)$$

and

$$(K^-)^{-1}\exp(-\tau\Delta_{\mathfrak{B}}^-)K^- = \exp(-\tau\Delta_d^-)$$

Since the heat operator is smoothing, we can apply Theorem 3.4.3 to conclude

$$K^- D_d^+ = D^+ K^+ \quad \text{and} \quad K^+ D_d^- = D^- K^-$$

By Prop 3.4.2 the cutting operators $K^{\pm}$ are flat with respect to the $L_2$ connection, thus

$$K^- \nabla D_d^+ \, D_d^- = \nabla D^+ \, D^- K^-$$

□

The main theorem now follows directly from Theorem 3.4.10 and Prop 3.4.9. We restate the theorem here for convenience.

**Theorem 3.4.10.** *There is a natural geometric isomorphism of the determinant lines $\mathcal{L}(D_d^+)$ and $\mathcal{L}(D^+, \mathfrak{B})$. The isometry maps the canonical section of $\mathcal{L}(D_d^+)$ to the canonical section of $\mathcal{L}(D^+, \mathfrak{B})$.*

*Proof.* Over $U_{[0,\alpha)}$, the determinant line bundle $\mathcal{L}(D_d^+)$ is naturally isomorphic to its local representative $\mathcal{L}_{[0,\alpha)}(D_d^+) := \det \mathcal{H}_{[0,\alpha)}(\Delta_d)$. Likewise $\mathcal{L}(D^+, \mathfrak{B})$ is naturally isomorphic to its local representative $\mathcal{L}_{[0,\alpha)}(D^+, \mathfrak{B})$. Prop. 3.4.7 shows that the cutting maps induce a super vector bundle map

$$K_\alpha : \mathcal{H}_{[0,\alpha)}(\Delta_d) \to \mathcal{H}_{[0,\alpha)}(\Delta_{\mathfrak{B}})$$

and therefore a map of $\mathcal{L}_{[0,\alpha)}(D_d^+)$ to $\mathcal{L}_{[0,\alpha)}(D^+, \mathfrak{B})$. Prop. 3.4.7 also implies that $K_\alpha$ is an $L_2$ isometry, and that the induced isometry maps the canonical section of $\mathcal{L}(D_d^+)$ to the canonical section of $\mathcal{L}(D^+, \mathfrak{B})$. On overlap sets $U_{(\alpha,\beta)}$, Prop. 3.4.7 implies that the transition functions, given by $\det(D^+_{(\alpha,\beta)})$, are conjugate by the induced map $K_{(\alpha,\beta)}$. Thus the maps $K_\alpha$ patch together to give an isomorphism of the line bundles.

Prop. 3.4.8 shows that the corrections to the metric defined by zeta-regularization are equal. To see that the isometry provided by the cutting maps is in fact flat note that we already showed that the cutting maps are flat with respect to the induced $L_2$ connection. Prop. 3.4.9 shows that the connection 1-forms defined by zeta regularization $\mathcal{L}_{[0,\alpha)}(D_d^+)$ and $\mathcal{L}_{[0,\alpha)}(D^+, \mathfrak{B})$ are equal. □

### 3.4.2 A family index theorem

Following [5], we recall the definition of the index bundle for a family $\mathsf{D}$ of Dirac operators associated to a family of compact manifolds without boundary over a compact base $B$. Let $\ker(\mathsf{D})$ denote the family of super vector spaces over $B$ defined by

$$\ker(\mathsf{D})_b := \ker(D_b^+) \oplus \ker(D_b^-)$$

In general, $\ker(\mathsf{D})$ is not a super vector bundle because the ranks of the kernels are not locally constant. In the special case where $\ker(\mathsf{D})$ has constant rank, they show that it is in fact a super vector bundle using the spectral projections $P_{[0,\alpha)}(\Delta^\pm)$. In that case, for every point $b$ there is a neighborhood of $b$ and a small enough $\epsilon > 0$ such that over the neighborhood $P_{[0,\epsilon)}(\Delta^\pm)$ are projections onto $\ker(D^{\pm})$. Compactness of $B$ then implies that there is an $\epsilon > 0$ such that $P_{[0,\epsilon)}(\Delta^{\pm})$ are projections over all of $B$ onto the kernels. Since this is a smooth family of smoothing operators of finite rank, its range is a smooth vector bundle. Thus $\ker(\mathsf{D})$ is a representative of a class $[\ker(\mathsf{D})] := [\ker(D^+)] - [\ker(D^-)]$ in $K(B)$. The index of $D^+$ is defined to be this class: $\mathrm{Ind}(D^+) := [\ker(\mathsf{D})]$.

If $\ker(\mathsf{D})$ is not constant rank, they construct finite rank perturbations $\tilde{\mathsf{D}}$ such that $\tilde{D}^+$ is surjective, and $\tilde{D}^-$ is injective. Specifically, they show that there exists an integer $N > 0$ and a monomorphism $R^+ : \mathbb{C}^N \to \pi_* S^-$ such that

$$\tilde{D}^+(\phi \oplus w) := D^+\phi + R^+ w$$

is surjective and, defining $R^- := (R^+)^*$,

$$\tilde{D}^-\psi = D^-\psi \oplus R^-\psi$$

is injective. Defining $\ker(\tilde{\mathsf{D}})$ as before, they show that it is a smooth vector bundle, and that the element of $K(B)$

$$\mathrm{Ind}(D^+) := [\ker(\tilde{\mathsf{D}})] - [\mathbb{C}^N]$$

is independent of $N$ and the other choices made in the construction of $R^+$.

The map $R^+$ is constructed as follows: at every point $b$ in $B$ there is an $\alpha > 0$ such that $b \in U_{[0,\alpha)}$. Over an open subset $U_b \subset U_{[0,\alpha)}$ containing $b$, $\mathcal{H}^-_{[0,\alpha)}$ is trivial. Fix a trivialization $r_b : U_b \times \mathbb{C}^{n_b} \to \mathcal{H}^-_{[0,\alpha)}|U_b$ and a cutoff function $\chi_b$ supported on $U_b$ and identically 1 near $b$. If $\mathcal{U}$ is a finite subcover of $\{U_b\}$, let $N = \sum_{\mathcal{U}} n_b$, and $R^+ : B \times \mathbb{C}^N \to \pi_* S^-$ be defined by

$$R^+(p,w) = \sum_{\mathcal{U}} \chi_b(p) r_b(w)$$

Clearly the construction of $R^+$ depends only on the existence of the super vector bundles $\mathcal{H}_{[0,\alpha)}$, the cover $U_{[0,\alpha)}$, and the compactness of $B$. As such, we can apply the same construction to the families of boundary value problems considered here. The definitions of $\tilde{D}^\pm$ given above work in this new context as well. The boundary condition applies in an obvious way to these new operators. We define

$$\mathrm{Ind}(D^+, \mathfrak{B}) := [\ker(\tilde{\mathsf{D}}, \mathfrak{B})] - [\mathbb{C}^N]$$

The proof given in [5] that this definition is independent of the choice of $R^+$ goes through verbatim.

**Theorem 3.4.11.** *We have*

$$\mathrm{Ind}(D_d^+) = \mathrm{Ind}(D^+, \mathfrak{B})$$

*Proof.* Let $R_d^+$ be constructed as above for $D_d^+$. Using Prop. 3.4.7, we see that the choices of data made in defining $R^+$ imply corresponding choices for constructing $R^+$ for $(D^+, \mathfrak{B})$: the underlying open covers $\{U_{[0,\alpha)}\}$ are the same, a local trivialization of $\mathcal{H}_{[0,\alpha)}(\Delta_d^-)$ induces a local trivialization of $\mathcal{H}_{[0,\alpha)}(\Delta^-)$ by composing with $K^-$, and we may take the cutoff functions to be the same. When $R^+$ is constructed with these choices, we see that

$$K^- R^+ = R_d^+$$

Extending $K^+$ to a map

$$\pi_*(\mathcal{S}_d^+ \otimes E_d) \oplus (B \times \mathbb{C}^N) \to \pi_*(\mathcal{S} \otimes E)^+ \oplus (B \times \mathbb{C}^N)$$

by making it the identity on $(B \times \mathbb{C}^N)$, we have

$$K^- \tilde{D}_d^+ = \tilde{D}^+ K^+$$

It follows that $K^+$ induces an isomorphism of kernels

$$K^+ : \ \ker(\tilde{\mathsf{D}}_d) \to \ker(\tilde{\mathsf{D}}, \mathfrak{B})$$

and since $\mathbb{C}^N$ is the same for both families, $\mathrm{Ind}(D_d^+) = \mathrm{Ind}(D^+, \mathfrak{B})$ $\square$

# Chapter 4

# The odd problem

In section 2.2.2.1 we introduced the odd problem, a class of local elliptic boundary value problems for the Dirac operator. We also showed that the odd problem is a geometric local boundary value problem. It follows from the results of that chapter that families of such boundary value problems have an associated determinant line bundle. When the geometry of the family is a product near the boundary (as we always assume) the boundary family has a well-defined determinant line bundle with Quillen metric and compatible Bismut-Freed connection. The boundary has even-dimensional compact closed fibers. The family connection induces a connection for the boundary family, and the spin-structure of the family induces a spin-structure on the boundary family.

The main theorem of this chapter asserts that there is a geometric isomorphism of the *square* of the determinant line bundle associated to the boundary value problem with the determinant line bundle of the boundary. Recall the defining data of the odd problem. Let $\pi : M \to B$ be a collared family of odd-dimensional spin manifolds with boundary, with each fiber diffeomorphic to $X$. Let $S$ denote the (ungraded) spinor bundle associated to the spin-structure on the vertical tangent bundle, and let $E \to M$ be an auxiliary complex vector bundle with hermetian metric and compatible connection. Let $D_M$ denote the Dirac operator for $S \otimes E$. We suppose that $E$ splits as an orthogonal direct sum over the collar neighborhood: $E|U \cong E^+ \oplus E^-$. We allow for the possibility that either $E^+$ or $E^-$ is rank zero (but not both). Let $S_{\partial M}$ be the associated bundle of boundary spinors; since the vertical tangent bundle of the boundary is even-rank, $S_{\partial M}$ is $\mathbb{Z}/2\mathbb{Z}$-graded into $\pm 1$-eigenspaces of the normalized relative volume form: $S_{\partial M} \cong S^+_{\partial M} \oplus S^-_{\partial M}$. Thus $S_{\partial M} \otimes E$ is $\mathbb{Z}/2\mathbb{Z}$-graded:

$$(S_{\partial M} \otimes E)^{\pm} := S^{\pm}_{\partial M} \otimes E^+ \oplus S^{\mp}_{\partial M} \otimes E^-$$

The boundary conditions $\mathfrak{B}_{\pm} : S_{\partial M} \otimes E \to S_{\partial M} \otimes E$ are orthogonal projection onto the $\pm$-graded subbundle. We distinguish the domain and codomain copies of $S \otimes E$ for the Dirac operator $D_M$ by grading the domain copy $+$ and the codomain copy $-$. Let $D^+ := D_M : C^\infty(S \otimes E)^+ \to C^\infty(S \otimes E)^-$ be the graded version of $D_M$, $D^- := (D^+)^*$ its formal adjoint, and let $D$ denote the direct sum $D := D^+ \oplus D^-$. Let $(D^+, \mathfrak{B})$ denote $D^+$ with the boundary condition $\mathfrak{B}_- = 0$ and let $(D^-, \mathfrak{B})$ denote $D^-$ with the boundary condition $\mathfrak{B}_+$. Let $D_\partial$ denote the graded boundary Dirac operator coupled to $E|\partial^\pi M$. That is

$$D^+_\partial : C^\infty(S^+_{\partial M} \otimes E^+ \oplus S^-_{\partial M} \otimes E^-) \to C^\infty(S^-_{\partial M} \otimes E^+ \oplus S^+_{\partial M} \otimes E^-)$$

The main theorem of this chapter is

**Theorem 4.0.12.** *There is a geometric isomorphism*

$$\mathcal{L}(D^+, \mathfrak{B})^2 \cong \mathcal{L}(D^+_\partial) = \mathcal{L}(D^+(E^+)) \otimes \mathcal{L}(D^+(E^-))^{-1}$$

To prove this result, we first consider special cases of the problem where we can compute the determinant line bundles directly. We then relate the general problem to these special cases using a construction known as the Bunke isometry. Using the Bunke isometry forces us to consider certain pertubations of the Dirac operator. We make a parallel transport argument to show that the determinant line bundle for such a perturbed Dirac operator is geometrically isomorphic to the determinant line for the unperturbed operator.

We also give a families index theorem for the odd problem:

**Theorem 4.0.13.** *We have*

$$\operatorname{Ind}(D^+, \mathfrak{B}) = \operatorname{Ind}(D_\partial(E^+) = -\operatorname{Ind}(D_\partial(E^-))$$

The method of proof is not related to the proof of the determinant line bundle theorem. Rather, it is based on the Agranovich-Dynin Theorem and is a slight generalization of the proof of Theorem B of [11].

## 4.1 Special cases

In this section, we consider two special cases of the odd-type boundary value problem. The first special case is that of a family of compact odd-dimensional manifolds *without* boundary. We show that the associated determinant line $\mathcal{L}(D^+)$ is trivial, and that its square is geometrically trivial. That is, there is a flat isometry of $\mathcal{L}(D^+)^2$ with the trivial line. If we further suppose that the family of manifolds is a family of spin-doubles, then we can strengthen that statement to say that $\mathcal{L}(D^+)$ is geometrically trivial. The second special case is that of a cylindrical family $M := [-1, 1] \times N$, where $N \to B$ is a family of closed even-dimensional manifolds. On the two boundary components of $M$, $N_\pm = \{\pm 1\} \times N$ we impose boundary conditions with the symmetry $\mathfrak{B}_- = (1 - \mathfrak{B}_+)$, where $\mathfrak{B}_\pm$ are the boundary conditions at $N_\pm$. We show that the determinant line bundle for $M$ is geometrically isomorphic to the square of the determinant line for $N$.

The results about the determinant line of a family of compact closed manifolds (but not the stonger statement about spin-doubles) are in [12]. To the best of our knowledge, this is the first place these results appear in the literature, although they may have been known previously.

### 4.1.1 Families of closed odd-dimensional manifolds

Throughout this section $\pi : M \to B$ is a Riemmanian fiber bundle with fiber diffeomorphic to $X$, a compact odd-dimensional manifold *without* boundary. The family connection is $H$. We suppose that the vertical tangent bundle has a spin-structure $\mathcal{F}_{\mathrm{Spin}}(M/B)$, and $\mathcal{S} = S(M/B)$ is the associated spinor bundle. Since the dimension of the fibers is odd, $\mathcal{S}$ is not graded. The construction of the determinant line bundle is based on graded Clifford module bundles and chiral Dirac operators. We construct a graded Clifford module $S$ by twisting $\mathcal{S}$ with a trivial graded vector bundle:

$$S = S^+ \oplus S^- := \mathcal{S} \otimes (\mathbb{C} \oplus \mathbb{C})$$

and let covectors $\xi$ act by

$$\mathrm{cl}(\xi) := c(\xi) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $c(\xi)$ is the Clifford action of $\xi$ on $\mathcal{S}$. Then $S$ is a graded Cliff$(M/B)$-module where the action of covectors is odd. Giving the factor $(\mathbb{C} \oplus \mathbb{C})$ the trivial connection, $S^+ \oplus S^-$ has a connection compatible with Clifford multiplication. Let $E \to M$ be an auxiliary complex vector bundle with hermetian metric and compatible connection. Let $D_M$ be the family of (ungraded) Dirac operators for $\mathcal{S} \otimes E$ and let

$$D^\pm : C^\infty(S^\pm \otimes E) \to C^\infty(S^\mp \otimes E)$$

be the chiral Dirac operators coupled to $E$.

Since the dimension of the fibers is odd, Clifford multiplication by the relative volume form $\mathrm{cl}(\mathrm{vol}_{M/B})$ is an odd endomorphism of $S \otimes E$ commuting with Clifford multiplication by covectors. Let $\omega := i^p \mathrm{cl}(\mathrm{vol}_{M/B})$, where $p = (\dim(X) + 1)/2$; $i^p$ is a normalizing factor chosen so that $\omega^2 = 1$. By our conventions, $i^p \mathrm{cl}(\mathrm{vol}_{M/B}) = 1$, so that

$$\omega = 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Let $\omega^{\pm} = \omega|_{S^{\pm}\otimes E}$ be the components of $\omega$. Since $\mathrm{vol}_{M/B}$ is flat with respect to the Levi-Civita connection $\nabla^{M/B}$ (in vertical directions), and since $\omega$ commutes with Clifford multiplication by vertical covectors, $\omega$ commutes with the Dirac operator in the sense that $\omega^- D^+ = D^- \omega^+$. Thus $(\omega^- D^+)^2 = \Delta^+$.

The key fact that we will use in this subsection is:

**Proposition 4.1.1.** *The imaginary part of the connection form vanishes on a closed odd-dimensional manifold.*

*Proof.* As a preliminary, we claim that $\omega$ commutes with each of the operators $D$, $\exp(-\tau\Delta)$, $P_{(\alpha,\infty)}$, and $\nabla D$. We already showed that $D$ commutes with $\omega$. It therefore commutes with $\Delta = D^2$ and the spectral operators $\exp(-\tau\Delta)$, and $P_{(\alpha,\infty)}$. Finally, it commutes with $\nabla D := [\nabla, D]$ because it is flat and commutes with $D$.

As explained in Chapter 2, the imaginary part of the connection 1-form $\mu_\alpha$ is given by

$$\mathrm{Im}\,\mu_\alpha = \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \mathrm{Str}\left\{P_{(\alpha,\infty)} \nabla D\, D \exp(-\tau\Delta)\right\}\, d\tau$$

Since $\omega^2 = 1$ and $\omega$ is odd, we have:

$$\begin{aligned} &= \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \mathrm{Str}\left\{\omega^2 P_{(\alpha,\infty)} \nabla D\, D \exp(-\tau\Delta)\right\}\, d\tau \\ &= \operatorname*{LIM}_{t\to 0^+} \int_t^\infty -\mathrm{Str}\left\{\omega P_{(\alpha,\infty)} \nabla D\, D \exp(-\tau\Delta)\omega\right\}\, d\tau \end{aligned}$$

Since $\omega$ commutes with $P_{(\alpha,\infty)}\nabla D\, D\exp(-\tau\Delta)$, we have:

$$= \underset{t\to 0^+}{\mathrm{LIM}} \int_t^\infty -\mathrm{Str}\left\{\omega^2 P_{(\alpha,\infty)}\nabla D\, D\exp(-\tau\Delta)\right\}\, d\tau$$

□

To show that $\mathcal{L}(D^+)$ we find nonvanishing sections of each $\mathcal{L}_{[0,\alpha)}(D^+)$ that agree under the transition functions. In fact we have nonvanishing sections of each $\mathcal{L}_{[0,\alpha)}(D^+)$: the normalized volume form $\omega$ induces isomorphisms

$$\omega^{\pm}_{[0,\alpha)} : \mathcal{H}^{\pm}_{[0,\alpha)} \xrightarrow{\sim} \mathcal{H}^{\mp}_{[0,\alpha)}$$

It follows that $\det(\omega^{+}_{[0,\alpha)})$ is a nonvanishing section of $\mathcal{L}_{[0,\alpha)}(D^+)$.

**Proposition 4.1.2.** *The sections*

$$s_\alpha := \frac{\det(\omega^{+}_{[0,\alpha)})}{\det_\zeta(\Delta^+,\alpha)^{\frac{1}{2}}}$$

*are unit-length and flat.*

*Proof.* By construction, the normalized volume form is flat in the $L_2$-connection, and unitary homomorphisms are unit-length in the $L_2$ norm. The proposition is therefore a corollary of Prop 4.1.1. □

The failure of these sections to patch together on overlaps $U_{[0,\alpha)}\cap U_{[0,\beta)}$,

$\alpha < \beta$ is measured by the ratio

$$\begin{aligned} s_\beta^{-1} \otimes s_\alpha \otimes \det(D^+_{(\alpha,\beta)}) \\ &= \left(\frac{\det_\zeta(\Delta^+,\beta)}{\det_\zeta(\Delta^+,\alpha)}\right)^{\frac{1}{2}} \det(\omega^+_{[0,\beta)})^{-1} \otimes \det(\omega^+_{[0,\alpha)}) \otimes \det(D^+_{(\alpha,\beta)}) \\ &= \frac{\det(\omega^- D^+_{(\alpha,\beta)})}{\det(\Delta^+_{(\alpha,\beta)})^{\frac{1}{2}}} \end{aligned}$$

This ratio is a real-valued function; at the point $b \in B$, its value is

$$\frac{\det(\omega^- D^+_{(\alpha,\beta)})}{\det(\Delta^+_{(\alpha,\beta)})^{\frac{1}{2}}} = \prod_{\substack{\lambda \in \operatorname{Spec} \omega^- D^+ \\ \alpha < \lambda^2 < \beta}} \frac{\lambda}{|\lambda|} = \prod_{\substack{\lambda \in \operatorname{Spec} \omega^- D^+ \\ -\sqrt{\beta} < \lambda < -\sqrt{\alpha}}} -1$$

**Theorem 4.1.3.** *The square of the determinant line for a family of compact, closed, odd-dimensional manifolds is geometrically trivial.*

*Proof.* By the foregoing calculation, the ratio of $s_\beta^2$ to $s_\alpha^2 \otimes \det(D^+_{(\alpha,\beta)})^2$ on the overlap set $U_{(\alpha,\beta)}$ is

$$\Big(\prod_{\substack{\lambda \in \operatorname{Spec} \omega^- D^+ \\ -\sqrt{\beta} < \lambda < -\sqrt{\alpha}}} -1\Big)^2 = 1$$

Thus the sections $s_\alpha^2$ of $\mathcal{L}(D^+_{[0,\alpha)})^2$ patch together. By Prop 4.1.2 these sections are unit-length and flat. □

The next goal is to show that $\mathcal{L}(D^+)$ is trivial, i.e., has a non-vanishing section. This will be a corollary of the following

**Proposition 4.1.4.** *There are smooth non-vanishing complex-valued functions* $\det_\zeta(\omega^- D^+_{(\alpha,\infty)})$ *on* $U_{[0,\alpha)}$, *with the property that*

$$\frac{\det_\zeta(\omega^- D^+_{(\alpha,\infty)})}{\det_\zeta(\omega^- D^+_{(\beta,\infty)})} = \det(\omega^- D^+_{(\alpha,\beta)}) = \det(\Delta^+_{(\alpha,\beta)})^{\frac{1}{2}} \cdot \prod_{\substack{\lambda \in \operatorname{Spec} \omega^- D^+ \\ -\sqrt{\beta} < \lambda < -\sqrt{\alpha}}} -1 \qquad (4.1.1)$$

We defer the proof.

**Corollary 4.1.5.** *The determinant line bundle $\mathcal{L}(D^+)$ has a nonvanishing section.*

*Proof.* The modified sections

$$\tilde{s}_\alpha := \frac{\det(\omega^+_{[0,\alpha)})}{\det_\zeta(\omega^- D^+_{(\alpha,\infty)})}$$

patch together into a nonvanishing section of $\mathcal{L}(D^+)$. □

Recall the definition of the zeta-function of $\Delta^+$:

$$\zeta(s; b, \Delta^+, \alpha) := \mathbf{M}\left[\mathrm{Tr}\left\{P^+_{(\alpha,\infty)} \exp(-t\Delta^+)\right\}_b\right](s)$$

The eta-function of $\omega^- D^+$ is defined similarly:

$$\eta(s; b, \omega^- D^+, \alpha) := \mathbf{M}\left[\mathrm{Tr}\left\{P^+_{(\alpha,\infty)} \omega^- D^+ \exp(-t\Delta^+)\right\}_b\right]\left(\frac{s+1}{2}\right)$$

These functions are meromorphic on $\mathbb{C}$, holomorphic at $s = 0$, and satisfy the functional identities

$$\zeta(s; b, \Delta^+, \alpha) - \zeta(s; b, \Delta^+, \beta) = \sum_{\substack{\lambda \in \mathrm{Spec}(\Delta^+_b) \\ \alpha < \lambda < \beta}} |\lambda|^{-s}$$

and

$$\eta(s; b, \omega^- D^+, \alpha) - \eta(s; b, \omega^- D^+, \beta) = \sum_{\substack{\lambda \in \mathrm{Spec}(\omega^- D^+_b) \\ \alpha < \lambda^2 < \beta}} \mathrm{sgn}(\lambda)|\lambda|^{-s}$$

Let $\zeta_\alpha(b) := \zeta(0; b, \Delta^+, \alpha)$, and let $\eta_\alpha(b) := \eta(0; b, \omega^- D^+, \alpha)$. From the fact that $(\omega^- D^+)^2 = \Delta^+$ and the functional identities above we have

$$\zeta_\alpha(b) - \zeta_\beta(b) = \sum_{\substack{\lambda \in \mathrm{Spec}(\omega^- D_b^+) \\ \alpha < \lambda^2 < \beta}} 1$$

and

$$\eta_\alpha(b) - \eta_\beta(b) = \sum_{\substack{\lambda \in \mathrm{Spec}(\omega^- D_b^+) \\ \alpha < \lambda^2 < \beta}} \mathrm{sgn}(\lambda)$$

Thus $\frac{1}{2}\left[(\zeta_\alpha(b) - \zeta_\beta(b)) - (\eta_\alpha(b) - \eta_\beta(b))\right]$ counts the multiplicity of the spectrum of $\omega^- D^+$ in $(-\sqrt{\beta}, -\sqrt{\alpha})$. That is,

$$\frac{1}{2}\left[(\zeta_\alpha(b) - \zeta_\beta(b)) - (\eta_\alpha(b) - \eta_\beta(b))\right] = \sum_{\substack{\lambda \in \mathrm{Spec}(\omega^- D_b^+) \\ -\sqrt{\beta} < \lambda < -\sqrt{\alpha}}} 1$$

*Proof of Prop 4.1.4.* Define

$$\det{}_\zeta(b, \omega^- D^+_{(\alpha,\infty)}) := \det{}_\zeta(b, \Delta^+, \alpha)^{\frac{1}{2}} \exp \frac{\pi i}{2}(\zeta_\alpha(b) - \eta_\alpha(b)) \qquad (4.1.2)$$

Then

$$\begin{aligned} \frac{\det_\zeta(b, \omega^- D^+_{(\alpha,\infty)})}{\det_\zeta(b, \omega^- D^+_{(\beta,\infty)})} &= \frac{\det_\zeta(b, \Delta^+_{(\alpha,\infty)})}{\det_\zeta(b, \Delta^-_{(\beta,\infty)})} \\ &\quad \cdot \exp \frac{\pi i}{2}\left[(\zeta_\alpha(b) - \zeta_\beta(b)) - (\eta_\alpha(b) - \eta_\beta(b))\right] \\ &= \prod_{\substack{\lambda \in \mathrm{Spec}\, \omega^- D^+ \\ \alpha < \lambda^2 < \beta}} |\lambda| \prod_{\substack{\lambda \in \mathrm{Spec}\, \omega^- D^+ \\ -\sqrt{\beta} < \lambda < -\sqrt{\alpha}}} -1 \\ &= \det(b, \omega^- D^+_{(\alpha,\beta)}) \end{aligned}$$

□

When the Dirac operator happens to be invertible, i.e., the kernel and cokernel of $D_b^+$ are trivial at every $b \in B$, we can strengthen these results.

**Theorem 4.1.6.** *Suppose that $D^+$ is a family of Dirac operators for a compact, closed, odd-dimensional manifold, and $D^+$ is invertible. Then $\mathcal{L}(D^+)$ is geometrically trivial. That is, there is a flat section of unit-length. In particular, this is true for the spin-double of a family of connected spin-manifolds with non-trivial boundary.*

*Proof.* Because the Dirac operator is invertible, $\det(D^+)$ is a non-vanishing section of $\mathcal{L}(D^+)$. The length of this section is

$$\|\det(D^+)\| = \langle \det(D^+_{[0,\alpha)}), \det(D^+_{[0,\alpha)})\rangle^{\frac{1}{2}} \cdot \det_\zeta(\Delta^+, \alpha)^{\frac{1}{2}} = \det_\zeta(\Delta^+, 0)^{\frac{1}{2}}$$

Thus $\frac{\det(D^+)}{\det_\zeta(\Delta^+,0)^{\frac{1}{2}}}$ is unit-length. By Prop 4.1.2, $\frac{\det(\omega^+_{[0,\alpha)})}{\det_\zeta(\Delta^+,\alpha)^{\frac{1}{2}}}$ is a unit-length, flat section of $\mathcal{L}(D^+)$ over $U_{[0,\alpha)}$. The ratio of these sections is

$$\begin{aligned}
\frac{\det_\zeta(\Delta^+, \alpha)^{\frac{1}{2}}}{\det_\zeta(\Delta^+, 0)^{\frac{1}{2}}} \det(\omega^+_{[0,\alpha)})^{-1} &\otimes \det(D^+_{[0,\alpha)}) \\
&= \det(\Delta^+_{[0,\alpha)})^{-\frac{1}{2}} \det(\omega^- D^+_{[0,\alpha)}) \\
&= \prod_{\substack{\lambda \in \operatorname{Spec} \omega^- D^+ \\ -\sqrt{\alpha} < \lambda < 0}} -1
\end{aligned}$$

This is clearly a constant function. It follows that $\frac{\det(D^+)}{\det_\zeta(\Delta^+,0)^{\frac{1}{2}}}$ is flat. □

### 4.1.2 Families of odd-dimensional cylinders

In this subsection, $\pi_N : N \to B$ is a family of manifolds, with fibers diffeomorphic to $Y$, a compact even-dimensional manifold without boundary. The family has a connection $H_N : TN \to T(N/B)$ and a spin-structure $\mathcal{F}_{\mathrm{Spin}}(N/B)$ on the vertical tangent bundle. Let $S_N$ be the associated spinor bundle; since the vertical tangent bundle is even-rank, $S_N$ is $\mathbb{Z}/2\mathbb{Z}$-graded into $\pm 1$-eigenspaces of the normalized relative volume form: $S_N \cong S_N^+ \oplus S_N^-$. Let $E_N \cong E_N^+ \oplus E_N^-$ be an auxiliary $\mathbb{Z}/2\mathbb{Z}$-graded complex vector bundle with metric and compatible connection. We allow for the possibility that either $E_N^+$ or $E_N^-$ is rank 0 (but not both). $S_N \otimes E_N$ is $\mathbb{Z}/2\mathbb{Z}$-graded:

$$(S_N \otimes E_N)^{\pm} := S_N^{\pm} \otimes E_N^+ \oplus S_N^{\mp} \otimes E_N^-$$

Let $\mathfrak{B}_{\pm} : S_N \otimes E_N \to S_N \otimes E_N$ be orthogonal projection onto the $\pm$-graded subbundle. Let $D_N : C^\infty(S_N \otimes E_N) \to C^\infty(S_N \otimes E_N)$ be the Dirac operator, and

$$D_N^{\pm} : C^\infty((S_N \otimes E_N)^{\pm}) \to C^\infty((S_N \otimes E_N)^{\mp})$$

its graded components. Let $\mathcal{L}(D_N^+)$ be the determinant line bundle for this family of Dirac operators.

We make a family of odd-dimensional manifolds with boundary by setting $M := [-1, 0] \times N$ and $\pi := \pi_N \circ \mathrm{pr}_2 : M \to B$, where $\mathrm{pr}_1$ and $\mathrm{pr}_2$ are the natural projections on the first and second factors of $[-1, 0] \times M$. Throughout this section, $u : [-1, 0] \to \mathbb{R}$ is the natural coordinate function. Identifying $T(M/B)$ with $T[-1, 0] \oplus T(N/B)$, we give $T(M/B)$ the product metric and spin-structure. Let $H := (\mathrm{pr}_1)_* \oplus H_N$ be the family connection. Let $S_M$ denote the associated spinor bundle. Let $E := \mathrm{pr}_2^{-1} E_N$, with the pullback metric and connection. Let $D_M : C^\infty(S_M \otimes E) \to C^\infty(S_\otimes E)$ be the associated Dirac operator.

**Proposition 4.1.7.** *As complex vector bundles with metric and connection,*

$$S_M \cong \mathrm{pr}_2^{-1} S_N$$

*Furthermore,*

$$D_M = \sigma\partial_u + D_N, \qquad \nabla D_M = \nabla D_N$$

*Proof.* The formulas $S \cong \mathrm{pr}_2^{-1} S_N$ and $D_M = \sigma\partial_u + D_N$ are standard for metric products $\mathbb{R} \times Y$ when the dimension of $Y$ is even. The last formula follows from $D_M = \sigma\partial_u + D_N$ and the fact that the family connection on $M$ is pulled back from the boundary, so that $\nabla(\sigma\partial_u) = 0$. □

As explained in section 4 and just as in the case of a family of closed odd-dimensional manifolds, we distinguish the domain and codomain copies of $S \otimes E$ for the Dirac operator $D_M$ by grading the domain copy + and the codomain copy −. Let $D^+ := D_M : C^\infty(S \otimes E)^+ \to C^\infty(S \otimes E)^-$ be the graded version of $D_M$, and $D^- := (D^+)^*$ its formal adjoint. (Since $D_M$ is formally self-adjoint, these differential operators are distinguished only by their grading. This will not be the case once we impose boundary conditions.) As usual $D$ denotes the direct sum $D := D^+ \oplus D^-$.

We impose odd-type boundary conditions on $D^+$ of a special kind. Let

$$N_- = \{-1\} \times N, \qquad N_+ = \{0\} \times N$$

denote the two ends of $\partial M$. By the outward normal first convention, $N_\pm$ have induced orientations and spin-structures, such that

$$N_+ \cong N, \qquad N_- \cong \overline{N} \tag{4.1.3}$$

where $\overline{N}$ indicates $N$ with the opposite orientation and spin-structure. It follows that

$$S_M \otimes E|N_- \overset{opp}{\cong} S_N \otimes E_N \cong S_M \otimes E|N_+ \tag{4.1.4}$$

where *opp* indicates the natural isomorphism of spinors for opposite spin-structures. Note that this map reverses the grading on spinors in even dimensions. Using these identifications, the boundary conditions $\mathfrak{B}$ we impose are

$$\mathfrak{B}_-(f|_{N_+}) = 0, \quad \mathfrak{B}_+(f|_{N_-}) = 0 \tag{4.1.5}$$

These are odd-type boundary conditions. Let $\mathcal{L}(D^+, \mathfrak{B})$ denote the associated determinant line bundle with Quillen metric and compatible Bismut-Freed connection. Our main goal in this subsection is to prove

**Theorem 4.1.8.** *There is a natural geometric isomorphism*

$$\mathcal{L}(D^+, \mathfrak{B}) \cong \mathcal{L}(D_N^+)$$

Note that this is a stronger result than our main theorem for odd-type problems would give.

The adjoint boundary conditions are

$$\mathfrak{B}_+(f|_{N_+}) = 0, \quad \mathfrak{B}_-(f|_{N_-}) = 0 \tag{4.1.6}$$

The adjoint boundary conditions are the natural boundary conditions for $D^-$. The natural boundary conditions for $\Delta^{\pm} := D^{\mp}D^{\pm}$ are

$$f \text{ satisfies } (D^{\pm}, \mathfrak{B}), \text{ and } D^{\pm}f \text{ satisfies } (D^{\mp}, \mathfrak{B}) \tag{4.1.7}$$

Again, for simplicity we denote these boundary conditions $(\Delta^{\pm}, B)$.

Let $\mathcal{H}^0(\Delta^{\pm})$ denote the $L_2$-closures of the $\pm$-graded copies of $\pi_*(S \otimes E)$, and let $\mathcal{H}^2(\Delta^{\pm}, \mathfrak{B})$ denote the $H^2$-sections satisfying the boundary condition of $(\Delta^{\pm}, \mathfrak{B})$. More generally, let $\mathcal{H}^{2n}(\Delta^{\pm}, \mathfrak{B})$ for $n$ in $\mathbb{Z}^{>1}$ denote the $H^{2n}$-sections $\psi$ satisfying the boundary condition of $(\Delta^{\pm}, \mathfrak{B})$ such that $\Delta^{+}\psi$ is in $\mathcal{H}^{n-1}(\Delta^{\pm}, \mathfrak{B})$. Finally, let

$$\mathcal{H}(\Delta^{\pm}, \mathfrak{B}) := \bigcap_{n \geq 1} \mathcal{H}^{2n}(\Delta^{\pm}, \mathfrak{B})$$

(To intersect these spaces, we identify them with their image in $\mathcal{H}^0(\Delta^{\pm})$ under the Sobolev embedding.) By definition, the image of $D^{\pm}$ on $\mathcal{H}(\Delta^{\pm}, B)$ is in $\mathcal{H}(\Delta^{\mp}, B)$. Eigensections of $\Delta_b^{\pm}$ and finite sums of eigensections are clearly in $\mathcal{H}(\Delta^{\pm}, \mathfrak{B})_b$. It follows that $\mathcal{H}(\Delta^{\pm}; (\alpha, \beta))$ is a subbundle of $\mathcal{H}(\Delta^{\pm}, B)$.

Our main tool for proving Theorem 4.1.8 will be $L_2$ unitary and flat, graded inclusions

$$U^{\pm} : \mathcal{H}(\Delta_N^{\pm}) \to \mathcal{H}(\Delta^{\pm}, B)$$

defined by a volume-scaled pullback from $N$ to $M$.

We have the following general situation. $M$ and $N$ are oriented Riemannian fiber bundles over $B$, and $\pi_N^M : M \to N$ is also an oriented Riemannian fiber bundle, compatible with $\pi_M$ and $\pi_N$ and the connections $H_M$ and $H_N$ in the sense that:

1. $\pi_M = \pi_N \pi_N^M$, so that $T(M/B) \cong T(M/N) \oplus (\pi_N^M)^{-1} T(N/B)$;
2. $H_M = H_{M/N} \oplus (\pi_N^M)^* H_N$; and
3. the metric and orientation of $T(M/B)$ is the direct sum metric and orientation.

Furthermore, there is a vector bundle $W$ over $N$, and $V := (\pi_N^M)^{-1} W$ has the pull-back metric and connection.

We want to relate the $L_2$-metric and connection of $(\pi_N)_* W$ to the $L_2$-metric and connection of $(\pi_M)_* V$. Recall that for any smooth family of vector bundles $V$, we defined the $L_2$-metric of $\pi_* V$ to be

$$\langle f, g \rangle_{L_2} := \int_{M/B} H^* \operatorname{vol}_{M/B} \langle f, g \rangle$$

and the $L_2$-connection to be $\nabla^{L_2} := \nabla^V + \frac{1}{2} k_{M/B}$ where $k_{M/B}$ is the horizontal 1-form defined implicitly by

$$k_{M/B} \wedge H^* \operatorname{vol}_{M/B} = d_M H^* \operatorname{vol}_{M/B}$$

modulo terms of lower vertical degree.

**Proposition 4.1.9.** *We have*

$$H_M^* \operatorname{vol}_{M/B} = H_{M/N}^* \operatorname{vol}_{M/N} \wedge (\pi_N^M)^* (H_{N/B}^* \operatorname{vol}_{N/B})$$

*and*

$$k_{M/B} = k_{M/N} + (\pi_N^M)^* k_{N/B}$$

*Proof.* The first identity follows because the metric and family connection are products. The second follows from the first; computing modulo terms of lower vertical degree,

$$\begin{aligned} d_M H_M^* \operatorname{vol}_{M/B} &= d_M \left[ H_{M/N}^* \operatorname{vol}_{M/N} \wedge (\pi_N^M)^* (H_{N/B}^* \operatorname{vol}_{N/B}) \right] \\ &= k_{M/N} \wedge H_{M/N}^* \operatorname{vol}_{M/N} \wedge (\pi_N^M)^* (H_{N/B}^* \operatorname{vol}_{N/B}) \\ &\quad + (-1)^{M/N} H_{M/N}^* \operatorname{vol}_{M/N} \wedge (\pi_N^M)^* (k_{N/B} \wedge H_{N/B}^* \operatorname{vol}_{N/B}) \\ &= (k_{M/N} + (\pi_N^M)^* k_{N/B}) \wedge H_M^* \operatorname{vol}_{M/B} \end{aligned}$$

□

The $M/N$-fiber volume is the smooth function on $N$ given by

$$\operatorname{vol}(M/N) := \int_{M/N} H_{M/N}^* \operatorname{vol}_{M/N}$$

We say that $k_{M/N}$ is constant along the fiber if

$$k_{M/N} = (\pi_N^M)^* \frac{d_N \operatorname{vol}(M/N)}{\operatorname{vol}(M/N)}$$

**Proposition 4.1.10.** *Suppose that $k_{M/N}$ is constant along the fiber. Then the volume-corrected pullback map $U_N^M : (\pi_N)_* W \to (\pi_M)_* V$ given by*

$$U_N^M : s \mapsto (\pi_N^M)^* \left[ \operatorname{vol}(M/N)^{-\frac{1}{2}} s \right]$$

*is $L_2$-unitary and flat.*

*Proof.* First, we check that the map is unitary:

$$\begin{aligned}
&\|(\pi_N^M)^* \left[\mathrm{vol}(M/N)^{-\frac{1}{2}} s\right]\|_{L_2}^2 \\
&= \int_{M/B} H_{M/B}^* \mathrm{vol}_{M/B} \frac{\|(\pi_N^M)^* s\|^2}{\mathrm{vol}(M/N)} \\
&= \int_{N/B} \int_{M/N} H_{M/N}^* \mathrm{vol}_{M/N} \wedge (\pi_N^M)^* (H_{N/B}^* \mathrm{vol}_{N/B}) \frac{\|(\pi_N^M)^* s\|^2}{\mathrm{vol}(M/N)} \\
&= \frac{\|s\|_{L_2}^2}{\mathrm{vol}(M/N)} \int_{M/N} H_{M/N}^* \mathrm{vol}_{M/N} = \|s\|_{L_2}^2
\end{aligned}$$

Second, we check that the map is flat:

$$\begin{aligned}
\nabla^{L_2} (\pi_N^M)^* \left[\mathrm{vol}(M/N)^{-\frac{1}{2}} s\right] &= -\frac{1}{2} k_{M/N} (\pi_N^M)^* \left[\mathrm{vol}(M/N)^{-\frac{1}{2}} s\right] \\
&\quad + \frac{1}{2} k_{M/B} (\pi_N^M)^* \left[\mathrm{vol}(M/N)^{-\frac{1}{2}} s\right] \\
&\quad + (\pi_N^M)^* \left[\mathrm{vol}(M/N)^{-\frac{1}{2}} \nabla^W s\right] \\
&= (\pi_N^M)^* \left[\mathrm{vol}(M/N)^{-\frac{1}{2}} (\nabla^W + \frac{1}{2} k_{N/B}) s\right]
\end{aligned}$$

□

In the particular case under consideration, where $M$ is a family of cylinders of constant length over $N$, $k_{M/N}$ vanishes, and is therefore constant along the fiber.

**Proposition 4.1.11.** *Suppose $\psi$ is a smooth section of $(S_N \otimes E_N)^{\pm}$. Then $U_N^M \psi$ satisfies the boundary condition for $(\Delta_M^{\pm}, \mathfrak{B})$. Furthermore,*

$$U_N^M D_N = D_M U_N^M$$

$$U_N^M \nabla D_N = \nabla D_M U_N^M$$

$$U_N^M \Delta_N = \Delta_M U_N^M$$

*Proof.* That $U_N^M D_N = D_M U_N^M$ is a consequence of the special form of the Dirac operator on an odd-dimensional cylinder, $D_M = \sigma\partial_u + D_N$ (Prop 4.1.7). $\sigma\partial_u$ vanishes on pullbacks, whereas $D_N$ commutes with pullbacks (the action of $D_N$ on sections over $M$ is defined like a partial derivative.) The second equation follows, because $U_N^M$ is flat. The third equation follows by the same reasoning as the first equation. On a metric cylinder, the Laplacian takes the special form $-\partial_u^2 + \Delta_N$.

Regarding the boundary conditions, suppose $\psi$ is a smooth section of $(S_N \otimes E_N)^+$, and let $\hat{\psi} := U_N^M \psi$. Using the boundary identification maps (4.1.3), we have $\hat{\psi}|N_+ = \psi$, and $\hat{\psi}|N_- = \mathfrak{R}\psi$, where $\mathfrak{R}$ is the spinor-reversal map $SN_- \to S\overline{N}_-$. Thus $\hat{\psi}|N_+$ is in the positively graded boundary bundle $(S_N \otimes E_N)^+$, and (since $\mathfrak{R}$ exchanges the chirality of spinors) $\hat{\psi}_\lambda|N_-$ is in the negatively graded boundary bundle $(S_N \otimes E_N)^-$. That is, $\hat{\psi}$ satisfies the boundary conditions (4.1.5). Since $D_M U_N^M = U_N^M D_N$, the previous argument then shows that $D_M\hat{\psi}$ satisfies the adjoint boundary conditions. Therefore $\hat{\psi}$ satisfies the boundary conditions for $(\Delta_M^+, \mathfrak{B})$. The same proof shows that $\hat{\psi}$ satisfies the boundary conditions for $(\Delta_M^-, \mathfrak{B})$ if $\psi$ is a section of $(S_N \otimes E_N)^-$. □

We define graded versions of the operator $U_N^M$. Let

$$U^\pm = U_N^M : \mathcal{H}(\Delta_N^\pm) \to \mathcal{H}(\Delta^\pm; \mathfrak{B})$$

be the graded version of $U_N^M$ (meaning the operators are equal if we forget their grading). Note that these are even operators. As usual, let $U := U^+ \oplus U^-$.

An immediate corollary of the last Proposition is

**Corollary 4.1.12.** *$U^{\pm}$ are well-defined and*

$$U^{\mp} D_N^{\pm} = D^{\pm} U^{\pm}$$

$$U^{\mp} \nabla D_N^{\pm} = \nabla D^{\pm} U^{\pm}$$

$$U^{\pm} \Delta_N^{\pm} = \Delta^{\pm} U^{\pm}$$

**Corollary 4.1.13.** *The map $U^{\pm}$ gives unitary, flat inclusions*

$$U^{\pm} : \mathcal{H}_{(\alpha,\beta)}(\Delta_N^{\pm}) \to \mathcal{H}_{(\alpha,\beta)}(\Delta^{\pm})$$

*Proof.* Let $\psi_\lambda$ be an eigensection in $\mathcal{H}_{(\alpha,\beta)}(\Delta_N^{+})_b$ of eigenvalue $\lambda$, $\|\psi_\lambda\|^2 = 1$. Then by Prop 4.1.10, $\hat{\psi}_\lambda := U^{+}\psi_\lambda$ is a unit-length section in $\mathcal{H}(\Delta^{+}, B)_b$. By Coro 4.1.12, $\hat{\psi}_\lambda$ satisfies the boundary condition for $(\Delta^{+}, \mathfrak{B})$, and is a $\lambda$-eigensection of $\Delta^{+}$. Using Prop 4.1.10 again, this map is unitary and flat.

The same proof works for $\mathcal{H}_{(\alpha,\beta)}(\Delta_N^{-})$. □

**Proposition 4.1.14.** *At $b$ in $B$, let $\left\{\psi_\lambda^{\pm}\right\}$ be an orthonormal basis of $L_2(S_N \otimes E_N)$ consisting of smooth eigensections of $\Delta_{N,b}^{\pm}$, and $\hat{\psi}_\lambda^{\pm} := U_N^M \psi_\lambda^{\pm}$. Then*

$$\{\hat{\psi}_\lambda^{+}\} \cup \{\sqrt{2}\cos(\pi m u)\hat{\psi}_\lambda^{+}, \sqrt{2}\sin(\pi m u)\hat{\psi}_\lambda^{-} \mid m \in \mathbb{Z}^{>0}\}$$

*is an orthonormal basis of $L_2(S \otimes E)$ consisting of smooth eigensections of $(\Delta_{M,b}, \mathfrak{B})$.*

*Proof.* First of all, it is clear that $\hat{\psi}_\lambda^{\pm}$ is smooth, since $\psi_\lambda^{\pm}$ is smooth and pullback takes smooth sections to smooth sections. Second, the special form of the Dirac operator, Prop 4.1.7 shows that $\Delta_{M,b} = -\partial_u^2 + \Delta_{N,b}$. Thus $\hat{\psi}_\lambda^+$, $\sqrt{2}\cos(\pi m u)\hat{\psi}_\lambda^+$, and $\sqrt{2}\sin(\pi m u)\hat{\psi}_\lambda^-$ are eigensections of $\Delta_{M,b}$. It can be checked directly that these sections are $L_2$-orthogonal and unit-length. It can also be verified directly that the sections listed above satisfy the boundary condition for $(\Delta^+, B)$, because $\psi_\lambda^\pm$ lies in $(S_N \otimes E_N)^\pm$.

We've shown that the proposed basis is an orthonormal system. To show that it is maximal, hence an orthonormal basis, we rely on the observation that $\mathcal{B}_1 = \{1, \sqrt{2}\cos(\pi m u) \mid m \in \mathbb{Z}, m > 0\}$ and $\mathcal{B}_2 = \{\sqrt{2}\sin(\pi m u) \mid m \in \mathbb{Z}, m > 0\}$ are orthonormal bases for $L_2[-1, 0]$. (There are unitary maps from functions on $[-1, 0]$ to the spaces of even or odd functions on $[-1, 1]$.) Then maximality follows immediately from the standard theorem that $L_2(X \times Y) = L_2(X) \otimes L_2(Y)$.

□

The proposition shows that there is an orthonormal basis of eigensections for $(\Delta^\pm, B)$ consisting of sections pulled back from the boundary and a secondary part that is the product of eigensections of $-\partial_u^2$ and boundary sections. Let $P_{\text{BP}}$ denote orthogonal projection onto the boundary part, the span of $\{U^+\psi_\lambda^+\} \cup \{U^-\psi_\lambda^-\}$.

**Corollary 4.1.15.** *If $\alpha < \beta < 1$, the graded volume-corrected pullback map*

$$U^\pm : \mathcal{H}_{(\alpha,\beta)}(\Delta_N^\pm) \to \mathcal{H}_{(\alpha,\beta)}(\Delta^\pm)$$

*is bijective. The determinant lines $\mathcal{L}_{(\alpha,\beta)}(D^+, \mathfrak{B})$ and $\mathcal{L}_{(\alpha,\beta)}(D_N^+)$, with the $L_2$-metric and connection, are geometrically isomorphic. The determinant lines*

*$\mathcal{L}(D^+)$ and $\mathcal{L}(D_N^+)$ are isomorphic.*

*Proof.* The eigenvalue of an eigenvector in

$$\{\sqrt{2}\cos(\pi m u)\psi_\lambda^+, \sqrt{2}\sin(\pi m u)\psi_\lambda^- \mid m \in \mathbb{Z}^{>0}\}$$

is at least $(\pi m)^2 > 1$. Thus, by Prop 4.1.14, $\mathcal{H}_{(\alpha,\beta)}(\Delta^\pm)_b$ is spanned by eigensections in the image of the graded volume-corrected pullback map. That is, the map is surjective. According to Corollary 4.1.13, the map is injective. This proves the first statement.

Since Corollary 4.1.13 also shows that the volume-corrected pullback map is unitary and flat for the $L_2$-metric and connection, the induced maps

$$U : \mathcal{L}_{(\alpha,\beta)}(D_N^+) \to \mathcal{L}_{(\alpha,\beta)}(D^+, \mathfrak{B})$$

are also unitary and flat, i.e., geometric isomorphisms. This proves the second statement.

Using Coro 4.1.12, the following diagram is seen to be commutative:

$$\begin{array}{ccc} \mathcal{L}_{(\alpha,\gamma)}(D_N^+) & \xrightarrow{\det(D^+_{N(\gamma,\beta)})} & \mathcal{L}_{(\alpha,\beta)}(D_N^+) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{L}_{(\alpha,\gamma)}(D^+, \mathfrak{B}) & \xrightarrow[\det(D^+_{(\gamma,\beta)})]{} & \mathcal{L}_{(\alpha,\beta)}(D^+, \mathfrak{B}) \end{array}$$

The vertical arrows (given by the volume-corrected pullback map) are bijections if $\beta < 1$. Because the spectrum of the Laplacians is discrete, and the sets $U_\alpha$ are open, the determinant line bundle can be constructed using only the data associated to spectral intervals $(\alpha, \beta)$ in with $\beta < 1$. The third statement follows. □

In view of Coro 4.1.15, there is a natural isomorphism

$$\mathcal{L}(D^+, \mathfrak{B}) \otimes \mathcal{L}(D_N^+)^{-1} \cong \mathbb{C}$$

However the induced metric and connection on $\mathbb{C}$ do not necessarily agree with the canonical metric and connection. The main theorem of the section, Theorem 4.1.8, can be restated as:

**Theorem 4.1.16.** *There is a unit-length, flat section of*

$$\mathcal{L}(D^+, \mathfrak{B}) \otimes \mathcal{L}(D_N^+)^{-1} \cong \mathbb{C}$$

*Proof.* Let $\mathbf{1}$ denote the canonical section of the trivial line bundle. Assume (for convenience only) that $D_N^+$ is invertible, and let $\alpha < 1$. Then over $U_\alpha$, $\mathbf{1}$ corresponds to $\det_{[0,\alpha)}(D^+, \mathfrak{B}) \otimes \det_{[0,\alpha)}(D_N^+)^{-1}$ via the canonical isomorphism . Thus,

$$\begin{aligned} \|\mathbf{1}\|^2 &= \frac{\|\det_{[0,\alpha)}(D^+, \mathfrak{B})\|^2}{\|\det_{[0,\alpha)}(D_N^+)\|^2} \\ &= \frac{\det_\zeta((\Delta^+, \mathfrak{B})_{[0,\alpha)})\det_\zeta((\Delta^+, \mathfrak{B})_{(\alpha,\infty)})}{\det_\zeta(\Delta^+_{N,[0,\alpha)})\det_\zeta(\Delta^+_{N,(\alpha,\infty)})} \\ &= \frac{\det_\zeta((\Delta^+, \mathfrak{B})_{(\alpha,\infty)})}{\det_\zeta(\Delta^+_{N,(\alpha,\infty)})} \end{aligned}$$

where we use Prop 4.1.14 to see that $\det_\zeta((\Delta^+, \mathfrak{B})_{[0,\alpha)}) = \det_\zeta(\Delta^+_{N,[0,\alpha)})$ if $\alpha < 1$. It follows that

$$\hat{\mathbf{1}} := \mathbf{1} \cdot \exp \frac{1}{2} \left\{ [-\zeta'(0; \Delta_N^+, \alpha)] - [-\zeta'(0; (\Delta_M^+, \mathfrak{B}), \alpha)] \right\}$$

is independent of $\alpha$, and defines a smooth unit-length section of $\mathcal{L}(D^+, \mathfrak{B}) \otimes \mathcal{L}(D_N^+)^{-1}$. Note that

$$\mathrm{Re}[\mu_{M,\alpha} - \mu_{N,\alpha}] = -d\frac{1}{2}\left\{[-\zeta'(0; \Delta_N^+, \alpha)] - [-\zeta'(0; (\Delta_M^+, \mathfrak{B}), \alpha)]\right\}$$

From this fact, and the fact that $\mathbf{1}$ is flat with respect to the canonical connection, it follows that $\hat{\mathbf{1}}$ is flat with respect to the connection

$$\nabla^{L_2} + \mathrm{Re}[\mu_{M,\alpha} - \mu_{N,\alpha}] = \nabla^{L_2} + [\mu_{M,\alpha} - \mu_{N,\alpha}] - i\,\mathrm{Im}[\mu_{M,\alpha} - \mu_{N,\alpha}]$$

In Prop 4.1.19 we will show that $\mathrm{Im}[\mu_{M,\alpha} - \mu_{N,\alpha}]$ vanishes, completing the proof. □

**Proposition 4.1.17.**

$$\mathrm{Im}\,\mu_{N,\alpha} = \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \mathrm{Str}\left\{P_{(\alpha,\infty)}(\Delta)\nabla D\, D \exp(-\tau\Delta) P_{\text{BP}}\right\}\, d\tau$$

*Proof.* We have

$$\mathrm{Im}\,\mu_{N,\alpha} = \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \mathrm{Str}\left\{P_{(\alpha,\infty)}(\Delta_N)\nabla D_N\, D_N \exp(-\tau\Delta_N)\right\}\, d\tau$$

Although $U$ is not surjective, hence not invertible, it is unitary onto its image. Therefore we can define $U^{-1}$ on the image. In particular, $U^{-1}$ is defined on the image of $P_{\text{BP}}$. Since $U^{-1}P_{\text{BP}}U$ is the identity on $\mathcal{H}_{(\alpha,\infty)}(\Delta_N)$, standard facts about trace imply

$$\begin{aligned}\mathrm{Str}&\left\{P_{(\alpha,\infty)}(\Delta_N)\nabla D_N\, D_N \exp(-\tau\Delta_N)\right\} \\ &= \mathrm{Str}\left\{U P(\Delta_N; (\alpha,\infty))\nabla D_N\, D_N \exp(-\tau\Delta_N) U^{-1} P_{\text{BP}}\right\}\end{aligned}$$

Using Coro 4.1.12,

$$\exp(-\tau\Delta_N)U^{-1}P_{\text{BP}} = U^{-1}\exp(-\tau\Delta)P_{\text{BP}}$$

and

$$UP_{(\alpha,\infty)}(\Delta_N)\nabla D_N\, D_N = P_{(\alpha,\infty)}(\Delta)\nabla D\, DU$$

The proposition follows. □

To evaluate $\mathrm{Im}[\mu_{M,\alpha} - \mu_{N,\alpha}]$, we will make use of the odd unitary operator

$$\nu := (-\partial_u^2)^{-\frac{1}{2}}(i\partial_u)$$

To make sense of this operator, we specify its action on the basis

$$\{\hat{\psi}_\lambda^+\} \cup \{\sqrt{2}\cos(\pi mu)\hat{\psi}_\lambda^+, \sqrt{2}\sin(\pi mu)\hat{\psi}_\lambda^- \mid m \in \mathbb{Z}^{>0}\}$$
$$\bigcup$$
$$\{\hat{\psi}_\lambda^-\} \cup \{\sqrt{2}\cos(\pi mu)\hat{\psi}_\lambda^-, \sqrt{2}\sin(\pi mu)\hat{\psi}_\lambda^+ \mid m \in \mathbb{Z}^{>0}\}$$

On $\hat{\psi}_\lambda^\pm$, $\nu$ acts as 0; on $\{\sqrt{2}\cos(\pi mu)\hat{\psi}_\lambda^\pm, \sqrt{2}\sin(\pi mu)\hat{\psi}_\lambda^\mp\}$, it acts as

$$\left\{\begin{matrix}\sqrt{2}\cos(\pi mu)\hat{\psi}_\lambda^\pm, \\ \sqrt{2}\sin(\pi mu)\hat{\psi}_\lambda^\mp\end{matrix}\right\} \mapsto \left\{\begin{matrix}-i\sqrt{2}\sin(\pi mu)\hat{\psi}_\lambda^\pm, \\ +i\sqrt{2}\cos(\pi mu)\omega\hat{\psi}_\lambda^\mp\end{matrix}\right\}$$

**Proposition 4.1.18.** *We have* $\nu^2 = 1 - P_{\text{BP}}$*, and* $\nu$ *commutes with* $P_{(\alpha,\infty)}$, $\exp(-\tau\Delta)$ *and* $\nabla D\, D$.

*Proof.* The above definition in terms of an eigenbasis shows that $\nu^2 = 1 - P_{\text{BP}}$ and that $\nu$ commutes with the spectral operators $P_{(\alpha,\infty)}$ and $\exp(-\tau\Delta)$. We claim that it also commutes with $\nabla D\, D$. Using Prop 4.1.7,

$$\nabla D\, D = \nabla D_N\,(\sigma\partial_u + D_N)$$

The differential operators $\nabla D_N$ and $D_N$ are pulled back from the boundary, hence commute with $(-\partial_u^2)^{-\frac{1}{2}}(i\partial_u)$; $\sigma\partial_u$ also commutes with $(-\partial_u^2)^{-\frac{1}{2}}(i\partial_u)$. $\square$

**Proposition 4.1.19.** *The imaginary part of the relative connection 1-form vanishes.*

$$\mathrm{Im}[\mu_{M,\alpha} - \mu_{N,\alpha}] = 0$$

*Proof.* The imaginary part of $\mu_{M,\alpha}$ is:

$$\mathrm{Im}\,\mu_\alpha := \underset{t\to 0^+}{\mathrm{LIM}} \int_t^\infty \mathrm{Str}\left\{P_{(\alpha,\infty)}\nabla D\, D\exp(-\tau\Delta)\right\}\, d\tau$$

Using this and Prop 4.1.17, we have

$$\begin{aligned}
\mathrm{Im}[\mu_{M,\alpha}&-\mu_{N,\alpha}] \\
&= \underset{t\to 0^+}{\mathrm{LIM}} \int_t^\infty \mathrm{Str}\left\{(1-P_{\mathrm{BP}})P_{(\alpha,\infty)}\nabla D\, D\exp(-\tau\Delta)\right\}\, d\tau \\
&= \underset{t\to 0^+}{\mathrm{LIM}} \int_t^\infty \mathrm{Str}\left\{\nu^2 P_{(\alpha,\infty)}\nabla D\, D\exp(-\tau\Delta)\right\}\, d\tau \\
&= \underset{t\to 0^+}{\mathrm{LIM}} \int_t^\infty -\mathrm{Str}\left\{\nu P_{(\alpha,\infty)}\nabla D\, D\exp(-\tau\Delta)\nu\right\}\, d\tau \\
&= \underset{t\to 0^+}{\mathrm{LIM}} \int_t^\infty -\mathrm{Str}\left\{\nu^2 P_{(\alpha,\infty)}\nabla D\, D\exp(-\tau\Delta)\right\}\, d\tau \\
&= -\mathrm{Im}[\mu_{M,\alpha} - \mu_{N,\alpha}]
\end{aligned}$$

$\square$

## 4.2 The Bunke isometry

In the previous section, we considered two special cases of the odd-type boundary value problem: a family of compact odd-dimensional manifolds without boundary, and a cylindrical family $M := [-1, 0] \times N$, where $N \to B$ is a family of closed even-dimensional manifolds, and the boundary conditions imposed are a certain special case of the odd-type boundary conditions.

In this section, we relate a general family of odd-type boundary-value problems to these special cases through *the Bunke isometry*. This is an isometry of Hilbert bundles that preserves boundary values and maps smooth sections to smooth sections, originally due to Bunke [9]. We follow the construction given in [10].

### 4.2.1 The Bunke Isometry

We begin by describing the Bunke construction in general before applying it to the particular problem at hand in order to avoid cluttering the presentation with details.

The underlying data of the construction consists of two families of manifolds with boundary, $M_1$ and $M_2$, with vector bundles $V_1 \to M_1$ and $V_2 \to M_2$, over a common base manifold $B$. These families satisfy the following conditions:

1. There is a family $Y \to B$ such that

$$\partial M_1 = \partial M_2 = Y \sqcup -Y$$

(as oriented families.)

2. $M_2$ is a disjoint union of families $C \sqcup M_2'$, where $C$ is a cylinder family $[-\epsilon, 0] \times Y$ and $\partial M_2' = \varnothing$.

3. Let $C_1, \ldots, C_4$ be four copies of $C$. There is an embedding of $C_1 \sqcup C_2$ into $M_1$ and an embedding of $C_3 \sqcup C_4$ into $M_2$ such that the embedding of $C_1 \sqcup C_2$ identifies $C_1$ with a collar neighborhood of $Y$ and $C_2$ with a collar neighborhood of $-Y$, and the embedding of $C_3 \sqcup C_4$ into $M_2$ identifies $C_4$ with $C$.

4. There is an isomorphism

$$M_1 - (C_1 \sqcup C_2) \cong M_2 - (C_3 \sqcup C_4)$$

5. These maps lift to the vector bundles $V_1$ and $V_2$. We denote the natural isomorphism $C_i \to C_j$ by $\psi_{i,j}$ and the corresponding lifts to the vector bundles by $\psi^*_{i,j}$.

Heuristically, $M_2$ is assembled from $M_1$ by cutting $C_1$ and $C_2$ through the submanifold $\{-\epsilon/2\} \times Y$ and gluing the interior piece of $C_1$, $[-\epsilon, -\epsilon/2] \times Y$, to the interior piece of $C_2$, $[-\epsilon/2, 0] \times Y$, to form $M_2'$, and gluing the two remaining cylinder pieces together to form $C$. The liftings of these embeddings are used to assemble $V_2$ from $V_1$ in the same fashion.

The Bunke map is a map of sections, $B : C^\infty(V_2) \to C^\infty(V_1)$. On the cylinders, this map has the form

$$B : \begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} \mapsto \begin{pmatrix} \psi^*_{1,3} f_L & \psi^*_{1,4} f_R \\ -\psi^*_{2,3} f_R & \psi^*_{2,4} f_L \end{pmatrix} \begin{pmatrix} \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} (B\phi)_1 \\ (B\phi)_2 \end{pmatrix}$$

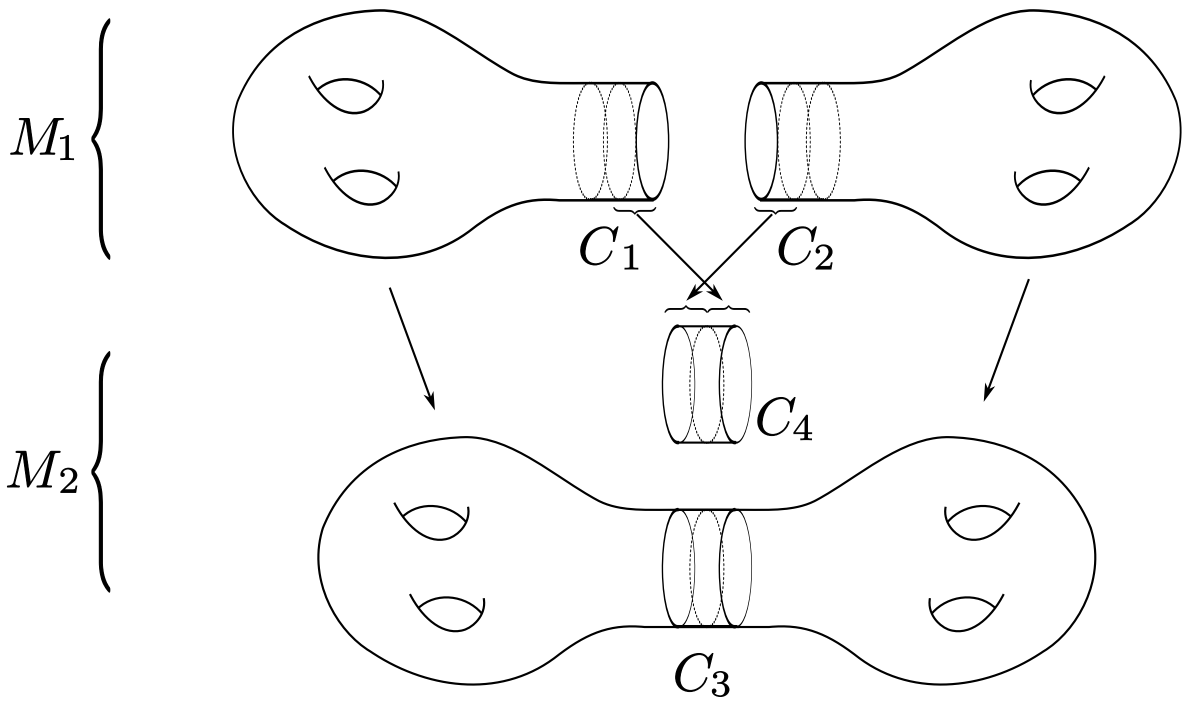


Figure 4.1: The Bunke construction

Here $f_L$ and $f_R$ are smooth cutoff functions defined on $C$ acting as multiplication operators and $\phi_i := \phi|C_i$.

The cutoff functions $f_L,\, f_R : [-1,0] \to [0,1]$ satisfy:

(i) $f_R(x) = f_L(-1-x)$,

(ii) $f_L([-1,-\frac{3}{4}]) = 1$, $f_L([-\frac{1}{4},0]) = 0$

(iii) $f_L^2 + f_R^2 = 1$,

After scaling by $\epsilon$, the length of the cylinder family, $f_L$ and $f_R$ extend to functions on $C_1, \ldots, C_4$. On the compliment of the cylinders, we use the identification $M_1 - (C_1 \sqcup C_2) \cong M_2 - (C_3 \sqcup C_4)$ (which lifts to the vector bundles) to pull back sections. It can easily be checked that this map takes smooth sections to smooth sections as claimed. It can also be seen that the inverse of the Bunke map also takes smooth sections to smooth sections.

**Proposition 4.2.1.** *The Bunke map preserves boundary values. That is,* $\phi|\partial M_2 = B\phi|\partial M_1$.

*Proof.* Recall that we assumed $\partial M_1 = \partial M_2 = Y \sqcup -Y$, and that this identification lifts to the vector bundles. As above, let $\phi_i$ mean $\phi|C_i$. The boundary values of $\psi = B\phi$ are $\psi_1|Y$ and $\psi_2|-Y$. From the definition,

$$\psi_1|Y = f_R(0)\,\phi_4|Y = \phi_4|Y$$

and

$$\psi_2|-Y = f_L(-1)\,\phi_4|-Y = \phi_4|-Y$$

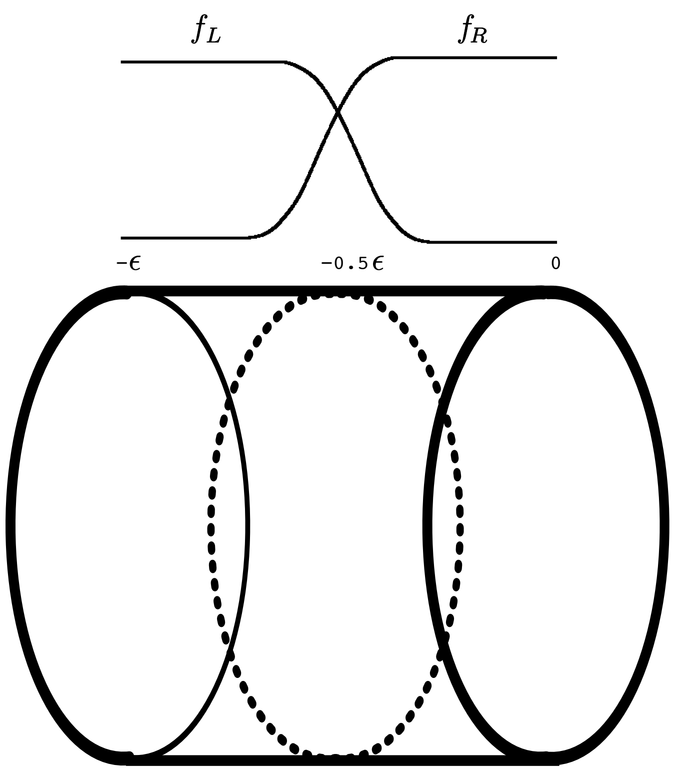


Figure 4.2: The cutoff functions $f_L$ and $f_R$

Thus $\psi$ and $\phi$ have the same boundary values. □

For the following two propositions we suppose that $M_1$ and $M_2$ are oriented Riemannian families, with family connections $H_1$ and $H_2$, and that all the geometry is a product on the cylinders $C_1, \ldots, C_4$. That is, the geometry of $Y$ determines the geometry on the cylinder subfamilies. We also assume that the embeddings of $C_1, \ldots, C_4$, and the lifts of these embeddings to the vector bundles are flat isometries.

**Proposition 4.2.2.** *The Bunke map is an $L_2$ isometry.*

*Proof.* The integral along the fibers defining the $L_2$ norm can be split into a sum of integrals over $M_1 - (C_1 \sqcup C_2) \cong M_2 - (C_3 \sqcup C_4)$ and over the cylinder families. On $M_1 - (C_1 \sqcup C_2)$ and $M_2 - (C_3 \sqcup C_4)$ the result is obvious, because there the Bunke map is simply the identity map. On $C_1 \sqcup C_2$ and $C_3 \sqcup C_4$ the $L_2$-unitarity of the Bunke map follows by an easy calculation from the fact that we are pulling back sections by isometric embeddings, and the fact that $f_L^2 + f_R^2 = 1$. □

**Proposition 4.2.3.** *The Bunke isometry is $L_2$ flat, i.e., for any smooth section $\phi$ of $V_2$,*

$$B\nabla^{L_2(V_2)}\phi = \nabla^{L_2(V_1)}B\phi$$

*Proof.* Recall that the $L_2$ connection can be written as the sum of the horizontal part of the connection on the vector bundle and the horizontal divergence of the volume form. In particular, for any vector field $X$ on $B$,

$$\nabla_X^{L_2(V_1)} = \left(\nabla^{V_1} + k_{M_1/B}\right)(X^{H_1})$$

and there is of course a similar expression for $\nabla^{L_2(V_2)}$.

The divergences of the volume forms are natural 1-forms determined entirely by local geometric data. In particular, pullback by $\psi_{i,j}^*$ preserves these forms, as does pullback by the identification

$$M_1 - (C_1 \sqcup C_2) \cong M_2 - (C_3 \sqcup C_4)$$

As in the last theorem we split the calculation into a part on the cylinder families and a trivial part on the remainders $M_1 - (C_1 \sqcup C_2)$ and $M_2 - (C_3 \sqcup C_4)$, where the Bunke map acts by pullback by the identity map. The commutator of $k_{M_1/B}(X^{H_1})$ with the cutoff functions (acting as a multiplication operators) vanishes.

Arguing similarly for the horizontal part of the connections on $V_1$ and $V_2$, we see that the problem is nontrivial only on the cylinder subfamilies, where we have to compute the commutator of the cutoff functions with the connection. Direct calculation shows that

$$B\nabla^{V_2/B}B^{-1} = \nabla^{V_1/B} + \begin{pmatrix} 0 & f_L df_R - f_R df_L \\ f_R df_L - f_L df_R & 0 \end{pmatrix}$$

Now we use the fact that, by construction, $df_L$ and $df_R$ vanish on horizontal vector fields; $df_L H_1 v = df_R H_1 v = 0$. Thus

$$B\nabla_{H_2 v}^{M_2/B} B^{-1} = \nabla_{H_1 v}^{M_1/B}$$

□

### 4.2.2 Auxiliary families

We now apply the Bunke contruction to computing the geometry of the determinant line bundle. Recall that such a problem is defined by the data of a family of compact odd-dimensional manifolds with boundary, $\pi : M \to B$ and an auxiliary complex vector bundle $E \to M$ with hermetian metric and compatible connection and a given splitting into orthogonal subbundles over the collar neighborhood of the boundary: $E|U \cong E^+ \oplus E^-$.

From each such family $M$, we construct several auxiliary families of manifolds. The first auxiliary family is the "weak double", $M_{wd} := \{-1, 0\} \times M$, where the vertical tangent bundle of $\{-1\} \times M$ is given the opposite orientation and spin-structure. The second is the double of the family, $M^d$. We construct the double of a collared family of spin manifolds with non-empty boundary using the same procedure described in Chapter 3. We define the double of a component of $M$ with vanishing boundary to simply be the weak double. The third is the cylinder family $M^c := [-\epsilon, 0] \times \partial M$. We choose $\epsilon$ small enough that $M^c$ embeds isometrically into the collar neighborhood of the family. (Thus it may be regarded as a collar neighborhood of the boundary of constant length. We suppose that such a constant-length collar neighborhood exists.)

There is one technical caveat: to double the subfamily with non-vanishing boundary, we double not across the boundary but across the submanifold consisting of points at distance $\frac{1}{2}\epsilon$ from the boundary. If we doubled across the boundary, the resulting family would be "too long in the middle", and the

Bunke map would fail to be an $L_2$ isometry. However, this creates no real complications, since the subfamily of points at distance greater than or equal to $\frac{1}{2}\epsilon$ from the boundary is naturally a collared family of spin-manifolds, and all the constructions go through as usual.

Let $S_{wd}$ be the spin-bundle of the weak double and let $E_{wd}$ be the vector bundle on the weak double which is the pullback of $E$ by projection on the second factor $\{-1, 0\} \times M \to M$.

Let $S_d$ be the spin-bundle for the double family, and $E_d$ the double of the vector bundle $E$ over $M^d$. By assumption, this geometric data on $M$ (the metric, the family connection, the spin-structure, and the splitting of $E_c$) is a product over the collar neighborhood of $\partial M$. The embedding of $M^c$ into $M$ defines a spin-structure on the vertical tangent bundle, a connection, and a split vector bundle $E_c \cong E_c^+ \oplus E_c^- \to M^c$.

Returning to the notation used to define the Bunke isometry, let

$$M_1 := M_{wd}, \qquad M_2 := M^d \sqcup M^c$$

Let $V_1 := S_{wd} \otimes E_{wd}$ and let $V_2 := \widehat{S} \otimes \widehat{E}$ be the vector bundle $S_d \otimes E_d \sqcup S_c \otimes E_c$ over $M_2$. Let $D_1$ and $D_2$ be the corresponding (graded) Dirac operators.

It can be verified that these families and their vector bundles satisfy all the conditions required for the Bunke construction and the subsequent propositions showing that the Bunke map respects the $L_2$ geometry. In particular if $C := M^c$, $C \sqcup C$ has an orientation-preserving embedding into the weak double as a (closed) collar neighborhood of the boundary, and $C$ has an orientation-preserving embedding into the double into $M^d$ that sends its centerline $\{u = -\frac{1}{2}\epsilon\}$ to the centerline of the double. Furthermore, these maps are spin-preserving.

Note that if $M$ had empty boundary, then this construction degenerates to constructing two copies of the weak double, and the corresponding Bunke map is simply pull back by the identity map. For the same reason, this construction is completely trivial for components of the original family that have empty boundary.

#### 4.2.2.1 Boundary conditions for the auxiliary families

The splitting of the vector bundle $E_c$ determines odd-type boundary conditions for the cylinder family. The boundary conditions for the cylinder family naturally determine boundary conditions for the weak double, as follows. The two families of manifolds, $M_{wd}$ and $M_2$ have a natural orientation-preserving identification of their boundaries, and there is a natural isomorphism of $S_c \otimes E_c|\partial M^c$ with $S_{wd} \otimes E_{wd}|\partial M_{wd}$ lifting this identification. It therefore makes sense to ask that the two families have the same boundary conditions. Since the Bunke map preserves boundary values, $B\phi$ will satisfy the boundary conditions if and only it $\phi$ does.

#### 4.2.2.2 The determinant line of the weak double

We will show that the determinant line bundle associated to the weak double (with the given boundary conditions) is the square of the determinant line bundle for the original family. That is,

$$\mathcal{L}(D^+, \mathfrak{B})^2 = \mathcal{L}(D_1^+, \mathfrak{B})$$

First we need the following

**Lemma 4.2.4.** *The determinant line $\mathcal{L}(-D^+, \mathfrak{B})$ is naturally geometrically isomorphic to the determinant line $\mathcal{L}(D^+, \mathfrak{B})$.*

*Proof.* First of all, we note the obvious fact that the associated Laplacians are equal: $(-D^{\mp})(-D^{\pm}) = D^{\mp}D^{\pm}$. Thus the underlying spectral data, consisting of open sets, vector bundles, and zeta-regularized determinant functions, are all equal for $D^+$ and $-D^+$.

For each $\alpha > 0$, let $c_\alpha := (-1)^{\operatorname{rank} \mathcal{H}^+_{[0,\alpha)}}$; this is a locally constant non-vanishing function on $U_{[0,\alpha)}$. Let $r_{\alpha\beta} := c_\beta / c_\alpha$; this is a locally constant function on $U_\alpha \cap U_\beta$. In fact, $r_{\alpha\beta} = \det(D^+_{(\alpha,\beta)}) \otimes \det(-D^+_{(\alpha,\beta)})^{-1}$, as is easy to show. Thus the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}_{[0,\alpha)}(D^+, \mathfrak{B}) & \xrightarrow{\det(D^+_{(\alpha,\beta)})} & \mathcal{L}_{[0,\beta)}(D^+, \mathfrak{B}) \\ {\scriptstyle c_\alpha}\downarrow & & \downarrow{\scriptstyle c_\beta} \\ \mathcal{L}_{[0,\alpha)}(-D^+, \mathfrak{B}) & \xrightarrow[\det(-D^+_{(\alpha,\beta)})]{} & \mathcal{L}_{[0,\beta)}(-D^+, \mathfrak{B}) \end{array}$$

In this diagram, the vertical arrows are multiplication by the function, and the horizontal arrows are the transition functions of $\mathcal{L}(D^+, \mathfrak{B})$ and $\mathcal{L}(-D^+, \mathfrak{B})$, given by tensor product by the section.

The commutativity of the diagram shows that the collection of functions $c_\alpha$ define an isomorphism of the determinant lines. Each $c_\alpha$ is obviously a flat isometry relative to the $L_2$ geometry, because it is locally $\pm 1$. Furthermore, in the formulas for the zeta-regularized metric and connection (see section 2.2.3) a minus sign that appears before $D^+$ and $D^-$ will cancel, since these operators appear in pairs in the formulas. Thus this is a geometric isomorphism.

$\square$

**Proposition 4.2.5.** *We have*

$$\mathcal{L}(D_1^+, \mathfrak{B})^2 = \mathcal{L}(D^+, \mathfrak{B})$$

*Proof.* The weak double family $M_1 = M_{wd}$ is the disjoint union of the positive and negative subfamilies. The corresponding Dirac operator can be written as the direct sum of the Dirac operator for the positive subfamily and the Dirac operator for the negative subfamily. That is, $D_1^+ = D_p^+ \oplus D_n^+$. The determinant line associated to such a direct sum of operators is the (geometric) tensor product of the determinant lines associated to the two subfamilies. The determinant line $\mathcal{L}(D_p^+, \mathfrak{B})$ associated to the positive subfamily is obviously isomorphic to $\mathcal{L}(D^+, \mathfrak{B})$, since the positive subfamily $\{0\} \times M$ is naturally isomorphic to $M$. Therefore we only need to show that the determinant line associated to the negative subfamily is also isomorphic to $\mathcal{L}(D^+, \mathfrak{B})$.

The negative subfamily has the opposite spin-structure and boundary condition. Pulling back a section satisfying the boundary condition via the obvious (spin-reversing) isometry from the positive subfamily to the negative subfamily gives a section of the opposite spin-bundle; applying the natural map

$\mathfrak{R}$ to this section gives a spinor section. (For a description of this map, see Appendix C.) Because $\mathfrak{R}$ exchanges the grading on the boundary bundles, the resulting section will satisfy the boundary condition on the positive subfamily.

Let $U$ denote the composition of these two maps (pullback from the negative subfamily of $M_{wd}$ followed by $\mathfrak{R}$). $U$ is a flat isometry for the $L_2$ metric and connection and we have $-D^+ = UD_n^+U^{-1}$ by Prop C.3.8. Thus $\mathcal{L}(-D^+, \mathfrak{B}) \cong \mathcal{L}(D_n^+, \mathfrak{B})$. Applying the previous lemma, we have $\mathcal{L}(D^+, \mathfrak{B}) \cong \mathcal{L}(D_n^+, \mathfrak{B})$.

□

### 4.2.3 The induced determinant line bundle isometry

Let $D_B^{\pm}$ denote the pullback operator $B^{-1}D_1^{\pm}B$. The following proposition is proven in [10].

**Proposition 4.2.6.** *We have*

$$D_B^{\pm} = D_2^{\pm} + G^{\pm}$$

*where $G^{\pm}$ are non-local odd bundle endomorphisms supported on the interior of $M_2$.*

The operator $(D_B^+, \mathfrak{B})$ has a well-defined determinant line bundle. This follows from the results of section A.4, where we showed that the usual heat operator constructions can be carried out for non-local perturbations of Dirac-type operators.

**Proposition 4.2.7.** *The determinant line bundle $\mathcal{L}(D_B^+, \mathfrak{B})$ has a well-defined Quillen metric and compatible Bismut-Freed connection. The regularized zeta determinants and connection 1-forms for the line bundles $\mathcal{L}(D_B^+)$ and $\mathcal{L}(D_1^+)$ are equal.*

*Proof.* We showed in Chapter 2 that the determinant line bundle $\mathcal{L}(D_B^+, \mathfrak{B})$ has a well-defined Quillen metric and compatible Bismut-Freed connection.

The regularized zeta-determinant functions are defined as derivatives of zeta functions:

$$\det{}_\zeta(b, \Delta, \alpha) = \exp\left\{-\frac{\partial}{\partial s}\Big|_{s=0}\zeta(s; b, \Delta, \alpha)\right\}$$

To show that $\det_\zeta(b, \Delta_1^+, \alpha) = \det_\zeta(b, \Delta_B^+, \alpha)$ it is therefore enough to show that the corresponding zeta functions are equal. However, these operators are unitarily equivalent, and therefore have the same spectrum. The equality of the zeta functions is an immediate consequence.

The connection 1-forms for the line bundle $\mathcal{L}(D^+)$ are defined as

$$\mu_\alpha := \operatorname*{LIM}_{t \to 0^+} \int_t^\infty \mathrm{Tr}\left\{P(\Delta^-; \alpha, \infty)\nabla D^+\, D^- \exp(-\tau\Delta^-)\right\}\, d\tau$$

Therefore to show that $\mu_\alpha(D_B^+) = \mu_\alpha(D_1^+)$, it is enough to show that

$$\begin{aligned}\mathrm{Tr}&\left\{P(\Delta_B^-; \alpha, \infty)\nabla D_B^+\, D_B^- \exp(-\tau\Delta_B^-)\right\} \\ &= \mathrm{Tr}\left\{P(\Delta_1^-; \alpha, \infty)\nabla D_1^+\, D_1^- \exp(-\tau\Delta_1^-)\right\}\end{aligned}$$

We claim that the operators within the traces are conjugate by $B$, and therefore have the same trace. $D_B^\pm$ is conjugate to $D_1^\pm$ by definition. The Bunke map is flat with respect to the $L_2$ connection, hence $\nabla D_B^+$ is conjugate to $\nabla D_1^+$. Finally, the projection operators and heat kernels are conjugate because the spectra of $\Delta_B^-$ and $\Delta_1^-$ are equal and their eigensections are related by $B$. $\square$

**Proposition 4.2.8.** *The Bunke map determines a natural flat isometry*

$$\mathcal{L}(D_B^+) \cong \mathcal{L}(D^+, \mathfrak{B})^2$$

*Proof.* We showed in Prop 4.2.5 that there is a natural flat isometry

$$\mathcal{L}(D_1^+, \mathfrak{B}) = \mathcal{L}(D^+, \mathfrak{B})^2$$

We will now show that $\mathcal{L}(D_B^+)$ is isomorphic to $\mathcal{L}(D_1^+, \mathfrak{B})$ by a flat isometry induced by the Bunke isometry.

To define this map, it suffices to work locally in $B$. Suppose $L_{[0,\alpha)}$, $0 < \alpha$ is a homomorphism

$$L_{[0,\alpha)} : \mathcal{H}_{[0,\alpha),b}(\Delta_B^+) \to \mathcal{H}_{[0,\alpha),b}(\Delta_B^-)$$

Conjugating with the restriction of the Bunke map, we obtain

$$B_{[0,\alpha)} L_{[0,\alpha)} B_{[0,\alpha)}^{-1} : \mathcal{H}_{[0,\alpha),b}(\Delta_1^+) \to \mathcal{H}_{[0,\alpha),b}(\Delta_1^-)$$

Let $\det B_{[0,\alpha)}$ be the map of determinant lines

$$\begin{aligned} \det B_{[0,\alpha)} : \mathcal{L}_{[0,\alpha)}(\Delta_B) &\to \mathcal{L}_{[0,\alpha)}(\Delta_1) \\ \det\left(L_{[0,\alpha)}\right) &\mapsto \det\left(B_{[0,\alpha)} L_{[0,\alpha)} B_{[0,\alpha)}^{-1}\right) \end{aligned}$$

If $\alpha < \beta$, then from the definition of $D_B^+$, and the fact that the Bunke map is unitary, we have

$$B_{[0,\beta)}\left(L_{[0,\alpha)} \oplus D^+_{B,[0,\beta)}\right) B^{-1}_{[0,\beta)} = \left(B_{[0,\alpha)} L_{[0,\alpha)} B^{-1}_{[0,\alpha)}\right) \oplus D_{1,[0,\beta)}$$

It follows that on the overlap set $U_{[0,\alpha)} \cap U_{[0,\beta)}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}_{[0,\alpha)}(D_B^+) & \xrightarrow{\det D^+_{B,(\alpha,\beta)}} & \mathcal{L}_{[0,\beta)}(D_B^+) \\ {\scriptstyle \det B_{[0,\alpha)}}\downarrow & & \downarrow{\scriptstyle \det B_{[0,\beta)}} \\ \mathcal{L}_{[0,\alpha)}(D_1^+) & \xrightarrow[\det D^+_{1,(\alpha,\beta)}]{} & \mathcal{L}_{[0,\beta)}(D_1^+) \end{array} \tag{4.2.1}$$

We showed that the Bunke map is a flat $L_2$ isometry. It follows immediately that the maps $\det B_{[0,\alpha)}$ are flat isometries when the line bundles $\mathcal{L}_{(\alpha,\beta)}(D_B^+, \mathfrak{B})$ have the $L_2$ geometry. The equality of the zeta-determinants of the Laplacians and the connection 1-forms shows that $\det B$ is also an isometry with respect to the zeta-corrected geometry on these line-bundles. $\square$

We summarize these results in the following

**Theorem 4.2.9.** *The operator $(D_B^+, \mathfrak{B})$ has a well-defined determinant line bundle with Quillen metric and Bismut-Freed connection. The Bunke map determines a flat isometry*

$$\mathcal{L}(D_B^+) \cong \mathcal{L}(D^+, \mathfrak{B})^2$$

## 4.3 The parallel transport isometry

In this section we complete the construction of an isometry between the square of the determinant line associated to a general family of odd type boundary value problems and the determinant line associated to an auxiliary family of special cases. Recall from the previous section that the Bunke isometry relates the square of the determinant line of a general family of odd-type boundary value problems to the determinant line of

$$D_B := D^+ + G^+$$

where $D^+$ is the Dirac operator of a related family of special cases and $G^+$ is an order zero non-local perturbation. Consider the interpolating family of operators

$$D_r^+ := D^+ + rG^+ \quad 0 \leq r \leq 1$$

over $\hat{B} := [0,1] \times B$, where the boundary conditions on $D_r^+$ are constant along the $r$-lines $p_x(r) = (r,x)$. We make a geometric family of manifolds with boundary

$$\hat{\pi} : \hat{M} \to \hat{B}$$

by pulling back $M$ and the family of vector bundles over it from $B$ to $\hat{B}$ using the natural projection map. We give $\hat{M}$ the natural family connection induced by the family connection $H$ on $M$. Note that the curvature of this induced connection vanishes in directions parallel to $\frac{\partial}{\partial r}$. We regard $(D_r^+)_b$ as acting on smooth sections over the fiber $\hat{M}_{(r,b)}$. In Chapter 2, we show that such a family of operators has a well-defined determinant line bundle with Quillen metric and compatible Bismut-Freed connection. Denote the determinant line of $D_r^+$ by $\hat{\mathcal{L}}$, denote the inclusion $B \to \hat{B}$ given by $b \mapsto (r, b)$ by $\iota_r$, and let

$$\mathcal{L}_r := \iota_r^{-1}(\hat{\mathcal{L}}|\{r\} \times B)$$

Thus $\{\mathcal{L}_r\}_{r\in[0,1]}$ is a family of line bundles on $B$. Parallel transport in $\hat{\mathcal{L}}$ along the $r$-lines gives a family of isometries, $\tau(r) : \mathcal{L}_0 \to \mathcal{L}_r$. In particular, $\tau(1)$ is an isometry from $\mathcal{L}_0$, the determinant line of $D_0^+ = D^+$, to $\mathcal{L}_1$, the determinant line of $D_1^+ = D^+ + G^+$. We will show that this isometry is flat, i.e., if $\nabla^r$ is the connection on $\mathcal{L}_r$, then

$$E(r) := \tau(r)^{-1}\nabla^r\tau(r) - \nabla^0$$

is identically zero. There is a simple criterion for this:

**Proposition 4.3.1.** *$E(r)$ vanishes if the curvature $\Omega^{\hat{\mathcal{L}}}$ of $\hat{\mathcal{L}}$ vanishes on all tangent 2-planes spanned by $\frac{\partial}{\partial r}$ and a vector field $(\iota_r)_*\frac{\partial}{\partial b}$ parallel to $B$, i.e., if*

$$\iota_r^*(\frac{\partial}{\partial r} \neg \Omega^{\hat{\mathcal{L}}}) \equiv 0$$

This is a simple consequence of a general result that expresses the total parallel transport around a closed curve bounding a simply connected region as the integral of the curvature over the region. We omit the easy proof. The main theorem for this section is that these components of the curvature do in fact vanish.

### 4.3.1 The curvature computation

Let $\mathsf{D}_r$ be the family of operators

$$\mathsf{D}_r := \begin{pmatrix} 0 & D_r^- \\ D_r^+ & 0 \end{pmatrix}$$

with the boundary conditions determined by $D_r^+$ and $D_r^-$. Note that the adjoint boundary condition for $\mathsf{D}_r$ is equal to its boundary condition. We first prove that the curvature components vanish under the following assumption:

**Assumption 4.3.1.** $\hat{B} = (-\epsilon, \epsilon)^2$, *and we are calculating* $\Omega^{\hat{\mathcal{L}}}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ *at* $(0,0)$*. The curvature of the family connection* $H$*, defined by*

$$\Omega^H(\xi_1, \xi_2) := [\xi_1, \xi_2]^H - [\xi_1^H, \xi_2^H]$$

*is zero. The family* $\mathsf{D}_r^2$ *has a spectral cut:* $\alpha > 0$ *is not in the spectrum at any point in* $\hat{B}$*. The boundary conditions are preserved by the connection.*

We show that this assumption can be made without loss of generality in section 4.3.2.

Let $\mathsf{P}_r^{[0,\alpha)}$ be the family of spectral projections associated to $\mathsf{D}_r^2$ relative to the assumed spectral cut, and $\mathsf{P}_r^{(\alpha,\infty)} := 1 - \mathsf{P}_r^{[0,\alpha)}$. The induced $L_2$ connection on the finite-dimensional superbundle $\mathcal{H}_{[0,\alpha)} := \operatorname{Ran} \mathsf{P}_r^{[0,\alpha)}$ is $\nabla^{[0,\alpha)} := \mathsf{P}_r^{[0,\alpha)} \circ \nabla \circ \mathsf{P}_r^{[0,\alpha)}$ The connection on the determinant line bundle may be written as the sum

$$\nabla := \nabla^{[0,\alpha)} + \mu_\alpha$$

where $\nabla^{[0,\alpha)}$ is the connection on the determinant line induced by the connection on $\mathcal{H}_{[0,\alpha)}$ and $\mu_\alpha$ is the connection form defined through zeta-regularization. The corresponding decomposition of the curvature is:

$$\Omega^{\hat{\mathcal{L}}} = \Omega^{[0,\alpha)} + d\mu_\alpha$$

We have

$$\Omega^{[0,\alpha)} = -\operatorname{Tr_s}\left(\nabla^{[0,\alpha)}\right)^2$$

The real part of $\mu_\alpha$ is exact. Therefore

$$\begin{aligned} d_{\hat{B}}\mu_\alpha &= d_{\hat{B}} \operatorname{Im} \mu_\alpha \\ &= -\frac{1}{2} d_{\hat{B}} \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \operatorname{Tr_s}\left\{\mathsf{D}_r \nabla \mathsf{D}_r \exp(-u\mathsf{D}_r^2)\mathsf{P}_r^{(\alpha,\infty)}\right\} du \end{aligned}$$

The curvature of the determinant line bundle is therefore

$$\Omega^{\hat{\mathcal{L}}} = -\operatorname{Tr_s}\left(\nabla^{[0,\alpha)}\right)^2 - \frac{1}{2} d_{\hat{B}} \operatorname*{LIM}_{t\to 0^+} \int_t^\infty \operatorname{Tr_s}\left\{\mathsf{D}_r \nabla \mathsf{D}_r \exp(-u\mathsf{D}_r^2)\mathsf{P}_r^{(\alpha,\infty)}\right\} du$$

Individually, these two terms are difficult to compute; in particular, the dependence of the spectral projections on the parameter $r$ is hard to analyze. However, the theory of determinant lines associated to Dirac operators on closed manifolds suggests that such dependences should cancel in the sum, for in that situation the curvature of the determinant line bundle is the 2-form part of the zeta-regularized Chern character of the Bismut superconnection.

#### 4.3.1.1 Two superconnections

We show that under Assumption 4.3.1, $\Omega^{\hat{\mathcal{L}}}$ is indeed the 2-form part of the zeta-regularized Chern character of the Bismut superconnection. We will then apply this result to show that $\Omega^{\hat{\mathcal{L}}}$ vanishes. In outline, the argument we present below is almost exactly the argument one might give for a family of closed odd-dimensional manifolds. The main technical difficulties and points where the argument differs from the closed case are in the proofs of Lemmas 4.3.2 and 4.3.3. In order not to obscure the presentation, we defer the proofs of these lemmas.

Let $\nabla^d$ be the diagonal part of the $L_2$ connection $\nabla$ relative to $\mathsf{P}_r^{[0,\alpha)}$,

$$\nabla^d := \mathsf{P}_r^{[0,\alpha)} \circ \nabla \circ \mathsf{P}_r^{[0,\alpha)} + \mathsf{P}_r^{(\alpha,\infty)} \circ \nabla \circ \mathsf{P}_r^{(\alpha,\infty)}$$

The connection $\nabla^d$ differs from the $L_2$ connection by a smoothing operator valued 1-form:

$$\begin{aligned} \nabla - \nabla^d &= \mathsf{P}_r^{[0,\alpha)} \circ \nabla \mathsf{P}_r^{[0,\alpha)} - \mathsf{P}_r^{(\alpha,\infty)} \circ \nabla \mathsf{P}_r^{[0,\alpha)} \\ &= 2\mathsf{P}_r^{[0,\alpha)} \circ \nabla \mathsf{P}_r^{[0,\alpha)} - \nabla \mathsf{P}_r^{[0,\alpha)} \end{aligned}$$

Consider the two families of superconnections:

$$\mathbb{A}_{1,u} = \nabla^d + \sqrt{u}\mathsf{D}_r, \qquad \mathbb{A}_{2,u} = [\nabla + u(\nabla^d - \nabla)] + \sqrt{t}\mathsf{D}_r$$

We define the corresponding heat operators using the formal Volterra series for a superconnection $\mathbb{A} = \tilde{\nabla} + L$. Let $\Sigma^k := \{(\sigma_0, \ldots, \sigma_k) \in \mathbb{R}^{k+1} \mid \sigma_i \geq 0,\, \sigma_0 + \cdots + \sigma_k = 1\}$ and $d\sigma$ be the measure on this simplex. Then the

Volterra series for $\tilde{\nabla} + L$ is:

$$
\begin{aligned}
\exp(-\mathbb{A}^2) := {} & \exp(-L^2) \\
& - \int_{\Sigma^1} \exp(-\sigma_0 L^2)(\tilde{\nabla}^2 + \tilde{\nabla} L) \exp(-\sigma_1 L^2)\, d\sigma \\
& + \int_{\Sigma^2} \exp(-\sigma_0 L^2)(\tilde{\nabla}^2 + \tilde{\nabla} L) \exp(-\sigma_1 L^2)(\tilde{\nabla}^2 + \tilde{\nabla} L) \exp(-\sigma_2 L^2)\, d\sigma \\
& - \cdots + \cdots
\end{aligned}
$$

For $\mathbb{A}_{1,u}$ we take $\tilde{\nabla}$ and $L$ to be $\nabla^d$ and $\sqrt{u}\mathsf{D}_r$ respectively, whereas for $\mathbb{A}_{2,u}$ we take them to be $[\nabla + u(\nabla^d - \nabla)]$ and $\sqrt{t}\mathsf{D}_r$. In either case, the sum is finite, having at most $\dim \hat{B} + 1$ terms. Any further terms in this expansion have degree greater than the maximum degree of the exterior algebra of the base, and therefore vanish. In particular, under Assumption 4.3.1, the Volterra series has only the three terms written in full above. Convergence of the Volterra series is therefore a question of the convergence of the individual terms. We demonstrate this convergence in Appendix B.

In the next section we will prove the following two lemmas

**Lemma 4.3.2.** *The curvature of the determinant line bundle is:*

$$\Omega^{\hat{\mathcal{L}}} = \operatorname*{LIM}_{t \to 0^+} \mathrm{Tr_s} \left\{ \exp(-\mathbb{A}_{1,t})^2 \right\}_{(2)}$$

**Lemma 4.3.3.** *We have*

$$0 = \frac{\partial}{\partial u} \mathrm{Tr_s} \left\{ \exp(-\mathbb{A}_{2,u})^2 \right\}_{(2)}$$

**Theorem 4.3.4.** *Under Assumption 4.3.1, the curvature of the determinant line bundle is the 2-form part of the zeta-regularized Chern character of the*

*Bismut superconnection. That is,*

$$\Omega^{\hat{\mathcal{L}}} = \operatorname*{LIM}_{t\to 0^+} \operatorname{Tr_s}\left\{\exp -(\nabla + \sqrt{t}\mathsf{D}_r)^2\right\}_{(2)}$$

*Remark* 4.3.1. Note that $\nabla + \sqrt{t}\mathsf{D}_r$ is indeed the Bismut superconnection; under Assumption 4.3.1 the 2-form part of this superconnection, which is proportional to the family curvature $\Omega^H$, vanishes.

*Proof.* For fixed $t > 0$, we have from Lemma 4.3.3,

$$\operatorname{Tr_s}\left\{\exp(-\mathbb{A}_{2,0})^2\right\}_{(2)} = \operatorname{Tr_s}\left\{\exp(-\mathbb{A}_{2,1})^2\right\}_{(2)}$$

We see directly from the form of $\mathbb{A}_{2,u} = [\nabla + u(\nabla^d - \nabla)] + \sqrt{t}\mathsf{D}_r$ that this equality is the same as

$$\operatorname{Tr_s}\left\{\exp -(\nabla + \sqrt{t}\mathsf{D}_r)^2\right\}_{(2)} = \operatorname{Tr_s}\left\{\exp(-\mathbb{A}_{1,t})^2\right\}_{(2)}$$

Applying Lemma 4.3.2, we have

$$\begin{aligned}\Omega^{\hat{\mathcal{L}}} &= \operatorname*{LIM}_{t\to 0^+} \operatorname{Tr_s}\left\{\exp(-\mathbb{A}_{1,t})^2\right\}_{(2)} \\ &= \operatorname*{LIM}_{t\to 0^+} \operatorname{Tr_s}\left\{\exp -(\nabla + \sqrt{t}\mathsf{D}_r)^2\right\}_{(2)}\end{aligned}$$

□

**Lemma 4.3.5.** *We have*

$$\iota_r^*(\frac{\partial}{\partial r}\neg\Omega^{\hat{\mathcal{L}}}) = -\operatorname*{LIM}_{t\to 0^+} t^{\frac{1}{2}} \operatorname{Tr_s}\left\{G\exp -(\nabla^B + \sqrt{t}\mathsf{D})^2\right\}_{(1)}$$

*where $\nabla^B$ denotes the $L_2$ connection on $B$.*

*Proof.* Applying Theorem 4.3.4 and the Volterra series expansion, we see that the 2-form part of the curvature, given by $\mathrm{Tr_s}\left\{\exp -(\nabla + \sqrt{t}\mathsf{D}_r)^2\right\}_{(2)}$ is

$$-t\,\mathrm{Tr_s}\int_{\Sigma^2} \exp(-t\sigma_0\mathsf{D}_r^2)[\nabla,\mathsf{D}_r]\exp(-t\sigma_1\mathsf{D}_r^2)[\nabla,\mathsf{D}_r]\exp(-t\sigma_2\mathsf{D}_r^2)\,d\sigma$$

By construction,

$$\frac{\partial}{\partial r}\neg[\nabla,\mathsf{D}_r] = G$$

Thus $\frac{\partial}{\partial r}\neg\,\mathrm{Tr_s}\left\{\exp -(\nabla + \sqrt{t}\mathsf{D}_r)^2\right\}_{(2)}$ is

$$\begin{aligned}-t\,\mathrm{Tr_s}\int_{\Sigma^2} &\exp(-t\sigma_0\mathsf{D}_r^2)G\exp(-t\sigma_1\mathsf{D}_r^2)[\nabla,\mathsf{D}_r]\exp(-t\sigma_2\mathsf{D}_r^2)\\ &+\exp(-t\sigma_0\mathsf{D}_r^2)[\nabla,\mathsf{D}_r]\exp(-t\sigma_1\mathsf{D}_r^2)G\exp(-t\sigma_2\mathsf{D}_r^2)\,d\sigma\end{aligned}$$

Using the trace property and then integrating down to $\Sigma^1$ along $(\sigma_1,\sigma_2)\mapsto\sigma_1$, we obtain

$$\begin{aligned}-t\,\mathrm{Tr_s}\int_{\Sigma^1} &\sigma_1[G\exp(-t\sigma_0\mathsf{D}_r^2)[\nabla,\mathsf{D}_r]\exp(-t\sigma_1\mathsf{D}_r^2)]\\ &+\sigma_1[G\exp(-t\sigma_1\mathsf{D}_r^2)[\nabla,\mathsf{D}_r]\exp(-t\sigma_0\mathsf{D}_r^2)]\,d\sigma\end{aligned}$$

Changing variables in the second term, we obtain

$$\begin{aligned}&-t\,\mathrm{Tr_s}\int_{\Sigma^1}(\sigma_0+\sigma_1)[G\exp(-t\sigma_0\mathsf{D}_r^2)[\nabla,\mathsf{D}_r]\exp(-t\sigma_1\mathsf{D}_r^2)]\,d\sigma\\ &= -t\,\mathrm{Tr_s}\,G\int_{\Sigma^1}\exp(-t\sigma_0\mathsf{D}_r^2)[\nabla,\mathsf{D}_r]\exp(-t\sigma_1\mathsf{D}_r^2)\,d\sigma\end{aligned}$$

Pulling back to $B$ with $\iota_r^*$, this gives the claim. □

We now prove the main theorem of this section.

**Theorem 4.3.6.** *We have*

$$\iota_r^*(\frac{\partial}{\partial r}\neg\Omega^{\hat{\mathcal{L}}}) \equiv 0$$

Before presenting the proof, we describe the proof strategy. The previous lemma provides our starting point:

$$\iota_r^*(\frac{\partial}{\partial r} \neg \Omega^{\hat{\mathcal{L}}}) = -\operatorname*{LIM}_{t \to 0^+} t \operatorname{Tr_s} G \int_{\Sigma^1} \exp(-t\sigma_0 \mathsf{D}_r^2)[\nabla^B, \mathsf{D}_r] \exp(-t\sigma_1 \mathsf{D}_r^2)\, d\sigma$$

It's easy to see that this expression involves only small time data for the heat kernels. Our strategy is to replace the heat kernels by approximate heat kernels constructed by the procedure we used in the proof of Theorem A.4.16. That is, we construct an interior approximation and a boundary approximation and patch them together using a system of sheltering functions. We estimate the error in making this replacement and determine that it is asymptotically negligible. There are two key observations. First, both the domain and range of $G$ are supported away from the boundary, and therefore we can arrange for the product of the boundary approximation and $G$ to vanish in either order. We use this observation to show that the contribution of the boundary approximation vanishes. Second, the interior approximation does not depend on the boundary condition, and so may be constructed to commute with the volume form. As we noted in section 4.1.1, on an odd-dimensional manifold the volume form is an odd involution that commutes with $G$ and $[\nabla^B, \mathsf{D}_r]$, but anticommutes with the grading operator. This symmetry shows that the contribution of the interior approximation vanishes.

*Proof of Theorem 4.3.6.* Let $\{\phi_0, \phi_1\}$ be a partition of unity for $\hat{M}$ such that $\phi_1$ has support disjoint from the boundary, and let $\{\psi_0, \psi_1\}$ be a corresponding system of sheltering functions. (See section A.4 for the definition of a sheltering function.) We choose the partition of unity and sheltering functions in such a way that $G\psi_0 = 0 = \psi_0 G$. This is possible because the support of the domain and range of $G$ are on the interior of $\hat{M}$.

Relative to this system of sheltering functions, let $E_0^{\pm}(t)$ be boundary approximate heat kernels for $(\Delta_r^{\pm}, \mathfrak{B})$, and let $E_1^{+}(t)$ be an interior approximate heat kernel for $\Delta_r^{+}$. (We work at a fixed $r$ in this calculation.) These approximations are constructed in section A.4 The interior approximation $E_1^{+}(t)$ is independent of the boundary condition. Furthermore, $\Delta_r^{+}$ intertwines with $\Delta_r^{-}$ via the volume form: $\Delta_r^{-}\omega\phi = \omega\Delta_r^{+}\phi$ for smooth sections $\phi$ supported on the interior. It follows that $E_1^{-}(t) := \omega E_1^{+}(t)\omega$ is an interior approximate heat kernel for $\Delta_r^{-}$. Finally, the boundary and interior approximations for $(\mathsf{D}_r^2, \mathfrak{B})$ are

$$E_0(t) := \begin{pmatrix} E_0^{+}(t) & \\ & E_0^{-}(t) \end{pmatrix}, \qquad E_1(t) := \begin{pmatrix} E_1^{+}(t) & \\ & E_1^{-}(t) \end{pmatrix}$$

Note that by construction $E_1(t)$ commutes with the volume form. Our approximate heat kernel is

$$E(t) := \psi_0 E_0(t)\phi_0 + \psi_1 E_1(t)\phi_1$$

From Lemma 4.3.5, we have

$$\iota_r^*(\frac{\partial}{\partial r}\neg\Omega^{\hat{\mathcal{L}}}) = -\operatorname*{LIM}_{t\to 0^+} t\,\mathrm{Tr_s}\, G\int_{\Sigma^1} \exp(-t\sigma_0\mathsf{D}_r^2)[\nabla^B, \mathsf{D}_r]\exp(-t\sigma_1\mathsf{D}_r^2)\, d\sigma$$

According to Coro. A.4.7, for all positive $t$ small enough, the difference in the norms of the kernels of the heat kernel and the approximate heat kernel is exponentially small. More precisely,

$$\|h(t) - e(t)\|_{C^l} \leq Ce^{-c/t}$$

This means that we can replace the heat operator with the approximate heat operator in the LIM above:

$$\begin{aligned}
\iota_r^*(\frac{\partial}{\partial r}\neg\Omega^{\hat{\mathcal{L}}}) &= -\operatorname*{LIM}_{t\to 0^+} t\,\mathrm{Tr_s}\,G\int_{\Sigma^1} E(-t\sigma_0)[\nabla^B, \mathsf{D}_r]E(-t\sigma_1)\,d\sigma \\
&= -\operatorname*{LIM}_{t\to 0^+} t\,\mathrm{Tr_s}\,G\int_{\Sigma^1} \psi_1 E_1(-t\sigma_0)\phi_1[\nabla^B, \mathsf{D}_r]\psi_1 E_1(-t\sigma_1)\phi_1\,d\sigma
\end{aligned}$$

The contribution of the boundary heat operator vanishes because by construction $G\psi_0 = 0 = \psi_0 G$. All the operators in the supertrace commute with $\omega$, which is an odd involution. The operator in the supertrace is odd. Thus

$$\begin{aligned}
\iota_r^*(\frac{\partial}{\partial r}\neg\Omega^{\hat{\mathcal{L}}}) &= -\operatorname*{LIM}_{t\to 0^+} t\,\mathrm{Tr_s}\,\omega^2 G\int_{\Sigma^1} \psi_1 E_1(-t\sigma_0)\phi_1[\nabla^B, \mathsf{D}_r]\psi_1 E_1(-t\sigma_1)\phi_1\,d\sigma \\
&= \operatorname*{LIM}_{t\to 0^+} t\,\mathrm{Tr_s}\,\omega G\int_{\Sigma^1} \psi_1 E_1(-t\sigma_0)\phi_1[\nabla^B, \mathsf{D}_r]\psi_1 E_1(-t\sigma_1)\phi_1\omega\,d\sigma \\
&= \operatorname*{LIM}_{t\to 0^+} t\,\mathrm{Tr_s}\,\omega^2 G\int_{\Sigma^1} \psi_1 E_1(-t\sigma_0)\phi_1[\nabla^B, \mathsf{D}_r]\psi_1 E_1(-t\sigma_1)\phi_1\,d\sigma \\
&= -\iota_r^*(\frac{\partial}{\partial r}\neg\Omega^{\hat{\mathcal{L}}})
\end{aligned}$$

We conclude $\iota_r^*(\frac{\partial}{\partial r}\neg\Omega^{\hat{\mathcal{L}}}) = 0$. □

#### 4.3.1.2 A transgression formula

On a compact closed manifold, integration by parts shows that the commutator of a smoothing operator with a differential operator is a smoothing operator, hence trace class. In fact, as one might expect, the trace of this commutator vanishes. This is not true on a compact manifold with boundary; the kernel of the commutator has a singular boundary part coming from integration by parts as well as a smooth part.

If the differential operator is first order, the kernel of the commutator still has a well-defined integral along the diagonal and we *define* the trace to be the integral along the diagonal. In order to compute such commutator terms, we will make use of Lemma 3.17 of [10]:

**Lemma 4.3.7.** *For $D$ the Dirac operator and $K$ a smoothing operator with smooth kernel $K(x, x')$ on a compact spin manifold $X$ with boundary we have*

$$\mathrm{Tr}[D, K] = -\int_{\partial X} \mathrm{tr}\, JK(y, y)\, dy$$

*where $J$ is the normal symbol of $D$, i.e., $J = \sigma_\nu(D)$.*

The proof comes down to an integration by parts, and as such applies equally well to families of Dirac operators and smoothing operators.

Recall that the adjoint boundary condition $\mathfrak{B}^-$ to a local boundary condition $\mathfrak{B}^+$ for the Dirac operator is defined by the boundary pairing: $\mathfrak{B}^-\phi = 0$ if and only if $\langle J\phi, \psi\rangle = 0$ for all $\psi$ such that $\mathfrak{B}^+\psi = 0$. If $\mathfrak{B}^+$ is a projection onto a subbundle of half-rank, we have $\mathfrak{B}^- = -J(1 - \mathfrak{B}^+)J$.

As the following lemma shows, $\mathrm{tr}(JK(y, y))$ is a kind of boundary self-pairing for odd bundle endomorphisms at the boundary.

**Proposition 4.3.8.** *Suppose $A$ is an odd smoothing operator acting from $S^+$ to $S^-$ (or vice versa) satisfying the adjoint local boundary conditions $\mathfrak{B}^\pm$ for* *the Dirac operator in the sense that* $\mathfrak{B}^- A = 0$ *and* $\mathfrak{B}^+ A^* = 0$*. Then*

$$\mathrm{tr}(JA(y,y)) = 0$$

*at all points* $y$ *in the boundary.*

*Proof.* The boundary conditions are local and 0-order. Using an approximation of the identity, it can be seen that the two conditions $\mathfrak{B}^- A = 0$ and $\mathfrak{B}^+ A^* = 0$ are satisfied if and only if the kernel of $A$ satisfies the corresponding conditions $\mathfrak{B}^- A(y,x) = 0$, $\mathfrak{B}^+ A(x,y)^* = 0$. Since $\mathfrak{B}^\pm$ are projections follows that $\mathrm{tr}(JA(y,y)) = \mathrm{tr}(J(1-\mathfrak{B}^-)A(y,y)(1-\mathfrak{B}^+))$ Using the relations $(1-\mathfrak{B}^+) = -J\mathfrak{B}^- J$, $J^2 = -1$, we have

$$\mathrm{tr}(JA(y,y)) = \mathrm{tr}(J\mathfrak{B}^-(1-\mathfrak{B}^-)A(y,y)) = 0$$

□

In Appendix B, we prove the following version of Duhamel's principle:

**Lemma 4.3.9.** *Let* $\exp(-\mathbb{A}_u^2)$ *be one of the two heat operators* $\exp(-\mathbb{A}_{1,u}^2)$ *and* $\exp(-\mathbb{A}_{2,u}^2)$ *defined above. The parts* $\exp(-\mathbb{A}_u^2)_{(i)}$ *of degree* $i$ *in the exterior algebra of* $\hat{B}$ *are smoothing operators satisfying the same boundary conditions that* $\exp(-\mathsf{D}^2)$ *satisfies, and we have*

$$\frac{\partial}{\partial u}\exp(-\mathbb{A}_u^2) = -\int_{\Sigma^1}\exp(-\sigma_0\mathbb{A}_u^2)\left(\frac{\partial}{\partial u}\mathbb{A}_u^2\right)\exp(-\sigma_1\mathbb{A}_u^2)\,d\sigma$$

We are ready to state our transgression formula.

**Theorem 4.3.10.** *Let $\mathbb{A}_u$ be one of the two superconnections defined above, $\exp(-\mathbb{A}_u^2)$ the corresponding heat operator. Let $L_u := \mathbb{A}_u - \nabla$ be the difference with the $L_2$ connection. We have*

$$-\frac{\partial}{\partial u}\operatorname{Tr_s}\left\{\exp(-\mathbb{A}_u^2)\right\} = d_{\hat{B}}\operatorname{Tr_s}\left\{\dot{L}_u\exp(-\mathbb{A}_u^2)\right\} + \operatorname{Tr_s}\left[L_u, \dot{L}_u\exp(-\mathbb{A}_u^2)\right]$$

*Proof.* From Lemma 4.3.9, we have

$$\frac{\partial}{\partial u}\exp(-\mathbb{A}_u^2) = -\int_{\Sigma^1}\exp(-\sigma_0\mathbb{A}_u^2)\left(\frac{\partial}{\partial u}\mathbb{A}_u^2\right)\exp(-\sigma_1\mathbb{A}_u^2)\,d\sigma$$

Taking the trace on both sides, we have

$$-\frac{\partial}{\partial u}\operatorname{Tr_s}\{\exp(-\mathbb{A}_u^2)\} = \int_{\Sigma^1}\operatorname{Tr_s}\left\{\exp(-\sigma_0\mathbb{A}_u^2)\left(\frac{\partial}{\partial u}\mathbb{A}_u^2\right)\exp(-\sigma_1\mathbb{A}_u^2)\right\}d\sigma$$

Applying the trace property,

$$\begin{aligned}
-\frac{\partial}{\partial u}\operatorname{Tr_s}\{\exp(-\mathbb{A}_u^2)\} &= \operatorname{Tr_s}\left\{\left(\frac{\partial}{\partial u}\mathbb{A}_u^2\right)\exp(-\mathbb{A}_u^2)\right\}\\
&= \operatorname{Tr_s}\left\{[\mathbb{A}_u, \dot{\mathbb{A}}_u]\ \exp(-\mathbb{A}_u^2)\right\}\\
&= \operatorname{Tr_s}\left[\mathbb{A}_u, \dot{\mathbb{A}}_u\ \exp(-\mathbb{A}_u^2)\right]\\
&= \operatorname{Tr_s}\left[\nabla + L_u, \dot{L}_u\ \exp(-\mathbb{A}_u^2)\right]\\
&= \operatorname{Tr_s}\left[\nabla, \dot{L}_u\ \exp(-\mathbb{A}_u^2)\right] + \operatorname{Tr_s}\left[L_u, \dot{L}_u\ \exp(-\mathbb{A}_u^2)\right]
\end{aligned}$$

Finally, we apply Prop. 2.1.16. □

Using Theorem 4.3.10, we compute $\frac{\partial}{\partial u}\operatorname{Tr_s}\exp(-\mathbb{A}_{1,u}^2)_{(2)}$.

**Proposition 4.3.11.** *We have*

$$\frac{\partial}{\partial u}\operatorname{Tr_s}\left\{\exp(-\mathbb{A}_{1,u}^2)\right\}_{[2]} = \frac{1}{2}d_{\hat{B}}\operatorname{Tr_s}\left\{\mathsf{D}_r\nabla\mathsf{D}_r\exp(-u\mathsf{D}_r^2)\right\}$$

*for $u > 0$.*

*Proof.* We have $L_u = (\nabla^d - \nabla) + \sqrt{u}\mathsf{D}_r$, $\dot{L}_u = \frac{1}{2\sqrt{u}}\mathsf{D}_r$, and

$$
\begin{aligned}
-\frac{\partial}{\partial u} \operatorname{Tr_s} \exp(-\mathbb{A}_{1,u})^2)_{(2)} =& d_{\hat{B}} \operatorname{Tr_s} \left\{ \frac{1}{2\sqrt{u}} \mathsf{D}_r \exp(-\mathbb{A}_{1,u}^2) \right\}_{(1)} \\
&+ \operatorname{Tr_s} \left\{ [(\nabla^d - \nabla) + \sqrt{u}\mathsf{D}_r, \frac{1}{2\sqrt{u}} \mathsf{D}_r \exp(-\mathbb{A}_{1,u}^2)] \right\}_{(2)}
\end{aligned}
$$

From the Volterra expansion, we see that

$$
[\exp(-\mathbb{A}_{1,u}^2)]_{(1)} = -\int_{\Sigma^1} \exp(-\sigma_0 u \mathsf{D}_r^2) \sqrt{u} \nabla \mathsf{D}_r \exp(-\sigma_1 u \mathsf{D}_r^2)\, d\sigma
$$

Thus the first term is

$$
\frac{1}{2} d_{\hat{B}} \operatorname{Tr_s} \left\{ \mathsf{D}_r \nabla \mathsf{D}_r \exp(-u\mathsf{D}_r^2) \right\}
$$

The proposition therefore comes down to showing that the commutator term vanishes. This term may be simplified by observing that $(\nabla^d - \nabla) + \sqrt{u}\mathsf{D}_r$ differs from $\mathsf{D}$ by a bounded odd operator. The supertrace of a supercommutator of a bounded operator with a trace-class operator vanishes. The second term is therefore

$$
\frac{1}{2} \operatorname{Tr_s} \left\{ [\mathsf{D}, \mathsf{D}_r \exp(-\mathbb{A}_{1,u}^2)_{[2]}] \right\}
$$

Applying the Volterra series again, we see that this is the sum

$$
\begin{aligned}
&\frac{1}{2} \operatorname{Tr_s} \left[ \mathsf{D}, \mathsf{D}_r \int_{\Sigma^1} \exp(-u\sigma_0 \mathsf{D}_r^2)(\nabla^d)^2 \exp(-u\sigma_1 \mathsf{D}_r^2)\, d\sigma \right] \\
&\quad + \frac{1}{2} u \operatorname{Tr_s} \left[ \mathsf{D}, \mathsf{D}_r \int_{\Sigma^2} \exp(-u\sigma_0 \mathsf{D}_r^2)(\nabla \mathsf{D}_r) \exp(-u\sigma_1 \mathsf{D}_r^2)(\nabla \mathsf{D}_r) \exp(-u\sigma_2 \mathsf{D}_r^2)\, d\sigma \right]
\end{aligned}
$$

For the moment, let $A(u; x, y)$ denote the kernel of the odd smoothing operator

$$
A_u = \mathsf{D}_r \int_{\Sigma^1} \exp(-u\sigma_0 \mathsf{D}_r^2)(\nabla^d)^2 \exp(-u\sigma_1 \mathsf{D}_r^2)\, d\sigma
$$

Lemma 4.3.7 implies that the first commutator term is given by the boundary integral

$$\frac{1}{2}\int_{\partial^\pi \hat{M}/\hat{B}} \mathrm{tr_s}\left(JA(u;y,y)\right)\,dy$$

We claim that $A_u$ satisfies the conditions of Prop. 4.3.8, so that the integrand vanishes pointwise. Clearly $A_u$ is an odd smoothing operator. Furthermore $\mathsf{D}_r \exp(-u\sigma_0\mathsf{D}_r^2)$ satisfies the boundary condition $\mathfrak{B}$ of $\mathsf{D}_r$ in the first variable, and $(\nabla^d)^2\exp(-u\sigma_1\mathsf{D}_r^2)$ satisfies the boundary condition in the second variable. Thus the kernel of the composite

$$\mathsf{D}_r \exp(-u\sigma_0\mathsf{D}_r^2)(\nabla^d)^2\exp(-u\sigma_1\mathsf{D}_r^2)$$

satisfies the boundary condition in the first and second variables. Finally, the integral over the simplex commutes with the boundary condition. Thus the first commutator term vanishes. Almost the same reasoning applies to the second commutator term. We conclude that it vanishes as well. □

We apply this result to the curvature computation. Observe that because $\mathsf{D}_r$ commutes with $\mathsf{P}_r^{[0,\alpha)}$ we have

$$\mathbb{A}_{1,u} = \mathbb{A}_{1,u}^{[0,\alpha)} + \mathbb{A}_{1,u}^{(\alpha,\infty)}$$

where $\mathbb{A}_{1,u}^{[0,\alpha)} = \mathsf{P}_r^{[0,\alpha)}\mathbb{A}_{1,u}\mathsf{P}_r^{[0,\alpha)}$ and $\mathbb{A}_{1,u}^{(\alpha,\infty)} = \mathsf{P}_r^{(\alpha,\infty)}\mathbb{A}_{1,u}\mathsf{P}_r^{(\alpha,\infty)}$. Furthermore, the heat kernel is the sum

$$\exp(-\mathbb{A}_{1,u}^2) = \exp\left\{-(\mathbb{A}_{1,u}^{[0,\alpha)})^2\right\} + \exp\left\{-(\mathbb{A}_{1,u}^{(\alpha,\infty)})^2\right\} \tag{4.3.1}$$

This can be seen directly from the Volterra expansion.

**Proposition 4.3.12.** *We have*

$$\frac{\partial}{\partial u}\operatorname{Tr_s}\exp -(\mathbb{A}_{1,u}^{(\alpha,\infty)})^2 = \frac{1}{2}d_{\hat{B}}\operatorname{Tr_s}\left\{\mathsf{D}_r\nabla\mathsf{D}_r\exp(-u\mathsf{D}_r^2)\mathsf{P}_r^{(\alpha,\infty)}\right\}$$

*Proof.* In view of Prop. 4.3.11 and Eq. 4.3.1, it is equivalent to show

$$\frac{\partial}{\partial u}\operatorname{Tr_s}\left\{\exp -(\mathbb{A}_{1,u}^{[0,\alpha)})^2\right\}_{[2]} = \frac{1}{2}d_{\hat{B}}\operatorname{Tr_s}\left\{\mathsf{D}_r\nabla\mathsf{D}_r\exp(-u\mathsf{D}_r^2)\mathsf{P}_r^{[0,\alpha)}\right\}$$

Since $\mathbb{A}_{1,u}^{[0,\alpha)}$ is a superconnection on a finite-dimensional superbundle, we have

$$\begin{aligned}\frac{\partial}{\partial u}\operatorname{Tr_s}\left\{\exp -(\mathbb{A}_{1,u}^{[0,\alpha)})^2\right\}_{[2]} &= -d_{\hat{B}}\operatorname{Tr_s}\left\{\dot{\mathbb{A}}_{1,u}^{[0,\alpha)}\exp -(\mathbb{A}_{1,u}^{[0,\alpha)})^2\right\}_{[1]}\\ &= \frac{1}{2}d_{\hat{B}}\operatorname{Tr_s}\left\{\mathsf{D}_r\nabla\mathsf{D}_r\exp(-u\mathsf{D}_r^2)\mathsf{P}_r^{[0,\alpha)}\right\}\end{aligned}$$

where in the last line we have used the fact that $\mathsf{D}_r$ and the spectral projection commute. □

*Proof of Lemma 4.3.2.* As we note above, the determinant line bundle curvature is the sum $\Omega^{[0,\alpha)} + d_{\hat{B}}\mu_\alpha$. Using Prop. 4.3.12, we have

$$\begin{aligned}d_{\hat{B}}\mu_\alpha &= -\frac{1}{2}d_{\hat{B}}\operatorname*{LIM}_{t\to 0^+}\int_t^\infty \operatorname{Tr_s}\left\{\mathsf{D}_r\nabla\mathsf{D}_r\exp(-u\mathsf{D}_r^2)\mathsf{P}_r^{(\alpha,\infty)}\right\}\,du\\ &= -\operatorname*{LIM}_{t\to 0^+}\int_t^\infty \frac{\partial}{\partial u}\operatorname{Tr_s}\left\{\exp -(\mathbb{A}_{1,u}^{(\alpha,\infty)})^2\right\}_{[2]}\,du\\ &= \operatorname*{LIM}_{t\to 0^+}\operatorname{Tr_s}\left\{\exp -(\mathbb{A}_{1,t}^{(\alpha,\infty)})^2\right\}_{[2]}\end{aligned}$$

The "small eigenvalue" part of the curvature is given by

$$\begin{aligned}\Omega^{[0,\alpha)} &= -\operatorname{Tr_s}\left((\mathsf{P}_r^{[0,\alpha)}\circ\nabla\circ\mathsf{P}_r^{[0,\alpha)})^2\right)\\ &= \operatorname*{LIM}_{t\to 0^+}\operatorname{Tr_s}\left\{\exp -\left(\mathsf{P}_r^{[0,\alpha)}(\nabla+\sqrt{t}\mathsf{D}_r)\mathsf{P}_r^{[0,\alpha)}\right)^2\right\}_{[2]}\\ &= \operatorname*{LIM}_{t\to 0^+}\operatorname{Tr_s}\left\{\exp -(\mathbb{A}_{1,t}^{[0,\alpha)})^2\right\}_{(2)}\end{aligned}$$

Therefore

$$\Omega^{\hat{\mathcal{L}}} = \operatorname*{LIM}_{t\to 0^+} \operatorname{Tr_s}\left\{\exp -(\mathbb{A}_{1,t}^{[0,\alpha)})^2\right\}_{(2)} + \operatorname*{LIM}_{t\to 0^+} \operatorname{Tr_s}\left\{\exp -(\mathbb{A}_{1,t}^{(\alpha,\infty)})^2\right\}_{(2)}$$
$$= \operatorname*{LIM}_{t\to 0^+} \operatorname{Tr_s}\left\{\exp(-\mathbb{A}_{1,t})^2\right\}_{(2)}$$

□

*Proof of Lemma 4.3.3.* From Theorem 4.3.10 we have

$$-\frac{\partial}{\partial u}\operatorname{Tr_s}\left\{\exp(-\mathbb{A}_u^2)\right\} = d_{\hat{B}}\operatorname{Tr_s}\left\{\dot{L}_u\exp(-\mathbb{A}_u^2)\right\} + \operatorname{Tr_s}\left\{[L_u, \dot{L}_u\exp(-\mathbb{A}_u^2)]\right\}$$

where $L_u = \mathbb{A}_{2,u} - \nabla = u(\nabla^d - \nabla) + \sqrt{t}\mathsf{D}_r$ is the difference of the superconnection and the $L_2$ connection. We are interested only in the 2-form part of this equation.

The 2-form part of the first term on the right hand side is

$$d_{\hat{B}}\operatorname{Tr_s}\left\{\dot{L}_u\exp(-\mathbb{A}_u^2)\right\} = d_{\hat{B}}\operatorname{Tr_s}\left\{(\nabla^d - \nabla)\exp(-\mathsf{D}_r^2)\right\}$$

The difference $\nabla^d - \nabla$ is off-diagonal with respect to the spectral projection $\mathsf{P}_r^{[0,\alpha)}$, i.e.,

$$\nabla^d - \nabla = \mathsf{P}_r^{[0,\alpha)}\circ\nabla\circ(1-\mathsf{P}_r^{[0,\alpha)}) + (1-\mathsf{P}_r^{[0,\alpha)})\circ\nabla\circ\mathsf{P}_r^{[0,\alpha)}$$

whereas $\exp(-\mathsf{D}_r^2)$ is diagonal with respect to the spectral projection. It follows that the first term vanishes.

The 2-form part of the second term is

$$\operatorname{Tr_s}\left\{[L_u, \dot{L}_u\exp(-\mathbb{A}_u^2)]\right\} = \operatorname{Tr_s}\left\{[u(\nabla^d - \nabla) + \sqrt{t}\mathsf{D}_r, (\nabla^d - \nabla)\exp(-\mathbb{A}_u^2)]\right\}$$
$$= \sqrt{t}\operatorname{Tr_s}\left\{[\mathsf{D}, (\nabla^d - \nabla)\exp(-\mathbb{A}_u^2)]\right\}_{(2)}$$

The second equality follows from the first because $L_u$ differs from $\mathsf{D}$ by a bounded odd operator, and the supertrace of the supercommutator of a bounded operator with a smoothing operator vanishes. The 1-form part of $\exp(-\mathbb{A}_u^2)$ is

$$-\int_{\Sigma^1} \exp(-\sigma_0 t\mathsf{D}_r^2)[\nabla - u(\nabla^d - \nabla), \sqrt{t}\mathsf{D}_r] \exp(-\sigma_1 t\mathsf{D}_r^2)\, d\sigma$$

Thus the second term fully written out is

$$-\operatorname{Tr_s}\left\{[\mathsf{D}, (\nabla^d - \nabla)\int_{\Sigma^1} \exp(-\sigma_0\mathsf{D}_r^2)[\nabla - u(\nabla^d - \nabla), \mathsf{D}_r] \exp(-\sigma_1\mathsf{D}_r^2)\, d\sigma]\right\}$$

Applying Lemma 4.3.7, this is

$$-\int_{\partial^\pi \hat{M}/\hat{B}} \operatorname{tr_s}\left(JA(u;y,y)\right)\, dy$$

where $A(u;x,y)$ is the kernel of the smoothing operator

$$(\nabla^d - \nabla)\int_{\Sigma^1} \exp(-\sigma_0\mathsf{D}_r^2)[\nabla - u(\nabla^d - \nabla), \mathsf{D}_r] \exp(-\sigma_1\mathsf{D}_r^2) d\sigma$$

We claim that $A_u$ satisfies the conditions of Prop. 4.3.8, so that the integrand vanishes pointwise. Recall that

$$\nabla - \nabla^d = 2\mathsf{P}_r^{[0,\alpha)} \circ \nabla\mathsf{P}_r^{[0,\alpha)} - \nabla\mathsf{P}_r^{[0,\alpha)}$$

The spectral projection satisfies the boundary condition. By Assumption 4.3.1, the connection preserves the boundary condition. It follows that $(\nabla^d - \nabla)\exp(-\sigma_0\mathsf{D}_r^2)$ satisfies the boundary condition $\mathfrak{B}$ of $\mathsf{D}_r$ in the first variable. Clearly

$$[\nabla - u(\nabla^d - \nabla), \mathsf{D}_r] \exp(-u\sigma_1\mathsf{D}_r^2)$$

satisfies the boundary condition in the second variable. Thus the kernel of the composite

$$(\nabla^d - \nabla)\exp(-\sigma_0\mathsf{D}_r^2)[\nabla - u(\nabla^d - \nabla), \mathsf{D}_r]\exp(-\sigma_1\mathsf{D}_r^2)$$

satisfies the boundary condition in the first and second variables. Finally, the integral over the simplex commutes with the boundary condition. Thus the first commutator term vanishes. □

### 4.3.2 Removing Assumption 4.3.1

In this section, we show that Theorem 4.3.6, proved above under Assumption 4.3.1, is true even without this assumption. The key point is that the determinant line bundle construction is functorial, as is curvature.

**Proposition 4.3.13.** *Suppose that for a point $(r, b)$ in $\hat{B}$ and a vector $\xi$ in $T_bB$, $q : (-\epsilon, \epsilon)^2 \to \hat{B}$ is an embedding such that*

$$q(0,0) = (r,b),\ q_*\frac{\partial}{\partial x_1}(0,0) = \frac{\partial}{\partial r},\ \text{and}\ q_*\frac{\partial}{\partial x_2}(0,0) = (\iota_r)_*\xi$$

*Then $q^{-1}\hat{M}$ has an associated determinant line bundle and there is a natural flat isometry*

$$q^{-1}\hat{\mathcal{L}} \cong \mathcal{L}(q^{-1}\hat{M}).$$

*Furthermore, we have*

$$\Omega^{\hat{\mathcal{L}}}(\frac{\partial}{\partial r}, (\iota_r)_*\xi)(r,b) = \Omega^{q^{-1}\hat{\mathcal{L}}}(\frac{\partial}{\partial r}, \frac{\partial}{\partial s})(0,0)$$

*Proof.* The existence of an associated determinant line bundle for $q^{-1}\hat{M}$ follows from the results of Chapter 2: the pullback family has a well-defined family connection, family of (perturbed) Dirac operators, boundary conditions, etc.

Prop. 2.1.7 implies that integrals along the fiber pull back nicely. In particular, it follows that the $L_2$ metric and connection for $(\hat{\pi}\circ q)_* q^{-1}V$ are the pull-back metric and connection from $\hat{\pi}_* V$ for any vector bundle $V \to \hat{M}$, and that the functions and one-forms $z_\alpha$, $\mu_\alpha$ used in the construction of the Quillen metric and Bismut-Freed connection for the pull-back family agree with $q^* z_\alpha$ and $q^*\mu_\alpha$. Thus the natural map from sections of $(\hat{\pi}\circ q)_* q^{-1}V$ to sections of $\hat{\pi}_* V$ (defined for any $V$) induces a flat isometry of determinant line bundles.

The second statement is a simple consequence of the functorality of curvature, $\Omega^{q^{-1}\hat{\mathcal{L}}} = q^*\Omega^{\hat{\mathcal{L}}}$. $\square$

Of course, for small enough $\epsilon$ we can always construct such an embedding about any interior point of $\hat{B}$. For example, if $z:\ (-\epsilon,\epsilon)\to B$ is a path in $B$ tangential to $\xi$ at $z(0)$, then we can take $q$ to be

$$q:\ (x_1,x_2)\mapsto(x_1+r, z(x_2))$$

In fact, the continuity of the curvature of the determinant line bundle implies that it is enough to consider only interior points of $\hat{B}$. This observation and Prop. 4.3.13 allow us to assume without loss of generality that $\hat{B}=(-\epsilon,\epsilon)^2$, we are calculating $\Omega^{\hat{\mathcal{L}}}(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2})$ at $(0,0)$, and furthermore the curvature of the family connection $H$, defined by

$$\Omega^H(\xi_1,\xi_2) := [\xi_1,\xi_2]^H - [\xi_1^H,\xi_2^H]$$

is zero. Shrinking $\epsilon$ if necessary, we can ensure that the image of $q$ is contained in one of the spectral sets $U_\alpha$, for $\alpha > 0$. Thus we may assume that there is a global spectral cut.

Finally, to see that we may assume the boundary conditions to be fixed, we use parallel transport by the $L_2$ connection in the $x_2$ direction and apply Lemma 2.2.4, which states that $L_2$ parallel transport trivializes geometric local boundary conditions. The $x_1$ direction corresponds to the $r$ direction in $\hat{M}$, and the boundary conditions in that direction are constant by construction. The $L_2$ curvature for $q^{-1}\hat{M}$ is zero because both the family connection and spinor connection are products in the $r$-direction. Thus parallel transport globally trivializes the boundary conditions.

## 4.4 The index of the odd problem

In this section we prove an index theorem for families of odd-type boundary value problems. The prototype for our index theorem is Freed's Theorem B in [11]. This is an index theorem for a special class of odd-type boundary value problems determined by subsets $I$ of the set of boundary components of a spin manifold with boundary. To explain this boundary condition, suppose $X$ is a spin manifold with boundary, $E$ is a trivial line bundle over $X$, and $D$ is the Dirac operator coupled to $E$. A subset of boundary components $I$ determines a decomposition of $E$ into $E^+ \oplus E^-$ by

$$E|_Y = \begin{cases} E^+|_Y & Y \in I \\ E^-|_Y & \text{otherwise} \end{cases}$$

We let $\Pi_I$ be the corresponding odd-type boundary condition.

As a special case, we have the classical chiral boundary condition, corresponding to $E = E^+$. We denote this boundary condition by $\Pi_+$.

**Proposition 4.4.1** ([8], Theorem 21.5)**.** *We have* $\mathrm{Ind}(D, \Pi_+) = 0$*. In fact, the kernel and cokernel of* $(D, \Pi_+)$ *are both zero.*

This result plays an important role in the proofs of both Theorem B of [11] and our index theorem.

**Theorem 4.4.2** (Theorem B of [11])**.** *For any subset $I$ of the components of the boundary family let $\hat{I}$ be the complement. Then*

$$\mathrm{Ind}(D, \Pi_I) = \sum_{Y \in I} \mathrm{Ind}(D_Y^+) = -\sum_{Y \in \hat{I}} \mathrm{Ind}(D_Y^+) \tag{4.4.1}$$

*where* $D_Y^+ : C^\infty(Y; \mathcal{S}^+) \to C^\infty(Y; \mathcal{S}^-)$ *is the chiral Dirac operator on the boundary component* $Y$*.*

The key analytic tool in Freed's proof of this result is the Agranovich-Dynin Theorem:

**Theorem 4.4.3** (Agranovich-Dynin)**.** *Suppose* $\mathfrak{B}_1$ *and* $\mathfrak{B}_2$ *are local elliptic boundary conditions for the Dirac operator* $D$ *and surjective projections onto subbundles* $V_1$*,* $V_2$ *of* $\mathcal{S}|_{\partial X}$*. Then*

$$\mathrm{Ind}(D, \mathfrak{B}_1) - \mathrm{Ind}(D, \mathfrak{B}_2) = \mathrm{Ind}(\mathfrak{B}_2 \mathcal{P}(D) \mathfrak{B}_1^*)$$

*where* $\mathcal{P}(D)$ *is the Calderon projector for* $D$*.*

The Agranovich-Dynin Theorem, combined with Prop. 4.4.1, reduces the index calculation to the boundary:

$$\mathrm{Ind}(D, \Pi_I) = \mathrm{Ind}(D, \Pi_I) - \mathrm{Ind}(D, \Pi_+) = \mathrm{Ind}(\Pi_I \mathcal{P}(D) \Pi_+^*)$$

The Atiyah-Singer families index theorem applies to the family of boundary operators $\mathfrak{B}_2 \mathcal{P}(D) \mathfrak{B}_1^*$. In particular, Prop. 2.4 and Theorem 3.1 of [3] imply that the index depends only on the restriction of the principal symbol to the unit cosphere bundle for the boundary family. Observing that the principal symbol of this family coincides with the principal symbol of the boundary Dirac operator $\oplus_{Y \in I} D_Y^+$ on the unit cosphere bundle, we have the theorem.

Our index theorem is a straightforward generalization of this result to the larger class of all odd-type boundary value problems with essentially the same method of proof.

**Theorem 4.4.4.** *Suppose $\mathfrak{B}$ is an odd-type boundary condition. Then*

$$\operatorname{Ind}(D, \mathfrak{B}) = \operatorname{Ind}(D_\partial^+(E^+)) = -\operatorname{Ind}(D_\partial^+(E^-)) \tag{4.4.2}$$

*where $D_\partial^+(E^+)$ and $D_\partial^+(E^-)$ are the boundary Dirac operator coupled to $E^+$ and $E^-$ respectively.*

*Proof.* We rely on Theorem 4.4.8, a version of the Agranovich-Dynin Theorem for families of local elliptic boundary value problems. This theorem implies that

$$\operatorname{Ind}(D, \mathfrak{B}) = \operatorname{Ind}(D, \mathfrak{B}) - \operatorname{Ind}(D, \Pi_+) = \operatorname{Ind} \mathfrak{B}\mathcal{P}(D)\Pi_+^*$$

Near the boundary, $D$ takes the form $J(\partial_u + D_\partial)$, where $D_\partial$ is the Dirac operator on the boundary coupled to $E_{|\partial X}$. The space $S \otimes E|\partial X$ decomposes as

$$(S^+ \otimes E^+) \oplus (S^- \otimes E^-) \oplus (S^+ \otimes E^-) \oplus (S^- \otimes E^+)$$

This decomposition of $S \otimes E|\partial X$ induces a corresponding decomposition of $D_\partial$. Let $D_{\partial ++} : C^\infty(\partial X; S^+ \otimes E^+) \to C^\infty(\partial X; S^- \otimes E^+)$ be the restriction of $D_\partial$ to $C^\infty(\partial X; S^+ \otimes E^+)$, and define $D_{\partial --}$, $D_{\partial +-}$, $D_{\partial -+}$ similarly. Let $a_{++}$, $a_{--}$, $a_{+-}$, and $a_{-+}$ be the corresponding principal symbols.

Let $p_+$ be the principal symbol of the Calderon projector $\mathcal{P}(D)$. Crucially, $p_+$ can be expressed in terms of the principal symbol of $A$ over the unit cosphere bundle (Prop. 14.2 and Coro. 14.3 of [8]). We have

$$p_+(z, \zeta) = \frac{1}{2} \begin{pmatrix} 1 & & & a_{-+} \\ & 1 & a_{+-} & \\ & a_{--} & 1 & \\ a_{++} & & & 1 \end{pmatrix}$$

We can write the principal symbols of $\mathfrak{B}$ and $\Pi_+$ as

$$\mathfrak{B} = \begin{pmatrix} & & 1 & \\ & & & 1 \end{pmatrix}$$
$$\Pi_+ = \begin{pmatrix} 1 & & & \\ & & 1 & \end{pmatrix}$$

Therefore the principal symbol of $\mathfrak{B}\mathcal{P}(D)\Pi_+^*$ is given by

$$\frac{1}{2}\begin{pmatrix} & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} 1 & & & a_{-+} \\ & 1 & a_{+-} & \\ & a_{--} & 1 & \\ a_{++} & & & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & \\ & 1 \\ & \end{pmatrix} = \frac{1}{2}\begin{pmatrix} & 1 \\ a_{++} & \end{pmatrix}$$

It follows that $\operatorname{Ind}(D, \mathfrak{B}) = \operatorname{Ind}(D_{\partial++})$. That is, $\operatorname{Ind}(D, \mathfrak{B}) = \operatorname{Ind}(D_\partial^+(E^+))$. To see that $\operatorname{Ind}(D, \mathfrak{B}) = -\operatorname{Ind}(D_\partial^+(E^-))$ it suffices to observe that

$$\operatorname{Ind}(D_\partial^+(E^+)) + \operatorname{Ind}(D_\partial^+(E^-)) = 0$$

by Prop. 4.4.1 □

It remains to prove Theorem 4.4.8, a version of the Agranovich-Dynin Theorem for families of local elliptic boundary value problems. We rely on standard results about the Calderon projector and the Poisson operator for $D$. Our reference for this material is [8].

Let $M$ be a family of spin manifolds with boundary, $\mathcal{S}$ the spinors associated to a spin-structure for $M$, $E \to M$ an auxiliary vector bundle, and $D$ the family of Dirac operators coupled to $E$. Let $\mathcal{H}(D)$ be the space of Cauchy data for smooth solutions of $D\phi = 0$ in the interior. For local elliptic boundary conditions $\mathfrak{B}$ for $D$ we let $\mathfrak{B}\mathcal{K}(D)$ denote $\mathfrak{B}$ restricted to the Cauchy data space.

**Lemma 4.4.5.** *Suppose $\mathfrak{B}$ is a local elliptic boundary condition for $D$ and an orthogonal projection onto a subbundle $V$ of $\mathcal{S} \otimes E|_{\partial X}$. Then the diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}(D) & \xrightarrow{\mathcal{K}(D)} & C^\infty(\mathcal{S}\otimes E) & \xrightarrow{D} & C^\infty(\mathcal{S}\otimes E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathfrak{B}\mathcal{K}(D)} & & \downarrow{\scriptstyle D\oplus\mathfrak{B}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & C^\infty(\partial X; V) & \longrightarrow & C^\infty(\mathcal{S}\otimes E)\oplus C^\infty(\partial X; V) & \longrightarrow & C^\infty(\mathcal{S}\otimes E) & \longrightarrow & 0 \end{array}$$

*commutes. The rows are short exact. The vertical operators have finite-dimensional kernels and cokernels, and we have*

$$\mathrm{Ker}(\mathfrak{B}\mathcal{K}(D)) \cong \mathrm{Ker}(D\oplus\mathfrak{B}), \qquad \mathrm{Coker}(\mathfrak{B}\mathcal{K}(D)) \cong \mathrm{Coker}(D\oplus\mathfrak{B})$$

*Proof.* The fact that $\mathfrak{B}\mathcal{K}(D)$ has finite-dimensional kernel and cokernel is an immediate consequence of Lemma 20.11 of [8]. The second row is trivially short exact. The exactness of the first row follows from Theorem 12.4 of [8], which shows that the Poisson operator is a bijection from $\mathcal{H}(D)$ onto the smooth kernel of $D$. Prop. 13.5 of [8] shows that the first square is commutative. The second square is trivially commutative. Finally, a diagram chase shows that we have natural isomorphisms

$$\mathrm{Ker}(\mathfrak{B}\mathcal{K}(D)) \cong \mathrm{Ker}(D\oplus\mathfrak{B}), \qquad \mathrm{Coker}(\mathfrak{B}\mathcal{K}(D)) \cong \mathrm{Coker}(D\oplus\mathfrak{B})$$

(Alternatively, Coro. 19.2 and Theorem 20.12 of [8] give these isomorphisms explicitly.) □

**Lemma 4.4.6.** *There exist finitely many smooth sections $\{(\phi_i, v_i)\}_{i=1}^N$ of $\pi_*(\mathcal{S}\otimes E)\oplus\pi_*(\partial X; V)$ such that*

$$\widetilde{D\oplus\mathfrak{B}} : C^\infty(\mathcal{S}\otimes E)\oplus\mathbb{C}^k \to C^\infty(\mathcal{S}\otimes E)\oplus C^\infty(\partial X; V)$$

*defined by*

$$\widetilde{D \oplus \mathfrak{B}} : \psi \oplus Z \mapsto (D \oplus \mathfrak{B})\psi + \sum_j Z^j \phi_j \oplus v_j$$

*is surjective.* $\operatorname{Ker} \widetilde{D \oplus \mathfrak{B}}$ *is a vector bundle on* $B$*. The* $K$*-theory element*

$$[\operatorname{Ker} \widetilde{D \oplus \mathfrak{B}}] - [\mathbb{C}^N]$$

*does not depend on* $N$ *or the choice of smooth sections satisfying the conditions above.*

*Proof.* We construct this finite-rank perturbation of $D \oplus \mathfrak{B}$ by a procedure similar to the one employed in the proof of Prop. 2.2 of [3]. At each point $b$ in $B$, the cokernel of $D \oplus \mathfrak{B}$ is a finite-rank subspace of $C^\infty(\mathcal{S} \otimes E) \oplus C^\infty(\partial X; V)$. We fix a basis $\{\phi_i \oplus v_i\}_{i=1}^{n_b}$ for the cokernel and use a local trivialization of $\pi_*(\mathcal{S} \otimes E) \oplus \pi_*(\partial X; V)$ over an open neighborhood $U_b$ of $b$ (by $L_2$ parallel transport along a radial vector field, for example) to extend the sections in the basis over $U_b$. The map

$$D \oplus \mathfrak{B} + r_b : C^\infty(\mathcal{S} \otimes E)|_{U_b} \oplus \mathbb{C}^k \to C^\infty(\mathcal{S} \otimes E) \oplus C^\infty(\partial X; V)$$

defined by

$$D \oplus \mathfrak{B} + r_b : \psi \oplus Z \mapsto (D \oplus \mathfrak{B})\psi + \sum_j Z^j \phi_j \oplus v_j$$

is surjective at $b$, and by continuity in a neighborhood of $b$ as well. Shrinking $U_b$ if necessary, we may assume that this map is surjective over $U_b$. We fix a cutoff function $\chi_b$ supported on $U_b$ and identically 1 near $b$. Carrying out this construction at each point of $B$, let $\mathcal{U}$ be a finite subcover of the cover $\{U_b\}$, $N = \sum_{\mathcal{U}} n_b$, and set

$$\widetilde{D \oplus \mathfrak{B}} : \psi \oplus Z \mapsto (D \oplus \mathfrak{B})\psi + \sum_j Z^j \chi_b \phi_j \oplus v_j$$

This map is a continuous family of surjective Fredholm operators by construction. The proof of the remaining statements now goes through exactly as in the proof of Prop. 2.2 of [3]. □

**Definition 4.4.1.** The index of $D \oplus \mathfrak{B}$ is the $K$-theory element

$$\operatorname{Ind} D \oplus \mathfrak{B} := [\operatorname{Ker} \widetilde{D \oplus \mathfrak{B}}] - [\mathbb{C}^N]$$

constructed above.

**Lemma 4.4.7.** *We have*

$$\operatorname{Ind} D \oplus \mathfrak{B} = \operatorname{Ind} \mathfrak{B}\mathcal{K}(D)$$

*Proof.* We make a finite-rank perturbation of the commutative diagram in Lemma 4.4.5 that makes all the vertical operators continuous families of surjective Fredholm operators.

Let $\{(\phi_i, v_i)\}_{i=1}^N$ and $\widetilde{D \oplus \mathfrak{B}}$ be constructed as in the previous lemma. By Lemma 4.4.5, there is a natural isomorphism $\operatorname{Coker}(\mathfrak{B}\mathcal{K}(D)) \cong \operatorname{Coker}(D \oplus \mathfrak{B})$. Let $\{w_j\}$ be the basis of $\operatorname{Coker}(\mathfrak{B}\mathcal{K}(D))$ corresponding to the chosen basis for $\operatorname{Coker}(D \oplus \mathfrak{B})$ under this isomorphism. Define $\widetilde{\mathfrak{B}\mathcal{K}(D)} : \mathcal{H} \oplus \mathbb{C}^N$ similarly to $\widetilde{D \oplus \mathfrak{B}}$ using the $\{w_j\}$. Then the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}(D) \oplus \mathbb{C}^N & \xrightarrow{\mathcal{K}(D)\oplus \mathrm{id}} & C^\infty(\mathcal{S}\otimes E)\oplus \mathbb{C}^N & \xrightarrow{D} & C^\infty(\mathcal{S}\otimes E) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{\mathfrak{B}\mathcal{K}(D)}} & & \downarrow{\scriptstyle \widetilde{D\oplus\mathfrak{B}}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & C^\infty(\partial X; V) & \longrightarrow & C^\infty(\mathcal{S}\otimes E)\oplus C^\infty(\partial X; V) & \longrightarrow & C^\infty(\mathcal{S}\otimes E) & \longrightarrow & 0 \end{array}$$

commutes by Lemma 4.4.5. The surjectivity of $\widetilde{D \oplus \mathfrak{B}}$ therefore implies the surjectivity of $\widetilde{\mathfrak{B}\mathcal{K}(D)}$, and we have

$$\operatorname{Ker} \widetilde{\mathfrak{B}\mathcal{K}(D)} \cong \operatorname{Ker} \widetilde{D \oplus \mathfrak{B}}$$

This isomorphism is induced by $\mathcal{K}(D)$, a smooth family of pseudodifferential operators. We conclude that $\operatorname{Ker} \widetilde{\mathfrak{B}\mathcal{K}(D)}$ is a vector bundle and

$$[\operatorname{Ker} \widetilde{D \oplus \mathfrak{B}}] - [\mathbb{C}^N] = [\operatorname{Ker} \widetilde{\mathfrak{B}\mathcal{K}(D)}] - [\mathbb{C}^N]$$

That is,

$$\operatorname{Ind} D \oplus \mathfrak{B} = \operatorname{Ind} \mathfrak{B}\mathcal{K}(D)$$

□

**Theorem 4.4.8** (Agranovich-Dynin for Families). *Suppose $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are families of local elliptic boundary conditions for the Dirac operator $D$ and surjective projections onto subbundles $V_1$, $V_2$ of $\mathcal{S} \otimes E|_{\partial X}$. Then*

$$\operatorname{Ind}(D, \mathfrak{B}_1) - \operatorname{Ind}(D, \mathfrak{B}_2) = \operatorname{Ind}(\mathfrak{B}_2 \mathcal{P}(D) \mathfrak{B}_1^*)$$

*Proof.* From Lemma 4.4.7 we have

$$\begin{aligned}\operatorname{Ind}(D, \mathfrak{B}_1) - \operatorname{Ind}(D, \mathfrak{B}_2) &= \operatorname{Ind} \mathfrak{B}_1 \mathcal{K}(D) - \operatorname{Ind} \mathfrak{B}_2 \mathcal{K}(D) \\ &= \operatorname{Ind} \mathfrak{B}_1 \mathcal{K}(D) + \operatorname{Ind} \mathcal{K}(D) \mathfrak{B}_2^* \\ &= \operatorname{Ind} \mathfrak{B}_1 \mathcal{K}(D) \mathfrak{B}_2^*\end{aligned}$$

□

# Appendices

# Appendix A

# Boundary value problems and heat operators

## A.1 Introduction

In this appendix, we review the theory of $p$-elliptic boundary value problems and associated heat operators. Following Greiner in [16], we introduce $p$-elliptic differential operators and $p$-elliptic boundary value problems. These are local boundary conditions determined by a boundary differential operator. We review the spectral theory of $p$-elliptic differential operators and boundary value problems, and the properties of the associated heat operator. In particular we see that there is an asymptotic expansion of the trace of the heat operator in powers $t^{k/2}$, $k \geq -\dim X$ as $t \to 0^+$, given by locally-determined coefficients that depend smoothly on the coefficients of the operator and its boundary condition. Furthermore, we have a similar kind of asymptotic expansion if we apply differential operators to the heat kernel before we take the trace.

## A.2 Heat operators for compact manifolds

Suppose $\hat{X}$ is a compact manifold without boundary, $E$ is a hermitian vector bundle on $\hat{X}$, and $P$ is a second-order differential operator of order 2,

$$P: \, C^\infty(E) \to C^\infty(E)$$

**Definition A.2.1.** We say that $P$ is $p$-elliptic if the eigenvalues of its principal symbol $p^0(x,\xi)$, $\lambda_i(x,\xi)$ at any point $(x,\xi)$ of the cosphere bundle satisfy

$$\operatorname{Re} \lambda_i(x,\xi) > \delta > 0$$

independent of $x$.

Thus a $p$-elliptic operator is both positive and uniformly elliptic.

We assume that $P$ is $p$-elliptic and formally self-adjoint.

**Theorem A.2.1.** *The closure of $P$ (which we also denote by $P$) is self-adjoint with domain $H^2(E)$. The spectrum of $P$ is a countable number of real eigenvalues converging to infinity (in particular, there is no finite accumulation point). There exists $R > 0$ such that $(P-\lambda)^{-1}$ exists if $\lambda \in \mathbb{C}$ and $\lambda \notin [-R,\infty)$. Furthermore,*

$$\|(P-\lambda)^{-1}\| < C \operatorname{dist}(\lambda, \operatorname{spec}(P))^{-1}$$

*There is a complete orthonormal basis of smooth eigensections.*

This theorem follows from Theorem 1.5.2 of Greiner.

A fundamental solution to the heat equation for $P$ is a family of operators $H(t)$ on $0 < t$, such that $Hf(t;x) := (H(t)f)(x)$ is continuous, $C^1$ in $t$ and $C^2$ in $x$,

$$[\frac{\partial}{\partial t} + P]H(t)f = 0$$

and

$$\lim_{t \to 0^+} H(t)f = f$$

For $t \geq 0$, define

$$\exp\{-tP\} := \frac{-1}{2\pi i} \int_\Gamma e^{-t\lambda}(P-\lambda)^{-1}\, d\lambda$$

where $\Gamma$ is a positively oriented contour following the rays $\arg(\lambda) = \pm\theta$, $0 < \theta < \pi/2$ for $|\lambda| > 2R$, and the arc $|\lambda| = 2R$, $\theta \leq \arg\lambda \leq 2\pi - \theta$. Theorem A.2.1 shows that the integrand exists and if $t > 0$ the integral converges to define a bounded operator on $L_2$ sections, solving the heat equation in $L_2$:

**Theorem A.2.2.** *For non-negative integral $k$, and $f$ in $L_2$,*

$$\left(\frac{\partial}{\partial t}\right)^k \exp\{-tP\}f = (-P)^k \exp\{-tP\}f = \exp\{-tP\}(-P)^k f$$

$$\lim_{t \to 0^+} \|\exp\{-tP\}f - f\|_{L_2} = 0$$

$$\exp\{-tP\}\exp\{-sP\} = \exp\{-(t+s)P\}$$

These results follow from Theorems 1.5.1 and 1.5.2 of Greiner. See the discussion following Equation 1.5.7.

In fact, $\exp\{-tP\}$ is a uniformly bounded family of operators on $L_2$. This can be seen by applying Parseval's Theorem relative to a complete orthonormal basis of eigensections:

$$\begin{aligned}\|\exp\{-tP\}f\|_{H^0}^2 &= \sum e^{-2t\lambda}|f_\lambda|^2 \\ &\leq \sum |f_\lambda|^2 = \|f\|_{H^0}^2\end{aligned}$$

**Theorem A.2.3.** *There exists a unique fundamental solution to the heat equation for $P$. $H(t)$ is given by a smooth kernel $h(t;x,x')$. In $L_2$,*

$$\exp\{-tP\}f = H(t)f$$

This theorem follows immediately from Theorems 1.4.3 and 1.5.4 of Greiner.

**Theorem A.2.4.** *For each $l \geq 0$ and $0 < T < \infty$, $H(t)$ is a uniformly bounded family of operators on $(0,T)$ from $C^{l+k}$ to $C^l$ for all $k$ such that $k > \dim(\hat{X})/2$ and $l+k$ is even.*

*Proof.* Suppose $s$ is a $C^{l+k}$ section. Let $n := \dim \hat{X}$. We apply the Sobolev embedding theorem and Garding's inequality:

$$\begin{aligned}\|H(t)s\|_{C^l} &\leq C\|H(t)s\|_{H^{l+k}} \\ &\leq C\left(\|P^{\frac{1}{2}(l+k)}H(t)s\|_{H^0} + \|H(t)s\|_{H^0}\right) \\ &= C\left(\|H(t)P^{\frac{1}{2}(l+k)}s\|_{H^0} + \|H(t)s\|_{H^0}\right)\end{aligned}$$

Since $H(t)$ is a uniformly bounded family of operators, we have

$$\begin{aligned}\|H(t)s\|_{C^l} &\leq C\left(\|P^{\frac{1}{2}(l+k)}s\|_{H^0} + \|s\|_{H^0}\right) \\ &\leq C\|s\|_{H^{l+k}} \\ &\leq C\|s\|_{C^{l+k}}\end{aligned}$$

□

*Remark* A.2.1. In fact, for Laplace-type operators $H(t)$ is a uniformly bounded family of operators from $C^l$ to $C^l$; this is shown in [5]. We introduce this argument because this result is strong enough for the patching argument we make below, and because a similar argument applies also to elliptic boundary value problems.

We conclude this summary with an estimate on the sup norm of the off-diagonal part of the heat kernel that we will need for our patching argument.

**Theorem A.2.5.** *Let $\psi$, $\phi$ be smooth functions on $\hat{X}$ with disjoint support. Then for some $T$, $0 < T < \infty$, and all non-negative $\alpha$, $\beta$, $p$,*

$$\sup_{\substack{x,y\in\hat{X} \\ 0<t<T}} |L(\frac{\partial}{\partial t})^p \psi(x)h(t;x,y)\phi(y)| < C\exp\left\{-\delta t^{-1}\right\}$$

*for some constant $\delta > 0$, where $L$ is a differential operator of order $\alpha$ in $x$, and order $\beta$ in $y$.*

This follows from Lemma 1.5.6 of Greiner.

## A.3 Heat operators for boundary value problems

As in the last section, $\hat{X}$ is a compact manifold without boundary, $E \to \hat{X}$ is a Hermitian vector bundle, and $P : C^\infty(E) \to C^\infty(E)$ is a formally self-adjoint elliptic differential operator of order 2. Suppose $X$ is a smooth compact manifold with boundary embedded in $\hat{X}$, so that $\partial X$ is an embedded hypersurface.

Next we give a definition of a local elliptic boundary value problem adapted to second order operators. Fix a collar neighborhood $N$ of $\partial X$ and a parametrization

$$u : (-1,1) \times \partial X \to N$$

such that $u(r,y)$ is in $X$ if $r < 0$. The connection on $E$ lifts this parametrization to a parametrization of $E|N$,

$$u_E : (-1,1) \times E|\partial X \to E|N$$

We define $u_E(r,e)$ to be the endpoint of parallel transport of $e$ along the path $\tau \mapsto u(\tau, \pi_E(e))$ as $\tau$ goes from 0 to $r$. This gives us a globally-defined normal derivative $\frac{\partial}{\partial r}$ for sections of $E$. Let $\underline{\gamma}$ be the Cauchy boundary operator,

$$\begin{aligned} \underline{\gamma} : \ & C^\infty(E) \to C^\infty(E|\partial X \oplus E|\partial X) \\ & s \mapsto s|_{\partial X} \oplus (-i \frac{\partial}{\partial r} s)_{|\partial X} \end{aligned}$$

Suppose $W$ is vector bundle over $\partial X$, $\operatorname{rank} E = \operatorname{rank} W$, and

$$\mathfrak{B} : C^\infty(E|N) \to C^\infty(W)$$

factors through $\underline{\gamma}$. That is, $\mathfrak{B} = \mathfrak{B}'\underline{\gamma}$, where $\mathfrak{B}'$ is a tangential differential operator. Furthermore, we assume that there is a decomposition of $W$, $W = W_0 \oplus W_1$ relative to which $\mathfrak{B}'$ is a lower-triangular matrix of differential operators,

$$\mathfrak{B}' = \begin{pmatrix} \mathfrak{B}'_{0,0} & 0 \\ \mathfrak{B}'_{0,1} & \mathfrak{B}'_{1,1} \end{pmatrix}$$

where $\mathfrak{B}'_{i,j}$ is of order at most $j - i$. (Thus $\mathfrak{B}$ has order at most 1.) We allow the terms of this decomposition to have rank 0. Let $\sigma_g(\mathfrak{B})(y,\zeta)$ denote the corresponding matrix of principal symbols,

$$\sigma_g(\mathfrak{B})_{i,j}(y,\zeta) = \begin{cases} \sigma(\mathfrak{B}'_{i,j})(y,\zeta)) & \text{if } \operatorname{ord}(\mathfrak{B}'_{i,j}) = j - i \\ 0 & \text{otherwise} \end{cases}$$

Let $(P, \mathfrak{B})$ denote $P$ acting on sections in $C^\infty(E|X)$ with the boundary condition $\mathfrak{B}s = 0$.

**Definition A.3.1.** A boundary value problem $(P, \mathfrak{B})$ is $p$-elliptic if $P$ is $p$-elliptic on $X$ and for each boundary point $y$ and pair $(\zeta, \lambda) \neq (0, 0)$ in $\mathbb{R}^{n-1} \times \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$, $u(r) = 0$ is the only bounded solution of the system of ODEs on $(-\infty, 0]$,

$$p^0(y, -i\frac{\partial}{\partial r}, \zeta)u(r) + \lambda u(r) = 0$$

subject to the boundary condition

$$\sigma_g(\mathfrak{B})(y,\zeta)\underline{\gamma}u = 0$$

(We have adapted this definition from the corresponding definitions in Greiner and Gilkey [14]. It is trivial to verify that this is a special case of the definition found in Greiner.)

We suppose that $(P, \mathfrak{B})$ is $p$-elliptic and formally self-adjoint. That is, the adjoint boundary value problem for $(P, \mathfrak{B})$ is $(P, \mathfrak{B})$. For such a boundary value problem, we have the analogs of Theorems A.2.1 and A.2.2.

**Theorem A.3.1.** *There is an orthonormal basis for $L_2(E)$ consisting of smooth eigensections of $P$ that satisfy the boundary condition $\mathfrak{B}$. There exists $R > 0$ such that the eigenvalues have no finite accumulation point and are contained in $[-R, \infty)$.*

Let $P_{\mathfrak{B}}$ denote the closure of $P$ in $L_2$ with the boundary condition $\mathfrak{B}$. As before, we construct a family of operators

$$\exp\{-tP_{\mathfrak{B}}\} : \; H^0(E) \to H^0(E)$$

as a Cauchy integral. (Or equivalently, by the spectral theorem.) This family will be uniformly bounded on the interval $[0, T)$ for each $0 < T < \infty$. Furthermore

**Theorem A.3.2.** *For all non-negative integral $k$ and $f$ in $H^0(E)$, $\exp\{-tP_{\mathfrak{B}}\}f$ is in the domain of $P^k$ determined by $(P, \mathfrak{B})$, i.e., $BP^j f = 0$, $0 \le j < k$, and*

$$\left(\frac{\partial}{\partial t}\right)^k \exp\{-tP_{\mathfrak{B}}\}f = (-P)^k \exp\{-tP_{\mathfrak{B}}\}f$$

$$\lim_{t \to 0^+} \|\exp\{-tP_{\mathfrak{B}}\}f - f\|_{L_2} = 0$$

$$\exp\{-tP_{\mathfrak{B}}\} \exp\{-sP_{\mathfrak{B}}\} = \exp\{-(t+s)P_{\mathfrak{B}}\}$$

*If $f$ is in the domain of $P^k$ determined by $(P, \mathfrak{B})$,*

$$(-P)^k \exp\{-tP_{\mathfrak{B}}\}f = \exp\{-tP_{\mathfrak{B}}\}(-P_{\mathfrak{B}})^k f$$

Garding's inequality generalizes to elliptic boundary value problems.

**Theorem A.3.3.** *Suppose $u$ is in $H^t(E|X)$, $Pu$ is in $H^s(E|X)$, and $\mathfrak{B}u$ is in $H^{s+1/2}(W)$, $s > 1/2$. Then $u$ is in $H^{s+2}(E)$, and*

$$\|u\|_{H^{s+2}} \leq C(\|Pu\|_{H^s} + \|\mathfrak{B}u\|_{H^{s+1/2}(W)} + \|u\|_{H^t}$$

This is Theorem VI.4 of Seeley [26].

**Corollary A.3.4.** *If $f$ is in the domain of $P^k$ determined by $(P, \mathfrak{B})$ and $0 < t < T < \infty$, then*

$$\|\exp\{-tP_{\mathfrak{B}}\}f\|_{H^{2k}} \leq C\|f\|_{H^{2k}}$$

*The constant depends only on $(P, \mathfrak{B})$, $T$, and $k$.*

*Proof.* By an easy induction argument, Garding's inequality for elliptic boundary value problems shows that $\|f\|_{H^{2k}} \leq C(\|P^k f\|_{H^0} + \|f\|_{H^0})$ for $f$ in the domain of $P^k$. Thus

$$\begin{aligned}\|\exp\{-tP_{\mathfrak{B}}\}f\|_{H^{2k}} &\leq C(\|P^k \exp\{-tP_{\mathfrak{B}}\}f\|_{H^0} + \|\exp\{-tP_{\mathfrak{B}}\}f\|_{H^0}) \\ &= C(\|\exp\{-tP_{\mathfrak{B}}\}P^k f\|_{H^0} + \|\exp\{-tP_{\mathfrak{B}}\}f\|_{H^0}) \\ &\leq C(\|P^k f\|_{H^0} + \|f\|_{H^0})\end{aligned}$$

We have used Theorem A.3.2 and the fact that $\exp\{-tP_{\mathfrak{B}}\}$ is a uniformly bounded family of operators on $H^0$ for $0 < t < T < \infty$. □

**Theorem A.3.5.** *For each $l \geq 0$, compact subset $K$ of the interior of $X$ and $0 < T < \infty$, $H(t)$ is a uniformly bounded family of operators on $(0, T)$ from $C_0^{l+k}(K)$ to $C^l(X)$ for all $k$ such that $k > \dim(\hat{X})/2$ and $l + k$ is even.*

*Proof.* Suppose $s$ is a $C^{l+k}$ section of $E$ supported in $K$. Then $s$ is in the domain of $P^{\frac{1}{2}(l+k)}$, i.e., $\mathfrak{B}P^j f = 0$, $0 \leq j < \frac{1}{2}(l+k)$. The proof is almost identical to the proof of Theorem A.2.4: we apply the Sobolev embedding theorem and the fact that $\exp\{-tP_{\mathfrak{B}}\}$ is uniformly bounded family of operators on $H^{l+k}$. □

**Theorem A.3.6.** *There exists a unique fundamental solution to the heat equation for* $(P, \mathfrak{B})$*.* $H(t)$ *is given by a smooth kernel* $h(t; x, x')$*. In* $L_2$*,*

$$\exp\{-tP\} f = H(t) f$$

This theorem follows immediately from Theorems 2.5.1 and 2.5.2 of Greiner.

Let $F$ be the restriction of the fundamental solution to the heat equation for $P$ on $\hat{X}$ to sections of $E|X$ and define $C(t)$ implicitly by

$$H(t) = F(t) - C(t)$$

In view of Theorem A.3.6, $C(t)$ is given by a smooth kernel, and

$$\mathfrak{B}F = \mathfrak{B}C$$

$$\lim_{t \to 0^+} \|C(t)s\|_{H^0} = 0$$

$C(t)$ is called the compensating factor associated to $(P, \mathfrak{B})$. Greiner constructs the heat kernel for $(P, \mathfrak{B})$ by constructing the compensating factor.

**Theorem A.3.7.** *There is an explicit formula for $C(t)$ in terms of $F(t)$ and finitely many jets of the symbols of $P$ and $\mathfrak{B}$. In particular, the kernel of $C(t)$ depends smoothly on $F$, $P$, and $\mathfrak{B}$.*

Equation 2.4.21 of Greiner is the explicit formula for $C(t)$. It follows that the kernel of $H$ depends smoothly on $F$, $P$, and $\mathfrak{B}$.

Since the heat operator for $P$ is unique, we may apply results from Gilkey-Smith, despite the fact that their construction of the heat kernel is not the same as the construction given in Greiner. The following is Lemma 2.6 of Gilkey-Smith [15].

**Theorem A.3.8.** *Suppose $Q$ is a differential operator of degree $d$, $(P, \mathfrak{B})$ is* p-*elliptic, and $H(t)$ is the fundamental solution to the heat equation. Then* $\mathrm{Tr}(QH(t))$ *has an asymptotic expansion as $t \to 0^+$ of the form:*

$$\mathrm{Tr}(QH(t)) \sim t^{-(n+d)/2} \sum_{k=0}^{\infty} t^{k/2} a_k(Q, P, \mathfrak{B})$$

*where $a_k(Q, P, \mathfrak{B})$ is given by a local formula:*

$$a_k(Q, P, \mathfrak{B}) = \int_X A_k(x, Q, P)\, \mathrm{dvol}_X + \int_{\partial X} A'_k(x, Q, P, \mathfrak{B})\, \mathrm{dvol}_{\partial X}$$

*where $A_k(x, Q, P)$ and $A'_k(x, Q, P, \mathfrak{B})$ are smooth local invariants of the jets of the symbols of the operators involved.*

## A.4 Heat operators for non-local differential operators

In Chapter 4 we relate the determinant line bundles for two different families of operators by parallel transport through the determinant line bundle of an auxiliary family of non-pseudodifferential operators. As we will see, the operators in this family fail to be differential in a very mild way. We call such "mildly non-differential" operators non-local differential operators. For non-local perturbations of Laplace-type differential operators (defined below), we define elliptic boundary conditions and extend the classical spectral theory and heat operator theory, arriving at results that generalize Theorems A.3.1, A.3.6, and A.3.8.

### A.4.1 Non-local differential operators

Intuitively, a non-local differential operator is an operator that is related to a differential operator by a quotient. To give a simple example, consider functions on $X = [0, 2\pi] \sqcup S^1$. We think of $S^1$ as $[0, 2\pi]/2\pi$. These may be written as $f \sqcup g$, and if $h$ is a function compactly supported on $(0, 2\pi)$, there is an obvious map $L : f \sqcup g \mapsto hg \sqcup hf$. Then $\frac{\partial}{\partial\theta} + L$ is an example of a non-local differential operator. Off the support of $h$, $L$ is differential, and on the support of $h$, we can consider the operator to be a differential operator acting on functions $(0, 2\pi) \to \mathbb{C}^2$. However, neither perspective can be adopted globally.

In general, we define a non-local differential operator as follows. Suppose $X$ is a manifold with boundary, and $E$ and $F$ are vector bundles over $X$. Suppose $U_1$ and $U_2$ are disjoint open subsets of $X$ with proper diffeomorphisms $q_i : \tilde{U} \to U_i$. Let $E_i := q_i^{-1}(E|U_i)$, $\tilde{E} := E_1 \oplus E_2$, and define $F_i$ and $\tilde{F}$

similarly. Let $q_E$ be the map of sections

$$q_E : \Gamma(E|U_1 \cup U_2) \to \Gamma(\tilde{E})$$
$$f_1 \cup f_2 \mapsto q_1^* f_1 \oplus q_2^* f_2$$

and define $q_F$ similarly. Since the underlying diffeomorphisms are proper, $q_E$, $q_F$ and their inverses are bounded in $C^l$ norm for all $l \geq 0$.

**Definition A.4.1.** If $G$ is a linear operator from $\Gamma(E) \to \Gamma(F)$, we say that $G$ is a non-local differential operator relative to $U_1$, $U_2$ if there is a closed subset $K \subset U_1 \cup U_2$ such that $G$ is a differential operator on $X - K$ and there is a differential operator $\tilde{G}$ from $\Gamma(\tilde{E}) \to \Gamma(\tilde{F})$ such that

$$Af = q_F^{-1} \tilde{A} q_E f$$

on all sections $f$ supported on $U_1 \cup U_2$. For points in the non-local support of $G$, we define the order of $G$ to be the order of its associated differential operator $\tilde{G}$. At other points, $G$ is differential, and its order is defined as usual.

**Proposition A.4.1.** *A non-local differential operator $G : \Gamma(E) \to \Gamma(F)$ of order $d$ is a bounded map from $C^l(E)$ to $C^{l-d}(F)$ for all $l \geq d$.*

*Proof.* This follows immediately from the corresponding result for differential operators and the fact that $q_E$ and $q_F^{-1}$ are bounded in $C^l$ norm. □

#### A.4.1.1 Non-local perturbations of Laplace-type operators.

Now suppose $X$ is Riemannian, $E$ and $F$ are Hermitian, and $D : C^\infty(E) \to C^\infty(F)$ is an operator of Dirac type with formal adjoint $D^*$, so that $D^*D$ is

Laplace-type, $p$-elliptic, and formally self-adjoint. Suppose $\mathfrak{B}$ is a boundary condition that makes $(D^*D, \mathfrak{B})$ $p$-elliptic and formally self-adjoint. In addition, suppose that $G: C^\infty(E) \to C^\infty(E)$ is a formally self-adjoint non-local differential operator of order at most 1 relative to an open subset $U = U_1 \cup U_2$. (As usual, we say that $G$ is formally self-adjoint if $(Gf, g) = (f, Gg)$ for all sections $f$, $g$ of $E$ supported on the interior of $X$.) For convenience, we assume that the support of $G$ is equal to its non-local support. In particular, its support is contained in $U$. Let $\Delta$ be the non-local differential operator

$$\Delta := D^*D + G$$

with boundary condition $\mathfrak{B}$. Let $\tilde{\Delta} := q_E \Delta q_E^{-1}$. Thus $\tilde{\Delta} = \widetilde{D^*D} + \tilde{G}$ on $\tilde{U}$, where $\widetilde{D^*D} := q_E D^*D q_E^{-1}$ and $\tilde{G}$ is the differential operator intertwining with $G$. $\widetilde{D^*D}$ acts diagonally with respect to the decomposition $\tilde{E} \cong E_1 \oplus E_2$; its leading symbol is therefore the direct sum of the pullbacks by $q_1$ and $q_2$ of the symbol of $D^*D$. Thus the principal symbol of $\tilde{\Delta}$ is "block-scalar". This implies $\tilde{\Delta}$ is a second order $p$-elliptic differential operator. In order to later apply the theory of $p$-elliptic operators on a manifold without boundary, we suppose that $\tilde{U}$ is a properly embedded open subset of a manifold without boundary, and that $\tilde{E}$ and $\tilde{\Delta}$ extend to this manifold in such a way that $\tilde{\Delta}$ is $p$-elliptic. (Alternatively, we could suppose $\tilde{U}$ is the interior of a manifold with boundary and impose Dirichlet boundary conditions.)

Both $D^*D$ and $G$ are formally self-adjoint; therefore so is $\Delta$. Furthermore, since $\Delta = D^*D$ in a neighborhood of the boundary, the usual notion of the adjoint boundary condition, defined by integration by parts in a collar neighborhood of the boundary, applies also to $(\Delta, \mathfrak{B})$. Since $(D^*D, \mathfrak{B})$ is formally self-adjoint, $(\Delta, \mathfrak{B})$ is also.

Garding's inequality for elliptic boundary value problems generalizes to this situation.

**Proposition A.4.2.** *For any section $s$ in $H^2(E)$, we have*

$$\|s\|_{H^2} \leq C(\|\Delta s\|_{H^0} + \|\mathfrak{B}s\|_{H^{1/2}(W)} + \|s\|_{H^0}$$

*Proof.* This follows from Garding's inequality for elliptic boundary value problems, Theorem A.3.3. We have

$$\begin{aligned} \|s\|_{H^2} &\leq C(\|D^*Ds\|_{H^0} + \|\mathfrak{B}s\|_{H^{1/2}(W)} + \|s\|_{H^0} \\ &\leq C(\|\Delta s\|_{H^0} + \|Gs\|_{H^0} + \|\mathfrak{B}s\|_{H^{1/2}(W)} + \|s\|_{H^0} \end{aligned}$$

The order of $G$ is at most 1, thus

$$\|Gs\|_{H^0} \leq C\|s\|_{H^1} \leq \epsilon\|s\|_{H^2} + C_\epsilon\|s\|_{H^0}$$

In the second bound $\epsilon$ is any positive number and $C_\epsilon$ depends only on $\epsilon$. (This is the Peter-Paul inequality.) Thus

$$\|s\|_{H^2} \leq \frac{1}{2}\|s\|_{H^2} + C(\|\Delta s\|_{H^0} + \|\mathfrak{B}s\|_{H^{1/2}(W)} + (C_\epsilon + 1)\|s\|_{H^0})$$

where we choose $\epsilon = 1/2C$. The proposition follows. □

**Proposition A.4.3.** *The operator $\Delta$, considered as an unbounded operator in $H^0(E)$ with domain all smooth sections satisfying $\mathfrak{B}$, is closable. The closure*

*$\Delta_{\mathfrak{B}}$ is self-adjoint, with domain all $H^2$ sections satisfying $\mathfrak{B}$ (considered as a subspace of $H^0(E)$.)*

*Proof.* Suppose $s$ is in the closure of the graph of $(\Delta, \mathfrak{B})$. That is, there is a sequence of smooth sections $s_i$ satisfying $\mathfrak{B}$ converging to $s$, and $\Delta s_i$ converges. Garding's inequality shows that $s$ is in $H^2(E)$ and $s_i$ converges to $s$ in $H^2$. Since $\Delta$ is continuous from $H^2$ to $H^0$,

$$\Delta_{\mathfrak{B}} s := \lim_{i \to \infty} \Delta s_i$$

is well-defined, i.e, $(\Delta, \mathfrak{B})$ is closable. Since $\mathfrak{B}$ is continuous from $H^2$ to $H^0$, $\mathfrak{B}s = 0$. Thus $\operatorname{Dom} \Delta_{\mathfrak{B}} = \{s \in H^2 \mid \mathfrak{B}s = 0\}$.

To see that $\Delta_{\mathfrak{B}}$ is self-adjoint, we apply the Kato-Rellich Theorem. Since $G$ is order 1, it is defined on $\operatorname{Dom} \Delta_{\mathfrak{B}}$, and considering it to be a symmetric unbounded operator with this domain, we have

$$\|Gs\| \leq C\|s\|_{H^2} \leq C(\|D^*Ds\| + \|s\|)$$

for all $s$ in $\operatorname{Dom} \Delta_{\mathfrak{B}}$. That is, $G$ is $D^*D$-bounded. Since the closure of $(D^*D, \mathfrak{B})$ is self-adjoint with domain equal to $\operatorname{Dom} \Delta_{\mathfrak{B}}$, the Kato-Rellich Theorem implies $\Delta_{\mathfrak{B}} = D^*D + G$ is self-adjoint. □

We will construct a heat kernel for $(\Delta, \mathfrak{B})$ by Duhamel's principle. This is the same procedure used in, for example, [8]. The first (and usually hardest) step in this procedure is to construct local heat kernels. The local heat kernels are then patched together to construct an approximate heat kernel. The heat kernel is then given by *Levi's sum*, a convergent series of corrections to the approximate heat kernel. In this section we begin the construction of a heat kernel for $(\Delta, \mathfrak{B})$ by constructing a system of local heat kernels. Where the operator is differential we can appeal to the standard theory, as outlined above. Where the operator is non-local, we constuct a heat kernel for the associated differential operator and pull back the resulting operator. The pulled-back operator is also given by a smooth kernel. In the next section, we review the construction of the heat kernel from an approximate heat kernel.

The notion of a sheltering function will be useful in the following patching constructions.

**Definition A.4.2.** Suppose $U$ is an open subset of a manifold. We say a function $\phi : U \to [0, 1]$ is a sheltering function for a function $\psi$ if $\phi\psi = \psi$ and $\operatorname{supp} d\phi$ and $\operatorname{supp} \psi$ are disjoint.

By Urysohn's Lemma, if $\operatorname{supp} f \subset U$ we can always find a sheltering function for $f$ supported on $U$.

**Definition A.4.3.** If $\phi$ shelters $\psi$, we say that a kernel $e_t$, with corresponding operator $E_t$, is a local heat kernel for $\Delta$ relative to $\phi$, $\psi$ if

(i) For each $T > 0$, $l \geq 0$, and compact subset $K$ of the interior, $E_t\psi$ is a uniformly bounded family of operators from $C_0^{l+k}(K)$ to $C^l(X)$ for all $k$ large enough, and

$$\lim_{t \to 0^+} \|E_t\psi g - \psi g\|_{C^l} = 0$$

for all smooth sections $g$ supported on $K$.

(ii) The associated remainder kernel

$$r(t;x,x') := (\partial_t + \Delta_x)\phi(x)e(t;x,x')\psi(x')$$

is given by

$$[\Delta, \phi_\alpha](x)e_\alpha(t,x,x')\psi_\alpha(x')$$

and bounded for $0 < t < T_0$ by

$$\|r(t)\|_{C^l} \leq C \exp\{-c/t\} \tag{A.4.1}$$

the constants depending on $l$.

We will now show that $\Delta := D^*D + G$ has a system of local heat kernels. Away from the support of the non-local part of $\Delta$, we will use the heat kernel of $D^*D$ on $X$ with boundary condition $\mathfrak{B}$. Denote this heat kernel by $e_0(t)$. Near the non-local support of $\Delta$, we will use the pullback of the heat kernel of the associated differential operator $\tilde{\Delta}$. This is a second-order self-adjoint $p$-elliptic differential operator on a manifold without boundary that equals $q_E\Delta q_E^{-1}$ on $\tilde{U}$. Let $\tilde{e}(t)$ denote the heat kernel of this operator, with corresponding operator $\tilde{E}(t)$, and let $e_1(t)$ be the kernel of $q_E^{-1}\tilde{E}(t)q_E$.

We have assumed that $\operatorname{supp} G = K \subset U$, where $U = U_1 \sqcup U_2$ is an open set separated from the boundary. Let $U_0 = X - K$. Let $\tilde{\psi} : \tilde{U} \to [0,1]$ be a smooth function of compact support that is identically 1 on $\tilde{K} := q_1^{-1}(K) \cup q_2^{-1}(K)$, and let $\tilde{\phi}$ be a sheltering function for $\tilde{\psi}$. Then we obtain smooth functions $\psi_i$ by extending $(q_i^{-1})^*\tilde{\psi}$ by zero to $X$. Define $\phi_1$, and $\phi_2$ similarly from $\tilde{\phi}$. Then $\phi_1$ shelters $\psi_1$ and $\phi_2$ shelters $\psi_2$. Let $\psi_0 = 1 - (\psi_1 + \psi_2)$; $\psi_0$ is supported on $U_0$, and $\{\psi_i\}$ is a partition of unity subordinate to the cover $\{U_i\}$. Let $\phi_0$ be a sheltering function for $\psi_0$ supported on $U_0$.

**Theorem A.4.4.** *The kernel $e_0(t)$ is a local heat kernel for $\Delta$ relative to $\phi_0$, $\psi_0$. The kernel $e_1(t)$ is smooth and a local heat kernel for $\Delta$ relative to $(\phi_1 + \phi_2)$, $(\psi_1 + \psi_2)$.*

*Proof.* The result for $e_0(t)$ is obvious, since $\Delta\phi_0 = D^*D\phi_0$ and a heat kernel for $(D^*D, \mathfrak{B})$ is a local heat kernel for $(D^*D, \mathfrak{B})$ relative to an pair $\phi_0$, $\psi_0$.

$E_1(t) = q_E^{-1}\tilde{E}(t)q_E$ is the composition of the smoothing operator $\tilde{E}(t)$ and operators that preserve the $C^l$ norms. It is therefore a smoothing operator. The smoothness of the corresponding kernel $e_1(t)$ follows immediately.

Note that our construction of $\psi_1$ and $\psi_2$ makes $(\psi_1 + \psi_2) = q_E^{-1}\tilde{\psi}q_E$ as a multiplication operator; likewise $(\phi_1 + \phi_2) = q_E^{-1}\tilde{\phi}q_E$. Thus

$$\begin{aligned}(\partial_t + \Delta)(\phi_1 + \phi_2)E_1(t)(\psi_1 + \psi_2) &= q_E^{-1}(\partial_t + \tilde{\Delta})\tilde{\phi}\tilde{E}(t)\tilde{\psi}q_E \\ &= q_E^{-1}[\tilde{\Delta}, \tilde{\phi}]\tilde{E}(t)\tilde{\psi}q_E \\ &= [\Delta, (\phi_1 + \phi_2)]E_1(t)(\psi_1 + \psi_2) \\ &= \sum_{i,j=1,2}[\Delta, \phi_i]E_1(t)\psi_j\end{aligned}$$

The "off-diagonal" terms $[\Delta, \phi_i]E_1(t)\psi_j$, $i \neq j$, vanish if $\Delta$ is differential. Let $y = q_i^{-1}(x)$, $y' = q_j^{-1}(x')$ for $x$, $x'$ in the support of $\phi_1 + \phi_2$. The calculation above implies

$$|[\Delta, \phi_i](x)e_1(t; x, x')\psi(x')| \leq C|[\tilde{\Delta}, \tilde{\phi}](y)\tilde{e}(t; y, y')\tilde{\psi}(y')|$$

Since the supports of $[\tilde{\Delta}, \tilde{\phi}]$ and $\tilde{\psi}$ are disjoint, Theorem A.2.5 implies that for all $t$, $0 < t < T < \infty$,

$$|[\Delta, \phi_i](x)e_1(t; x, x')\psi(x')| \leq C \exp\{-c/t\}$$

uniformly in $x$, $x'$. The same theorem shows that a similar estimate holds on the derivatives in $x$, $x'$, and $t$. Thus the $C^l$ norm of the remainder kernel is exponentially decaying as $t \to 0^+$ for each $l \geq 0$.

Let $N$ be a compact subset of the interior of $X$. Then $\tilde{N} := q_i^{-1}(N)$ is a compact subset of $\tilde{U}$. By Theorem A.2.2, for each $T > 0$, $l \geq 0$, $\tilde{E}(t)$ is a uniformly bounded family of operators from $C_0^{l+k}(\tilde{N})$ to $C^l$ for all $k$ large enough, and

$$\lim_{t \to 0^+} \|\tilde{E}(t)f - f\|_{C^l} = 0$$

for all smooth sections $f$ supported on $\tilde{N}$. This holds in particular, if $g$ is in $C_0^{l+k}(N)$ and $f = q_E^{-1}(\psi_1 + \psi_2)g$. Thus

$$\lim_{t \to 0^+} \|E_1(t)(\psi_1 + \psi_2)g - (\psi_1 + \psi_2)g\|_{C^l} = \lim_{t \to 0^+} \|q_E(\tilde{E}(t)f - f)\|_{C^l} = 0$$

since $q_E$ is bounded in $C^l$ norm.

□

Let $E(t)$ be the operator with kernel

$$e(t; x, x') := \phi_0(x)e_0(t; x, x')\psi_0(x') + (\phi_1 + \phi_2)(x)e_1(t; x, x')(\psi_1 + \psi_2)(x')$$

**Theorem A.4.5.** *Suppose $Q$ is a non-local differential operator of order $d$ relative to $U_1$, $U_2$, with non-local support contained in $K$. Then* $\mathrm{Tr}(QE(t))$ *has an asymptotic expansion as $t \to 0^+$ of the form:*

$$\mathrm{Tr}(QE(t)) \sim t^{-(n+d)/2} \sum_{k=0}^{\infty} t^{k/2} a_k(Q, P, \mathfrak{B})$$

*Proof.* Let $p(t; x, x') := Q_x e(t; x, x')$, i.e., $p(t)$ is the kernel of $QE(t)$. We have

$$\begin{aligned} p(t; x, x') = {} & [Q, \phi_0](x) e_0(t, x, x') \psi_0(x') \\ & + [Q, (\phi_1 + \phi_2)](x) e_1(t, x, x')(\psi_1 + \psi_2)(x') \\ & + \phi_0(x) Q_x e_0(t, x, x') \psi_0(x') \\ & + (\phi_1 + \phi_2)(x) Q_x e_1(t, x, x')(\psi_1 + \psi_2)(x') \end{aligned}$$

As usual, to compute the trace of $QE(t)$ we integrate the local trace $\mathrm{tr}\, p(t; x, x)$ over the diagonal. We claim that the two commutator terms above vanish along the diagonal. By construction, all the functions $\phi_\alpha$ are constant on $K$, and by assumption the non-local support of $Q$ is contained in $K$. Thus $[Q, \phi_0]$ and $[Q, (\phi_1 + \phi_2)]$ are purely differential operators. The support of $[Q, \phi_0]$ contained in the support of $d\phi_0$, and is therefore disjoint from the support of $\psi_0$. Thus $[Q, \phi_0](x) e_0(t, x, x') \psi_0(x')$ vanishes along the diagonal. Likewise the other commutator term vanishes along the diagonal.

We have

$$\begin{aligned}\operatorname{Tr}(QE(t)) &= \int_X \operatorname{tr} p(t;x,x)\,\mathrm{dvol}_X \\ &= \int_X \operatorname{tr}\left[\phi_0(x)Q_x e_0(t,x,x')\psi_0(x')\right]_{x=x'}\mathrm{dvol}_X \\ &\quad + \int_X \operatorname{tr}\left[(\phi_1+\phi_2)(x)Q_x e_1(t,x,x')(\psi_1+\psi_2)(x')\right]_{x=x'}\mathrm{dvol}_X\end{aligned}$$

Using the commutativity of the trace, and the fact that $\psi_\alpha(x)\phi_\alpha(x) = \psi_\alpha(x)$, we have

$$\begin{aligned}\operatorname{Tr}(QE(t)) &= \int_X \operatorname{tr}\left[\psi_0(x)Q_x e_0(t,x,x')\right]_{x=x'}\mathrm{dvol}_X \\ &\quad + \int_X \operatorname{tr}\left[(\psi_1+\psi_2)(x)Q_x e_1(t,x,x')\right]_{x=x'}\mathrm{dvol}_X\end{aligned}$$

The first term equals $\operatorname{Tr}(\psi_0 Q E_0(t))$, and by Theorem A.3.8 this has an asymptotic expansion of the required form. Recall that $(\psi_1+\psi_2) = q_E^{-1}\tilde{\psi}q_E$. Since $Q$ is non-local relative to $U_1$, $U_2$, there is a differential operator $\tilde{Q}$ on $\tilde{U}$ such that $(\psi_1+\psi_2)(x)Q_x = q_E^{-1}\tilde{\psi}\tilde{Q}q_E$. Thus the second term equals $\operatorname{Tr}(\tilde{\psi}\tilde{Q}\tilde{E}(t))$. Applying Theorem A.3.8 again, we see that this has a asymptotic expansion of the required form. □

#### A.4.1.2 Patching local heat kernels

We construct the heat kernel of $\Delta$ using the system of local heat kernels constructed in the last section: $\{(e_\alpha,\phi_\alpha,\psi_\alpha)\}_{\alpha\in\mathcal{O}}$ has been given relative to a finite open cover $\mathcal{O}$. That is, $\{\psi_\alpha\}$ is a partition of unity subordinate to $\mathcal{O}$, $\phi_\alpha$ shelters $\psi_\alpha$ and is supported on the same element of the cover, and $e_\alpha$ is a local heat kernel for $\Delta$ relative to $\phi_\alpha$, $\psi_\alpha$. We assume that each function $\psi_\alpha$ in this partition of unity is either identically 1 or identically 0 on each component of the boundary, and that each $\phi_\alpha e_\alpha$ satisfies the boundary condition $\mathfrak{B}$, i.e., the associated operator $\phi_\alpha E_\alpha$ is a map into sections satisfying the boundary condition. Then our approximate heat kernel for $\Delta$ is

$$e(t;x,x') := \sum_{\alpha\in\mathcal{O}} \phi_\alpha(x)e_\alpha(t,x,x')\psi_\alpha(x')$$

Since the boundary condition is linear and we assumed each $\phi_\alpha e_\alpha$ satisfies the boundary condition, $e$ satisfies the boundary condition. Let $r_1(t;x,x')$ be the remainder kernel,

$$r_1(t;x,x') := (\partial_t + \Delta_x)e(t;x,x')$$

If $a_t$ and $b_t$ are two kernels defined for $0 < t$, define

$$(a * b)(t;x,x') := \int_0^t \int_X a(s;x,z)b(t-s;z,x')\,dzds$$

Define $r_j$ for $j > 1$ inductively by

$$r_j := r_1 * r_{j-1}$$

and let

$$r := \sum_{j=1}^{\infty} (-1)^j r_j$$

We will show this is a convergent sum and that

$$h := e + e * r$$

is a heat kernel for $\Delta$.

Before we begin the proof, note that the support in $x$ of $r_1(t;x,x')$ (i.e., the closure of the union of the supports in $x$ over all $x'$) is contained in a compact subset of the interior. Specifically, the support in $x$ is contained in the union of the supports of $d\phi_\alpha$, $\alpha \in \mathcal{O}$. Let $K_{\mathcal{O}}$ denote this compact set. Likewise, the support of each $r_j(t;x,x')$, $j > 1$, is contained in $K_{\mathcal{O}}$, as is the support in $x$ of $r(t;x,x')$. We will need this fact in the proof.

**Lemma A.4.6.** *For each $l > 0$, there are constants $C$, $c$, and $T_0$ such that for all $j$ and $0 < t < T_0$,*

$$\|r_j(t)\|_{C^l} \leq \frac{C(C \operatorname{vol} X)^{j-1} t^{j-1}}{(j-1)!} \exp\{-c/t\} \tag{A.4.2}$$

*The sum defining $r$ converges in $C^l$ for $0 < t < T_0$, and there are constants $C$, $c$ such that for all $0 < t < T_0$,*

$$\|r(t)\|_{C^l} \leq C \exp\{-c/t\} \tag{A.4.3}$$

*Proof.* For $r_1$, this is a consequence of the corresponding estimates for each term in the sum defining $e$ assumed in Definition A.4.3.

Proceeding inductively, we estimate

$$\begin{aligned} \|r_j(t)\|_{C^l} &= \|\int_0^t \int_X r_1(s;x,z) r_{j-1}(t-s;z,x')\, dz ds\|_{C^l} \\ &\leq \int_0^t \int_X C \exp\{-c/s\} \\ &\qquad \cdot \frac{C(C \operatorname{vol} X)^{j-2}(t-s)^{j-2}}{(j-2)!} \exp\{-c/(t-s)\}\, dz ds \end{aligned}$$

Using the elementary inequality $c/t < c/s + c/(t-s)$,

$$\begin{aligned} &\leq C \exp\{-c/t\} (C \operatorname{vol} X)^{j-1} \int_0^t \frac{(t-s)^{j-2}}{(j-2)!} \, ds \\ &= \frac{C(C \operatorname{vol} X)^{j-1} t^{j-1}}{(j-1)!} \exp\{-c/t\} \end{aligned}$$

This is the claimed bound (A.4.2). To show that the sum defining $r_t$ converges,

$$\begin{aligned} \|r(t)\|_{C^l} &\leq \sum_{j=1}^{\infty} \|r_j(t)\|_{C^l} \\ &\leq C \exp\{-c/t\} \sum_{j=1}^{\infty} \frac{(C \operatorname{vol} X)^{j-1} t^{j-1}}{(j-1)!} \\ &= C e^{C(\operatorname{vol} X)t} \exp\{-c/t\} \\ &\leq C e^{C(\operatorname{vol} X)T_0} \exp\{-c/t\} \end{aligned}$$

We obtain (A.4.3). □

**Corollary A.4.7.** *For all* $0 < t \leq T_0$,

$$\|h(t) - e(t)\|_{C^l} \leq C e^{-c/t}$$

*Proof.* We have $\|h(t) - e(t)\|_{C^l} = \|(e * r)(t)\|_{C^l}$. Part of our definition of a local heat kernel $e_\alpha(t)$ is that it defines a uniformly bounded family of operators from $C_0^{l+k}(K)$ to $C^l(X)$ for each compact subset $K$ of the interior and $k$ large enough. As we noted above, the support in $x$ of $r(t; x, x')$ is contained in $K_{\mathcal{O}}$, a compact subset of the interior. Thus $\|(e * r)(t)\|_{C^l} \leq C' \|r(t)\|_{C_0^{l+k}(K_{\mathcal{O}})} \leq C' C e^{-c/t}$. □

**Corollary A.4.8.** *For all $0 < t \leq T_0$, the operators $H_t$ with kernels $h_t$ are a uniformly bounded family of operators from $C_0^{l+k}(K)$ to $C^l(X)$ for each compact subset $K$ of the interior and $k$ large enough.*

*Proof.* We have for all $0 < t \leq T_0$,

$$\|h(t) - e(t)\|_{C^l} \leq Ce^{-c/t}$$

Using the estimate above and the Banach-Steinhaus Theorem, it is easy to check that the family of operators with kernel $h(t) - e(t)$ is uniformly bounded from $C^0(X)$ to $C^l(X)$, hence from $C_0^{l+k}(K)$ to $C^l(X)$. The kernel $e(t)$ defines a uniformly bounded family of operators in the $C_0^{l+k}(K)$-norm for each compact subset $K$ of the interior and $k$ large enough. It follows that $H_t$ is uniformly bounded from $C_0^{l+k}(K)$ to $C^l(X)$. $\square$

**Theorem A.4.9.** *The kernel $h$ is a heat kernel for $(\frac{\partial}{\partial t} + \Delta, \mathfrak{B})$ for $0 < t < T_0$.*

*Proof.* As we noted above, the boundary condition is linear and we assumed each $\phi_\alpha e_\alpha$ satisfies the boundary condition (possibly by vanishing at the boundary.) Thus $e$ satisfies the boundary condition. It follows that $e * r$ also satisfies the boundary condition.

Furthermore, the corresponding operator $H(t)$ approaches the identity as $t \to 0^+$ because $H(t) - E(t)$ approaches 0 and by definition

$$E(t) = \sum_{\alpha \in \mathcal{O}} \phi_\alpha E_\alpha(t) \psi_\alpha$$

approaches $\sum_\alpha \phi_\alpha \psi_\alpha = 1$.

To see that $h$ solves the heat equation, we calculate formally,

$$\begin{aligned}(\frac{\partial}{\partial t} + \Delta)[e * r] &= r + r_1 * r \\ &= r + -(r + r_1) = -r_1\end{aligned}$$

and

$$\begin{aligned}(\frac{\partial}{\partial t} + \Delta)h_t &= r_1 + (\frac{\partial}{\partial t} + \Delta)[e * r] \\ &= r_1 - r_1 \\ &= 0\end{aligned}$$

The difficulty in the first step of this formal calculation is that $e$ develops a singularity as $t \to 0^+$. However, for $0 < t < T_0$, the integrand is at least $C^2$. Thus if $T_0 > t > a > 0$, and $a > \epsilon > 0$,

$$\begin{aligned}&(\frac{\partial}{\partial t} + \Delta)\int_0^{t-\epsilon}\int_X e(t-s; x, z)r(s; z, x')\,dzds \\ &\quad= \int_X e(\epsilon; x, z)r(t-\epsilon; z, x')\,dz + \int_0^{t-\epsilon}\int_X r_1(t-s; x, z)r(s; z, x')\,dzds\end{aligned}$$

From the definition of $e$, and the fact that the support of $r(t; z, x')$ in $z$ is contained in a compact subset of the interior of $X$ independent of $x'$, the first integral converges uniformly to $r(t; x, x')$. It follows from our estimates in Lemma A.4.6 that the second converges uniformly to $(r_1 * r)(t; x, x')$. Thus we conclude

$$\begin{aligned}(\frac{\partial}{\partial t} + \Delta)e * r(t; x, x') &= \lim_{\epsilon \to 0^+}(\frac{\partial}{\partial t} + \Delta)\int_0^{t-\epsilon}\int_X e(t-s; x, z)r(s; z, x')\,dzds \\ &= r(t; x, x') + (r_1 * r)(t; x, x')\end{aligned}$$

The second step in the formal calculation is justified since the sum defining $r$ converges absolutely by Lemma A.4.6. Thus we may exchange summation and convolution in $r_1 * r = r_1 * \sum_{j=1}^{\infty}(-1)^j r_j$ to conclude $r_1 * r = -r_1 - r$. $\square$

Now that we have a $C^l$ heat kernel defined up to some finite positive time $T_0$, we can easily produce a $C^l$ heat kernels defined on $0 < t < \infty$ as follows. Let $h^1 := h$ be the heat kernel constructed above on $0 < t < T_0$, and take $T_0 > T > 0$ somewhat smaller than $T$ so that the kernel is defined up to $T$.

**Proposition A.4.10.** *For each integer $N \geq 2$ there is a $C^l$ heat kernel $h^N$ defined on $0 < t \leq NT$ extending $h^{N-1}$.*

*Proof.* If $N = 2$, we define $h^N(t)$ to be the kernel of $H^1(t)$ if $0 < t \leq T$, and if $T < t \leq 2T$, we take the kernel to be the kernel of $H^1(t - T_0)H^1(T_0)$. Clearly $h^2$ is $C^l$ away from $t = T_0$. The kernel is $C^l$ at $t = T_0$ since $H^1(\epsilon)H^1(T_0)$ converges in $C^l$ norm to $H^1(T_0)$ as $\epsilon \to 0^+$. Clearly $(\frac{\partial}{\partial t} + \Delta)H^2(t) = 0$. Therefore we can conclude that $h^2$ is at least $C^{\lfloor l/2 \rfloor}$ in $t$. It is also clear that it satisfies the boundary condition. Since this kernel extends the kernel of $H^1$, $H^2$ approaches the identity in $C^l$-norm as $t \to 0^+$.

We define the kernel of $H^N$ for $N > 2$ similarly assuming that $H^{N-1}$, has been constructed. In the same way as for $H^2$, it can be shown that the kernel $h^N$ is a heat kernel for $\Delta$ extending $h^{N-1}$. $\square$

#### A.4.1.3 Properties of the heat kernel for $(\Delta, \mathfrak{B})$

In view of Proposition A.4.10, for each $l > 1$ we have a $C^l$ heat kernel for $(\Delta, \mathfrak{B})$ defined for $0 < t < \infty$. Following [5] closely, we show that a (sufficiently smooth) heat kernel is in fact unique. We conclude that there is a unique smooth heat kernel for $(\Delta, \mathfrak{B})$. It follows easily from this uniqueness theorem and the fact that $(\Delta, \mathfrak{B})$ is formally self-adjoint that as usual the heat operator is a self-adjoint semigroup of smoothing operators.

**Proposition A.4.11.** *Suppose $p(t; x, x')$ is the kernel for a family of operators $P(t) : H^0(E) \to H^0(E)$, $t > 0$, and that: $p$ is continuous, $C^2$ in $(x, x')$, and $C^1$ in $t$; satisfies the heat equation for $\Delta$ and the boundary condition $\mathfrak{B}$; for any section $s$, $\lim_{t\to 0^+} P(t)s = 0$ in $H^0$. Then $p$ is zero.*

*Proof.* We have shown that there exists a heat kernel $h(t; x, x')$ for $(\Delta, \mathfrak{B})$ that is continuous, $C^2$ in $(x, x')$, and $C^1$ in $t$; let $H(t)$ denote the corresponding family of operators, and let

$$f(\theta) := (P(t-\theta)s_1, H(\theta)s_2)_{H^0}$$

for any two fixed sections $s_1$ and $s_2$. Then $f$ is $C^1$ on $(0, t)$, continuous on $[0, t]$, and

$$f(0) = (P(t)s_1, s_2), \quad f(t) = 0$$

$$\frac{\partial}{\partial\theta} f(\theta) = (\Delta P(t-\theta)s_1, H(\theta)s_2) - (P(t-\theta)s_1, \Delta H(\theta)s_2) = 0$$

The derivative vanishes because $P(t)s_1$ and $H(t)s_2$ are $C^2$ and satisfy $\mathfrak{B}$, and $(\Delta, \mathfrak{B})$ is formally self-adjoint. Thus $f(0) = f(t)$, i.e.,

$$(P(t)s_1, s_2) = 0$$

Since $p$ is continuous, this implies that it is identically zero. $\square$

In view of the last proposition, there is a unique smooth heat kernel $H(t)$ for $(\Delta, \mathfrak{B})$.

**Proposition A.4.12.** *(The semigroup property) For all $\infty > t_1,\ t_2 \geq 0$, we have*

$$H(t_1 + t_2) = H(t_1)H(t_2)$$

*Proof.* The theorem is trivial if either $t_1$ or $t_2 = 0$. Therefore we assume $t_1$ and $t_2$ are positive.

Let $f$ be an $H^0(E)$ section. Then

$$s(t) := [H(t + t_2) - H(t)H(t_2)]f$$

satisfies the heat equation for $(\Delta, \mathfrak{B})$ and $\lim_{t\to 0^+} s(t) = 0$. Thus $s(t)$ is identically zero. Since $f$ was arbitrary, $H(t + t_2) - H(t)H(t_2)$ is zero for all $t > 0$ by Proposition A.4.11. In particular, $H(t_1 + t_2) = H(t_1)H(t_2)$. $\square$

**Proposition A.4.13.** *The heat kernel of $(\Delta, \mathfrak{B})$ is self-adjoint.*

*Proof.* For any two sections $s_1$, $s_2$ and $t > 0$, let $f(\theta) := (H(t-\theta)s_1, H(\theta)s_2)_{H^0}$. The function $f$ is $C^1$ on $(0, t)$. We have

$$\begin{aligned}\frac{\partial}{\partial\theta}f(\theta) &= (\Delta H(t-\theta)s_1, H(\theta)s_2) - (H(t-\theta)s_1, \Delta H(\theta)s_2) \\ &= 0\end{aligned}$$

because $(\Delta, \mathfrak{B})$ is formally self-adjoint, and $H(t)$ is smoothing and satisfies $\mathfrak{B}$. Thus $f$ is constant on $(0, t)$ and we have

$$(H(t)s_1, s_2) = \lim_{\theta\to 0^+} f(\theta) = \lim_{\theta\to t^-} f(\theta) = (s_1, H(t)s_2)$$

□

**Proposition A.4.14.** *For all $t > 0$, $H(t)$ is positive, trace-class, and smoothing. There is an orthonormal basis of $H^0(E)$ diagonalizing the family $H(t)$ consisting of smooth sections that satisfy the boundary condition $\mathfrak{B}$.*

*Proof.* Since the heat kernel is smooth, and the underlying manifold is compact, $H(t)$ is Hilbert-Schmidt. Using the semigroup property, we conclude $H(t) = H(t/2)H(t/2)$ is in fact trace-class. The semigroup property and the fact that $H(t)$ is self-adjoint shows that $H(t)$ is positive. Since $H(t)$ is trace-class, positive, and self-adjoint, we conclude that: the spectrum is real, non-negative, and bounded; the non-zero part of the spectrum is discrete, consisting of eigenvalues of finite multiplicity; there is an orthonormal basis of $H^0(E)$ simultaneously diagonalizing the family $H(t)$. If $s$ is a unit-length eigensection of $H(t)$, $s = H(t)s/\|H(t)s\|$ in $H^0$ for all $t \geq 0$. Thus $s$ is smooth (i.e., has a smooth representative) and satisfies the boundary condition $\mathfrak{B}$. □

**Corollary A.4.15.** *There is an orthonormal basis of $H^0(E)$ diagonalizing $\Delta$ consisting of smooth sections that satisfy $\mathfrak{B}$. The eigenvalues are real and positive, and the corresponding eigenspaces are finite-dimensional.*

*Proof.* Fixing an eigensection $s$ of unit length, define $f(t) := (H(t)s, s)$; clearly $H(t)s = f(t)s$. It follows that $f$ is $C^1$, and $f(0) = 1$. Differentiating $f$, and using the semigroup property again, we have $\frac{\partial}{\partial t} f(t) = -\lambda f(t)$, where $-\lambda := \frac{\partial}{\partial t} f(0)$. Thus $f(t) = \exp\{-\lambda t\}$. Since $\frac{\partial}{\partial t} H(t)s = -\lambda H(t)s = -\Delta H(t)s$, $s$ is an eigensection of $(\Delta, \mathfrak{B})$ of eigenvalue $\lambda$. □

**Theorem A.4.16.** *Suppose $Q$ is a non-local differential operator of order $d$ relative to $U_1$, $U_2$, with non-local support contained in $K$. Then* $\mathrm{Tr}(QH(t))$ *has an asymptotic expansion as $t \to 0^+$ of the form:*

$$\mathrm{Tr}(QH(t)) \sim t^{-(n+d)/2} \sum_{k=0}^{\infty} t^{k/2} a_k(Q, P, \mathfrak{B})$$

*Proof.* Let $E(t)$ be the approximate heat kernel we constructed in section A.4.1.1, and let $e(t)$ be its kernel.By Theorem A.4.5, $\mathrm{Tr}(QE(t))$ has an asymptotic expansion of the claimed form. By Corollary A.4.7,

$$\|h(t) - e(t)\|_{C^l} \leq Ce^{-c/t}$$

for all $t$ small enough. We showed that non-local differential operators of order $d$ are bounded as maps from $C^{l+d}$ to $C^l$. It follows that the difference between the kernels of $QH(t)$ and $QE(t)$ decays exponentially at small times, and therefore $\mathrm{Tr}(QH(t))$ and $\mathrm{Tr}(QE(t))$ have the same asymptotic expansion.

The existence and form of this asymptotic expansion then follows from Theorem A.4.5. □

## A.5 Heat kernels depending on a parameter

In this section, we show that the heat kernel depends smoothly on the operator $P$ and the boundary condition $\mathfrak{B}$. As in Appendix A, let $\hat{X}$ be a compact manifold without boundary, containing $X$ as an embedded manifold of the same dimension, so that $\partial X$ is an embedded hypersurface. Let $\Delta_z$ be a family of non-local perturbations of a Laplace-type operator on $X$ extending to $\hat{X}$ as a Laplace-type operator on the complement of $X$. Let $F_z(t)$ be the family of heat kernels on $\hat{X}$ for $\Delta_z$. By essentially the same methods as the proof of Theorem 2.48 in [5], we have Duhamel's principle for $\Delta_z$:

**Theorem A.5.1.** *The heat kernel $F_z(t)$ for the family $\Delta_z$ is given by a kernel $f_z(t;x,x')$, smooth in $(z,t,x,x')$ for all $t>0$. Its derivative with respect to $z$ is given by*

$$\frac{\partial}{\partial z} f_z(t;x,x') = -\int_0^t \int_{\hat{X}} f_z(t-s;x,y)(\frac{\partial}{\partial z}\Delta_z) f_z(s;y,x') \,\mathrm{dvol}_{\hat{X}}(y) ds$$

Now suppose $(\Delta_z, \mathfrak{B}_z)$, $z \in \mathbb{R}$ be a smooth family of self-adjoint second-order $p$-elliptic boundary value problems on $X$. Let $H_z(t)$ be the family of associated heat operators, and $h_z(t;x,x')$ the family of heat kernels.

**Proposition A.5.2.** *The kernel $h_z(t;x,x')$ of $H$ is smooth in $(t;x,x';z)$.*

*Proof.* Recall that Theorem A.3.7 asserts that there is an explicit formula for the compensating factor $C_z(t)$, defined by

$$H_z(t) = F_z(t) - C_z(t)$$

and that in particular the kernel of $C_z(t)$ depends smoothly on $F_z(t)$ and the symbols of $P_z$ and $\mathfrak{B}_z$ near the boundary. The same formula for the compensating factor applies to a family of Laplace-type non-local differential operators $\Delta_z$; since the non-local perturbation is supported in the interior, the symbol is well-defined in a boundary chart. Combining this result with Theorem A.5.1, we conclude that $H_z(t)$ is a smooth family of smoothing operators for all $t > 0$. □

The heat kernel depends on the boundary condition. Thus when the boundary conditions vary, we cannot expect a formula for the derivative of the heat kernel like the formula given in Theorem A.5.1. Nevertheless, we have Duhamel's principle when the boundary conditions are fixed.

**Theorem A.5.3.** *Suppose that* $(\Delta_z, \mathfrak{B})$*, is a smooth family of self-adjoint second-order* p-*elliptic boundary value problems on* $X$*. Let* $H_z(t)$ *be the family of associated heat operators, and* $h_z(t; x, x')$ *the family of heat kernels. Then*

$$\frac{\partial}{\partial z} h_z(t; x, x') = -\int_0^t \int_{\hat{X}} h_z(t-s; x, y)[\frac{\partial}{\partial z}, \Delta_z] h_z(s; y, x') \text{ dvol}_{\hat{X}}(y) ds$$

*Proof.* Given a smooth section $\phi$ satisfying the boundary condition $\mathfrak{B}$, let $f(t; z) := [\frac{\partial}{\partial z}, \Delta_z] \exp(-t\Delta_z)\phi$. Then by Prop. A.5.2, $f(t; z)$ is smooth. Furthermore, $\Delta_z \exp(-t\Delta_z)\phi$ satisfies $\mathfrak{B}$, and since the boundary condition is independent of $z$, so does $f(t;z)$.

Applying Prop. 9.10 of Taylor [27], the unique solution to

$$(\frac{\partial}{\partial t} + \Delta_z)u = -f(t;z), \quad \lim_{t\to 0^+} u = 0$$

is

$$\begin{aligned} u(z;t) &:= -\int_0^t \exp(-(t-s)\Delta_z) f(s;z)\, ds \\ &= -\int_0^t \exp(-(t-s)\Delta_z)[\frac{\partial}{\partial z}, \Delta_z] \exp(-s\Delta_z)\phi\, ds \end{aligned}$$

On the other hand,

$$(\frac{\partial}{\partial t} + \Delta_z)\frac{\partial}{\partial z}\exp(-t\Delta_z)\phi = -f(t;z)$$

and

$$\lim_{t\to 0^+} \frac{\partial}{\partial z}\exp(-t\Delta_z)\phi = \frac{\partial}{\partial z}\phi = 0$$

Thus

$$\frac{\partial}{\partial z}\exp(-t\Delta_z)\phi = u(z;t)$$

This gives us equality of the operators on the domain of $\Delta_z$. Both sides have smooth kernels, so we get this equality on the level of kernels as well. □

## A.6 Spectral projections

Following [5], we show that the spectral projections associated to $(\Delta, \mathfrak{B})$are smooth families of smoothing operators.

Let $H_z := H_z(1)$, the heat operator at $t = 1$. By the results of the last section, this is a smooth family of self-adjoint smoothing operators. For $\mu > 0$ not in the spectrum of $H(z)$, let

$$\Pi_\mu(z) := \frac{1}{2\pi i} \int_{\mathcal{C}} (\lambda - H_z)^{-1} \, d\lambda$$

Here $\mathcal{C}$ is a positively-oriented circle in the complex plane with center on the real axis and crossing the real axis at $e^{-\mu}$ and $1 + \varepsilon$, $\varepsilon > 0$. The spectral theorem for self-adjoint compact operators implies that $\Pi_\mu(z)$ is orthogonal projection onto the subspace of $H^0(E)$ spanned by eigensections of $H_z$ with eigenvalues in the interval $(e^{-\mu}, 1 + \varepsilon)$. This is a finite-dimensional subspace spanned by smooth sections. Thus each $\Pi_\mu(z)$ is a smoothing operator.

Fix $z_0$ and $\mu > 0$ not in the spectrum of $H_{z_0}$. Since the family $H(z)$ is continuous and invertibility is an open condition, there is an open interval around $z_0$ where $\Pi_\mu(z)$ is defined. We will show that in fact $\Pi_\mu(z)$ is a smooth family of smoothing operators over this interval.

**Lemma A.6.1.** *If $Q$ is a smoothing operator, then $Q(\lambda - H_z)^{-1}$ and $(\lambda - H_z)^{-1}Q$ are smoothing operators.*

*Proof.* Note that $\lambda$ is never zero on the contour $\mathcal{C}$. We have

$$\lambda(\lambda - H_z)^{-1} = 1 + (\lambda^{-1} H_z) \left(1 - (\lambda^{-1} H_z)\right)^{-1}$$

so that

$$\begin{aligned} Q(\lambda - H_z)^{-1} &= \lambda^{-1} Q + \lambda^{-2} Q H_z \left(1 - (\lambda^{-1} H_z)\right)^{-1} \\ &= \lambda^{-1} Q + \lambda^{-2} Q \left(1 - (\lambda^{-1} H_z)\right)^{-1} H_z \end{aligned}$$

and

$$(\lambda - H_z)^{-1}Q = \lambda^{-1}Q + \lambda^{-2}H_z \left(1 - (\lambda^{-1}H_z)\right)^{-1} Q$$

Since a sum of smoothing operators is smoothing and $H_z$ is smoothing, it suffices to show that $Q_1(1 - (\lambda^{-1}H_z))^{-1}Q_2$ is smoothing if $Q_1$ and $Q_2$ are. Since $(1 - (\lambda^{-1}H_z))^{-1}$ is bounded on $H^0$ this is clear. □

**Theorem A.6.2.** *On the open intervals where it is defined, $\Pi_\mu(z)$ is a smooth family of smoothing operators.*

*Proof.* We have

$$\frac{\partial}{\partial z}(\lambda - H_z)^{-1} = (\lambda - H_z)^{-1}(\frac{\partial}{\partial z}H_z)(\lambda - H_z)^{-1}$$

Applying Lemma A.6.1, we see that $\frac{\partial}{\partial z}(\lambda - H_z)^{-1}$ is a family of smoothing operators, since $\frac{\partial}{\partial z}H_z$ is a smooth family of smoothing operators. Similarly, higher derivatives in $z$ of $(\lambda - H_z)^{-1}$ are families of smoothing operators. $\Pi_\mu$ is a family of smoothing operators. Furthermore, we see that

$$\frac{\partial}{\partial z}\Pi_\mu(z) = \frac{1}{2\pi i}\int_{\mathcal{C}}(\lambda - H_z)^{-1}\frac{\partial}{\partial z}H_z(\lambda - H_z)^{-1}d\lambda$$

is a family of smoothing operators. Similarly, any higher derivative in $z$ is a family of smoothing operators. Thus $\Pi_\mu$ is a smooth family of smoothing operators. □

# Appendix B

# Heat kernels for superconnections

Recall the two families of superconnections considered in Chapter 4,

$$\mathbb{A}_{1,u} = \nabla^d + \sqrt{u}\mathsf{D}_r, \qquad \mathbb{A}_{2,u} = [\nabla + u(\nabla^d - \nabla)] + \sqrt{t}\mathsf{D}_r$$

The corresponding heat operators are defined using the formal Volterra series for a superconnection $\mathbb{A} = \tilde{\nabla} + L$:

$$\begin{aligned}\exp(-\mathbb{A}^2) := \exp(-L^2)& \\ &- \int_{\Sigma^1} \exp(-\sigma_0 L^2)(\tilde{\nabla}^2 + \tilde{\nabla} L) \exp(-\sigma_1 L^2)\, d\sigma \\ &+ \int_{\Sigma^2} \exp(-\sigma_0 L^2)(\tilde{\nabla}^2 + \tilde{\nabla} L) \exp(-\sigma_1 L^2)(\tilde{\nabla}^2 + \tilde{\nabla} L) \exp(-\sigma_2 L^2)\, d\sigma \\ &- \cdots + \cdots\end{aligned}$$

For $\mathbb{A}_{1,u}$ we take $\tilde{\nabla}$ and $L$ to be $\nabla^d$ and $\sqrt{u}\mathsf{D}_r$ respectively, whereas for $\mathbb{A}_{2,u}$ we take them to be $[\nabla + u(\nabla^d - \nabla)]$ and $\sqrt{t}\mathsf{D}_r$. As we have already noted, under Assumption 4.3.1 the Volterra series has only the three terms written in full above and thus the convergence of the Volterra series depends only on the convergence of the individual terms. Under Assumption 4.3.1, we demonstrate this and prove Lemma 4.3.9, a version of Duhamel's principle for superconnections.

## B.1 Convergence of the Volterra series

Recall Assumption 4.3.1:

**Assumption B.1.1.** $\hat{B} = (-\epsilon, \epsilon)^2$, *and we are calculating* $\Omega^{\hat{\mathcal{L}}}(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ *at* $(0,0)$. *The curvature of the family connection* $H$*, defined by*

$$\Omega^H(\xi_1, \xi_2) := [\xi_1, \xi_2]^H - [\xi_1^H, \xi_2^H]$$

*is zero. The family* $\mathsf{D}_r^2$ *has a spectral cut:* $\alpha > 0$ *is not in the spectrum at any point in* $\hat{B}$*. The boundary conditions are preserved by the connection.*

As in Chapter 4, let $\mathsf{P}^{[0,\alpha)}$ denote the spectral projection associated to the spectral cut $\alpha$, $\nabla$ the $L_2$ connection on $\pi_* E$, and

$$\nabla^d := \mathsf{P}^{[0,\alpha)} \nabla \mathsf{P}^{[0,\alpha)} + \mathsf{P}^{(\alpha,\infty)} \nabla \mathsf{P}^{(\alpha,\infty)}$$

**Lemma B.1.1.** $(\nabla^d - \nabla)$ *is a* 1*-form on* $B$ *valued in even smoothing operators.*

*Proof.* We have

$$\nabla^d - \nabla = 2\mathsf{P}^{[0,a)}[\nabla, \mathsf{P}^{[0,a)}] - [\nabla, \mathsf{P}^{[0,a)}]$$

The commutator $[\nabla, \mathsf{P}^{[0,a)}]$ is a 1-form on $B$ valued in even smoothing operators. □

**Lemma B.1.2.** *Suppose* $E \to M$ *is a complex vector bundle with metric and compatible connection. Let* $\mathsf{R}^E$ *denote the curvature tensor of* $E$*. Under Assumption 4.3.1, the curvature* $\nabla$ *of* $\pi_* E$ *is a two-form on* $B$ *valued in vertical zero-order differential operators,*

$$\nabla^2(\xi_1, \xi_2) = \mathsf{R}^E(\xi_1^H, \xi_2^H)$$

*Proof.* In general, we have

$$(\nabla^{\pi_* E})^2(\xi_1, \xi_2) = \nabla^{\pi_* E}_{\Omega^H(\xi_1,\xi_2)} + \mathsf{R}^E(\xi_1^H, \xi_2^H)$$

This is a simple calculation:

$$\begin{aligned}(\nabla^{\pi_* E})^2(\xi_1, \xi_2) &= \nabla^E_{\xi_1^H}\nabla^E_{\xi_2^H} - \nabla^E_{\xi_2^H}\nabla^E_{\xi_1^H} - \nabla^E_{[\xi_1,\xi_2]^H} \\ &= \mathsf{R}^E(\xi_1^H, \xi_2^H) + \nabla^E_{[\xi_1^H,\xi_2^H]} - \nabla^E_{[\xi_1,\xi_2]^H} \\ &= \mathsf{R}^E(\xi_1^H, \xi_2^H) + \nabla^E_{\Omega^H(\xi_1,\xi_2)}\end{aligned}$$

Under Assumption 4.3.1, $\Omega^H(\xi_1, \xi_2)$ vanishes. □

**Lemma B.1.3.** *The adjoint boundary condition of* $\nabla \mathsf{D}_r := [\nabla, \mathsf{D}_r]$ *equals the boundary condition of* $\mathsf{D}_r$.

*Proof.* Suppose $\phi$ and $\psi$ are sections satisfying the boundary condition for $\mathsf{D}_r$, so that

$$\langle \mathsf{D}_r\phi, \psi\rangle - \langle \phi, \mathsf{D}_r\psi\rangle = 0$$

This is a geometric local boundary condition (Defn. 2.2.3). In particular, we have $[\nabla^{L_2(S|\partial^\pi M)}, \mathfrak{B}^\pm] = 0$. Thus $\nabla_\xi\phi$ and $\nabla_\xi\psi$ are sections satisfying the boundary conditions for any vector field $\xi$ on $B$. We have

$$\begin{aligned}0 &= \xi\left(\langle \mathsf{D}_r\phi, \psi\rangle - \langle \phi, \mathsf{D}_r\psi\rangle\right) \\ &= \langle \nabla_\xi \mathsf{D}_r\phi, \psi\rangle + \langle \mathsf{D}_r\phi, \nabla_\xi\psi\rangle - \langle \nabla_\xi\phi, \mathsf{D}_r\psi\rangle - \langle \phi, \nabla_\xi \mathsf{D}_r\psi\rangle \\ &= \langle [\nabla_\xi, \mathsf{D}_r]\phi, \psi\rangle - \langle \phi, [\nabla_\xi, \mathsf{D}_r]\psi\rangle\end{aligned}$$

□

The following lemma is a version of Lemma 9.47 of [5], modified to work in the context of heat operators with boundary conditions.

**Lemma B.1.4.** *Suppose* $Q := \tilde{\nabla}^2 + \tilde{\nabla} L$*, where we take* $\tilde{\nabla} = \nabla + u(\nabla^d - \nabla)$ *and* $L = \sqrt{t}\mathsf{D}_r$*, and suppose* $K$ *is a smoothing operator. Then there exists a constant* $C > 0$ *such that for all* $t$*,* $0 < t \leq T$*, where* $T$ *is a positive real number, and* $k$ *large enough,*

$$\|Q \exp(-t\mathsf{D}_r^2) K\|_l \leq C\|K\|_{l+k} \tag{B.1.1}$$

*On the domain of* $\mathsf{D}_r^2$ *we have*

$$K \exp(-t\mathsf{D}_r^2) Q = (Q^* \exp(-t\mathsf{D}_r^2) K^*)^*$$

*and the estimate,*

$$\|K \exp(-t\mathsf{D}_r^2) Q\|_l \leq C\|K\|_{l+k} \tag{B.1.2}$$

*Proof.* Estimate (B.1.1) follows almost immediately from the uniform boundedness estimate of Coro. A.4.8.

Regarding estimate (B.1.2), consider $Q^* \exp(-t\mathsf{D}_r^2) K^*$, where by $Q^*$ we mean the formal adjoint found by integration by parts. $K^*$ is a smoothing operator, $Q^*$ is a differential operator of order $l$, thus estimate B.1.1 applies to the composition $Q^* \exp(-t\mathsf{D}_r^2) K^*$. Since this composition is a smoothing operator, the estimate also applies to its adjoint $(Q^* \exp(-t\mathsf{D}_r^2) K^*)^*$. We claim that the difference

$$K \exp(-t\mathsf{D}_r^2) Q - (Q^* \exp(-t\mathsf{D}_r^2) K^*)^*$$

vanishes on sections $\phi$ satisfying the boundary conditions of $\mathsf{D}_r^2$ provided that the adjoint boundary condition of $Q$ is the boundary condition of $\mathsf{D}_r^2$. Observe that

$$(Q^* \exp(-t\mathsf{D}_r^2)K^*)^* = K\left(Q^* \exp(-t\mathsf{D}_r^2)\right)^*$$

Thus, it is enough to show that

$$\exp(-t\mathsf{D}_r^2)Q - (Q^* \exp(-t\mathsf{D}_r^2))^*$$

vanishes on sections satisfying the boundary condition of $\mathsf{D}_r^2$. Let $\phi$ be such a section, and let $\psi$ be another smooth section. Then we have

$$\begin{aligned}
&\langle \psi, \exp(-t\mathsf{D}_r^2)Q\phi\rangle - \langle \psi, (Q^* \exp(-t\mathsf{D}_r^2))^*\phi\rangle \\
&= \langle \exp(-t\mathsf{D}_r^2)\psi, Q\phi\rangle - \langle Q^* \exp(-t\mathsf{D}_r^2)\psi, \phi\rangle \\
&= \langle \tilde{\psi}, Q\phi\rangle - \langle Q^*\tilde{\psi}, \phi\rangle
\end{aligned}$$

where $\tilde{\psi} := \exp(-t\mathsf{D}_r^2)\psi$ is a smooth section satisfying the boundary condition of $\mathsf{D}_r^2$. By definition the difference $\langle \tilde{\psi}, Q\phi\rangle - \langle Q^*\tilde{\psi}, \phi\rangle$ vanishes exactly when $\tilde{\psi}$ satisfies the adjoint boundary conditions of $Q$, i.e., when the adjoint boundary condition is equal to the boundary condition of $\mathsf{D}_r^2$. □

**Lemma B.1.5.** *Let* $\tilde{\nabla} = \nabla + u(\nabla^d - \nabla)$*,* $L = \sqrt{t}\mathsf{D}_r$*, and*

$$K = \exp(-\sigma_0 L^2)(\tilde{\nabla}^2 + \tilde{\nabla}L)\exp(-\sigma_1 L^2)\cdots(\tilde{\nabla}^2 + \tilde{\nabla}L)\exp(-\sigma_n L^2)$$

*Then for all* $\sigma_0 > \delta > 0$ *and fixed* $t > 0$*,* $u$ *and* $r$*,* $K$ *is a smoothing operator with a uniformly bounded kernel.*

*Proof.* Suppose that $u = 0$, so that $\tilde{\nabla}^2 + \tilde{\nabla} L = \nabla^2 + \nabla L$. Lemma B.1.2 implies that the $C^l$-norms of the kernel of $\exp(-\sigma_i L^2)\nabla^2 \exp(-\sigma_{i+1} L^2)$ are uniformly bounded by the $C^l$-norms of the kernel of $\exp(-\sigma_i L^2)\exp(-\sigma_{i+1} L^2)$. Thus, for the purposes of computing the kernel norms of $K$, it suffices to estimate the norms of the kernel of

$$\exp(-\sigma_0 L^2)(\nabla L)\exp(-\sigma_1 L^2)\cdots \nabla L \exp(-\sigma_n L^2)$$

Observing that $\exp(-\sigma_n L^2)$ satisfies the boundary conditions, and using Lemma B.1.3, we see that the second part of Lemma B.1.4 applies to this kernel, giving us the required estimate.

Finally, note that Lemma B.1.2 implies that $\tilde{\nabla}^2 + \tilde{\nabla} L$ differs from $\nabla^2 + \nabla L$ by a smoothing operator satisfying the same boundary conditions as $\exp(-\sigma_0 L^2)$. Thus the same proof applies when $u > 0$. □

**Proposition B.1.6.** *For $\mathbb{A}_{1,u}$ and $\mathbb{A}_{2,u}$, each of the terms in the Volterra series exists.*

*Proof.* Since $\mathbb{A}_{2,1} = \mathbb{A}_{1,t}$, it suffices to consider Volterra series for $\mathbb{A}_{2,u}$ with $0 \leq u \leq 1$. The leading term of this series is $\exp(-t\mathsf{D}_r^2)$. There is no question about the existence of this term. The $n$th term of this series following the leading term is

$$(-1)^n \int_{\Sigma^n} \exp(-\sigma_0 L^2)(\tilde{\nabla}^2 + \tilde{\nabla} L)\exp(-\sigma_1 L^2)\cdots(\tilde{\nabla}^2 + \tilde{\nabla} L)\exp(-\sigma_n L^2)\, d\sigma$$

where $\tilde{\nabla} = \nabla + u(\nabla^d - \nabla)$ and $L = \sqrt{t}\mathsf{D}_r$. We proceed as in [5], Theorem 9.48. On the simplex $\Sigma^n$, one of the $\sigma_i$ is greater than $1/(n+1)$. For $\sigma_i > 1/(n+1)$ and fixed $t$, $\exp(-\sigma_i L^2) = \exp(-\sigma_i t\mathsf{D}_r^2)$ has a uniformly bounded kernel. Let

$$K = \exp(-\sigma_i L^2)(\tilde{\nabla}^2 + \tilde{\nabla}L)\exp(-\sigma_{i+1}L^2)\cdots(\tilde{\nabla}^2 + \tilde{\nabla}L)\exp(-\sigma_n L^2)$$

By Lemma B.1.5, $K$ is a smoothing operator with a uniformly bounded kernel. Thus $(\tilde{\nabla}^2 + \tilde{\nabla}L)K$ also has a uniformly bounded kernel. Writing the integrand above as

$$\exp(-\sigma_0 L^2)\cdots\exp(-\sigma_{(i-1)}L^2)(\tilde{\nabla}^2 + \tilde{\nabla}L)K$$

an iterative application of the first part of Lemma B.1.4 shows that the integrand has a uniformly bounded kernel. Thus the integral converges.

□

## B.2 Duhamel's principle for superconnections

**Definition B.2.1.** Suppose $A_u$ is the infinitesimal generator of a semigroup $\exp(-A_u)$ that is a family of smoothing operators smooth in $u$. (That is, the corresponding family of kernels is smooth in $u$.) Then we say $\exp(-A_u)$ satisfies Duhamel's principle if

$$\frac{\partial}{\partial u}\exp(-A_u) = -\int_{\Sigma^1}\exp(-\sigma_0 A_u)\frac{\partial}{\partial u}A_u\exp(-\sigma_1 A_u)\,d\Sigma^1$$

(This integral is to be understood as an integral of the corresponding kernels.)

**Proposition B.2.1.** *Suppose $A_u$ and $B_u$ are $C^1$ operator-valued functions of $u$ and $\exp(-(A_u + B_u))$ is given by a Volterra series with leading term*

$\exp(-A_u)$, *a family of smoothing operators smooth in* $u$. *Furthermore, suppose the Volterra series can be differentiated term-by-term and the leading term satisfies Duhamel's principle. Then* $\exp(-(A_u + B_u))$ *satisfies Duhamel's principle.*

*Proof.* We will show that

$$\begin{aligned}
&\frac{\partial}{\partial u} \exp(-(A_u + qB_u)) \\
&\qquad = -\int_{\Sigma^1} \exp(-\sigma_0(A_u + qB_u) \frac{\partial}{\partial u}(A_u + qB_u) \exp(-\sigma_1(A_u + qB_u))\, d\Sigma^1
\end{aligned}$$

where $q$ is a formal parameter. Of course, setting $q = 1$ gives the proposition.

Consider the Volterra series for $\exp(-t(A_u + qB_u))$,

$$\exp(-t(A_u + qB_u)) := \sum_{k=0}^{\infty} (-1)^k q^k I_k(t)$$

where

$$\begin{aligned}
I_0(t) &:= \exp(-tA_u) \\
I_k(t) &:= t^k \int_{\Sigma^k} \exp(-\sigma_0 t A_u) B_u \exp(-\sigma_1 t A_u) \cdots B_u \exp(-\sigma_k t A_u)\, d\Sigma^k \\
&= \int_{\Sigma^k(t)} \exp(-\sigma_0 A_u) B_u \exp(-\sigma_1 A_u) \cdots B_u \exp(-\sigma_k A_u)\, d\Sigma^k(t)
\end{aligned}$$

and $I_k := I_k(1)$. Differentiating term-by-term, we see that the term of degree $k$ in $q$ of $\frac{\partial}{\partial u} \exp(-(A_u + qB_u))$ is $(-1)^k \frac{\partial}{\partial u} I_k$. The term of degree $k$ in $q$ of

$$-\int_{\Sigma^1} \exp(-\sigma_0(A_u + qB_u) \frac{\partial}{\partial u}(A_u + qB_u) \exp(-\sigma_1(A_u + qB_u))\, d\Sigma^1$$

is

$$(k=0) \qquad -\int_{\Sigma^1} \exp(-\sigma_0 A_u)\frac{\partial}{\partial u}A_u \exp(-\sigma_1 A_u)\, d\Sigma^1$$

$$(k>0) \qquad (-1)^{k-1}\sum_{j=0}^{k}\int_{\Sigma^1} I_{k-j}(\sigma_0)\frac{\partial}{\partial u}A_u I_j(\sigma_1)\, d\Sigma^1 + (-1)^k \sum_{j=0}^{k-1}\int_{\Sigma^1} I_{k-1-j}(\sigma_0)\frac{\partial}{\partial u}B_u I_j(\sigma_1)\, d\Sigma^1$$

The degree 0 terms are equal because $\exp(-A_u)$ satisfies Duhamel's principle. The equality of terms of higher degree follows from the degree 0 case and an application of Fubini's Theorem. □

**Lemma B.2.2.** *Let* $\exp(-\mathbb{A}_u^2)$ *be one of the two heat operators* $\exp(-\mathbb{A}_{1,u}^2)$ *and* $\exp(-\mathbb{A}_{2,u}^2)$ *defined above. The parts* $\exp(-\mathbb{A}_u^2)_{(i)}$ *of degree* $i$ *in the exterior algebra of* $\hat{B}$ *are smoothing operators satisfying the same boundary conditions that* $\exp(-\mathsf{D}^2)$ *satisfies, and we have*

$$\frac{\partial}{\partial u}\exp(-\mathbb{A}_u^2) = -\int_{\Sigma^1} \exp(-\sigma_0 \mathbb{A}_u^2)\left(\frac{\partial}{\partial u}\mathbb{A}_u^2\right)\exp(-\sigma_1 \mathbb{A}_u^2)\, d\sigma$$

*Proof.* As we pointed out above, this series has finitely many non-zero terms; we can therefore differentiate it term-by-term. Furthermore, we have Duhamel's principle for the leading term of the series, $\exp(-L_u^2)$. Applying Prop. B.2.1 we have the statement. □

# Appendix C

# Doubling

## C.1 Gluing manifolds

**Definition C.1.1.** If $A$ and $B$ are two topological spaces, $U \subset A$ and $V \subset B$ are open, and $\phi : U \to V$ is a homeomorphism, then $A \sqcup_\phi B$ is the quotient of $A \sqcup B$ by the relation $a \sim b$ if $\phi(a) = b$.

We give a sufficient condition for $A \sqcup_\phi B$ to be a smooth manifold.

**Proposition C.1.1.** *If $A$ and $B$ are smooth manifolds and $\phi$ is a diffeomorphism, then $A \sqcup_\phi B$ is a smooth manifold if the image of $\bar{U} - U$ is Hausdorff separated from the image of $\bar{V} - V$. That is, there are disjoint open sets $N_U$, $N_V$ containing the images of $\bar{U} - U$ and $\bar{V} - V$ respectively in $A \sqcup_\phi B$.*

*Proof.* To show that $A \sqcup_\phi B$ is a manifold, we have to check that $A \sqcup_\phi B$ is locally Euclidean, second countable, and Hausdorff. Smoothness will then be a direct consequence of the smoothness of $A$ and $B$ and the fact that $\phi$ is smooth.

Let $\pi_Q : A \sqcup B \to A \sqcup_\phi B$ be the quotient map. By the definition of the quotient topology, a subset of $A \sqcup_\phi B$ is open if and only if the preimage of that subset by $\pi_Q$ is open. If $O$ is a open subset of $A$, $\pi_Q^{-1}(\pi_Q(O)) = O \sqcup \phi(O \cap U)$.

Since $U$ is open and $\phi$ is a diffeomorphism, $\pi_Q(O)$ is open. Similarly, $\pi_Q$ is open if $O$ is an open subset of $B$. Thus $\pi_Q$ is an open map. Furthermore, $\pi_Q$ is by definition a homeomorphism when restricted to $A$ or $B$, hence a local homeomorphism. From these facts and the fact that $A$ and $B$ are manifolds it follows easily that $A \sqcup_\phi B$ is locally Euclidean and second-countable.

To see that $A \sqcup_\phi B$ is Hausdorff, suppose that $p$ and $q$ are two points in the space. If either $p$ or $q$ has more than one preimage in $A \sqcup B$, we can suppose without loss of generality that both points have preimages on $A$. Then the fact that $A$ is Hausdorff and $\pi_Q$ is a homeomorphism when restricted to $A$ shows that $p$ and $q$ are Hausdorff separated. Thus we can assume $p$ has a unique preimage $\tilde{p}$ on $A - U$ and $q$ a unique preimage $\tilde{q}$ on $B - V$. If $\tilde{p}$ is not in $\bar{U}$, then $\pi_Q(A - \bar{U})$ is open, contains $p$, and is disjoint from $\pi_Q(B)$, so that $p$ and $q$ are Hausdorff separated. Thus we can assume that $\tilde{p}$ is in $\bar{U} - U$ and $\tilde{q}$ is in $\bar{V} - V$. But then $N_U$ and $N_V$ Hausdorff separate $p$ and $q$ by assumption.

□

Unfortunately, this proposition does not apply directly in the situation where we would most like to use it: when the manifolds $A$ and $B$ have boundary. However, the technique of elongation, which embeds a manifold with boundary in a manifold without boundary, provides a workaround in that case.

The following definition and theorem can be found in [17].

**Definition C.1.2.** If $X$ is a manifold with boundary, a collar on $X$ is an embedding

$$f:\ (-1,0]\times\partial X\to X$$

such that $f(0,x)=x$.

**Theorem C.1.2.** *Every manifold with boundary has a collar.*

**Corollary C.1.3.** *Every manifold with boundary $X$ is embedded in a smooth manifold without boundary.*

*Proof.* Let $f$ be a collar for $X$, choose $0<\epsilon<1/2$, and let $U$ be $f^{-1}[(-\epsilon,0]\times\partial X]$. Note that $\partial U=\partial X$. Let $X':=X-\partial X$, $U'=U-\partial X$, and $\phi:=f^{-1}|U'$. Set

$$X^e:=X'\sqcup_\phi[(-\epsilon,\epsilon)\times\partial X]$$

To show that $X^e$ is a smooth manifold we apply Prop C.1.1 with $N_U=f^{-1}[(-2\epsilon/3,-\epsilon/3)\times\partial X]$ and $N_V=(-\epsilon/3,\epsilon/3)\times\partial X$.

Finally, $X'$ is embedded into $X^e$ by the quotient map $\pi_Q$ and $U$ is embedded into $X^e$ by $\pi_Q\circ\phi$ on $U$; these two maps agree on the overlap $U'$, giving an embedding $\psi:\ X\to X^e$. $\square$

Suppose $A$ and $B$ are manifolds with boundary and $\phi:\ \partial A\to\partial B$ is a diffeomorphism. Corollary C.1.3 then shows that $A$ and $B$ can be elongated to incomplete manifolds $A^e$, $B^e$ by gluing cylinders to them. The boundary diffeomorphism $\phi$ extends to a diffeomorphism of the cylinders, $\hat{\phi}:\ (u,y)\mapsto(-u,\phi(y))$. Proposition C.1.1 then shows that $A^e\sqcup_{\hat{\phi}}B^e$ is a smooth manifold, and that the image of $A$ meets the image of $B$ along their boundaries. This procedure will be used repeatedly in what follows.

## C.2 Constructing metric doubles

**Definition C.2.1.** For a Riemannian manifold with boundary $X$, a metric collar (of length $\epsilon$) is a neighborhood of the boundary isometric to $(-\epsilon, 0] \times \partial X$.

Not every Riemannian manifold with boundary has a metric collar; a hemisphere has no metric collar of its boundary.

**Definition C.2.2.** A collared Riemannian manifold is a Riemannian manifold with boundary and an isometry $(r, \pi_{\partial X}) : U \to (-\epsilon, 0] \times \partial X$, where $U$ is a neighborhood of the boundary.

**Definition C.2.3.** A metric double of a Riemannian manifold with boundary $X$ is a Riemannian manifold $X^d$ containing two isometric images $X_l$, $X_r$ of $X$ such that $X^d = X_l \cup X_r$ and $\partial X \cong X_l \cap X_r$.

We will make extensive use of the following fact:

**Proposition C.2.1.** *Collared Riemannian manifolds $X$ have smooth metric doubles.*

*Proof.* The first step is to isometrically embed $X$ in a manifold without boundary, $X^e$. Just as in Corollary C.1.3, let $U^d := (-\epsilon, \epsilon) \times \partial X$, $X' = X - \partial X$, and

$$X^e := X' \sqcup_{(r,\pi_{\partial X})} U^d$$

We use the fact that the gluing map is an isometry to give $X^e$ a metric. Let $\psi$ be the isometric embedding of $X$ in $X^e$.

Identify $U^d$ with its image in $X^e$. The reflection map on $U^d$ given by $\rho : (r, y) \to (-r, y)$ glues two copies of $X^e$ (called the left and right copies) into a smooth manifold:

$$X^d := X^e \sqcup_\rho X^e$$

To see that $X^d$ is a manifold we apply Prop C.1.1 directly. Using the fact that $\rho$ is an isometry, $X^d$ is naturally Riemannian.

For the left and right copies of $X^e$ we have isometric embeddings $\iota_l$ and $\iota_r$, and from the definition of these embeddings (given in the proof of C.1.3), it is clear that $\iota_l(X) \cup \iota_r(X) = X^d$ and $\iota_l(X) \cap \iota_r(X) \cong \partial X$. Thus $X^d$ is a metric double. □

## C.3 Constructing spin doubles

**Definition C.3.1.** A collared spin manifold is a collared Riemannian manifold with a spin structure.

In this section we will show how to canonically double the spin-structure on a collared spin manifold. Throughout this section $X$ will denote a collared spin manifold with metric collar $(r, \pi_{\partial X}) : U \to (-\epsilon, 0] \times X$ and spin structure

$$\mathcal{F}_{\mathrm{Spin}}(X) \xrightarrow{\Lambda_X} \mathcal{F}_{\mathrm{SO}}(X)$$

In the doubling procedure outlined below, we will make frequent use of the *opposite* of a spin-structure. If $Y$ is a manifold with boundary having spin-structure $(\mathcal{F}_{\mathrm{Spin}}(Y), \Lambda_Y)$, then

$$\mathcal{F}_{\mathrm{Pin}}(Y) := \mathcal{F}_{\mathrm{Spin}}(Y) \times_{\mathrm{Spin}(n)} \mathrm{Pin}(n)$$

with structure map $\Lambda_Y \times_{\mathrm{Spin}(n)} \lambda$ is a pin-structure for $Y$. Here $\lambda$ is the covering homomorphism $\mathrm{Pin}(n) \to \mathrm{O}(n)$.

**Definition C.3.2.** The opposite spin-structure for $Y$ is the subbundle

$$\mathcal{F}_{\mathrm{Spin}}(Y) \times_{\mathrm{Spin}(n)} [\mathrm{Pin}(n) - \mathrm{Spin}(n)]$$

with structure map $\Lambda_Y \times_{\mathrm{Spin}(n)} \lambda$.

Let $\mathrm{Pin}^-(n) := \mathrm{Pin}(n) - \mathrm{Spin}(n)$. An element of the opposite spin bundle can be written as $\tilde{f} \times_{\mathrm{Spin}(n)} r$, where $\tilde{f}$ is a spin-frame and $r$ is an element of $\mathrm{Pin}^-(n)$. Because the product of two elements of $\mathrm{Pin}^-(n)$ is in $\mathrm{Spin}(n)$, taking the opposite twice leads back to the original spin-structure:

$$(\tilde{f} \times_{\mathrm{Spin}(n)} r) \times_{\mathrm{Spin}(n)} r' = \tilde{f} \cdot (rr')$$

Reversing spin-structure corresponds to reversing orientation. A diffeomorphism $\phi$ reverses spin structure if there is a lift of $\phi$ to the opposite spin bundle. Such a diffeomorphism necessarily reverses orientation.

**Definition C.3.3.** The spin double of a spin manifold $X$ is the metric double $X^d$ of $X$ given a spin-structure

$$\mathcal{F}_{\mathrm{Spin}}(X^d) \xrightarrow{\Lambda_X^d} \mathcal{F}_{\mathrm{SO}}(X^d)$$

compatible with the spin-structure on $X$ in the sense that the two embeddings $\iota_l$ and $\iota_r$ of $X$ in $X^d$ preserve and reverse the spin-structure on $X$ respectively.

There is a small technical point to consider before we describe the spin-doubling procedure in general, namely, we need to show that all the geometry of a collared spin manifold is a product over its metric collar. In order to show this, we need to discuss the induced spin-structure on the boundary.

The spin-structure on $X$ induces a spin-structure on the boundary of $X$ as follows. The oriented frame bundle of the boundary can be identified in a natural way with the subbundle of $\mathcal{F}_{\mathrm{SO}}(X)|\partial X$ consisting of frames that send the first basis vector $e_1$ of $\mathbb{R}^n$ to the outward unit normal vector $\frac{\partial}{\partial r}$. (This choice of orientation for the boundary is called the *outward normal first* convention.) $\mathcal{F}_{\mathrm{Spin}}(\partial X)$ is then defined to be the inverse image of this subbundle by the structure map $\Lambda_X$, and $\Lambda_{\partial X}$ to be $\Lambda_X|\mathcal{F}_{\mathrm{Spin}}(\partial X)$. This convention also defines the spin-structure of the cylinder $(-\epsilon, 0] \times \partial X$ as the product spin-structure.

**Proposition C.3.1.** *The collar neighborhood isometry* $(r, \pi_{\partial X}) : U \to (-\epsilon, 0] \times \partial X$ *has a canonical lift to the spin bundles*

$$\widetilde{(r, \pi_{\partial X})} : \mathcal{F}_{\mathrm{Spin}}(X)|U \to \mathcal{F}_{\mathrm{Spin}}((-\epsilon, 0] \times \partial X)$$

*Proof.* Giving $(-\epsilon, 0] \times \partial X$ the product orientation, $(r, \pi_{\partial X})$ is an orientation-preserving isometry. Thus $d(r, \pi_{\partial X})$ is a lift of $(r, \pi_{\partial X})$ to $\mathcal{F}_{\mathrm{SO}}(X)|U$. On $(-\epsilon, 0] \times \partial X$ there is a natural identification of an orthonormal frame at $(r, p)$

with an orthonormal frame at $(0,p)$ in the boundary. This lifts the paths $r \mapsto (r,p)$ to $\mathcal{F}_{\mathrm{SO}}((-\epsilon,0]\times\partial X)$. Using $d(r,\pi_{\partial X})$, this path-lifting can be pulled back to $U$. That the path can be further lifted to the double cover $\mathcal{F}_{\mathrm{Spin}}(X)|U$ is then immediate. Thus path-lifting provides a canonical identification of a spin frame at $(r,p)$ with a spin frame at the boundary point $(0,p)$.

Since the spin-structures of $X$ and $(-\epsilon,0]\times X$ are identified at the boundary by definition, $d(r,\pi_{\partial X})$ lifts canonically.

□

**Corollary C.3.2.** *The elongation of a collared spin-manifold has a natural spin-structure extending $\mathcal{F}_{\mathrm{Spin}}(X)$.*

*Proof.* Let $X^e$ and $U^d$ be as in Prop C.2.1. Giving $U^d$ the product spin-structure, the lift of $(r,\pi_{\partial X})$ described in the last proposition glues together a spin-structure on $X^e$

$$\mathcal{F}_{\mathrm{Spin}}(X^e) := \mathcal{F}_{\mathrm{Spin}}(X) \sqcup_{\widetilde{(r,\pi_{\partial X})}} \mathcal{F}_{\mathrm{Spin}}(U^d)$$

□

**Proposition C.3.3.** *The orientation of $X$ induces an orientation of $X^d$, and*

$$\mathcal{F}_{\mathrm{SO}}(X^d) := \mathcal{F}_{\mathrm{SO}}(X^e)|U^d \sqcup_{\rho_*} \mathcal{F}_{\mathrm{SO}}(\bar{X}^e)|\bar{U}^d$$

*Proof.* Let $X^d$ be the metric double of $X$, and $\rho$ the gluing map $(r,y)\to(-r,y)$. Orienting $\bar{X}^e$ opposite to $X^e$ makes $\rho$ orientation-preserving. Thus $X^d$ is oriented. □

We are ready for the main theorem of this subsection.

**Theorem C.3.4.** *There is a natural lift* $\widetilde{\rho_*}$ *of* $\rho_*$ *that makes the following diagram commute:*

$$\begin{array}{ccc} \mathcal{F}_{\mathrm{Spin}}(X^e)|U^d & \xrightarrow{\widetilde{\rho_*}} & \mathcal{F}_{\mathrm{Spin}}(\bar{X}^e)|\bar{U}^d \\ \Lambda_{X^e}\downarrow & & \downarrow\Lambda_{\bar{X}^e} \\ \mathcal{F}_{\mathrm{SO}}(X^e)|U^d & \xrightarrow{\rho_*} & \mathcal{F}_{\mathrm{SO}}(\bar{X}^e)|\bar{U}^d \end{array}$$

*Then the spin-structure of the spin-double is the spin-bundle*

$$\mathcal{F}_{\mathrm{Spin}}(X^d) := \mathcal{F}_{\mathrm{Spin}}(X^e) \sqcup_{\widetilde{\rho_*}} \mathcal{F}_{\mathrm{Spin}}(\bar{X}^e)$$

*with structure map*

$$\Lambda_{X^d} : \ (\tilde{f}_l, \tilde{f}_r) \mapsto (\Lambda_{X^e}\tilde{f}_l, \Lambda_{\bar{X}^e}\tilde{f}_r)$$

*Proof.* We double the spin-structure by lifting $\rho_*$ to the spin-bundle of $U^d$. Then $\widetilde{\rho_*}$ will glue together a spin-structure on $X^d$. Let

$$R_\nu : \ \mathcal{F}_{\mathrm{SO}}(X)|\partial X \to \mathrm{O}(n)$$

denote the equivariant map corresponding to reflection across the plane perpendicular to $\frac{\partial}{\partial r}$; $\rho_*$ sends $(u, f)$ to $(-u, f \cdot R_\nu(f))$. Let $\nu : \ \mathcal{F}_{\mathrm{Spin}}(X)|\partial X \to \mathbb{R}^n$ be the spin-equivariant map corresponding to $\frac{\partial}{\partial r}$. $\nu$ transforms as $\nu(\tilde{f} \cdot h) = \lambda(h^{-1})\nu(\tilde{f})$. The quantization map $c$ that includes $\mathbb{R}^n$ in $\mathrm{Cliff}(\mathbb{R}^n)$ takes $S^n$ into $\mathrm{Pin}(n)$. Define $\widetilde{\rho_*}$ by

$$\widetilde{\rho_*}(u, \tilde{f}) := (-u, \tilde{f} \cdot c(\nu(\tilde{f})))$$

Claim: $\widetilde{\rho_*}$ lifts $\rho_*$. On the subbundle of spin frames covering oriented orthonormal frames where the first basis vector is the outward unit normal, it is clear that Clifford multiplication by $e_1$, the first basis vector of $\mathbb{R}^n$, covers reflection across the plane perpendicular to the outward unit normal. Since $\nu \equiv e_1$ on this subbundle, $\widetilde{\rho_*}$ lifts $\rho_*$ there. Any other spinframe $\tilde{f}$ can be moved onto this subbundle by some element $h$ of Spin$(n)$, thus the full claim follows if we just check that $c(\nu(\tilde{f}))$ transforms spin-equivariantly as a map into Pin$(n)$.

Let $h = x_1 \cdots x_{2k}$ be a product of an even number of unit vectors in Cliff$(n)$. Any element of Spin$(n)$ has many such representations, and then $h^{-1} = x_{2k} \cdots x_1$. Then

$$\begin{aligned} c(\nu(\tilde{f} \cdot h)) &= c(\lambda(h^{-1})\nu(\tilde{f})) \\ &= x_{2k} \cdots x_1 \cdot c(\nu(\tilde{f})) \cdot x_1 \cdots x_{2k} \\ &= h^{-1} c(\nu(\tilde{f}))h \end{aligned}$$

□

To double the pin-bundle, we must choose the *inward* unit normal as the gluing map on the complement of the spin bundle. An element $h$ of Pin$^-(n)$ can be written as a product of an odd number of unit vectors $h = x_1 \cdots x_{2k+1}$, and then $h^{-1} = -x_{2k+1} \cdots x_1$. Repeating the above calculation, we have $c(\nu(\tilde{f} \cdot h)) = -h^{-1} c(\nu(\tilde{f}))h$. In other words, $c(\nu(\tilde{f}))$ does not transform pin-equivariantly.

**Proposition C.3.5.** *The opposite of* $\mathcal{F}_{\text{Spin}}(X^d)$ *is naturally isomorphic to*

$$\mathcal{F}_{\text{Spin}}(\bar{X}^e) \sqcup_{-\widetilde{\rho_*}} \mathcal{F}_{\text{Spin}}(X^e)$$

*Proof.* By definition the opposite of $\mathcal{F}_{\text{Spin}}(X^d)$ is $\mathcal{F}_{\text{Spin}}(X^d) \times_{\text{Spin}(n)} \text{Pin}^-(n)$, that is,

$$\left[\mathcal{F}_{\text{Spin}}(X^e) \times_{\text{Spin}(n)} \text{Pin}^-(n)\right] \sqcup_{\widetilde{\rho_*} \times \text{id}} \left[\mathcal{F}_{\text{Spin}}(\bar{X}^e) \times_{\text{Spin}(n)} \text{Pin}^-(n)\right]$$

But this is the same as $\mathcal{F}_{\text{Spin}}(\bar{X}^e) \sqcup_{\widetilde{\rho_*} \times \text{id}} \mathcal{F}_{\text{Spin}}(X^e)$. Therefore we just need to check that the gluing maps are the same. If $\tilde{f} \cdot r$ is in $\mathcal{F}_{\text{Spin}}(\bar{X}^e)$, then

$$\begin{aligned} -\widetilde{\rho_*}(\tilde{f} \cdot r) &= \tilde{f} \cdot r \cdot (-\nu(\tilde{f} \cdot r)) \\ &= \tilde{f} \cdot r \cdot r^{-1} \nu(\tilde{f}) \cdot r \\ &= \tilde{f} \cdot \nu(\tilde{f}) \cdot r = \widetilde{\rho_*}(\tilde{f}) \cdot r \end{aligned}$$

Therefore the gluing maps agree. □

Choosing the gluing map to be the inward normal at every boundary component gives a spin-structure equivalent to choosing the outward normal at every boundary component. For $S^2$, which has a unique spin-structure for its round metric, this is clear. However, if the inward unit normal is selected as a gluing map on some boundary components and the outward on others, then the resulting spin-structure may not be equivalent to the doubled spin-structure. An example of this is doubling the spin-structure of the closed interval $[-\pi, 0]$ to get a spin-structure on the circle.

### C.3.1 The Dirac operator of the spin double

Recall that the spinor bundle $SX$ of a spin manifold of dimension $n$ is the vector bundle associated to $\mathbb{S} = \mathbb{S}_n$, the Spin-representation of Spin$(n)$. Thus a point in the spinor bundle is an equivalence class of pairs $[\tilde{f}, s] \in \mathcal{F}_{\text{Spin}}(X^e) \times \mathbb{S}$, with $[\tilde{f}, s] \sim [\tilde{f} \cdot h^{-1}, hs]$ for any $h \in \text{Spin}(n)$.

The associated Dirac operator is a first-order differential operator

$$D : \, C^\infty(SX) \to C^\infty(SX)$$

defined as follows. The Levi-Civita connection for the tangent bundle induces a connection on $\mathcal{F}_{\text{Spin}}(X)$, called the spin connection. Using the spin-connection, a vector field on $X$ can be lifted to the tangent bundle of the frame bundle. The *canonical vector fields* on $\mathcal{F}_{\text{Spin}}(X)$ are the lifts $\partial_k(\tilde{f})$ of $\lambda(\tilde{f})(e_k)$. The Dirac operator of $X^d$ is then defined to be $D := c(e^k)\partial_k$. From the definition, it is apparent that the Dirac operator acts on $\mathbb{S}$-valued functions $\tilde{\phi}$ on $\mathcal{F}_{\text{Spin}}(X^d)$. The following proposition is standard.

**Proposition C.3.6.** *If $\psi : \, \mathcal{F}_{\text{Spin}}(X^d) \to \mathbb{S}$ is spin-equivariant, so is $D\psi$.*

Therefore the Dirac operator descends to a differential operator on sections of $SX$; this operator is also called the Dirac operator. In particular, for $\mathcal{F}_{\text{Spin}}(X^e)$, its opposite $\mathcal{F}_{\text{Spin}}(\bar{X}^e)$, and the double $\mathcal{F}_{\text{Spin}}(X^d)$, there are corresponding spinor-bundles $SX^e$, $S\bar{X}^e$, and $SX^d$, and Dirac operators $D_{X^e}$, $D_{\bar{X}^e}$, and $D_{X^d}$. As one might expect, these operators are related in a simple way.

**Proposition C.3.7.** *The spinor bundle $SX^e$ is isomorphic to $S\bar{X}^e$ via the map*

$$\mathfrak{R} : [\tilde{f}, s] \mapsto [\tilde{f} \times_{\mathrm{Spin}(n)} r^{-1}, rs]$$

*where $r$ is any element of $\mathrm{Pin}^-(n)$. This map is canonical; in particular, it does not depend on the choice of $r$.*

*Proof.* We have to show this is map is well-defined, independant of the choice of $r$ and the representative of $[\tilde{f}, s]$. Fixing $r$, any element of $\mathrm{Pin}^-(n)$ can be written as $gr$ where $g$ is in $\mathrm{Spin}(n)$. Therefore making a different choice of representative for $[\tilde{f}, s]$ is equivalent to choosing $r$ differently. In either case we have:

$$[\tilde{f} \times_{\mathrm{Spin}(n)} (gr)^{-1}, grs] = [\tilde{f} \times_{\mathrm{Spin}(n)} r^{-1}g^{-1}, grs] = [\tilde{f} \times_{\mathrm{Spin}(n)} r^{-1}, rs]$$

□

Using $\mathfrak{R}$, we can compare the Dirac operators for $X^e$ and $\bar{X}^e$.

**Proposition C.3.8.** *We have*

$$D_{\bar{X}^e} = -\mathfrak{R} D_{X^e} \mathfrak{R}^{-1}$$

*Proof.* We need two observations. The first is that there is a natural Pin action on the canonical vector fields, defined by letting $\lambda(r) \cdot \partial_k$ be the horizontal lift of $\Lambda(\tilde{f})(\lambda(r) \cdot e_k)$. Using the basis $\{e_k\}$ to write $\lambda(r)$ as an orthogonal matrix, this definition is obviously equivalent to matrix multiplication of $\lambda(r)$ on $\partial_k$. The second is that

$$Ds = c(e^k)\partial_k s = c(\lambda(g) \cdot e^k)\lambda(g) \cdot \partial_k s$$

This is a simple consequence of the fact that $\lambda(g)$ acts on elements of the cobasis $\{e^k\}$ by the adjoint action.

We prove the proposition in the form

$$-\mathfrak{R}(D_{X^e} s) = D_{\bar{X}^e}(\mathfrak{R}s)$$

By definition,

$$\mathfrak{R}(D_{X^e} s)|_{\tilde{f}\times r^{-1}} = r \cdot D_{X^e} s|_{\tilde{f}} = r \cdot c(e^k)[\partial_k s]_{\tilde{f}}$$

Thus

$$\begin{aligned}
-\mathfrak{R}(D_{X^e} s)|_{\tilde{f}\times r^{-1}} &= -r \cdot c(e^k) r^{-1} \cdot r \cdot [\partial_k s]_{\tilde{f}} \\
&= c(\lambda(r) \cdot e^k) r \cdot [\partial_k s]_{\tilde{f}} \\
&= c(\lambda(r) \cdot e^k)[\partial_k r \cdot s]_{\tilde{f}} \\
&= c(\lambda(r) \cdot e^k)[\partial_k r^* \mathfrak{R}s]_{\tilde{f}} \\
&= c(\lambda(r) \cdot e^k)[(r_* \partial_k)\mathfrak{R}s]_{\tilde{f}\times r^{-1}} \\
&= c(\lambda(r) \cdot e^k)[(\lambda(r) \cdot \partial_k)\mathfrak{R}s]_{\tilde{f}\times r^{-1}}
\end{aligned}$$

Using our second observation, we see that this is the right hand side of the proposed equality. □

**Proposition C.3.9.** *We have*

$$SX^d = SX^e \sqcup_{\widetilde{\rho_*}} S\bar{X}^e$$

*Proof.*

$$\begin{aligned} SX^d &= \mathcal{F}_{\mathrm{Spin}}(X^d) \times_{\mathrm{Spin}(n)} \mathbb{S} \\ &= \left(\mathcal{F}_{\mathrm{Spin}}(X^e) \sqcup_{\widetilde{\rho_*}} \mathcal{F}_{\mathrm{Spin}}(\bar{X}^e)\right) \times_{\mathrm{Spin}(n)} \mathbb{S} \\ &= SX^e \sqcup_{\widetilde{\rho_*}} S\bar{X}^e \end{aligned}$$

□

In view of the last proposition, to each section $s$ of $SX^d$ we can uniquely associate a pair of sections $(s_l, s_r)$ of $SX^e$ and $S\bar{X}^e$ respectively.

The double $X^d$ of a manifold $X$ has an obvious reflection symmetry $\mu$. If $\iota_l$ and $\iota_r$ are the left and right embeddings of $X^e$ in $X^d$, $\mu$ is the map that exchanges $\iota_l(x)$ and $\iota_r(x)$. The fixed point set of $\mu$ is $\iota_l(\partial X)$. Note that $\mu$ is an orientation-reversing isometry. It therefore commutes with the Levi-Civita connection.

Using $\mathfrak{R}$ this symmetry can be lifted to spinor sections.

$$\begin{aligned} \tilde{\mu} :\ & C^\infty(SX^d) \to C^\infty(SX^d) \\ & (s_l, s_r) \mapsto (\mathfrak{R}^{-1}s_r, -\mathfrak{R}s_l) \end{aligned}$$

**Proposition C.3.10.** *The map $\tilde{\mu}$ has the following properties:*

1. *$\tilde{\mu}(s_l, s_r)$ is a section of $SX^d$, i.e., $\widetilde{\rho_*} \circ \mathfrak{R}^{-1}s_r = -\mathfrak{R}s_l \circ \rho$.*

2. *Over the fixed point set $\iota_l(\partial X)$, $\tilde{\mu}$ acts by Clifford multiplication by the outward unit normal (pointing from the left side into the right side).*

3. *$\tilde{\mu}$ anticommutes with the Dirac operator.*

4. *$\tilde{\mu}^2 = -1$*

*Proof.* We make a preliminary claim. Let $(u, \tilde{f})$ be a spinframe over $U^d$ in $X^d$. Then

$$(\mathfrak{R}^{-1} \circ \widetilde{\rho_*})([(u, \tilde{f}), s]) = [(-u, \tilde{f}), c(\nu(-u, \tilde{f})) \cdot s] \tag{C.3.1}$$

This is trivial–just apply $\mathfrak{R}$ to both sides of the equation, choosing $r = -c(\nu(-u, \tilde{f}))$.

To show Claim 1, note that $\widetilde{\rho_*} \circ s_l \circ \rho = s_r | U^d$ since $(s_l, s_r)$ is a section of $SX^d$. Therefore $\widetilde{\rho_*} \circ \mathfrak{R}^{-1} s_r = -\mathfrak{R} s_l \circ \rho$ is equivalent to

$$\widetilde{\rho_*} \circ \mathfrak{R}^{-1} \widetilde{\rho_*} \circ s_l \circ \rho = -\mathfrak{R} s_l \circ \rho$$

Factoring out $s_l \circ \rho$, we must show that $\widetilde{\rho_*} \circ \mathfrak{R}^{-1} \widetilde{\rho_*} = -\mathfrak{R}$. But this is equivalent to $(\mathfrak{R}^{-1} \circ \widetilde{\rho_*})^2 = -1$ which follows directly from (C.3.1).

To show Claim 2, let $\iota_l(0, y)$ be a boundary point and $\tilde{f}$ a spinframe over $(0, y)$. Then

$$\begin{aligned}
\tilde{\mu}(s_l, s_r)(0, y) &= (\mathfrak{R}^{-1} s_r, -\mathfrak{R} s_l)(\iota_l(0, y)) = (\mathfrak{R}^{-1} s_r)(0, y) \\
&= \mathfrak{R}^{-1}(s_r \circ \rho)(0, y) = \mathfrak{R}^{-1}(\widetilde{\rho_*} \circ s_l)(0, y) \\
&= [\tilde{f}, c(\nu(\tilde{f})) \cdot \tilde{s}_l(\tilde{f})] = \sigma s_l(0, y)
\end{aligned}$$

where $\sigma$ is Clifford multiplication by the outward unit conormal. Claim 3 follows directly from Prop C.3.8, and Claim 4 is trivial. $\square$

We can now relate the Dirac operator on $X^d$ to the Dirac operators on $X^e$ and $\bar{X}^e$.

**Proposition C.3.11.**

$$D_{X^d}(s_l, s_r) = (D_{X^e} s_l, D_{\bar{X}^e} s_r)$$

*Proof.* The Dirac operator $D_{X^d}$ is a well-defined differential operator on spinors over $X^d$, and $s = (s_l, s_r)$ is a well-defined spinor section. Therefore $g = D_{X^d}(s_l, s_r)$ can be written in the form $(g_l, g_r)$, where $g_l$ is a spinor section over $X^e$ and $g_r$ is a spinor section over $\bar{X}^e$. It suffices to check that

$$g_l = D_{X^e} s_l \quad \text{and} \quad g_r = D_{\bar{X}^e} s_r$$

By definition, $g_l = \iota_l^* g$ and $s_l = \iota_l^* s$. Thus the first equation is equivalent to

$$\iota_l^* D_{X^d} s = D_{X^e} \iota_l^* s$$

which follows by the naturality of the Dirac operator under spin-preserving diffeomorphisms. To show that the second equation holds, we use the first equation, Prop. C.3.8, and part 3 of Prop. C.3.10. Specifically, we use the relations $D_{\bar{X}^e} = -\mathfrak{R} D_{X^e} \mathfrak{R}^{-1}$, and $D_{X^d}\tilde{\mu} = -\tilde{\mu} D_{X^d}$. From the first equation and the definition of $\tilde{\mu}$, we have

$$(D_{X^d}\tilde{\mu} s)_l = D_{X^e} \mathfrak{R} s_r$$

Using this and the fact that $D_{X^d}\tilde{\mu} = -\tilde{\mu} D_{X^d}$, we have

$$D_{X^e}\mathfrak{R}^{-1}s_r = -(\tilde{\mu} D_{X^d} s)_l = -\mathfrak{R}^{-1}(D_{X^d} s)_r$$

That is,

$$D_{X^e}\mathfrak{R}^{-1}s_r = -\mathfrak{R}^{-1}g_r$$

Equivalently, $D_{\bar{X}^e} s_r = -\mathfrak{R} D_{X^e}\mathfrak{R}^{-1}s_r = g_r$ This is the second equation. □

The following theorem is closely related to the Theorem of the Invertible Double, given in [8]. That theorem shows that for a Dirac-type operator on a manifold with boundary acting on sections of a Clifford module, the Clifford module and the operator can be extended to the double of the manifold, and the extension of the operator is invertible in the sense that its kernel and cokernel are trivial. The theorem we prove below applies only to the Dirac operator on a collared spin-manifold and shows that in that case, the Dirac operator of the spin-double is invertible. The main technique of the proof, to apply the unique continuation property, is the same.

The following theorem is the combination of Theorem 8.2 and Corollary 8.3 of [8].

**Theorem C.3.12.** *(The Unique Continuation Theorem) Let $X = X^+ \cup X^-$ be a connected partitioned manifold with $X^+ \cap X^- = \partial X^\pm = Y$. Let $S$ be a* $\mathrm{Cliff}(X)$*-module with compatible connection $\nabla$, and $D$ the corresponding Dirac operator. Then for any smooth section $s$ with $Ds = 0$ and $s|_Y = 0$, $s = 0$.*

**Theorem C.3.13.** *Suppose $X$ is compact, connected and $\partial X \neq \varnothing$. Then the Dirac operator on the spin-double is invertible. That is, its kernel and cokernel are trivial.*

*Proof.* The spin-double $X^d$ is a partitioned manifold, partitioned by $\partial X$.

Suppose $\phi$ is in the kernel of $D$. Then $\phi$ is necessarily smooth by elliptic regularity. Let $P_\pm := \frac{1}{2}(1 \pm i\tilde{\mu})$, and let $\phi_\pm = P_\pm \phi$. Since $\tilde{\mu}$ anticommutes with the Dirac operator, both $\phi_+$ and $\phi_-$ are in the kernel as well. Clearly $\phi_+ + \phi_- = \phi$. We will show that both $\phi_+$ and $\phi_-$ are zero, implying the theorem.

The key point is that since $(\tilde{\mu}\phi)|_{\partial X} = \sigma\phi|_{\partial X}$, we have $\sigma\phi_\pm|_{\partial X} = \mp i\phi_\pm|_{\partial X}$. Let $\phi_\pm = (s_l, s_r)$. Then $\sigma s_l|_{\partial X} = \mp i s_l|_{\partial X}$ and $\sigma s_r|_{\partial X} = \mp i s_r|_{\partial X}$.

Since $D$ is formally self-adjoint, Green's Theorem shows that

$$\begin{aligned} 0 &= \langle D s_l, s_l \rangle - \langle s_l, D s_l \rangle \\ &= \int_{\partial X} \langle \sigma s_l, s_l \rangle \\ &= \int_{\partial X} \langle \mp i s_l, s_l \rangle = \mp i \| s_l|_{\partial X} \|^2 \end{aligned}$$

Thus $s_l|_{\partial X} = 0$. Thus $\phi_\pm|_{\partial X} = 0$. The Unique Continuation Theorem implies $\phi_\pm = 0$. □

*Remark* C.3.1. In a sense, the last theorem can be regarded as the origin for the two classes of boundary conditions we consider. As the proof shows, if $\phi$ is a smooth section of $S$ on $X^d$, $\phi_+ = \frac{1}{2}(1 + i\tilde{\mu})\phi$ and $s_l$ is defined by

$\phi_+ = (s_l, s_r)$, then $s_l$ satisfies the boundary condition $\frac{1}{2}(1 - i\sigma)s_l|_{\partial X} = 0$. That is, $s_l|_{\partial X} = i\sigma s_l|_{\partial X}$.

If $X$ is odd-dimensional, the boundary condition implies that $s_l|_{\partial X}$ lies in $S^+|\partial X$, which we define to be the $+1$-eigenspace of $i\sigma$. This is obviously an example of the odd-type boundary value problem. If $X$ is even-dimensional, the boundary condition says that $s_l|_{\partial X}$ is in the graph of $i\sigma : S^+|\partial X \to S^-|\partial X$. Taking $E$ to be the trivial graded vector bundle $E \cong \mathbb{C} \oplus \mathbb{C}$, (so that $S^+E^+ + S^-E^- = S$) and $\tau$ to be

$$\tau = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

we see that this is an even-type boundary value problem.

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# Index

# Vita

Matthew Gregory Scholl was born in Silver Spring, Maryland on Dec 3, 1974. He received the Bachelor of Science degree in Mathematics from the Ohio State University. He applied to the University of Texas at Austin for enrollment in their math program. He was accepted and started graduate studies in August, 1998.

Permanent address: 5516 Grover Ave., Apt. 101
Austin, Texas 78756

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