The Dissertation Committee for James Patrick Kelliher
certifies that this is the approved version of the following dissertation:

The vanishing viscosity limit for incompressible fluids in two dimensions

Committee:

Mikhail M. Vishik, Supervisor

Rafael de la Llave

William Beckner

Philip J. Morrison

Hans Koch

The vanishing viscosity limit for incompressible fluids in two dimensions

by

James Patrick Kelliher, B. S., M. Sc.

Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

May 2005

Dedicated, like every good thing, to my parents.

Acknowledgments

My path in life to a PhD in mathematics has been longer than most; so too will be my acknowledgements.

My parents I must thank for everything, but most particularly for demonstrating to me by the example of their lives that knowledge and rationality are virtues, while ignorance is not.

I thank my brother, Dennis Kelliher, for being 2-1/2 years older and smarter than I. My brother's early fascination with the infinite, which he passed on to me, was the beginning of my interest in mathematics. I also thank him for always being there as someone to strive to beat.

I would also like to thank Dr. John C. Kelly, my boss of many years, for forcing me to grow as a person and giving me the opportunity to see a good part of the world. Not all my years before math were a wasteland.

There are three professors who had a particular influence on me at George Mason University, where I received my Masters degree. I would like to thank Professor David Walnut for encouraging me to pursue a doctorate in pure mathematics and for looking out for me in many ways at George Mason. I would also like to thank Professor Flavia Colonna, my Master's thesis advisor, and Professor L. Brian Lawrence, from whom I took several classes, for the time they invested in me.

My first year at the University of Texas was the most challenging and the most fun year of my life. I would particularly like to acknowledge the role played by

Jason Deblois, Richard Kent, and Ben McReynolds in both aspects of that year. I also want to acknowledge Riad Masri for the many and fruitful conversations about math that I have had with him starting in my second year, conversations I expect will continue beyond our graduations.

There is a culture among graduate students at UT Austin of friendliness and collaboration. How this started is a bit of a mystery, but it was there before I came, and I expect it will continue after I leave. It has been passed down from generation to generation of students, but is also fostered by many professors and staff members. This role has been most visibly played, in my experience, by Nancy Lamm, Professor Bruce Palka, and Professor Alan Reid (though it is certainly not restricted to them). I thank each of them for the many graces, large and small, that they have granted me during my years at UT. I also thank Professor Rafael de la Llave for his encouragement and for establishing Wednesday nights at the Possee.

Most of all I thank my advisor, Professor Misha Vishik. I thank him for believing that I was an intelligent person capable of understanding and producing mathematics, even when I believed no such thing; for expecting me to do things on my own, but always being there for advice and support; for looking out for my financial well being while his student; for helping me find my own direction in mathematics, without dominating me. He has given me a comfortable space in which to grow, and I only hope that I can find another such space in the future.

James Patrick Kelliher

The University of Texas at Austin
May 2005

The vanishing viscosity limit for incompressible fluids in two dimensions

Publication No. ____________________

James Patrick Kelliher, Ph.D.
The University of Texas at Austin, 2005

Supervisor: Mikhail M. Vishik

The Navier-Stokes equations describe the motion of an incompressible fluid of constant density and constant positive viscosity. With zero viscosity, the Navier-Stokes equations become the Euler equations. A question of longstanding interest to mathematicians and physicists is whether, as the viscosity goes to zero, a solution to the Navier-Stokes equations converges, in an appropriate sense, to a solution to the Euler equations: the so-called "vanishing viscosity" or "inviscid" limit. We investigate this question in three settings: in the whole plane, in a bounded domain in the plane, and for radially symmetric solutions in the whole plane.

Working in the whole plane and in a bounded domain, we assume a particular bound on the growth of the L^p-norms of the initial vorticity (curl of the velocity)

with p, and obtain a bound on the convergence rate in the vanishing viscosity limit. This is the same class of initial vorticities considered by Yudovich and shown to imply uniqueness of the solution to the Euler equations in a bounded domain lying in Euclidean space of dimension 2 or greater.

For radially symmetric initial vorticities we obtain a more precise bound on the convergence rate as a function of the smoothness of its initial vorticity as measured by its norm in a Sobolev space or in certain Besov spaces.

We also consider the questions of existence, uniqueness, and regularity of solutions to the Navier-Stokes and Euler equations, as necessary, to make sense of the vanishing viscosity limit. In particular, we investigate properties of the flow for solutions to the Euler equations in the whole plane. We construct a specific example of an initial vorticity for which there exists a unique solution to the Euler equations whose associated flow lies in no Hölder space of positive exponent for any positive time. This example is an adaptation of a bounded-vorticity example of Bahouri and Chemin's, which they show has a flow lying in no Hölder space of exponent greater than an exponentially decreasing function of time.

Contents

Chapter 1

Introduction

1.1 Overview

The Navier-Stokes equations,

$$(NS) \quad \begin{cases} \partial_t v_\nu + v_\nu \cdot \nabla v_\nu - \nu \Delta v_\nu = -\nabla p_\nu \\ \operatorname{div} v_\nu = 0 \\ v_\nu|_{t=0} = v^0, \end{cases}$$

describe the motion in $\mathbb{R}^d$, $d \geq 2$, of an incompressible fluid of constant density and constant viscosity, ν. The velocity of the fluid at time t and position x in space is $v_\nu(t, x)$ with corresponding pressure $p(t, x)$. We will also consider a fluid that is constrained to a bounded domain of $\mathbb{R}^d$ (for $d = 2$), in which case boundary conditions are imposed as well (see Chapter 3).

The Navier-Stokes equations can be derived directly from Newton's laws under the assumptions that the fluid is incompressible (which means, in effect, that the speed of sound in the fluid is infinite) and that the shear stress tensor varies linearly with the rate of strain tensor. These issue are explained, for instance, in Chapter 1 of [10], but will not concern us here.

With zero viscosity, the The Navier-Stokes equations become the Euler equations:

$$(E) \quad \begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p \\ \operatorname{div} v = 0 \\ v|_{t=0} = v^0. \end{cases}$$

In brief, the subject of this thesis is whether the solutions to (NS) converge, in an appropriate sense, to a solution to (E) as ν goes to zero: the so-called "vanishing viscosity" or "inviscid" limit. We investigate this question in three settings: all of $\mathbb{R}^2$ in Chapter 2, a bounded domain in the plane in Chapter 3, and radially symmetric solutions in the plane in Chapter 4.

We also consider the questions of existence, uniqueness, and regularity of solutions to (NS) and (E), as necessary, to make sense of the vanishing viscosity limit. In particular, we investigate properties of the flow for solutions to (E) in Chapter 5.

What precisely it means to solve (NS) and (E) depends upon the function spaces in which the initial velocity lies; we will give an explicit definition of what we mean by a solution in each of the chapters that follow. As a general statement, however, we will always be assuming sufficiently strong conditions on the initial velocity to insure the existence and uniqueness of weak solutions to both (NS) and (E) within the particular class of solutions that we are studying. In this regard, (E) is more of an issue than (NS), in that existence as well as uniqueness are known for (NS) under considerably weaker assumptions than for (E). Our assumptions will be, for the most part, nearly as weak as possible to insure uniqueness of the Euler equations, given what is known (or established here).

The study of the vanishing viscosity limit in various settings has a long history. Temam has a discussion of this in Appendix III of [33]. See also Kato's remarks in [15]. Briefly, convergence of smooth solutions in $\mathbb{R}^d$ is well understood. Much less is known about convergence of weak solutions in $\mathbb{R}^d$ or the convergence of solutions, weak or smooth, in a domain with boundaries. In the introduction to each of the chapters that follow, we discuss the existing state of knowledge in each of the settings we consider.

1.2 Chapter summaries

We now summarize the results of this thesis in more detail, chapter-by-chapter.

Chapter 2: The plane. (The results in this chapter appear in [16].) In [36], Yudovich established the uniqueness of solutions to the Euler equations in a bounded domain Ω of $\mathbb{R}^d$ for $d \geq 2$ when the initial vorticity ω_0 is in $L^\infty(\Omega)$. He extended this uniqueness result in [37] to to a class of initial velocity fields, $\mathbb{Y}$. This is the class of all divergence-free vector fields v whose vorticity ω lies in $L^p(\Omega)$ for all p in $[p_0, \infty)$ for some p_0 in $[1, \infty)$ and for which $\int_0^1 (\beta(x))^{-1}\, dx = \infty$, where $\beta(x) = \inf\{M^\epsilon(x^{1-\epsilon}/\epsilon)\, \|\omega\|_{L^{1/\epsilon}} : \epsilon \in (0, 1/p_0]\}$ and M is a positive real number. The space $\mathbb{Y}$ is independent of the choice of M (however, see below). These conditions are satisfied, for instance, if $\|\omega\|_{L^p} \leq C \log p$, but not for L^p–norms that grow much faster than logarithmically. Such growth in the L^p–norms arises, for instance, for a compactly supported vorticity with a point singularity of type $\log\log|x|^{-1}$.

In [4], Chemin shows that for initial vorticity in $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, solutions to the Navier-Stokes equations converge strongly to a solution to the Euler equations in the inviscid limit, and gives a bound on the rate of convergence. In Chapter 2, we combine the approaches of Yudovich in [37] and Chemin in [4] to establish strong convergence in the inviscid limit for initial velocity in $\mathbb{Y}$ (with $p_0 \leq 2$), and give a bound on the rate of convergence that depends on the function β described above.

Specifically, if u_ν is a solution to the Navier-Stokes equations and u is a solution to the Euler equations, both with initial velocity u_0, then $\|u_\nu - u\|_{L^\infty([0,t];L^2)} \le \rho(\nu, t)$, where ρ is defined implicitly by

$$\int_{R\nu t}^{\rho(\nu,t)} \frac{dx}{\beta(x)} = t, \tag{1.2.1}$$

where $R = C\|\omega_0\|_{L^2}^2$ and M is an upper bound on $4\,\|u_0\|_{L_\infty}^2$ (which is finite for u_0 in $\mathbb{Y}$).

An interesting feature of this bound is that one can always find initial vorticities for which the bounded rate of convergence is arbitrarily slow. This is in accord with Yudovich's assumptions on the initial vorticity being among the weakest assumptions known to imply uniqueness.

Chapter 3: A bounded domain. (The results in this chapter appear in [17].) The approach in Chapter 2 to the inviscid limit encounters the well known obstacle of boundary layer effects when applied to a bounded domain in $\mathbb{R}^2$ with the usual no-slip boundary condition for the Navier-Stokes equations (see, for instance, [15]). Using Navier boundary conditions for the Navier-Stokes equations, however, boundary layer effects are manageable, as shown in [7], where existence and uniqueness of solution to the Navier-Stokes equations along with weak convergence in the inviscid limit to a solution to the Euler equations is established when the initial vorticity is in $L^\infty(\mathbb{R}^2)$. In [27], existence, uniqueness, and convergence is extended to initial vorticities in $L^p(\mathbb{R}^2)$ for some $p > 2$. Both [7] and [27] treat only the case of a simply connected domain.

Navier boundary conditions can be expressed for a bounded domain in $\mathbb{R}^2$ with a C^2-boundary Γ as $\omega(v) = (2\kappa - \alpha)v \cdot \boldsymbol{\tau}$ and $v \cdot \mathbf{n} = 0$ on Γ, where v is the velocity, $\omega(v)$ the vorticity, $\mathbf{n}$ a unit normal vector, $\boldsymbol{\tau}$ a unit tangent vector, and α is in $L^\infty(\Gamma)$. Such boundary conditions are an idealization of a rough boundary, with the function α measuring the degree of roughness.

In Chapter 3, we extend the results of [7] and [27] to non-simply connected domains, and establish strong convergence in the inviscid limit with the same bound on the convergence rate as in all of $\mathbb{R}^2$ established in [16]. In [7] and [27], only the case where α is nonnegative is treated, since for such solutions energy is nonincreasing. Taking a different approach to the existence and uniqueness proofs, we also extend the results of [7] and [27] to general α. This has the advantage of allowing one to view the boundary conditions $\omega(v) = 0$ and $v \cdot n = 0$ on Γ, which were studied by J. L. Lions in [25] and later by P. L. Lions in [26], as special cases of Navier boundary conditions for bounded domains that are not necessarily convex.

Finally, we show in Chapter 3 that in the limit as $\alpha \to +\infty$ on the boundary, when the boundary is C^3 and the initial velocity lies in $H^3(\Omega)$, solutions of the Navier-Stokes equations with Navier boundary conditions converge in the $L^\infty([0,T]; L^2(\Omega))$ norm to the solution of the Navier-Stokes equations with no-slip boundary conditions. We also give a bound on the rate of convergence.

Chapter 4: Radial symmetry. We specialize to radially symmetric initial vorticity in the plane, assuming that the initial velocity is in E_m (a space described in Appendix 2A). Because of radial symmetry, there will always be a steady state solution to (E), and the solution to (NS) will be the same as the solution to the heat equations. This allows us to work with weaker assumptions on the initial vorticity than those of Chapter 2 (so the steady state solution to (E) may not be unique). We obtain upper bounds on the convergence rate in the vanishing viscosity limit under various assumptions on the initial vorticity.

In particular, we show that if the initial vorticity is in $\dot{H}^\eta$ for η in $(-1, 1]$, then for all $\nu t \geq 0$, the solution to (E) approaches the solution to (NS) in the L^2-norm at a rate bounded by

$$\sqrt{2}\, \|\omega_0\|_{\dot{H}^\eta} \, (\nu t)^{(1+\eta)/2}.$$

We also adapt an approach of [1] to obtain a bound on the convergence rate when the initial vorticity is radially symmetric and lies in the Besov space $B^{\eta}_{2,\infty}$ for η in $(-1, 1)$.

We then consider a superposition of disjoint circular patches of vorticity (eddies) in the plane, each with zero total vorticity: this gives a steady state solution to the Euler equations. Let the initial vorticity be such a superposition of a finite number of compactly supported eddies with a positive minimum distance between any two patches. We employ an inequality established in [28], where the same superposition of confined eddies is studied for initial vorticity that is a particular subclass of bounded measures, to obtain a bound on the combined convergence rate for the eddies. We show that for any $T > 0$, the corresponding solutions to the Navier-Stokes equations will converge in $L^\infty([0, T]; L^2(\mathbb{R}^2))$ to the steady state solution to the Euler equations at a rate that is the same, over a short time, as for a single eddy, though with a larger constant.

Chapter 5: The flow. We return to the setting of Chapter 2 and establish the properties of the flow for the solution to the Euler equations in the plane given Yudovich initial vorticity. These properties are stated, in slightly different form, by Yudovich in [37], though not proved there. The properties are interesting in themselves, but can also be used in the proof of existence of weak solutions to (E) and (NS) as we defined them in Chapter 2.

We extend an example of Bahouri's and Chemin's described in [3] and in Section 5.3 of [5] to show that the upper bound on the modulus of continuity of the vector field can be achieved, to within a constant at time zero, at least for the sequence of unbounded vorticities in [37]. We then obtain a lower bound for the modulus of continuity of the vector field over time, and hence, for that of the flow. Finally, we show that for certain initial velocities it is possible to have a unique solution to the Euler equations with an associated flow that lies, for all positive time, in no Hölder space of positive exponent.

1.3 Notational conventions

We follow the convention that C is always an unspecified constant that may vary from expression to expression, even across an inequality (but not across an equality). When we wish to emphasize that a constant depends, at least in part, upon the parameters $x_1, \ldots, x_n$, we write $C(x_1, \ldots, x_n)$. When we need to distinguish between unspecified constants, we use C and C'.

For vectors u and v in $\mathbb{R}^2$, we alternately write $\nabla v u$ and $u \cdot \nabla v$, by both of which mean $u^i \partial_i v^j \mathbf{e}_j$, where $\mathbf{e}_1, \mathbf{e}_2$ are basis vectors, and we define $\nabla u \cdot \nabla v = u^{ij} v^{ij}$. Here, as everywhere in this paper, we follow the common convention that repeated indices are summed—whether or not one is a superscript and one is a subscript.

If X is a function space and k a positive integer, we define $(X)^k$ to be

$$\{(f_1, \ldots, f_k) : f_1 \in X, \ldots, f_k \in X\}.$$

For instance, $(H^1(\Omega))^2$ is the set of all vector fields, each of whose components lies in $H^1(\Omega)$. To avoid excess notation, however, we always suppress the superscript k when it is clear from the context whether we are dealing with scalar-, vector-, or tensor-valued functions.

1.4 A comment on units of measure

It will occasionally be useful to perform a dimensional analysis to at least determine the plausibility of our results, or those in the literature. Toward this end, we observe that, from (NS):

1. viscosity ν has units $(\text{distance})^2/(\text{time})$;

2. vorticity ω has units (velocity)/(distance) = 1/(time);

3. $\sqrt{\nu t}$ has units (distance).

Chapter 2

Vanishing viscosity in $\mathbb{R}^2$

2.1 Introduction

The motion of an incompressible fluid of constant density and constant viscosity is governed by the Navier-Stokes equations, and without viscosity by the Euler equations. These are equations (NS) and (E) introduced in Chapter 1.

In this chapter, we restrict our attention to fluids extending throughout $\mathbb{R}^2$, with the initial velocity belonging, for some real number m, to the space E_m of [4] and [5]. This space is described in detail in Appendix 2A, but, in brief, a vector v belongs to E_m if it is divergence-free and can be written in the form $v = \sigma + v'$, where v' is in $L^2(\mathbb{R}^2)$ and where σ is a *stationary vector field*, meaning that σ is of the form,

$$\sigma = \left(-\frac{x^2}{r^2} \int_0^r \rho g(\rho)\, d\rho,\ \frac{x^1}{r^2} \int_0^r \rho g(\rho)\, d\rho \right), \tag{2.1.1}$$

where g is in $C_0^\infty(\mathbb{R})$. The subscript m is the integral over all space of the vorticity of v, the vorticity of v being $\partial_1 v^2 - \partial_2 v^1$. We use the notation $\omega(v)$, or just ω when v is understood, for the vorticity of v. The initial vorticity we denote by ω^0.

E_m is an affine space; fixing an origin, σ, in E_m we can define a norm by

$\|\sigma + v'\|_{E_m} = \|v'\|_{L^2}$. Convergence in E_m is equivalent to convergence in the L^2–norm to a vector in E_m.

The following is a fundamental result of Yudovich's ([36]), as adapted by Chemin in [6] from bounded domains to all of $\mathbb{R}^2$ (see [5]):

Theorem 2.1.1 (Yudovich's theorem). *Let v^0 be in E_m, with ω^0 belonging to $L^a(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ for some $1 < a < \infty$. Then there exists a unique solution v of (E) belonging to $C(\mathbb{R}; E_m)$ such that $\omega(v)$ is in $L^\infty(\mathbb{R} \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}; L^a(\mathbb{R}^2))$.*

In [37], Yudovich, in the setting of a bounded domain in $\mathbb{R}^n$ with impermeable boundary, weakens the conditions on the initial vorticity in Theorem 2.1.1, allowing unbounded vorticity, and is still able to obtain uniqueness. (Similar results have been obtained by Serfati in [31].) Chemin shows in [4] that with the assumptions on the initial data in Theorem 2.1.1 with $a = 2$, solutions $(v_\nu)_{v>0}$ of (NS) converge in the L^2–norm uniformly over a finite time interval as $\nu \to 0$ to the unique solution v of (E) given by Theorem 2.1.1. We establish the same convergence as Chemin, but with the initial vorticity of Yudovich.

To describe Yudovich's conditions on the initial vorticity, we need the following definition:

Definition 2.1.2. Let $\theta : [p_0, \infty) \to \mathbb{R}$ for some $p_0 > 1$. We say that θ is *admissible* if the function $\beta : (0, \infty) \to [0, \infty)$ defined, for some $M > 0$, by[1]

$$\beta(x) := \beta_M(x) := 2C_0 x \inf \left\{ (M^\epsilon x^{-\epsilon}/\epsilon)\theta(1/\epsilon) : \epsilon \text{ in } (0, 1/p_0] \right\}, \tag{2.1.2}$$

where C_0 is a fixed absolute constant, satisfies

$$\int_0^1 \frac{dx}{\beta(x)} = \infty. \tag{2.1.3}$$

[1]The definition of β in Equation (2.1.2) differs from that in [16] in that it directly incorporates the factor of p that appears in the constant in the Calderón-Zygmund inequality of Theorem 2.1.6.(iv); in [16] this factor is included in the equivalent of Equation (2.1.3).

Because $\beta_M(x) = M\beta_1(x/M)$, this definition is independent of the value of M. Also, β is a monotonically increasing continuous function, with $\lim_{x \to 0^+} \beta(x) = 0$. In Section 2.2 we give examples of admissible functions and discuss how our definition relates to the equivalent definition in [37].

Yudovich proves in [37] that for a bounded domain in $\mathbb{R}^n$ with impermeable boundary (which adds the condition to (E) that the normal component of the velocity on the boundary is zero), if $\|\omega^0\|_{L^p} \le \theta(p)$ for some admissible function θ, then at most one solution to (E) exists. Because of this, we call the class of all such vorticities, *Yudovich vorticity*:

Definition 2.1.3. We say that a vector field v has *Yudovich vorticity* if $p \mapsto \|\omega(v)\|_{L^p(\Omega)}$ is an admissible function.

For our purposes, we define (weak) solutions to (E) and (NS) as follows:

Definition 2.1.4. Fix an arbitrary m in $\mathbb{R}$. A time-varying vector field $v\colon \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$ is a weak solution to (E) or (NS) if at all times t in $\mathbb{R}^+$ there exists a tempered distribution $p(t)$ such that (E) or (NS) hold in the sense of tempered distributions and if, in addition,

(i) v is in $L^\infty_{loc}(\mathbb{R}^+; E_m)$ for some real m, and

(ii) there exists a $p_0 > 1$ such that $\|\omega(t)\|_{L^p} \le \|\omega^0\|_{L^p}$ for all p in (p_0, ∞) and all $t > 0$.

A weak solution of (E) or (NS) will also lie in $C(\mathbb{R}; E_m)$ (after possibly changing their values on a set of measure zero), but we do not use this fact. Also, because of Lemma 2.5.1, it is possible to show that we could replace the requirement in Definition 2.1.4 that (E) hold in the sense of tempered distributions with the requirement that

$$\int_{\mathbb{R}^2} \varphi(T,x) \cdot v(T,x)\,dx - \int_{\mathbb{R}^2} \varphi(0,x) \cdot v_0\,dx$$
$$= \int_0^T \int_{\mathbb{R}^2} \partial_t \varphi \cdot v + \nabla\varphi \cdot (v \otimes v)\,dx\,dt$$

for all $T > 0$ and all test functions $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ that are smooth, compactly supported, and divergence-free in the space variables. Here, $(v \otimes v)^{ij} = v^i v^j$, and $\nabla\varphi \cdot (v \otimes v) = \partial_i \varphi^j v^i v^j$. A similar equivalence holds for solutions to (NS).

We combine the techniques of Chemin and Yudovich to prove the following theorem:

Theorem 2.1.5. *Fix m in $\mathbb{R}$, let v^0 be in E_m, and let ω^0 be in $L^p(\mathbb{R}^2)$ for all p in $[2,\infty)$, with $\|\omega^0\|_{L^p} \le \theta(p)$ for some admissible function θ. Then:*

(i) There exists a unique solution v of (E).

(ii) For all $\nu > 0$, there exists a unique solution v_ν of (NS).

(iii) $\|v_\nu(t) - v(t)\|_{L^2} \to 0$ in $L^2(\mathbb{R}^2)$ uniformly on $[0,T]$ as $\nu \to 0^+$.

We prove only the uniqueness statements of (i) and (ii), a proof of existence following from the bounds we obtain on the L^2–norm of the difference between two solutions, much as in the proof of Theorem 2.1.1 (see Section 2.6).

Given an initial velocity in E_m, there exists a unique solution in the sense of distributions to (NS) in $C([0,T]; E_m) \cap L^2([0,T]; \dot{H}^1)$ for all $T > 0$. This is essentially a classical result of Leray, Ladyženskaja, Lions, and Prodi ([21], [23], [22], [19], [20], [24]), which can be proved, for instance, by straightforward modifications of the proofs of Theorems 3.1 and 3.2 of Chapter 3 of [33]. (See [18] for details.) Additional assumptions, such as those of Theorem 2.1.5, are required, however, to conclude that the velocity is in $L^\infty([0,T]; \dot{H}^1)$, not just in $L^2([0,T]; \dot{H}^1)$.

The rate of convergence in the inviscid limit is also of interest. Constantin and Wu in [8] show that the L^2–rate of convergence of the velocity for a vortex patch in $\mathbb{R}^2$ with smooth boundary is $O(\sqrt{\nu t})$ uniformly over any finite time interval, and remark that this same result holds when ∇v is in $L^1_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$, where v is the solution to (E). Chemin in [4] gives essentially the same bound on the convergence rate as that in [8], assuming that v is in $L^\infty_{loc}(\mathbb{R}^+; Lip)$, which implies the condition in [8] that ∇v lie in $L^1_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$.

Chemin goes on to establish bounds on the rate of convergence given initial vorticity in $L^2 \cap L^\infty$, the bounded rate of convergence always being slower than $O(\sqrt{\nu})$, but approaching that order for small time intervals. The approach we take leads, in the special case of $L^2 \cap L^\infty$, to the same bound on the rate of convergence as Chemin. In the general case of unbounded vorticity, however, the bounded rate of convergence can be arbitrarily slow.

In [9], Constantin and Wu consider an initial vorticity in $\mathbb{R}^2$ lying in the space $\mathbf{Y}$ of bounded, compactly supported functions. They also assume that the initial vorticity lies in certain Besov spaces, and establish convergence of the *vorticity* in every L^p–norm for $p \geq 2$, with the rate of convergence increasing with increasing p.

We also note that given the uniqueness of the solution to (E) in $\mathbb{R}^2$ established in Theorem 2.1.5, the compactness argument on p. 131-133 of [26] would imply the strong convergence in (iii) of Theorem 2.1.5. A bound on the rate of convergence does not follow from that approach, however.

We will need the following theorem, which summarizes some known facts:

Theorem 2.1.6. *Let v be a solution to (NS) or (E) as defined in Definition 2.1.4, and let σ be any stationary vector field in E_m. Then:*

(i) $v - \sigma$ is in $L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$ (i.e., the L^2–norm of $v - \sigma$ is bounded over any finite time interval), the norm being bounded over $\{\nu > 0\}$;

(ii) v is in $L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$, the norm being bounded over $\{\nu > 0\}$;

(iii) there exists a constant C such that for all $p \geq 2$, $\|\nabla v\|_{L^p} \leq C_0 p \|\omega\|_{L^p}$ when ω is in L^p, where C_0 is an absolute constant.

In Theorem 2.1.6, (i) comes from energy estimates, as does (ii) after decomposing $v - \sigma$ into high and low frequencies (see Lemma 2B.1 and Lemma 2B.2). Equality holds in (iii) for solutions to (E). Statement (iii) is a result from harmonic analysis that applies to all divergence-free vector fields in $\mathbb{R}^n$.

We will also need Osgood's lemma, the proof of which can be found, for example, on p. 92 of [5].

Lemma 2.1.7 (Osgood's lemma). *Let L be a measurable positive function and γ a positive locally integrable function, each defined on the domain $[t_0, t_1]$. Let $\mu\colon [0,\infty) \to [0,\infty)$ be a continuous nondecreasing function, with $\mu(0) = 0$. Let $a \geq 0$, and assume that for all t in $[t_0, t_1]$,*

$$L(t) \leq a + \int_{t_0}^{t} \gamma(s)\mu(L(s))\, ds.$$

If $a > 0$, then

$$-\mathcal{M}(L(t)) + \mathcal{M}(a) \leq \int_{t_0}^{t} \gamma(s)\, ds, \text{ where } \mathcal{M}(x) = \int_x^1 \frac{ds}{\mu(s)}.$$

If $a = 0$ and $\mathcal{M}(0) = \infty$, then $L \equiv 0$.

2.2 Yudovich's unbounded vorticity

Definition 2.1.2 is equivalent to requiring that

$$\psi(x) := \inf\{(x^\epsilon/\epsilon)\theta(1/\epsilon) : \epsilon \in (0, 1/p_0]\} \tag{2.2.1}$$

satisfy

$$\int_1^\infty \frac{dx}{x\psi(x)} = \infty,$$

which is essentially the same as the condition in [37]. The functions ψ and β are related by $\psi(x) = x\beta(1/x)$ when $M = 1$.

Choosing $\epsilon = 1/\log x$ in Equation (2.2.1) gives $\psi(x) \le e(\log x)\theta(\log x)$ when $x \ge \exp(p_0)$. It follows that

$$\int_1^\infty \frac{dx}{x\psi(x)} \ge \int_{e^{p_0}}^\infty \frac{dx}{ex(\log x)\theta(\log x)} = \frac{1}{e}\int_{p_0}^\infty \frac{dp}{p\theta(p)}. \tag{2.2.2}$$

For θ to be admissible, it is sufficient, though not necessary, that the final integral in Equation (2.2.2) be infinite. Thus we can say, as a rough measure only, that the L^p–norm of the initial vorticity can grow in p only slightly faster than $\log p$ and still be handled by our approach. Such growth in the L^p–norm arises, for example, from a point singularity of the type $\log\log(1/x)$.

Define, as in [37], the sequence of admissible bounds on vorticity,

$$\theta_0(p) = 1, \theta_1(p) = \log p, \ldots, \theta_m(p) = \log p \cdot \log\log p \cdots \log^m p, \tag{2.2.3}$$

where $\log^m$ is log composed with itself m times. These are each admissible since $\psi(x) \le e(\log x)\theta_m(\log x) = e\theta_{m+1}(x)$, and a repeated change of variables shows that the final integral in Equation (2.2.2) is infinite for $\theta = \theta_m$.

2.3 Proof of Theorem 2.1.5

We take a unified approach to proving the three parts of Theorem 2.1.5. Let each of $(v_\nu)_{\nu>0}$ and $(v'_\nu)_{\nu>0}$ be either a family of solutions to (NS) parameterized by the viscosity ν or a single solution to (E). In the latter case, the solution is independent of the value of ν. All solutions in $(v_\nu)_{\nu>0}$ and $(v'_\nu)_{\nu>0}$ share the same initial velocity

v^0, which lies in E_m and satisfies the vorticity bounds assumed in Theorem 2.1.5. Let

$$w_\nu = v_\nu - v'_\nu.$$

Theorem 2.3.1. *Under the assumptions of Theorem 2.1.5, for all $t \geq 0$,*

$$\int_{\mathbb{R}^2} |w_\nu(t,x)|^2 \, dx \leq R\nu t + 2\int_0^t \int_{\mathbb{R}^2} \left|\nabla v'_\nu(s,x)\right| |w_\nu(s,x)|^2 \, dx\, ds.$$

$R = 0$ when w_ν is the difference between two solutions to (NS) and when w_ν is the difference between two solutions to (E). $R = C\|\omega^0\|^2_{L^2(\mathbb{R}^2)} > 0$ when w_ν is the difference between a solution to (NS) and a solution to (E).

Proof. See Section 2.5. □

Proof of Theorem 2.1.5. Let

$$M = \sum_{\nu > 0} \| \, |w_\nu|^2 \, \|_{L^\infty(\mathbb{R} \times \mathbb{R}^2)} = \sum_{\nu > 0} \| \, |v_\nu|^2 - 2 v_\nu v'_\nu + \left|v'_\nu\right|^2 \|_{L^\infty(\mathbb{R} \times \mathbb{R}^2)},$$

which is finite by Theorem 2.1.6.(ii). Let s be in $[0, T]$, and

$$A = |w_\nu(s,x)|^2, \quad B = \left|\nabla v'_\nu(s,x)\right|, \quad L_\nu(s) = \|w_\nu\|^2_{L^2}.$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^2} \left|\nabla v'_\nu(s,x)\right| |w_\nu(s,x)|^2 \, dx &= \int_{\mathbb{R}^2} AB = \int_{\mathbb{R}^2} A^\epsilon A^{1-\epsilon} B \leq M^\epsilon \int_{\mathbb{R}^2} A^{1-\epsilon} B \\
&\leq M^\epsilon \left\|A^{1-\epsilon}\right\|_{L^{1/(1-\epsilon)}} \|B\|_{L^{1/\epsilon}} = M^\epsilon \|A\|^{1-\epsilon}_{L^1} \|B\|_{L^{1/\epsilon}} \\
&= M^\epsilon L_\nu(s)^{1-\epsilon} \left\|\nabla v'\right\|^{L^{1/\epsilon}} \leq C M^\epsilon L_\nu(s)^{1-\epsilon} \frac{1}{\epsilon} \|\omega^0\|_{L^{1/\epsilon}} \\
&\leq C_0 M^\epsilon L_\nu(s)^{1-\epsilon} \frac{1}{\epsilon} \theta(1/\epsilon).
\end{aligned}$$

Since this is true for all ϵ in $[1/p_0, \infty)$, it follows that

$$2 \int_{\mathbb{R}^2} \left| \nabla v'_\nu(s,x) \right| |w_\nu(s,x)|^2 \, dx \leq \beta(L_\nu(s))$$

and thus, from Theorem 2.3.1, that

$$L_\nu(t) \leq R\nu t + \int_0^t \beta(L_\nu(s)) \, ds.$$

Since Equation (2.1.3) holds, by Osgood's lemma, when $R = 0$, $L_\nu(t) \equiv 0$, giving (i) and (ii) of Theorem 2.1.5. When $R > 0$,

$$\int_{R\nu t}^{L_\nu(t)} \frac{ds}{\beta(s)} = \left(- \int_{L_\nu(t)}^1 + \int_{R\nu t}^1 \right) \frac{ds}{\beta(s)} \leq \int_0^t ds = t. \tag{2.3.1}$$

It follows that for all t in $(0, T]$,

$$\int_{R\nu t}^1 \frac{ds}{\beta(s)} \leq T + \int_{L_\nu(t)}^1 \frac{ds}{\beta(s)}. \tag{2.3.2}$$

As $\nu \to 0^+$, the left side of Equation (2.3.2) becomes infinite; hence, so must the right side. But this implies that $L_\nu(t) \to 0$ as $\nu \to 0^+$, and that the convergence is uniform over $[0, T]$. □

2.4 Rates of convergence

Define $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ implicitly by

$$\int_x^{f(x)} \frac{ds}{\beta(s)} = T. \tag{2.4.1}$$

As x decreases to zero, $f(x)$ monotonically decreases (to zero) because β is positive. Also, because of Equation (2.3.1),

$$L_\nu(t) \leq f(R\nu t) \leq f(R\nu T), \tag{2.4.2}$$

giving an expression for a uniform bound on the convergence rate. When $1/\beta$ can be explicitly integrated, a bound on the rate can sometimes be determined in closed form. For the case of bounded vorticity, one obtains essentially the same bound on the rate as in [4], as we show below. The sequence of bounds on vorticity in Equation (2.2.3) can also be handled this way, using the upper bound on the corresponding β functions that Yudovich derives in [37]. In the notation of Section 2.2 this is $\beta(x) = x\psi(1/x) \leq ex\theta_{m+1}(1/x)$.

In general, though, one can bound the initial vorticity by an admissible function that will yield an arbitrarily slow bounded rate of convergence. This is because the function f, which was defined implicitly in terms of β, can, conversely, be used to define β, and we can choose f so that it approaches zero arbitrarily slowly.

To derive the rate of convergence for bounded vorticity, let M be defined as in the proof of Theorem 2.1.5 and let $A = \|\omega^0\|_{L^{p_0} \cap L^\infty}$ and . Then

$$\begin{aligned}\beta(x) &= 2C_0 x \inf\left\{(M^\epsilon x^{-\epsilon}/\epsilon)A : \epsilon \text{ in } (0, 1/p_0]\right\} \\ &= 2C_0 A \inf\left\{g(\epsilon) : \epsilon \text{ in } (0, 1/p_0]\right\},\end{aligned}$$

where $g(\epsilon) = M^\epsilon x^{1-\epsilon}/\epsilon$. Then

$$g'(\epsilon) = 2C_0 \frac{M^\epsilon x^{1-\epsilon}(\epsilon \log(M/x) - 1)}{\epsilon^2},$$

which is zero when $\epsilon = \epsilon_0 := 1/(\log(M/x))$ if $x < M$ and $\epsilon_0 < 1/p_0$, and never zero otherwise. But

$$\begin{aligned}\epsilon_0 < 1/p_0 &\iff \frac{1}{\log(M/x)} < \frac{1}{p_0} \iff \log(M/X) > p_0\\ &\iff \frac{M}{x} > e^{p_0} \iff x < Me^{-p_0},\end{aligned}$$

so the condition $x < M$ is redundant.

Assume that $x < Me^{-p_0}$. Then $g(\epsilon)$ approaches infinity as ϵ approaches either zero or infinity; hence, ϵ_0 minimizes g. Thus,

$$\begin{aligned}\beta(x) &= BM^{\epsilon_0}x^{1-\epsilon_0}/\epsilon_0 = Bx(M/x)^{\epsilon_0}\log(M/x)\\ &= Bxe^{\log(M/x)\epsilon_0}\log(M/x) = -Bex\log(x/M),\end{aligned}$$

where $B = 2C_0A$.

From Equation (2.4.1),

$$\begin{aligned}-\int_x^{f(x)} \frac{ds}{Bes\log(s/M)} &= \frac{1}{Be}\int_{f(x)/M}^{x/M} \frac{M\,du}{Mu\log u}\\ &= \frac{1}{Be}\Big[\log\log u\Big]_{f(x)/M}^{x/M} = \frac{1}{Be}\left[\log\log(x/M) - \log\log(f(x)/M)\right]\\ &= \frac{1}{Be}\log\left[\log(x/M)/\log(f(x)/M)\right] = 2C_0T,\end{aligned}$$

so

$$\begin{aligned}\log(x/M)/\log(f(x)/M) = e^{BeT} &\implies \log(f(x)/M) = \log(x/M)e^{-BeT}\\ &\implies f(x) = M\left(\frac{x}{M}\right)^{e^{-BeT}},\end{aligned}$$

It follows from Equation (2.4.2) that

$$L_\nu(t) \le M\left(\frac{R\nu T}{M}\right)^{e^{-BeT}},$$

or,

$$\|v_\nu - v\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \le M^{1/2} \left(\frac{R\nu T}{M}\right)^{\frac{1}{2}e^{-BeT}}, \tag{2.4.3}$$

when $\nu t < (M/R)e^{-p_0}$, where $R = C\|\omega^0\|^2_{L^2(\mathbb{R}^2)}$. This is similar to the bound on the convergence rate in Theorem 1.4 of [4].

As a check on the validity of Equation (2.4.3), M has units of $(\text{velocity})^2$, so $M^{1/2}$ has the correct units of (velocity). The factor $R = C\|\omega^0\|^2_{L^2}$ has units $(\text{vorticity})^2 = 1/(\text{time})^2$, since C is an absolute constant, so the units of $R\nu T$ are $(\text{distance})^2/(\text{time})^2 = (\text{velocity})^2$ (see Section 1.4), and $R\nu T/M$ is unitless, as it must be. Finally, A has units of (vorticity) = 1/(time), so AeT is unitless, as it also must be.

Also, observe from the definition of $\beta(x)$ in Equation (2.1.2), that x must have the same units as M, and that Equation (2.1.2) has the units (units of x)(units of θ). But as applied in Equation (2.4.1), θ has units (vorticity) = 1/(time), and x must have the same units as s in Equation (2.4.1)—namely, $(\text{velocity})^2$. Thus, $\beta(x)$ has units of $(\text{velocity})^2/(\text{time})$, and the integral in the left-hand side of Equation (2.4.1) has units $(\text{velocity})^2/((\text{velocity})^2/(\text{time}))$ = (time), in agreement with its right-hand side. Finally, M, having the same units as x, always has units $(\text{velocity})^2$.

2.5 Proof of Theorem 2.3.1

In this section we establish Theorem 2.3.1, which is a standard energy inequality used by Chemin in [4]. We will need the following lemma, which establishes the membership of the various terms in (E) and (NS) in certain Lebesgue spaces.

Lemma 2.5.1. *Let (v, p) be a solution to (E), and (v_ν, p_ν) be a solution to (NS) in the sense of Definition 2.1.4. Then $v \cdot \nabla v$, $v_\nu \cdot \nabla v_\nu$, ∇p, ∇p_ν, and $\partial_t v$ all lie $L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$, while $\partial_t v_\nu$ and Δv_ν lie in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$.*

Proof. First, if we can prove that $v \cdot \nabla v$ and ∇p both lie in $L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$, then the same follows for $\partial_t v$, since $\partial_t v = -\nabla p - (v \cdot \nabla v)$.

By Lemma 2B.1,

$$\begin{aligned} \|v(t) - \sigma\|_{L^\infty} &\le C(\|v(t) - \sigma\|_{L^2} + \|\omega(v(t) - \sigma)\|_{L^a}) \\ &\le C(\|v\|_{L^\infty_{loc}(\mathbb{R}; E_m)} + \|\omega\|_{L^\infty_{loc}(\mathbb{R}; L^a)} + \|\omega(\sigma)\|_{L^a} < \infty, \end{aligned}$$

so $v - \sigma$ belongs to $L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$. But $v - \sigma$ also belongs to $L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$ since v is in $L^\infty_{loc}(\mathbb{R}; E_m)$. Then if we let $2 < p \le \infty$, then since $\|v - \sigma\|_{L^p} \le \max\{\|v - \sigma\|_{L^2}, \|v - \sigma\|_{L^\infty}\}$ (for instance, see Ex. 4(d) p. 71 of [30]), $v - \sigma$ is in $L^\infty_{loc}(\mathbb{R}; L^p(\mathbb{R}^2))$. (The issue is not just membership of $v(t)$ in $L^p(\mathbb{R}^2)$, but having a uniform bound on the $L^p(\mathbb{R}^2)$-norm over any finite time interval.) But σ is bounded and decays like $1/r$, so σ is in $L^p(\mathbb{R}^2)$ and hence v is in $L^\infty_{loc}(\mathbb{R}; L^p(\mathbb{R}^2))$. This in turn means that v^2 is in $L^\infty_{loc}(\mathbb{R}; L^q(\mathbb{R}^2))$ for all $q > 1$.

Because $a > 2$, $|\nabla v|^2$ is in $L^\infty_{loc}(\mathbb{R}; L^b(\mathbb{R}^2))$ where $b = a/2$, and

$$\begin{aligned} \|v \cdot \nabla v\|_{L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))} &\le \| \, |v|^2 \, |\nabla v|^2 \, \|^{1/2}_{L^\infty_{loc}(\mathbb{R}; L^1(\mathbb{R}^2))} \\ &\le \| \, |v|^2 \, \|^{1/2}_{L^\infty_{loc}(\mathbb{R}; L^{b'}(\mathbb{R}^2))} \| \, |\nabla v|^2 \, \|^{1/2}_{L^\infty_{loc}(\mathbb{R}; L^b(\mathbb{R}^2))} < \infty, \end{aligned}$$

where b' is such that $1/b' + 1/b = 1$. This shows that $v \cdot \nabla v$ lies in $L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$, and the same argument applies for $v_\nu \cdot \nabla v_\nu$.

We now consider ∇p_ν. Using $\operatorname{div} v_\nu = 0$, we have

$$\begin{aligned}
p_\nu = \Delta^{-1}\Delta p_\nu = \Delta^{-1}(\partial_t \operatorname{div} v_\nu + \Delta p_\nu) &= \Delta^{-1}\left(\partial_t \sum_j \partial_j v_\nu^j + \sum_j \partial_j^2 p_\nu\right) \\
&= \sum_j \Delta^{-1}(\partial_j \partial_t v_\nu^j + \partial_j \partial_j p_\nu) = \sum_j \partial_j \Delta^{-1}(\partial_t v_\nu^j + \partial_j p_\nu) \\
&= -\sum_j \partial_j \Delta^{-1}(v_\nu \cdot \nabla v_\nu^j - \nu \Delta v_\nu^j) = -\sum_{j,k} \partial_j \Delta^{-1}(v_\nu^k \partial_k v_\nu^j) + \nu \sum_j \partial_j v_\nu^j, \\
&= -\sum_{j,k} \partial_j \Delta^{-1}(v_\nu^k \partial_k v_\nu^j),
\end{aligned}$$

so

$$\nabla p_\nu = -\nabla \sum_{j,k} \partial_j \Delta^{-1}(v_\nu^k \partial_k v_\nu^j). \tag{2.5.1}$$

We used above the fact that partial derivatives commute with the inverse Laplacian, since they commute with the Laplacian itself.

We can write a term in Equation (2.5.1) as

$$\begin{aligned}
-\nabla \partial_j \Delta^{-1}(v_\nu^k \partial_k v_\nu^j) &= -\nabla \partial_j \mathcal{F}^{-1}\left(-\frac{1}{|\xi|^2}\mathcal{F}(v_\nu^k \partial_k v_\nu^j)\right) \\
&= \nabla \mathcal{F}^{-1}\left(\frac{\xi^k}{|\xi|^2}\mathcal{F}(v_\nu^k \partial_k v_\nu^j)\right) \\
&= \sum_{n=1}^{2} \mathcal{F}^{-1}\left(\frac{\xi^n \xi^k}{|\xi|^2}\mathcal{F}(v_\nu^k \partial_k v_\nu^j)\right),
\end{aligned}$$

where we used the expression for Δ^{-1} (the left inverse of the Laplacian) on p. 14 of [5], and where we ignored possible factors of $i = \sqrt{-1}$.

It follows (for instance, from Comment 4.5 p. 78 of [32]) that $\xi^n \xi^k / |\xi|^2$ is the multiplier for a higher Riesz transform (it is the product of two Riesz multipliers). Therefore, each term in Equation (2.5.1) is in $L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$, so ∇p_ν is in

$L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$. The argument is identical for ∇p, except now the Laplacian term (which was eliminated by our argument) never appears.

Since $\partial_t v = -\nabla p - (v \cdot \nabla v)$, it follows that $\partial_t v$ is also in $L^\infty_{loc}(\mathbb{R}; L^2(\mathbb{R}^2))$. Since v_ν is in $L^\infty_{loc}(\mathbb{R}; H^1(\mathbb{R}^2))$, Δv_ν is in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ as, then, is $\partial_t v_\nu$, since $\partial_t v_\nu = -\nabla p_\nu - (v_\nu \cdot \nabla v_\nu) + \nu \Delta v_\nu$. □

Proof of Theorem 2.3.1. We consider three cases: 1) both v_ν and v'_ν are solutions to (NS); 2) v_ν is a solution to (NS) while v'_ν is a solution to (E); 3) both v_ν and v'_ν are solutions to (E).

Consider case 1. Taking the inner product of both sides of the first equation in (NS) with w_ν and subtracting the resulting equations for v_ν and v'_ν gives

$$\begin{aligned} &w_\nu \cdot \partial_t w_\nu + w_\nu \cdot (v_\nu \cdot \nabla w_\nu) \\ &\qquad = -w_\nu \cdot \nabla(p_\nu - p'_\nu) + \nu w_\nu \cdot \Delta w_\nu - w_\nu \cdot (w_\nu \cdot \nabla v'_\nu). \end{aligned} \tag{2.5.2}$$

Integrating both sides of Equation (2.5.2) over $[0, T] \times \mathbb{R}^2$, the pressure term disappears because w_ν is divergence-free both $w_\nu(t)$ and $\nabla(p_\nu - p'_\nu)(t)$ belong to $L^2(\mathbb{R}^2)$ by Lemma 2.5.1.

Similarly, the term $w_\nu \cdot (v_\nu \cdot \nabla w_\nu)$ disappears because

$$\int_{\mathbb{R}^2} w_\nu \cdot (v_\nu \cdot \nabla w_\nu) = \int_{\mathbb{R}^2} w_\nu \cdot (\sigma \cdot \nabla w_\nu) + \int_{\mathbb{R}^2} w_\nu \cdot (\overline{v}_\nu \cdot \nabla w_\nu),$$

where σ is a stationary solution to (E) with total vorticity equal to that of v_ν and $\overline{v}_\nu$ is in $H^1(\mathbb{R}^2)$. The second term above is zero, since $\overline{v}_\nu$ and w_ν are both in $H^1(\mathbb{R}^2)$, so we can approximate each vector field by smooth functions and integrate by parts (see, for instance, Lemma 1.3 p. 109 of [33]). The first term we can handle similarly, by multiplying σ by a series of cutoff functions supported on larger and larger balls. Each integral is zero by the same argument as above for the second term, and the integral itself is finite by Hölder's inequality; therefore, the integral itself is zero.

It follows that

$$\int_0^T \int_{\mathbb{R}^2} w_\nu \cdot \partial_t w_\nu \, dx \, dt = \int_0^T \int_{\mathbb{R}^2} \nu w_\nu \cdot \Delta w_\nu - w_\nu \cdot (w_\nu \cdot \nabla v'_\nu) \, dx \, dt.$$

But w_ν in $L^\infty_{loc}(\mathbb{R}; H^1(\mathbb{R}^2))$ and $\partial_t w_\nu$ in $L^\infty_{loc}(\mathbb{R}; H^{-1}(\mathbb{R}^2))$ is sufficient to conclude (see, for instance, Lemma 1.2 p. 176 of [33]) that

$$\int_0^T \int_{\mathbb{R}^2} w_\nu \cdot \partial_t w_\nu \, dx \, dt = \frac{1}{2} \|w_\nu(T)\|_{L^2}^2 ,$$

where we have used $w_\nu(0) = 0$.

It follows that

$$\textbf{1:} \quad \|w_\nu(T)\|_{L^2}^2 = 2 \int_0^T \int_{\mathbb{R}^2} \nu w_\nu \cdot \Delta w_\nu - w_\nu \cdot (w_\nu \cdot \nabla v'_\nu) \, dx \, dt. \tag{2.5.3}$$

Because of the absolute continuity of the integral, $\|w_\nu(T)\|_{L^2}^2$ is an absolutely continuous function of T.

Following a similar procedure for the other two cases, we obtain

$$\textbf{2:} \quad \|w_\nu(T)\|_{L^2}^2 = 2 \int_0^T \int_{\mathbb{R}^2} \nu w_\nu \cdot \Delta v_\nu - w_\nu \cdot (w_\nu \cdot \nabla v'_\nu) \, dx \, dt, \tag{2.5.4}$$

$$\textbf{3:} \quad \|w_\nu(T)\|_{L^2}^2 = -2 \int_0^T \int_{\mathbb{R}^2} w_\nu \cdot (w_\nu \cdot \nabla v'_\nu) \, dx \, dt. \tag{2.5.5}$$

For the term common to Equation (2.5.3)-Equation (2.5.5),

$$-2 \int_0^T \int_{\mathbb{R}^2} w_\nu \cdot (w_\nu \cdot \nabla v'_\nu) \, dx \, dt \leq 2 \int_0^T \int_{\mathbb{R}^2} |w_\nu|^2 \left|\nabla v'_\nu\right|^2 \, dx \, dt.$$

Since $w(t)$ is in $H^1(\mathbb{R}^2)$ for all time t,

$$\int_0^T \int_{\mathbb{R}^2} w_\nu \cdot \Delta w_\nu \, dx \, dt = - \int_0^T \int_{\mathbb{R}^2} |\nabla w_\nu|^2 \, dx \, dt \leq 0. \tag{2.5.6}$$

Similarly,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^2} w_\nu \cdot \Delta v_\nu \, dx \, dt &= -\int_0^T \int_{\mathbb{R}^2} \nabla w_\nu \cdot \nabla v_\nu \, dx \, dt \\ &= -\int_0^T \int_{\mathbb{R}^2} |\nabla v_\nu|^2 \, dx \, dt + \int_0^T \int_{\mathbb{R}^2} \nabla v_\nu \cdot \nabla v'_\nu \, dx \, dt \\ &\leq \|\nabla v_\nu\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} \left\|\nabla v'_\nu\right\|_{L^\infty([0,T];L^2(\mathbb{R}^2))} T \\ &\leq C\|\omega^0\|^2_{L^2(\mathbb{R}^2)} T.\end{aligned}$$

Putting this all together gives Theorem 2.3.1 with, for the three cases,

$$\textbf{1:}\quad R = 0, \quad \textbf{2:}\quad R = C\|\omega^0\|^2_{L^2(\mathbb{R}^2)} > 0, \quad \textbf{3:}\quad R = 0.$$

In case 1, we only know that R, which comes from Equation (2.5.6), is negative or equal to 0; we cannot choose, a priori, a specific constant other than 0.

Remark: If v_ν and v'_ν had different initial velocities, then Equation (2.5.3) and Equation (2.5.5) would have the additional term $\|\omega_\nu(0)\|^2_{L^2}$ on the right-hand side. Modifying the argument in Section 2 to incorporate this term is the basis of the proof of existence in Theorem 2.1.5.

2.6 Brief comments on existence

There are at least two approaches to the proof of existence; we briefly comment on these approaches here.

The first approach is that followed by [29] p. 311-319. The key step in their proof of existence is establishing the properties of the flow for smooth solutions to (E) or (NS), then making a limiting argument for a sequence of the flows corresponding to the smooth solutions for a sequence of mollified initial velocities. In the classical case of initial vorticity in $L^1 \cap L^\infty$, the $L^1 \cap L^\infty$–norm is used to show that the sequence of smooth velocities are log-Lipschitz and hence obtain the modulus of continuity for the sequence of smooth flows, the modulus of continuity being essentially $r \mapsto Cr(1 - \log r)$ for $r \le 1$. This same approach works in our setting—for both solutions to (E) and to (NS)—though now using function β of Equation (2.1.2) in place of the $L^1 \cap L^\infty$–norm. This is because the resulting modulus of continuity has all the properties of $r \mapsto r(1 - \log r)$ that are required to establish convergence. This aspect of the argument is given in Section 5.2 through Section 5.3, though our interest there is wholly in characterizing the properties of the flow given that a weak solution is known to exist. (The same argument applies, of course, for smooth solutions.)

An alternate approach is to use an energy inequality (see the first remark following the proof of Theorem 2.3.1 in Section 2.5) to establish the convergence of the same smooth solutions as in the previous paragraph in the $L^\infty_{loc}(\mathbb{R}; E_m)$–norm. This gives (i) of Definition 2.1.4 quite easily. Establishing (ii) is much harder, and would seem to require general results for renormalized solutions to transport equations.

The first approach has the advantage of its relative brevity (when fully elucidated) and its more elementary nature. The merit of the second approach is its focus on the central issue of convergence in $C([0, T]; E_m)$, leaving the bound on the L^p–norms of vorticity as a side issue to be resolved through technical means.

Appendices

2A The space E_m

The most natural function space for a velocity field v that is a solution to (NS) or (E) is probably $L^2_{sol}(\mathbb{R}^2)$, the space of all divergence-free (in the sense of a distribution) vector fields in $L^2(\mathbb{R}^2)$, since this space is exactly the space of solutions of finite energy. (One could argue that also assuming some level of smoothness is more natural or even required.) A deficiency of this space, however, is that, under fairly weak additional assumptions on v in $L^2_{sol}(\mathbb{R}^2)$, it necessarily follows that $\int_{\mathbb{R}^2} \omega(v) = 0$ (see Theorem 2A.5). There are, however, applications in which one wishes to consider nonnegative or nonpositive measures; for instance, when studying vortex patches.

A vortex patch is the solution to (E) in which the initial vorticity ω^0 is the characteristic function of a bounded domain. The velocity v^0 associated to such a vorticity is given by the Biot-Savart law:

$$v^0 = K * \omega^0, \tag{2A.1}$$

where $K(x) = (-x^2, x^1)/\left|x\right|^2$. Since vorticity is transported by the flow for a solution to (E), $\omega(t)$ remains nonnegative for all time, and $v(t)$ is never in $L^2_{sol}(\mathbb{R}^2)$. Thus, we need a larger space than $L^2_{sol}(\mathbb{R}^2)$ when dealing with vortex patches. It turns out, as we will see below, that the velocity associated with a vortex patch is in the space E_m that we briefly introduced in Section 2.1, where m is the total vorticity—that is, $\int_{\mathbb{R}^2} \omega$.

Because the E_m spaces are central to all of our results in the plane, we chose to give a self-contained and careful elucidation of all of their properties that we will use. This appendix can be seen as a fleshing out of the account of these spaces given by Chemin in Chapter 1 of [5].

E_m is defined in Chapter 1 of [5] p. 12 as the space of all divergence-free vector fields that are the sum of a stationary vector field σ of total vorticity m (that is, $\int_{\mathbb{R}^2} \omega(\sigma) = m$) and an L^2 vector field. A stationary vector field is defined in [5] p. 11 as a vector field in the plane of the form

$$\sigma = \left(-\frac{x^2}{r^2} \int_0^r \rho g(\rho)\, d\rho, \frac{x^1}{r^2} \int_0^r \rho g(\rho)\, d\rho \right), \tag{2A.2}$$

where $g \in C_0^\infty(\mathbb{R})$.

Majda and Bertozzi in [29] p. 93 call this way of decomposing a vector in E_m the *radial-energy decomposition*, though they only make the assumption that g is smooth and vanishes at infinity, not that it is compactly supported (see the comment following Corollary 2A.4). We use whichever of Chemin's or Majda and Bertozzi's terminology seems more convenient at the time.

Remark: Equation (2A.2) is what results when one applies the Biot-Savart law to the radially symmetric vorticity g, giving the velocity field σ.

The following lemma is Proposition 1.3.1 p. 9 of [5]:

Lemma 2A.1. *Two vector fields whose coefficients are tempered distributions and whose divergence and vorticity are equal, equal each other up to a vector field with harmonic polynomials as coefficients.*

An immediate corollary of Lemma 2A.1 is the following:

Corollary 2A.2. *Let v_1 and v_2 be two vector fields lying in $L^p + L^q$ for some p, q in $[1, \infty]$ for which $\operatorname{div} v_1 = \operatorname{div} v_2$ and $\omega(v_1) = \omega(v_2)$. Then $v_1 = v_2$.*

Proof. Let v_1 and v_2 be two such vector fields. It follows from Lemma 2A.1 that they differ by a vector field with harmonic polynomials as coefficients. But the only polynomial in $L^p + L^q$ is the zero polynomial; hence, $v_1 = v_2$. □

Remark: It follows, in particular, from Corollary 2A.2 that a divergence-free vector field in E_m is uniquely determined by its vorticity, since $E_m \subseteq L^2 + L^{2+\epsilon}$ for all $\epsilon > 0$.

The following theorem is a combination of Corollary 2A.2 and Proposition 1.3.2 p. 11 of [5]:

Theorem 2A.3. *Let σ be a stationary vector field. Then σ has the following properties:*

1. *σ is smooth;*

2. *$\omega(\sigma)$ is in $C_0^\infty(\mathbb{R}^2)$ and is radially symmetric;*

3. *$\operatorname{div}\sigma = 0$;*

4. *$|\sigma|$ is radially symmetric and for all sufficiently large r,*

$$|\sigma|(r) = \frac{m}{2\pi r},$$

   where $m = \int_{\mathbb{R}^2} \omega(\sigma)$;

5. *σ is a solution to the time-independent Euler equations*

Conversely, properties (1) through (4) are enough to insure that a vector field is a stationary vector field with $g = \omega(\sigma)$; in fact, this still holds if property (4) is weakened to the assumption that the vector field vanishes at infinity. (To be more explicit, if properties (1) through (3) hold and σ vanishes at infinity, then σ is given by Equation (2A.2) with $g = \omega(\sigma)$, and all five properties hold.)

Remark: Let σ be a stationary vector field with total vorticity m. Then because σ is divergence-free by Theorem 2A.3, it follows that if $\sigma + v$ is in E_m, then v is also divergence-free.

Corollary 2A.4. *The space E_m is independent of the particular choice of stationary vector field of the form Equation (2A.2) that is used to define it.*

Proof. If σ and σ' are two stationary vector fields with the same total vorticity m, then it follows from property (4) of Theorem 2A.3 that $\sigma - \sigma'$ is in $L^2(\mathbb{R}^2)$ (in fact, in $C_0^\infty(\mathbb{R}^2)$). Even if the two stationary vector fields do not share the same origin about which their vorticities are circularly symmetric, property (4) insures that their difference decays like $1/r^2$ and so is in $L^2(\mathbb{R}^2)$. Hence, the space E_m does not depend upon the specific choice of the stationary vector field. □

Remark: The independence in Corollary 2A.4 would not follow simply from assuming that the function g vanishes at infinity: compact support of g is stronger than required, but an exact statement of the required decay is not immediately clear.

The following is Lemma 1.3.1 p. 12 of [5]:

Theorem 2A.5. *Let μ be a finite measure such that $(1+|x|)\,|\mu|$ is also finite. If μ is in $H^{-1}(\mathbb{R}^2)$, then there exists a unique divergence-free vector field v in E_m, where*

$$m = \int_{\mathbb{R}^2} d\mu,$$

and such that $\omega(v) = \mu$.

Remark: Given a vector in E_m, all we can immediately say about its vorticity is that it lies in $H^{-1}(\mathbb{R}^2)$ (in particular, it is generically not in any L^p-spaces). Using Theorem 2A.5 we could define a space, or at least a set, E'_m of all vectors in E_m whose vorticity satisfies the conditions of Theorem 2A.5. These sets would have the properties that $E'_m \cap E'_{m'} = \emptyset$ when $m \neq m'$, and that if $v = \sigma + \overline{v}$ is a radial-energy decomposition of v in E'_m, then $\int_{\mathbb{R}^2} \overline{v} = 0$ (only the first property is shared by the spaces E_m).

Remark: Let v be in E_m. If $\omega(v)$ is compactly supported and smooth, then it follows from the Biot-Savart law that v decays like $m/((2\pi)r)$. (For instance, see [29] p. 92.) It follows that such a v will be in $L^2(\mathbb{R}^2)$ if and only if $m = 0$; that is, if and only if its total vorticity is zero. Theorem 2A.5 can be seen as a broad generalization of this result in which the condition of smoothness and compact support of $\omega(v)$ are replaced with much weaker conditions.

The following is Lemma 5.1.2 p. 89 of [5].

Lemma 2A.6. *Let (ρ_n) be an approximation to the identity and σ a stationary vector field in E_m. Then $\sigma - \rho_n * \sigma$ is in L^2 and*

$$\lim_{n\to\infty} \|\sigma - \rho_n * \sigma\|_{L^2} = 0.$$

Proof. Being an approximation to the identity means that $\rho_n(x) = (1+n)^2\rho((1+n)x)$ where ρ is in $\mathcal{S}(\mathbb{R}^2)$ and integrates to 1. We let $\sigma_n = \rho_n * \sigma$.

By the mean value theorem, given $\xi \in \mathbb{R}^2$, there exists an η on the line segment between the origin and $\xi/(n+1)$ (see, for instance, Theorem 8.4 p. 254 of [2]) such that

$$\frac{\widehat{\rho}(\xi/(n+1)) - \widehat{\rho}(0)}{|\xi/(n+1)|} = \widehat{\xi} \cdot \nabla\widehat{\rho}(\eta),$$

where $\widehat{\xi}$ is a unit vector in the direction of ξ. From this it follows that

$$\left|1 - \widehat{\rho}\left(\frac{\xi}{n+1}\right)\right| \leq \frac{|\xi|}{1+n} \|D\widehat{\rho}\|_{L^\infty}.$$

Even though σ is not in L^2, we still have

$$\begin{aligned}
\|\widehat{\sigma} - \widehat{\sigma_n}\|_{L^2} &= \left\|\widehat{\sigma} - \widehat{\rho_n * \sigma}\right\|_{L^2} = \|\widehat{\sigma} - \widehat{\rho_n}\widehat{\sigma}\|_{L^2} = \|\widehat{\sigma}(1-\widehat{\rho_n})\|_{L^2} \\
&= \left\|\widehat{\sigma}(\xi)\left(1 - \widehat{\rho}\left(\frac{\xi}{1+n}\right)\right)\right\|_{L^2} \leq \left\|\widehat{\sigma}(\xi)\frac{|\xi|}{1+n}\|D\widehat{\rho}\|_{L^\infty}\right\|_{L^2} \\
&= \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\xi\widehat{\sigma}(\xi)\|_{L^2} = \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\left\|\widehat{\nabla\sigma}\right\|_{L^2} \\
&= \frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\nabla\sigma\|_{L^2} \leq C\frac{\|D\widehat{\rho}\|_{L^\infty}}{n+1}\|\omega(\sigma)\|_{L^2},
\end{aligned}$$

the last inequality being by Theorem 3.1.1 p. 45 of [5]. Letting $n \to 0$ completes the proof.

□

2B Bounding the L^∞–norm of velocity

Lemma 2B.1. *Let v be a divergence-free vector field in L^2 with vorticity ω lying in L^a for some a and b, $2 < a < b \le \infty$. Then v is in $L^2 \cap L^\infty$ and*

$$\begin{aligned} \|v\|_{L^b} &\le C\left(\|v\|_{L^2} + \frac{a^2}{(a-1)(1-2^{2/a-2/b-1})}\|\omega\|_{L^a}\right) \\ &\le C\left(\|v\|_{L^2} + \frac{a^2}{(a-1)(1-2^{2/a-1})}\|\omega\|_{L^a}\right). \end{aligned} \tag{2B.1}$$

(When $b = \infty$, we let $1/b = 0$.)

Proof. Let a and b be such that $2 < a < b \le \infty$. Then

$$\|v\|_{L^b} \le \|\chi(D)v\|_{L^b} + \|(\mathrm{Id} - \chi(D))v\|_{L^b},$$

where we are using the definitions for the Littlewood-Paley operators and associated functions from [5]. In these definitions, χ is a smooth radially symmetric nonnegative function supported in a ball of radius 4/3.

The first term is bounded by

$$\begin{aligned} \|\chi(D)v\|_{L^b} &\le C\|\chi(D)v\|_{L^2} = C\|\chi\widehat{v}\|_{L^2} \le C\|\chi\|_{L^\infty}\|v\|_{L^2} \\ &\le C\|v\|_{L^2} < \infty, \end{aligned}$$

where we used Bernstein's lemma (Lemma 2.1.1 of [5] with $\lambda = 1$) for the first inequality and the fact that $\chi \in L^\infty$ to absorb its norm into the constant, C. We note that $b > 2$ was needed to apply Bernstein's lemma.

For the second term, we have

$$\|(\mathrm{Id} - \chi(D))v)\|_{L^b} = \left\| \sum_{j=0}^{\infty} \Delta_j v \right\|_{L^b} \leq \sum_{j=0}^{\infty} \|\Delta_j v\|_{L^b} .$$

Applying Bernstein's inequality twice, we have

$$\begin{aligned}
\|\Delta_j v\|_{L^b} &\leq C2^{-j} \|\partial_k \Delta_j v\|_{L^b} \leq C2^{-j} \|\nabla \Delta_j v\|_{L^b} \\
&\leq C2^{-j}(2^j)^{2(1/a-1/b)} \|\nabla \Delta_j v\|_{L^a} \\
&\leq C2^{j(2/a-2/b-1)} \frac{a^2}{a-1} \|\omega(\Delta_j v)\|_{L^a} \\
&= C2^{j(2/a-2/b-1)} \frac{a^2}{a-1} \|\Delta_j \omega\|_{L^a} ,
\end{aligned}$$

where the last inequality is the Calderon-Zygmund inequality. The first application of Bernstein's inequality required that the Fourier transform of $\Delta_j v$ be supported in an annulus; this is why we decomposed v into low and high frequencies before bounding its norm.

In any case, we then have

$$\begin{aligned}
\|\Delta_j \omega\|_{L^a} &= \left\|\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\widehat{\omega})\right\|_{L^a} \\
&= \left\|\mathcal{F}^{-1}\left(\mathcal{F}(\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)))\widehat{\omega}\right)\right\|_{L^a} \\
&= \left\|\mathcal{F}^{-1}\left(\mathcal{F}(\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)) * \omega)\right)\right\|_{L^a} \\
&= \left\|\mathcal{F}^{-1}(\varphi(2^{-j}\cdot)) * \omega\right\|_{L^a} \leq \left\|\mathcal{F}^{-1}(\varphi(2^{-j}\cdot))\right\|_{L^1} \|\omega\|_{L^a} \\
&= \left\|\mathcal{F}^{-1}(\varphi)\right\|_{L^1} \|\omega\|_{L^a} = C\|\omega\|_{L^a} .
\end{aligned}$$

That $\left\|\mathcal{F}^{-1}(\varphi(2^{-j}\cdot))\right\|_{L^1} = \left\|\mathcal{F}^{-1}(\varphi)\right\|_{L^1}$ follows by a change of variables.

Thus,

$$
\begin{aligned}
\|(\mathrm{Id} - \chi(D))v\|_{L^\infty} &\le \sum_{j=0}^{\infty} C 2^{j(2/a-2/b-1)} \frac{a^2}{a-1} \|\omega\|_{L^a} \\
&\le C \frac{a^2}{(a-1)(1-2^{2/a-2/b-1})} \|\omega\|_{L^a}
\end{aligned}
$$

as long as $2/a - 2/b - 1 < 0$—that is, $1/a - 1/b < 1/2$; this follows from $2 < a < b \le \infty$.

Combining these results gives Equation (2B.1). Since v is in L^2 by assumption, and v is in L^∞ (the case $b = \infty$, above), v is in $L^2 \cap L^\infty$, and the proof is complete. □

Lemma 2B.2. *Let u be a solution to (E) or (NS). Fix an origin σ in the affine space E_m. Then*

$$
\|v(t)\|_{E_m} \le \left[\|v(0)\|_{E_m} + \nu C \|\nabla\sigma\|_{L^2} \|\omega^0\|_{L^2} \right] e^{2\|\nabla\sigma\|_{L^\infty} t}, \tag{2B.2}
$$

where, of course, $\nu = 0$ when u is a solution to (E).

Proof. Assume first that u is a solution to (E). Then we can write $u(t) = \sigma + v(t)$ where $v(t)$ is in L^2, and we have

$$
\begin{aligned}
&\partial_t(\sigma + v) + (\sigma + v) \cdot \nabla(\sigma + v) + \nabla p = 0 \\
&\quad \Longrightarrow\ \partial_t v + \sigma \cdot \nabla\sigma + \sigma \cdot \nabla v + v \cdot \nabla\sigma + v \cdot v + \nabla p = 0.
\end{aligned} \tag{2B.3}
$$

But $\sigma \cdot \nabla\sigma = \nabla q$ for some q in $C^\infty \cap H^1$, because σ is a stationary solution to the Euler equations, as we showed in Theorem 2A.3, and because σ is in L^∞ and $\nabla\sigma$ is in L^2. Thus, taking the inner product of the above equality with v and integrating over space, and arguing as in the proof of Theorem 2.3.1, we have

$$\frac{1}{2}\frac{d}{dt}\left\|v\right\|_{L^2}^2 = -\int_{\mathbb{R}^2}(v\cdot\nabla\sigma)\cdot v \le \left\|\nabla\sigma\right\|_{L^\infty}\left\|v\right\|_{L^2}^2,$$

which leads to Equation (2B.2) after integrating.

If u is a solution to (NS), then instead of 0, the right-hand side of Equation (2B.3) becomes

$$\begin{aligned}\nu\int_{\mathbb{R}^2}\Delta(\sigma+v)\cdot v &= -\nu\int_{\mathbb{R}^2}\nabla\sigma\cdot\nabla v - \nu\int_{\mathbb{R}^2}\left|\nabla v\right|^2 \le \nu\left\|\nabla\sigma\right\|_{L^2}\left\|\nabla v\right\|_{L^2}\\ &\le \nu C\left\|\nabla\sigma\right\|_{L^2}\left\|\omega^0\right\|_{L^2}.\end{aligned}$$

Applying Gronwall's lemma gives Equation (2B.2). □

An important, but subtle conclusion of Equation (2B.2) is that our solutions to (NS) and (E) for initial velocity in E_m remain in E_m for all time. This is true even if the initial vorticity is not in L^1. (The condition that $\int_{\mathbb{R}^2}\omega = m$ in the definition of E_m is an integral over a measure in $H^{-1}(\mathbb{R}^2)$ that is assumed to be finite. If ω is in L^1, then it is the same as the integral of ω as an L^1–function.)

For the solution to (NS) to remain in E_m, though, we require that its initial vorticity lie in L^2, because its L^2–norm appears directly in Equation (2B.2). A solution to (E) on the other hand would remain in E_m even if the initial vorticity were only in a higher L^p–space. Thus, the restriction we made in Theorem 2.1.5 that ω^0 lie in $L^p(\mathbb{R}^2)$ for all p in $[2,\infty)$ was not just so that the energy argument in the proof of Theorem 2.3.1 would work (which gave, ultimately, both the uniqueness of the solutions to (E) and (NS), but the vanishing viscosity limit), but because the basic existence proof for (NS) requires it.

Chapter 3

Vanishing viscosity in a bounded domain with Navier boundary conditions

3.1 Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$ with a boundary Γ consisting of a finite number of connected components. We always assume that Γ is at least as smooth as C^2, but will assume additional smoothness as needed.

We consider the existence and uniqueness of a solution u to the Navier-Stokes equations under *Navier boundary conditions*; namely,

$$v \cdot \mathbf{n} = 0 \text{ and } 2D(v)\mathbf{n} \cdot \boldsymbol{\tau} + \alpha v \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma, \tag{3.1.1}$$

where α is in $L^\infty(\Gamma)$, $\mathbf{n}$ and $\boldsymbol{\tau}$ are unit normal and tangent vectors, respectively, to

Γ, and $D(v)$ is the rate-of-strain tensor,

$$D(v) = \frac{1}{2}\left[\nabla v + (\nabla v)^T\right].$$

We follow the convention that $\mathbf{n}$ is an outward normal vector and that the ordered pair $(\mathbf{n}, \boldsymbol{\tau})$ gives the standard orientation to $\mathbb{R}^2$. (We give an equivalent form of Navier boundary conditions in Corollary 3.4.2.)

J.L. Lions in [25] p. 87-98 and P.L. Lions in [26] p. 129-131 consider the following boundary conditions, which we call *Lions* boundary conditions:

$$v \cdot \mathbf{n} = 0 \text{ and } \omega(v) = 0 \text{ on } \Gamma,$$

where $\omega(v) = \partial_1 v^2 - \partial_2 v^1$ is the vorticity of v. Lions boundary conditions are the special case of Navier boundary conditions in which $\alpha = 2\kappa$, as we show in Corollary 3.4.3.

J.L. Lions, in Theorem 6.10 p. 88 of [25], proves existence and uniqueness of a solution to the Navier-Stokes equations in the special case of Lions boundary conditions, but includes the assumption that the initial vorticity is bounded. With the same assumption of bounded initial vorticity, the existence and uniqueness is established in Theorem 4.1 of [7] for Navier boundary conditions, under the restriction that α is positive (and in $C^2(\Gamma)$). This is the usual restriction, which is imposed to insure the conservation of energy. Mathematically, negative values of α present no real difficulty, so we do not make that restriction (until the last section). The only clear gain from removing the restriction, however, is that it allows us to view Lions boundary conditions as a special case of Navier boundary conditions for more than just convex domains (nonnegative curvature).

P.L. Lions establishes an energy inequality on p. 130 of [26] that can be used in place of the usual one for no-slip boundary conditions. He argues that existence and uniqueness can then be established—with no assumption on the initial vorticity—exactly as was done for no-slip boundary conditions in the earlier sections of his text. As we will show, P.L. Lions's energy inequality applies to Navier boundary conditions in general, which gives us the same existence and uniqueness theorem as for no-slip boundary conditions. (P.L. Lions's comment on the regularity of $\frac{\partial u}{\partial t}$ does not follow as in [26], though, because (4.18) of [26] is not valid for general Navier boundary conditions.) We include a proof of existence and uniqueness in Section 3.6, based on the classical proofs as they appear in [25] and [33]. In Section 3.7, we extend the existence, uniqueness, regularity, and convergence results of [7] and [27] to non-simply connected domains.

It is shown in [27] that if the initial vorticity is in $L^p(\Omega)$ for some $p > 2$, then after extracting a subsequence, solutions to the Navier-Stokes equations with Navier boundary conditions converge in $L^\infty([0, T]; L^2(\Omega))$ to a solution to the Euler equations (with the usual boundary condition of tangential velocity on the boundary) as $\nu \to 0$. This extends a result in [7] for initial vorticity in $L^\infty(\Omega)$, and because the solution to the Euler equations is unique in this case, it follows that the convergence is strong in $L^\infty([0, T]; L^2(\Omega))$—that is, does not require the extraction of a subsequence.

The convergence in [27] also generalizes the similar convergence established for the special case of Lions boundary conditions on p. 131 of [26] (though not including the case $p = 2$). The main difficulty faced in making this generalization is establishing a bound on the L^p-norms of the vorticity, a task that is much easier for Lions boundary conditions (see p. 91-92 of [25] or p. 131 of [26]). In contrast, nearly all of [7] and [27], including the structure of the existence proofs, is directed toward establishing an analogous bound.

The methods of proof in [7] and [27] do not yield a bound on the rate of convergence. With the assumptions in [27], such a bound is probably not possible.

We can, however, make an assumption that is weaker than that of [7] but stronger than that of [27] and achieve a bound on the rate of convergence. Specifically, we assume, as in Chapter 2, that the L^p-norms of the initial vorticity grow sufficiently slowly with p (Definition 2.1.3) and establish the bound given in Theorem 3.8.3. To achieve this result, we also assume additional regularity on α and Γ.

The bound on the convergence rate in $L^\infty([0, T]; L^2(\Omega))$ in Theorem 3.8.3 is the same as that obtained for $\Omega = \mathbb{R}^2$ in Chapter 2. In particular, it gives a bound on the rate of convergence for initial vorticity in $L^\infty(\Omega)$ proportional to

$$(\nu t)^{\frac{1}{2} \exp\left(-C \|\omega^0\|_{L^2 \cap L^\infty} t\right)},$$

where C is a constant depending on Ω and α, and ω^0 is the initial vorticity. This is essentially the same bound on the convergence rate as that for $\Omega = \mathbb{R}^2$ appearing in [4].

Another interesting question is whether solutions to the Navier-Stokes equations with Navier boundary conditions converge to a solution to the Navier-Stokes equations with the usual no-slip boundary conditions if we let the function α grow large. We show in Section 3.9 that such convergence does take place for initial velocity in $H^3(\Omega)$ and Γ in C^3 when we let α approach $+\infty$ uniformly on Γ. This type of convergence is, in a sense, an inverse of the derivation of the Navier boundary conditions from no-slip boundary conditions for rough boundaries discussed in [13] and [14].

3.2 Function spaces

Let

$$E(\Omega) = \left\{ v \in (L^2(\Omega))^2 : \operatorname{div} v \in L^2(\Omega) \right\}, \tag{3.2.1}$$

as in [33], with the inner product,

$$(u, v)_{E(\Omega)} = (u, v) + (\operatorname{div} u, \operatorname{div} v).$$

We will use several times the following theorem, which is Theorem 1.2 p. 7 of [33].

Lemma 3.2.1. *There exists a continuous linear operator $\gamma_{\mathbf{n}}$ mapping $E(\Omega)$ into $H^{-1/2}(\Gamma)$ such that*

$$\gamma_{\mathbf{n}} v = \text{ the restriction of } v \cdot \mathbf{n} \text{ to } \Gamma, \text{ for every } v \text{ in } (\mathcal{D}(\overline{\Omega}))^2.$$

Also, the following form of the divergence theorem is true for all vector fields v in $E(\Omega)$ and scalar functions h in $H^1(\Omega)$:

$$\int_\Omega v \cdot \nabla h + \int_\Omega (\operatorname{div} v) h = \int_\Gamma \gamma_{\mathbf{n}} v \cdot \gamma_0 h.$$

We always suppress the trace function γ_0 in our expressions, and we write $v \cdot \mathbf{n}$ in place of $\gamma_{\mathbf{n}} v$.

Define the following function spaces as in [7]:

$$\begin{aligned} H &= \left\{ v \in (L^2(\Omega))^2 : \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}, \\ V &= \left\{ v \in (H^1(\Omega))^2 : \operatorname{div} v = 0 \text{ in } \Omega \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\}, \\ \mathcal{W} &= \left\{ v \in V \cap H^2(\Omega) : v \text{ satisfies } \textit{Equation } (3.1.1) \right\}. \end{aligned} \tag{3.2.2}$$

We give $\mathcal{W}$ the H^2-norm, H the L^2-inner product and norm, which we symbolize by $(\cdot, \cdot)$ and $\|\cdot\|_{L^2(\Omega)}$, and V the H^1-inner product,

$$(u, v)_V = \sum_i (\partial_i u, \partial_i v),$$

and associated norm. This norm is equivalent to the H^1-norm, because Poincaré's inequality holds on V, as we show in Lemma 3.2.2.

Lemma 3.2.2 (Poincaré's inequality). *For all v in V and all p in* $[1, \infty]$,

$$\|v\|_{L^p(\Omega)} \le C(\Omega) \|\nabla v\|_{L^p(\Omega)} .$$

Proof. Let v be in V and let $h = x^i$, $i = 1, 2$, on Ω. By Lemma 3.2.1,

$$0 = \int_\Gamma (v \cdot \mathbf{n}) h = \int_\Omega v \cdot \nabla h + \int_\Omega (\operatorname{div} v) h = \int_\Omega v^i.$$

But it is a classical result that Poincaré's inequality holds for any scalar function in $H^1(\Omega)$ whose average value is zero, so Poincaré's inequality holds for each component of v and hence for v itself. □

Corollary 3.2.3. *For all u and v in V,*

$$\|u \cdot v\|_{L^1(\Gamma)} \le C(\Omega) \|u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} ,$$

and

$$\|v\|_{L^2(\Gamma)} \le C(\Omega) \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2} .$$

Proof. Using Lemma 3.2.1, Lemma 3.2.2, $\operatorname{div} u = 0$, and the standard trace theorem, we have

$$\begin{aligned} \|u \cdot v\|_{L^1(\Gamma)} &\le \|u\|_{H^{-1/2}(\Gamma)} \|v\|_{H^{1/2}(\Gamma)} \le C(\Omega) \|u\|_{E(\Omega)} \|v\|_{H^1(\Omega)} \\ &\le C(\Omega) \|u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} . \end{aligned}$$

Letting $u = v$ gives the second inequality in the statement of the theorem. □

We need, as in the classical case, Ladyzhenskaya's inequality ([20]), although we now have a domain-dependent constant.

Lemma 3.2.4 (Ladyzhenskaya's inequality). *For any v in V,*

$$\|v\|_{L^4(\Omega)} \le C(\Omega) \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2} . \tag{3.2.3}$$

Proof. Let $\mathcal{E}$ be an extension operator from V to $H^1(\mathbb{R}^2)$ with the property that $\|\mathcal{E}v\|_{H^k(\mathbb{R}^2)} \le C \|v\|_{H^k(\Omega)}$, $k = 0, 1$. (See, for instance, Theorem 5 p. 181 of [32].) Then

$$\begin{aligned} \|\nabla(\mathcal{E}v)\|_{L^2(\mathbb{R}^2)} &\le \|\mathcal{E}v\|_{H^1(\mathbb{R}^2)} \le C \|v\|_{H^1(\mathbb{R}^2)} \\ &= C \|v\|_{L^2(\Omega)} + C \|\nabla v\|_{L^2(\Omega)} \le C \|\nabla v\|_{L^2(\Omega)} , \end{aligned}$$

where we used Lemma 3.2.2 in the last inequality.

So for v in V,

$$\begin{aligned} \|v\|_{L^4(\Omega)} &\le \|\mathcal{E}v\|_{L^4(\mathbb{R}^2)} \le 2^{-1/4} \|\mathcal{E}v\|_{L^2(\mathbb{R}^2)}^{1/2} \|\nabla(\mathcal{E}v)\|_{L^2(\mathbb{R}^2)}^{1/2} \\ &\le C \|v\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)}^{1/2} , \end{aligned}$$

where we used Ladyzhenskaya's inequality for vector fields in $H^1(\Omega')$ (for instance, see (2.8) p. 32 of [11]). $\square$

Define, as is customary, the trilinear function,

$$b(u, v, w) = \int_\Omega u^i \partial_i v^j w^j = \int_\Omega (u \cdot \nabla v) \cdot w.$$

The property that $b(u, v, w) = -b(u, w, v)$ for all u, v, and w in V holds as it does classically. This property along with Ladyzhenskaya's inequality are sufficient to establish the following bound on $|b(u, v, w)|$, from Lemma 3.4 p. 198 of [33]:

Lemma 3.2.5. *For all u, v, and w in V,*

$$|b(u,v,w)| \leq C(\Omega) \|u\|_{L^2(\Omega)}^{1/2} \|\nabla u\|_{L^2(\Omega)}^{1/2} \|\nabla v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}^{1/2} \|\nabla w\|_{L^2(\Omega)}^{1/2}.$$

3.3 Hodge decomposition of H

Only simply connected domains are considered in [7] and [27]. To handle non-simply connected domains we will need a portion of the Hodge decomposition of $L^2(\Omega)$. We briefly summarize the pertinent facts, drawing mostly from Appendix I of [33].

Let $\{\Sigma_1, \dots, \Sigma_N\}$ be one-manifolds with boundary that generate the one-dimensional real homology class of Ω relative to its boundary Γ.

We can decompose the space H into two subspaces, $H = H_0 \oplus H_c$, where

$$\begin{aligned} H_0 &= \{v \in H : \text{all internal fluxes are zero}\}, \\ H_c &= \{v \in H : \omega(v) = 0\}. \end{aligned}$$

An internal flux is a value of $\int_{\Sigma_i} v \cdot \mathbf{n}$. Then $H_0 = H_c^\perp$ and there is an orthonormal basis $\nabla q_1, \dots, \nabla q_N$ for $H_c \subseteq C^\infty(\overline{\Omega})$ consisting of the gradients of N harmonic functions, $q_1, \dots, q_N$. (Each q_i is multi-valued in Ω, but ∇q_i is single-valued.)

If v is in V, then v is also in H so there exists a unique v_0 in H_0 and v_c in H_c such that $v = v_0 + v_c$; also, $(v_0, v_c) = 0$. But v_c is in $C^\infty(\overline{\Omega})$ and so in V; hence, v_0 also lies in V. This shows that $V = (V \cap H_0) \oplus H_c$, though this is not an orthogonal decomposition of V.

The following is a result of Yudovich's:

Lemma 3.3.1. *For any p in $[2, \infty)$ and any v in $V \cap H_0$,*

$$\|\nabla v\|_{L^p(\Omega)} \leq C(\Omega) p \|\omega(v)\|_{L^p(\Omega)}.$$

Proof. Let v be in $V \cap H_0$. Since v has no harmonic component, $v = \nabla^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)$ for some stream function ψ, which we can assume vanishes on Γ. Applying Corollary 1 of [35] with the operator $L = \Delta$ and $r = 0$ gives

$$\|\nabla v\|_{L^p(\Omega)} \le \|\psi\|_{H^{2,p}(\Omega)} \le C(\Omega) p \, \|\Delta \psi\|_{L^p(\Omega)} = C(\Omega) p \, \|\omega(v)\|_{L^p(\Omega)} .$$

□

For Ω simply connected, $H = H_0$, and Lemma 3.3.1 applies to all of V.

Corollary 3.3.2. *For any p in $[2, \infty)$ and any v in V,*

$$\|\nabla v\|_{L^p(\Omega)} \le C(\Omega) p \, \|\omega(v)\|_{L^p(\Omega)} + C'(\Omega) \, \|v\|_{L^2(\Omega)} ,$$

the constants $C(\Omega)$ and $C'(\Omega)$ being independent of p.

Proof. Let v be in V with $v = v_0 + v_c$, where v_0 is in $V \cap H_0$ and v_c is in H_c, and assume that ∇v is in $L^p(\Omega)$. Let $v_c = \sum_{i=1}^N c_i \nabla q_i$ and $r = \|v_c\|_{L^2(\Omega)} = (\sum_i c_i^2)^{1/2}$. Then

$$\begin{aligned}
\|\nabla v_c\|_{L^p(\Omega)} &= \sum_{i=1}^N |c_i| \, \|\nabla \nabla q_i\|_{L^p(\Omega)} \le \sum_{i=1}^N r \, |\Omega|^{1/p} \, \|\nabla \nabla q_i\|_{L^\infty(\Omega)} \\
&\le r \max \left\{ 1, |\Omega|^{1/2} \right\} \sum_{i=1}^N \|\nabla \nabla q_i\|_{L^\infty(\Omega)} \le C \, \|v_c\|_{L^2(\Omega)} ,
\end{aligned}$$

where we used the smoothness of ∇q_i. But, $H_0 = H_c^\perp$, so $\|v\|_{L^2(\Omega)} = \|v_0\|_{L^2(\Omega)} + \|v_c\|_{L^2(\Omega)}$ and thus $\|v_c\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}$. Therefore,

$$\begin{aligned}
\|\nabla v\|_{L^p(\Omega)} &\le \|\nabla v_0\|_{L^p(\Omega)} + \|\nabla v_c\|_{L^p(\Omega)} \\
&\le C(\Omega) p \, \|\omega(v)\|_{L^p(\Omega)} + C'(\Omega) \, \|v\|_{L^2(\Omega)}
\end{aligned}$$

by virtue of Lemma 3.3.1. □

3.4 Vorticity on the boundary

If we parameterize each component of Γ by arc length, s, it follows that

$$\frac{\partial \mathbf{n}}{\partial \boldsymbol{\tau}} := \frac{d\mathbf{n}}{ds} = \kappa \boldsymbol{\tau},$$

where κ, the curvature of Γ, is continuous because Γ is C^2.

The second part of the following theorem is Lemma 2.1 of [7].

Lemma 3.4.1. *If v is in $(H^2(\Omega))^2$ with $v \cdot \mathbf{n} = 0$ on Γ, then*

$$\nabla v \mathbf{n} \cdot \boldsymbol{\tau} = \omega(v) - \kappa v \cdot \boldsymbol{\tau}, \tag{3.4.1}$$

and

$$D(v)\mathbf{n} \cdot \boldsymbol{\tau} = \frac{1}{2}\omega(v) - \kappa v \cdot \boldsymbol{\tau}. \tag{3.4.2}$$

Proof. Because $C^\infty(\overline{\Omega}) \cap \{v : v \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$ is dense in $H^2(\Omega) \cap \{v : v \cdot \mathbf{n} = 0 \text{ on } \Gamma\}$ and the trace operator of Lemma 3.2.1 is continuous, it is sufficient to establish Equation (3.4.1) and Equation (3.4.2) for smooth velocity fields on $\overline{\Omega}$.

Because $v \cdot \mathbf{n}$ has a constant value (of zero) along Γ,

$$0 = \frac{\partial}{\partial \boldsymbol{\tau}}(v \cdot \mathbf{n}) = \frac{\partial v}{\partial \boldsymbol{\tau}} \cdot \mathbf{n} + v \cdot \frac{\partial \mathbf{n}}{\partial \boldsymbol{\tau}} = \nabla v \boldsymbol{\tau} \cdot \mathbf{n} + \kappa v \cdot \boldsymbol{\tau}.$$

Letting

$$\mathbf{n} = \begin{pmatrix} n^1 \\ n^2 \end{pmatrix}, \quad \boldsymbol{\tau} = \begin{pmatrix} -n^2 \\ n^1 \end{pmatrix}$$

with $(n^1)^2 + (n^2)^2 = 1$, we have

$$
\begin{aligned}
&\nabla v \mathbf{n} \cdot \boldsymbol{\tau} - \nabla v \boldsymbol{\tau} \cdot \mathbf{n} \\
&\quad = \left(\begin{pmatrix} \partial_1 v^1 & \partial_2 v^1 \\ \partial_1 v^2 & \partial_2 v^2 \end{pmatrix} \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} \right) \cdot \begin{pmatrix} -n^2 \\ n^1 \end{pmatrix} - \left(\begin{pmatrix} \partial_1 v^1 & \partial_2 v^1 \\ \partial_1 v^2 & \partial_2 v^2 \end{pmatrix} \begin{pmatrix} -n^2 \\ n^1 \end{pmatrix} \right) \cdot \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} \\
&\quad = \begin{pmatrix} \partial_1 v^1 n^1 + \partial_2 v^1 n^2 \\ \partial_1 v^2 n^1 + \partial_2 v^2 n^2 \end{pmatrix} \cdot \begin{pmatrix} -n^2 \\ n^1 \end{pmatrix} - \begin{pmatrix} -\partial_1 v^1 n^2 + \partial_2 v^1 n^1 \\ -\partial_1 v^2 n^2 + \partial_2 v^2 n^1 \end{pmatrix} \cdot \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} \\
&\quad = -\partial_1 v^1 n^1 n^2 - \partial_2 v^1 (n^2)^2 + \partial_1 v^2 (n^1)^2 + \partial_2 v^2 n^1 n^2 \\
&\qquad + \partial_1 v^1 n^1 n^2 - \partial_2 v^1 (n^1)^2 + \partial_1 v^2 (n^2)^2 - \partial_2 v^2 n^1 n^2 \\
&\quad = \left[(n^1)^2 + (n^2)^2 \right] \left[\partial_1 v^2 - \partial_2 v^1 \right] = \omega(v).
\end{aligned}
$$

Thus,

$$
\nabla v \mathbf{n} \cdot \boldsymbol{\tau} = \omega(v) + \nabla v \boldsymbol{\tau} \cdot \mathbf{n} = \omega(v) - \kappa v \cdot \boldsymbol{\tau}, \tag{3.4.3}
$$

establishing Equation (3.4.1).

To establish Equation (3.4.2), observe that

$$
\begin{aligned}
D(v)\mathbf{n} \cdot \boldsymbol{\tau} &= \frac{1}{2}\omega(v) - \kappa(v \cdot \boldsymbol{\tau}) = -\frac{1}{2}\omega(v) + \omega(v) - \kappa(v \cdot \boldsymbol{\tau}) \\
\iff D(v)\mathbf{n} \cdot \boldsymbol{\tau} &= -\frac{1}{2}\left[\nabla v \mathbf{n} \cdot \boldsymbol{\tau} - \nabla v \boldsymbol{\tau} \cdot \mathbf{n} \right] + \nabla v \mathbf{n} \cdot \boldsymbol{\tau} \\
\iff D(v)\mathbf{n} \cdot \boldsymbol{\tau} &= \frac{1}{2}\left[\nabla v \mathbf{n} \cdot \boldsymbol{\tau} + \nabla v \boldsymbol{\tau} \cdot \mathbf{n} \right].
\end{aligned}
$$

This last identity can be verified by direct calculation. □

Corollary 3.4.2. *A vector v in $V \cap H^2(\Omega)$ satisfies Navier boundary conditions (that is, lies in $\mathcal{W}$) if and only if*

$$
\omega(v) = (2\kappa - \alpha) v \cdot \boldsymbol{\tau} \text{ and } v \cdot \mathbf{n} = 0 \text{ on } \Gamma. \tag{3.4.4}
$$

Also, for all v in $\mathcal{W}$ and u in V,

$$\nabla v \mathbf{n} \cdot u = (\kappa - \alpha) v \cdot u \text{ on } \Gamma. \tag{3.4.5}$$

Proof. Let v be in $V \cap H^2(\Omega)$. Then from Equation (3.4.2),

$$2D(v)\mathbf{n} \cdot \boldsymbol{\tau} + 2\kappa(v \cdot \boldsymbol{\tau}) = \omega(v). \tag{3.4.6}$$

If v satisfies Navier boundary conditions, then Equation (3.4.4) follows by subtracting $2D(v)\mathbf{n}\cdot\boldsymbol{\tau}+\alpha v\cdot\boldsymbol{\tau} = 0$ from Equation (3.4.6). Conversely, substituting the expression for $\omega(v)$ in Equation (3.4.4) into Equation (3.4.6) gives $2D(v)\mathbf{n} \cdot \boldsymbol{\tau} + \alpha v \cdot \boldsymbol{\tau} = 0$.

If v is in $\mathcal{W}$, then from Equation (3.4.1),

$$\nabla v \mathbf{n} \cdot \boldsymbol{\tau} = \omega(v) - \kappa v \cdot \boldsymbol{\tau} = (2\kappa - \alpha) v \cdot \boldsymbol{\tau} - \kappa v \cdot \boldsymbol{\tau} = (\kappa - \alpha) v \cdot \boldsymbol{\tau},$$

and Equation (3.4.5) follows from this, since u is parallel to τ on Γ. □

Corollary 3.4.3. *For initial velocity in $H^2(\Omega)$, Lions boundary conditions are the special case of Navier boundary conditions where*

$$\alpha = 2\kappa.$$

That is, any solution of (NS) with Navier boundary conditions where $\alpha = 2\kappa$ is also a solution to (NS) with Lions boundary conditions.

3.5 Weak formulations

We give two equivalent formulations of a weak solution to the Navier-Stokes equations with Navier boundary conditions, in analogy with Problems 3.1 and 3.2 p. 190-191 of [33].

For all u in $\mathcal{W}$ and v in V,

$$\begin{aligned}\int_\Omega \Delta u \cdot v &= \int_\Omega (\operatorname{div} \nabla u^i) v^i = \int_\Gamma (\nabla u^i \cdot \mathbf{n}) v^i - \int_\Omega \nabla u^i \cdot \nabla v^i \\ &= \int_\Gamma (\nabla u \mathbf{n}) \cdot v - \int_\Omega \nabla u \cdot \nabla v = \int_\Gamma (\kappa - \alpha) u \cdot v - \int_\Omega \nabla u \cdot \nabla v,\end{aligned} \tag{3.5.1}$$

where we used Equation (3.4.5) of Corollary 3.4.2. This motivates our first formulation of a weak solution.

Definition 3.5.1. Given a viscosity $\nu > 0$ and initial velocity u^0 in H, u in $L^2([0,T]; V)$ is a weak solution to the Navier-Stokes equations (without forcing) if $u(0) = u^0$ and

$$\frac{d}{dt} \int_\Omega u \cdot v + \int_\Omega (u \cdot \nabla u) \cdot v + \nu \int_\Omega \nabla u \cdot \nabla v - \nu \int_\Gamma (\kappa - \alpha) u \cdot v = 0$$

for all v in V. (We will make sense of the initial condition $u(0) = u^0$ as in [33].)

Our formulation of a weak solution is equivalent to that in (2.11) and (2.12) of [7]. This follows from the identity,

$$2 \int_\Omega D(u) \cdot D(v) = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \kappa u \cdot v,$$

which holds for all u and v in V. This identity can be derived from Equation (3.4.3) and Lemma 3.2.1, and the density of $H^2(\Omega) \cap V$ in V.

Our second formulation of a weak solution will be identical to that of Problem 3.2 p. 191 of [33], except that the operator A of [33] will also include the boundary term of Equation (3.5.1). Accordingly, we define A by

$$\langle Au, v \rangle_{V,V'} = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma (\kappa - \alpha) u \cdot v$$

for all u and v in V.

By Corollary 3.2.3,

$$
\begin{aligned}
|\langle Au, v\rangle_{V,V'}| &\le \left|\int_\Omega \nabla u\cdot\nabla v\right| + \left|\int_\Gamma (\kappa-\alpha)u\cdot v\right| \\
&\le \|u\|_V \|v\|_V + C\|u\cdot v\|_{L^1(\Gamma)} \\
&\le \|u\|_V \|v\|_V + C\|u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \le C\|u\|_V \|v\|_V .
\end{aligned}
\tag{3.5.2}
$$

Thus, if u is in $L^2([0,T];V)$, then Au is in $L^2([0,T];V')$: this is the fact we need to argue as in [33] p. 191 that the following formulation of a weak solution is equivalent to that of Definition 3.5.1:

Definition 3.5.2. Given a viscosity $\nu > 0$ and initial velocity u^0 in H, u in $L^2([0,T];V)$ is a weak solution to the Navier-Stokes equations if $u(0) = u^0$ and

$$
\begin{cases}
u' \in L^1([0,T];V'), \\
u' + \nu Au + Bu = 0 \text{ on } (0,T), \\
u(0) = u^0,
\end{cases}
$$

where $u' := \partial_t u$.

From here on we will refer to either of the formulations in Definitions 3.5.1 and 3.5.2 as (NS).

If a is the symmetric bilinear form on $V\times V$ defined by $a[u,v] = \langle Au, v\rangle_{V,V'}$, then by Equation (3.5.2), $|a[u,v]| \le C\|u\|_V \|v\|_V$. Also, a is (V,H)-coercive, since

$$
\begin{aligned}
|a[u,u]| &\ge \|\nabla u\|^2_{L^2(\Omega)} - \|\kappa-\alpha\|_{L^\infty(\Gamma)} \|u\|^2_{L^2(\Gamma)} \\
&\ge \|\nabla u\|^2_{L^2(\Omega)} - C\|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\
&\ge \|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} - C\|u\|^2_{L^2(\Omega)} \\
&= \frac{1}{2}\|u\|^2_V - C\|u\|^2_H ,
\end{aligned}
$$

where we used Corollary 3.2.3 and Young's inequality. This is enough to insure the existence of an orthonormal basis for V consisting of eigenvectors of A, with eigenvalues $\lambda_1 \le \lambda_2 \le \dots$, where $\lambda_j \to \infty$ as $j \to \infty$.

3.6 Existence and uniqueness

Our proof of the existence of a solution to (NS) proceeds as in the first proof of existence in [25] p. 75-77, though using the analog of the energy inequality on p. 130 of [26]. The proofs of regularity in time and space and of uniqueness proceed as in the proof of Theorem 3.2 p. 199 of [33].

Theorem 3.6.1. *Assume that Γ is C^2 and α is in $L^\infty(\Gamma)$. Let u^0 be in H and let $T > 0$. Then there exists a solution u to (NS). Moreover, u is in $L^2([0,T];V) \cap C([0,T];H)$, u' is in $L^2([0,T];V')$, and we have the energy inequality,*

$$\|u(t)\|_{L^2(\Omega)} \le e^{C(\alpha)\nu t}\|u^0\|_{L^2(\Omega)}, \tag{3.6.1}$$

where the constant $C(\alpha) = 0$ if α is nonnegative on Γ.

Proof. **Existence**: We follow [25], but use the basis of Corollary 3A.3. Because this basis is also a basis for H, if we let u^{0m} be the projection in H of u^0 onto the span of the first m basis vectors, then $u^{0m} \to u^0$ in $L^2(\Omega)$. Because the basis is in $H^2(\Omega)$, the approximate solution u_m is in $C^1([0,T];H^2(\Omega))$.

Definition 3.5.1 leads to the following replacement for (3.27) p. 193 of [33]:

$$(u_m'(t), u_m(t)) + \nu \|\nabla u_m(t)\|^2_{L^2(\Omega)} = \nu \int_\Gamma (\kappa - \alpha) u_m \cdot u_m.$$

Using Equation (3.4.5) of Corollary 3.4.2 and Lemma 1.2 p. 176 of [33], we conclude that

$$\frac{1}{2}\frac{d}{dt}\|u_m\|^2_{L^2(\Omega)} + \nu \|\nabla u_m\|^2_{L^2(\Omega)} \le C\nu \|u_m\|^2_{L^2(\Gamma)}, \tag{3.6.2}$$

where $C = \sup_\Gamma |\kappa - \alpha|$. Except for the value of the constant, Equation (3.6.2) is identical to the first inequality on p. 130 of [26], which is for the special case of Lions boundary conditions.

Arguing exactly as in [26], it follows that

$$\frac{d}{dt} \|u_m\|_{L^2(\Omega)}^2 + \nu \|\nabla u_m\|_{L^2(\Omega)}^2 \leq C\nu \|u_m\|_{L^2(\Omega)}^2 .$$

Integrating over time gives

$$\begin{aligned} \|u_m(t)\|_{L^2(\Omega)}^2 &+ \nu \int_0^t \|\nabla u_m(s)\|_{L^2(\Omega)}^2 \, ds \\ &\leq \|u^{0m}\|_{L^2(\Omega)}^2 + C\nu \int_0^t \|u_m(s)\|_{L^2(\Omega)}^2 \, ds. \end{aligned} \tag{3.6.3}$$

The energy bound,

$$\|u_m(t)\|_{L^2(\Omega)}^2 \leq e^{C\nu t} \|u^{0m}\|_{L^2(\Omega)}^2 \leq e^{C\nu t} \|u^0\|_{L^2(\Omega)}^2, \tag{3.6.4}$$

then follows from Gronwall's lemma, showing that the right side of Equation (3.6.3) is bounded uniformly on $[0, T]$. It follows from Equation (3.6.3) and Equation (3.6.4) that

$$\{u_m\} \text{ is bounded in } L^2([0,T]; V) \cap L^\infty([0,T]; H),$$

from which Equation (3.6.1) will follow. (If α is nonnegative, then, in fact, energy is conserved—in the absence of forcing—so $C(\alpha) = 0$. This follows from the equation preceding (2.16) of [7].)

The proof of existence is completed by showing that $\{u_m'\}$ is bounded in $L^2([0,T]; V')$; this is done as in [25] without change. Also, we make sense of the initial conditions as in [33]. (Specifically, see Lemma 1.2 of Section III.1.4 and the argument at the end of section III.1.3 of [33].)

Regularity in time: This follows exactly as in part (i) of the proof of Theorem 3.2 p. 199 of [33].

Uniqueness: We argue as in part (ii) of the proof of Theorem 3.2 p. 199 of [33], but using our first formulation of a weak solution rather than the second.

Assume that u_1 and u_2 are two solutions to (NS), and let $u = u_1 - u_2$. Using the test function $v = u$ in Definition 3.5.1 applied to u_1 and u_2, and subtracting the two equalities gives

$$\begin{aligned}\frac{d}{dt} \|u(t)\|^2_{L^2(\Omega)} &+ 2\nu \|\nabla u(t)\|^2_{L^2(\Omega)} \\ &= -2b(u(t), u_2(t), u(t)) + \nu \int_\Gamma (\kappa - \alpha) |u|^2 .\end{aligned}$$

Using Corollary 3.2.3,

$$\begin{aligned}\left| \nu \int_\Gamma (\kappa - \alpha) |u|^2 \right| &\leq C\nu \|u(t)\|_{L^2(\Omega)} \|\nabla u(t)\|_{L^2(\Omega)} \\ &\leq C\nu \|u(t)\|^2_{L^2(\Omega)} + \nu \|\nabla u(t)\|^2_{L^2(\Omega)} ,\end{aligned}$$

and from Lemma 3.2.5 (note the typographical error in the corresponding inequality in [33]),

$$\begin{aligned}|2b(u(t), u_2(t), u(t))| &\leq C' \|u(t)\|_{L^2(\Omega)} \|\nabla u(t)\|_{L^2(\Omega)} \|\nabla u_2(t)\|_{L^2(\Omega)} \\ &\leq \nu \|\nabla u(t)\|^2_{L^2(\Omega)} + \frac{C'}{\nu} \|u(t)\|^2_{L_2(\Omega)} \|\nabla u_2(t)\|^2_{L^2(\Omega)} .\end{aligned}$$

From these last three relations, we have

$$\begin{aligned}\frac{d}{dt} \|u(t)\|^2_{L^2(\Omega)} &\leq C\nu \|u(t)\|^2_{L^2(\Omega)} + \frac{C'}{\nu} \|u(t)\|^2_{L_2(\Omega)} \|\nabla u_2(t)\|^2_{L^2(\Omega)} \\ &= \left[C\nu + \frac{C'}{\nu} \|\nabla u_2(t)\|^2_{L^2(\Omega)} \right] \|u(t)\|^2_{L_2(\Omega)} ,\end{aligned}$$

so

$$\frac{d}{dt}\left[\exp\left(-\int_0^t \left[C\nu + \frac{C'}{\nu}\left\|\nabla u_2(s)\right\|_{L^2(\Omega)}^2\right] ds\right)\left\|u(t)\right\|_{L^2(\Omega)}^2\right] \le 0. \tag{3.6.5}$$

The integral in this expression is finite because u_2 is in $L^2([0,T];V)$ by the existence portion of this theorem. Since $u(0) = 0$, we conclude that $\|u(t)\|_{L^2(\Omega)}^2 \le 0$ for all t in $[0,T]$, so $u_1 = u_2$ and the solution is unique. □

3.7 Additional regularity

In this section we establish an existence theorem suited to addressing the issue of convergence of a solution to (NS) to a solution to the Euler equations, where we always impose stronger regularity on the initial velocity.

If we assume extra regularity on the initial velocity, that regularity will be maintained for all time. Our proof of this is an adaptation of the proof of Theorem 3.5 p. 202-204 of [33] to establish the regularity of u', combined with the second half of the proof of Theorem 2.3 of [7] to establish the regularity of u.

Definition 3.7.1. A vector field v in $\mathcal{W}$ is called *compatible* if $\omega(v)$ is in $L^\infty(\Omega)$.

Definition 3.7.1 is as in [27], except that we define the vector field to be compatible instead of the vorticity.

Theorem 3.7.2. *Assume that Ω is a bounded domain with a $C^{2,1/2+\epsilon}$ boundary Γ and that α is in $H^{1/2+\epsilon}(\Gamma) + C^{1/2+\epsilon}(\Gamma)$ for some $\epsilon > 0$. Let u^0 be in $\mathcal{W}$ with initial vorticity ω^0, and let u be the unique solution to (NS) given by Theorem 3.6.1 with corresponding vorticity ω. Let $T > 0$. Then*

$$u' \in L^2([0,T];V) \cap C([0,T];H).$$

If, in addition, ω^0 is in $L^\infty(\Omega)$ (so u^0 is compatible), then

$$u \in C([0,T]; H^2(\Omega)),\ \omega \in C([0,T]; H^1(\Omega)) \cap L^\infty([0,T] \times \overline{\Omega}).$$

Proof. We prove the regularity of u' in three steps as in the proof of Theorem 3.5 p. 202-204 of [33]. The only change in step (i) is that we use the basis of Corollary 3A.3 rather than the basis in [33].

No change to step (ii) is required, because (3.88) of [33] still holds.

In step (iii), an additional term of

$$\nu \int_\Gamma (\kappa - \alpha) |u_m'|^2$$

appears on the right side of (3.94) of Temam's proof, which we bound by

$$C\nu \left\| u_m' \right\|_{L^2(\Omega)} \left\| \nabla u_m' \right\|_{L^2(\Omega)} \le \frac{\nu}{2} \left\| \nabla u_m' \right\|_{L^2(\Omega)}^2 + C\nu \left\| u_m' \right\|_{L^2(\Omega)}^2 .$$

Then (3.95) of Temam's proof becomes

$$\frac{d}{dt} \left\| u_m'(t) \right\|_{L^2(\Omega)}^2 \le \phi_m(t) \left\| u_m'(t) \right\|_{L^2(\Omega)}^2 ,$$

where

$$\phi_m(t) = \left(\frac{2}{\nu} + C\nu \right) \left\| u_m(t) \right\|_{L^2(\Omega)}^2 ,$$

and the proof of the regularity of u' is completed as in [33], along with the observation in [7] that u' is then in $C([0,T]; H)$.

To prove the regularity of u and ω, we follow the argument in the second half of the proof of Theorem 2.3 in [7] (which does not rely on α being nonnegative). We must, however, impose additional regularity on Γ and on α over that assumed in Theorem 3.6.1. This is to insure that u lying in $C^{1/2}([0,T]; (H^1(\Omega))^2)$ implies that

$(\kappa - \alpha/2) u \cdot \boldsymbol{\tau}$ lies in $C^{1/2}([0,T]; H^1(\Omega))$. Our conditions on Γ and α are sufficient, though not necessary (see, for instance, Theorem 1.4.1.1 p. 21 and Theorem 1.4.4.2 p. 28 of [12]).

Then, after it is shown that u is in $C([0,T]; (H^{2,q}(\Omega))^2)$, we know by Sobolev embedding that u is in $C([0,T] \times \Omega)$. Thus,

$$\|u \cdot \nabla u(t)\|_H \leq \|u\|_{L^\infty([0,T]\times\Omega)} \|u(t)\|_V ,$$

and since we already have u in $C([0,T]; V)$, it follows that $u \cdot \nabla u$ and also Φ are in $C([0,T]; H)$. Then curl Φ is in $C([0,T]; H^{-1}(\Omega))$, and another pass through the argument in [7], this time with $q = 2$, gives u in $C([0,T]; (H^2(\Omega))^2)$. Because the increase in regularity of the solution arises from the equation $-\Delta\psi = w$ with the boundary condition $\psi = 0$, no regularity on Γ or on α beyond that we have assumed is required.

(The argument in [7] is for a simply connected domain. We can easily adapt it, though, by using the equivalent of Lemma 2.5 p. 26 of [33], which gives a stream function ψ in $C([0,T] \times \overline{\Omega})$ that is constant on each boundary component, which is good enough to apply Grisvard's result (Theorem 2.5.1.1 p. 128 of [12]) to conclude that ψ is in $C([0,T]; H^{3,q}(\Omega))$.) □

With Theorem 3.7.2, we have a replacement for Theorem 2.3 of [7] that applies regardless of the sign of α. Since the nonnegativity of α is used nowhere else in [7] and [27], all the results of both of those papers apply for simply connected domains as well regardless of the sign of α, but with the extra regularity assumed on Γ (and the lower regularity assumed on α).

To remove the restriction on the domain being simply connected, it remains only to show that Lemmas 3.2 and 4.1 of [27] remain valid for non-simply connected domains. We show this for Lemma 3.2 of [27] in Theorem 3A.2. As for Lemma

4.1 of [27], we need only use Corollary 3.3.2 to replace the term $\|\omega(\cdot,t)\|_{L^p(\Omega)}^{1-\theta}$ with $(\|\omega(\cdot,t)\|_{L^p(\Omega)} + \|u(\cdot,t)\|_{L^2(\Omega)})^{1-\theta}$ in the proof of Lemma 4.1 in [27]. Lemma 4.1 of [27] then follows with no other changes in the proof—only the value of the constant C changes.

Fix $q > 2$ and suppose that ω^0 is in $L^p(\Omega)$ for some $p \geq q$. The argument in the proof of Lemma 4.1 of [27] can be used to bound $\Lambda = \|(2\kappa - \alpha)u \cdot \boldsymbol{\tau}\|_{L^\infty(\Omega)}$ in terms of $\|\omega^0\|_{L^q(\Omega)}$, and this in turn gives a bound on $\|\omega(t)\|_{L^p(\Omega)}$ that has no direct dependence on p. This gives a bound on $\|\omega(t)\|_{L^p(\Omega)}$ very similar to that for the Euler equations. The result is Theorem 3.7.3, which is only a slight modification of Proposition 5.2 of [27].

Theorem 3.7.3. *Assume that Ω and α are as in Theorem 3.7.2. Let q be in $(2,\infty]$, and assume that u^0 is in V with initial vorticity ω^0 in $L^p(\Omega)$ for some p in $[q,\infty]$. Let $T > 0$. Then there exists a unique solution u to (NS) with corresponding vorticity ω, and for all p in $[q,\infty]$,*

$$\|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} + C_0 \tag{3.7.1}$$

for almost all t in $[0,T]$, where

$$C_0 = C(T,\alpha,\kappa,q)e^{C(\alpha)\nu T} \max\{|\Omega|^{1/2}, 1\} \left(\|u^0\|_{L^2(\Omega)} + \|\omega^0\|_{L^q(\Omega)}\right),$$

C_0 being independent of p.

Also, u is in $L^\infty([0,T]; C(\overline{\Omega})) \cap L^\infty([0,T]; V)$, the norm of u in this space being bounded over any finite range of viscosity ν.

Proof. Approximate u^0 by a sequence of compatible vector fields via Theorem 3A.2, and let u_n be the corresponding solutions to (NS) given by Theorem 3.7.2. The bound in Equation (3.7.1) holds for each u_n via the minor modification of Lemma 4.1 of [27] described above, and holds for the solution u in the limit, as in the proof of Proposition 5.2 in [27]. (The constant $C(T, \alpha, \kappa, q)$ approaches infinity as q approaches 2, so it is not possible to extend this result to $p = 2$.)

Finally, using Sobolev interpolation and Corollary 3.3.2,

$$\begin{aligned}\|u(t)\|_{C(\overline{\Omega})} &\le C \|u(t)\|_{L^2(\Omega)}^{\theta} \|u(t)\|_{H^{1,p}(\Omega)}^{1-\theta} \\ &\le C \|u(t)\|_{L^2(\Omega)}^{\theta} (\|\omega(t)\|_{L^p(\Omega)} + \|u(t)\|_{L^2(\Omega)})^{1-\theta},\end{aligned} \tag{3.7.2}$$

where $\theta = (p-2)/(2p-2)$. This norm is finite by Equation (3.6.1), so u is also in $L^\infty([0,T]; C(\overline{\Omega}))$ and its norm is uniformly bounded over any finite range of viscosity, as is its norm in $L^\infty([0,T]; V)$. Explicitly,

$$\begin{aligned}\|u\|_{L^\infty([0,T];V)} = \|\nabla u\|_{L^\infty([0,T];L^2(\Omega))} &\le C \|\nabla u\|_{L^\infty([0,T];L^q(\Omega))} \\ &\le C(\|\omega\|_{L^\infty([0,T];L^q(\Omega))} + \|u\|_{L^\infty([0,T];L^2(\Omega))}) \\ &\le C(T, \alpha, \kappa) e^{C(\alpha)\nu T},\end{aligned} \tag{3.7.3}$$

a bound we will use in Section 3.8. In the second inequality above we used Corollary 3.3.2. □

It is in the proof of Theorem 4.1 of [7] (which is extended in Proposition 5.2 of [27], upon which Theorem 3.7.3 is based) where a marked departure is made from the classical approach. The approach in [7] relies upon Lemma 4.2 of [7] (whose extension is Lemma 3.2 of [27]), which has no classical analog (fundamentally, because the classical space V is not dense in our space V). Also, there is no known classical analog to Equation (3.7.1), making impossible the compactness argument in the proofs in [7] and [27] in the classical case.

3.8 Vanishing viscosity

We showed in Chapter 2 that having Yudovich initial vorticity (see Definition 2.1.3), we can derive a bound on the rate of convergence in $L^\infty([0,T]; L^2(\mathbb{R}^2))$ of solutions to the Navier-Stokes equations in all of $\mathbb{R}^2$ to the unique solution to the Euler equations. In this section we extend this result to bounded domains when the Navier-Stokes equations have Navier boundary conditions.

Definition 3.8.1. Given an initial velocity u^0 in V, u in $L^2([0,T];V)$ is a weak solution to the Euler equations if $u(0) = u^0$ and

$$\frac{d}{dt}\int_\Omega u\cdot v + \int_\Omega (u\cdot\nabla u)\cdot v = 0$$

for all v in V.

The existence of a weak solution to the Euler equations under the assumption that the initial vorticity ω^0 is in $L^p(\Omega)$ for some $p > 1$ (a weaker assumption than that of Definition 3.8.1 when $1 < p < 2$) was proved in [36]. These solutions have the property that $\omega(u)$ is in $L^\infty_{loc}(\mathbb{R}; L^p(\Omega))$. It is shown in [37] that Yudovich initial vorticity is enough to insure uniqueness of solutions for which $\omega(u)$ and $\partial_t u$ are in $L^\infty_{loc}(\mathbb{R}; L^p(\Omega))$ for all p in $[1,\infty)$. (Yudovich's uniqueness result in [37] applies to a bounded domain in $\mathbb{R}^d$, although existence is not known for $d > 2$. His approach works, with only very minor changes, when applied to all of $\mathbb{R}^d$; the particular case of $d = 2$ we proved in Chapter 2.)

Lemma 3.8.2. *Assume that Ω and α are as in Theorem 3.7.2. Let u^0 be in V with initial vorticity ω^0 in $L^p(\Omega)$ for some p in $(2,\infty]$, and let u be the unique solution to (NS) given by Theorem 3.7.3. Let $T > 0$ and v be in $L^2([0,T];V)$. Then for all t in $(0,T)$,*

$$\int_\Omega \partial_t u\cdot v + \int_\Omega (u\cdot\nabla u)\cdot v + \nu\int_\Omega \nabla u\cdot\nabla v - \nu\int_\Gamma (\kappa-\alpha)u\cdot v = 0. \qquad (3.8.1)$$

Similarly, if $\overline{u}$ is a solution to the Euler equations, then

$$\int_\Omega \partial_t \overline{u} \cdot v + \int_\Omega (\overline{u} \cdot \nabla \overline{u}) \cdot v = 0. \tag{3.8.2}$$

Proof. Since v is in $L^2([0,T]; V)$, we can write

$$v = \sum_{k=1}^{\infty} g_k(t) w_k,$$

where w_k is an orthonormal basis of V and $\{g_k\}$ is in $L^2([0,T]; l^2)$. Then by Definition 3.5.1,

$$\int_\Omega \partial_t u \cdot w_k + \int_\Omega (u \cdot \nabla u) \cdot w_k + \nu \int_\Omega \nabla u \cdot \nabla w_k - \nu \int_\Gamma (\kappa - \alpha) u \cdot w_k = 0.$$

Multiplying the above by $g_k(t)$ and summing gives

$$\int_\Omega \partial_t u \cdot v_m + \int_\Omega (u \cdot \nabla u) \cdot v_m + \nu \int_\Omega \nabla u \cdot \nabla v_m - \nu \int_\Gamma (\kappa - \alpha) u \cdot v_m = 0,$$

where

$$v_m = \sum_{k=1}^{m} g_k(t) w_k.$$

But $v_m \to v$ in $L^2([0,T]; V)$, from which Equation (3.8.1) follows (using Corollary 3.2.3 for the boundary integral). The equality in Equation (3.8.2) follows similarly. □

Theorem 3.8.3. *Assume that Ω and α are as in Theorem 3.7.2. Fix $T > 0$, let u^0 be in V, and assume that ω^0 is in $L^p(\mathbb{R}^2)$ for all p in $[2, \infty)$, with $\|\omega^0\|_{L^p} \le \theta(p)$ for some admissible function θ. Let $\{u_\nu\}_{\nu > 0}$ be the solutions to (NS) given by Theorem 3.7.3 and $\overline{u}$ be the unique weak solution to the Euler equations for which*

$\omega(\overline{u})$ and $\partial_t \overline{u}$ are in $L^\infty_{loc}(\mathbb{R}; L^p(\Omega))$, $\overline{u}$ and each u_ν having initial velocity u^0. Then

$$u_\nu(t) \to \overline{u}(t) \text{ in } L^\infty([0,T]; L^2(\Omega) \cap L^2(\Gamma)) \text{ as } \nu \to 0.$$

Also, there exists a constant $R = C(T, \alpha, \kappa)$, such that if we define the function $f : [0, \infty) \to [0, \infty)$ by

$$\int_{R\nu}^{f(\nu)} \frac{dr}{\beta(r)} = cT,$$

where $c > 1$ and β is defined as in Equation (2.1.2), then

$$\begin{aligned} \|u_\nu - \overline{u}\|_{L^\infty([0,T];L^2(\Omega))} &\le f(\nu)^{1/2} \text{ and} \\ \|u_\nu - \overline{u}\|_{L^\infty([0,T];L^2(\Gamma))} &\le C'(T,\alpha,\kappa) f(\nu)^{1/4} \end{aligned} \tag{3.8.3}$$

for all ν in $(0, 1]$.

Proof. We let

$$w = u_\nu - \overline{u},$$

apply both Equation (3.8.1) and Equation (3.8.2) of Lemma 3.8.2 with $v = w$, and subtract to obtain

$$\begin{aligned} \int_\Omega w \cdot \partial_t w + \int_\Omega w \cdot (u_\nu \cdot \nabla w) + \int_\Omega w \cdot (w \cdot \nabla \overline{u}) \\ = \nu \int_\Gamma (\kappa - \alpha) u_\nu \cdot w - \nu \int_\Omega \nabla u_\nu \cdot \nabla w. \end{aligned} \tag{3.8.4}$$

Both $\partial_t u_\nu$ and $\partial_t \overline{u}$ are in $L^2([0,T]; V')$, so (see, for instance, Lemma 1.2 p. 176 of [33]),

$$\int_\Omega w \cdot \partial_t w = \frac{1}{2} \frac{d}{dt} \|w\|^2_{L^2(\Omega)}.$$

Applying Lemma 3.2.1,

$$\begin{aligned}
&\int_\Omega w \cdot (u_\nu \cdot \nabla w) \\
&\quad = \int_\Omega w^i u_\nu^j \partial_j w^i = \frac{1}{2} \int_\Omega u_\nu^j \partial_j \sum_i (w^i)^2 = \frac{1}{2} \int_\Omega u_\nu \cdot \nabla \left| w \right|^2 \\
&\quad = \frac{1}{2} \int_\Gamma (u_\nu \cdot \mathbf{n}) \left| w \right|^2 - \frac{1}{2} \int_\Omega (\operatorname{div} u_\nu) \left| w \right|^2 = 0,
\end{aligned}$$

since $u_\nu \cdot \mathbf{n} = 0$ on Γ and $\operatorname{div} u_\nu = 0$ on Ω. Thus, integrating Equation (3.8.4) over time,

$$\|w(t)\|_{L^2(\Omega)}^2 \le A + 2 \int_0^t \int_\Omega \left| w \right|^2 \left| \nabla \overline{u} \right|, \tag{3.8.5}$$

where

$$A = 2\nu \int_0^t \left[\int_\Gamma (\kappa - \alpha) u_\nu \cdot w - \int_\Omega \nabla u_\nu \cdot \nabla w \right].$$

Using Corollary 3.2.3, Equation (3.7.3), Equation (3.6.1), and the conservation of energy for $\overline{u}$, we have

$$\begin{aligned}
\left| \int_\Gamma (\kappa - \alpha) u_\nu \cdot w \right| &\le \|\kappa - \alpha\|_{L^\infty(\Gamma)} \|u_\nu \cdot w\|_{L^1(\Gamma)} \\
&\le \|\kappa - \alpha\|_{L^\infty(\Gamma)} \|\nabla u_\nu\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \le C(T, \alpha, \kappa) e^{C(\alpha)\nu T}.
\end{aligned} \tag{3.8.6}$$

By Equation (3.7.3) we also have

$$\left| \int_\Omega \nabla u_\nu \cdot \nabla w \right| \le \|\nabla u_\nu\|_{L^2(\Omega)} \|\nabla w\|_{L^2(\Omega)} \le C(T, \alpha, \kappa) e^{C(\alpha)\nu T}, \tag{3.8.7}$$

so $A \le C(T, \alpha, \kappa) e^{C(\alpha)\nu T} \nu$.

By Theorem 3.7.3, $\|u_\nu\|_{L^\infty([0,T]\times\Omega)} \le C$ for all ν in $(0, 1]$. It is also true that

$\overline{u}$ is in $L^\infty([0,T]\times\Omega)$ (arguing, for instance, exactly as in Equation (3.7.2)). Thus,

$$M = \sup_{\nu\in(0,1]} \|\,|w|^2\,\|_{L^\infty([0,T]\times\Omega)}$$

is finite.

Also, because vorticity is conserved for $\overline{u}$, we have, by Corollary 3.3.2,

$$2\,\|\nabla\overline{u}(t)\|_{L^p(\Omega)} \le Cp\|\omega^0\|_{L^p(\Omega)} + C\,\|\overline{u}\|_{L^2(\Omega)} \le Cp[\theta(p)+1/p] \tag{3.8.8}$$

for all $p \ge 2$. Because θ is admissible, so is $p \mapsto C[\theta(p)+1/p]$, and its associated β function—call it $\overline{\beta}$—is bounded by a constant multiple of that associated to θ. That is, $\overline{\beta} \le c\beta$, where $c = C(\|\overline{u}\|_{L^2(\Omega)}) > 1$. Then, arguing as in Chapter 2, we have

$$2\int_\Omega |w|^2\,|\nabla\overline{u}| \le \overline{\beta}(\|w\|_{L^2}^2) \le c\beta(\|w\|_{L^2}^2).$$

Letting $L(t) = \|w(t)\|_{L^2(\Omega)}^2$, we have

$$L(t) \le A + c\int_0^t \beta(L(r))\,dr. \tag{3.8.9}$$

Using Osgood's lemma as in Chapter 2, we conclude that

$$\int_A^{L(t)} \frac{dr}{\beta(r)} \le ct, \tag{3.8.10}$$

and that as $\nu \to 0$, $A \to 0$, and $L(t) \to 0$ uniformly over any finite time interval. The rate of convergence given in $L^\infty([0,T];L^2(\Omega))$ in Equation (3.8.3) can be derived from Equation (3.8.10) precisely as in Chapter 2.

By Corollary 3.2.3,

$$\begin{aligned}\|u_\nu - \overline{u}\|_{L^2(\Gamma)} = \|w\|_{L^2(\Gamma)} &\le C \|\nabla w\|_{L^2(\Omega)}^{1/2} \|w\|_{L^2(\Omega)}^{1/2} \\ &\le C(T, \alpha, \kappa) e^{C(\alpha)\nu T} L(t)^{1/4},\end{aligned}$$

from which the convergence rate for $L^\infty([0, T]; L^2(\Gamma))$ in Equation (3.8.3) follows. □

The convergence rate in $L^\infty([0, T]; L^2(\Omega))$ established in Theorem 3.8.3 is the same as that established for the entire plane in Chapter 2, except for the presence of the constant c and the value of the constant R, which now increases more rapidly with time.

3.9 No-slip boundary conditions

As long as α is non-vanishing, we can reexpress the Navier boundary conditions in Equation (3.1.1) as

$$v \cdot \mathbf{n} = 0 \text{ and } 2\gamma D(v)\mathbf{n} \cdot \boldsymbol{\tau} + v \cdot \boldsymbol{\tau} = 0 \text{ on } \Gamma, \tag{3.9.1}$$

where $\gamma = 1/\alpha$. When γ is identically zero, we have the usual no-slip boundary conditions. An obvious question to ask is whether it is possible to arrange for γ to approach zero in such a manner that the corresponding solutions to the Navier-Stokes equations with Navier boundary conditions approach the solution to the Navier-Stokes equations with the usual no-slip boundary conditions in $L^\infty([0, T]; L^2(\Omega))$.

Let u^0 be an initial velocity in V, and assume that $\gamma > 0$ lies in $L^\infty(\Gamma)$. Fix a $\nu > 0$ and let

$$\begin{aligned} u_{\nu,\gamma} &= \text{the unique solution to the Navier-Stokes equations} \\ & \text{with Navier boundary conditions for } \alpha = 1/\gamma \text{ and} \\ \widetilde{u}_\nu &= \text{the unique solution to the Navier-Stokes equations} \\ & \text{with no-slip boundary conditions,} \end{aligned}$$

in each case with the same initial velocity u^0. (In Theorem 3.8.3 we wrote $u_{\nu,\gamma}$ as u_ν.)

If we let γ approach 0 uniformly on the boundary, we automatically have some control over $u_{\nu,\gamma}$ on the boundary.

Lemma 3.9.1. *For sufficiently small* $\|\gamma\|_{L^\infty(\Gamma)}$,

$$\|u_{\nu,\gamma}\|_{L^2([0,T];L^2(\Gamma))} \le \frac{\|u^0\|_{L^2(\Omega)}}{\sqrt{\nu}} \|\gamma\|_{L^\infty(\Gamma)}^{1/2}. \tag{3.9.2}$$

Proof. Assume that $\|\gamma\|_{L^\infty(\Gamma)}$ is sufficiently small that $\alpha > \kappa$ on Γ. Then, as in the proof of Theorem 3.6.1, we have

$$\frac{1}{2}\frac{d}{dt}\|u_{\nu,\gamma}(t)\|_{L^2(\Omega)}^2 + \nu\|\nabla u_{\nu,\gamma}(t)\|_{L^2(\Omega)}^2 = \nu\int_\Gamma (\kappa - \alpha)u_{\nu,\gamma}\cdot u_{\nu,\gamma},$$

so,

$$\|u_{\nu,\gamma}(t)\|_{L^2(\Omega)}^2 \le \|u^0\|_{L^2(\Omega)}^2 + 2\nu\int_0^t\int_\Gamma (\kappa - \alpha)u_{\nu,\gamma}\cdot u_{\nu,\gamma}.$$

But,

$$\int_\Gamma (\kappa - \alpha)u_{\nu,\gamma}\cdot u_{\nu,\gamma} \le -\inf_\Gamma\{\alpha - \kappa\}\|u_{\nu,\gamma}(t)\|_{L^2(\Gamma)}^2,$$

so

$$\|u_{\nu,\gamma}(t)\|^2_{L^2(\Omega)} \le \|u^0\|^2_{L^2(\Omega)} - 2\nu \inf_\Gamma \{\alpha - \kappa\} \|u_{\nu,\gamma}\|^2_{L^2([0,t];L^2(\Gamma))}$$

and

$$\|u_{\nu,\gamma}\|^2_{L^2([0,t];L^2(\Gamma))} \le \|u^0\|^2_{L^2(\Omega)}/(2\nu \inf_\Gamma \{\alpha - \kappa\}).$$

Since $\|\gamma\|_{L^\infty(\Gamma)} \inf_\Gamma \{\alpha - \kappa\} \to 1$ as $\|\gamma\|_{L^\infty(\Gamma)} \to 0$, Equation (3.9.2) follows. □

If we assume enough smoothness of the initial data and of Γ, we can use Equation (3.9.2) to establish convergence of $u_{\nu,\gamma}$ to $\widetilde{u}_\nu$ as $\|\gamma\|_{L^\infty(\Gamma)} \to 0$.

Theorem 3.9.2. *Fix $T > 0$, assume that u^0 is in $V \cap H^3(\Omega)$ with $u^0 = 0$ on Γ, and assume that Γ is C^3. Then for any fixed $\nu > 0$,*

$$u_{\nu,\gamma} \to \widetilde{u}_\nu \text{ in } L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; L^2(\Gamma)) \tag{3.9.3}$$

as $\gamma \to 0$ in $L^\infty(\Gamma)$.

Proof. First, $u_{\nu,\gamma}$ exists and is unique by Theorem 3.6.1; the existence and uniqueness of $\widetilde{u}_\nu$ is a classical result. Because u^0 is in $H^3(\Omega)$ and Γ is C^3, $\widetilde{u}_\nu$ is in $L^\infty([0,T]; H^3(\Omega))$ by the argument on p. 205 of [33] following the proof of Theorem 3.6 of [33]. Hence, $\nabla\widetilde{u}_\nu$ is in $L^\infty([0,T]; H^2(\Omega))$ and so in $L^\infty([0,T]; C(\Omega))$.

By the classical analog of Lemma 3.8.2 with $w = u_{\nu,\gamma} - \widetilde{u}_\nu$ in place of v,

$$\begin{aligned}
&\int_\Omega \partial_t \widetilde{u}_\nu \cdot w + \int_\Omega (\widetilde{u}_\nu \cdot \nabla \widetilde{u}_\nu) \cdot w - \nu \int_\Omega \Delta \widetilde{u}_\nu \cdot w \\
&\quad = \int_\Omega \partial_t \widetilde{u}_\nu \cdot w + \int_\Omega (\widetilde{u}_\nu \cdot \nabla \widetilde{u}_\nu) \cdot w + \nu \int_\Omega \nabla \widetilde{u}_\nu \cdot \nabla w - \nu \int_\Gamma (\nabla \widetilde{u}_\nu \mathbf{n}) \cdot w = 0.
\end{aligned}$$

Subtracting Equation (3.8.1) with w in place of v, we obtain

$$\begin{aligned}\int_\Omega \partial_t w \cdot w + \int_\Omega w \cdot (u_{\nu,\gamma} \cdot \nabla w) + \int_\Omega w \cdot (w \cdot \nabla \widetilde{u}_\nu) + \int_\Omega \nabla w \cdot \nabla w \\ - \nu \int_\Gamma (\kappa - \alpha) u_{\nu,\gamma} \cdot w + \nu \int_\Gamma (\nabla \widetilde{u}_\nu \mathbf{n}) \cdot w = 0.\end{aligned}$$

But $\widetilde{u}_\nu = 0$ on Γ so $w = u_{\nu,\gamma}$ on Γ, and

$$\begin{aligned}\int_\Omega \partial_t w \cdot w + \int_\Omega w \cdot (w \cdot \nabla \widetilde{u}_\nu) + \int_\Omega |\nabla w|^2 + \nu \int_\Gamma (\alpha - \kappa) |u_{\nu,\gamma}|^2 \\ + \nu \int_\Gamma (\nabla \widetilde{u}_\nu \mathbf{n}) \cdot u_{\nu,\gamma} = 0.\end{aligned}$$

Then, for $\|\gamma\|_{L^\infty(\Gamma)}$ sufficiently small that $\alpha = 1/\gamma > \kappa$ on Γ,

$$\|w(t)\|_{L^2(\Omega)}^2 \le A + 2 \int_0^t \int_\Omega |w|^2 |\nabla \widetilde{u}_\nu|, \tag{3.9.4}$$

where

$$A = -2\nu \int_0^t \int_\Gamma (\nabla \widetilde{u}_\nu \mathbf{n}) \cdot u_{\nu,\gamma}.$$

By Equation (3.4.1), $(\nabla \widetilde{u}_\nu \mathbf{n}) \cdot \boldsymbol{\tau} = \omega(\widetilde{u}_\nu) - \kappa \widetilde{u}_\nu \cdot \boldsymbol{\tau} = \omega(\widetilde{u}_\nu)$ on Γ. But $u_{\nu,\gamma}$ is parallel to $\boldsymbol{\tau}$ on Γ, so $(\nabla \widetilde{u}_\nu \mathbf{n}) \cdot u_{\nu,\gamma} = \omega(\widetilde{u}_\nu) u_{\nu,\gamma} \cdot \boldsymbol{\tau}$. Thus,

$$\begin{aligned}-\int_\Gamma (\nabla \widetilde{u}_\nu \mathbf{n}) \cdot u_{\nu,\gamma} = -\int_\Gamma \omega(\widetilde{u}_\nu) u_{\nu,\gamma} \cdot \boldsymbol{\tau} \le \|\omega(\widetilde{u}_\nu)\|_{L^2(\Gamma)} \|u_{\nu,\gamma} \cdot \boldsymbol{\tau}\|_{L^2(\Gamma)} \\ \le C \|\widetilde{u}_\nu\|_{H^2(\Omega)} \|u_{\nu,\gamma} \cdot \boldsymbol{\tau}\|_{L^2(\Gamma)},\end{aligned}$$

so

$$A \le C\nu \|\widetilde{u}_\nu\|_{L^2([0,T];H^2(\Omega))} \|u_{\nu,\gamma}\|_{L^2([0,T];L^2(\Gamma))}.$$

By Theorem 3.10 p. 213 of [33], $\|\widetilde{u}_\nu\|_{L^2([0,T];H^2(\Omega))}$ is finite (though the bound on it in [33] increases to infinity as ν goes to 0), so by Lemma 3.9.1,

$$A \le C_1(\nu)\,\|\gamma\|_{L^\infty(\Gamma)}^{1/2}. \tag{3.9.5}$$

Because $\nabla\widetilde{u}_\nu$ is in $L^\infty([0,T];C(\overline{\Omega}))$,

$$\int_0^t\int_\Omega |w|^2\,|\nabla\widetilde{u}_\nu| \le C_2(\nu)\int_0^t \|w(s)\|_{L^2(\Omega)}^2\,ds,$$

where $C_2(\nu) = \|\nabla\widetilde{u}_\nu\|_{L^\infty([0,T]\times\Omega)}$, and Equation (3.9.4) becomes

$$\|w(t)\|_{L^2(\Omega)}^2 \le C_1(\nu)\,\|\gamma\|_{L^\infty(\Gamma)}^{1/2} + C_2(\nu)\int_0^t \|w(s)\|_{L^2(\Omega)}^2\,ds.$$

By Gronwall's Lemma,

$$\|w(t)\|_{L^2(\Omega)}^2 \le C_1(\nu)\,\|\gamma\|_{L^\infty(\Gamma)}^{1/2}\,e^{C_2(\nu)t},$$

and the convergence in $L^\infty([0,T];L^2(\Omega))$ follows immediately. Convergence, then, in $L^2([0,T];L^2(\Gamma))$ follows directly from Lemma 3.9.1, since $\widetilde{u}_\nu = 0$ on Γ. $\square$

We cannot prove convergence in $L^\infty([0,T];L^2(\Gamma))$ as we did in Theorem 3.8.3, because we do not have a bound on the vorticity of $u_{\nu,\gamma}$ that is uniform over sufficiently small values of $\|\gamma\|_{L^\infty(\Gamma)}$. But if we did have such a bound, we could also establish convergence in $L^\infty([0,T];L^2(\Omega)\cap L^2(\Gamma))$ when u^0 in $V\cap H^2(\Omega)$ has Yudovich initial vorticity by combining the approaches in the proofs of Theorem 3.8.3 and Theorem 3.9.2.

Appendices

3A Compatible Sequences

For p in $(1, \infty)$, define the spaces

$$X_0^p = H_0 \cap H^{1,p}(\Omega) \text{ and } X^p = H \cap H^{1,p}(\Omega) = X_0^p \oplus H_c, \qquad (3A.1)$$

each with the $H^{1,p}(\Omega)$-norm.

Lemma 3A.1. *Let p be in $(1, \infty]$. For $p < 2$ let $\widehat{p} = p/(2-p)$, for $p > 2$ let $\widehat{p} = \infty$, and for $p = 2$ let $\widehat{p}$ be any value in $[2, \infty]$. Then for any v in X_0^p,*

$$\|v\|_{L^{\widehat{p}}(\Gamma)} \le C(p) \|\omega(v)\|_{L^p(\Omega)}.$$

Proof. For $p < 2$ and any v in X_0^p, we have

$$\begin{aligned} \|v\|_{L^{\widehat{p}}(\Gamma)} &\le C(p) \|v\|_{L^p(\Omega)}^{1-\lambda} \|\nabla v\|_{L^p(\Omega)}^{\lambda} \le C(p) \|\nabla v\|_{L^p(\Omega)} \\ &\le C(p) \|\omega(v)\|_{L^p(\Omega)}, \end{aligned}$$

where $\lambda = 2(\widehat{p} - p)/(p(\widehat{p} - 1)) = 1$ if $p < 2$ and $\lambda = 2/p$ if $p \ge 2$. The first inequality follows from Theorem 3.1 p. 42 of [11], the second follows from Lemma 3.2.2, and the third from Lemma 3.3.1. □

Given a vorticity ω in $L^p(\Omega)$ with p in $(1, \infty)$, the Biot-Savart law gives a vector field v in H whose vorticity is ω. (That v is in $L^2(\Omega)$ follows as in the proof of Lemma 3A.1, Ω being bounded.) Let $v = v_0 + v_c$, where v_0 is in H_0 and v_c is in H_c. Then $\omega(v_0) = \omega$ as well, so we can define a function K_Ω: $L^p(\Omega) \to H_0$ by $\omega \mapsto v_0$ having the property that $\omega(K_\Omega(\omega)) = \omega$. By Lemma 3.2.2 and Lemma 3.3.1, v_0 is also in $H^{1,p}(\Omega)$, so in fact, K_Ω: $L^p(\Omega) \to X_0^p$ and is the inverse of the function ω. It is continuous by the same two lemmas.

Theorem 3A.2. *Assume that Γ is C^2 and α is in $L^\infty(\Gamma)$. Let $\overline{v}$ be in X^p for some p in $(1, \infty)$ and have vorticity $\overline{\omega}$. Then there exists a sequence $\{v_i\}$ of compatible vector fields (Definition 3.7.1) whose vorticities converge strongly to $\overline{\omega}$ in $L^p(\Omega)$. The vector fields $\{v_i\}$ converge strongly to $\overline{v}$ in X^p and, if $p \geq 2$, also in V.*

Proof. We adapt the proof of Lemma 3.2 of [27]. Suppose that $\overline{v} = \overline{v}_0 + \overline{v}_c$ with $\overline{v}_0 \in X_0^p$ and $\overline{v}_c$ in H_c. Define β as in Equation (4) of [27], but let $v = K_\Omega[\beta] + \overline{v}_c$ and start the iteration with $\omega_1 = \overline{\omega}$. Then the fixed point argument goes through unchanged because $v_1 - v_2$ is in X_0^p and we can apply Lemma 3A.1. The only further change is the estimate on $\|G^n\|_{L^{\hat{p}}(\Gamma)}$, which becomes

$$\begin{aligned}\|G^n\|_{L^{\hat{p}}(\Gamma)} &\leq \|2\kappa - \alpha\|_{L^\infty} \|K_\Omega[\omega_n] + \overline{v}_c\|_{L^{\hat{p}}(\Gamma)} \\ &\leq C_p(\|\omega\|_{L^p(\Omega)} + \|\overline{v}_c\|_{L^{\hat{p}}(\Gamma)}) + \frac{1}{2}\|G^n\|_{L^{\hat{p}}(\Gamma)},\end{aligned}$$

for n sufficiently large, which is still sufficient to imply the required bound that insures convergence of ω_n to $\overline{\omega}$ in $L^p(\Omega)$.

Letting $v_n = K_\Omega[\omega_n] + \overline{v}_c$, we have

$$\begin{aligned}\|\nabla\overline{v} - \nabla v_n\|_{L^p(\Omega)} &= \|\nabla\overline{v}_0 + \nabla\overline{v}_c - (\nabla K_\Omega[\omega_n] + \nabla\overline{v}_c)\|_{L^p(\Omega)} \\ &= \|\nabla(\overline{v}_0 - K_\Omega[\omega_n])\|_{L^p(\Omega)} \\ &\leq Cp\,\|\omega(\overline{v}_0 - K_\Omega[\omega_n])\|_{L^p(\Omega)} = Cp\,\|\overline{\omega} - \omega_n\|_{L^p(\Omega)},\end{aligned}$$

where we used Lemma 3.3.1. Then by Lemma 3.2.2, v_n converges strongly to $\overline{v}$ in X^p as well. Convergence in V for $p \geq 2$ follows since Ω is bounded. □

We only require Theorem 3A.2 for $p \geq 2$. We include all the cases, however, for the same reason as in [27]: in the hope that if the vorticity bound in Lemma 4.1 of [27] can be extended to p in $(1, 2)$, then the convergence in Proposition 5.2 of [27] can also be extended (for non-simply connected Ω).

Corollary 3A.3. *Assume that Γ is C^2, and α is in $L^\infty(\Gamma)$. Then there exists a basis for V lying in $\mathcal{W}$ that is also a basis for H.*

Proof. The space $V = (V \cap H_0) \oplus H_c$ is separable because $V \cap H_0$ is the image under the continuous function K_Ω of the separable space $L^2(\Omega)$ and H_c is finite-dimensional. Let $\{v_i\}_{i=1}^\infty$ be a dense subset of V. Applying Theorem 3A.2 to each v_i and unioning all the sequences, we obtain a countable subset $\{u_i\}_{i=1}^\infty$ of $\mathcal{W}$ that is dense in V. Selecting a maximal independent set gives us a basis for V and for H as well, since V is dense in H. $\square$

Chapter 4

Vanishing viscosity for radially symmetric initial vorticity in $\mathbb{R}^2$

4.1 Introduction

In this chapter we investigate the vanishing viscosity limit for radially symmetric initial vorticity. We will obtain a bound on the convergence rate of order $(\nu t)^\alpha$, where α in $(0,1]$ reflects an appropriate measure of smoothness of the initial vorticity. We will then show that this same convergence rate holds for a finite superposition of such radial vorticities (eddies), as long as each eddy is compactly supported with total vorticity zero, and is a finite distance from the other eddies.

Because we will be dealing with the Navier-Stokes equations, the Euler equations, as well as the heat equations in this chapter, we will depart from the notation used in the other chapters. We will use v^N, v^E, and v^H for solutions to the Navier-

Stokes, Euler, and heat equations, respectively:

$$(NS) \quad \begin{cases} \partial_t v^N + v^N \cdot \nabla v^N - \nu \Delta v^N = -\nabla p^N \\ \operatorname{div} v^N = 0 \\ v^N|_{t=0} = v_0, \end{cases}$$

$$(E) \quad \begin{cases} \partial_t v^E + v^E \cdot \nabla v^E = -\nabla p^E \\ \operatorname{div} v^E = 0 \\ v^E|_{t=0} = v_0, \end{cases}$$

$$(H) \quad \begin{cases} \partial_t v^H - \nu \Delta v^H = 0 \\ v^H|_{t=0} = v_0. \end{cases}$$

We let $\omega^N = \omega(v^N)$, $\omega^E = \omega(v^E)$, and $\omega^H = \omega(v^H)$ be the corresponding vorticities. We will view the velocities v^N and v^E as being the solutions to (NS) and (E), ignoring the pressures p^N and p^E, which will make no direct appearance in this chapter.

The solution to (H) is given by $v^H = p_{\nu t} * v_0$ or, in vorticity form, by $\omega^H = p_{\nu t} * \omega_0$, where p_λ is the heat kernel,

$$p_\lambda(x) = \frac{1}{4\pi\lambda} e^{-|x|^2/4\lambda}. \tag{4.1.1}$$

Throughout this chapter we assume that the initial velocity lies in E_m for some m. As we observed in Chapter 2 (shortly after the statement of Theorem 2.1.5), this by itself is enough to insure that there exists a unique solution to (NS). More is required to ensure the uniqueness of a solution to (E) (this was, in part, the topic of Chapter 2); however, for radially symmetric initial vorticity, existence of a solution to (E) is assured, since $v^E(t) = v^0$ is a stationary solution.

4.2 Bounds on convergence rate for circular symmetry

We start first with a simple, but important, case.

Theorem 4.2.1. *Assume that ω_0 is radially symmetric and that v_0 is in E_m with ω_0 in L^2. Let v^N be the unique solution to (NS). Then v^N is also the solution to the homogeneous heat equation, and*

$$\|v^N(t) - v^E\|_{L^2} = \|v^H(t) - v^E\|_{L^2} \le C\|\omega_0\|_{L^2}\sqrt{\nu t}. \tag{4.2.1}$$

Proof. The velocity field, $v^E(t) = v_0$ is the steady state solution to (E) (for the same reason that σ of Appendix 2A is a steady state solution). Also, $\omega^N = \omega^H = p_{\nu t} * \omega_0$ is the solution to the homogeneous heat equation, as can be seen by using the vorticity formulation of (NS) and observing that $v^N \cdot \nabla \omega^N \equiv 0$.

Then $v^N = K * \omega^N$, where K is the Biot-Savart kernel, so $v^N = K * p_{\nu t} * \omega_0 = p_{\nu t} * K * \omega_0 = p_{\nu t} * v_0$.

Then,

$$\begin{aligned}
\partial_t \|v^N - v^E\|_{L^2}^2 &= \int \partial_t (v^N - v^E) \cdot (v^N - v^E) \\
&= \int \partial_t v^N \cdot (v^N - v^E) = \nu \int \Delta v^N \cdot (v^N - v^E) \\
&= -\nu \int \nabla v^N \cdot (\nabla v^N - \nabla v^E) \le \nu \int \nabla v^N \cdot \nabla v^E \\
&\le \nu \left\|\nabla v^N\right\|_{L^2} \left\|\nabla v^E\right\|_{L^2} = \nu \left\|p_{\nu t} * \nabla v_0\right\|_{L^2} \left\|\nabla v_0\right\|_{L^2} \\
&\le \nu \left\|p_{\nu t}\right\|_{L^1} \left\|\nabla v_0\right\|_{L^2} \left\|\nabla v_0\right\|_{L^2} = \nu \left\|\nabla v_0\right\|_{L^2}^2 \le C\nu \left\|\omega_0\right\|^2.
\end{aligned} \tag{4.2.2}$$

We justify the first equality as in the proof of Theorem 2.3.1: $\partial_t(v^N - v^E) = \partial_t v^N = \nu \Delta v^N$ is in $L^\infty(\mathbb{R}^+; H^{-1})$ and v^N is in $L^\infty(\mathbb{R}^+; H^1)$, so we can apply Lemma 1.2

p. 176 of [33]. Also, the use of the divergence theorem is valid because $v^N(t)$ and $v^E(t)$ are in H^1.

Integrating Equation (4.2.2) over time gives Equation (4.2.1). □

Theorem 4.2.2 is a natural generalization of Theorem 4.2.1.

Theorem 4.2.2. *Assume that v_0 is in E_m and that ω_0 is in $\dot{H}^\eta$ for η in $(-1, 1]$. Then for all $\nu t \geq 0$,*

$$\left\| v^N - v^E \right\|_{L^2} \leq \sqrt{2} \left\| \omega_0 \right\|_{\dot{H}^\eta} (\nu t)^{(1+\eta)/2}. \tag{4.2.3}$$

Proof. We have,

$$\begin{aligned}
\left\| v^N - v^E \right\|_{L^2}^2 &= \left\| \mathcal{F}(v^N - v^E) \right\|_{L^2}^2 = \left\| \mathcal{F}(p_{\nu t} * v_0) - \mathcal{F}(v_0) \right\|_{L^2}^2 \\
&= \left\| (1 - e^{-\nu t |\xi|^2}) \widehat{v_0}(\xi) \right\|_{L^2}^2 = \int_{\mathbb{R}^2} (1 - e^{-\nu t |\xi|^2})^2 \left| \widehat{v_0}(\xi) \right|^2 \, d\xi \\
&\leq \int_{\mathbb{R}^2} \left| (1 - e^{-\nu t |\xi|^2})^{(1+\eta)/2} \widehat{v_0}(\xi) \right|^2 \, d\xi \\
&\leq \int_{\mathbb{R}^2} \left| (\nu t \left| \xi \right|^2)^{(1+\eta)/2} \widehat{v_0}(\xi) \right|^2 \, d\xi = (\nu t)^{1+\eta} \int_{\mathbb{R}^2} \left| \left| \xi \right|^{1+\eta} \widehat{v_0}(\xi) \right|^2 \, d\xi \\
&\leq (\nu t)^{1+\eta} \left\| \left| \xi \right|^{1+\eta} \widehat{v_0}(\xi) \right\|_{L^2}^2 \leq 2 (\nu t)^{1+\eta} \left\| r^\eta \widehat{\omega_0} \right\|_{L^2}^2 \\
&= 2 (\nu t)^{1+\eta} \left\| \omega_0 \right\|_{\dot{H}^\eta}^2 ,
\end{aligned}$$

where we used Lemma 4.2.3. From this, Equation (4.2.3) follows.

□

Lemma 4.2.3. *If $\widehat{\omega}$ is a regular tempered distribution, then*

$$\frac{\sqrt{2}}{2} r |\widehat{v}| \leq |\widehat{\omega}| \leq r |\widehat{v}| \tag{4.2.4}$$

almost everywhere. (This inequality does not require that ω be radially symmetric.)

Proof. As a distribution,

$$0 = \mathcal{F}(\operatorname{div} v) = \mathcal{F}(\partial_1 v^1 + \partial_2 v^2) = i(\xi_1 \widehat{v^1} + \xi_2 \widehat{v^2}),$$

so $\xi_1 \widehat{v^1} = -\xi_2 \widehat{v^2}$. Then,

$$\begin{aligned}\widehat{\omega} &= \mathcal{F}(\partial_1 v^2 - \partial_2 v^1) = i(\xi_1 \widehat{v^2} - \xi_2 \widehat{v^1}) = i\left(\xi_1 \widehat{v^2} - \frac{\xi_2 \xi_1 \widehat{v^1}}{\xi_1}\right) \\ &= i\left(\xi_1 \widehat{v^2} + \frac{\xi_2 \xi_2 \widehat{v^2}}{\xi_1}\right) = i\frac{r^2}{\xi_1}\widehat{v^2}.\end{aligned}$$

Similarly,

$$\widehat{\omega} = i(\frac{\xi_1 \xi_2 \widehat{v^2}}{\xi_2} - \xi_2 \widehat{v^1}) = -i\left(\frac{\xi_1 \xi_1 \widehat{v^1}}{\xi_2} + \xi_2 \widehat{v^1}\right) = -i\frac{r^2}{\xi_2}\widehat{v^1}.$$

Thus, $|\widehat{\omega}| \geq r|\widehat{v^j}|$ almost everywhere for $j = 1, 2$, and

$$|\widehat{\omega}| = \frac{\sqrt{2}}{2}\sqrt{|\widehat{\omega}|^2 + |\widehat{\omega}|^2} \geq \frac{\sqrt{2}}{2}\sqrt{r^2(\widehat{v}^1)^2 + r^2(\widehat{v}^2)^2} \geq \frac{\sqrt{2}}{2} r|\widehat{v}|$$

almost everywhere. Also, though,

$$|\widehat{\omega}| = \left|\xi_1 \widehat{v}^2 - \xi_2 \widehat{v}^1\right| \leq |\xi \cdot \widehat{v}| \leq r\,|\widehat{v}|$$

almost everywhere, and we conclude that Equation (4.2.4) holds. □

The condition in Theorem 4.2.2 that $\dot{H}^\eta$ be in ω_0 is equivalent to requiring that $|D|^{1+\eta}\, v_0$ be in L^2. It is natural to ask whether it is possible to weaken this assumption to $|D|^{1+\eta}\, v_0$ being in weak-L^2 or, perhaps, $r^{1+\eta}\widehat{v_0}(r)$ being in weak-L^2, which is not quite the same thing. This is probably not possible, because we would need to simultaneously control the size of $(1 - e^{-\nu t r^2})$ and the size of $\widehat{v_0}$, and weak-L^2 is a rearrangement-invariant space. But we can make an assumption that is stronger than $r^{1+\eta}\widehat{v_0}(r)$ being in weak-L^2, though it does not imply that $r^{1+\eta}\widehat{v_0}(r)$ is in L^2 (nor is it implied by it), and still obtain the same convergence rate as in Theorem 4.2.2—though not for $\eta = 1$, and with a constant that is, in general, worse than that of Theorem 4.2.2 (and that becomes infinite in the limit as $\eta \to 1$). This result is contained in Theorem 4.2.4, the proof of which is motivated by the proof in [1] that $C(\nu t)^{3/4}$ is a *lower* bound for the convergence rate for a vortex patch.

Theorem 4.2.4. *Assume that v_0 is in E_m and that $r^{2+\eta}\widehat{v_0}$ is in L^∞ for some η in $(-1, 1)$. Then for all $\nu t \geq 0$,*

$$\left\|v^N - v^E\right\|_{L^2} \leq \sqrt{\frac{2\pi}{1-\eta^2}}\|r^{2+\eta}\widehat{v_0}\|_{L^\infty}(\nu t)^{(1+\eta)/2}. \tag{4.2.5}$$

Proof. Let

$$\gamma = \left(\frac{A}{\nu t}\right)^x,$$

where we will choose the values of A and x later. Then as in the proof of Theorem 4.2.2,

$$\begin{aligned}
\left\|v^N - v^E\right\|_{L^2}^2 &= \int_{\mathbb{R}^2} (1 - e^{-\nu t|\xi|^2})^2 \left|\widehat{v_0}(\xi)\right|^2 \, d\xi \\
&= 2\pi \int_0^\infty (1 - e^{-\nu t r^2})^2 \left|\widehat{v_0}(r)\right|^2 r \, dr \\
&\leq 2\pi(\nu t)^2 \int_0^\gamma r^4 \left|\widehat{v_0}(r)\right|^2 r \, dr + 2\pi \int_\gamma^\infty \left|\widehat{v_0}(r)\right|^2 r \, dr \\
&\leq 2\pi(\nu t)^2 \|r^{2+\eta}\widehat{v_0}\|_{L^\infty}^2 \int_0^\gamma r^{5-2(2+\eta)} \, dr + 2\pi\|r^{2+\eta}\widehat{v_0}\|_{L^\infty}^2 \int_\gamma^\infty \frac{1}{r^{3+2\eta}} \, dr \\
&= C(\nu t)^2 \frac{\gamma^{2(1-\eta)}}{2(1-\eta)} + C\frac{(1/\gamma)^{2(1+\eta)}}{2(1+\eta)} \\
&= C\left[(\nu t)^{2-2x(1-\eta)} A^{2x(1-\eta)} + (\nu t)^{2x(1+\eta)} A^{-2x(1+\eta)}\right],
\end{aligned}$$

where $C = 2\pi\|r^{2+\eta}\widehat{v_0}\|_{L^\infty}^2$.

To obtain the best convergence rate, we must maximize the minimum of $2 - 2x(1-\eta)$ and $2x(1+\eta)$, which is the same as setting them equal. This gives $x = 1/2$, independently of the value of η, so

$$\left\|v^N - v^E\right\|_{L^2}^2 \leq \frac{C}{2}(\nu t)^{1+\eta}\left[\frac{A^{1-\eta}}{1-\eta} + \frac{A^{-1-\eta}}{1+\eta}\right],$$

which is minimized when $A = 1$, independently of the value of η, giving

$$\begin{aligned}\left\|v^N - v^E\right\|_{L^2}^2 &\leq \frac{C}{2}(\nu t)^{1+\eta}\left[\frac{1}{1-\eta} + \frac{1}{1+\eta}\right] \\ &= C(\nu t)^{1+\eta}\frac{1}{1-\eta^2},\end{aligned}$$

and Equation (4.2.5) follows. □

In [1] it is proved that the convergence rate for a circular vortex patch is of order $(\nu t)^{3/4}$. For a circular vortex patch of radius 1, we can explicitly calculate,

$$\widehat{\omega_0}(\xi) = \frac{2\pi}{|\xi|}J_1(|\xi|), \quad \widehat{v_0}(\xi) = -\frac{i\xi^\perp}{|\xi|^3}J_1(|\xi|),$$

where J_1 is the Bessel function of the first kind of order 1 and $\xi^\perp = (-\xi_2, \xi_1)$. Because J_1 is bounded, has a zero of order 1 at the origin, and $J^1(r) \approx \sin r/\sqrt{r}$ as $r \to \infty$, it follows that $\widehat{v_0}$ has a singularity of order $1/r$ at the origin and decays like $1/r^{5/2}$ at infinity. Such a vortex patch thus does not quite meet the requirements of Theorem 4.2.2 with $\eta = 1/2$, but does meet the requirements of Theorem 4.2.4 with $\eta = 1/2$. As we will see in Section 4.3, its vorticity is also in the Besov space $B^\eta_{2,\infty}$, but this by itself is insufficient to obtain the convergence rate of Theorem 4.2.2.

This example also shows that it is possible for $\widehat{v_0}$ to have a singularity at the origin and still satisfy the conditions of Theorem 4.2.4. The integral around the origin in the proof of Theorem 4.2.4, however, is not controlling the contribution of this singularity to $\left\|v^N - v^E\right\|_{L^2}^2$, but is controlling the contribution due to the rate of decay of $\widehat{v_0}$ at infinity, as, too, is the second integral. Therefore, we should not expect to be able to improve this result by treating the nature of any singularity of $\widehat{v_0}$ at the origin differently from the decay of $\widehat{v_0}$ at infinity. (Also, as we mentioned earlier, it is shown in [1] that the convergence rate of $(\nu t)^{3/4}$ for a vortex patch is optimal.)

4.3 Besov spaces

We define inhomogeneous Besov spaces as in [1]:

Definition 4.3.1. For u in $\mathcal{S}'$, let

$$I_q u = \mathcal{F}^{-1}(\psi_q \mathcal{F} u) \text{ for } q = 0, 1, 2, \ldots,$$

where

$$\psi_0 = \mathbf{1}_{\{|\xi| \le 1\}},\ \psi_q = \mathbf{1}_{\{2^{q-1} \le |\xi| \le 2^q\}} \text{ for } q = 1, 2, \ldots.$$

Then the *inhomogeneous Besov space* $B^s_{2,\infty}$ is the set of all u in $\mathcal{S}'$ such that

$$\|u\|_{B^s_{2,\infty}} := \sup_{q \ge 0} 2^{qs} \|I_q u\|_{L^2} < \infty.$$

We will also use paradifferential operators Δ_q, $q = -1, 0, 1, 2, \ldots$, defining them as Chemin does in [5]. These operators are defined similarly to I_q of Definition 4.3.1, though in place of ψ_q, Chemin uses a C^∞-function supported on a ball for $q = -1$ and supported on an annulus of inner and outer radius proportional to 2^q for $q \ge 0$. The ball and the annuli cover all of $\mathbb{R}^2$ and no more than 3 of them intersect at any point. (We make the change in indexing between I_q and Δ_q to be consistent with [5] and [1].)

The operators Δ_q and I_q can be used interchangeably for defining L^2-based

Besov spaces. Also, $H^s = B^s_{2,2}$ for all real s.

It will also be convenient to define the following spaces as in [1]:

Definition 4.3.2. For α in $\mathbb{R}$ and ρ in $[1, \infty)$, define the norms,

$$\|u\|_{\widetilde{L}^\rho_t(B^\alpha_{2,\infty})} = \sup_q 2^{q\alpha} \left(\int_0^t \|\Delta_q u(\tau)\|^\rho_{L^2} \, d\tau \right)^{1/\rho}$$

and

$$\|u\|_{\widetilde{L}^\rho_t(H^\alpha)} = \left(\sum_q 2^{2q\alpha} \left(\int_0^t \|\Delta_q u(\tau)\|^\rho_{L^2} \, d\tau \right)^{2/\rho} \right)^{1/2},$$

and the associated subspaces, $\widetilde{L}^\rho_t(B^\alpha_{2,\infty})$ and $\widetilde{L}^\rho_t(H^\alpha)$ of $\mathcal{S}'$. We also make the similar definitions for $\rho = \infty$.

We observe, as in [1], that

$$\|\nabla u\|_{\widetilde{L}^\rho_t(B^{\alpha-1}_{2,\infty})} \le C \|u\|_{\widetilde{L}^\rho_t(B^\alpha_{2,\infty})} \text{ and } \|\nabla u\|_{\widetilde{L}^\rho_t(H^{\alpha-1})} \le C \|u\|_{\widetilde{L}^\rho_t(H^\alpha)}, \tag{4.3.1}$$

that

$$\|u\|_{\widetilde{L}^\rho_t(H^\alpha)} \le C \|u\|^\theta_{\widetilde{L}^\rho_t(B^{\alpha_1}_{2,\infty})} \|u\|^{1-\theta}_{\widetilde{L}^\rho_t(B^{\alpha_2}_{2,\infty})}, \tag{4.3.2}$$

where θ is in $(0, 1)$, $\alpha = \theta\alpha_1 + \theta\alpha_2$, and C depends only on θ, and, finally, that

$$\|\nabla u\|_{\widetilde{L}^1_T(H^1)} \le C \|\omega\|_{\widetilde{L}^1_T(H^1)}. \tag{4.3.3}$$

Lemma 4.3.3 relates our condition that $r^{2+\eta}\widehat{v_0}$ be in L^∞ to membership in $B^\eta_{2,\infty}$.

Lemma 4.3.3. *Assume that v is in E_m and that $r^{2+\eta}\widehat{v}(r)$ is in L^∞ for some η in*

$(-\infty, 0) \cup (0, \infty)$. *Then* $\omega = \omega(v)$ *is in* $B^{\eta}_{2,\infty}$*, and*

$$\|\omega\|_{B^{\eta}_{2,\infty}} \le \begin{cases} \frac{\sqrt{2\pi}}{|\eta|} \left\| r^{1+\eta} \widehat{\omega} \right\|_{L^\infty}, & \text{if } \eta < 0, \\ \frac{\sqrt{2\pi}}{\eta} (2^{2\eta} - 1)^{1/2} \left\| r^{1+\eta} \widehat{\omega} \right\|_{L^\infty} + \|\omega_0\|_{L^2}, & \text{if } \eta > 0. \end{cases}$$

(We are not requiring that ω *be radially symmetric.)*

Proof. For $q \ge 1$,

$$\begin{aligned} \|I_q \omega\|_{L^2} &= \left\| \mathcal{F}^{-1}(\mathbf{1}_{\{2^{q-1} \le |\xi| \le 2^q\}} \mathcal{F}\omega) \right\|_{L^2} \\ &= \left\| \mathbf{1}_{\{2^{q-1} \le |\xi| \le 2^q\}} \mathcal{F}\omega \right\|_{L^2} = \left(\int_{B_{2^q} \setminus B_{2^{q-1}}} |\widehat{\omega}|^2 \, d\xi \right)^{1/2} \\ &= \left(\int_{B_{2^q} \setminus B_{2^{q-1}}} \left| |\xi|^{1+\eta} \, \widehat{\omega} \right|^2 |\xi|^{-2(1+\eta)} \, d\xi \right)^{1/2} \\ &\le \left\| r^{1+\eta} \widehat{\omega} \right\|_{L^\infty} \left(2\pi \int_{2^{q-1}}^{2^q} r^{-2(1+\eta)} r \, dr \right)^{1/2} \\ &= \frac{C}{-\eta} \left(\left[r^{-2\eta} \right]_{2^{q-1}}^{2^q} \right)^{1/2} = \frac{C}{\eta} \left(2^{-2(q-1)\eta} - 2^{-2q\eta} \right)^{1/2} \\ &= \frac{C}{|\eta|} \left(2^{-2q\eta} \left| 1 - 2^{2\eta} \right| \right)^{1/2} = \frac{C}{|\eta|} \left| 1 - 2^{2\eta} \right|^{1/2} 2^{-q\eta}, \end{aligned}$$

where $C = \sqrt{2\pi} \left\| r^{1+\eta} \widehat{\omega} \right\|_{L^\infty}$. We used Lemma 4.2.3 to conclude that $r^{1+\eta}\widehat{\omega}$ is in L^∞.

If $\eta < 0$, then the above argument also works for $q = -1$, since $\left[r^{-2\eta} \right]_0^1 = 1$. It follows that for all $q \ge 0$,

$$\|I_q \omega\|_{L^2} \le \frac{C}{|\eta|} 2^{-q\eta},$$

so

$$\|\omega\|_{B^{\eta}_{2,\infty}} = \sup_{q \ge 0} 2^{q\eta} \|I_q \omega\|_{L^2} \le \frac{C}{|\eta|} \sup_{q \ge 0} 2^{q\eta} 2^{-q\eta} = \frac{C}{|\eta|}$$

is finite, and we obtain the stated bound on $\|\omega\|_{B^\eta_{2,\infty}}$.

Assume that $\eta > 0$, and let $v = \sigma + u$ where σ is a stationary vector field and u is in L^2, as in Appendix 2A. Then $\widehat{u}$ is in L^2 so $r\widehat{u}$ is in $L^2(B_1(0))$. Also, $r^{2+\eta}\widehat{v} = r^{2+\eta}\widehat{\sigma} + r^{2+\eta}\widehat{u}$ in L^∞ implies that $r^{1+\eta}(r\widehat{u})$ is in L^∞ (since $\widehat{\sigma}$ is smooth and compactly supported), so $r\widehat{u}$ is in $L^2(\mathbb{R}^2 \setminus B_1(0))$, $1+\eta$ being greater than 1; therefore, $r\widehat{u}$ is in L^2. But $|\widehat{\omega}(u)| \le r\,|\widehat{u}|$ by Lemma 4.2.3, so $\omega(u)$ and hence $\omega(v)$ are in L^2. But then $\|I_0\omega\|_{L^2} \le \|\omega\|_{L^2}$, and again $\|\omega\|_{B^\eta_{2,\infty}}$ is finite and has the stated bound. □

4.4 Convergence in Besov spaces

Theorem 4.4.1 follows from Theorem 1.1 of [1]; this theorem does *not* contain the assumption that the initial vorticity is radially symmetric.

Theorem 4.4.1. *Let ω_0 be in $L^2 \cap L^\infty \cap B^\eta_{2,\infty}$ with η in $[0,1)$, and assume that ∇v^E and ∇v^N are in $L^\infty_{loc}(\mathbb{R}^+; L^\infty)$. Then for all $\nu t \le 1$,*

$$\left\|(v^N - v^E)(t)\right\|_{L^2} \le C(t)(\|\omega_0\|_{B^\eta_{2,\infty}} + \|\omega_0\|_{L^2})(\nu t)^{(1+\eta)/2}.$$

More than this is established in Theorem 1.1 of [1], but the facts in Theorem 4.4.1 are all that concern us in relation to radially symmetric vorticities. As Abidi and Danchin observe in [1], a vortex patch with a C^1-boundary belongs to $B^{1/2}_{2,\infty}$ and that for a vortex patch with a $C^{1+\epsilon}$-boundary, $\epsilon > 0$, ∇v^N and ∇v^E are both in $L^\infty_{loc}(\mathbb{R}^+; L^\infty)$. Thus, a vortex patch with a $C^{1+\epsilon}$-boundary has a bound on the convergence rate given by Theorem 4.4.1 of order $(\nu t)^{3/4}$. This result applies, in particular, for a circular vortex patch.

We observed toward the end of Section 4.2 that for a circular vortex patch, ω_0 is bounded and decays like $1/r^{5/2}$ at infinity. It follows from Lemma 4.3.3 that the vorticity for a circular vortex patch lies in $B^{1/2}_{2,\infty}$ (since $\eta = 1/2$) and hence by

Theorem 4.4.1 that the convergence rate in the vanishing viscosity limit is order $(\nu t)^{3/4}$, as we concluded more directly in Section 4.2.

Because $B^\eta_{2,\infty} \subseteq H^{\eta-\epsilon}$ for all $\epsilon > 0$, we can obtain the same bound on the convergence rate as in Theorem 4.2.2 for initial vorticity in $B^\eta_{2,\infty}$, but with a loss of ϵ in the exponent. As it turns out, though, we can obtain the convergence rate without any loss in the exponent by specializing the proof of Theorem 1.1 of [1] to radially symmetric initial vorticity: the result is Theorem 4.4.2.

Theorem 4.4.2. *Assume that v_0 is in E_m and that ω_0 is in $B^\eta_{2,\infty}$ for η in $(-1, 1)$. Then for all $\nu t \geq 0$,*

$$\left\| v^N - v^E \right\|_{L^2} \leq C \left(\|\omega_0\|_{B^\eta_{2,\infty}} + \nu t \left\|\Delta_{-1}\omega_0\right\|_{L^2} \right) (\nu t)^{(1+\eta)/2}. \tag{4.4.1}$$

Remark: When η is in $[0, 1)$, it follows from Equation (4.4.1) that

$$\left\| v^N - v^E \right\|_{L^2} \leq C \left(\|\omega_0\|_{B^\eta_{2,\infty}} + \nu t \left\|\omega_0\right\|_{L^2} \right) (\nu t)^{(1+\eta)/2}.$$

To prove Theorem 4.4.2 we must first establish two lemmas.

Lemma 4.4.3. *Let v_0 be as in Theorem 4.4.2. Then there exists a constant κ such that for all $t \geq 0$,*

$$\left\|\omega^N(t)\right\|_{B^\eta_{2,\infty}} + \kappa\nu \left\|\omega^N\right\|_{\widetilde{L}^1_t(B^{2+\eta}_{2,\infty})} \leq \|\omega_0\|_{B^\eta_{2,\infty}} + \kappa\nu \int_0^t \left\|\Delta_{-1}\omega^N(\tau)\right\|_{L^2} d\tau.$$

Proof. Because $\omega^N = p_{\nu t} * \omega^0$ and $p_{\nu t}$ is Schwartz-class in time and space for all

$t > 0$, $\Delta_q \omega^N$ is C^∞ in time and space. Thus we can write,

$$\begin{aligned}
\partial_t \omega^N = \nu \Delta \omega^N &\implies \partial_t \Delta_q \omega^N = \nu \Delta \Delta_q \omega^N \\
&\implies \partial_t \Delta_q \omega^N \cdot \Delta_q \omega^N = \nu \Delta \Delta_q \omega^N \cdot \Delta_q \omega^N \\
&\implies \frac{1}{2} \frac{d}{dt} \left\| \Delta_q \omega^N \right\|_{L^2}^2 + \nu \left\| \nabla \Delta_q \omega^N \right\|_{L^2}^2 = 0,
\end{aligned} \tag{4.4.2}$$

where we used the divergence theorem in the last step.

But by Bernstein's inequality, there exists a constant $\kappa > 0$ such that

$$\left\| \nabla \Delta_q \omega^N \right\|_{L^2}^2 \geq \kappa 2^{2q} \left\| \Delta_q \omega^N \right\|_{L^2}^2$$

for all $q \geq 0$. Thus, for all $q \geq -1$,

$$\left\| \Delta_q \omega^N \right\|_{L^2} \frac{d}{dt} \left\| \Delta_q \omega^N \right\|_{L^2} + \kappa \nu 2^{2q} \left\| \Delta_q \omega^N \right\|_{L^2}^2 \leq \kappa \nu \delta_{-1,q} 2^{2q} \left\| \Delta_q \omega^N \right\|_{L^2}^2 .$$

If $\|\Delta_q \omega^N\|_{L^2}$ is nonzero, we can divide both sides of the above inequality by $\|\Delta_q \omega^N\|_{L^2}$ and multiply by $2^{2\eta q}$ to obtain

$$\begin{aligned}
&\frac{d}{dt} 2^{2\eta q} \left\| \Delta_q \omega^N \right\|_{L^2} + \kappa \nu 2^{(\eta+2)q} \left\| \Delta_q \omega^N \right\|_{L^2} \\
&\qquad \leq \kappa \nu \delta_{-1,q} 2^{(\eta+2)q} \left\| \Delta_q \omega^N \right\|_{L^2} .
\end{aligned} \tag{4.4.3}$$

Integrating Equation (4.4.3) over time then taking the supremum of both sides completes the proof.

Now suppose that $\|\Delta_q \omega^N\|_{L^2}$ is zero at least at one point in $(0, \infty)$. By our initial observation that $\Delta_q \omega^N$ is C^∞ in time and space, $f(t) := \|\Delta_q \omega^N(t)\|_{L^2}^2$ is infinitely differentiable for all $t > 0$. It is also, of course, nonnegative, and it follows from Equation (4.4.2) that $f'(t) < 0$ except, possibly, when $f(t) = 0$.

Now suppose that $f(t_0) = 0$. Then we must have $f(t) = 0$ for all $t \geq t_0$, else $f'(t)$ would by necessity be positive for some $t > t_0$ by the mean value theorem.

Thus, f is positive except possibly in an interval $[a, \infty)$ for some $a \geq 0$. On the interval $[0, a)$, Equation (4.4.3) holds by our argument above since we avoid division by zero. But on the interval $[a, \infty)$, both $f(t)$ and $f'(t)$ are identically zero and Equation (4.4.3) holds trivially, except possible at $t = a$.

(Actually, more can be said. In fact, it follows from Equation (4.4.2) that $\|\Delta_q \omega^N\|_{L^2}$ is either positive for all time, or is identically zero.) □

Lemma 4.4.4. *Let v_0 be as in Theorem 4.4.2. Then*

$$\left\|v^N(t) - v^E(t)\right\|_{L^2} \leq C\nu \left\|\Delta v^N\right\|_{\widetilde{L}^1_t(H^0)}. \tag{4.4.4}$$

Proof. Let $w = v^N - v^E$. Subtracting (E) from (NS), applying Δ_q, taking the inner product with $\Delta_q w$, and integrating over $\mathbb{R}^2$ gives

$$\int_{\mathbb{R}^2} \partial_t \Delta_q w \cdot \Delta_q w + \int_{\mathbb{R}^2} \nabla(\Delta_q(p^N - p^E)) \cdot \Delta_q w = \nu \int_{\mathbb{R}^2} \Delta\Delta_q v^N \cdot \Delta_q w,$$

where there are no nonlinear terms since the solutions are radially symmetric. We can treat the first integral exactly as in the proof of Theorem 4.2.1, and the second integral on the left-hand side is zero because $\operatorname{div} \Delta_q w = \Delta_q \operatorname{div} w = 0$, so

$$\begin{aligned}
\|\Delta_q w\|_{L^2} \frac{d}{dt} \|\Delta_q w\|_{L^2} = \frac{1}{2}\frac{d}{dt} \|\Delta_q w\|^2_{L^2} &\leq \nu \|\Delta_q w\|_{L^2} \left\|\Delta\Delta_q v^N\right\|_{L^2} \\
\implies \frac{d}{dt} \|\Delta_q w\|_{L^2} &\leq \nu \left\|\Delta\Delta_q v^N\right\|_{L^2} \\
\implies \|\Delta_q w(t)\|_{L^2} &\leq \nu \int_0^t \left\|\Delta\Delta_q v^N(\tau)\right\|_{L^2} d\tau.
\end{aligned}$$

Now, for any L^2-function u, $u \mapsto \left(\sum_q \|\Delta_q u\|^2_{L^2}\right)^{1/2}$ gives a norm that is equivalent to the L^2-norm (see, for instance, Section 2.2 of [5]). Thus, squaring our last inequality and summing over q, we have

$$\|w(t)\|_{L^2}^2 \le C\nu^2 \sum_q \left(\int_0^t \left\|\Delta\Delta_q v^N(\tau)\right\|_{L^2} \, d\tau \right)^2 = C\nu^2 \left\|\Delta v^N\right\|_{\widetilde{L}_t^1(H^0)}^2 .$$

□

Proof of Theorem 4.4.2: Using Equation (4.3.1), Equation (4.3.2) with $\theta = (1+\eta)/2$, $\alpha_1 = \eta$, and $\alpha_2 = 2+\eta$ (so that $\alpha = 1$), and Equation (4.3.3), we have

$$\begin{aligned}
\left\|\Delta v^N\right\|_{\widetilde{L}_t^1(H^0)} &\le C \left\|\nabla v^N\right\|_{\widetilde{L}_t^1(H^1)} \le C \left\|\omega^N\right\|_{\widetilde{L}_t^1(H^1)} \\
&\le C \left\|\omega^N\right\|_{\widetilde{L}_t^1(B_{2,\infty}^\eta)}^{(1+\eta)/2} \left\|\omega^N\right\|_{\widetilde{L}_t^1(B_{2,\infty}^{2+\eta})}^{(1-\eta)/2} \\
&\le C t^{(1+\eta)/2} \left\|\omega^N\right\|_{\widetilde{L}_t^\infty(B_{2,\infty}^\eta)}^{(1+\eta)/2} \left\|\omega^N\right\|_{\widetilde{L}_t^1(B_{2,\infty}^{2+\eta})}^{(1-\eta)/2},
\end{aligned}$$

where we used Hölder's inequality in the last step.

We now apply Lemma 4.4.5 with $\theta = (1+\eta)/2$, $a = \left\|\omega^N\right\|_{\widetilde{L}_t^\infty(B_{2,\infty}^\eta)}$, $b = \left\|\omega^N\right\|_{\widetilde{L}_t^1(B_{2,\infty}^{2+\eta})}$, and $\delta = \kappa\nu$ to conclude that

$$\begin{aligned}
\kappa\nu \left\|\Delta v^N\right\|_{\widetilde{L}_t^1(H^0)} &\le C(\kappa\nu t)^{(1+\eta)/2} \left(\left\|\omega^N\right\|_{\widetilde{L}_t^\infty(B_{2,\infty}^\eta)} + C'\kappa\nu \left\|\omega^N\right\|_{\widetilde{L}_t^1(B_{2,\infty}^{2+\eta})} \right) \\
&\le C(\kappa\nu t)^{(1+\eta)/2} \left(\left\|\omega^N\right\|_{\widetilde{L}_t^\infty(B_{2,\infty}^\eta)} + \kappa\nu \left\|\omega^N\right\|_{\widetilde{L}_t^1(B_{2,\infty}^{2+\eta})} \right).
\end{aligned}$$

From Lemma 4.4.3 and Lemma 4.4.4 it then follows that

$$\begin{aligned}
&\left\|v^N(t) - v^E(t)\right\|_{L^2} \le C\nu \left\|\Delta v^N\right\|_{\widetilde{L}_t^1(H^0)} \\
&\quad \le C(\kappa\nu t)^{(1+\eta)/2} \left(\left\|\omega^N\right\|_{\widetilde{L}_t^\infty(B_{2,\infty}^\eta)} + \kappa\nu \left\|\omega^N\right\|_{\widetilde{L}_t^1(B_{2,\infty}^{2+\eta})} \right) \\
&\quad \le C(\kappa\nu t)^{(1+\eta)/2} \left(\|\omega_0\|_{B_{2,\infty}^\eta} + \kappa\nu \int_0^t \left\|\Delta_{-1}\omega^N(\tau)\right\|_{L^2} \, d\tau \right) \\
&\quad \le C(\kappa\nu t)^{(1+\eta)/2} \left(\|\omega_0\|_{B_{2,\infty}^\eta} + \nu t \left\|\Delta_{-1}\omega_0\right\|_{L^2} \right).
\end{aligned}$$

□

Remark: The proof of Theorem 4.4.2 fails for $\eta = 1$ because the interpolation inequality, Equation (4.3.2), cannot be applied.

Lemma 4.4.5. *Let θ be in $(0, 1)$ and a, b, and δ be positive real numbers. Then*

$$\delta a^\theta b^{1-\theta} \le \delta^\theta a + C\delta^{1+\theta} b = \delta^\theta (a + C\delta b),$$

where $C = 1 - \theta$.

Proof. We apply Young's inequality in the form

$$xy \le \frac{x^p}{p} + \frac{y^q}{q},$$

for $1/p + 1/q = 1$, with $p = 1/\theta$, $q = 1/(1-\theta)$, $x = \delta^{\theta^2} a^\theta / \theta^\theta$, $y = \delta^{1-\theta^2} b^{1-\theta} / \theta^\theta$, giving

$$\delta a^\theta b^{1-\theta} = xy \le \theta x^{1/\theta} + (1-\theta) y^{1/(1-\theta)} = \theta \frac{\delta^\theta a}{\theta} + (1-\theta) \frac{\delta^{1+\theta} b}{\theta^{\theta/(1-\theta)}}.$$

□

4.5 A superposition of confined eddies

We now consider superposition of a finite number of eddies, an *eddy* being circularly symmetric vorticity whose total integral is zero.

Define the *extent* of an eddy to be the smallest closed annulus or disk that contains the support of the eddy. For instance, if an eddy is supported on two concentric annuli, then the extent is the union of the two annuli. If the vorticity on each annulus integrates to zero, then we could alternately consider the eddy to be two distinct eddies.

We assume that the extent of each eddy is separated by a finite distance from the extent of each of the other eddies. We also assume that each eddy has compact support. (In the special case of one eddy, the assumption of compact support and of zero total vorticity can be dropped.)

There is not a unique way to decompose a superposition of radially symmetric vorticities into eddies, but this will have no effect on our analysis.

We let $v_{0,j}$ and $\omega_{0,j} = \omega(v_{0,j})$ be the initial velocity and vorticity, respectively, of the j-th eddy, and we let v_0 and $\omega_0 = \omega(v_0)$ be the total initial velocity and vorticity, respectively.

The most striking feature of such a superposition of eddies is that it is a stationary solution to the Euler equations. This is because each eddy is a stationary solution (for the same reason that σ of Appendix 2A is a stationary solution), and because by Equation (2A.2) and the remark that follows it, each eddy makes no contribution to the velocity field outside its extent, its total vorticity being zero, or inside its extent (assuming there is an inside—that is, its extent does not include a disk).

It is not sufficient to assume that the supports of each eddy are separated by a finite distance rather than the extents being so separated. For example, consider the superposition of two eddies, with one eddy supported on concentric annuli, the inner one of which has positive constant vorticity and the outer one of which has negative constant vorticity, and with the other eddy lying entirely between the annuli of the first eddy at a positive distance from each annuli. The supports of the two eddies are separated by a positive distance, but the extents are not. This is not a steady state solution to the Euler equations, since the velocity at points in the second eddy will experience a nonzero contribution from the inner annulus of the first eddy, but no contribution from the outer eddy.

In [28], the strong convergence in L^2 of the Navier-Stokes solutions for the superposition of eddies to the steady state Euler solution is established for a specific subclass of vorticities that are bounded measures in the plane. We combine the bounds on convergence rates we obtained for radially symmetric initial vorticity in the previous sections with an estimate from [28] for the interaction of the eddies to obtain Theorem 4.5.1.

Theorem 4.5.1. *Let ω_0 be a superposition of confined eddies, as described above, and assume that v_0 and ω_0 satisfy the conditions of Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.2, or Theorem 4.4.2. Then for any $\delta > 0$ there exists a $T > 0$ such that the same bound on the convergence rate of $\|v^N - v^E\|_{L^2}$ in each of these theorems applies over the time range $[0, T]$, but with the constants in the convergence rates of these theorems increased by δ.*

Proof. The existence and uniqueness of the solution to (NS) is a classical result (see, for instance, Theorems 3.1 and 3.2 of Chapter 3 of [33]). Because $\int \omega_{0,j} = 0$, $v_{0,j}$ vanishes outside the extent of the j-th eddy (this is essentially (4) of Theorem 2A.3). Then, since each $\omega_{0,j}$ is a steady state solution to the Euler equations, it follows that $v^E(t) = v_0$ is a steady state solution to the Euler equations.

Since the heat equation is linear, $v^H = \sum_i v_i^H$, where v_i^H is the solution of the heat equation for the i-th eddy by itself. Thus,

$$\begin{aligned} \|v^H - v^E\|_{L^2} &= \|\sum_i v_i^H - \sum_i v_{0,i}\|_{L^2} = \|\sum_i (v_i^H - v_{0,i})\|_{L^2} \\ &\leq \sum_i \|v_i^H - v_{0,i}\|_{L^2}. \end{aligned}$$

But each of the bounds in Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.2, and Theorem 4.4.2 is sublinear in ω_0 and v_0. For instance, it follows for Theorem 4.2.1

that

$$\|v^H - v^E\|_{L^2} \le \sum_i C\|\omega_{0,i}\|_{L^2}\sqrt{\nu t} = C\|\omega_0\|_{L^2}\sqrt{\nu t},$$

the last equality holding because the supports of the $\omega_{0,i}$ are disjoint. We conclude that each of Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.2, and Theorem 4.4.2 continue to hold without change for the superposition of eddies, if we use v^H in place of v^N.

In Theorem 4.5.2, we establish the rate at which $\|v^N - v^H\|_{L^2}$ converges to 0 by using a technique that appears in [28]. The convergence rate to 0 that we obtain depends upon whether we are assuming the initial data of Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.2, or Theorem 4.4.2, but in all cases is faster, over some finite range [0, T], than that of $\|v^H - v^E\|_{L^2}$ to 0, and the theorem follows from the observation that

$$\|v^N - v^E\|_{L^2} \le \|v^H - v^E\|_{L^2} + \|v^N - v^H\|_{L^2}.$$

□

Theorem 4.5.2. *For a superposition of confined eddies as we described above, assume that v_0 and ω_0 satisfy the conditions of Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.2, or Theorem 4.4.2. Then $\|v^N - v^H\|_{L^2} \to 0$ exponentially fast over a sufficiently small time interval. More explicitly, there exists an $\alpha < 1$ such that for any $T > 0$, we have, for all t in $[0, T]$,*

$$\left\|v^N - v^H\right\|_{L^2} \le \frac{\sqrt{T}\sqrt{\nu t}}{\sqrt{C_2}} \exp\left(\frac{T}{2} + \frac{C\left\|\omega_0\right\|_{\dot{H}^{-\alpha}} T}{(\nu t)^{(1+\alpha)/2}} - \frac{C_2}{2\nu t}\right). \tag{4.5.1}$$

Proof. Whether or not our initial data satisfies the conditions of Theorem 4.2.1, Theorem 4.2.2, Theorem 4.2.2, or Theorem 4.4.2, in each case ω_0 is in $\dot{H}^{-\alpha}$ for some

$\alpha < 1$. Then,

$$\begin{aligned}\|p_{\nu s} * \omega_{0,j}\|_{L^\infty} &= |\langle p_{\nu s}(x - \cdot), \omega_{0,j}(\cdot)\rangle| \le \|p_{\nu s}\|_{\dot{H}^\alpha} \|\omega_{0,j}\|_{\dot{H}^{-\alpha}} \\ &= C(\nu s)^{-(1+\alpha)/2} \|\omega_{0,j}\|_{\dot{H}^{-\alpha}}.\end{aligned}$$

It is proven in [28] (see the equation following Equation (10) of [28]) that for ω_0 in $\mathbb{P}$, if we let $f(t) := \|v^N - v^H\|_{L^2}^2$, then

$$f(t) \le \int_0^t (C(\nu, s) + 1) f(s)\, ds + \int_0^t C_1 e^{-C_2/(\nu s)}\, ds, \tag{4.5.2}$$

where $C(\nu, s) = \|p_{\nu s} * \omega_0\|_{L^\infty}$. We have,

$$\begin{aligned}C(\nu, s) &\le \sum_{j=1}^N \|p_{\nu s} * \omega_{0,j}\|_{L^\infty} \le C \sum_{j=1}^N (\nu s)^{-(1+\alpha)/2} \|\omega_{0,j}\|_{\dot{H}^{-\alpha}} \\ &= C(\nu s)^{-(1+\alpha)/2} \|\omega_0\|_{\dot{H}^{-\alpha}},\end{aligned}$$

the final equality holding because the supports of the eddies are disjoint. Then by Gronwall's lemma,

$$f(t) \le \left(\int_0^t C_1 e^{-C_2/(\nu s)}\, ds\right) \exp\left(\int_0^t (C(\nu, s) + 1)\, ds\right).$$

Integrating our expression for $C(\nu, s)$ and using $(1 + \alpha)/2 < 1$ gives

$$\int_0^t (C(\nu, s) + 1)\, ds = t + C \|\omega_0\|_{\dot{H}^{-\alpha}} \frac{t}{(\nu t)^{\frac{1+\alpha}{2}}}.$$

Making the change of variables $u = C_2/(\nu s)$,

$$\int_0^t C_1 e^{-C_2/(\nu s)}\, ds = -\frac{C_2}{\nu} \int_{\infty}^{C_2/(\nu t)} \frac{1}{u^2} e^{-u}\, du = \frac{C_2}{\nu} \int_{C_2/(\nu t)}^{\infty} \frac{1}{u^2} e^{-u}\, du$$
$$\leq \frac{C_2}{\nu} \left(\frac{\nu t}{C_2}\right)^2 \int_{C_2/(\nu t)}^{\infty} e^{-u}\, du = \frac{\nu t^2}{C_2} e^{-C_2/(\nu t)}.$$

Thus,

$$f(t) \leq \frac{\nu t^2}{C_2} \exp\left(t + C \left\|\omega_0\right\|_{\dot{H}^{-\alpha}} \frac{t}{(\nu t)^{\frac{1+\alpha}{2}}} - \frac{C_2}{\nu t}\right),$$

so

$$\left\|v^N - v^H\right\|_{L^2} \leq \frac{\sqrt{\nu t}}{\sqrt{C_2}} \exp\left(\frac{t}{2} + C \left\|\omega_0\right\|_{\dot{H}^{-\alpha}} \frac{t}{(\nu t)^{\frac{1+\alpha}{2}}} - \frac{C_2}{2\nu t}\right)$$
$$= \frac{\sqrt{t}\sqrt{\nu t}}{\sqrt{C_2}} \exp\left(\frac{t}{2} + \frac{C \left\|\omega_0\right\|_{\dot{H}^{-\alpha}} t}{(\nu t)^{(1+\alpha)/2}} - \frac{C_2}{2\nu t}\right).$$

From this, Equation (4.5.1) follows.

We see that the bound in Equation (4.5.1) decreases exponentially fast to 0 as $\nu t \to 0$, because the term $-C_2/(2\nu t)$ dominates in the exponential for small νt, $(1+\alpha)/2$ being less than 1. □

Chapter 5

Properties of the flow for solutions to the Euler equations in $\mathbb{R}^2$

5.1 Introduction

We return to the setting of deterministic solutions in the plane of Chapter 2 to investigate the properties of the flow associated to the velocity field v that is a solution to (E).

In [37], Yudovich establishes the uniqueness of solutions to (E) under the assumptions on the initial vorticity that we described in Chapter 2. Yudovich is working in a bounded domain in $\mathbb{R}^d$, but the essence of his argument is unchanged when applied to all of $\mathbb{R}^2$. Also, his uniqueness argument is the basis of the proof of the existence of a solution (v, p) to (E) when working in $\mathbb{R}^2$.

Yudovich also proves in [37] that there exists a flow associated with the velocity field v, though he does not explicitly state the regularity of the flow (though the regularity would follow from his argument). This approach, in the special case of bounded vorticity, is worked out in some detail by Chemin in Section 5.2 of [5]. It is also worked out in detail for bounded vorticity by Majda and Bertozzi in [29], though there it is done in the context of establishing existence of a solution to (E) (see Section 2.6). We modify the argument at it appears in [5] to prove the second part of Theorem 5.1.1, below. For completeness, we include, in the terminology of Chapter 2, a complete statement of the main result proved by Yudovich in [37] as it applies to solutions in $\mathbb{R}^2$. In this form, it is a generalization of the statement of Theorem 5.1.1 of [5] from bounded to unbounded vorticity.

Theorem 5.1.1 (Yudovich's Theorem for Unbounded Vorticity). ***First part:*** *Let v_0 in E_m be a divergence-free vector field whose vorticity ω_0 is bounded in L^p-norm by $\theta(p)$ for some admissible function θ (see Definition 2.1.2). Then there exists a unique weak solution v of (E) (see Definition 2.1.4). Moreover, v is in $C(\mathbb{R}; E_m) \cap L^\infty_{loc}(\mathbb{R}; L^\infty(\mathbb{R}^2))$. Also,*

$$\|\omega(t)\|_{L^p} = \|\omega^0\|_{L^p} \text{ for all } 1 \leq p \leq \infty. \tag{5.1.1}$$

Second part: *Moreover, the vector field has a flow. More precisely, there exists a unique mapping ψ, continuous from $\mathbb{R} \times \mathbb{R}^2$ to $\mathbb{R}^2$, such that*

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x))\, ds.$$

If $\Gamma_t \colon [0, \infty) \to [0, \infty)$ is defined by

$$\int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{\beta_{1,\phi}(r)} = t,$$

then $\delta \mapsto \Gamma_t(\delta)$ is an upper bound on the modulus of continuity of the flow at time t.

See Chapter 2 for the precise definition of a weak solution to the Euler equations and for a proof of uniqueness in the first part of the theorem. In Section 5.2 through Section 5.4, we prove the second part (what Chemin refers to as the "Lagrangian" part) of the theorem.

5.2 Replacement for log-Lipschitzian property of the velocity

The first step in the classical approaches of [5] and [29] to deriving the existence, uniqueness, and regularity of the flow for bounded vorticity is to show that the velocity field is log-Lipschitzian. This gives a function $\mu(r) = Cr(1 - \log r)$ that satisfies the following five properties:

1. $\mu\colon [0,1] \to [0,\infty)$,
2. $|v(t,x) - v(t,x')| \le \mu(|x - x'|)$ for all $|x - x'| \le 1$ and $t \ge 0$,
3. $\mu(0) = 0$,
4. μ is nondecreasing, and
5. μ satisfies:

$$\int_0^1 \frac{da}{\mu(a)} = \infty. \tag{5.2.1}$$

(We can also express the second property as saying that the function μ is a bound on the modulus of continuity of $v(t)$.)

These properties of μ allow us to apply the machinery of Osgood's lemma to prove the existence and uniqueness of a continuous flow associated with the velocity field.

Our goal in this section is to establish that a function μ with all these properties also exists in the case of the solutions to (E) of Theorem 5.1.1. The properties we use of this solution are that the L^p-norms of the vorticity are non-increasing over time for all p in $[1, \infty]$ (of course, some of these norms might be infinite), and that the initial vorticity is of Yudovich class (see Definition 2.1.3). Thus, our argument applies equally well to the solutions to (NS) given by Theorem 2.1.5.

Before proceeding, we make a comment on the first property. To apply Osgood's lemma, we need μ to be defined on the domain $[0, \infty)$. We can always extend μ to $[0, \infty)$, however, in such a way that it continues to obey the other four properties by defining $\mu(x) = \mu(x - n) + n\mu(1)$, where n is the greatest integer less-than-or-equal-to x. This clearly satisfies properties 3 through 5. To see that μ satisfies property 2—for all $|x - x'| \geq 0$—let n be the greatest integer less-than-or-equal-to $|x - x'|$ and let $\mathbf{e} = (x - x')/|x - x'|$ for $|x - x'| > 0$. Then

$$\begin{aligned}
&\left|v(t,x) - v(t,x')\right| \\
&\quad \leq \left|v(t,x) - v(t,x' + n\mathbf{e})\right| + \sum_{j=1}^{n} \left|v(t,x' + j\mathbf{e}) - v(t,x' + (j-1)\mathbf{e})\right| \\
&\quad \leq \mu(\left|x - x'\right| - n) + n\mu(1) = \mu(\left|x - x'\right|).
\end{aligned}$$

From the Biot-Savart law, we have

$$\begin{aligned}
I :&= \left|v(t,x) - v(t,x')\right| \\
&= \frac{1}{2\pi} \int \omega(t,y) \left[\frac{(x-y)^\perp}{|x-y|^2} - \frac{(x'-y)^\perp}{|x'-y|^2} \right] dy \\
&\leq \frac{1}{2\pi} \sqrt{(I_1)^2 + (I_2)^2},
\end{aligned}$$

where

$$I_2 := \int |\omega(t,y)| \left| \frac{x^1 - y^1}{|x-y|^2} - \frac{(x')^1 - y^1}{|x'-y|^2} \right| dy,$$

and where I_1 is defined similarly. The technique we use to bound I_2 will clearly apply equally well to I_1, so we will deal only with I_2.

Let $a = |x - x'|/2$, and let R and λ be fixed positive real numbers. We will specify the values of R and λ later, but it will be true that $R \geq \lambda$. Assume also that $|x - x'| \leq 1$. Then we can split I_2 into three integrals:

$$I_2 = J + K + L,$$

where

$$J := \int_{B_{\lambda a}} |\omega(t, y)| \, f(y) \, dy, \quad K := \int_{B_R \setminus B_{\lambda a}} |\omega(t, y)| \, f(y) \, dy,$$

and

$$L := \int_{\mathbb{R}^2 \setminus B_R} |\omega(t, y)| \, f(y) \, dy,$$

with

$$f(y) := \left| \frac{x^1 - y^1}{|x - y|^2} - \frac{(x')^1 - y^1}{|x' - y|^2} \right|.$$

We bound J, K, and L differently.

Let $2 < p < \infty$ and $1 < q < 2$ be such that $1/p + 1/q = 1$. Then

$$\begin{aligned} J &\leq \int_{B_{\lambda a}} |\omega(t, y)| \left[\frac{|x^1 - y^1|}{|x - y|^2} + \frac{|(x')^1 - y^1|}{|x' - y|^2} \right] dy \\ &\leq 2 \int_{B_{2\lambda a}} |\omega(t, y)| \frac{|y^1|}{|y|^2} \, dy \leq 2 \, \|\omega(t)\|_{L^p} \, \|1/|y|\|_{L^q(B_{2\lambda a})} . \end{aligned}$$

But,

$$\begin{aligned} \|1/|y|\|_{L^q(B_{2\lambda a})} &= \left(\int_{B_{2\lambda a}} \frac{1}{|y|^q} \, dy \right)^{1/q} = \left(2\pi \int_0^{2\lambda a} \frac{1}{r^q} r \, dr \right)^{1/q} \\ &= \left(2\pi \left[\frac{r^{2-q}}{2-q} \right]_0^{2\lambda a} \right)^{1/q} = \left(\frac{2\pi (2\lambda)^{2-q}}{2-q} \right)^{1/q} a^{2/q - 1}. \end{aligned}$$

Thus,

$$J \le C_1(q) \left\| \omega^0 \right\|_{L^p} a^{2/q-1},$$

where

$$C_1(q) = 2 \left(\frac{2\pi (2\lambda)^{2-q}}{2-q} \right)^{1/q}.$$

To bound K, place the origin halfway between x and x', with x' placed at $(a, 0)$ and x at $(-a, 0)$. Then

$$\begin{aligned} (x')^1 - y^1 &= a - r\cos\theta, \quad |x' - y|^2 = a^2 + r^2 - 2ar\cos\theta, \\ x^1 - y^1 &= -a - r\cos\theta, \quad |x - y|^2 = a^2 + r^2 + 2ar\cos\theta, \end{aligned}$$

so

$$\begin{aligned} f(y) = f(r, \theta) &= \left| \frac{-a - r\cos\theta}{a^2 + r^2 + 2ar\cos\theta} - \frac{a - r\cos\theta}{a^2 + r^2 - 2ar\cos\theta} \right| \\ &= \left| \frac{2a(a^2 + r^2 - 2r^2\cos^2\theta}{a^4 + 2a^2r^2 + r^4 - 4a^2r^2\cos^2\theta} \right|, \end{aligned} \tag{5.2.2}$$

and

$$K \le \|\omega(t)\|_{L^p} \left(\int_0^{2\pi} \int_{\lambda a}^R f(r, \theta)^q r \, dr \, d\theta \right)^{1/q}. \tag{5.2.3}$$

Now, the minimum value of r in the integrand of K is λa, so if we choose $\lambda \ge 1$, then $a \le r$ so

$$\left| 2a(a^2 + r^2 - 2r^2\cos^2\theta) \right| \le \left| 2a(r^2 + r^2 - 2r^2\cos^2\theta) \right| \le 8ar^2.$$

If, further, we choose λ so that

$$a^4 + 2a^2r^2 + r^4 - 4a^2r^2 = a^4 - 2a^2r^2 + r^4 \geq \frac{1}{2}r^4 \tag{5.2.4}$$

for all $r \geq \lambda a$, then

$$a^4 + 2a^2r^2 + r^4 - 4a^2r^2 \cos^2\theta \geq a^4 + 2a^2r^2 + r^4 - 4a^2r^2 \geq \frac{1}{2}r^4, \tag{5.2.5}$$

and it follows from Equation (5.2.2) that

$$f(r,\theta) \leq \frac{8ar^2}{\frac{1}{2}r^4} = \frac{16a}{r^2}. \tag{5.2.6}$$

The condition Equation (5.2.4) is equivalent to

$$1 - 2\left(\frac{r}{a}\right)^2 + \left(\frac{r}{a}\right)^4 \geq \frac{1}{2}\left(\frac{r}{a}\right)^4,$$

or

$$\left(\frac{r}{a}\right)^4 - (5/2)\left(\frac{r}{a}\right)^2 + 1 \geq 0,$$

which, by the quadratic equation, will follow if $r/a \geq \sqrt{2}$, this in turn being insured by the choice $\lambda = \sqrt{2}$. We can then choose R to be any fixed real number greater than or equal to λ, so that $R \geq \lambda a$, a being no greater than 1. It will be convenient to choose

$$R = \lambda = \sqrt{2}.$$

Then from Equation (5.2.3) and Equation (5.2.6),

$$\begin{aligned}
K &\le \|\omega(t)\|_{L^p} \left(2\pi \int_{\lambda a}^{\lambda} \left(\frac{16a}{r^2}\right)^q r dr\right)^{1/q} \\
&\le 16 \left\|\omega^0\right\|_{L^p} (2\pi)^{1/q} a \left(\int_{\lambda a}^{\lambda} r^{1-2q}\, dr\right)^{1/q} \\
&= 16 \left\|\omega^0\right\|_{L^p} (2\pi)^{1/q} a \left(\left[\frac{r^{2-2q}}{2-2q}\right]_{\lambda a}^{\lambda}\right)^{1/q} \\
&= 16 \left\|\omega^0\right\|_{L^p} \left(\frac{\pi}{1-q}\right)^{1/q} a \left(\lambda^{2-2q} - (\lambda a)^{2-2q}\right)^{1/q} \\
&= 16 \left\|\omega^0\right\|_{L^p} \left(\frac{\pi}{q-1}\right)^{1/q} \lambda^{2/q-2} a \left(a^{2-2q} - 1\right)^{1/q} \\
&= C_2(q) \left\|\omega^0\right\|_{L^p} a \left(\frac{a^{2-2q}-1}{q-1}\right)^{1/q},
\end{aligned}$$

where

$$C_2(q) = 16\pi^{1/q} \lambda^{2/q-2}.$$

As with J and K, we bound L using Hölder's inequality, but with a fixed choice of complementary exponents $\overline{p}$ and $\overline{q}$, with $p_0 \le \overline{p} < \infty$, where p_0 is as in Definition 2.1.2. This gives

$$L \le \|\omega(t)\|_{L^{\overline{p}}(\mathbb{R}^2 \setminus B_R)} \|f\|_{L^{\overline{q}}(\mathbb{R}^2 \setminus B_R)} \le 16a\|\omega^0\|_{L^{\overline{p}}} \left\|\frac{1}{r^2}\right\|_{L^{\overline{q}}(\mathbb{R}^2 \setminus B_R)},$$

where we used Equation (5.2.6). But,

$$\begin{aligned}
\left\|\frac{1}{r^2}\right\|_{L^{\overline{q}}(\mathbb{R}^2 \setminus B_R)} &= \left(2\pi \int_{\lambda}^{\infty} \frac{1}{r^{2\overline{q}}} r\, dr\right)^{1/\overline{q}} \\
&= (2\pi)^{1/\overline{q}} \left(\left[\frac{r^{2-2\overline{q}}}{2-2\overline{q}}\right]_{\lambda}^{\infty}\right)^{1/\overline{q}} = \left(\frac{\pi}{\overline{q}-1}\right)^{1/\overline{q}} \lambda^{2/\overline{q}-2}.
\end{aligned}$$

Thus,

$$L \leq C_3 a,$$

where

$$C_3 = 16\|\omega^0\|_{L^{\overline{p}}} \left(\frac{\pi}{\overline{q}-1}\right)^{1/\overline{q}} \lambda^{2/\overline{q}-2}.$$

From our bounds on J, K, and L, we have

$$I_2 \leq C_1(q) \left\|\omega^0\right\|_{L^p} a^{2/q-1} + C_2(q) \left\|\omega^0\right\|_{L^p} a \left(\frac{a^{2-2q}-1}{q-1}\right)^{1/q} + C_3 a.$$

With the identical bound on I_1, we can write

$$\begin{aligned} I &\leq \frac{1}{2\pi}\sqrt{(I_1)^2 + (I_2)^2} \\ &\leq \frac{1}{\pi}\left[C_1(q) \left\|\omega^0\right\|_{L^p} a^{2/q-1} + C_2(q) \left\|\omega^0\right\|_{L^p} a \left(\frac{a^{2-2q}-1}{q-1}\right)^{1/q} + C_3 a\right]. \end{aligned} \tag{5.2.7}$$

We will reexpress this bound in terms of p rather than q, because it will allow us to recognize the connection with Yudovich's bounds on the L^p–norms more easily.

Because the condition on admissibility in Definition 2.1.2 does not depend on the choice of p_0, we can assume that $p_0 > 2$, in which case $C_1(q)$ and $C_2(q)$ are bounded over p in $[p_0, \infty)$ by constants that we will simply call C_1 and C_2.

Then, using

$$\frac{1}{q} = \frac{p-1}{p}, \quad q = \frac{p}{p-1}, \quad q - 1 = \frac{1}{p-1},$$

we have

$$\begin{aligned}
a\left(\frac{a^{2-2q}-1}{q-1}\right)^{1/q} &= a\left(\frac{a^{-2/(p-1)}-1}{1/(p-1)}\right)^{(p-1)/p} \\
&= a\,(p-1)^{(p-1)/p}\left(a^{-2/(p-1)}-1\right)^{(p-1)/p} \\
&\leq a(p-1)\left(a^{-2/(p-1)}-1\right)^{(p-1)/p} \\
&\leq a(p-1)\left(a^{-2/(p-1)}\right)^{(p-1)/p} \\
&= (p-1)a^{1-2/p} \leq pa^{1-2/p}.
\end{aligned}$$

But

$$\frac{2}{q}-1=\frac{2p-2}{p}-\frac{p}{p}=\frac{p-2}{p}=1-\frac{2}{p},$$

so from Equation (5.2.7), we have

$$\begin{aligned}
I &\leq \frac{1}{\pi}\left[C_1\left\|\omega^0\right\|_{L^p}a^{1-2/p}+C_2\left\|\omega^0\right\|_{L^p}pa^{1-2/p}+C_3a\right] \\
&\leq C\left\|\omega^0\right\|_{L^p}pa^{1-2/p}\leq Ca^{1-2/p}p\theta(p),
\end{aligned} \tag{5.2.8}$$

where $\left\|\omega^0\right\|_{L^p}\leq\theta(p)$ as in Definition 2.1.2, and where we used the fact that for a in $(0,1/2]$, $a\leq a^{1-2/p}\leq pa^{1-2/p}$.

Since Equation (5.2.8) holds for all p in $[p_0,\infty)$, it follows that

$$I\leq\inf\left\{Ca^{1-2/p}p\theta(p):p\in[p_0,\infty)\right\},$$

or,

$$\left|v(t,x)-v(t,x')\right|\leq\mu(\left|x-x'\right|), \tag{5.2.9}$$

where

$$\mu(r) := \inf \left\{ C \left(\frac{r}{2} \right)^{1-2/p} p\theta(p) : p \in [p_0, \infty) \right\}. \tag{5.2.10}$$

The first three required properties of μ follow immediately from Equation (5.2.10).

To show that μ is nondecreasing, first observe that $r \mapsto (r/2)^{1-2\epsilon}$ is an increasing function for all ϵ in $(0, 1/p_0]$, since, having assumed that $p_0 > 2$, it follows that $1 - 2\epsilon > 0$. Thus, if $r < s$ then

$$\begin{aligned}\mu(r) &= \inf \left\{ C(r/2)^{1-2\epsilon}\theta(1/\epsilon)/\epsilon : \epsilon \in (0, 1/p_0] \right\} \\ &\leq \inf \left\{ C(s/2)^{1-2\epsilon}\theta(1/\epsilon)/\epsilon : \epsilon \in (0, 1/p_0] \right\} = \mu(s).\end{aligned}$$

Finally, to verify Equation (5.2.1), we express μ in terms of the function β of Definition 2.1.2 with $M = 1$:

$$\begin{aligned}\mu(r) &= \inf \left\{ C(r/2)^{1-2\epsilon}\theta(1/\epsilon)/\epsilon : \epsilon \in (0, 1/p_0] \right\} \\ &= (C/r) \inf \left\{ (r/2)^{2-2\epsilon}\theta(1/\epsilon)/\epsilon : \epsilon \in (0, 1/p_0] \right\} \\ &= (C/r) \inf \left\{ (r^2/4)^{1-\epsilon}\theta(1/\epsilon)/\epsilon : \epsilon \in (0, 1/p_0] \right\} \\ &= (C/r)\beta(r^2/4).\end{aligned} \tag{5.2.11}$$

Then, making the change of variables $u = r^2/4$,

$$\begin{aligned}\int_0^1 \frac{dr}{\mu(r)} = \int_0^1 \frac{dr}{(C/r)\beta(r^2/4)} &= C \int_0^1 \frac{r\,dr}{\beta(r^2/4)} \\ = C \int_0^{1/4} \frac{du}{\beta(u)} &> \infty,\end{aligned} \tag{5.2.12}$$

since β is an admissible function by assumption. Thus, μ satisfies the final of its required properties. As we show in Section 5.4, this is enough to establish the existence and uniqueness of a continuous flow.

It is worth observing that the function μ is independent of time, because the L^p–norms of the vorticity are conserved and because the constant M that is used in the definition of β in Equation (2.1.2) equals 1 for all time. Also, since $\mu(0) = 0$, the vector field $v(t)$ is uniformly continuous (see, for instance, exercise 17 p. 14 of [34]).

Finally, if the vorticity is bounded, then

$$\beta(r) := -e \left\|\omega^0\right\|_{L^{p_0} \cap L^\infty} r \log r,$$

for $r \le e^{-p_0}$ as in Section 2.4, where now $M = 1$. This gives

$$\begin{aligned}\mu(r) &= (C/r)\beta(r^2/4) = (C/r)\left(-e \left\|\omega^0\right\|_{L^{p_0} \cap L^\infty} (r^2/4) \log(r^2/4)\right) \\ &= -C \left\|\omega^0\right\|_{L^{p_0} \cap L^\infty} r\,(2 \log r - \log 4) \le C \left\|\omega^0\right\|_{L^{p_0} \cap L^\infty} r(1 - \log r),\end{aligned}$$

in accord with Chemin's result for bounded vorticity in Section 5.2 of [5].

5.3 Existence and uniqueness of the flow

We now give the details of the proof that given a function μ with the five properties of Section 5.2, that a continuous flow associated with the velocity field v exists and is unique. This approach is based upon that in Section 5.2 of [5], though it is slightly less abstract.

Theorem 5.3.1. *For all x_0 in $\mathbb{R}^2$, there exists a unique continuous function x mapping $[0, \infty)$ to $\mathbb{R}^2$—the trajectory of the point x_0—such that*

$$x(t) = x_0 + \int_0^t v(s, x(s))\, ds. \tag{5.3.1}$$

Proof. First we prove uniqueness. Suppose x and x' are two solutions to Equation (5.3.1). Let

$$\rho(t) = \left|x(t) - x'(t)\right|.$$

Then

$$\begin{aligned}\rho(t) = \left|x(t) - x'(t)\right| &= \left|\int_0^t v(s, x(s)) - v(s, x'(s))\, ds\right| \\ &\leq \int_0^t \left|v(s, x(s)) - v(s, x'(s))\right|\, ds \leq \int_0^t \mu(\left|x(s) - x'(s)\right|)\, ds \\ &= \int_0^t \mu(\rho(s))\, ds.\end{aligned}$$

By Osgood's lemma it follow that ρ is identically zero on $[0, t]$. Since t is an arbitrary positive number, it follows that any solution that exists to *Equation* (5.3.1) is unique on $[0, \infty)$.

To establish existence, we use a classical Picard scheme. We construct a series of trajectories, $\{x_k\}_{k=1}^{\infty}$, inductively by

$$x_1(t) = x_0 + \int_0^t v(s, x_0)\, ds$$

and

$$x_{k+1}(t) = x_0 + \int_0^t v(s, x_k)\, ds.$$

We restrict ourselves to the interval $[0, T]$, where $T > 0$ is arbitrary. Since we know that v is bounded over such an interval by some constant, M,

$$|x_{k+1}(t)| \leq |x_0| + \int_0^t |v(s, x_k(s))|\, ds \leq |x_0| + MT,$$

so the sequence $\{x_k\}$ is bounded over the interval $[0, T]$.

We now show that $\{x_k\}$ is a Cauchy sequence in the space of continuous functions from the interval $[0,T]$ to $\mathbb{R}^2$. Let

$$\rho_{k+1,n}(t) = |x_{k+1+n}(t) - x_{k+1}(t)| .$$

Then

$$\begin{aligned}
0 \le \rho_{k+1,n}(t) &= \left| \int_0^t v(s, x_{k+n}) - v(s, x_k)\, ds \right| \\
&\le \int_0^t |v(s, x_{k+n}) - v(s, x_k)|\, ds \le \int_0^t \mu(|x_{k+n}(s) - x_k(s)|)\, ds \\
&= \int_0^t \mu(\rho_{k,n}(s))\, ds.
\end{aligned}$$

Letting

$$\rho_k(t) = \sup_{n \ge 0} \rho_{k+1,n}(t),$$

we have

$$0 \le \rho_{k+1}(t) \le \sup_{n\ge 0} \int_0^t \mu(\rho_{k,n}(s))\, ds \le \int_0^t \sup_{n \ge 0} \mu(\rho_{k,n}(s))\, ds,$$

the last inequality being by Fatou's lemma. Because μ is increasing, it follows that

$$\widetilde{\rho}(t) := \limsup_{k\to\infty} \rho_k(t) \le \int_0^t \mu(\widetilde{\rho}(s))\, ds.$$

By Osgood's lemma, it follows that $\widetilde{\rho}$ is identically zero on the interval $[0,T]$, which means that the sequence $\{x_k\}$ is Cauchy in the complete space of continuous functions from the interval $[0,T]$ to $\mathbb{R}^2$. Hence, $x(t) = \lim_{k\to\infty} x_k(t)$ exists and is continuous—and also must satisfy Equation (5.3.1).

Since $T > 0$ was arbitrary, it follows that a unique continuous x satisfying

Equation (5.3.1) exists on the interval $[0, \infty)$. □

The flow is now defined by

$$\psi(t, x_0) := x(t).$$

The continuity of ψ in time follow from Theorem 5.3.1, the continuity in space by the results of Section 5.4.

5.4 Bounding the modulus of continuity of the flow

The flow ψ that results from the argument in Section 5.3 can be written in the form

$$\psi(t, x) = x + \int_0^t v(s, \psi(s, x))\, ds. \tag{5.4.1}$$

To bound the modulus of continuity of the flow, we want to determine how far apart two nearby points at time zero can become after time t. Toward this end, let x_1 and x_2 be two points in the plane and let $x_i(t) = \psi(t, x_i)$, $i = 1, 2$, be the corresponding trajectories of these points along the flow. (So $x_i = x_i(0)$, which is just slightly confusing notation.) Then from Equation (5.4.1),

$$\begin{aligned} |x_1(t) - x_2(t)| &\le |x_1 - x_2| + \int_0^t |v(s, \psi_1(s, x)) - v(s, \psi_2(s, x))|\, ds \\ &= |x_1 - x_2| + \int_0^t |v(s, x_1(s)) - v(s, x_2(s))|\, ds \\ &\le |x_1 - x_2| + \int_0^t \mu(x_1(s) - x_2(s))\, ds. \end{aligned}$$

Applying Osgood's lemma in the the form of Lemma 2.1.7 with $\rho(t) =$

$|x_1(t) - x_2(t)|$, $a = |x_1 - x_2|$, and $\gamma(t) = 1$, we conclude that

$$\begin{aligned} -\int_{|x_1(t)-x_2(t)|}^{1} \frac{dr}{\mu(r)} + \int_{|x_1-x_2|}^{1} \frac{dr}{\mu(r)} &= \int_{|x_1-x_2|}^{|x_1(t)-x_2(t)|} \frac{dr}{\mu(r)} \\ &= \int_{|x_1-x_2|/4}^{|x_1(t)-x_2(t)|/4} \frac{dr}{\beta(r)} \le t, \end{aligned} \tag{5.4.2}$$

where in the last integration we changed variables as in Equation (5.2.12).

Let $\Gamma_t \colon [0, \infty) \to [0, \infty)$ be defined by

$$\int_{s}^{\Gamma_t(s)} \frac{dr}{\mu(r)} = \int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{\beta(r)} = t.$$

It follows that $|x_1(t) - x_2(t)| \le \Gamma_t(|x_1 - x_2|)$, so $\delta \mapsto \Gamma_t(\delta)$ is an upper bound on the modulus of continuity of the flow at time t. Arguing much as in Section 2.4, it follows that $|x_1(t) - x_2(t)| \le \Gamma_t(|x_1 - x_2|) \to 0$ as $|x_1 - x_2| \to 0$, but that the bound on the convergence can be arbitrarily slow for initial vorticities that satisfy Yudovich bounds on the growth of their vorticity.

5.5 An example at time zero

In Section 5.2 we showed that the modulus of continuity of the vector field, given Yudovich bounds on the initial vorticity, is bounded by the same function (with a change of variables—see Equation (5.2.11)), β, that was derived from the Yudovich bounds. The analogous result for bounded vorticities is the log-Lipschitzian property of the vector field derived, for instance, in Section 5.2 of [5]. The obvious question arises of whether it is possible to achieve an analogous result to that of Section 5.3 of [5], where Chemin gives an explicit example of a bounded initial vorticity that *is* log-Lipschitzian—and no better—for all nonnegative time.

We will show that such an analogous result does hold, at time zero, in that, given any of Yudovich's sequence of example admissible L^p-norm bounds,

$$\begin{aligned}&\theta_0(p) := 1,\ \theta_1(p) := \log p,\ \theta_2(p) := \log p \cdot \log\log p, \dots,\\&\theta_m(p) := \theta_{m-1}(p) \log^m p,\end{aligned} \tag{5.5.1}$$

that there exists an initial vorticity with such a bound (asympotically with p) for which the resulting vector field has a modulus of continuity at the origin equal to a constant times the corresponding β_m function.

We will prove the following theorem, which is a generalization of Proposition 5.3.1 of Chemin:

Theorem 5.5.1. *Assume that ω_0 is symmetric by quadrant and, in the first quadrant, is nonnegative, square-symmetric, and non-increasing as a function of the distance from the origin. (By square-symmetric in the first quadrant we mean that $\omega_0(x_1, x_2) = \omega_0(\max\{x_1, x_2\}, 0)$.) Then for any λ in $(0, 1)$ and any λ' in $(0, \lambda)$ there exists a right-neighborhood of the origin on which*

$$v_0^1(x_1, 0) \geq -\frac{2(1-\lambda')}{\pi}\omega_0(x_1^\lambda, 0) x_1 \log x_1. \tag{5.5.2}$$

Proof. We proceed by considering an initial vorticity that is symmetric by quadrant, then progressively adding more and more of the symmetry assumptions.

Symmetry by quadrant. Assume that

1. $\omega_0(x) = \omega_0(x_1, x_2)$ is odd in x_1 and x_2; that is, $\omega_0(-x_1, x_2) = -\omega_0(x_1, x_2)$ and $\omega_0(x_1, -x_2) = -\omega_0(x_1, x_2)$—so also $\omega_0(-x) = \omega_0(x)$.
2. $\omega_0 \geq 0$ in Q_1.

Here we number the quadrants Q_1 through Q_4, starting with

$$Q_1 = \{(x_1, x_2) : x_1 \geq 0, x_2 \geq 0\},$$

and moving counterclockwise through the quadrants.

Then by the Biot-Savart law,

$$v_0 = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega_0(y)\, dy,$$

so

$$\begin{aligned} v_0^1(x_1, 0) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_2}{|x-y|^2} \omega_0(y)\, dy = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega_0(y)\, dy \\ &= \frac{1}{2\pi} \sum_{j=1}^{4} \int_{Q_j} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega_0(y)\, dy. \end{aligned}$$

Making the changes of variables, $u = (-y_1, y_2)$, $u = -y$, and $u = (y_1, -y_2)$ on Q_2, Q_3, and Q_4, respectively, in all cases the determinant of the Jacobian is ± 1, and we obtain

$$\begin{aligned} v_0^1(x_1, 0) = \frac{1}{2\pi} &\left[\int_{Q_1} \frac{y_2}{(x_1 - y_1)^2 + y_2^2} \omega_0(y)\, dy - \int_{Q_1} \frac{u_2}{(x_1 + u_1)^2 + u_2^2} \omega_0(u)\, du \right. \\ &\left. + \int_{Q_1} \frac{u_2}{(x_1 + u_1)^2 + u_2^2} \omega_0(u)\, du - \int_{Q_1} \frac{u_2}{(x_1 - u_1)^2 + u_2^2} \omega_0(u)\, du \right] \\ = \frac{1}{\pi} &\int_{Q_1} \left(f_1(x_1, y) - f_2(x_1, y) \right) \omega_0(y)\, dy, \end{aligned}$$

where

$$f_1(x_1, y) = \frac{y_2}{(x_1 - y_1)^2 + y_2^2}, \quad f_2(x_1, y) = \frac{y_2}{(x_1 + y_1)^2 + y_2^2}.$$

It follows from $(x_1 - y_1)^2 + y_2^2 \leq (x_1 + y_1)^2 + y_2^2$ on Q_1 that $f_1(x_1, y) \geq f_2(x_1, y)$, equality holding only when $y_1 = 0$ or $y_2 = 0$. Therefore,

$$v_1^0(x_1, 0) \geq 0. \tag{5.5.3}$$

Since $\omega_0(x_1, -x_2) = -\omega_0(x_1, x_2)$, if $\omega(t, x_1, x_2)$ is a solution to (E) then $-\omega(t, x_1, -x_2)$ is also a solution. Then by the Biot-Savart law of Equation (2A.1),

$$\begin{aligned}
v^2(t, x_1, -x_2) &= (K^2 * \omega)(t, x_1, x_2) \\
&= \int_{\mathbb{R}^2} K^2(x_1 - y_1, -x_2 - y_2)\omega(t, y_1, y_2)\, dy \\
&= -\int_{\mathbb{R}^2} K^2(x_1 - y_1, x_2 + y_2)\omega(t, y_1, -y_2)\, dy \\
&= -\int_{\mathbb{R}^2} K^2(x_1 - y_1, x_2 - (-y_2))\omega(t, y_1, -y_2)\, dy \\
&= -v^2(t, x_1, x_2),
\end{aligned}$$

where we used the fact that $K^2(x_1, -x_2) = K^2(x_1, x_2)$ and the symmetry in ω observed above. A similar calculation shows that $v^1(t, -x_1, x_2) = -v^1(t, x_1, x_2)$. Thus, the velocity along the x-axis is directed along the x-axis, and the velocity along the y-axis is directed along the y-axis, so the axes are preserved by the flow. In particular, the origin is fixed, and $v(t, 0, 0) = 0$ for all $t \geq 0$.

Constant on squares, and symmetric by quadrant. Now make the additional assumption that $\omega_0 = 2\pi$ on the square $[0, r] \times [0, r]$. Then

$$\begin{aligned}
v_0^1(x_1, 0) &= 2\pi \frac{1}{\pi} \int_{[0,r]\times[0,r]} (f_1(x_1, y) - f_2(x_1, y))\ dy \\
&= 2\int_0^r \int_0^r \frac{y_2}{(x_1 - y_1)^2 + y_2^2}\, dy_2\, dy_1 - 2\int_0^r \int_0^r \frac{y_2}{(x_1 + y_1)^2 + y_2^2}\, dy_2\, dy_1.
\end{aligned}$$

Both of the inner integrals above are of the form

$$\begin{aligned}
\int_0^r \frac{y_2}{a^2 + y_2^2}\, dy_2 &= \frac{1}{2}\int_0^{r^2} \frac{du}{a^2 + u}\, du = \frac{1}{2}\left[\log(a^2 + u)\right]_0^{r^2} \\
&= \frac{1}{2}\log(a^2 + r^2) - \frac{1}{2}\log(a^2),
\end{aligned}$$

where a is $x_1 - y_1$ and $x_1 + y_1$ in the two integrals. This gives

$$\begin{aligned} v_0^1(x_1,0) = 2\int_0^r \Big[\frac{1}{2}\log((x_1-y_1)^2+r^2) - \frac{1}{2}\log((x_1-y_1)^2) \\ -\frac{1}{2}\log((x_1+y_1)^2+r^2) + \frac{1}{2}\log((x_1+y_1)^2)\Big]\, dy_1 \\ = \widetilde{v}_0^1(x_1,0) + \overline{v}_0^1(x_1,0), \end{aligned} \tag{5.5.4}$$

where

$$\begin{aligned} \widetilde{v}_0^1(x_1,0) &= \int_0^r \log((x_1+y_1)^2)\, dy_1 - \int_0^r \log((x_1-y_1)^2)\, dy_1, \\ \overline{v}_0^1(x_1,0) &= \psi_r(x_1) = \int_0^r \log\frac{(x_1-y_1)^2+r^2}{(x_1+y_1)^2+r^2}\, dy_1. \end{aligned} \tag{5.5.5}$$

But,

$$\int_0^r \log((x_1-y_2)^2)\, dy_1 = 2(r-x_1)\log|r-x_1| + 2x_1\log x_1 - 2r,$$

which we can verify by integrating separately for the case $r \le x_1$ and for $r \ge x_1$ (where we split the integral into two parts), and

$$\int_0^r \log((x_1+y_2)^2)\, dy_1 = 2(r+x_1)\log(r+x_1) - 2x_1\log x_1 - 2r,$$

so

$$\begin{aligned} \widetilde{v}_0^1(x_1,0) &= -4x_1\log x_1 + 2(r+x_1)\log(r+x_1) - 2(r-x_1)\log|r-x_1| \\ &= -4x_1\log x_1 + \phi_r(x_1), \end{aligned}$$

where

$$\phi_r(x_1) = 2(r+x_1)\log(r+x_1) - 2(r-x_1)\log|r-x_1|\,. \tag{5.5.6}$$

So far, our analysis is much as in the proof of Proposition 5.3.1 p. 95 of [5], the significant difference being that for Chemin $r = 1$. This allows Chemin to conclude that $v_0^1(x_1, 0) \geq -2x_1 \log x_1$ in some right-neighborhood of the origin, but we will find that for an arbitrary r in $(0, 1)$, the size of this neighborhood shrinks with r. This will be of importance when we ultimately consider an initial vorticity that is a sum of vorticities that are square-symmetric by quadrant. We seek both to refine Chemin's bound slightly and to establish a lower bound on the size of the right-neighborhood.

By Equation (5.5.4), Equation (5.5.5), and Equation (5.5.6),

$$v_0^1(x_1, 0) = -4x_1 \log x_1 + \phi_r(x_1) + \psi_r(x_1). \tag{5.5.7}$$

Suppose we can show for x_1 lying in $(0, 1)$ but also bounded by some function of r, that

$$\phi_r(x_1) + \psi_r(x_1) \geq C_0 x_1 \log x_1, \tag{5.5.8}$$

where $0 < C_0 < 4$. (Observe that $x_1 \log x_1 < 0$.) This will insure, that

$$v_0^1(x_1, 0) \geq -C_1 x_1 \log x_1 \tag{5.5.9}$$

for the given range of x_1, where $C_1 = 4 - C_0$ is a positive constant.

We will first show that $\psi_r(x_1)$ is negligible compared to the other two terms in Equation (5.5.7). Then we will show that $\phi_r(x_1)$, while comparable in magnitude to $-4x_1 \log x_1$, can be made sufficiently smaller than it so that Equation (5.5.8) will hold, as long as we restrict x_1 so that $x_1 < r^\lambda$ for some λ in $(0, 1)$.

We have,

$$
\begin{aligned}
\frac{\partial}{\partial r}\psi_r(x_1) &= -2\tan^{-1}\left(\frac{x_1 - r}{r}\right) - 2\tan^{-1}\left(\frac{x_1 + r}{r}\right) + 4\tan^{-1}\left(\frac{x_1}{r}\right) \\
&\quad + \log\left(\frac{x_1^2 + 2r^2 - 2x_1 r}{x_1^2 + 2r^2 + 2x_1 r}\right) \\
&= -2\tan^{-1}(w-1) - 2\tan^{-1}(w+1) + 4\tan^{-1}(w) \\
&\quad + \log\left(\frac{w^2 + 2 - 2w}{w^2 + 2 + 2w}\right) \\
&=: h(w),
\end{aligned}
$$

where $w = x_1/r < 1$, since $0 \le x_1 < r$. Also,

$$
h'(w) = 4\frac{w^2(w^2-4)}{(w^2+2-2w)(w^2+2+2w)(1+w^2)},
$$

which is is negative for w in $(0,2)$, is zero at $w = 2$, and is positive for w in $(2,\infty)$. This means that $h(w)$ decreases from a value of 0 at 0 to a negative value at $w = 2$, then monotonically increases. However, it is also true that $\lim_{w\to\infty} h(w) = 0$, which means that $h(w) < 0$ for all $w > 0$.

What we have shown is that that for all $x_1, r > 0$,

$$
\frac{\partial}{\partial r}\psi_r(x_1) = h(w) < 0,
$$

so for a fixed value of x_1, $\psi_r(x_1)$ is a decreasing function of r. Restricting r to lie in the range $(0,1]$, we can therefore bound $\psi_r(x_1)$ from below by $\psi_1(x_1)$. But $\psi_1(x_1)$ is a smooth function equal to zero at the origin, so

$$
\psi_r(x_1) \ge \psi_1(x_1) \ge -C_3 x_1
$$

for some positive constant C_3 in a fixed neighborhood of the the origin that is independent of the value of r. This means that by restricting x_1 to a possibly smaller neighborhood of the origin, we can insure that the contribution of $\psi_r(x_1)$ to the left-hand side of Equation (5.5.8) is as small as required. Hence, we can replace Equation (5.5.8) by the requirement that

$$\phi_r(x_1) \geq C_0 x_1 \log x_1 \tag{5.5.10}$$

for all $x_1 < r$, where $0 < C_0 < 4$, and from this Equation (5.5.9) will follow.

We have,

$$\begin{aligned}
\frac{\phi_r(x_1)}{2} &= (r + x_1)\log(r + x_1) - (r - x_1)\log(r - x_1) \\
&= r(1 + w)\log(r(1 + w)) - r(1 - w)\log(r(1 - w)) \\
&= r(1 + w)\log r - r(1 - w)\log r + r(1 + w)\log(1 + w) \\
&\qquad - r(1 - w)\log(1 - w) \\
&= r\log r + rw\log r - r\log r + rw\log r \\
&\qquad + r\left[\log(1 + w) - \log(1 - w)\right] + rw\left[\log(1 + w) + \log(1 - w)\right] \\
&= 2rw\log r + r\log\left(\frac{1 + w}{1 - w}\right) + rw\log(1 - w^2).
\end{aligned}$$

Since

$$\begin{aligned}
&r\log\left(\frac{1 + w}{1 - w}\right) + rw\log(1 - w^2) \\
&\quad \geq r\log\left(\frac{1 + w}{1 - w}\right) + r\log(1 - w^2) = r\log\left(\frac{1 + w}{1 - w}(1 + w)(1 - w)\right) \\
&\quad = 2r\log(1 + w) \geq 0,
\end{aligned}$$

it follows that

$$\phi_r(x_1) \geq 4rw \log r = 4x_1 \log r.$$

We now add the restriction that

$$x_1^\lambda \leq r$$

for some λ in $(0, 1)$. Then

$$\phi_r(x_1) \geq 4x_1 \log r \geq 4x_1 \log x_1^\lambda = 4\lambda x_1 \log x_1,$$

which is Equation (5.5.10) with $C_0 = 4\lambda$. It follows that

$$v_0^1(x_1, 0) \geq -(4 - 4\lambda)x_1 \log x_1 - C_3 x_1. \tag{5.5.11}$$

Square symmetry by quadrant We now let our initial vorticity be a sum of initial vorticities each defined as above on successively smaller squares. We will write this as

$$\omega_0(x) = 2\pi \sum_{k=1}^{\infty} a_k \mathbf{1}_{[0,r_k]\times[0,r_k]}(x) \tag{5.5.12}$$

on Q_1, where $r_k \searrow 0$ as $k \to \infty$, and where each $a_k \geq 0$. This implies that $\omega_0(x_1, 0)$ is a non-increasing function of $x_1 > 0$ and is "square-symmetric" by quadrant. We also assume, for convenience in the bounds we obtain below, that $r_1 < 1$, though any positive value of r_1 would work with a slight modification to our argument.

It follows from Equation (5.5.11) and Equation (5.5.3) that

$$v_0^1(x_1, 0) \geq \left(\sum_{k \leq \eta_\lambda(x_1)} a_k \right) (-4(1-\lambda) x_1 \log x_1 - C_3 x_1), \tag{5.5.13}$$

where

$$\eta_\lambda(x_1) = \max_{k \in \mathbb{Z}^+} \left\{ r_k \geq x_1^\lambda \right\}.$$

We can write ω_0 as

$$\omega_0(x) = 2\pi \int_0^1 \alpha(s) 1_{[0,s]\times[0,s]}(x)\, ds, \tag{5.5.14}$$

for some measurable, nonnegative function $\alpha \colon (0,1) \to [0, \infty)$. This means that

$$\omega_0(x_1, 0) = 2\pi \int_{x_1}^1 \alpha(s)\, ds. \tag{5.5.15}$$

Approximating ω_0 in Equation (5.5.14) by the sum in Equation (5.5.12), the expression corresponding to Equation (5.5.13) is

$$v_0^1(x_1, 0) \geq \left(\int_{x_1^\lambda}^1 \alpha(s)\, ds \right) (-4(1-\lambda) x_1 \log x_1 - C_3 x_1).$$

The bound in Equation (5.5.2) then follows from Equation (5.5.15). (Choosing $\lambda' < \lambda$ allows us to ignore the factor $-C_3 x_1$.) $\square$

Yudovich's sequence of initial vorticities. The difficulty in applying Theorem 5.5.1 lies in choosing a function ω_0 that both gives a bound in Equation (5.5.2) that is an admissible function in the sense of β in Definition 2.1.2, and is such that we can obtain an asymptotic formula for its L^p-norms that obeys Yudovich bounds on the vorticity. We will show that each of the initial vorticities in Equation (5.5.1) satisfy both these requirements.

That the first of the two requirements is satisfied is easily seen for $m = 2$, the argument for $m > 2$ being essentially the same, just more difficult to write down. We have from Equation (5.5.2) that

$$\begin{aligned} v_0^1(x_1, 0) &\geq -\frac{2(1-\lambda')}{\pi} \log\log\left(1/(x_1^\lambda)\right) x_1 \log x_1 \\ &= -\frac{2(1-\lambda')}{\pi} \log\left(\lambda \log(1/x_1)\right) x_1 \log x_1 \\ &= -\frac{2(1-\lambda')}{\pi} \left[(\log\lambda) x_1 \log x_1 + \left(\log\log(1/x_1)\right) x_1 \log x_1\right] \\ &\geq C x_1 \log(1/x_1) \cdot \log\log(1/x_1), \end{aligned}$$

the last bound applying for sufficiently small x_1, where $0 < C < 1$.

That the second of the two requirements is satisfied is shown in the following theorem.

Theorem 5.5.2. *Let ω_0 have the symmetry described in Theorem 5.5.1 with*

$$\omega_0(x_1, 0) = \theta_m(1/x_1)/\log(1/x_1) = \log^2(1/x_1) \cdots \log^m(1/x_1),$$

for $0 < x_1 < \exp^{m-1}(-1)$, and ω_0 equal to zero elsewhere in the first quadrant. Then for all sufficiently large p,

$$e^{-2} \log p \cdots \log^{m-1} p \leq \|\omega_0\|_{L^p} \leq C \log p \cdots \log^{m-1} p,$$

where $C = 1$ for $m \geq 2$ and is asymptotically equal to 1 for large p when $m = 1$.

Proof. Because of the symmetry of ω_0,

$$\begin{aligned} \|\omega_0\|_{L^p}^p &= 4 \int_0^{\exp^{m-1}(-1)} 2 \int_0^{x_1} (\omega_0(x_1, 0))^p \, dx_2 \, dx_1 \\ &= 8 \int_0^{\exp^{m-1}(-1)} x_1 \left[\log^2(1/x_1) \cdots \log^m(1/x_1)\right]^p \, dx_1. \end{aligned} \tag{5.5.16}$$

Making the change of variables, $u = \log(1/x_1) = -\log x_1$, it follows that $x_1 = e^{-u}$ and $du = -(1/x_1)\,dx$ so $dx_1 = -e^{-u}\,du$. Thus,

$$\begin{aligned}
\|\omega_0\|_{L^p}^p &= 8\int_{\infty}^{\exp^{m-2}(1)} e^{-u}\left[\log u \cdots \log^{m-1} u\right]^p (-e^{-u})\,du \\
&= 8\int_{\exp^{m-2}(1)}^{\infty} e^{-2u}\left[\log u \cdots \log^{m-1} u\right]^p\,du.
\end{aligned}$$

Making the further change of variables $x = u/p$, so that $u = px$ and $du = p\,dx$, we have

$$\|\omega_0\|_{L^p}^p = 8p\int_{\exp^{m-2}(1)/p}^{\infty} e^{-2xp}\left[\log(xp)\cdots\log^{m-1}(xp)\right]^p\,dx. \tag{5.5.17}$$

(Note that p is eventually large enough that the lower bound of integration in Equation (5.5.17) is less than 1.)

Obtaining an upper bound on $\|\omega_0\|_{L^p}$ is easy. For $x \geq 1$,

$$\log(xp)\cdots\log^{m-1}(xp) \geq \log p \cdots \log^{m-1} p,$$

so

$$\begin{aligned}
\|\omega_0\|_{L^p}^p &\geq 8p\int_1^{\infty} e^{-2xp}\left[\log p \cdots \log^{m-1} p\right]^p\,dx \\
&= 8p\left[\log p \cdots \log^{m-1} p\right]^p \int_1^{\infty} e^{-2xp}\,dx \\
&= 8p\left[\log p \cdots \log^{m-1} p\right]^p \left(-\frac{1}{2p}\right)\left[e^{-2xp}\right]_1^{\infty} \\
&= 4\left[e^{-2}\log p \cdots \log^{m-1} p\right]^p.
\end{aligned}$$

Thus, for all sufficiently large p,

$$\|\omega_0\|_{L^p} \geq e^{-2}\log p \cdots \log^{m-1} p.$$

We now obtain a lower bound on $\|\omega_0\|_{L^p}$. For $x \leq 1$,

$$\log(xp) \cdots \log^{m-1}(xp) \leq \log p \cdots \log^{m-1} p,$$

while for $x \geq 1$ and sufficiently large p, Equation (5.5.18) holds. Thus,

$$\begin{aligned}
\|\omega_0\|_{L^p}^p &= 8p \left(\int_{\exp^{m-2}(1)/p}^{1} + \int_1^\infty \right) e^{-2xp} \left[\log(xp) \cdots \log^{m-1}(xp) \right]^p \, dx \\
&\leq 8p \int_{\exp^{m-2}(1)/p}^{1} e^{-2xp} \left[\log p \cdots \log^{m-1} p \right]^p \, dx \\
&\qquad + 8p \int_1^\infty e^{-2xp} \left[\left[\log p \cdots \log^{m-1} p \right] e^{x-1} \right]^p \, dx \\
&\leq 8p \left[\log p \cdots \log^{m-1} p \right]^p \left[\int_{\exp^{m-2}(1)/p}^{1} e^{-2xp} \, dx + e^{-p} \int_1^\infty e^{-xp} \, dx \right] \\
&= 8p \left[\log p \cdots \log^{m-1} p \right]^p \left[\frac{1}{2p} \left(e^{-p \exp^{m-1}(1)/p} - e^{-p} \right) + e^{-p} \frac{e^{-p}}{p} \right] \\
&\leq \frac{8}{\exp^m(1)} \left[\log p \cdots \log^{m-1} p \right]^p .
\end{aligned}$$

It follows that for sufficiently large p,

$$\|\omega_0\|_{L^p} \leq \left(\frac{8}{\exp^m(1)} \right)^{1/p} \log p \cdots \log^{m-1} p,$$

which completes the proof. □

Remark: Essentially the same proof gives the same result for radial or other symmetries (except that e^{-2} might become a different constant), as well as for any spatial dimension. In dimension n with radial symmetry, for example, the factor of x_1 in the original integration in Equation (5.5.16) would become r^n and the factor of 8 would become the surface area of the unit sphere. The r^n factor would become e^{-nxp} (even for $n = 1$) in Equation (5.5.17), which does not materially affect the rest of the argument.

Lemma 5.5.3. *Let m be a positive integer. Then for sufficiently large p,*

$$\log(xp) \cdots \log^m(xp) \leq [\log p \cdots \log^m p] \, e^{x-1}. \tag{5.5.18}$$

Proof. We prove this for $m = 2$, the proof for other values of m being entirely analogous. First, Equation (5.5.18) holds if and only if

$$f(x) := \log\log(xp) + \log\log\log(xp) \leq g(x) := \log(\log p \log\log p) + x - 1.$$

Because equality holds for $x = 1$, our result will follow if we can show that $f' \leq g'$ for all $x \geq 1$ and sufficiently large p. This is, in fact, true, since

$$f' = \frac{1}{x\log(xp)} + \frac{1}{x\log(xp)\log\log(xp)} \leq 1 = g'$$

for all $x \geq 1$ and $p \geq e^e$. □

5.6 The same example over time

Theorem 5.6.1. *Assume that ω_0 is symmetric by quadrant and, in the first quadrant, is nonnegative, radially-symmetric, and non-increasing as a function of the distance from the origin. Then for any λ in $(0, 1)$ and any λ' in $(0, \lambda)$ there exists a right-neighborhood $\mathcal{N}$ of the origin on which*

$$v^1(t, x_1, 0) \geq -\frac{2(1-\lambda')}{\pi}\omega_0(2^{\lambda/2}\Gamma_t(x_1)^\lambda)x_1 \log x_1 \tag{5.6.1}$$

for all time $t \geq 0$, where Γ_t is defined as in Theorem 5.1.1.

Further, if $x_1(t)$ is the solution to

$$\frac{dx_1(t)}{dt} = -\frac{2(1-\lambda')}{\pi}\omega_0(2^{\lambda/2}\Gamma_t(x_1(t))^\lambda)x_1(t)\log x_1(t)$$

with $x_1(0) = a$ *in* $\mathcal{N}$, *then* $\psi^1(t, a, 0) \geq x_1(t)$ *for all* $t \geq 0$.

Remark: We state the theorem for ω_0 radially-symmetric within each quadrant rather than square-symmetric because the derivation is simpler. But it is clear that a nearly identical result holds for square-symmetric functions as well. In fact, using reasoning entirely analogous to that of the proof, a similar result holds as long as the level sets of ω_0 have a nonzero lower bound and a finite upper bound on the ratio of their closest distance to the origin to the their furthest distance from the origin.

Proof of Theorem 5.6.1. First observe that $\omega_0 \geq \widetilde{\omega}_0$ where $\widetilde{\omega}_0$ is square-symmetric and

$$\widetilde{\omega}_0(x_1, 0) = \omega_0(\sqrt{2}\,|x_1|).$$

Examining the proof of Theorem 5.5.1, we see that

$$v_0^1(x_1, 0) \geq -\frac{2(1-\lambda')}{\pi}\widetilde{\omega}_0(x_1^\lambda, 0)x_1 \log x_1,$$

because the positivity of ω_0 in the first quadrant, and the symmetry by quadrant, makes all the estimates in the proof of Theorem 5.5.1 with $\widetilde{\omega}_0$ in place of ω_0 underestimates for $v_0^1(x_1, 0)$. Thus,

$$v_0^1(x_1, 0) \geq -\frac{2(1-\lambda')}{\pi}\omega_0(2^{\lambda/2}x_1^\lambda)x_1 \log x_1 \tag{5.6.2}$$

is the equivalent of Theorem 5.5.1 for initial vorticities radially-symmetric within each quadrant.

Our approach will be to show that the flow can only move the vorticity by a limited amount over time, and that the maximum possible movement in the worst possible direction results in the bound in Equation (5.6.1).

From the second part of Theorem 5.1.1,

$$\psi(t,x) = x + \int_0^t v(s,\psi(s,x))\, ds.$$

Because $v^1(t,0,0) = 0$ for all time $t \geq 0$, it follows from Equation (5.2.9) that

$$|v(s,\psi(s,x))| = |v(s,\psi(s,x)) - v(s,0,0)| \leq \mu(|\psi(s,x)|).$$

We can then obtain upper and lower bounds on $\psi(t,x)$ using the triangle inequality:

$$|\psi(t,x)| \leq |x| + \int_0^t |v(s,\psi(s,x))|\; ds \leq |x| + \int_0^t \mu(|\psi(s,x)|)\, ds,$$

and

$$|\psi(t,x)| \geq |x| - \int_0^t |v(s,\psi(s,x))|\; ds \geq |x| - \int_0^t \mu(|\psi(s,x)|)\, ds. \tag{5.6.3}$$

To achieve a lower bound on $\omega_0(t)$ and hence a lower bound on $v_0^1(x_1,0)$, we assume that Equation (5.6.3) holds for all x and t. This will minimize the lower bound on $v_0^1(x_1,0)$ in Equation (5.5.2), since ω_0 decreases with the distance from the origin. (This is a physically unrealizable flow, since it is not measure preserving. It still gives us a perfectly valid point-by-point bound, however.) Letting $a = |x|$ and $\rho(t) = \psi(t,x)$, it follows in this worst case that

$$\rho(t) = a - \int_0^t \mu(\rho(s))\, ds,$$

with the restriction that ρ remain continuous (which would follow from the continuity of the integral even if ρ remained only measurable).

Since ρ is continuous and μ is continuous, ρ is, in fact, differentiable, with

$$\rho'(t) = -\mu(\rho(t)).$$

Thus,

$$\begin{aligned}
\frac{d}{dt}\rho(t) = -\mu(\rho(t)) &\implies \frac{d\rho}{\mu(\rho)} = -dt \\
&\implies \int_{\rho(0)}^{\rho(t)} \frac{ds}{\mu(s)} = \int_{a}^{\rho(t)} \frac{ds}{\mu(s)} = -t \\
&\implies \int_{\rho(t)}^{a} \frac{ds}{\mu(s)} = t,
\end{aligned}$$

or in general, since this is the worst case estimate, that

$$\int_{|\psi(t,x)|}^{|x|} \frac{ds}{\mu(s)} \le t.$$

Inverting the roles of x and $\psi(t,x)$, it follows that

$$\int_{|x|}^{\left|\psi^{-1}(t,x)\right|} \frac{ds}{\mu(s)} = \int_{|x|/4}^{\left|\psi^{-1}(t,x)\right|/4} \frac{dr}{\beta(r)} \le t,$$

as in Equation (5.4.2). Thus,

$$\left|\psi^{-1}(t,x)\right| \le \Gamma_t(|x|),$$

where Γ_t is as in the statement of Theorem 5.1.1, and

$$\omega(t,x) = \omega_0(\left|\psi^{-1}(t,x)\right|) \ge \omega_0(\Gamma_t(|x|)),$$

since ω_0 decreases with increasing distance from the origin. Therefore, arguing as in Equation (5.6.2), we obtain Equation (5.6.1).

The lower bound on the flow, $\psi^1(t, a, 0) \geq x_1(t)$, follows from using the minimum possible value of $v^1(t, x_1, 0)$ in Equation (5.6.1), setting it equal to $dx_1(t)/dt$, and integrating over time. $\square$

We now apply Theorem 5.6.1 to the first two in the sequence of Yudovich's vorticity bounds in Section 2.2. The first of these is for bounded vorticity—say $|\omega_0| = \mathbf{1}_{B_{1/2}}$ so that $\|\omega_0\|_{L^1 \cap L^\infty} = 1$. This gives (see Section 2.4)

$$\beta(r) = -er \log r$$

for $r < 1$, so for $x_1 > 0$,

$$\begin{aligned} \int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{\beta(r)} = -\int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{er \log r} &= -\frac{1}{e} \left[\log(-\log r)\right]_{s/4}^{\Gamma_t(s)/4} = t \\ &\implies \log(-\log(s/4)) - \log(-\log(\Gamma_t(s)/4)) = et \\ &\implies \Gamma_t(s) = 4(s/4)^{e^{-et}}. \end{aligned}$$

Thus, Theorem 5.6.1 gives

$$\begin{aligned} v^1(t, x_1, 0) &\geq -\frac{2(1-\lambda')}{\pi} \omega_0(2^{\lambda/2} 4(x_1/4)^{\lambda e^{-et}}) x_1 \log x_1 \\ &\geq -\frac{2(1-\lambda')}{\pi} x_1 \log x_1 \end{aligned}$$

as long as $2^{\lambda/2} 4(x_1/4)^{\lambda e^{-et}} < 1/2$. (This also implies that $4\Gamma_t(x_1^\lambda/4) < 1$, which is required for our expression for β to be valid.)

Solving $dx_1(t)/dt = -(2(1-\lambda')/\pi) x_1 \log x_1$ with $x_1(0) = a$ gives

$$\psi^1(t, a, 0) \geq x_1(t) = a^{\exp(-2(1-\lambda')t/\pi)}.$$

Since $\psi(t,0,0)=0$,

$$\frac{|\psi(t,a,0)-\psi(t,0,0)|}{a^\alpha} \geq \frac{\left|\psi^1(t,a,0)\right|}{a^\alpha} \geq a^{\exp(-2(1-\lambda')t/\pi)-\alpha},$$

so the flow can be in no Hölder space with exponent $\alpha > \exp(-2(1-\lambda')t/\pi)$. (This is essentially as in Theorem 5.3.1 p. 94 of [5].)

The second example bound in Yudovich's sequence is $\theta(p) = \log p$, which is produced asymptotically when $\omega_0(x) = \log(-\log x)$ on the unit ball. (This is by Theorem 5.5.2 and the comment following it.) As mentioned in Section 2.4 (using Yudovich's argument),

$$\beta(r) \leq er\log(1/r)\log\log(1/r).$$

Then,

$$\begin{aligned}
-\frac{1}{e}\left[\log\log(-\log r)\right]_{s/4}^{\Gamma_t(s)/4} &= \int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{er\log(1/r)\log\log(1/r)} \\
&\leq \int_{s/4}^{\Gamma_t(s)/4} \frac{dr}{\beta(r)} = -\frac{1}{e}\left[\log\log(-\log r)\right]_{s/4}^{\Gamma_t(s)/4} = t \\
\implies \log(-\log(\Gamma_t(s)/4)) &\geq e^{-et}\log(-\log(s/4)).
\end{aligned}$$

Ignoring the factors of $1/4$, the factor $2^{\lambda/2}$, and using x_1 in place of x_1^λ would give, for $x_1 > 0$,

$$\begin{aligned}
v^1(t,x_1,0) &\geq -\frac{2(1-\lambda')}{\pi}\omega_0(\Gamma_t(x_1))x_1\log x_1 \\
&= -\frac{2(1-\lambda')}{\pi}\log(-\log(\Gamma_t(x_1)))x_1\log x_1 \\
&\geq -\frac{2(1-\lambda')}{\pi}e^{-et}\log(-\log x_1)x_1\log x_1,
\end{aligned}$$

which we note is e^{-et} times the lower bound for $v_0^1(x_1,0)$. (Without ignoring the factors above, we introduce a small additional time-varying factor.)

It would be reasonable to think that this exponentially decreasing lower bound on $v^1(t, x_1, 0)$ would persist for the rest of Yudovich's example vorticity bounds, but it does not: at least it does not follow from Theorem 5.6.1.

Solving for

$$\frac{dx_1(t)}{dt} = -\frac{2(1-\lambda')}{\pi} e^{-et} \log(-\log x_1) x_1 \log x_1$$

with $x_1(0) = a$, we get

$$\log\log(-\log x_1(t)) = \log\log(-\log a) + \frac{2(1-\lambda')}{\pi e}\left(e^{-et} - 1\right),$$

so

$$\begin{aligned}\psi^1(t, a, 0) \geq x_1(t) &= \exp\left(-(-\log a)^{\exp\left(2(1-\lambda')(e^{-et}-1)/\pi e\right)}\right) \\ &= e^{-(-\log a)^\gamma},\end{aligned}$$

where $\gamma = \exp\left(2(1-\lambda')(e^{-et} - 1)/\pi e\right)$.

Observe that $\gamma < 1$ for all $t > 0$. Thus, for any α in $(0, 1)$ and all $t > 0$,

$$\begin{aligned}\|\psi - \mathrm{Id}\|_{C^\alpha} &\geq \lim_{a \to 0^+} \frac{\psi^1(t, a, 0) - \psi^1(t, 0, 0)}{a^\alpha} \geq \lim_{a \to 0^+} \frac{x_1(t)}{a^\alpha} \\ &= \lim_{a \to 0^+} \frac{e^{-(-\log a)^\gamma}}{e^{(-\log a)\alpha}} = \lim_{u \to \infty} \frac{e^{-u^\gamma}}{e^{-\alpha u}} = \lim_{u \to \infty} e^{\alpha u - u^\gamma} = \infty.\end{aligned}$$

We conclude that the flow lies in no Hölder space of positive exponent for all positive time, a result that we state explicitly as a corollary of Theorem 5.6.1.

Corollary 5.6.2. *There exists initial velocities satisfying the conditions of Theorem 2.1.5 for which the unique solution to (E) has an associated flow lying, for all positive time, in no Hölder space of positive exponent.*

Bibliography

[1] H. Abidi and R. Danchin. Optimal bounds for the inviscid limit of Navier-Stokes equations. *Asymptot. Anal.*, 38(1):35–46, 2004.

[2] Tom M. Apostol. *Calculus. Vol. II: Multi-variable calculus and linear algebra, with applications to differential equations and probability.* Second edition. Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1969.

[3] H. Bahouri and J.-Y. Chemin. Équations de transport relatives á des champs de vecteurs non-lipschitziens et mécanique des fluides. *Arch. Rational Mech. Anal.*, 127(2):159–181, 1994.

[4] Jean-Yves Chemin. A remark on the inviscid limit for two-dimensional incompressible fluids. *Comm. Partial Differential Equations*, 21(11-12):1771–1779, 1996.

[5] Jean-Yves Chemin. *Perfect incompressible fluids*, volume 14 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1995 French original by Isabelle Gallagher and Dragos Iftimie.

[6] J.-Y. Chemin. Sur le mouvement des particules d'un fluide parfait incompressible bidimensionnel. *Invent. Math.*, 103(3):599–629, 1991.

[7] Thierry Clopeau, Andro Mikelić, and Raoul Robert. On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions. *Nonlinearity*, 11(6):1625–1636, 1998.

[8] Peter Constantin and Jiahong Wu. Inviscid limit for vortex patches. *Nonlinearity*, 8(5):735–742, 1995.

[9] Peter Constantin and Jiahong Wu. The inviscid limit for non-smooth vorticity. *Indiana Univ. Math. J.*, 45(1):67–81, 1996.

[10] Charles R. Doering and J. D. Gibbon. *Applied analysis of the Navier-Stokes equations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1995.

[11] Giovanni P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, New York, 1994. Linearized steady problems.

[12] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[13] Willi Jäger and Andro Mikelić. On the roughness-induced effective boundary conditions for an incompressible viscous flow. *J. Differential Equations*, 170(1):96–122, 2001.

[14] Willi Jäger and Andro Mikelić. Couette flows over a rough boundary and drag reduction. *Comm. Math. Phys.*, 232(3):429–455, 2003.

[15] Tosio Kato. Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary. In *Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983)*, volume 2 of *Math. Sci. Res. Inst. Publ.*, pages 85–98. Springer, New York, 1984.

[16] James P. Kelliher. The inviscid limit for two-dimensional incompressible fluids with unbounded vorticity. *Math. Res. Lett.*, 11(4):519–528, 2004.

[17] James P. Kelliher. Navier-stokes equations with Navier boundary conditions for a bounded domain in the plane. *Submitted.*

[18] James P. Kelliher. Some calculations pertaining to Roger Temam's "Navier-Stokes" equations. *Available at http://www.ma.utexas.edu/users/kelliher/.*

[19] O. A. Ladyženskaja. Solution "in the large" to the boundary-value problem for the Navier-Stokes equations in two space variables. *Soviet Physics. Dokl.*, 123 (3):1128–1131 (427–429 Dokl. Akad. Nauk SSSR), 1958.

[20] O. A. Ladyženskaja. Solution "in the large" of the nonstationary boundary value problem for the Navier-Stokes system with two space variables. *Comm. Pure Appl. Math.*, 12:427–433, 1959.

[21] Jean Leray. Etude de diversesequations integrales non lineaires et de quelques problèmes que pose l'hydrodynamique. *J. Math. Pures Appl.*, 12:1–82, 1933.

[22] Jean Leray. Essai sur les mouvements plans d'un liquide visqueux que limitent des parois. *J. Math. Pures Appl.*, 13:341–418, 1934.

[23] Jean Leray. Essai sur le mouvement d'un liquide visqueux emplissant l'espace, acta math. *Acta Math.*, 63:193–248, 1934.

[24] Jacques-Louis Lions and Giovanni Prodi. Un théorème d'existence et unicité dans les équations de Navier-Stokes en dimension 2. *C. R. Acad. Sci. Paris*, 248:3519–3521, 1959.

[25] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, 1969.

[26] Pierre-Louis Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.

[27] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and G. Planas. On the inviscid limit for 2d incompressible flow with Navier friction condition. *SIAM Math Analysis*, 36(4):1130 – 1141, 2005.

[28] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and Yuxi Zheng. Convergence of the vanishing viscosity approximation for superpositions of confined eddies. *Comm. Math. Phys.*, 201(2):291–304, 1999.

[29] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 2002.

[30] Walter Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[31] P. Serfati. Pertes de régularité pour le Laplacien et l'équation d'Euler sur $\mathbb{R}^n$. *preprint, 15 pp.*

[32] Elias M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[33] Roger Temam. *Navier-Stokes equations*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[34] Richard L. Wheeden and Antoni Zygmund. *Measure and integral*. Marcel Dekker Inc., New York, 1977. An introduction to real analysis, Pure and Applied Mathematics, Vol. 43.

[35] V. I. Yudovich. Some bounds for solutions of elliptic equations. *Mat. Sb. (N.S.)*, 59 (101)(suppl.):229–244 (Russian); English translation in Amer. Math. Soc. Transl. (2) 56 (1966), 1962.

[36] V. I. Yudovich. Non-stationary flows of an ideal incompressible fluid. *Ž. Vyčisl. Mat. i Mat. Fiz.*, 3:1032–1066 (Russian), 1963.

[37] V. I. Yudovich. Uniqueness theorem for the basic nonstationary problem in the dynamics of an ideal incompressible fluid. *Math. Res. Lett.*, 2(1):27–38, 1995.

Vita

James Patrick Kelliher was born on 26 June 1958 in the city of Washington, District of Columbia, the son of John Joseph Kelliher and Anne Moffitt Kelliher. He graduated from Magruder High School in Olney, Maryland in 1976, received a Bachelor of Science degreee in Physics and Mathematics from the Univerity of Maryland, College Park in May 1980, and a Master of Science degree in Mathematics from George Mason University in May 2000. He entered graduate school at the University of Texas at Austin in August 2000.

Permanent Address: 4515B Avenue G
Austin, TX 78751

This dissertation was typeset with $\LaTeX\,2_\varepsilon$[1] by the author.

[1] $\LaTeX\,2_\varepsilon$ is an extension of LaTeX. LaTeX is a collection of macros for TeX. TeX is a trademark of the American Mathematical Society. The macros used in formatting this dissertation were written by Dinesh Das, Department of Computer Sciences, The University of Texas at Austin, and extended by Bert Kay, James A. Bednar, and Ayman El-Khashab.