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Contact Structures and Open Books

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Contact Structures and Open Books

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Dedicated to my father.

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Contact Structures and Open Books

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We explore the correspondence between open books and contact structures on three-manifolds. We begin with the necessary definitions and proofs for the correspondence; then we obtain technical results to understand the relationship between compatibility, Murasugi sum, and homotopy classes of plane fields. We use these to prove Harer's conjecture on fibered links. Finally we explore the topological meaning of overtwistedness. We define sobering arcs in an open book and give a condition, then a criterion, for an open book to be overtwisted. Using this condition we give examples of overtwisted open books which come from positive configuration graphs. Finally we explore the limits of our sobering arc technique.

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Chapter 1

Introduction

This work deals with the interplay of the topology of three-manifolds and the contact structures which may be constructed upon them.

Contact structures were first introduced from the physics of geometric optics. After a long formative period, in which the classical flexibility results (Darboux and Gray theorems) were established, Eliashberg began the modern study by establishing the first rigidity results. Eliashberg showed that contact structures fall into two classes, tight and overtwisted, and that it is the tight that are both rare and interesting. It became reasonable to expect tight contact structures to be related to the topology of three-manifolds in much the same way that symplectic structures are in four dimensions. While much progress has been made in understanding, and classifying, tight contact structures, almost no connections have been made with pure topology. (A notable exception is the work of Rudolph, in which gauge theoretic and contact techniques are used to bound the slice genus of knots.) We will prove, by contact considerations, a topological result: that all fibered links are equivalent by Hopf-plumbing (this result was obtained independently by Emmanuel Giroux).

We connect contact structures to topology through open books. Open books began as a way to decompose three-manifolds, after the initial work of Alexander and another burst of interest in the seventies, most authors ignored open books in favor of Heegaard splittings and the geometric program of Thurston. Open books first appeared in connection with contact structures in the 1975 paper of Thurston and Winkelnkemper, in which they gave an alternate proof that there is a contact structure on each three-manifold. This they did by constructing a contact structure from an open book. In 2000 Emanuel Giroux announced the proof of the converse: that every contact structure arises in this way from an open book. Giroux was also able to show that two different open books giving rise to the same contact structure are related by positive Hopf-plumbing. To the best of our knowledge the proofs of these results haven't appeared in print. We give a sketch of the former and a proof of the latter, in the spirit of the proofs sketched by Giroux in his lectures.

We should point out that similar ideas were considered by I. Torisu, who considered contact Heegaard splittings. Indeed, Torisu was aware of the important properties relating Murasugi sum to contact connect sum (albeit in different language).

The final part of this work attempts to bring the connection between contact structures and open books up to speed with modern contact topology by understanding tightness for open books. We have mixed success: we give a complete condition, but it is difficult to apply, and we give a sufficient condition which is useful, but not necessary.

In the second chapter we provide necessary background on open books and on contact topology. In the third chapter we begin to study the correspondence between these two structures. We define compatibility, go through the Thurston-Winkelnkemper construction of contact structures from open books, then prove uniqueness of compatible contact structures. Next we go through Giroux's theorems, giving proofs in varying level of detail. We consider the interaction of Murasugi sum with compatibility, and give a useful convexity result. In the fourth chapter we prove Harer's conjecture. To do this we first spend a while understanding homotopy classes of compatible contact structures. After the proof, we consider generalizations in several directions. In chapter five we study overtwisted open books. We develop the idea of sobering arcs, which gives a sufficient condition for overtwistedness. We give a criterion for tightness. We use these ideas to do a number of examples, and then find the limits of our techniques, by finding overtwisted open books with no sobering arc.

Chapter 2

Background

Throughout this chapter of background material we will omit proofs that can be easily found in the literature. In particular, there are now many good references on contact geometry, see [7].

2.1 Topological Notions and Conventions

Unless we explicitly state otherwise all manifolds are three dimensional and oriented, and all surfaces are also oriented.

By a C^n -small perturbation of a submanifold $S \subset M$ we will mean a way of choosing an isotopy in any neighborhood of the given inclusion map $S \subset M$, in the C^n topology on the set of maps $S \hookrightarrow M$.

The *disk* D^2 is the set of complex numbers $\{a \in \mathbb{C} \mid a\bar{a} \leq 1\}$, while the *n-ball* $B^n = \{(x_i) \in \mathbb{R}^n \mid \sum_i x_i^2 \leq 1\}$.

If $S \subset M$ is a surface with boundary embedded in three-manifold M , and T another surface so that $\partial S \subset T$, then the *framing difference* for ∂S between S and T , written $Fr(\partial S; S, T)$ or just $Fr(S, T)$, is the oriented number of intersections between S and a push-off of ∂S along T . To compute the intersection number we orient S and give ∂S the boundary orientation, but

the framing doesn't depend on the chosen orientation.

An isotopy relative to a submanifold is an isotopy which fixes the submanifold point-wise.

An automorphism of a surface is a bijective self-homeomorphism. The *mapping torus* of a surface automorphism, $\phi : \Sigma \rightarrow \Sigma$, is the three-manifold $\Sigma \times_{\phi} S^1 = (\Sigma \times I)/(p, 1) \sim (\phi(p), 0)$.

Given a simple closed curve $c \subset \Sigma$ we can define an automorphism $D_c : \Sigma \rightarrow \Sigma$, which has support only near c , as follows. Let N be a neighborhood of c which is identified (by oriented coordinate charts) with the annulus $\{a \in \mathbb{C} | 1 \leq \|a\| \leq 2\}$ in \mathbb{C} . Then D_c is the map $a \mapsto e^{-i2\pi(\|a\|-1)}a$ on N , and the identity on $\Sigma \setminus N$. We call D_c a right handed, or positive, Dehn twist and often write it D_c^+ . The inverse $D_c^- = D_c^{-1}$ is a left handed, or negative, Dehn twist.

2.2 Open Books

An *open book* is a surface Σ , with non-empty boundary, together with an automorphism $\phi : \Sigma \rightarrow \Sigma$ which fixes $\partial\Sigma$ point-wise.

Open books (Σ_i, ϕ_i) , $i = 1, 2$, are *equivalent* if $\Sigma_1 = \Sigma_2$ and ϕ_1 is isotopic to $h^{-1} \circ \phi_2 \circ h$ for some automorphism h .

Associated to each open book there is a three-manifold, $M_{\Sigma, \phi} = (\Sigma \times_{\phi} S^1) \cup_{\partial} (D^2 \times \partial\Sigma)$, where the union is taken by gluing boundary tori in such a way that the curve $\{p\} \times S^1$, for $p \in \partial\Sigma$, glues to meridian $\partial(D^2 \times \{p\})$, and

$\partial\Sigma \times \{\theta\}$ glues to $\{e^{i\theta}\} \times \partial\Sigma$.

The link $\{0\} \times \partial\Sigma \subset D^2 \times \partial\Sigma \subset M_{\Sigma,\phi}$ is called the *binding*, sometimes we'll write it $\partial\Sigma \subset M_{\Sigma,\phi}$.

There is an alternative, equivalent, way to define $M_{\Sigma,\phi}$: Cap off each component $\partial\Sigma$ with a disk to get a closed surface, $\tilde{\Sigma}$. Extend ϕ to $\tilde{\Sigma}$ by the identity on each disk. Let $c_i \in \tilde{\Sigma}$ be the centers of the disks, and let $t_i \in T_{c_i}\tilde{\Sigma}$ be fixed vectors. Then the knot $c_i \times S^1 \subset \tilde{\Sigma} \times_{\phi} S^1$ is framed by the lift of v_i (defined since ϕ fixes a neighborhood of c_i). The manifold $M_{\Sigma,\phi}$ is obtained from $\tilde{\Sigma} \times_{\phi} S^1$ by zero-surgery on the c_i (relative to the given framing). This description is useful when doing Kirby calculus with open books.

Lemma 2.2.1. *Equivalent open books give homeomorphic $(M_{\Sigma,\phi}, \partial\Sigma)$.*

A *fibred link* is a link $L \subset M$ whose complement $M \setminus N(L)$ fibers over S^1 , in such a way that $\partial N(L)$ intersects a fiber in a curve isotopic to L .

Theorem 2.2.2. *If L is a fibred link in M , then there is an open book (Σ, ϕ) such that $M_{\Sigma,\phi} = M$, the binding is isotopic to L , and Σ is homeomorphic to the fiber.*

Proof. Removing a neighborhood of L we get a surface fibration $\pi : M \setminus N(L) \rightarrow S^1$, with fiber $\Sigma = \pi^{-1}(0)$. The curve $\partial\Sigma$ defines a longitude on $\partial N(L)$, hence a framing of L . Doing zero surgery on L , with respect to this framing, gives a surface bundle since the fiber is capped off by a meridional disk of $N(L)$. This exhibits the open book structure, using the alternative definition. □

The converse of this theorem is clear: the binding of an open book is a link in $M_{\Sigma, \phi}$, and its complement is a mapping torus, which fibers over S^1 . One can think in this way of an open book decomposing a three manifold, and indeed it is traditional to call the an open book an "open book decomposition". Another piece of terminology is also made clear by this lemma: the binding $\partial\Sigma \subset M$ of an open book is called a *fibred link* (note that the binding is oriented as the boundary of the page, so these are oriented links).

Some examples of fibred links in S^3 are the un-knot, U , the Hopf links, H^\pm , the trefoils, T^\pm , and the figure-eight knot E . For a proof that T^\pm and E are fibred, see Rolfsen[21]. We now demonstrate the other examples:

If we view S^3 as the unit sphere in \mathbb{C}^2 , then the un-knot is the pre-image of $0 \in \mathbb{C}$ under the map $(a, b) \mapsto a$, and the fibering of the complement is given by $(a, b) \mapsto a/||a||$.

As fibred links the H^\pm are given as the pre-images of $0 \in \mathbb{C}$ of the maps $(a, b) \mapsto a \cdot b$ and $(a, b) \mapsto \bar{a} \cdot b$, respectively. The fibration on the complement of the links is given by $(a, b) \mapsto (a \cdot b)/(||a \cdot b||)$, and similarly for the negative link. These open books are called *Hopf bands*, also written H^\pm . Figure 2.1 show's H^\pm in S^3 .

2.2.1 Murasugi Sum

The main operation which we'll consider on open books is the Murasugi sum. Unless stated otherwise our Murasugi sum will be the simplest kind, defined below in terms of rectangles. A more general Murasugi sum, also called

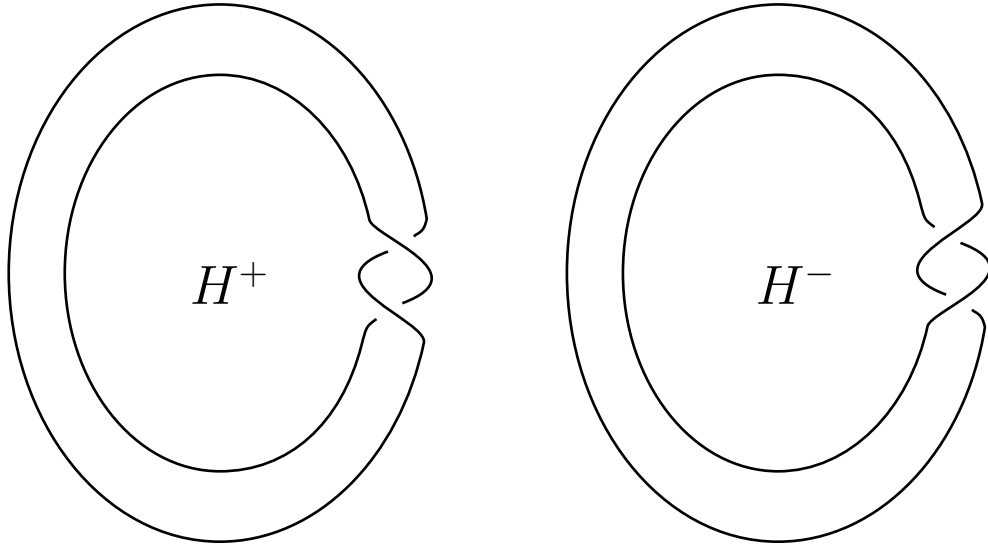


Figure 2.1: Hopf bands.

generalized plumbing or $*$ -product in the literature, can be defined similarly by using polygons with $2m$ sides.

Definition 2.2.1. Let (Σ_i, ϕ_i) , $i = 1, 2$, be open books, $l_i \subset \Sigma_i$ properly embedded arcs, and R_i rectangular neighborhoods of l_i such that $R_i \cap \partial\Sigma_i$ is two arcs of ∂R_i . The surface $\Sigma_1 * \Sigma_2$ is the union $\Sigma_1 \cup_{R_i} \Sigma_2$ with R_1 identified to R_2 in such a way that $\partial R_1 \cap \partial\Sigma_1 = \partial R_2 \setminus \partial\Sigma_2$, and vice versa. (This is the identification which gives a new surface, rather than a branched surface.) The *Murasugi sum* of the (Σ_i, ϕ_i) along l_i is the open book $(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2) = (\Sigma_1 * \Sigma_2, \phi_1 \circ \phi_2)$ (where each ϕ_i is understood to extend to $\Sigma_1 * \Sigma_2$ by the identity on $(\Sigma_1 * \Sigma_2) \setminus \Sigma_i$). We will often suppress mention of the attaching arcs l_i when they are clear or irrelevant.

Note that $\partial(\Sigma_1 * \Sigma_2)$ will be connected if $\partial\Sigma_i$ are both connected.

Lemma 2.2.3. $\Sigma_1 * \Sigma_2$ is homeomorphic to $\Sigma_2 * \Sigma_1$, and $(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)$ is equivalent to $(\Sigma_2, \phi_2) * (\Sigma_1, \phi_1)$.

Proof. Both are immediate from the definitions. □

Theorem 2.2.4. The manifold $M_{(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)}$ is $M_{\Sigma_1, \phi_1} \# M_{\Sigma_2, \phi_2}$.

Proof. We first exhibit the reducing sphere S in $M_{(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)}$, then show that each component of $M_{(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)} \setminus S$ is one of the $M_{\Sigma_i, \phi_i} \setminus B^3$. In this deconstruction we'll be able to follow the fibrations on each piece. This will help to prepare us for later chapters.

Divide the interval as $I_1 = [0, 0.5]$ and $I_2 = [0.5, 1]$, for notational convenience, and consider $R = R_i$ and Σ_i as submanifolds of $\Sigma_1 * \Sigma_2$. Let s_i be $R \cap \partial\Sigma_i$, the sides of R which meet l_i . Each s_i is the disjoint union of two properly embedded arcs in $\Sigma_1 * \Sigma_2$, and $\partial s_i \in \partial(\Sigma_1 * \Sigma_2)$ is four points – the corners of the rectangle. (Note $\partial s_1 = \partial s_2$.)

The surfaces $s_i \times I_i$ are each two disks in $(\Sigma_1 * \Sigma_2) \times I$, which can be viewed as disks in $(\Sigma_1 * \Sigma_2) \times_{\phi_1 \circ \phi_2} S^1$. $R \times \{0\}$ and $R \times \{0.5\}$ are also disks in $(\Sigma_1 * \Sigma_2) \times_{\phi_1 \circ \phi_2} S^1$. We glue $\bigcup_{i=1,2} s_i \times I_i$ to $(R \times \{0\}) \cup (R \times \{0.5\})$, along the edges $\bigcup_{i=1,2} (s_i \times \{0\}) \cup (s_i \times \{0.5\})$, to get a surface S' with boundary $\partial S' = \partial s_1 \times S^1$. We cap off these boundary components with the four disks $D^2 \times \partial s_1$, across the binding, to get a closed surface S . S is a sphere.

Now consider $M_{(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)} \setminus S$; let us understand the pieces of this manifold by building up the mapping torus in layers, then adding the binding. The mapping torus $(\Sigma_1 * \Sigma_2) \times_{\phi_1 \circ \phi_2} S^1$ is equivalent to $(\Sigma_1 * \Sigma_2 \times I_1) \cup (\Sigma_1 * \Sigma_2 \times I_2)$ where the union identifies $\Sigma_1 * \Sigma_2 \times \{0.5\}$ to $\Sigma_1 * \Sigma_2 \times \{0.5\}$ by ϕ_1 , and $\Sigma_1 * \Sigma_2 \times \{1\}$ to $\Sigma_1 * \Sigma_2 \times \{0\}$ by ϕ_2 . Removing S , we have $(\Sigma_1 \sqcup (\Sigma_2 \setminus R)) \times I_1$ glued to $(\Sigma_2 \sqcup (\Sigma_1 \setminus R)) \times I_2$. Since ϕ_1 is the identity on Σ_2 , and vice versa, we get two pieces after gluing: $(\Sigma_i \times_{\phi_i} S^1) \setminus (R \times I_i)$.

What of the binding? The solid tori $D^2 \times \partial(\Sigma_1 * \Sigma_2)$ are cut by S along the disks $D^2 \times \partial s_i$. Gluing the binding in the usual way we get pieces $M_{(\Sigma_i, \phi_i)} \setminus ((R \times I_i) \cup (D^2 \times s_i))$.

Thus we've seen that $M_{(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)}$ decomposes as a connect sum along the sphere S into $M_{(\Sigma_i, \phi_i)}$ less the three-balls $B_i = (R \times I_i) \cup (D^2 \times s_i)$. The fibration on each component is the expected one, and on $B_i \subset M_{(\Sigma_i, \phi_i)}$ it is the restriction. \square

If we have an open book presented as a fibered link, as in theorem 2.2.2, how can we see a Murasugi sum? Notice that the sphere S , in the previous proof, intersected the fiber $(\Sigma_1 * \Sigma_2) \times \{0\}$ in a rectangle R with corners on the binding $\partial(\Sigma_1 * \Sigma_2)$. (The rectangle on S , as presented above had corners on disks across the binding, but by shrinking the neighborhood of the binding we can get the corners as close to the binding as we like.) It turns out that this configuration is equivalent to being a Murasugi sum:

Lemma 2.2.5. *If $L \subset M$ is a fibered link with a fiber $\Sigma \subset M$, S a sphere*

which separates M as $M_1 \#_S M_2$, and $S \cap \Sigma = R$ is a rectangle with corners on L , then the open book associated to L is a Murasugi sum of open books on M_i . The surface $(\Sigma \cap M_i) \cup R$ is a page in each of these open books.

Proof. See [10]. □

In the special case $B * H^\pm$ of Murasugi sum with a Hopf band we introduce some special notation: In this case the sphere S will be a neighborhood of the *core disk* which has boundary on the center of the annulus $H^\pm = (\Sigma \cap S^3) \cup R$. An arc across this annulus, parallel to ∂R , is called a *transverse arc*. A repeated Murasugi sum with Hopf bands (ie. $B * H^\pm * \dots * H^\pm$) is called *Hopf plumbing*.

2.3 Contact Structures

A *plane field* on a three-manifold M is a smoothly varying choice of two-dimensional subspace $\xi \in T_p M$. Any plane field is locally the kernel of a one-form; a plane field is *co-orientable* if it is the kernel of a globally defined one-form. For us a plane field will always mean an oriented, co-oriented plane field.

We wish to define a plane field which is nowhere tangent to an embedded surface, or in the jargon, a plane field which is nowhere integrable. There are a number of ways to do this, but we take as definition: a *contact structure* is a plane field $\xi = \ker(\alpha)$, where the one-form α satisfies $\alpha \wedge d\alpha > 0$ (that

is, this three-form is a volume form for the oriented three-manifold). Such a one-form is called a *contact one-form* for ξ .

A manifold with a contact structure is a *contact manifold*, which we'll often write (M, ξ) .

The usual example is the one-form $\alpha = dz - ydx$ on \mathbb{R}^3 (in cartesian coordinates). The unit ball $B^3 \subset \mathbb{R}^3$ with this contact structure is the *standard contact ball*. Related to this (by compactifying \mathbb{R}^3) is the *standard structure* on S^3 which is the plane field of complex tangencies to S^3 viewed as the unit sphere in \mathbb{C}^2 .

A *contactomorphism* is a diffeomorphism $f : (M_1, \xi_1) \rightarrow (M_2, \xi_2)$ such that $f_*(\xi_1) = \xi_2$. An ambient isotopy $f_t : M \rightarrow M$ generates a contactomorphism from ξ to $(f_t)_*(\xi)$, which can also be thought of as a homotopy of plane fields through contact structures. In fact these are equivalent:

Theorem 2.3.1 (Gray's Theorem). *If ξ_t is a homotopy of plane fields, such that each ξ_t is a contact structures on M , then there is an ambient isotopy $f_t : M \rightarrow M$, $f_0 = Id$, with the property: $(f_t)_*(\xi_0) = \xi_t$.*

We will usually be interested in isotopy classes of contact structures.

Theorem 2.3.2 (Darboux's Theorem). *Each point $p \in (M, \xi)$ has a neighborhood contactomorphic to the standard contact ball.*

A ball in a contact manifold with a given contactomorphism with the standard contact ball is called a *Darboux ball*, and a cover by such balls is a

Darboux cover. (Clearly such a cover exists: take as many points as needed so that their Darboux neighborhoods cover.)

Fix a contact manifold (M, ξ) . A vector field whose flow is a symmetry of ξ is called a *contact vector field*. If we additionally fix a contact form α , then the *Reeb field* of α is the contact vector field R such that $\alpha(R) = 1$. Equivalently a Reeb field is a vector field R such that $R \lrcorner d\alpha = 0$ and $\alpha(R) = 1$.

A one dimensional submanifold $L \subset (M, \xi)$ is *Legendrian* if $T_p L \subset \xi$ at each $p \in L$, and is *transverse* if $T_p L \not\subset \xi$. Any curve in a contact manifold can be made Legendrian or transverse by a C^0 -small perturbation. A *Legendrian graph* is a graph embedded in a contact manifold with Legendrian edges.

Similar to Darboux's theorem for a point, there are standard neighborhood theorems for Legendrian and transverse curves. In particular any two transverse curves have contactomorphic neighborhoods.

Given an embedded surface $S \subset (M, \xi)$ the contact planes restrict to a (singular) line field on S . This singular line field integrates into a singular foliation (since any line field is integrable), which we call the *characteristic foliation*, $\xi|_S$.

2.3.1 Overtwisted Vs. Tight

For $S \subset M^3$ a surface with ∂S Legendrian, $tw(\partial S, S)$ is the difference in framing of ∂S between ξ and S . (That is the intersection number of S with a push-off of ∂S along ξ .)

A contact structure is *overtwisted* if there is an embedded disk D with Legendrian boundary and $tw(\partial D, D) = 0$. A contact structure that's not overtwisted is *tight*.

Theorem 2.3.3 (The Bennequin Inequality). *If (M, ξ) is a tight contact manifold, $S \subset M$ is an embedded surface, and ∂S is Legendrian, then:*

$$tw(\partial S, S) \leq -\chi(S) - |r| \leq -\chi(S).$$

Lutz and Martinet:

Theorem 2.3.4. *There is an overtwisted contact structure in each homotopy class of plane fields.*

The next theorem, due to Eliashberg [5], is key to our work in that it reduces classification of overtwisted contact structures to classification of plane fields up to homotopy.

Theorem 2.3.5. *Overtwisted contact structures are isotopic if and only if they are homotopic as plane fields.*

Another important result of Eliashberg[5]:

Lemma 2.3.6. *Any two tight contact structures on B^3 which induce the same characteristic foliation on ∂B^3 are isotopic relative to ∂B^3 . Further, if (B_i, ξ_i) are tight contact three-balls, and $(B_1, \xi_1) \subset (M, \xi)$, then there is an isotopy of M sending (B_1, ξ_1) to (B_2, ξ_2) .*

This sets the stage for translating the usual topological connect sum construction to contact manifolds. Indeed, a *contact connect sum* is a connect sum of contact three-manifolds, (M_1, ξ_1) and (M_2, ξ_2) , along tight contact three-balls (such that the contact structures glue). The contact structure on $M_1 \# M_2$ is called $\xi_{M_1 \# M_2}$.

By taking the balls small enough (eg. inside a Darboux chart) any topological connect sum will happen along balls on which the contact structure is tight. Lemma 2.3.6 then implies that after an isotopy the contact structures glue, so this becomes a contact connect sum. Results of Colin[3] state that any two contact connect sums of given contact manifolds give the same contact structure on the connect sum.

2.3.2 Convex Surfaces

A surface $S \subset (M, \xi)$ is *convex* if there is a contact vector field transverse to S . If v is such a vector field, the curves $\Gamma = \{p \in S | v(p) \in \xi\}$ are called the *dividing curves* (or *divides*) of S . For a convex surface with Legendrian boundary $tw(\partial S, S)$ is negative half the number of intersection points in $\Gamma \cap \partial S$.

In [12] Giroux gives a useful characterization of convex surfaces. A characteristic foliation with isolated singularities is called *Morse-Smale* if all singularities are hyperbolic (in the dynamical systems sense) and there are no saddle-to-saddle connections. (There is another technical requirement in the definition of Morse-Smale, it will be satisfied if there are no closed orbits, and

this will be enough for us.) A surface $S \subset (M, \xi)$ is convex if and only if $\xi|_S$ is Morse-Smale. For details see [12].

Lemma 2.3.7. *Any closed surface $S \subset (M, \xi)$ can be made convex by a C^0 -small perturbation. A surface with Legendrian boundary can be made convex by a C^0 -small perturbation, relative to ∂S , if $tw(\partial S, S) \leq -1$ for each component of ∂S .*

We'll frequently need the following lemma, called the Legendrian Realization Principal (LeRP), due to Ko Honda [16]:

Lemma 2.3.8 (LeRP). *Let $c \subset S$ be a set of curves (possibly with $\partial c \subset S$), on a convex surface S . If each component of $S \setminus c$ intersects the dividing curves of S , then S may be perturbed, by a C^0 -small perturbation through convex surfaces, to make c Legendrian.*

We will often use "LeRP" as a verb, meaning "to apply lemma 2.3.8 to a set of curves".

The folding trick, implicit in [16]:

Lemma 2.3.9. *Let $c \subset S$ be a non-separating collection of closed curves on convex surface S which don't intersect the divides $\Gamma \subset S$, and c' a parallel copy of c . Then S can be perturbed to a convex surface divided by $\Gamma \cup c \cup c'$. This perturbation is C^0 -small, but not through convex surfaces.*

Chapter 3

Contact Structures and Open Books

In this chapter we will explore the surprising connection between contact structures and open books. We will first consider an old construction which uses an open book to build a contact structure. Motivated by this construction we'll define the compatibility of a contact structure and an open book, and investigate existence and uniqueness of contact structures compatible with a given open book. Then we'll turn to the theorems of Giroux: existence and equivalence for open books compatible with a given contact structure. Finally we'll relate the Murasugi sum of open books to the contact connect sum.

3.1 A Construction of Contact Structures

In [23] Thurston and Winkelnkemper gave a simple proof of the fact, due to Martinet and Lutz, that every three-manifold has a contact structure. Their construction starts with an open book decomposition (Σ, ϕ) of the three manifold M , which was known to exist by the work of Alexander [2]. We now describe the construction.

First, since ϕ fixes $\partial\Sigma$ we can assume it fixes a collar neighborhood, C ,

(of each component) of the boundary, say C has coordinates (r, θ) such that $r = 0$ is the boundary. Let $d\theta$ be the coordinate one-form for the θ direction (it isn't exact!). Choose $\lambda \in \Omega^1(\Sigma)$ which satisfies: (i) λ is primitive (that is, $d\lambda$ is an area form on Σ), (ii) λ is $(1+r)d\theta$ near the boundary (see [23] for a quick construction of such a one-form).

Since ϕ fixes the collar neighborhood, $\phi^*\lambda$ satisfies (ii), and since $d\phi^*\lambda = \phi^*d\lambda$ it satisfies (i). The family of one-forms $(1-t)\lambda + t\phi^*\lambda$, $0 \leq t \leq 1$, also satisfies (i) and (ii) (since the sum of primitives is primitive). The product structure on $\Sigma \times I$ allows us to extend λ_t to a one form on $\Sigma \times I$ which glues into a one-form, α on the mapping torus $\Sigma \times_\phi S^1$, which satisfies (ii) (notice $(1-r)d\theta$ is naturally defined on $C \times S^1$), and (i) on each leaf.

Since $(\alpha + Kds) \wedge d(\alpha + Kds) = \alpha \wedge d\alpha + Kds \wedge d\alpha$, the one-form $\omega = \alpha + Kds$ will be positive, for large enough K , as long as $ds \wedge d\alpha$ is positive. This is true by the next lemma, hence ω is a contact form on $\Sigma \times_\phi S^1$.

Lemma 3.1.1. *Let $\pi : M^3 \rightarrow S^1$ be an oriented fibration. If α is a one-form which restricts to a primitive form on each fiber $\pi^{-1}(s)$, and ds is the pull-back of a positive one-form on the base, then $ds \wedge d\alpha$ is positive.*

Proof. For $p \in S = \pi^{-1}(s) \subset M$, let x, y be a basis of T_pS , and v a vector in the kernel of $d\alpha$ at p . Since $d\alpha$ is non-zero on T_pS , v is transverse: $d\pi(v) \neq 0$. Assume (v, x, y) form a positive basis, so $d\pi(v) > 0$, and $d\alpha(x, y) > 0$ (using that $d\alpha$ is an area form on S). Then, since we also have $ds(x) = ds(y) = 0$,

$ds \wedge d\alpha(v, x, y) = ds(v)d\alpha(x, y) > 0$. Thus $ds \wedge d\alpha$ is positive at each point p . □

Now we have to fill in the contact structure across the binding. Put coordinates (ρ, ϕ, θ) on the solid torus $D^2 \times S^1$ near (a component of) the binding, so that (ρ, ϕ) are coordinates of D^2 . Assume that this is set up with $C \times S^1 = \{1 \leq \rho < 2\} \times S^1$, so our new coordinates are related to our old ones by $\theta = \theta, \rho = (1 + r)$. In these coordinates $\omega = \rho d\theta + K d\phi$ on $C \times S^1$. The form $-d\theta + \rho^2 d\phi$ is contact and smoothly defined for the entire solid torus. We wish to interpolate between these two forms.

Let $(f_1(\rho), f_2(\rho)), 0 < \rho \leq 2$, be smooth functions such that: $(f_1(\rho), f_2(\rho)) = (-1, \rho^2)$ near $\rho = 0$, $(f_1(\rho), f_2(\rho)) = (\rho, K)$ on $1 \leq \rho \leq 2$, and $\det(f_i, f'_i) < 0$. A quick computation verifies that $\tilde{\omega} = f_1 d\theta + f_2 d\phi$ is contact. Filling the solid torus with this contact structure gives a globally defined contact structure and completes the construction.

Note that $d\tilde{\omega}$ restricted to the page, $\phi = \text{constant}$, is $f'_1 dr \wedge d\theta$, so we can guarantee that $\tilde{\omega}$ is primitive on each page. This then gives a contact form which is primitive on each page for the entire manifold (less the binding).

3.2 Compatibility and Uniqueness

The contact structure constructed in the previous section could be made as close to tangent to the pages as we liked, except near the binding which was always transverse. This motivates our first definition of compatibility.

Definition 3.2.1. A contact structure ξ and an open book (Σ, ϕ) , on M^3 , are called *compatible* if there is a homotopy ξ_t of plane fields such that:

1. $\xi_1 = \xi$,
2. ξ_t is smooth and contact for $t > 0$,
3. $T\partial\Sigma \not\subset \xi_0$ and $\xi_0|_{M \setminus \partial\Sigma} = T\Sigma_s$ (ie. ξ_0 is tangent to the leaves, and transverse to the binding).

This definition encodes an intuitive idea of compatibility that we glean from the Thurston-Winkelnkemper construction: the contact structure can be "pushed into" the pages of the open book, away from the binding, (or conversely that the contact structure arises as a perturbation of the foliation, away from the binding). It also codifies the similar idea of "concentrating the twisting" of a contact structure near a curve. Of course the homotopy above will be discontinuous at $t = 0$, but this isn't a problem.

The next definition, used by other authors, is often more useful for proofs, but less intuitive (and possibly more problematic in examples where one wishes for flexibility).

Definition 3.2.2. A contact structure ξ and an open book (Σ, ϕ) , on M , are called *compatible* if there is a one-form α for ξ and a Reeb vector field v for α such that: $v|_{\partial\Sigma} \in T\partial\Sigma$ and $v|_{M \setminus \partial\Sigma} \notin T\Sigma_s$ (ie. v is tangent to the binding and transverse to the leaves).

However, the two definitions are equivalent in as strong a sense as a contact geometer could wish:

Theorem 3.2.1. *Definitions 3.2.1 and 3.2.2 are equivalent up to isotopy. (That is, if ξ satisfies definition 3.2.1 then it is isotopic to ξ' satisfying 3.2.2, and vice versa.)*

To prove this theorem, and those below, we'll need an easy lemma:

Lemma 3.2.2. *If α is a contact one-form with Reeb field v transverse ($ds(v) > 0$) to a fibration over S^1 , where ds is the pull-back of a positive one-form on the base, then α restricts to a primitive form on each fiber, and $\alpha + Kds$ is contact for constant $K \geq 0$.*

Proof. Since v is in the kernel of $d\alpha$, by definition, the kernel is transverse, hence $d\alpha$ restricts to a non-zero form. It is positive since since $\alpha \wedge d\alpha(v, x, y) = \alpha(v)d\alpha(x, y) > 0$ and $ds(v) > 0$. For the second claim: $(\alpha + Kds) \wedge d(\alpha + Kds) = \alpha \wedge d\alpha + Kds \wedge d\alpha$, by lemma 3.1.1 $ds \wedge d\alpha$ is positive, hence $\alpha + Kds$ is contact. \square

Proof of Theorem 3.2.1. Given an α satisfying definition 3.2.2, the form $\alpha + (1/t)ds$ is contact for $t > 0$ in the complement of the binding, by lemma 3.2.2. We can re-scale to get one form $t\alpha + ds$, which has the same kernel as $\alpha + (1/t)ds$ hence is also contact. This homotopy extends smoothly, for $t > 0$, to the identity on the binding, and gives the homotopy required for definition 3.2.1.

On the other hand, assuming we have a homotopy ξ_t as in definition 3.2.1, choose a family of one-forms α_t . Eventually (for some t) the characteristic foliation on each leaf will be positive, hence expanding. A one form with expanding flow is primitive for a symplectic, hence area, form. However, if this α_T restricts to be primitive, then the kernel of $d\alpha_T$ can't lie in the page, so the Reeb field is transverse. Note that the binding must eventually be transverse. After this point in the homotopy we may adjust a standard neighborhood (whose width decreases as we go along) to have the standard one-form, which provides an appropriate Reeb field near the binding. \square

The contact structure created from an open book in the Thurston-Winkelnkemper construction, above, is compatible with the open book, as expected. Indeed, the form constructed, ν , restricts to a primitive form on each page. This implies that the kernel of $d\nu$, and hence the Reeb field of ν , is transverse to the pages. Since ν is radially symmetric about the binding, its Reeb field must be tangent to the binding. This gives an existence theorem:

Theorem 3.2.3. *For any given open book (Σ, ξ) , there is a contact structure compatible with (Σ, ξ) .*

At this point one would be inclined to start classifying contact structures compatible with a fixed open book. This proves to be quite easy – they're all the same up to isotopy:

Theorem 3.2.4 (Uniqueness). *If ξ_0 and ξ_1 are both compatible with open book (Σ, ϕ) then they are isotopic.*

Proof. Since the above definitions of compatibility are equivalent up to isotopy, we can assume that we are using the second definition; let α_i be the contact form for ξ_i . Choose standard neighborhoods of the binding, and isotope to get these neighborhoods to match. Now remove the binding, and forget about it.

The homotopy $\alpha_i + kds$, for $0 \leq k \leq K$, gives a homotopy through contact forms (lemma 3.2.2).

Let $\alpha_t = (1 - t)(\alpha_0 + Kds) + t(\alpha_1 + Kds)$, $0 \leq t \leq 1$. We now show that this straight line homotopy is contact for K sufficiently large:

$$\begin{aligned} d\alpha_t &= (1 - t)d\alpha_0 + td\alpha_1, \\ \alpha_t \wedge d\alpha_t &= (1 - t)^2\alpha_0 \wedge d\alpha_0 + t^2\alpha_1 \wedge d\alpha_1 \\ &\quad + t(1 - t)\alpha_0 \wedge d\alpha_1 + t(1 - t)\alpha_1 \wedge d\alpha_0 \\ &\quad + K((1 - t)ds \wedge d\alpha_0 + tds \wedge d\alpha_1) \end{aligned}$$

By lemmas 3.2.2 and 3.1.1, $ds \wedge d\alpha_i$ is positive, so $(1 - t)ds \wedge d\alpha_0 + tds \wedge d\alpha_1$ is also positive. If K is big enough this final, positive, term will dominate the sum to make $\alpha_t \wedge d\alpha_t$ positive.

We now have a sequence of homotopies, through contact one-forms, from α_0 to α_1 . By Gray's theorem there is an isotopy which generates this homotopy (at least on the level of plane fields). \square

We will often write $\xi_{(\Sigma, \phi)}$ for the contact structure compatible with open book (Σ, ϕ) on manifold $M_{(\Sigma, \phi)}$. (Properly, we only have an isotopy class of such contact structures: we chose one. Where this might cause confusion we are more explicit.)

Now let us give a few examples.

Lemma 3.2.5. *The un-knot and H^+ are both compatible with the standard tight structure, ξ_0 , on S^3 .*

Proof. To prove this we'll view S^3 as the unit sphere in \mathbb{C}^2 and ξ_0 as the complex tangencies to S^3 . If N is a unit vector field normal to S^3 , then $R = iN$ lies tangent to S^3 and is a symmetry of the complex tangents, hence a contact vector field of ξ_0 . Since R is orthogonal to ξ_0 , it is a Reeb field for some contact one-form for ξ_0 .

The Hopf map $h : \mathbb{C}^2 \rightarrow \mathbb{C}\mathbb{P}^1$, defined by $h(a, b) = [a, b]$, provides a fibration of S^3 over $\mathbb{C}\mathbb{P}^1 \cong S^2$. The fibers $h^{-1}([a, b]) = e^{i\theta}(a, b)$ have (oriented) tangent vectors $ie^{i\theta}(a, b)$. Since $e^{i\theta}(a, b)$ (as a vector) is normal to S^3 at $e^{i\theta}(a, b)$, and of unit length, it is N and $R = ie^{i\theta}(a, b)$ is (oriented) tangent to the fibers of h .

Consider the projection $P : \mathbb{C}^2 \rightarrow \mathbb{C}$ given by $(a, b) \mapsto a$. Zero is a regular value with pre-image $U = (0, e^{i\theta})$, which is a fiber of h . In the complement of U the coordinate map Θ on \mathbb{C}^2 ($\Theta(a) = a/||a||$) gives a fibration over S^1 by disks – an open book with binding the un-knot. Writing $a = r_a e^{i\Theta_a}$, $d\Theta \circ dP(R) = \Theta_a$ when $r \neq 0$, so R is transverse to the disks of the open book (and tangent to the binding, since the binding is a fiber of h). We conclude that ξ_0 is compatible with (the open book for) the un-knot.

Now consider the map $B : \mathbb{C}^2 \rightarrow \mathbb{C}$ given by $(a, b) \mapsto ab$. Again, the origin is a regular value, with pre-image $(0, e^{i\theta})$ and $(e^{-i\theta}, 0)$, two fibers of h .

It is strait-forward to check that $\Theta \circ B$ gives a fibration of the complement to $B^{-1}(0)$, in this case by left-handed Hopf-bands, and that R is transverse to the fibers.... \square

3.3 Existence

In the previous section we showed that for an open book there is a unique compatible contact structure, in this section we'll show that for a contact structure there is always a compatible open book. This result is due to Giroux, and we sketch a proof similar in spirit to his, though phrased in terms of general cellular decompositions instead of triangulations. We avoid the language of globally convex contact structures by following the outline of Gabai's result on fibered links (see the introduction for more history and perspective).

Definition 3.3.1. A *contact cellular decomposition* of the contact manifold (M, ξ) is a cellular decomposition of M with the following properties:

1. The 1-skeleton of the complex is a Legendrian graph.
2. Each 2-cell, D , has $tw(\partial D, D) = -1$.
3. The restriction of ξ to any 3-cell is tight.

Note that since each 2-cell is a disk with $tw(\partial D, D) = -1$ it can be made convex by a small perturbation, fixing the boundary, and is then divided by a single arc. However, we must be carefull: the perturbation shouldn't require twisting the disk around a vertex (since this can destroy the cellular

decomposition). To avoid this we need each one-cell on the boundary of D to have $tw \leq -1$. This can be arranged by stabilizing (adding small spirals to) the Legendrian one-cells. (It is always possible to decrease tw of a Legendrian arc – it is increasing that can be tricky.) Once we have arranged this we may have decreased the total $tw(\partial D, D)$ too much, then we'll need to subdivide D (by adding arcs in D that don't intersect the divides, and LeRPing to make them Legendrian). At the end of all this we have arranged, as we hoped, that each face of the decomposition is convex with $tw(\partial D, D) = -1$. We will often use this technique below, and generally assume that the two-cells are already convex.

Theorem 3.3.1. *Any (M, ξ) has a contact cellular decomposition.*

Proof. Take a cover by Darboux balls, and refine any (topological) cellular decomposition until each 3-cell is inside a Darboux ball. Make the 1-skeleton Legendrian, by a C^0 -small perturbation. Since each 3-cell is in a Darboux ball it is tight, so (1) and (3) of the definition are satisfied.

Take a 2-cell D ; since it is a disk in a Darboux ball it can be made convex and the dividing set, Γ , is a union of properly embedded arcs. (If tw of an edge is non-negative we can Legendrian-stabilize the edge so that D can be made convex without wrapping around any vertices, see above comments.) Now sub-divide the disk by a union A of properly embedded arcs such that $A \cap \Gamma = \emptyset$, and each piece of $D \setminus A$ contains exactly one component of Γ . LeRP A and refine the decomposition by adding these arcs as 1-cells (sub-dividing D).

The boundary of each 2-cell in $D \setminus A$ intersects Γ at two points, so it has $tw = -1$.

Subdividing each face in this way gives (2) of the definition, and completes the construction. \square

Definition 3.3.2. The *ribbon* of a Legendrian graph $\Gamma \subset (M^3, \xi)$ is a surface R which deformation retracts to Γ and has $T_p R = \xi$ for points $p \in \Gamma \subset R$.

This definition is designed to emphasize the topology of the ribbon, but the ribbon also fits in nicely with the contact geometry:

Lemma 3.3.2. *There is a small convex neighborhood $N(\Gamma)$ of Legendrian graph Γ divided by $\partial R \subset \partial N(\Gamma)$.*

Proof. The construction of such a neighborhood is straight forward: a small enough neighborhood will be tight, and such a neighborhood can be separated as tubular neighborhoods of the edges and balls around the vertices. If the tubes are small enough they will be contained in a standard neighborhood, hence divided by only two curves, which define the bands of the ribbon. The balls are tight, so each is divided by a single curve – which is the boundary of a disk of the ribbon. To show that the the bands glue properly to the disks it is necessary only to show that the dividing curves glue as expected – for this we can use the corner smoothing lemma of [16]. We omit the details. \square

In what follows *the ribbon* will refer to this construction inside a convex neighborhood.

In the language of Gabai [10] the annuli $\partial R \times I$ on a product neighborhood of a ribbon R are called the *suture*. In the case of the ribbon of the 1-skeleton of a contact cellular decomposition, each 2-cell, D , intersects the ribbon twice after isotopy (since $tw = -1$) and the disk $D \setminus R$ crosses the suture of a product neighborhood of the ribbon in two places. In the language of Gabai these are c-product disks and, since the 2-cells decompose the complement of the ribbon into 3-cells, they form a complete c-product decomposition. The main theorem of [10] then implies:

Lemma 3.3.3. *The ribbon of the 1-skeleton of a contact cellular decomposition is a fibered surface (that is, a page of an open book).*

To prove this Gabai considers the hierarchy given by cutting along the c-product disks, there is a natural fibration on the bottom of the hierarchy (a union of spheres), and Gabai shows that this fibration can be glued at each stage. This gives a fibration of the complement to the product neighborhood, which can be glued to the product neighborhood in a way that respects the fibration (away from the suture, which becomes the boundary of a neighborhood of the binding). We will essentially follow this proof in what follows, adapting it to our needs by keeping track of a compatible contact structure.

Theorem 3.3.4 (Giroux). *For any contact structure, ξ , there is an open book compatible with ξ .*

Sketch of Proof: Take a contact cellular decomposition for ξ , let R be its ribbon. We remove a convex neighborhood, N , of the 1-skeleton with $\partial R \subset N$

the dividing set. Cutting the remainder along the 2-cells we get 3-cells divided by a single circle (since each is tight). Removing these divides gives a natural identification of each 3-cell with $D^2 \times I$ such that the contact structure is compatible with the fibration by disks.

Next we reverse the process: show that when gluing 3-cells as above not only the fibration, but the compatibility of the contact structure can be extended across the gluing.

The neighborhood N is fibered by the ribbon cross interval, and since the dividing curves are ∂R the contact structure is compatible with this fibration.

Now we have a pair of handle-bodies with sutures, each fibered over I , and a gluing which preserves the fibering. The glued contact structure remains compatible, and we have only to glue back the dividing set of $N = \partial R$, the binding. □

Corollary 3.3.5. *For each contact structure ξ , there is an open book with connected binding compatible with ξ .*

Proof. Take an open book (Σ, ϕ) compatible with ξ , and a set of $n - 1$ arcs connecting the n boundary components of Σ . Summing an H^+ along each arc we get an open book with connected binding. In section 3.5 we'll see that the contact structure compatible with the resulting open book is isotopic to the original ξ . □

3.4 Stabilization Equivalence

The result of the previous section gave an open book for each contact structure, but this open book is far from unique. In this section we'll show that an open book compatible with a contact structure is unique up to stabilization (by Murasugi sum with H^+) and de-stabilization.

Lemma 3.4.1. *An open book (Σ, ϕ) compatible with contact structure ξ is isotopic after appropriate H^+ -stabilization to the ribbon of a contact cellular decomposition for ξ . This stabilization can also be done with T^+ .*

Proof. Take a set of arcs a_i which cut Σ into disks, and look at the suspension disks $A_i = a_i \times I \subset M_{(\Sigma, \phi)}$ (suitably smoothed off of $\partial\Sigma$, as in section 5.2). The A_i each intersect $\partial\Sigma$ in two points. These are the 2-cells of a cellular decomposition where a graph G onto which Σ deformation retracts is the 1-skeleton. This graph can be LeRPed to be Legendrian and, since it doesn't intersect the dividing set, the contact framing on any cycle in G is the Σ framing. We can then view Σ as the ribbon of G . Each disk A_i intersects the boundary of the ribbon Σ in two points, so it has $tw(\partial A_i, A_i) \in \{-1, +1, 0\}$. If disk A_i has $tw(\partial A_i, A_i) \in \{+1, 0\}$ make an H^+ -stabilization along a boundary parallel arc near the positive half-twist(s) as in figure 3.1. The disk across the added Hopf-band, D_H , has $tw(\partial D_H, D_H) = -1$. Add D_H to the complex, and replace A_i with $D' = A_i \cup D_H$ (in this union the two disks are glued along the attaching arc). Since $tw(\partial D_H, D_H) = -1$ and $tw(\partial D', D') = tw(\partial A_i, A_i) - 1$, this procedure will give a complex with 2-cells of $tw = -1$.

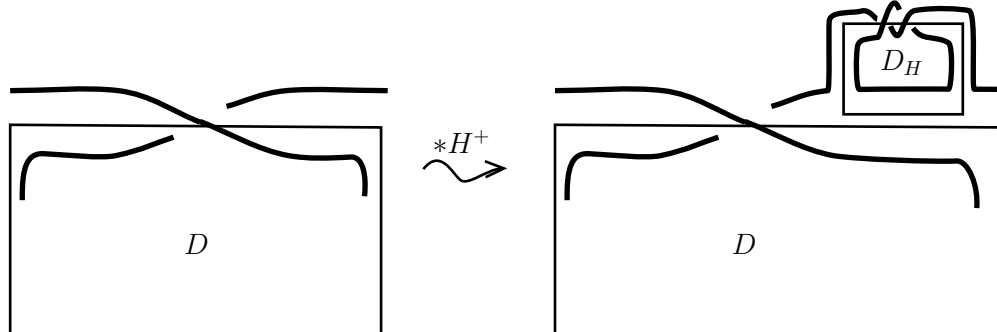


Figure 3.1: Reversing a half twist by adding a Hopf-band. The disk $D' = D \cup D_H$ has framing $Fr(D', R * H^+) = Fr(D, R) - 1$.

The 3-cells are fibered as $D^2 \times I$, and this fibration, since it comes from the open book by cutting along the 2-cells, is compatible with the contact structure. Hence the 3-cells are tight.

We could add an additional H^+ attached across each of the Hopf-bands added above. By including the disk across the new H^+ in the decomposition we still have a contact cellular decomposition, but to get there we've added $H^+ * H^+ = T^+$ at each step. \square

Lemma 3.4.2. *Two contact cellular decompositions for ξ have a common refinement. This refinement can be built from either decomposition by applying the following moves:*

1. *Subdividing an existing 2-cell, D , by a Legendrian arc $I \subset D$ which, when D is made convex, intersects the dividing set in one point.*

2. Adding a 2-cell, D , and 1-cell, I' , with $\partial D = I \cup I'$ a union of intervals (I in the existing 1-skeleton), which can be made convex so that the single dividing curve goes from I to I' . (This also changes a 3-cell.)
3. Adding a 2-cell whose boundary is already in the 1-skeleton. (Subdividing a 3-cell.)

Proof. We begin by doing a small perturbation to assure that the two complexes intersect transversely. For the two (and three) cells this is purely topological; for the one skeleton we use the fact that Legendrian graphs may be made transverse (ie. non-intersecting) by a C^0 -small perturbation.

Let the two complexes be called C_i , and their k -skeletons C_i^k . If D is a 2-cell of C_1^2 , then the intersection $D \cap C_2^2$ is a collection of embedded arcs in D , which form a properly embedded 1-complex $A \subset D$, and some simple closed curves (coming from an internal intersection of D and a 2-cell of C_2^2). By adding an arc running from each of these simple closed curves to A (iteratively if the closed curves are nested) we get a 1-complex A' such that $D \cap C_2^2 \subset A'$. Topologically we may add A' to the complex C_1 , by subdividing D , which happens (forgetting the framings!) by move (1).

Now we wish to show that A' , or a refinement, can be added to the *contact* complex C_1 . Assume, without loss of generality (see earlier comments), that D is convex with a single dividing arc Γ . Arrange that each edge of A' intersects Γ transversely and non-empty (we are free to isotope the faces of C_2^2). Now, on each component of $D \setminus A'$ which intersects Γ in n components,

subdivide by $n - 1$ properly embedded arcs which don't intersect Γ . This provides A'' , further refining D , such that each component of $D \setminus A''$ intersects Γ in exactly one component. A'' can now be LeRPed, so that we may add the Legendrian 1-complex A'' to C_1^1 , and subdivide D – getting a new complex by applying move (1) several times.

Applying this procedure to each face of C_1 we have a new contact complex C'_1 such that $C_1^2 \cap C_2^2$ is contained in C_1^1 . We can similarly arrive at C'_2 with the same property (only we needn't LeRP or isotope the intersection, since it has already been made Legendrian and the faces convex). Each C'_i is obtained from C_i by move (1).

Now we wish to add the 2-cells of C'_2 to C'_1 (and vice-versa, but the procedure is identical). Let B be a 3-cell of C'_1 . Then C'_2 intersects B in a cellular decomposition of B . Topologically this can be achieved by applying move (2) (forgetting framings) until the one-skeleton is filled in, then applying move (3) to add the remaining 2-cells (to see this induct on the edges that must be added to the one-skeleton). However, since each 1-cell of $C'_2 \cap B$ already belongs to a contact cell decomposition, it is Legendrian. Similarly each 2-cell already has the required framing, so the moves applied to subdivide B are in fact the contact moves (2) and (3).

Applying this subdivision to each 3-cell of C'_1 and C'_2 gives identical contact cellular decompositions (since we've added each cell of one, properly subdivided, to the other). This intersection complex is obtained from either C_i by moves (1), (2) and (3). □

Note that move (1) can be constructed from move (3), its inverse, and move (2). Indeed moves (2), (3), and a 1-cell retraction move, provide a complete set of moves to get between two contact cellular decompositions – not surprising as the same set of moves, forgetting framings, gets between two topological cellular decompositions. (It is interesting that adding move (2'), the same but with framing +1, we can get between *any* two framed cellular decompositions.)

Lemma 3.4.3. *The effect of the moves of lemma 3.4.2 on the ribbon of the 1-skeleton can be realized, topologically, by H^+ -stabilization.*

Proof. Let Δ be the Legendrian graph which is the 1-skeleton of the contact cellular decomposition. Move (3) doesn't change Δ or its ribbon, so can be ignored.

In each of the other moves we add an edge A to Δ in such a way that there is a convex disk, D , with $\partial D \subset A \cup \Delta$, connected dividing set Γ , and $A \cap \Gamma = \{p\}$ is a single point. The ribbon is changed by adding a one-handle along A , which has a left-handed half twist relative to the disk at p . Indeed, since the ribbon is framed according to D except at $\partial D \cap \Gamma$, the one-handle can be slipped down into a small neighborhood of $\Delta \cap \Gamma$. Sliding one of the attaching points of the one-handle across the half-twist, as in figure 3.2, we see the one handle becomes a Hopf-band. This shows that the added one-handle is isotopic to a positive Hopf-band plumbed to the ribbon near $\Delta \cap \Gamma$. (Note that the attaching arc of this Hopf-band is not boundary parallel.) \square

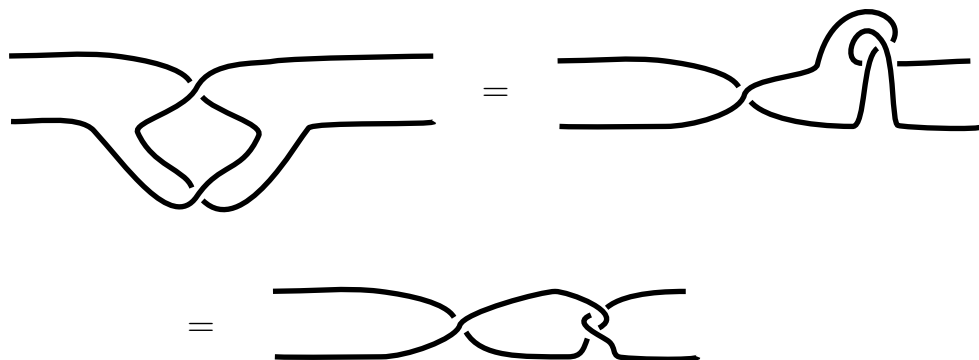


Figure 3.2: Sliding a half-twisted band past a half-twist to make a Hopf-band.

Theorem 3.4.4 (Giroux). *Two open books B and B' compatible with the same contact structure are H^+ -stably equivalent: $B * H^+ * \cdots * H^+ = B' * H^+ * \cdots * H^+$, for some Murasugi sums.*

Proof. The proof is immediate from the lemmas: by lemma 3.4.1 B and B' can be stabilized to the ribbons of two contact cell decompositions, by lemma 3.4.2 there is a common refinement of these decompositions which can be reached by a series of moves shown in lemma 3.4.3 to be stabilizations. \square

Lemma 3.4.5. *Each 2-cell added (or created) in the refinement of lemma 3.4.2 can be split in two, leading to a further common refinement. The effect of this on the ribbon is to stabilize by T^+ , instead of H^+ .*

Proof. In moves (1) and (2) of lemma 3.4.2 a properly embedded arc across the convex disk, intersecting the dividing set in one point, can be LeRPed and added to the cell complex (sub-dividing the face into two). Applying figure

3.2 twice shows that we've now added $H^+ * H^+ = T^+$ instead of just H^+ . \square

Using this lemma in place of lemma 3.4.2 in the proof of 3.4.4 proves:

Theorem 3.4.6. *Two open books B and B' , with connected binding, which are compatible with the same contact structure are T^+ -stably equivalent: $B * T^+ * \dots * T^+ = B' * T^+ * \dots * T^+$, for some Murasugi sums.*

Putting together all the theorems of this chapter, so far, we get an equivalence between open books and contact structures:

Theorem 3.4.7. *The set of contact structures on a three-manifold M is in bijective correspondence with the set of open books up to H^+ -stabilization; also with the set of open books with connected binding up to T^+ -stabilization.*

3.5 Murasugi Sum

In the previous sections we used Murasugi sums with H^+ , the stabilizations, which left the resulting open book still compatible with the original contact structure (we didn't need to show this explicitly, since these sums happened in terms of contact cellular decompositions). Now let's consider the effect of an arbitrary Murasugi sum on the compatible contact structure.

The following results were largely contained in Torisu[24], though in a very different language.

Lemma 3.5.1. *The contact structure $\xi_{(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)}$ compatible with a Murasugi sum $(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)$ of open books is isotopic to the contact connect*

sum, $\xi_{(\Sigma_1, \phi_1)} \# \xi_{(\Sigma_2, \phi_2)}$.

Proof. By theorem 2.2.4 the connect sum happens in a way compatible with the open book on each piece. We'll show that a compatible contact structure induced on each $M_{(\Sigma_i, \phi_i)} \setminus B_i$, from one on $M_{(\Sigma_1, \phi_1) * (\Sigma_2, \phi_2)}$, extends across the ball B_i .

We resume the notation and constructions from the proof of theorem 2.2.4. Then B_i is $(R \times I_i) \cup (D^2 \times s_i)$. Let ξ_λ be the compatible structure built by the Thurston-Winkelnkemper construction from a one form λ on $\Sigma_1 * \Sigma_2$, as usual, but with the homotopy extending λ to the mapping torus chosen to be $\phi_1^*(\lambda)$ on $(\Sigma_1 * \Sigma_2) \times \{0.5\}$. Let λ_i be the restriction of λ to Σ_i . The Thurston-Winkelnkemper construction then gives a contact structure, ξ_{λ_i} , on $M_{(\Sigma_i, \phi_i)}$. By following the steps of the construction one sees that ξ_{λ_i} restricted to each piece $M_{(\Sigma_i, \phi_i)} \setminus ((R \times I_i) \cup (D^2 \times s_i))$ is the same contact structure as ξ_λ restricted to $M_{(\Sigma_i, \phi_i)} \setminus ((R \times I_i) \cup (D^2 \times s_i))$. This shows that the compatible contact structure can be extended across the ball $(R \times I_i) \cup (D^2 \times s_i)$ to give another compatible structure, as required.

Finally, in order for this to be a contact connect sum, we need to show that $\xi_{(\Sigma_i, \phi_i)}|_{B_i}$ is tight. This ball, with this compatible contact structure, occurs in tight manifolds (such as (M_{H^+}, ξ_0)) when they are Murasugi summed, so it must be tight. \square

Lemma 3.5.2. $\xi_{(\Sigma, \phi) * H^+}$ is isotopic to $\xi_{(\Sigma, \phi)}$.

Proof. ξ_{H^+} is the standard tight contact structure, ξ_0 , on S^3 , so $\xi_{(\Sigma,\phi)*H^+}$ is isotopic to $\xi_{(\Sigma,\phi)}\#\xi_0$ on $M_{(\Sigma,\phi)}\#S^3 = M_{(\Sigma,\phi)}$.

Let us use \mathbb{C}^2 coordinates for S^3 and view ξ_0 as the complex tangencies, as usual. Let $B = \{(a, b) \in S^3 | \text{Im}(a) \leq 0\}$ and $B' = \{(a, b) \in S^3 | \text{Im}(a) \geq 0\}$, as (tight!) contact balls. We can arrange that the connect sum happens along a ball in $M_{(\Sigma,\phi)}$ contactomorphic to B (see lemma 2.3.6 and subsequent remarks), and along B' in S^3 , by the map $(a, b) \mapsto (-a, b)$ (which is a contactomorphism). Then the contact connect sum simply removes B from $M_{(\Sigma,\phi)}$ and replaces it with $S^3 \setminus B' = B$. Thus the contact structure is unchanged, up to an isotopy. \square

Lemma 3.5.3. $\xi_{(\Sigma,\phi)*H^-}$ is overtwisted for any open book (Σ, ϕ) .

Proof. We defer proof to section 5.3, where new techniques make it quick and easy. \square

3.6 Convexity of Pages

Lemma 3.6.1. Let (Σ, ϕ) be an open book, $\pi : M_{\Sigma,\phi} \setminus \partial\Sigma \rightarrow S^1$ the fibration of the complement to the binding. The closure of $\pi^{-1}(0) \cup \pi^{-1}(0.5)$ is convex and divided by the binding, $\partial\Sigma = \partial(\pi^{-1}(0))$, for some compatible contact structure.

Proof. First note that the Reeb field, v , is a contact vector field transverse to both $\xi_{\Sigma,\phi}$ and the pages $\pi^{-1}(t)$, so each page is convex, divided on its boundary. (The pages, though, have transverse, not Legendrian boundary, so this is minimally useful.)

Let S be the closure of $\pi^{-1}(0) \cup \pi^{-1}(0.5)$ (oriented by $\pi^*(ds)$ on $\pi^{-1}(0)$ and $\pi^*(-ds)$ on $\pi^{-1}(0.5)$), we take a compatible contact structure built as in the Thurston-Winkelnkemper construction from a one-form λ . We may choose the homotopy extending λ to the mapping torus in such a way that the characteristic foliation on S looks like $\ker(\lambda + \phi^*(\lambda))$ on $\pi^{-1}(0.5)$ (and like $\ker(\lambda)$ on $\pi^{-1}(0)$). By a small perturbation of this homotopy (eg. by perturbing ϕ) we can assure that the saddles on $\pi^{-1}(0)$ connect to the sources/sinks on $\pi^{-1}(0.5)$, and vice versa. This means that the characteristic foliation is Morse-Smale. By the results in [12], a surface with such a characteristic foliation is convex. The dividing curves separate the regions where the contact one-form is positive on a positive normal vector from those where it is negative. Since the contact structure is compatible, $\pi^{-1}(0)$ is positive while $\pi^{-1}(0.5)$ is negative (with the natural orientation: $\pi^*(ds)$, on each), hence the dividing curve is exactly the binding. \square

Chapter 4

Fibered Links

In this section we use the term *fibered link* to mean an open book for S^3 (or the binding of such an open book).

Not every fibered link is a Hopf-plumbing of the un-knot. Indeed, Gabai argued in [10] that the links shown in figure 4.1 cannot be written as a Murasugi sum in any way, including as Hopf plumbings. However, in 1982[15] Harer showed that all fibered links could be constructed from the un-knot by H^\pm -stabilization, de-stabilization, and a complicated twist move (generalizing a construction of Stallings). In that paper he conjectured that the twist move could be removed from this result. In this chapter we'll prove Harer's conjecture, and some generalizations.

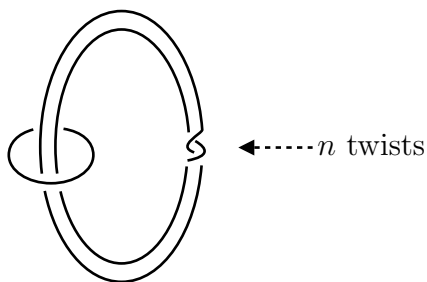


Figure 4.1: The links in this family, for n full twists ($n \neq 0$), cannot be written as a Murasugi sum.

4.1 Homotopy Classes

Gompf has given in [13] complete invariants for the homotopy class of a plane field. There are both two and three-dimensional invariants, which can be defined in terms of spin structures and bounding four-manifolds. Fortunately we will be concerned mostly with comparison of the homotopy classes of two given contact structures, for this we can simplify things greatly by fixing a trivialization of the tangent bundle to the manifold. We begin with proposition 4.1 of [13]:

Lemma 4.1.1. *Let M be a closed connected three-manifold. Then any trivialization τ of the tangent bundle of M determines a map Γ_τ sending homotopy classes of oriented 2-plane fields on M to $H_1(M; \mathbb{Z})$. For a fixed $x \in H_1(M; \mathbb{Z})$ the set $\Gamma_\tau^{-1}(x)$ has a natural \mathbb{Z} -action and is isomorphic to $\mathbb{Z}/d(\xi)$ as a \mathbb{Z} -space. ($d(\xi)$ is an integer, the divisibility of the chern class, independent of τ or the choice of $\xi \in \Gamma_\tau^{-1}(x)$.)*

Rather than prove this here, we will indicate the construction of the relevant objects, used below. A fixed trivialization τ of the tangent bundle identifies a contact structure ξ with its unit normal vector field: $\phi_\xi : M \mapsto S^2$. By the Thom-Pontrjagin construction, see [18], homotopy classes of such maps correspond bijectively to framed cobordism classes of framed links: the correspondence sends ϕ_ξ to $\phi_\xi^{-1}(p)$ for any regular value $p \in S^2$, framed by pulling back a basis of $T_p S^2$. $\phi_\xi^{-1}(p)$ is oriented by the orientation of M (from τ), and the homology class $\Gamma_\tau(\xi)$ is defined to be $[\phi_\xi^{-1}(p)] \in H_1(M, \mathbb{Z})$ (this

is independent of p , and depends on ξ and τ only through their homotopy classes).

For a fixed $x \in H_1(M, \mathbb{Z})$ the set $\Gamma_\tau^{-1}(x)$ is identified with framed cobordism classes of framed links representing x . The \mathbb{Z} -action on this set is clear: $n \in \mathbb{Z}$ acts by adding n right twists to the framing.

The integer $d(\xi)$ is defined to be the divisibility of the chern class $c_1(\xi) \in H^2(M; \mathbb{Z})$. If $c_1(\xi)$ is of finite order (eg. if $H^2(M; \mathbb{Z})$ vanishes) then $d(\xi) = 0$.

We will often refer to $\Gamma_\tau(\xi)$ as the 'two-dimensional invariant', and the element of the $\mathbb{Z}/d(\xi)$ -torsor as the 'three-dimensional invariant' keeping in mind that this assignment only makes sense after fixing a trivialization.

On S^3 there is a natural choice of trivialization given by the Hopf fibration. In particular, viewing S^3 as the unit quaternions we get a trivialization as follows: let N be the unit normal to $S^3 \subset \mathbb{H}$, then the vectors iN , jN and kN are orthogonal and tangent to the sphere. We refer to this trivialization as τ_H , the Hopf framing. Using this choice, and the natural framings on S^3 , we can remove the fuzziness in the above invariants:

Lemma 4.1.2. *Fixing the the trivialization τ_H of S^3 , the set of homotopy classes of plane fields is $\Gamma_{\tau_H}^{-1}(0)$ which is a \mathbb{Z} -torsor, and it can be naturally identified with \mathbb{Z} by letting Seifert surfaces define the 0-framing. There are no other choices in this assignment of each ξ to an integer, we write it $\lambda(\xi)$. If L is a fibered link compatible with ξ , define: $\lambda(L) = \lambda(\xi)$.*

Proof. Because $H_1(S^3; \mathbb{Z}) = 0$ all homotopy classes of plane fields are in $\Gamma_{\tau_H}^{-1}(0)$,

and because $H^2(S^3; \mathbb{Z}) = 0$, $d(\xi) = 0$ so $\Gamma_{\tau_H}^{-1}(0)$ is a \mathbb{Z} -torsor. In S^3 every link is naturally framed by its Siefert surface, and this framing is preserved under framed cobordism, giving a natural identification $\lambda : \Gamma_{\tau}^{-1}(0) \rightarrow \mathbb{Z}$ (the difference between the Hopf framing and the Siefert framing). \square

The quantity $\lambda(L)$ for a fibered link was introduced, in a different way, in the work of Rudolph and Neumann [20][22], where it was called the *refined Milnor number*. We will need a few example computations later:

Lemma 4.1.3. $\lambda(\xi_{H^+}) = 0$.

Proof. We saw in the previous chapter that $\xi_{H^+} = \xi_0$, the standard structure which arises as complex tangencies of $S^3 \subset \mathbb{C}^2$. In particular the Hopf field iN , where N is normal to S^3 , is the orthogonal vector field to ξ_0 (which is $\text{span}(jN, kN)$ in quaternionic language). So, ϕ_{ξ_0} maps to a single point, and any other point in S^2 is (vacuously) a regular value with empty pre-image. Thus the empty link represents $\Gamma_{\tau_H}(\xi_0)$, and $\lambda(\xi_0) = 0$. \square

Lemma 4.1.4. $\lambda(\xi_{H^-}) = 1$.

Proof. The Hopf fibration $S^3 \rightarrow \mathbb{C}\mathbb{P}^1$ gives an open book whose binding is the pre-image of $[0, 1]$ and $[1, 0]$, and whose pages are pre-images of the lines $\{[t \cdot a, b] | t \in \mathbb{R}\}$. The pages are right-twisted annuli, so this open book is H^- . The compatible contact structure has an isotopy in which the contact planes tend to the tangents to the pages, since the Hopf field iN is tangent to the pages, this implies that the contact planes are orthogonal to iN (in the Hopf

frame) only near the binding (after the isotopy). Indeed, by symmetry near the binding the contact planes can be orthogonal only on the binding (where they must be orthogonal). Since the Hopf field agrees with the boundary orientation of the pages only on one of the two components, the framed link representing the homotopy class of ξ_{H^-} is a single component of the negative Hopf link ∂H^- , framed (as one can verify directly) by H^- . This framing differs from the Siefert framing (of the un-knot) by $+1$, hence $\lambda(\xi_{H^-}) = 1$. \square

Next we'd like to investigate how the homotopy classes of plane fields change under connect sum. In general this operation is ill defined, as the resulting plane field will depend closely on the spheres of the sum. However, as with contact connect sum, we can remedy this problem by demanding that the trivialization be standard on each ball. In particular, we'll say a ball equipped with a contact structure and trivialization of the tangent bundle is *standard* if it looks like one of the balls $\{(a, b) \in S^3 | \text{Im}(a) \geq 0\} \subset S^3$ or $\{(a, b) \in S^3 | \text{Im}(a) \leq 0\} \subset S^3$ with Hopf framing (iN, jN, kN) and contact structure ξ_0 coming from complex tangencies. Note that these two balls are diffeomorphic under the involution $V : S^3 \rightarrow S^3$ given by $(a, b) \mapsto (-a, b)$ in \mathbb{C}^2 coordinates (which reverses orientation on the boundary). We will use this identification below to define a standard form connect sum.

A contact connect sum can be arranged to happen via the gluing V between two standard balls, forgetting the trivialization, since each of these is tight, any two tight balls are isotopic, and contact connect sums along

isotopic balls give isotopic contact structures (Colin's result [3]). We'll need a preliminary lemma which allows us to arrange this for the trivializations, too:

Lemma 4.1.5. *If (M, ξ) is a contact manifold with trivialization τ , and $B \subset M$ is a ball on which ξ is standard, then there is trivialization σ homotopic to τ such that B is standard.*

Proof. Briefly, all trivializations on a three-ball are homotopic, since the three-ball is contractible. In more detail: Let h_t be the homotopy of trivializations between $\tau|_B$ and a standard trivialization $\sigma|_B$ (in coordinates for which ξ is standard). We claim that this extends to a homotopy \tilde{h}_t on all of M . To see this take a neighborhood $S^2 \times I \subset M$ with $\partial B = S^2 \times \{0\}$. The homotopy h_t extends to the neighborhood as $\tilde{h}_t(p, s) = h_{t(1-s)}(p)$ for $p \in \partial B$. (This makes sense as the product structure gives an identification of $T_{(p,s)}(S^2 \times I)$ with $T_{(p,s')}(S^2 \times I)$.) Now by defining \tilde{h}_t as the identity outside of $B \cup (S^2 \times I)$, we have a globally defined homotopy taking τ to a trivialization $\sigma = \tilde{h}_1$ which restricts to a standard trivialization on B . \square

On manifolds equipped with both contact structures and trivializations, a connect sum along standard balls, via the map V , will be referred to as a *standard connect sum*. In particular, V provides a canonical way to glue the trivializations τ_i along the boundary of the standard balls to get a trivialization τ' on the connect sum. The above lemma shows that any contact connect sum, of manifolds with trivialization, can be made into a standard connect sum by homotoping the trivializations.

Just as happened when we computed $\lambda(\xi_0)$, the homology class representing the homotopy class of a contact structure must miss each standard ball:

Lemma 4.1.6. *Let (M, ξ) be a contact manifold with trivialization τ , $B \subset M$ a standard ball. Then $\phi_\xi^{-1}(p) \cap B = \emptyset$ for any regular value p .*

Proof. As in the proof of lemma 4.1.3 the map ϕ_ξ on the standard ball B will map to the single point represented by basis vector iN , this point is not a regular value, but all others can be. For every other point the pre-image is empty on B . □

Lemma 4.1.7. *For a standard connect sum of (M, ξ_1, τ) with (S^3, ξ_2, τ_H) the final trivialization τ' on $M \# S^3 = M$ is homotopic to τ .*

Proof. On the trivializations the connect sum identifies the boundary of the standard ball $B = \{(a, b) \in S^3 | \text{Im}(a) \geq 0\} \subset S^3$ with that of the standard ball $S^3 \setminus B = \{(a, b) \in S^3 | \text{Im}(a) \leq 0\}$ by the involution $(a, b) \mapsto (-a, b)$. Since the complement of the latter ball is again B we replace one standard ball B by another, not changing the trivialization at all (though we may have significantly changed the contact structure!). □

Putting together the above discussion we get:

Lemma 4.1.8. *For standard connect sum on contact manifolds with trivializations, under the identification $H_1(M_1 \# M_2, \mathbb{Z}) = H_1(M_1, \mathbb{Z}) \oplus H_1(M_2, \mathbb{Z})$ we*

have:

$$\Gamma_{\tau'}(\xi_1 \# \xi_2) = \Gamma_{\tau_1}(\xi_1) + \Gamma_{\tau_2}(\xi_2).$$

Further, the natural \mathbb{Z} -action on $\Gamma_{\tau'}^{-1}(\Gamma_{\tau'}(\xi_1 \# \xi_2))$ is generated by the \mathbb{Z} -action on the homotopy class of either ξ_i .

Proof. The map $\phi_{\xi_1 \# \xi_2}$ restricts to the maps ϕ_{ξ_i} on the respective pieces of the connect sum. Next, the links $x_i = \phi_{\xi_i}^{-1}(p)$ (p a regular value of both maps ϕ_{ξ_i}) miss the connect sum: $x_i \cap B_i = \emptyset$ by lemma 4.1.6, since each B_i is standard. From this $x = \phi_{\xi_1 \# \xi_2}^{-1}(p) = x_1 \cup x_2$. In homology: $[x] = [x_1] + [x_2]$.

If we add a right twist to the framing of either piece x_i of the link x , we change the total framing by a right twist. The framed link x_i represents the homotopy class of ξ_i on M_i , so the \mathbb{Z} -action on either piece, ξ_i , generates the total \mathbb{Z} -action. \square

Lemma 4.1.9. *Fixing a trivialization τ of $TM_{(\Sigma, \phi)}$, $\Gamma_{\tau}(\xi_{(\Sigma, \phi) * H^-}) = \Gamma_{\tau}(\xi_{(\Sigma, \phi)})$. The map $\xi_{(\Sigma, \phi)} \mapsto \xi_{(\Sigma, \phi) * H^-}$, acting on homotopy classes, generates the \mathbb{Z} -action on $\Gamma_{\tau}^{-1}(\Gamma_{\tau}(\xi_{(\Sigma, \phi)}))$.*

Proof. The statements of the lemma make sense since $\xi_{(\Sigma, \phi) * H^-}$ is a contact structure on $M \# S^3 = M$. By lemma 3.5.1 the Murasugi sum is accomplished by a contact connect sum on the associated structures $\xi_{(\Sigma, \phi)}$ and ξ_{H^-} , which is a standard connect sum, after homotopy of τ .

By lemma 4.1.8 $\Gamma_{\tau'}(\xi_{(\Sigma, \phi) * H^-}) = \Gamma_{\tau}(\xi_{(\Sigma, \phi)}) + \Gamma_{\tau_H}(\xi_{H^-}) = \Gamma_{\tau}(\xi_{(\Sigma, \phi)})$. We have taken the trivialization τ_H for S^3 so lemma 4.1.7 says that τ' is in

fact homotopic to τ . Since everything is homotopy invariant, $\Gamma_\tau(\xi_{(\Sigma,\phi)*H^-}) = \Gamma_\tau(\xi_{(\Sigma,\phi)})$.

Let L be a framed link with $[L] = [x_{(\Sigma,\phi)}]$. We wish to compare L with $L \cup x_{H^-}$, up to framed cobordism. Now, from lemma 4.1.4 the framing of x_{H^-} differs from the Siefert surface framing by $+1$. We apply -1 (by the \mathbb{Z} -action) to x_{H^-} , and can then remove this link by a framed cobordism. By lemma 4.1.8 we can cancel the effect, in the connect sum, of the -1 -action by acting on L with $+1$. Thus the framed link $L \cup x_{H^-}$, is equivalent to L with framing increased by one right twist. The connect sum generates the natural \mathbb{Z} -action on $\Gamma_\tau^{-1}([x_{(\Sigma,\phi)}])$. \square

Lemma 4.1.10. *For (S^3, ξ_1) and (S^3, ξ_2) , $\lambda(\xi_1 \# \xi_2) = \lambda(\xi_1) + \lambda(\xi_2)$. In particular $\lambda(L_1 * L_2) = \lambda(L_1) + \lambda(L_2)$ for fibered links L_i .*

Proof. The Siefert surface for the (un-linked) disjoint union of links is the union of Siefert surfaces. So, for framed links, the total difference between the given framing and the standard (Siefert) framing is the sum of the differences for each link. By lemma 4.1.8 (and its proof) the framed link x representing the homotopy class $\xi_1 \# \xi_2$ is the disjoint union of links x_i for ξ_i . Since the trivialization is unchanged (lemma 4.1.7) the differences of framings on x_i add under connect sum, and so do the λ . \square

4.2 Proof of Harer's Conjecture

We can now prove Harer's conjecture. The proof relies on the homotopy considerations of the previous section, on the stabilization theorems of the last chapter, and on Eliashberg's uniqueness result for overtwisted contact structures. In this sense it is an essentially contact proof, not just topology in contact clothing.

Theorem 4.2.1. *All fibered links are equivalent by Hopf-plumbing. In particular, if B, B' , are any two open books for S^3 , then $B * H^- * H^+ * \dots * H^+ = B' * H^- * \dots * H^+ * \dots * H^+$ for some Murasugi sums (or vice versa: one side requires only one H^- summand).*

Proof. Without losing generality we can assume $\lambda(B) \geq \lambda(B')$. Since H^- -stabilization acts by the natural \mathbb{Z} -action on the homotopy class of the compatible contact structure, we can H^- -stabilize to get the compatible contact structures of the two open books in the same homotopy class. In particular Murasugi sum H^- to B' , $\lambda(B) - \lambda(B')$ times, to get $\lambda(B' * H^- * \dots * H^-) = \lambda(B)$.

One more H^- -stabilization on each link guarantees that the two contact structures are overtwisted; still in the same homotopy class they are isotopic by Eliashberg's theorem (2.3.5). Now applying theorem 3.4.4 the two open books are H^+ -stably equivalent. \square

Any fibered link can be made into a knot by H^+ stabilization, and knots are often simpler to consider. In the above proof the number of components of the link may change at each stabilization, however this is not necessary:

Theorem 4.2.2. *Any two fibered knots, K and K' , are equivalent after plumbing with some number of trefoils and a figure-eight knot. In particular each stage of the equivalence is a knot. (At most one figure-eight is needed.)*

Proof. Since $T^- = H^- * H^-$ and the figure-eight is $E = H^- * H^+$: $\lambda(T^-) = 2$ and $\lambda(E) = 1$. Assume $\lambda(K) \geq \lambda(K')$. If $\lambda(K) - \lambda(K')$ is odd we begin by plumbing a figure-eight to K' to get $\lambda(K) - \lambda(K' * E)$ even. Then plumb $(\lambda(K) - \lambda(K' * E))/2$ negative trefoils to get: $\lambda(K) = \lambda(K' * E * T^- * \dots * T^-)$.

Now plumb T^- to each knot, making the compatible contact structures overtwisted, and still homotopic. By theorem 2.3.5 the contact structures are isotopic, and then by theorem 3.4.6 the knots are equivalent after some T^+ -plumbings.

The Murasugi sum of two knots gives another knot, so any two fibered knots are equivalent as claimed. □

4.3 Grothendieck Groups of Fibered Links

The Grothendieck group \mathcal{G} of fibered links in S^3 is the group generated from the set of isotopy classes of fibered links in S^3 by the operation $*$ which is Murasugi sum. To be more precise, let \mathcal{F} be the free group generated by the set of fibered links, up to isotopy, then \mathcal{G} is this group modulo the relations: $B * B' = C$ if C is some Murasugi sum of B and B' . In the same way we can define a group \mathcal{G}_k from fibered knots. Note that the un-knot U is the identity since $U * B = B$, for any B , and that the group is commutative.

Neumann and Rudolph re-phrased, and expanded, Harer's original questions in terms of this Grothendieck group (it is in this form that they appear in problem 1.83 of Kirby's list [1]). We answer these questions in what follows.

Corollary 4.3.1. *The group \mathcal{G} is generated by the two elements H^\pm . The group \mathcal{G}_k is generated by T^\pm and E .*

Proof. From theorem 4.2.1 any fibered link is equivalent to the unlink by: $B * H^- * H^+ * \dots * H^+ = U * H^- * \dots * H^+ * \dots * H^+$. In \mathcal{G} this relation gives: $B = H^- * \dots * H^+ * \dots * H^+ * (H^- * H^+ * \dots * H^+)^{-1}$. Similarly for knots. \square

The elements $H^\pm \in \mathcal{G}$ are infinite order (since $\dim H_1(\Sigma)$ is additive under Murasugi sum), and, because of commutativity, this completely determines the group (ie. $\mathcal{G} \simeq \mathbb{Z} \times \mathbb{Z}$).

\mathcal{G}_k is similarly an abelian group generated by T^\pm and E . T^\pm again are infinite order, with no relations between them, however there is the relation $E^2 = T^+ * T^-$. To see this identity consider (on the topological level) that $E * E = (H^+ * H^-) * (H^+ * H^-)$, so the compatible contact structure ξ_E is isotopic to T^- . Since each of these is a fibered knot, this implies $E * E * T^+ * \dots * T^+ = T^- * T^+ * \dots * T^+$. By counting genus we see that there must be one more copy of T^+ on the right than the left, so, in the group: $E * E = T^- * T^+$.

4.4 General Manifolds

The techniques used above are applicable to open books on arbitrary three-manifolds. In what follows we will talk about two two-plane fields having

the same two-dimensional invariant if $\Gamma_\tau(\xi_1) = \Gamma_\tau(\xi_2)$ for some trivialization τ . For another trivialization τ' , $\Gamma_{\tau'}(\xi_1) = \Gamma_{\tau'}(\xi_2)$ iff $\Gamma_\tau(\xi_1) = \Gamma_\tau(\xi_2)$ (see [13]), so it doesn't matter which trivialization we chose. Since the contact structure, ξ_B , associated with an open book, B , is unique up to homotopy, we may define the two and three dimensional invariants of B to be those of ξ_B .

Theorem 4.4.1. *Two open books, B and B' , on a manifold M^3 are H^\pm -stably equivalent if and only if B and B' have the same two-dimensional invariant.*

Proof. If $\Gamma_\tau(\xi_B) = \Gamma_\tau(\xi'_B)$ the three dimensional invariants belong to the same \mathbb{Z}/m -torsor (for some m). Since Murasugi sum with H^- generates the \mathbb{Z} -action (lemma 4.1.9), we can bring ξ_B and ξ'_B into the same homotopy class by H^- -stabilization. (Indeed, in the case when $m \neq 0$ we may do this with at most $m - 1$ stabilizations applied to either open book.) One further sum with H^- assures that both contact structures are overtwisted (lemma 3.5.3), and by Eliashberg's theorem they are isotopic. Then by theorem 3.4.4 the open books are equivalent by H^+ -stabilization.

On the other hand, Murasugi sum with H^\pm doesn't change $\Gamma_\tau(\xi_B)$. So if the two-dimensional invariants differ the open books cannot be H^\pm -stably equivalent. \square

In the same vein we may consider the 'mirror invariants' of $B = (\Sigma, \phi)$, which we define as the invariants of (Σ, ϕ^{-1}) . These are in fact the invariants of the negative contact structure which can be built from the open book (eg. by taking a negative one-form $-ds$ in the Thurston-Winkelnkemper construction).

The mirror invariants can sometimes distinguish open books which have the same homotopy invariants, for instance B and $B * H^+$. It is interesting to note that the three dimensional invariant and it's mirror may be as different as one likes (take the example of Hopf plumbings). We don't yet know of any useful applications of the mirror invariants.

4.5 Geography and Genus

For a surface Σ , $\mu(\Sigma)$ denotes the dimension of $H_1(\Sigma)$; for an open book B , or its binding L (eg. a fibered link) $\mu(B)$ and $\mu(L)$ refers to μ of the page (Siefert surface).

We've already introduced $\lambda(L)$, and there is another invariant, call it $t(L)$ with is 0 if ξ_L is tight and 1 otherwise. On S^3 , $\lambda(L)$ and $t(L)$ determine ξ_L , since there is only one tight structure and the overtwisted ones are determined by their homotopy class, determined by λ .

It is natural to wonder about the geography of the (μ, λ, t) invariants, that is, which values are realized by fibered links.

Lemma 4.5.1. *If there is a fibered link L with $\mu(L) = n$, and $m \geq n$, then there is a link L' with $\mu(L') = m$, $\lambda(L') = \lambda(L)$ and $t(L') = t(L)$.*

Proof. Plumb $m - n$ copies of H^+ to L . □

Thus questions of geography are really questions about the minimal possible μ .

Definition 4.5.1. $m(\lambda, t)$ is the minimum of $\mu(L)$ over all fibered links, L , with $\lambda(L) = \lambda$ and $t(L) = t$. More generally, $m(M, \xi)$ is the minimum $\mu(B)$ over open books, B , on M with compatible contact structure (isotopic to) ξ .

Some values are immediate: $m(\lambda, t) = 0$ iff $t = 0$ (note $t = 0$ implies $\lambda = 0$), realized by the un-knot. $m(1, 1) = 1$ realized by H^- , and in fact $m(\lambda, 1) > 1$ if $\lambda \neq 0, 1$. Indeed, $m(n, 1) \leq n$ by an H^- plumbing ($n \geq 0$), etc. However, Neumann and Rudolph show in [20] that $m(\lambda, t) \leq 2$. This settles the geography questions for S^3 , but the question of general manifolds remains interesting. In particular, the Heegaard genus $g(M)$ of a manifold M , defined as the minimum genus of a Heegaard splitting of M , is known to be an interesting and difficult invariant. Since an open book gives a Heegaard splitting we get lower bound $g(M) \leq m(M, \xi)$. The question then is how close is this bound to being tight? For example, one might conjecture, following the S^3 case, that there is a number $U(M)$ such that $m(M, \xi) \leq g(M) + U(M)$

Chapter 5

Overtwisted Open Books

We now begin to study contact structures by using open books. The first important consideration about a contact structure, as shown by the fundamental theorem of Eliashberg (2.3.5), is whether it is tight or overtwisted. It is this property we consider in this chapter. Of course, because of the uniqueness theorem 3.2.4, the contact structures compatible with a given open book are either all tight or all overtwisted. This leads us to:

Definition 5.0.2. An open book (Σ, ϕ) is called *overtwisted* if a contact structure $\xi_{\Sigma, \phi}$ compatible with it is overtwisted.

5.1 First Examples and Results

A few open books can be shown to be overtwisted using general considerations or special techniques. An example is the Poincaré homology-sphere with reversed orientation. Etnyre and Honda show in [8] that every contact structure on this manifold is overtwisted, so any open book decomposition for this manifold must be overtwisted. The Poincaré homology sphere arises as -1 -surgery on the left handed trefoil T^- , so reversing orientation gives $+1$ -surgery on T^+ . This surgery changes the open book structure of T^+ by

adding a boundary parallel negative Dehn twist. (To see this push the surgery curve onto a page then verify directly that an arc passing the surgery experiences a negative Dehn twist). Explicitly: let Σ_1 be the punctured torus, δ a boundary parallel curve, and a, b curves which intersect in a point. Then the open book $(\Sigma_1, D_\delta^{-1}D_b^{+1}D_a^{+1})$ represents the Poincare homology-sphere with reversed orientation, and hence is overtwisted. We'll show this more directly later.

The next lemma could be proven by careful calculation and the results relating Murasugi sum to contact connect sum, as in lemma 3.5.1, however we'll give a far simpler proof in section 5.3.

Lemma 5.1.1. *The Murasugi sum $(\Sigma, \phi) * H^-$, of any open book (Σ, ϕ) , along any attaching curve, is overtwisted.*

Proof. See section 5.3. □

From this lemma we expect the presence of H^- summands to be intimately related to the overtwisted property. Indeed, using theorem 3.4.4 we can give the following criterion, up to stabilization, for an open book to be overtwisted. (Recall that two open books are H^+ -stably equivalent if there is a sequence of plumbings and de-plumbings of Hopf-bands, H^+ , which moves from the first to the second.)

Lemma 5.1.2. *An open book (Σ, ϕ) is overtwisted if and only if it is H^+ -stably equivalent to an open book with an H^- summand.*

Proof. If: Since the final open book has an H^- summand it will be overtwisted, by lemma 5.1.1. By lemma 3.5.2 the contact structure remains unchanged at each sum of the equivalence, in particular it remains overtwisted, so (Σ, ϕ) is overtwisted.

Only If: Fix a trivialization τ of the tangent bundle to $M_{(\Sigma, \phi)}$. Let ξ_{-1} be the homotopy class of plane fields arrived at from the homotopy class of $\xi_{\Sigma, \phi}$ by acting with -1 using the natural \mathbb{Z} -action. There is a contact structure in the homotopy class ξ_{-1} , since Eliashberg has shown that there is an overtwisted contact structure in each homotopy class of plane fields. Let ξ_{-1} also note a choice of such a contact structure. Let (Σ_{-1}, ϕ_{-1}) be an open book compatible with ξ_{-1} (which must exist by theorem 3.3.4).

Now, using lemma 4.1.9, $\xi_{(\Sigma_{-1}, \phi_{-1}) * H^-}$ is homotopic to $\xi_{\Sigma, \phi}$. Both contact structures are overtwisted, so by Eliashberg's classification (theorem 2.3.5), $\xi_{(\Sigma_{-1}, \phi_{-1}) * H^-}$ is isotopic to $\xi_{\Sigma, \phi}$. Then by theorem 3.4.4 they are H^+ -stably equivalent. \square

In practice this criterion is very difficult to apply. In the next section we give a more general sufficient condition for an open book to be overtwisted. This, in turn, gives the criterion of corollary 5.2.5, which can be easier to apply.

5.2 Sobering Arcs

Definition 5.2.1. Let $\alpha, \beta \subset \Sigma$ be properly embedded oriented arcs which intersect transversely, Σ an oriented surface. If $p \in \alpha \cap \beta$ and T_α, T_β are tangent vectors to α and β (respectively) at p , then the intersection p is considered positive if the ordered basis (T_α, T_β) agrees with the orientation of Σ at p . Sign conventions are illustrated in Fig. 5.1. We define the following intersection numbers:

1. The *algebraic intersection number*, $i_{alg}(\alpha, \beta)$, is the oriented sum over interior intersections.
2. The *geometric intersection number*, $i_{geom}(\alpha, \beta)$, is the unsigned count of interior intersections, minimized over all boundary fixing isotopies of α and β .
3. The *boundary intersection number*, $i_\partial(\alpha, \beta)$, is one-half the oriented sum over intersections at the boundaries of the arcs, after the arcs have been isotoped, fixing boundary, to minimize geometric intersection.

In particular, for an arc $\alpha \subset \Sigma$ in the page of an open book (Σ, ϕ) we may consider the intersection numbers: $i_{alg}(\alpha, \phi(\alpha)), i_{geom}(\alpha, \phi(\alpha)), i_\partial(\alpha, \phi(\alpha))$. Here $\phi(\alpha)$ is oriented by reversing a pushed forward orientation on α . Since reversing the orientation of α will also reverse that on $\phi(\alpha)$, leaving the intersection signs fixed, it is irrelevant which orientation we choose.

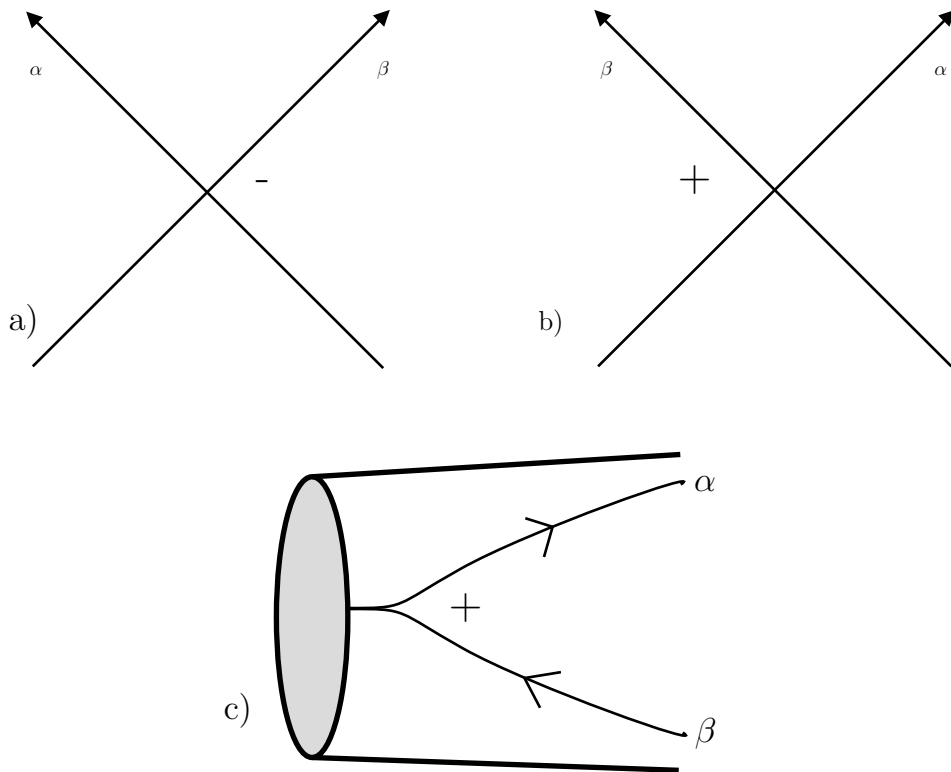


Figure 5.1: The sign conventions for the intersection of curves (α, β) : a) negative intersection, b) positive, c) a positive intersection as it appears at the boundary of the surface.

Much subtle information about a surface automorphism can be gleaned by comparing these intersection numbers for properly embedded arcs. For instance the algebraic monodromy (the induced map $\phi_* : H_*(\Sigma) \mapsto H_*(\Sigma)$) is unable to distinguish between (Σ, ϕ) and (Σ, ϕ^{-1}) , since ϕ_* is related to ϕ_*^{-1} by conjugation (through the change of basis which reverses the orientation of each basis vector). However, $i_{\partial}(\alpha, \phi(\alpha)) = -i_{\partial}(\alpha, \phi^{-1}(\alpha))$ is a geometrically well-defined quantity which can make this distinction.

Further explorations, guided by the theorem which follows, lead to the definition:

Definition 5.2.2. A properly embedded arc $\alpha \subset \Sigma$ is *sobering*, for a monodromy ϕ , if $i_{alg}(\alpha, \phi(\alpha)) + i_{\partial}(\alpha, \phi(\alpha)) + i_{geom}(\alpha, \phi(\alpha)) \leq 0$, and α is not isotopic to $\phi(\alpha)$.

In particular, since $i_{\partial} \geq -1$, there can be no intersections with positive sign: each positive intersection contributes 2 to the sum of intersection numbers. So we can reinterpret the definition:

Lemma 5.2.1. *An arc, α , is sobering if and only if $i_{\partial} \geq 0$, after minimizing geometric intersections, there are no positive (internal) intersections of α with $\phi(\alpha)$, and α isn't isotopic to $\phi(\alpha)$.*

There is a good reason to be interested in such arcs:

Theorem 5.2.2. *If there is a sobering arc $\alpha \subset \Sigma$ for ϕ , then the open book (Σ, ϕ) is overtwisted.*

To prove this theorem we will construct a surface with Legendrian boundary which violates the Bennequin inequality, theorem 2.3.3. Since this inequality is a necessary condition for tightness, we'll conclude that $\xi_{\Sigma, \phi}$ is overtwisted. In doing so, however, we will not find an overtwisted disk – such a disk must exist, but will, in general, not be nicely positioned with respect to the open book.

We begin our construction by suspending α in the mapping torus: $\alpha \times I \subset \Sigma \times_{\phi} I$. Let $p \in \partial\Sigma$ be an endpoint of α . Since ϕ fixes the boundary, $p \times I$ is a meridian around the binding. Glue a disk across each of the meridians $\partial\alpha \times I$ – extending the surface across the binding in $M_{\Sigma, \phi}$. We now have a disk, call it D_{α} , which has $\partial D_{\alpha} \subset \Sigma$. D_{α} is embedded on its interior, but possibly immersed on its boundary.

We isotope $\phi(\alpha)$, relative to the boundary, to have minimal and transverse intersection with α . This induces an isotopy of D_{α} , which intersects itself at transverse double points on its boundary. To proceed we need to remove the double points of D_{α} . However, if we simply separate the edges of the disk meeting at an intersection, the boundary of the disk won't stay on Σ , eliminating any control we might have had over the framing. Instead we will opt to resolve the double points by decreasing the Euler characteristic of the surface:

Definition 5.2.3. Let F be an orientable surface which is embedded on $Int(F)$ and has isolated double points on ∂F . The *resolution* of F is constructed by thickening each double point into a half-twisted band in such a

way that the resulting surface remains oriented. The local model is shown in figure 5.2.

Let S be the resolution of D_α , with additionally ∂D_α pushed into $Int(\Sigma)$ at the points $\partial\alpha$ (see figure 5.3). Figure 5.2b) shows, on Σ , how the boundary ∂D_α changes into $\partial S \subset \Sigma$ at a resolution.

Lemma 5.2.3. $\chi(S) = 1 - i_{geom}(\alpha, \phi(\alpha))$.

Proof. The construction of S starts with the disk D_α which has $i_{geom}(\alpha, \phi(\alpha))$ double points on its boundary. At each double point we glue a band from the disk to itself. The counting follows since a disk has $\chi = 1$, and adding a band reduces the Euler characteristic by 1. \square

Lemma 5.2.4. *The difference in framings of ∂S with respect to S and Σ is:*
 $Fr(\partial S; S, \Sigma) = -i_{alg}(\alpha, \phi(\alpha)) - i_\partial(\alpha, \phi(\alpha))$.

Proof. The relative framing can be computed as the oriented intersection number of S with a push-off of ∂S along Σ . To choose the push-off first choose an orientation for S , and let s be a tangent vector to ∂S which gives the boundary orientation. Choose vector $t \in T\Sigma$ so that (s, t) provide an oriented basis agreeing with the orientation of Σ . Let L be the push-off of ∂S in the t direction.

Now, $L \subset \Sigma$ can intersect S only on $Int(S) \cap \Sigma$. This set consists of arcs where intersections were resolved in the construction of S (including the boundary intersections, which are smoothed, as in figure 5.3). For each of the

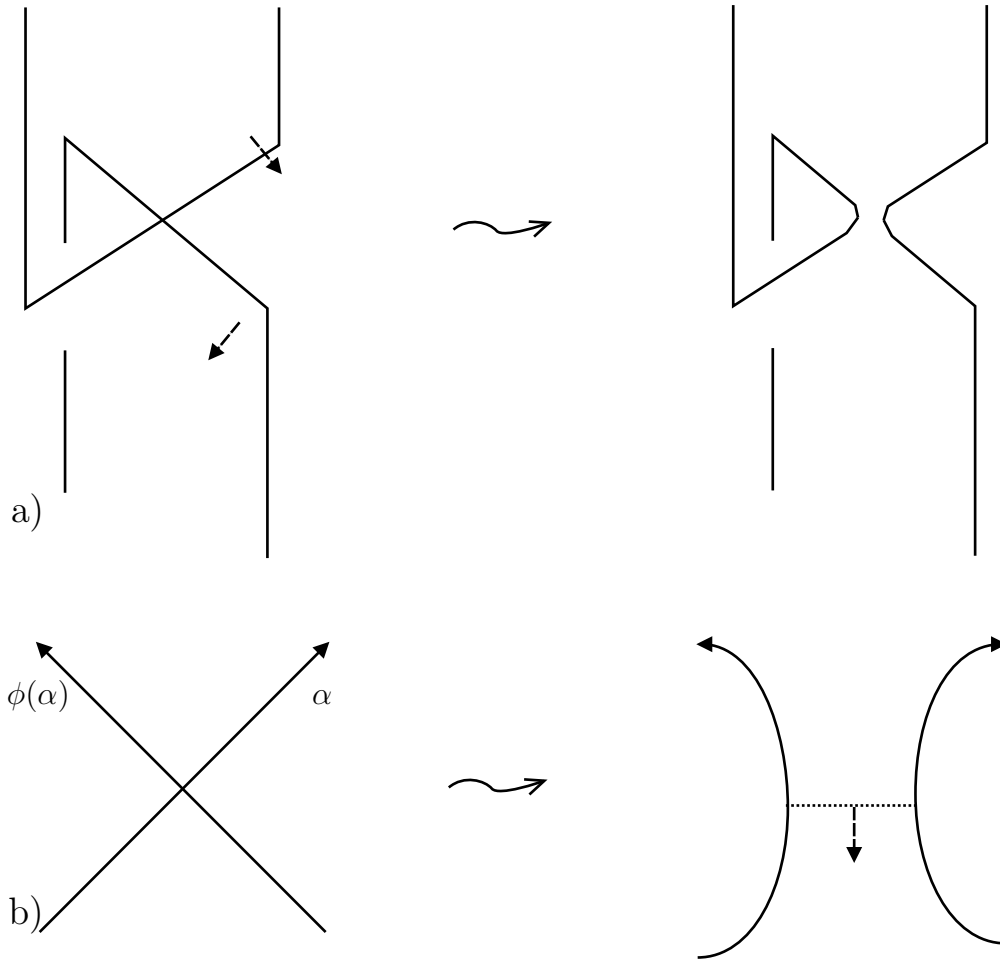


Figure 5.2: The local model for resolutions of immersed points on the boundary of a surface. a) the surface, with dashed arrows indicating orientations, b) the resolution of $\alpha \cap \phi(\alpha)$, drawn on Σ , for the case of D_α (the dashed line represents $D_\alpha \cap \Sigma$).

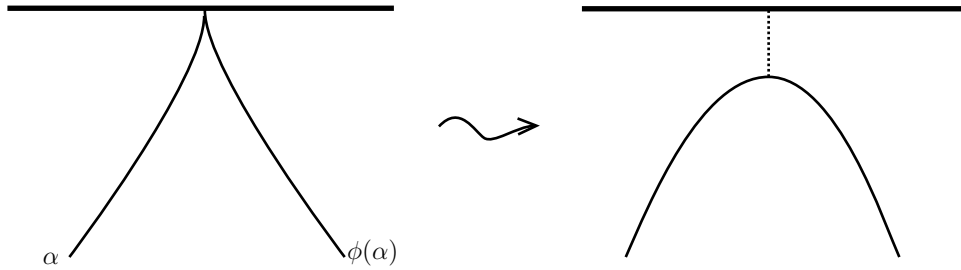


Figure 5.3: Smoothing the boundary of the suspension disk, D_α . The dashed line is $D_\alpha \cap \Sigma$.

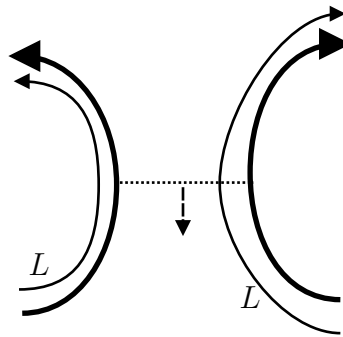


Figure 5.4: Framing at the resolution of a positive intersection. The dashed line is $S \cap \Sigma$, while the arrow is the orientation of S . L is a push-off of ∂S .

internal intersections of α with $\phi(\alpha)$, L intersects S once with sign opposite that of the intersection, as shown in figure 5.4.

It remains to account for the boundary intersections. Since ϕ fixes $\partial\Sigma$ (as for any open book), there are exactly two intersection points in $\partial\alpha \cap \partial\phi(\alpha) = \partial\alpha$ (so $i_\partial(\alpha, \phi(\alpha)) \in \{+1, -1, 0\}$). There are four possibilities for the intersection of L with S at these resolutions, according to the signs of the intersections at the beginning and end of α . One can compute each of these possibilities, as in the example of figure 5.5, the result is shown in table 5.1.

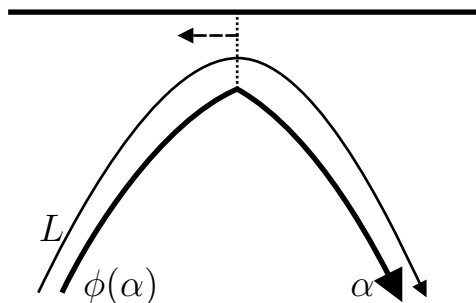


Figure 5.5: Framing near boundary: an example of computing the intersection of L with S , on Σ . Here a positive boundary intersection at the start of α contributes -1 to the framing. The dotted line is $S \cap \Sigma$, while the dashed arrow is the orientation of S .

There will be one intersection near the boundary, with sign ∓ 1 , if $i_{\partial}(\alpha, \phi(\alpha)) = \pm 1$, and either two with opposite signs, or none, if $i_{\partial}(\alpha, \phi(\alpha)) = 0$. Therefore the boundary intersections contribute $-i_{\partial}(\alpha, \phi(\alpha))$ to the framing, $Fr(S, \Sigma)$.

Finally, in the above analysis we chose an orientation on α (which then oriented $\phi(\alpha)$, D_{α} and S). Reversing this orientation is equivalent to reversing the orientation of ∂S , and the effect on the framing computation is to take the push-off of ∂S along Σ but to the other side of S . However, the relative framing $Fr(S, \Sigma)$ is well-defined independent of which push-off is used, so our result doesn't depend on the chosen orientation. (The independence can also be seen directly by repeating the calculation using the other push-off.)

□

We are now ready to prove the main theorem of this section.

	start α	end α
pos.	-1	0
neg.	0	+1

Table 5.1: The framing intersections contributed by a boundary intersection of the given sign at the beginning or end of α . Computations were made as in figure 5.5.

Proof of Thm 5.2.2. We must first rule out any components of ∂S which bound on disks Σ .

Assume there is a disk $D \subset \Sigma$ such that $\partial D \subset \partial S$. The intersection signs at the corners of a disk (where resolutions have occurred on ∂S) must alternate: see figure 5.6. However, by the sobering condition, there is at most one positive intersection, which occurs at the boundary. This implies that the only possible disk is a bi-gon with a corner on the boundary, but in this case either the intersection $\alpha \cap \phi(\alpha)$ isn't minimal, or the other corner of the bi-gon is also on the boundary. The former contradicts the minimal set-up we've arranged, the latter contradicts the definition (α isn't isotopic to $\phi(\alpha)$), so there can be no such disk.

Inside $M_{\Sigma, \phi}$ the closed surface $\bar{\Sigma} = \Sigma_0 \cup \Sigma_{0.5}$, consisting of two pages of the open book, is a convex surface divided by the binding $\partial \Sigma$ (lemma 3.6.1). We want to apply the LeRP (lemma 2.3.8) to the curves ∂S , but we must insure that there is a component of the dividing set in each component of $\bar{\Sigma} \setminus \partial S$. To achieve this, let Γ contain a simple closed curve in each component of $\bar{\Sigma} \setminus \partial S$. Using the folding lemma 2.3.9 along Γ we perturb $\bar{\Sigma}$, changing the dividing set to $\partial \Sigma \cup \Gamma \cup \Gamma'$ (Γ' a parallel copy of Γ). Now we can apply the

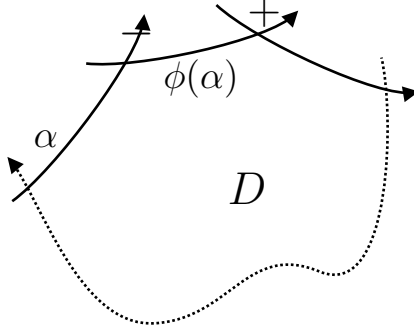


Figure 5.6: If disk $D \subset \Sigma$ was created by resolving $\alpha \cap \phi(\alpha)$, the signs at the intersections would have to alternate.

LeRP, perturbing $\bar{\Sigma}$ again to make the curves ∂S Legendrian.

By the definition of convex surfaces there is a contact vector field v normal to $\bar{\Sigma}$. Since ∂S doesn't cross the dividing set, $v|_{\partial S} \notin \xi_{\Sigma, \phi}$, so the framing of S with respect to $\xi_{\Sigma, \phi}$ (the twisting number) may be computed by using a push-off of ∂S in the v direction. But v is normal to $\bar{\Sigma}$ so a push-off along $\bar{\Sigma}$ is equivalent, for framing purposes, to the push off along v . Hence, the framing coming from $\xi_{\Sigma, \phi}$ is the same as that from Σ : $tw(\partial S, S) = Fr(\partial S; S, \Sigma)$.

We computed above that $Fr(S, \Sigma) = -i_{alg}(\alpha, \phi(\alpha)) - i_{\partial}(\alpha, \phi(\alpha))$. So,

$$tw(\partial S, S) + \chi(S) \geq -i_{alg}(\alpha, \phi(\alpha)) - i_{\partial}(\alpha, \phi(\alpha)) + 1 - i_{geom}(\alpha, \phi(\alpha)).$$

The sobering condition $-i_{alg}(\alpha, \phi(\alpha)) - i_{\partial}(\alpha, \phi(\alpha)) - i_{geom}(\alpha, \phi(\alpha)) \geq 0$ then implies:

$$tb(\partial S, S) + \chi(S) > 0,$$

violating the Bennequin inequality (theorem 2.3.3), which must hold if $\xi_{\Sigma,\phi}$ is tight. Hence, (Σ, ϕ) is overtwisted. \square

Remark 5.2.1. A word about the use of arcs instead of closed curves inside Σ . Our construction, above, would fail for these curves for simple Euler characteristic reasons, or we may rule out such considerations more directly: in the complement of the binding $\xi_{\Sigma,\phi}$ is a perturbation of the taught foliation given by the leaves of the open book. From the results of Eliashberg and Thurston [6] such a contact structure is symplectically fillable, and hence tight. Thus, no surface entirely in the complement of the binding, such as the suspension of a closed curve on Σ , can violate the Bennequin inequality.

We can immediately prove another criterion for tightness, up to stabilization:

Corollary 5.2.5. *An open book is overtwisted if and only if it is H^+ -stably equivalent to an open book with a sobering arc.*

Proof. If: The final open book of the equivalence has a sobering arc so its compatible contact structure is overtwisted. Neither summing nor de-summing H^+ changes the contact structure (lemma 3.5.2), so the original open book is overtwisted.

Only If: From lemma 5.1.2 an overtwisted open book is H^+ -stably equivalent to one with an H^- summand. Such an open book has a sobering arc, across the Hopf-band, as discussed below. \square

We can slightly extend the class of monodromies for which sobering arcs are useful by a little cleverness:

Theorem 5.2.6. *If an arc $\alpha \subset \Sigma$ has:*

$$i_{alg}(\alpha, \phi(\alpha)) + i_{\partial}(\alpha, \phi(\alpha)) + i_{geom}(\alpha, \phi(\alpha)) = -1,$$

then each open book (Σ, ϕ^n) , for $n > 0$, is overtwisted.

Proof. The open book (Σ, ϕ^n) can be built by gluing n -copies of $\Sigma \times I$ to each other by ϕ (including the bottom of the first to the top of the last) to get the mapping torus $\Sigma \times_{\phi^n} S^1$, then adding the binding as usual. In each $\Sigma \times I$ we have a disk $\alpha \times I$. The top of one disk intersects the bottom of the next as $\alpha \cap \phi(\alpha)$. If we resolve these intersections, as in the previous proofs, we get a surface with $\chi = n - ni_{geom}(\alpha, \phi(\alpha))$ and framing $ni_{alg}(\alpha, \phi(\alpha))$. Next we add in the binding and cap off the surface with disks $D^2 \times \partial\alpha$, to get a surface S with $\chi(S) = 2 - n - ni_{geom}(\alpha, \phi(\alpha))$ and framing $ni_{\partial}(\alpha, \phi(\alpha)) + ni_{alg}(\alpha, \phi(\alpha))$.

To finish the proof we proceed exactly as in the proof of theorem 5.2.2: remove components of ∂S which bound disks on a page, then fold and LeRP to make ∂S Legendrian (we'll have to do this along several pages, but they won't interfere since folding and LeRPing can be done in small neighborhoods). Finally we get a surface with $tb(\partial S, S) + \chi(S) = -n(i_{alg}(\alpha, \phi(\alpha)) + i_{\partial}(\alpha, \phi(\alpha))) + 2 - n$. Then under the condition $i_{alg}(\alpha, \phi(\alpha)) + i_{\partial}(\alpha, \phi(\alpha)) + i_{geom}(\alpha, \phi(\alpha)) = -1$ we get: $tb(\partial S, S) + \chi(S) = n + 2 - n = 2 > 0$, so the open book is overtwisted.

□

We can apply this theorem to get some interesting examples:

Corollary 5.2.7. *Let Σ_g be a surface with genus g and one boundary component, and let $\delta \subset \Sigma_g$ be parallel to the boundary. The open book $(\Sigma_g, D_\delta^{-n})$ is overtwisted ($n > 0$).*

Proof. Let $\{a_i, b_i\}_{i=1, \dots, g}$ be simple closed curves on Σ_g such that a_i intersects b_i in one point, b_i intersects a_{i+1} in one point, and the curves are otherwise disjoint. One can show, as in [17], that $D_\delta = (\prod_i D_{a_i} D_{b_i})^{4g+2}$.

Then $D_\delta^{-n} = (D_{b_g}^- D_{a_g}^- \cdots D_{b_1}^- D_{a_1}^-)^{n(4g+2)}$. An arc which intersects only b_g will be sobering for $\phi = D_{b_g}^- D_{a_g}^- \cdots D_{b_1}^- D_{a_1}^-$. \square

On the other hand, since the open books (Σ_g, D_δ^n) are positive, they are Stein fillable, and hence tight. (The connection between fillable contact structures and positive monodromy was found by many people, see [9] for a review.)

Returning to the open book given in section 5.1 for the Poincare homology-sphere with reversed orientation, the technique of the corollary shows that the monodromy can be written $\phi = (D_b^- D_a^-)^6 (D_b^+ D_a^+) = (D_b^- D_a^-)^5$, and that this is overtwisted.

5.3 A Special Case

Let us consider the simplest open books. In a disk all arcs are boundary parallel, so we turn our attention to annuli, and especially to the Hopf-bands,

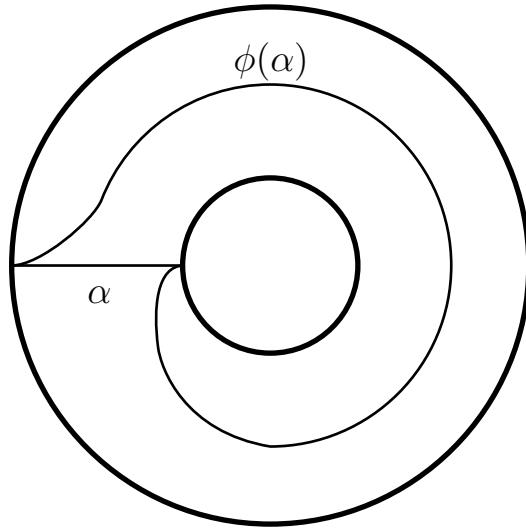


Figure 5.7: A negative Hopf-band, H^- , with the transverse arc, α , and its image.

H^\pm , which are annuli with a single right or left-handed Dehn twist along the center of the annulus. If we let α be an arc which crosses the annulus, we see from figure 5.7 that $i_{geom}(\alpha, \phi(\alpha)) = i_{alg}(\alpha, \phi(\alpha)) = 0$, while $i_\partial(\alpha, \phi(\alpha)) = \pm 1$ for H^\pm , respectively. Since $\phi(\alpha)$ isn't isotopic to α , α is sobering for H^- (but not for H^+ !).

Indeed, we can immediately prove lemma 5.1.1, that $(\Sigma, \psi)*H^-$ is over-twisted: Take an arc across H^- , outside the attaching region. It is unmoved by ψ , so will have the same intersection properties relative to $D^- \circ \psi$ as the arc α , above, had relative to ϕ . So this arc will be sobering in the sum.

How special are the intersection properties of the arcs we've just considered? It turns out that they uniquely specify the geometric situation of having a Hopf-band Murasugi summand.

Theorem 5.3.1. *If $i_{geom}(\alpha, \phi(\alpha)) = 0$ and $i_{\partial}(\alpha, \phi(\alpha)) = \pm 1$, then α is the transverse arc to a positive (respectively negative) Hopf-band. That is $(\Sigma, \phi) = (\Sigma', \phi') * H^{\pm}$ where α is parallel to the attaching curve in H^{\pm} .*

Proof. Let $c \subset \Sigma$ be the curve which comes from smoothing $\alpha \cup \phi(\alpha)$. Since $i_{geom}(\alpha, \phi(\alpha)) = 0$, c will be a simple closed curve. Note that $\phi(\alpha) = D_c^{\pm}(\alpha)$ up to isotopy, where the sign of the Dehn-twist agrees with $i_{\partial}(\alpha, \phi(\alpha))$.

Let $\phi' = D_c^{\mp} \circ \phi$ on the surface $\Sigma' = \Sigma \setminus \alpha$. This makes sense since $\phi'(\alpha) = D_c^{\mp} \circ \phi(\alpha) = D_c^{\mp} \circ D_c^{\pm}(\alpha) = \alpha$ (up to isotopy, so for some representative of the isotopy class of ϕ').

If we plumb a Hopf-band onto (Σ', ϕ') along $c \setminus \alpha \subset \Sigma'$, we get the map $D_c^{\pm} \circ \phi' = \phi$ on Σ . Thus $(\Sigma, \phi) = (\Sigma', \phi') * H^{\pm}$, and α crosses the one-handle, as required. \square

Of course if $(\Sigma, \phi) = (\Sigma', \phi') * H^{\pm}$, then there is such an arc α , described above, so we in fact have a necessary and sufficient condition for the existence of a Hopf-band summand in an open book. This criterion can be used to show that certain fibered links are not Hopf plumbings, and it is often helpful in finding stabilizations in otherwise difficult examples, as in the next section.

5.4 Boundaries of Symplectic Configurations

Following [11] we define a *symplectic configuration graph* to be a labelled graph G with no edges from a vertex to itself and with each vertex v_i labelled

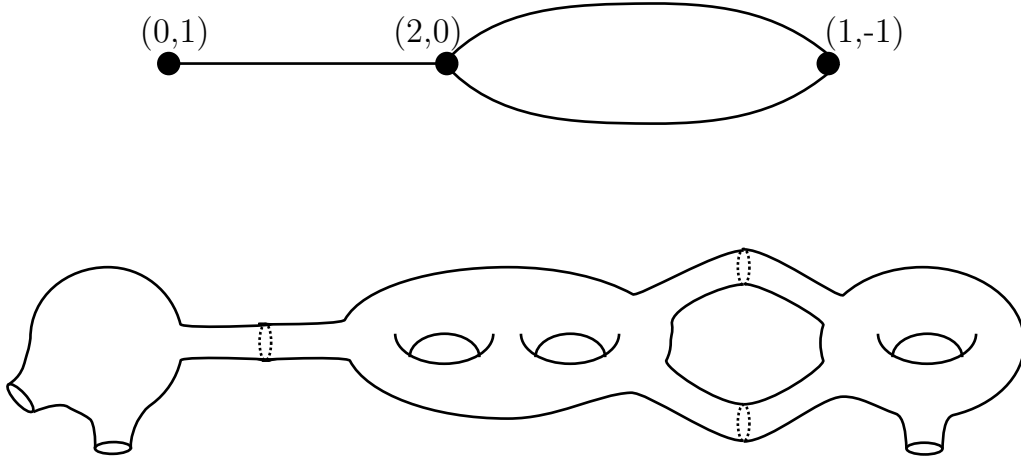


Figure 5.8: A positive configuration graph (the a_i have been suppressed), and the surface $\Sigma(G)$. The dotted circles are the e_{ij} .

with a triple (g_i, m_i, a_i) , where $g_i \in \{0, 1, 2, \dots\}$, $m_i \in \mathbb{Z}$ and $a_i \in (0, \infty)$. (a_i doesn't enter into our purely three-dimensional concerns, so we'll often omit it.) Let d_i denote the degree of vertex v_i . A configuration graph is called *positive* if $m_i + d_i > 0$ for every vertex v_i .

Given a positive configuration graph G we define an open book $(\Sigma(G), \phi(G))$ as follows: For each vertex v_i let F_i be a surface of genus g_i with $m_i + d_i$ boundary components. $\Sigma(G)$ is the surface obtained by connect sum of the F_i , with one connect sum between F_i and F_j for each edge connecting v_i to v_j . See figure 5.8 for an example. For each edge in G there is a circle e_{ij} in $\Sigma(G)$. Let $\sigma(G) = \prod D_{e_{ij}}^+$, a right-handed Dehn twist at each connect sum. Let $\delta(G)$ be the product of one right-handed Dehn twist around each circle of $\partial\Sigma(G)$. Finally, $\phi(G) = \sigma(G)^{-1} \circ \delta(G)$.

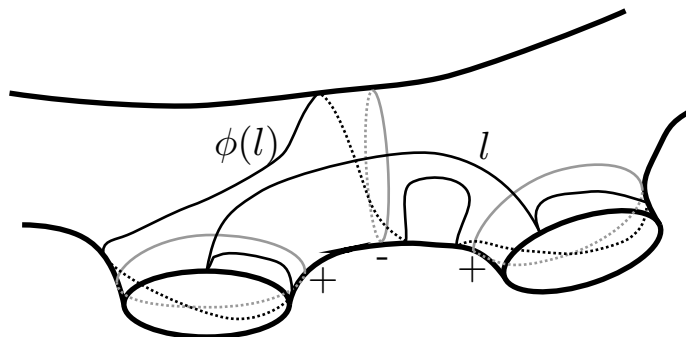


Figure 5.9: The arc $l \subset \Sigma(G)$, its image under $\phi(G)$, and the twisting curves it crosses.

In [11] Gay shows that the open books $(\Sigma(G), \phi(G))$ arise as the concave boundary of certain symplectic manifolds. In that paper he asks the question:

Question 1. *Are there any positive configuration graphs G for which we can show that $(\Sigma(G), \phi(G))$ is overtwisted (and hence conclude that a symplectic configuration with graph G cannot embed in a closed symplectic 4-manifold)?*

We will now use the techniques of the previous sections to give such examples.

First, note that if l is a properly embedded arc in $\Sigma(G)$ which crosses from one F_i to an adjacent one, so crosses one negative Dehn twist and two positive, then $i_{geom}(l, \phi(l)) = 0$ and $i_{\partial}(l, \phi(l)) = +1$ (see figure 5.9). From theorem 5.3.1 we know that there is an H^+ summand. To remove it we'll first re-write the monodromy using the lantern relation (this is a standard fact of mapping class groups, see [4]):

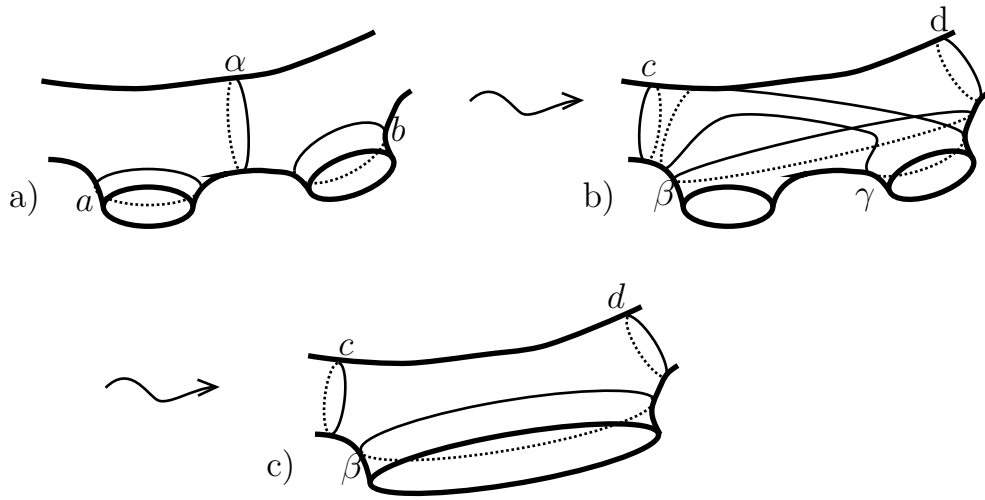


Figure 5.10: De-plumbing an H^+ from $(\Sigma(G), \phi(G))$. The first step is an application of the lantern lemma to go from $D_a^+ \circ D_b^+ \circ D_\alpha^-$ to $D_c^- \circ D_d^- \circ D_\beta^+ \circ D_\gamma^+$, the second is the removal of a Hopf-band along γ .

Lemma 5.4.1. *The following relation holds in the mapping class group of a four-times punctured sphere (fixing the boundary point-wise), referring to figure 5.10(a) and (b):*

$$D_a^+ \circ D_b^+ \circ D_c^+ \circ D_d^+ = D_\alpha^+ \circ D_\beta^+ \circ D_\gamma^+.$$

Applying this relation to the region of figure 5.9 around arc l , we can rewrite the monodromy as in figure 5.10 (a) and (b): $D_a^+ \circ D_b^+ \circ D_\alpha^- = D_c^- \circ D_d^- \circ D_\beta^+ \circ D_\gamma^+$ (Dehn twists about a, b, c, d commute with each other and with α, β , and γ , since these pairs don't intersect). It is then clear, since l crosses only the final Dehn twist D_γ^+ , how to remove a Hopf-band: remove D_γ^+ and cut along l , figure 5.10(c). We arrive at a surface with one fewer boundary component, and $D_c^- \circ D_d^- \circ D_\beta^+$ replacing $D_a^+ \circ D_b^+ \circ D_\alpha^-$ in the monodromy.

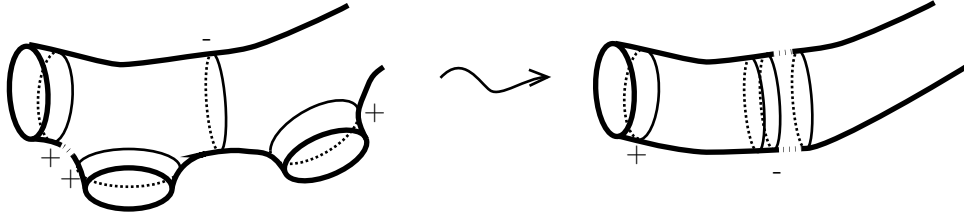


Figure 5.11: Removing almost all boundary components by applying figure 5.10 to remove H^+ .

We'll use this move repeatedly to prove the next lemma. Note that this move doesn't necessarily return an open book $(\Sigma(G'), \phi(G'))$, since the resulting open book may have several negative Dehn twists around the same curve.

Lemma 5.4.2. *If G is a positive configuration graph with a vertex v_1 such that $d_1 = 1$ and $g_1 = 0$, but G is not a graph with one edge, $m_1 + m_2 \leq 1$ and $g_1 = g_2 = 0$, then $(\Sigma(G), \phi(G))$ is overtwisted.*

Proof. First we apply the move illustrated in figure 5.10 repeatedly, removing positive Hopf-bands until all of the boundary components of F_1 are gone, leaving a boundary component with one positive Dehn twist and $m_1 + 2$ negative twists. These, of course, are equivalent to $m_1 + 1$ negative Dehn twists. See figure 5.11.

Now, let α be an arc from this new boundary component, with $m_1 + 1$ negative Dehn twists, to another boundary component. It will cross the $m_1 + 1$ negative Dehn twists, possibly some more negative twists, and finally a positive Dehn twist (see figure 5.12). For this curve, if G has more than one edge, $g_i \neq 0$ or $m_2 > 1$, then $i_{\partial}(\alpha, \phi(\alpha)) = 0$, $i_{alg}(\alpha, \phi(\alpha)) = -i_{geom}(\alpha, \phi(\alpha)) \geq -m_1$. Thus

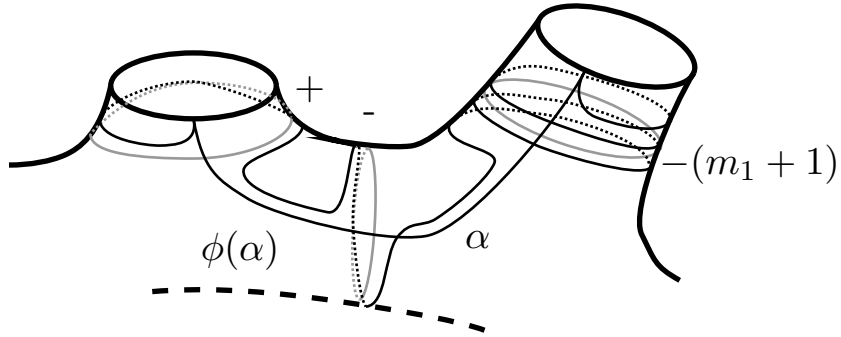


Figure 5.12: Schematic of the sobering curve.

α is sobering. In the other cases $\Sigma(G)$ becomes a cylinder after removing all the Hopf-bands, the monodromy $\phi(G)$ is $-(m_1 + m_2 - 1)$ twists around the cylinder. α is still sobering unless $m_1 + m_2 \leq 1$.

By theorem 5.2.2, the presence of a sobering arc implies that the open book is overtwisted. When stabilizing, to get back the open book $(\Sigma(G), \phi(G))$, the compatible contact structure remains overtwisted (in fact, the same up to isotopy, by lemma 3.5.2), so $(\Sigma(G), \phi(G))$ is overtwisted. \square

Remark 5.4.1. Being overtwisted these contact manifolds can't be the convex boundary of a symplectic four-manifold. Corollary 2.1 of [11] also shows this for some (but not all) of these graphs. The techniques used there are four-dimensional – a clever application of the adjunction inequality.

5.5 Protected Boundaries

In the proof of lemma 5.4.2 it appeared necessary to de-stabilize before finding a sobering arc. In this section we'll develop some tools to show that a given open book has no sobering arcs, then give an example of an open book which is overtwisted but has no sobering arcs.

Definition 5.5.1. A boundary component $\sigma \subset \partial\Sigma$ of an open book (Σ, ϕ) is *protected* if for every properly embedded arc $\alpha \subset \Sigma$ with $\#(\sigma \cap \alpha) = 1$, the sign of $\alpha \cap \phi(\alpha) \cap \sigma$ is positive (after minimizing the geometric intersection number over isotopies of $\phi(\alpha)$). An open book in which each boundary component is protected has *protected boundary*.

If an open book has protected boundary then $i_{\partial}(\alpha, \phi(\alpha)) = +1$. Putting this together with the fact $i_{alg}(\alpha, \phi(\alpha)) + i_{geom}(\alpha, \phi(\alpha)) \geq 0$ we immediately get:

Lemma 5.5.1. *An open book with protected boundary has no sobering arcs.*

Next we show that the set of open books with protected boundary is not entirely dull.

Theorem 5.5.2. *Let Σ be a surface with boundary, but not a cylinder, $a_i \subset \Sigma$ be non-intersecting simple closed curves which are not boundary parallel, and $\sigma_i \subset \partial\Sigma$ the boundary components. Then σ_j is protected in the open book (Σ, ϕ) , where $\phi = D_{\sigma_j}^+ \circ \prod_i D_{a_i}^{t_i} \circ \prod_{i \neq j} D_{\sigma_i}^{s_i}$, and $t_i, s_i \in \mathbb{Z}$.*

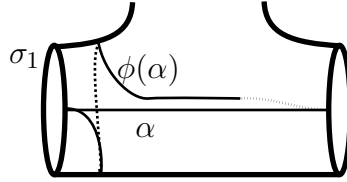


Figure 5.13: The base case for the proof of theorem 5.5.2, a thrice punctured sphere.

Proof. For convenience re-number the σ_i so that $j = 1$, so we're trying to show that σ_1 is protected. All arcs are properly embedded, and $\#(l \cap k)$ indicates the geometric count of internal intersections. We'll prove the theorem by induction on the genus of Σ . The base case is a sphere with at least 3 punctures, and monodromy containing a single positive twist about the first puncture. Fixing an α we collect all the punctures which don't intersect $\partial\alpha$ and replace them with a single puncture (this clearly doesn't effect $\phi(\alpha)$ since α never intersects the boundary parallel monodromy around these components). Now we have the case of a three-times punctured sphere, where the claim can be verified directly: see figure 5.13.

For the induction step fix an arc $\alpha \subset \Sigma$, with one end point on σ_1 . Let Σ have genus g ; choose a curve $c \subset \Sigma$ which intersects none of the a_i , and isn't boundary parallel – for instance a parallel copy of a_1 . Assume, by isotoping α as necessary, that α intersects c minimally (that is, there are no bi-gons bounded by α and c). Note that $\phi(c) = c$, so $\phi(\alpha)$ also has this property. The surface $\Sigma \setminus c$ has genus $g - 1$ and boundary $c_+ \cup c_- \cup \partial\Sigma$. By the induction hypothesis, the theorem applies to open books $(\Sigma \setminus c, D_{c_+}^{m_+} \circ D_{c_-}^{m_-} \circ \phi)$, where

$m_{\pm} \in \mathbb{Z}$.

Claim: If $\delta, \gamma \in \Sigma \setminus c$ are arcs with one end on σ_1 , such that δ is isotopic to $\phi(\gamma)$ relative to $\partial\Sigma$, and $\#(\gamma \cap \delta)$ is minimal over all such isotopies, then $\delta \cap \gamma \in \sigma_1$ is a positive intersection.

Let δ_t be an isotopy from $\delta_0 = \phi(\gamma)$ to $\delta_1 = \delta$. If $\gamma \cap (c_+ \cup c_-) = \emptyset$ then δ_t is an isotopy relative to $c_+ \cup c_- \cup \partial\Sigma$. If not, say $\gamma \cap c_+ \neq \emptyset$, arrange that $\delta \cap c_+ = \gamma \cap c_+$ (by changing the isotopy δ_t by a partial rotation of c_+ – this doesn't change $\#(\gamma \cap \delta)$). Now Let m_+ be the winding number of $\delta_t \cap c_+$ about c_+ , for $t \in [0, 1]$, then δ is isotopic to $D_{c_+}^{m_+} \phi(\gamma)$ relative to $c_+ \cup c_- \cup \partial\Sigma$. In either case the hypothesis leads us to conclude that $\delta \cap \gamma \in \sigma_1$ is a positive intersection, proving the claim.

Now, to apply this, for an arc $\beta \subset \Sigma$, with one end on σ_1 define the arc $I_c(\beta) \subset (\Sigma \setminus c)$ as the segment of β between σ_1 and (the first intersection of β with) c . If $\#(\beta \cap \alpha)$ is minimal (with respect to isotopies of β relative to $\partial\Sigma$) then there are no bi-gons in $\Sigma \setminus (\alpha \cup \beta)$, and so no bi-gons in $(\Sigma \setminus c) \setminus (I_c(\alpha) \cup I_c(\beta))$. This implies that $\#(I_c(\alpha) \cap I_c(\beta))$ is minimal over isotopies of $I_c(\beta)$ relative to $\partial\Sigma$.

If β is isotopic to $\phi(\alpha)$ such that $\#(\beta \cap \alpha)$ is minimal, we may assume that β intersects c minimally, ie. no bi-gons bounded by β and c : if there is such a bigon we may isotope β across it, without increasing intersection with α (since α bounds no bi-gons with c). Since β and $\phi(\alpha)$ both intersect c minimally, the isotopy between them can be arranged to always intersect

c minimally (any intermediary bi-gon is inessential and can be removed). In particular the isotopy from β to $\phi(\alpha)$ will always be transverse to c . In this case $I_c(\beta)$ is isotopic to $I_c(\phi(\alpha))$ relative to $\partial\Sigma$, and it has minimal intersection by the earlier comments. By the claim $I_c(\alpha) \cap I_c(\beta) \in \sigma$ is a positive intersection, so $\alpha \cap \beta \in \sigma$ also is a positive intersection. This completes the induction step. \square

Since each boundary component of the open book coming from a positive configuration graph satisfies theorem 5.5.2:

Corollary 5.5.3. *For any positive configuration graph G , such that $\Sigma(G)$ is not a cylinder and there is no i for which $g_i = m_i = 0$ and $d_i = 1$, the open book $(\Sigma(G), \phi(G))$ has protected boundary.*

We showed in the previous section that some of these open books are overtwisted. This provides examples of open books which are overtwisted but have no sobering arcs.

5.6 Sobering Arcs Under Stabilization

A sobering arc need not persist after stabilization, but the surface, S , constructed in the proof of theorem 5.2.2 is still present, topologically. How can we see this surface in the open book picture?

Say $S_{\Sigma, \phi}$ is the overtwisted surface constructed in section 5.2 for a sobering arc α in a particular open book (Σ, ϕ) . Assume that we plumb H^+

to (Σ, ϕ) along an arc $l \subset \Sigma$ which intersects $\phi(\alpha)$ in one point. Let D_{H^+} be the core disk spanning the Hopf-band in $M_{(\Sigma, \phi)*H^+}$. The surface $S_{\Sigma, \phi}$ will be homotopic to $S_{(\Sigma, \phi)*H^+}$ glued to D_{H^+} along the core of the one handle. Let us just indicate how to see this: the Hopf band lies just 'below' the attaching arc l , so the original surface $S_{\Sigma, \phi}$ goes on its merry way and intersects the edge of the Hopf band in the two points above $l \cap \phi(\alpha)$. On the other hand, $S_{(\Sigma, \phi)*H^+}$ doesn't intersect the Hopf band at all (since the end points of α are well away from l), and D_{H^+} intersects in two points. So both $S_{(\Sigma, \phi)*H^+} \cup D_{H^+}$ and $S_{\Sigma, \phi}$ have "framing" two near the hopf band. They clearly are the same away from the added Hopf band, so they must be homotopic.

When $\phi(\alpha)$ intersects l more than once the surface $S_{\Sigma, \phi}$ will simply be $S_{(\Sigma, \phi)*H^+}$ glued to several parallel copies of D_{H^+} .

We see here how unwanted intersections of α with $\phi(\alpha)$ can sometimes be removed by keeping track of auxiliary curves, and gluing together the entire set of suspension disks (in some way) before resolving. However, the surfaces constructed in this way will generally be immersed, and it is quite difficult (unless we have apriori knowledge, as above) to show that they may be homotoped, rel ∂ , to an embedded surface.

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Vita

He was born on Wednesday when it rained in the desert, the first of February 1978 in Tucson. His brother nursed a stomach ache from eating an entire plate of brownies, and one may assume that his parents, Allan S. Goodman and Barbara G. Goodman, were exhausted and pleased. It was the brother that named him, Noah Daniel Goodman. In second grade they moved to the community of the Desert Ashram, and about the same time he was first told he "needed more math" to understand the quantum mechanics and general relativity that fascinated him. He set out to get "more math".

He was educated at Sam Hughes Elementary, Safford Middle, Kino, and St. Gregory Schools. In junior high he began taking classes at Pima Community College. In 1994 he graduated from high school and enrolled at the University of Arizona. He graduated in 1997 with degrees in math and physics, and entered the Ph.D. program in mathematics at the University of Texas at Austin. In the six years of his graduate studies he was resident at the American Institute of Mathematics and Stanford University one semester, and at the University of Pennsylvania another. He traveled more than necessary and seems to have written a dissertation. He supposes that he now knows "more math".

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