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The Wonderful Compactification for Quantum Groups

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The Wonderful Compactification for Quantum Groups

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The Wonderful Compactification for Quantum Groups

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This thesis studies the asymptotics of quantum groups using an approach centered on the wonderful compactification. The wonderful compactification of a semisimple group was introduced by De Concini and Procesi, and has become an important tool in geometric representation theory. We provide an exposition of several constructions of the wonderful compactification in order to illustrate how it links the geometry of the group to the geometry of its partial flag varieties, and how it encodes the asymptotics of matrix coefficients for the group. We then construct quantum group versions of the wonderful compactification, its associated Vinberg semigroup, its stratification by $G \times G$ orbits, and its algebra of differential operators. A key technical aspect of our approach is the notion of a noncommutative projective scheme associated to a ring graded by a lattice. We provide explicit descriptions of our constructions in the case of SL_2, explain connections to previous work on the flag variety of a quantum group, and discuss conjectural applications of the newly-defined objects that appear in this thesis.

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Chapter 1

Introduction

New geometric perspectives on familiar representation-theoretic objects often illuminate deep unifying structures among diverse phenomena, and lead to the resolution of long-standing algebraic problems. This thesis synthesizes two distinct sources of insight in representation theory: (1) the study of a group 'at infinity,' that is, of compactifications of a group, and (2) the role of quantum groups as symmetries in noncommutative geometry. The interaction of these perspectives leads to results on the asymptotics of quantum groups, and to a deeper understanding of various central objects in representation theory.

To place our work in context, we recall that a given semisimple algebraic group has a distinguished compactification, known as the 'wonderful' compactification. It is a projective variety, introduced by de Concini and Procesi, that links the geometry of the group with the geometry of its partial flag varieties and Levi subgroups [DCP]. In addition, it captures the equivariant degenerations of the group and encodes the asymptotics of matrix coefficients. Related to the wonderful compactification of a group is the associated Vinberg semigroup, introduced by Vinberg [Vi1]. The latter is an affine variety that (generically) forms a multi-cone over the wonderful compactification. As such, the Vinberg semigroup can be regarded as a linear version of the wonderful compactification where various structures simplify, and where the rational degenerations of the group become more apparent.

The wonderful compactification has recently emerged as a powerful tool in geometric representation theory. For example, in the setting of p-adic groups, Bezrukavnikov and Kazhdan have illustrated how the wonderful compactification leads to a geometric understanding of the second adjointness theorem [BeKa] (see also [SV]). Drinfeld and Gaitsgory use the Vinberg semigroup in order to establish adjunctions that relate categories of $\mathcal{D}$-modules on certain moduli stacks of bundles; their results contribute to the geometric Langlands program [DG]. The Vinberg semigroup is also crucial in the proof of a dimension formula for a group version of affine Springer fibers, due to Bouthier [Bo]. The wonderful compactification has been used in the theory of character sheaves by several authors [BFO, Gi1, He, Lu, Sp], while Lu, Yakimov, and others have studied the Poisson geometry of the wonderful compactification and related varieties [LY1, LY2].

Another, similarly fruitful, source of new perspectives in representation theory is the study of noncommutative geometry emanating from quantum groups. Since the inception of quantum groups by Drinfeld and Jimbo in the 1980s, much work has been devoted to the construction of q-deformations of classical varieties in order to understand categories of representations. These constructions often take the form of q-deformations of algebras (i.e. as global rather than formal quantizations) and pivot on the structure of the quantum group as a q-deformation of the universal enveloping algebra of a Lie algebra. Just like quantum groups themselves, the q-deformations that appear in representation theory have remarkable connections to various other areas of mathematics. For example, there are strong parallels between the behavior of an object's q-deformation when q is a root of unity and the geometry of the classical object in positive characteristic.

The spirit of quantum geometric representation theory is embodied in work of Backelin and Kremnitzer on quantum flag varieties and differential operators, as well as related results by Lunts and Rosenberg and by Tanisaki [BaKr, LR, T]. Their work demonstrates that the category of quasicoherent sheaves on the flag variety of a reductive group admits a q-deformation that can be regarded as the category of sheaves on the quantum flag variety, and is a noncommutative projective scheme in the sense of Artin and Zhang [AZ]. Moreover, quantum versions of differential operators on flag varieties encode representations of quantum groups, giving rise to a quantum version of the Beilinson-Bernstein localization theorem.

In this thesis, we take inspiration from previous work on quantum flag varieties in order to establish the wonderful compactification for quantum groups. Our approach is ultimately rooted in Peter-Weyl theorem and the asymptotics of matrix coefficients, and employs the formalism of noncommutative projective schemes. Moreover, we consider quantum versions of the wonderful compactification's stratification by $G \times G$ orbits, its associated Vinberg semigroup, and its algebra of differential operators. Our perspectives relate directly to the aforementioned quantum flag varieties, and connect with many central themes in geometric representation theory.

In the next section, we give an overview of the key properties of the wonderful compactification and the Vinberg semigroup. In Section 1.2, we explain the context behind our construction of the wonderful compactification for quantum groups. We list the main results of this thesis in Section 1.3, before providing further motivation for our work in the form of on-going and future projects in Section 1.4.

1.1 What is the wonderful compactification?

Let G be a connected semisimple group over $\mathbb{C}$, and let $G^{\mathrm{ad}} = G/Z(G)$ denote the adjoint group of G. The wonderful compactification is a certain projective variety $\overline{G^{\mathrm{ad}}}$ that contains G^{ad} as an open subvariety. We recall the precise definition of $\overline{G^{\mathrm{ad}}}$ in Chapter 3. For the purposes of this overview, here we simply highlight the key properties of $\overline{G^{\mathrm{ad}}}$:

1. The variety $\overline{G^{\mathrm{ad}}}$ is stratified by the orbits of a $G \times G$ action. These orbits are indexed by subsets I of the set Δ of positive simple roots.

2. The orbit corresponding to $I = \Delta$ is a copy of the adjoint group G^{ad} with $G \times G$ action by left and right multiplication. This is the unique open orbit.

3. The orbit corresponding to $I = \emptyset$ is the square of the flag variety, $G/B \times B^{-}\backslash G$, with $G \times G$ action induced by outermost left and right multiplication. This is the unique closed orbit.

4. The other orbits are related to partial flag varieties, Levi subgroups, rational degenerations of G, and wonderful compactifications of groups of smaller rank. The rich structure of the orbits distinguishes the wonderful compactification from other compactifications[1] of $\overline{G^{\mathrm{ad}}}$.

Example 1.1.1. The wonderful compactification of PSL_2 is $\mathbb{CP}^3$. This can be seen by realizing $\mathbb{CP}^3$ as the projective space of the space of two by two matrices. The flag variety

[1] The adjective 'wonderful' is a technical term. A 'wonderful variety' for an algebraic group H refers to a smooth, connected, complete H-variety with an open orbit and whose boundary divisors have normal crossings and satisfy additional properties. For more details, see [Lun]. In the case of the wonderful compactification, the group H is $G \times G$.

for PSL_2 is $\mathbb{CP}^1$, and $\mathbb{CP}^1 \times \mathbb{CP}^1$ embeds in $\mathbb{CP}^3$ via the Segre map. The image of this embedding is a closed $\mathrm{PSL}_2 \times \mathrm{PSL}_2$-orbit, and forms the complement in $\mathbb{CP}^3$ of the copy of PSL_2. However, for $n \geq 3$, the wonderful compactification of PSL_n is not $\mathbb{CP}^{n^2-1}$.

The wonderful compactification captures the asymptotics of matrix coefficients in the following way. By the Peter-Weyl theorem, the coordinate algebra $\mathcal{O}(G)$ is spanned by matrix coefficients of representations of G. The fact that the isomorphism classes of irreducible finite-dimensional representations of G are labeled by dominant weights leads to the definition of a (multi-)filtration on $\mathcal{O}(G)$ by the weight lattice Λ of G. We refer to this filtration as the Peter-Weyl filtration on $\mathcal{O}(G)$.

Theorem 1.1.2. *[Br, Theorems 2.2.3 and 3.2.3] The following algebras are isomorphic as Λ-graded algebras:*

1. *The (multi-)Rees algebra of $\mathcal{O}(G)$ for the Peter-Weyl filtration.*
2. *The total coordinate ring, or Cox ring, of the wonderful compactification $\overline{G^{\mathrm{ad}}}$. In particular, the Picard group of $\overline{G^{\mathrm{ad}}}$ is identified with Λ.*
3. *The coordinate ring $\mathcal{O}(\mathbb{V}_G)$ of the Vinberg semigroup $\mathbb{V}_G$ of G.*

We recall the definition of the Vinberg semigroup in Chapter 3. For now, one can think of $\mathbb{V}_G$ as the affine variety associated to either of the isomorphic rings in (1) or (2) of Theorem 1.1.2. As the name indicates, the variety $\mathbb{V}_G$ carries a canonical semigroup structure, and the group of units is, up to a finite group, the direct product of G with a maximal torus T of G. The weight lattice Λ of G can be realized as the character lattice of T, and so the Λ-grading on $\mathcal{O}(\mathbb{V}_G)$ corresponds to an action of T on $\mathbb{V}_G$.

Theorem 1.1.3. *[MT, Theorem 5.3] Let λ be a regular dominant weight, regarded as a character of T. The wonderful compactification is isomorphic to the geometric invariant theory (GIT) quotient of $\mathbb{V}_G$ by T along λ:*

$$\overline{G^{\mathrm{ad}}} = \mathbb{V}_G /\!\!/_{\lambda} T.$$

Remark 1.1.4. The wonderful compactification admits a realization as a moduli space of certain framed bundle chains, as demonstrated by Martens and Thaddeus [MT]. This perspective precipitates the construction of a distinguished smooth stack that compactifies any given semisimple group. If the group has trivial center, this stack coincides with the wonderful compactification.

1.2 What is the wonderful compactification for quantum groups?

A key tenet of algebraic geometry, due to Grothendieck, asserts that a space can be completely understood through its category of sheaves. Furthermore, q-deformations of the category of sheaves can be viewed as categories of sheaves on a (nonexistent) quantum version of the original space. Thus, replacing a space with a category provides more flexibility in producing deformations; this is starting point of much of noncommutative geometry.

Applying this philosophy to the case of the wonderful compactification, we seek a category $\mathsf{QCoh}_q(\overline{G^{\mathrm{ad}}})$ that forms a q-deformation of the category $\mathsf{QCoh}(\overline{G^{\mathrm{ad}}})$ of quasicoherent sheaves on the wonderful compactification $\overline{G^{\mathrm{ad}}}$. Here q is a nonzero complex number that is not a root of unity.

To obtain the desired q-deformation, we take inspiration from a result of Serre, which describes quasicoherent sheaves on a projective variety in terms of graded modules for the homogeneous coordinate ring. Specifically, let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be commutative graded ring, and let X be the associated projective scheme with twisting sheaf $\mathcal{O}(1)$. A graded A-module is called torsion if every element is annihilated by $A_{\geq N}$ for some N. The quotient of the category of graded modules for A by the full subcategory of torsion modules produces an abelian category denoted $\mathsf{Proj}(A)$.

Theorem 1.2.1. *[Se] If A is finitely generated by elements of degree one over a field, then the functor of graded global sections*

$$\Gamma_* : \mathsf{QCoh}(X) \to \mathsf{Proj}(A)$$
$$\mathcal{F} \mapsto \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n})$$

is an equivalence of categories.

The definition of $\mathsf{Proj}(A)$ works just as well when A is a noncommutative noetherian graded ring. Moreover, rather than just considering rings graded by the integers, one can consider rings graded by the weight lattice Λ of a semisimple group G. For such a ring R, one can make sense of a torsion graded R-module, and form the quotient $\mathsf{Proj}(R)$ of the category of graded R-modules by the subcategory of graded torsion modules.

We now outline the construction of the quantum wonderful compactification:

1. Use Theorem 1.1.2 to establish an equivalence between the category $\mathsf{QCoh}(\overline{G^{\text{ad}}})$ of quasicoherent sheaves on the wonderful compactification $\overline{G^{\text{ad}}}$ and the category $\mathsf{Proj}(\mathcal{O}(\mathbb{V}_G))$, where $\mathcal{O}(\mathbb{V}_G)$ is the algebra of functions on the Vinberg semigroup.

2. Produce a q-deformation $\mathcal{O}_q(\mathbb{V}_G)$ of the ring $\mathcal{O}(\mathbb{V}_G)$, compatible with relevant structures. These structures include: a grading by Λ, a $U_q(\mathfrak{g} \times \mathfrak{g})$-action, and a Poisson structure.

3. Define the category of quasicoherent sheaves on the quantum wonderful compactification as the category $\mathsf{QCoh}_q\left(\overline{G^{\mathrm{ad}}}\right) := \mathsf{Proj}\left(\mathcal{O}_q(\mathbb{V}_G)\right)$.

1.3 Main results

Fix q to be a nonzero complex number that is not a root of unity. A version of the Peter-Weyl theorem holds for the quantum coordinate algebra $\mathcal{O}_q(G)$ and we obtain a filtration on $\mathcal{O}_q(G)$ by Λ, which we refer to as the Peter-Weyl filtration.

Proposition 1.3.1. *The (multi-)Rees algebra of $\mathcal{O}_q(G)$ for the Peter-Weyl filtration is a q-deformation of the coordinate ring of the Vinberg semigroup $\mathbb{V}_G$ of G, and quantizes a certain Poisson structure on $\mathbb{V}_G$. We denote this Rees algebra by $\mathcal{O}_q(\mathbb{V}_G)$.*

Observe that the algebra $\mathcal{O}_q(\mathbb{V}_G)$ carries a grading by Λ. As it is a q-deformation of the total coordinate ring of the wonderful compactification, we make the following definition, which recovers the category of quasicoherent sheaves on $\overline{G^{\mathrm{ad}}}$ when $q = 1$.

Definition 1.3.2. The category of quasicoherent sheaves on the quantum wonderful compactification is defined as

$$\mathsf{QCoh}_q\left(\overline{G^{\mathrm{ad}}}\right) = \mathsf{Proj}\left(\mathcal{O}_q(\mathbb{V}_G)\right).$$

Recall that the variety $\overline{G^{\mathrm{ad}}}$ is stratified by the orbits of a $G \times G$ action, and these orbits are indexed by subsets of the set of positive simple roots. Given such a subset I, we write Orb_I for the corresponding orbit, and Λ_I for the sublattice of Λ spanned by the roots in I. There is a filtration on $\mathcal{O}_q(G)$ by the quotient lattice Λ/Λ_I, which is, in a sense, coarser than the Peter-Weyl filtration. Let $\mathrm{gr}_I(\mathcal{O}_q(G))$ denote the associated graded algebra.

Definition 1.3.3. Fix a subset I of positive simple roots. The category of quasicoherent sheaves on the quantum orbit corresponding to I is defined as

$$\mathsf{QCoh}_q\left(\mathrm{Orb}_I\right) = \mathsf{Proj}\left(\mathrm{gr}_I(\mathcal{O}_q(G))\right).$$

When I comprises all positive simple roots, we obtain the category of modules for the quantum coordinate algebra $\mathcal{O}_q(G^{\mathrm{ad}})$ of the adjoint group of G. At the other extreme, when $I = \emptyset$, the category of sheaves on the corresponding quantum orbit is given in terms of the associated graded algebra $\mathrm{gr}(\mathcal{O}_q(G))$ for $\mathcal{O}_q(G)$ with the Peter-Weyl filtration.

A subset I of positive simple roots determines a parabolic subgroup $P = P_I$ of G and an opposite parabolic P^-. These have a common Levi group L, whose Lie algebra is denoted $\mathfrak{l}$. Let $\mathfrak{u}$ and $\mathfrak{u}^-$ be the Lie algebras of the unipotent radicals of P and P^-.

Theorem 1.3.4. *There is an isomorphism of Λ/Λ_I-graded algebras*

$$\mathrm{gr}_I(\mathcal{O}_q(G)) = \mathcal{O}_q(G \times G)^{U_q(\mathfrak{u}\times\mathfrak{l}\times\mathfrak{u}^-)}$$

Consequently, the quantum orbits can also be described in terms of certain graded subalgebras of $\mathcal{O}_q(G\times G)$, which are given as invariants for a quantum group action. When I consists of all positive simple roots, the subalgebra in question is the 'diagonal' copy of $\mathcal{O}_q(G)$. At the other extreme, when $I = \emptyset$, the multihomogeneous coordinate ring of the classical orbit $G/B\times B^-\backslash G$ is the algebra of functions on the asymptotic cone $(G/N\times N^-\backslash G)/T$, which quantizes precisely to $\mathcal{O}_q(G \times G)^{U_q(\mathfrak{n} \times \mathfrak{t} \times \mathfrak{n}^-)}$. By Theorem 1.3.4, this algebra is isomorphic to the full associated graded algebra $\mathrm{gr}(\mathcal{O}_q(G))$ for the Peter-Weyl filtration.

The algebra of quantum differential operators on the group G is defined as the smash product of the quantum coordinate algebra and the quantum group: $\mathcal{D}_q(G) = \mathcal{O}_q(G) \star U_q(\mathfrak{g})$. The Peter-Weyl filtration on $\mathcal{O}_q(G)$ defines a filtration on $\mathcal{D}_q(G)$, and we have:

Definition 1.3.5. The category of $\mathcal{D}$-modules on the quantum wonderful compactification is defined as the Proj category for the (multi-)Rees algebra of $\mathcal{D}_q(G)$:

$$\mathcal{D}_q\left(\overline{G^{\mathrm{ad}}}\right)\text{-}\mathsf{mod} = \mathsf{Proj}\left(\mathrm{Rees}(\mathcal{D}_q(G))\right).$$

1.4 Motivation and future directions

This thesis forms the first step in a program to understand the role of the wonderful compactification in quantum geometric representation theory. We remark on several goals in this program that provide motivation for the current work.

1.4.1 Beilinson-Bernstein localization via asymptotics

A fundamental result of geometric representation theory is the Beilinson-Bernstein localization theorem, which describes representations of any semisimple Lie algebra in terms of modules for the algebra of differential operators on the associated flag variety. In joint work with D. Ben-Zvi and D. Nadler, we aim to place this theorem within the framework of the wonderful compactification and asymptotics of matrix coefficients. Precursors to our work appear in work of Ben-Zvi and Nadler, and of Emerton, Nadler, and Vilonen [BN1, ENV]. We expect our techniques, together with the newly-introduced quantum wonderful compactification, to place the quantum Beilinson-Bernstein within the same framework of asymptotics of matrix coefficients.

1.4.2 Quantum character sheaves

Another source of motivation is the development of the theory of quantum character sheaves as a contribution to the study of harmonic analysis on quantum groups. Character sheaves were invented by Lusztig in order to adapt classical constructions from the theory of finite groups to the setting of algebraic groups. The theory of character sheaves for quantum groups will involve:

1. The construction of the appropriate q-deformation of the Hecke category of Borel equivariant $\mathcal{D}$-modules on the flag variety. Related constructions appear in work of Backelin and Kremnitzer on quantum flag varieties [BaKr].

2. The formulation of certain functoriality properties for quantum $\mathcal{D}$-modules. These properties are necessary in order to elevate various links between character sheaves and the wonderful compactification (see, e.g. [BFO]) to the quantum level.

3. An alignment with the quantum geometric Langlands program. In particular, we expect direct relations with topological field theories arising from quantum groups and their categories of representations [BBJ].

1.4.3 Root of unity behavior

When the quantum parameter q is a root of unity, radical changes occur in the structure and representation theory of quantum groups. For example, at $q = \epsilon$ a root of unity, the quantum coordinate algebra $\mathcal{O}_\epsilon(G)$ is finite dimensional over its center, and contains the unquantized coordinate algebra $\mathcal{O}(G)$ as a central sub-Hopf algebra. An exposition of these results is given in Brown and Goodearl [BG, Part III]. We expect the category $\mathsf{QCoh}_\epsilon(\overline{G^{\mathrm{ad}}})$ to be category of modules for a certain sheaf of algebras $\mathcal{A}$ on $\overline{G^{\mathrm{ad}}}$, and that $\mathcal{A}$ will have Azumaya properties over the closures of certain double Bruhat cells.

1.4.4 Smaller questions

The following questions are of smaller scope than the ones mentioned above, but still of technical importance:

1. What is the correct notion of quantum differential operators on the Vinberg semigroup? A possibly correct definition is as the smash product of $\mathcal{O}_q(\mathbb{V}_G)$ with $U_q(\mathfrak{g} \times \mathfrak{t})$. This smash product would contain the Rees algebra $\mathrm{Rees}(\mathcal{D}_q(G))$ as a proper subalgebra.

2. What is the quantum version of the Verdier specialization functor

$$\mathcal{D}_q(G^{\mathrm{ad}})\text{-}\mathsf{mod} \to \mathcal{D}_q((G/N \times N^-\backslash G)/T)\text{-}\mathsf{mod}\ ?$$

   Answering this question involves defining the category $\mathcal{D}_q((G/N \times N^-\backslash G)/T)\text{-}\mathsf{mod}$, but we expect the quantum Verdier specialization functor to be, roughly, a functor that associates certain filtered modules to their associated graded modules.

3. Fix a regular dominant weight $\lambda \in \Lambda$. Conjecture 5.2.1 in Chapter 5 below proposes an equivalence of categories

$$\mathsf{Proj}\left(\mathcal{O}_q(\mathbb{V}_G)\right) \xrightarrow{\sim} \mathsf{Proj}\left(\bigoplus_{n \geq 0} \mathcal{O}_q(\mathbb{V}_G)_{n\lambda}\right).$$

   We discuss progress towards this conjecture in Section 5.2.

1.5 Outline

We now describe the contents of this thesis.

Chapter 2 contains preliminary material and can be skipped on first reading and referred to as necessary. It includes background on matrix coefficients for Hopf algebras (Section 2.1), notation for algebraic groups (Section 2.2), background on quantum groups (Section 2.3), and the definition of the smash product (Section 2.4). The final section (Section 2.5) gives background on localization of abelian categories with respect to dense subcategories, and the particular example of Proj categories for a graded ring.

Chapter 3 gives an expository account of the construction of the Vinberg semigroup (Section 3.1) and wonderful compactification (Section 3.2). Although there are no new results in Chapter 3, our approach is somewhat more algebraic than that of other authors, and provides insight on the quantum case. We describe the stratification of the wonderful compactification by $G \times G$ orbits in Section 3.3, and examine the case of SL_2 in Section 3.4.

The main construction of this thesis requires the formalism of Proj categories for non-commutative, multi-graded algebras. The appropriate generalization is described in Chapter 4. The chapter begins with basic definitions related to Λ-graded rings (Section 4.1) and torsion graded modules (Section 4.2). We highlight special behavior in the case where Λ is the weight lattice in Section 4.3. In Section 4.4, we recall a characterization of Proj categories established by Artin-Zhang [AZ], and related work by other authors. Finally, Section 4.5 collects results on quantum flag varieties established in [BaKr].

Chapter 5 forms the heart of this thesis. We begin the chapter with the construction of the quantum Vinberg semigroup and the quantum wonderful compactification (Section

5.1). We briefly describe work-in-progress on different descriptions of the quantum wonderful compactification (Section 5.2). We introduce filtrations on the quantum coordinate algebra $\mathcal{O}_q(G)$ in Section 5.3 and use these to describe the quantum orbits in Section 5.4. In Section 5.5, we define quantum differential operators on the wonderful compactification, and prove basic properties.

Chapter 6 examines the general constructions of Chapter 5 in the case of $G = \mathrm{SL}_2$. We include basic background (Section 6.1) followed by a description of the Peter-Weyl filtration for $\mathcal{O}_q(\mathrm{SL}_2)$ (Section 6.2). We state results on the quantum Vinberg semigroup and wonderful compactification for SL_2 (Section 6.3), and discuss the algebra $\mathcal{D}_q(\mathrm{SL}_2)$ of quantum differential operators on SL_2 (Section 6.4).

Chapter 2

Preliminaries

This chapter collects background, notation, and other preliminary material that will be used in subsequent chapters. The reader may skip this chapter on first reading and refer to it as necessary. Throughout the remainder of this thesis, unless specified otherwise, the ground field is $\mathbb{C}$.

2.1 Matrix coefficients

The following discussion follows Section 1.9 of [BG].

Definition 2.1.1. The finite or Hopf dual of an algebra A is defined as

$$A^\circ = \{f \in A^* \mid f(I) = 0 \text{ for some ideal } I \text{ of } A \text{ with } \dim(A/I) < \infty\}.$$

Lemma 2.1.2. *Let A be an algebra with multiplication m and unit η. The Hopf dual A° is a coalgebra with $\Delta = m^*$ and $\epsilon = \eta^*$, and there is an equivalence between the category of locally finite-dimensional left A-modules and the category of right A°-comodules. Moreover, if $H = (H, m, \eta, \Delta, \epsilon)$ is a Hopf algebra, then $H^\circ = (H^\circ, \Delta^*, \epsilon^*, m^*, \eta^*)$ is a Hopf algebra.*

Definition 2.1.3. Let M be a (left) module over H. For $v \in M$ and $f \in M^*$ define the coordinate function $c^M_{f,v} \in H^*$ as $c^M_{f,v}(h) = f(hv)$ for $h \in H$. Thus, we have a map $c^M : M^* \otimes M \to H^*$ taking $v \otimes f$ to $c^M_{f,v}$. The image of c^M is called the set of 'matrix coefficients' for M.

Lemma 2.1.4. *We collect the following basic properties of matrix coefficients:*

1. *If M is finite-dimensional, then its matrix coefficients lie in H°. In fact, every element of H° is the coordinate function of a finite-dimensional H-module.*

2. *The map $c^M : M^* \otimes M \to H^*$ is $H \times H$-equivariant. Consequently, if M is an irreducible H-module, then c^M is injective.*

3. *Let M and N be finite dimensional modules for H, let $v \in M$, $f \in M^*$, $w \in N$ and $g \in N^*$, and let v_i and f_i be dual bases of M and M^*. Then:*

$$c^M_{f,v} + c^N_{g,w} = c^{M \oplus N}_{(f,g),(v,w)} \qquad c^M_{f,v} c^N_{g,w} = c^{M \otimes N}_{f \otimes g, v \otimes w}$$

$$\Delta(c^V_{f,v}) = \sum_i c^M_{f,v_i} \otimes c^M_{f_i,v} \qquad \epsilon(c^M_{f,v}) = f(v) \qquad S(c^M_{f,v}) = c^{M^*}_{v,f}.$$

In particular, the coproduct on H° sends $c^M(M^* \otimes M)$ to $c^M(M^* \otimes M) \otimes c^M(M^* \otimes M)$.

4. *As an $H \times H$-module, H° is isomorphic to the directed union of the matrix coefficients for finite-dimensional irreducible H-modules M:*

$$\bigoplus_{M \text{ fin. dim. irr.}} M^* \otimes M \xrightarrow{\sim} H^\circ.$$

5. *Suppose $\phi : M \to N$ is an H-equivariant homomorphism, and let $\phi^* : N^* \to M$ be the dual homomorphism. Then $c^N_{f,\phi(m)} = c^M_{\phi^*(f),m}$ for any $m \in M$ and $f \in N^*$.*

Let F be a family of finite-dimensional H-modules, and let $\hat{F}$ denote the closure of F under finite direct sums and tensor products.

Lemma 2.1.5. *Let A be the subalgebra of H° generated by all matrix coefficients of elements in F. Then A is a sub-bialgebra of H°, and, as an $H \times H$-module, is the directed union of the spaces of matrix coefficients for $M \in \hat{F}$. Moreover, if F is closed under duals, then A is a sub-Hopf algebra of H°.*

2.2 Notation related to algebraic groups

Let G be a connected semisimple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. Write $\mathfrak{b}$ and $\mathfrak{h}$ for the corresponding Lie subalgebras of $\mathfrak{g}$. The Borel subgroup B has unipotent radical N, with Lie algebra $\mathfrak{n}$, and it has an opposite Borel subgroup B^- uniquely characterized by the property that $B \cap B^- = T$. Let r be the rank of G.

The weight lattice Λ_W of $\mathfrak{g}$ is generated by the fundamental weights $\omega_1, \ldots, \omega_r$. The weight lattice contains the cone Λ_W^+ of dominant weights. The interior of Λ_W^+ comprises the regular dominant weights. Thus, dominant weights comprise the nonnegative linear combinations of the fundamental weights, and regular dominant weights comprise the positive linear combinations of fundamental weights. Fix a set of positive simple roots $\{\alpha_1, \ldots, \alpha_r\}$ of T relative to B. These generate the root lattice Λ_R, and we use the set $\Delta = \{1, \ldots, r\}$ to index the positive simple roots.

Definition 2.2.1. Define a partial order on Λ by setting $\lambda \leq \mu$ whenever $\lambda - \mu$ is a non-negative multiple of positive simple roots α_i. Similarly, we write $\lambda < \mu$ if $\lambda \leq \mu$ and $\lambda \neq \mu$.

The weight lattice Λ_G of G is the character lattice $X^*(T)$ of the maximal torus. We have inclusions of lattices: $\Lambda_R \subseteq \Lambda_G \subseteq \Lambda_W$. The set of isomorphism classes of finite-dimensional irreducible representations of G are in bijection with points in the cone of dominant weights $\Lambda_G^+ = \Lambda_W^+ \cap \Lambda_G$. We denote by V_λ the irreducible representation corresponding to $\lambda \in \Lambda^+$. Points in the interior of Λ_G^+ are called regular dominant weights.

Given a subset $I \subseteq \Delta$, denote by P_I the parabolic subgroup of G whose Lie algebra $\mathfrak{p}_I$ is generated by $\mathfrak{b}$ and the root vectors corresponding to the roots $-\alpha_i$ for $i \in I$. Let U_I be the unipotent radical of P_I (with Lie algebra $\mathfrak{u}_I$), and let L_I denote the subgroup of P_I whose Lie algebra $\mathfrak{l}_I$ is generated by $\mathfrak{h}$ and the root vectors corresponding to the roots $\pm\alpha_i$ for $i \in I$. Then L_I is a maximal reductive subgroup of P_I, and is called a Levi subgroup of G. We have $P_I = U_I \rtimes L_I$. Similarly, we define the opposite parabolic P_I^- and its unipotent radical U_I^-. Observe that $P_I \cap P_I^- = L_I$. Write $\mathfrak{p}_I^-$ and $\mathfrak{u}_I^-$ for the corresponding Lie algebras.

Lemma 2.2.2. *If V_ν appears as an irreducible subrepresentation of the tensor product $V_\lambda \otimes V_\mu$, then $\nu \leq \lambda + \mu$ for the partial order defined above.*

Write $Z = Z(G)$ for the center of G, and $G^{\text{ad}} = G/Z(G)$ for the adjoint group of G. The quotient Λ/Λ_R is naturally isomorphic to the dual $\widehat{Z(G)}$ of $Z(G)$ (see section 23.1 of [FH]).

Let $(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $(x, y) = \operatorname{Tr}(\operatorname{ad}(x) \circ \operatorname{ad}(y))$ denote the Killing form on $\mathfrak{g}$. The Killing from is nondegenerate and restricts to a nondegenerate form on $\mathfrak{h}$ and allows us to identify $\mathfrak{g}$ and $\mathfrak{h}$ with their duals. The Cartan matrix $C = (a_{ij})$ is defined as $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$. For $i = 1, \ldots, r$, set $d_i = (\alpha_i, \alpha_i)/2$.

2.3 Quantum groups

For the definition of the quantized enveloping algebra, we follow [BG, Chapter I.6] and [KS, Chapter 6]. Fix $q \in \mathbb{C}^\times$. We will assume throughout that q is not a root of unity. Set $q_i = q^{d_i}$ for $i = 1, \ldots, n$. For $q \in \mathbb{C}^\times$ and an integer n, define the corresponding quantum integer $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$. Define

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q [n-1]_q \cdots [1]_q}{[r]_q [r-1]_q \cdots [1]_q [n-r]_[n-r-1]_q \cdots [1]_q}.$$

Definition 2.3.1. The Drinfeld-Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ is defined as the algebra generated by elements $E_1, \ldots, E_n, F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$, with relations

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j \qquad K_i K_j = K_j K_i$$

$$K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j \qquad E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \begin{bmatrix} 1 - a_{ij} \\ \ell \end{bmatrix}_{q_i} E_i^{1-a_{ij}-\ell} E_j E_i^\ell = 0 \qquad (i \neq j)$$

$$\sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \begin{bmatrix} 1 - a_{ij} \\ \ell \end{bmatrix}_{q_i} F_i^{1-a_{ij}-\ell} F_j F_i^\ell = 0 \qquad (i \neq j)$$

Proposition 2.3.2. *The algebra $U_q(\mathfrak{g})$ has the following Hopf algebra structure:*

$$\Delta(K_i) = K_i \otimes K_i \qquad \epsilon(K_i) = 1 \qquad S(K_i) = K_i^{-1}$$

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \qquad \epsilon(E_i) = 0 \qquad S(E_i) = -K_i^{-1} E_i$$

$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \qquad \epsilon(F_i) = 0 \qquad S(F_i) = -F_i K_i$$

Definition 2.3.3. We define the following subalgebras of $\mathcal{U}_q(\mathfrak{g})$:

- $U_q(\mathfrak{p}_I)$ is generated by $K_i^{\pm 1}$ and E_i for $i = 1, \ldots, n$, as well as F_i for $i \in I$.

- $U_q(\mathfrak{p}_I^-)$ is generated by $K_i^{\pm 1}$ and F_i for $i = 1, \ldots, n$, as well as E_i for $i \in I$.

- $U_q(\mathfrak{u}_I)$ is generated by E_i for $i \notin I$.

- $U_q(\mathfrak{u}_I^-)$ is generated by F_i for $i \notin I$.

- $U_q(\mathfrak{l}_I)$ is generated by $K_i^{\pm 1}$ for $i = 1, \ldots, n$, as well as E_i and F_i for $i \in I$.

The one-dimensional representations of $U_q(\mathfrak{g})$ are naturally in bijection with the set $\mathrm{Hom}(\Lambda_R, \mathbb{Z}/2\mathbb{Z})$. Let V be a finite-dimensional $U_q(\mathfrak{g})$-module. For $\sigma \in \mathrm{Hom}(\Lambda_R, \mathbb{Z}/2\mathbb{Z})$ and $\lambda \in \Lambda^+$, define

$$V_{\lambda,\sigma} = \{v \in V \mid K_\mu v = \sigma(\mu) q^{(\mu,\lambda)} v \text{ for all } \mu \in \Lambda_R\}.$$

As a vector space, V decomposes as $V = \bigoplus_{\sigma,\lambda} V_{\lambda,\sigma}$ where the sum ranges $\sigma \in \mathrm{Hom}(\Lambda_R, \mathbb{Z}/2\mathbb{Z})$ and $\lambda \in \Lambda^+$. We call $\lambda \in \Lambda^+$ a *weight* of V if $V_{\lambda,\sigma}$ is nonzero. The vectors in $V_{\lambda,\sigma}$ are called *weight vectors*, and a weight vector v is a *highest weight vector* if $E_i v = 0$ for $i = 1, \ldots, n$.

Definition 2.3.4. Let V be a finite-dimensional module for $U_q(\mathfrak{g})$. We say that V is of type **1** if the $V_{\lambda,\sigma} = 0$ for any nontrivial $\sigma \in \mathrm{Hom}(Q, \mathbb{Z}/2\mathbb{Z})$.

The category $\mathcal{C}_q(\mathfrak{g})$ of finite-dimensional $U_q(\mathfrak{g})$-modules of type **1** has strong parallels with the category of finite-dimensional $U(\mathfrak{g})$-modules.

Proposition 2.3.5. *The category $\mathcal{C}_q(\mathfrak{g})$ is a semisimple rigid tensor subcategory of $U_q(\mathfrak{g})$-mod whose irreducible objects are in bijection with the set Λ_W^+ of dominant weights.*

We write $V_\lambda \leftrightarrow \lambda$ for this bijection. This notation conflicts with the notation for representations of classical Lie algebras and Lie groups. In subsequent sections, the context will indicate the intended meaning of V_λ. The representation V_λ has a highest weight vector of weight λ.

Lemma 2.3.6. *If V_ν appears as an irreducible subrepresentation of the tensor product $V_\lambda \otimes V_\mu$, then $\nu \leq \lambda + \mu$.*

Let $\mathcal{C}_q(G)$ denote the subcategory of $\mathcal{C}_q(\mathfrak{g})$ generated irreducible representations whose highest weights lie in Λ_G^+. We use this category to define the quantized coordinate algebra $\mathcal{O}_q(G)$ of G, which is a flat deformation of the algebra $\mathcal{O}(G)$.

Definition 2.3.7. The quantized coordinate algebra $\mathcal{O}_q(G)$ of G is defined as sub-Hopf algebra of $U_q(\mathfrak{g})^\circ$ generated by the matrix coefficients of all modules in $\mathcal{C}_q(G)$.

2.4 Smash products

Our references for this section are [BaKr, Jo1, VV]. Let H be a Hopf algebra. We use Sweedler notation for coproducts. The category H-mod is a tensor category.

Definition 2.4.1. [VV, 1.4] Let A be an algebra in H-mod. The smash product $A \star H$ is defined as the vector space $A \otimes H$ with algebra structure determined by the following conditions:

- $A \simeq A \otimes 1$ and $H \simeq 1 \otimes H$ are subalgebras.
- The cross relations for $h \in H$ and $a \in A$ are given by $ha = \left(h_{(1)} \triangleright a\right) h_{(2)}$.

Algebras in H mod arise from so-called dual pairings of Hopf algebras.

Definition 2.4.2. [BG, Section I.9.22] A dual pairing or Hopf pairing of two Hopf algebras H and K over $\mathbb{C}$ is a bilinear form $\kappa : H \times K \to \mathbb{C}$ that satisfies, for all $h, h' \in H$ and $k, k' \in K$,

$$\kappa(h, kk') = \kappa(h_{(1)}, k)\kappa(h_{(2)}, k'), \qquad \kappa(hh', k) = \kappa(h, k_{(1)})\kappa(h', k_{(2)}),$$

$$\kappa(h, 1_K) = \epsilon_H(h), \qquad \text{and} \qquad \kappa(1_H, k) = \epsilon_K(k)$$

The pairing is called perfect (or nondegenerate) if the induced maps $H \to K^*$ and $K \to H^*$ are both injective.

Remark 2.4.3. The conditions above imply that $\kappa(S(h), k) = \kappa(h, S(k))$ for all $h \in H$ and $k \in K$.

Example 2.4.4. There is a perfect pairing between $U(\mathfrak{g})$ and $\mathcal{O}(G)$ for any semisimple group G with Lie algebra $\mathfrak{g}$. Moreover, if G is simply-connected, then $\mathcal{O}(G)$ can be identified with $U(\mathfrak{g})^\circ$. On the other hand, it is not true $\mathcal{O}(G)^\circ$ equals $U(\mathfrak{g})$.

Example 2.4.5. The evaluation pairing $\kappa : U_q(\mathfrak{g}) \times \mathcal{O}_q(G)$ is given by

$$\kappa(x, c_{f,v}) = c_{f,v}(x) = \langle f, x \cdot v \rangle$$

and defines a perfect pairing for any semisimple group G with Lie algebra $\mathfrak{g}$.

Suppose there is a dual pairing $\kappa : H \times K \to \mathbb{C}$ between Hopf algebras H and K. Then there is a left action of H on K given by:

$$h \triangleright k = k_{(1)}\kappa(h, k_{(2)})$$

Moreover, K is an algebra in the tensor category H-mod (see [BG, Section I.7.13]), and we can form the smash product $K \star H$.

2.5 Localization of abelian categories

The reference for this section is [Pop, Sections 4.3 and 4.4].

Definition 2.5.1. A full subcategory $\mathcal{T}$ of an abelian category $\mathcal{A}$ is called dense if it is closed under extensions. In other words, for any short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of objects in $\mathcal{A}$, the object M belongs to $\mathcal{T}$ if and only if M' and M'' both belong to $\mathcal{T}$.

Dense subcategories are also known as Serre subcategories. Given a dense subcategory $\mathcal{T}$ of an abelian category $\mathcal{A}$, one defines a category $\mathcal{A}/\mathcal{T}$ whose objects are the objects of $\mathcal{A}$, and whose morphisms are obtained via localization with respect to the multiplicative system

$$\Sigma = \{s : M \to N \mid \ \ker(s) \text{ and } \operatorname{coker}(s) \text{ are objects of } \mathcal{T}\}.$$

Theorem 2.5.2. *Let $\mathcal{A}$ be an abelian category and $\mathcal{T}$ a dense subcategory.*

- *[Pop, Theorem 4.3.8] The category $\mathcal{A}/\mathcal{T}$ is abelian, and the canonical functor $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{T}$ is exact.*

- *[Pop, Proposition 4.5.2] The functor π admits a right adjoint $\omega : \mathcal{A}/\mathcal{T} \to \mathcal{A}$ when the following two conditions are satisfied: (a) for any object M of $\mathcal{A}$, the collection of subobjects of M belonging to $\mathcal{T}$ has a maximal object, and (b) the category $\mathcal{A}$ has injective envelopes.*

- *[Pop, Proposition 4.4.3] When π admits a right adjoint ω, the counit of the adjunction $\pi\omega \to \mathrm{Id}$ is a natural isomorphism, and the unit of the adjunction $X \to \omega\pi(X)$ has torsion kernel and cokernel for any object X of $\mathcal{A}$.*

Example 2.5.3. Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a noetherian ring graded by the integers. A graded module $M = \bigoplus M_n$ is called torsion if for every $m \in M$, there exists an integer N such that $am = 0$ for any $a \in A_{\geq N} = \bigoplus_{n \geq N} A_n$. The full subcategory $\mathsf{Tors}(A)$ of torsion graded modules forms a dense subcategory of the category $\mathsf{Grmod}(A)$ of all graded modules, and the quotient $\mathsf{Grmod}(A)/\mathsf{Tors}(A)$ is the proj category $\mathsf{Proj}(A)$ of A. More details can be found in [AZ, Se].

Chapter 3

The wonderful compactification

In this chapter, we give an exposition of the construction of the Vinberg semigroup and the wonderful compactification. We begin by recalling the Peter-Weyl theorem, which describes the coordinate algebra $\mathcal{O}(G)$ in terms of matrix coefficients for representations of G. We define Vinberg semigroup as the Rees construction for the coordinate algebra $\mathcal{O}(G)$ with a filtration arising from the Peter-Weyl theorem (Section 3.1). We define the wonderful compactification of the adjoint group of G as a GIT quotient of the Vinberg semigroup by an action of a torus, as explained in Section 3.2. We also mention another, perhaps more standard construction of the wonderful compactification as the closure of the image of the adjoint group in the projective space $\mathbb{P}(\mathrm{End}(V))$, where V is an irreducible finite-dimensional representation of G of regular highest weight lambda. Finally, we discuss the $G \times G$ orbits in Section 3.3 and the case of SL_2 in Section 3.4.

3.1 The Vinberg semigroup

The Peter-Weyl theorem forms the starting point for our discussion.

Theorem 3.1.1 (Peter-Weyl Theorem)**.** *The map of matrix coefficients gives an isomorphism of $U\mathfrak{g}$-bimodules $\phi : \bigoplus_{\lambda \in \Lambda_G^+} V_\lambda^* \otimes V_\lambda \xrightarrow{\sim} \mathcal{O}(G)$, where the sum runs over all finite-dimensional irreducible representations of G.*

The Peter-Weyl theorem endows $\mathcal{O}(G)$ with a grading by Λ_G. While the algebra structure on $\mathcal{O}(G)$ does not respect this grading, one instead defines a filtration on $\mathcal{O}(G)$ by Λ_G that is compatible with the algebra structure. This filtration relies on the partial order on Λ_G from Definition 2.2.1, and is known as the Peter-Weyl filtration.

Definition 3.1.2. For $\lambda \in \Lambda$, define the following subspace of $\mathcal{O}(G)$:

$$\mathcal{O}(G)_{\leq \lambda} = \phi \left(\sum_{\mu \leq \lambda} V_\mu^* \otimes V_\mu \right).$$

Proposition 3.1.3. *The subspaces $\mathcal{O}(G)_{\leq \lambda}$ endow $\mathcal{O}(G)$ with the structure of a Λ_G-filtered algebra. The associated graded algebra has λ-graded piece equal to $\phi\left(V_\lambda^* \otimes V_\lambda\right)$.*

The proof of the preceding proposition is a consequence of Lemma 2.2.2 and basic properties of matrix coefficients. Let $\mathbb{C}[z^\lambda] = \mathbb{C}[z^\lambda \mid \lambda \in \Lambda_G]$ denote the algebra generated by the formal variables z^λ for $\lambda \in \Lambda_G$ with relations $z^\lambda z^\mu = z^{\lambda+\mu}$. The Rees algebra for $\mathcal{O}(G)$ with the Peter-Weyl filtration is defined as $\mathrm{Rees}(G) = \bigoplus_{\lambda \in \Lambda_G} \mathcal{O}(G)_{\leq \lambda} z^\lambda$. By definition, it is a Λ_G-graded subalgebra of $\mathcal{O}(G) \otimes \mathbb{C}[z^\lambda]$.

Lemma 3.1.4. *The $U\mathfrak{g}$-bimodule structure on $\mathcal{O}(G)$ extends to a $U\mathfrak{g}$-bimodule structure on* $\mathrm{Rees}(G)$*. The Hopf algebra structure on $\mathcal{O}(G)$ induces a bialgebra structure on* $\mathrm{Rees}(G)$.

Proof. To obtain the $U\mathfrak{g}$-bimodule structure, let $U\mathfrak{g}$ act trivially on $\mathbb{C}[z^\lambda]$ and note that, by the Peter-Weyl theorem, each subset $\mathcal{O}(G)_{\leq \lambda}$ of $\mathcal{O}(G)$ is stable under the action of $U\mathfrak{g} \otimes U\mathfrak{g}$. The coproduct on $\mathcal{O}(G)$ restricts to a map $\mathcal{O}(G)_{\leq \lambda} \to \mathcal{O}(G)_{\leq \lambda} \otimes \mathcal{O}(G)_{\leq \lambda}$. The coproduct and counit on $\mathrm{Rees}(G)$ are given in terms of the coproduct and counit on $\mathcal{O}(G)$, namely, for $f \in \mathcal{O}(G)_{\leq \lambda}$, we have $\Delta(f z^\lambda) = \Delta_{\mathcal{O}(G)}(f) \cdot z^\lambda \otimes z^\lambda$ and $\epsilon(f z^\lambda) = \epsilon_{\mathcal{O}(G)}(f)$. □

Definition 3.1.5. The Vinberg semigroup $\mathbb{V}_G$ for G is defined as the spectrum of the Rees algebra for $\mathcal{O}(G)$ with the Peter-Weyl filtration:

$$\mathbb{V}_G = \operatorname{Spec}\left(\bigoplus_{\lambda\in\Lambda} \mathcal{O}(G)_{\leq\lambda} z^{\lambda}\right).$$

Henceforth, we denote the Rees algebra as $\mathcal{O}(\mathbb{V}_G)$. Lemma 3.1.4 implies that $\mathbb{V}_G$ is a semigroup with an action of $G \times G$. Observe that $z^{\alpha_i} \in \mathcal{O}(\mathbb{V}_G)$ for any positive root α_i, so there is an inclusion $\mathbb{C}[z^{\alpha_i}] \hookrightarrow \mathcal{O}(\mathbb{V}_G)$, where $\mathbb{C}[z^{\alpha_i}]$ is the polynomial subalgebra of $\mathbb{C}[z^{\lambda}]$ generated by the elements z^{α_i} for $i \in \Delta$. The induced surjective map

$$\pi : \mathbb{V}_G \to \mathbb{A}$$

is the abelianization map of [Vi1], where $\mathbb{A}$ denotes the spectrum of $\mathbb{C}[z^{\alpha_i}]$. Hence, $\mathbb{A}$ is an r-dimensional affine space, and the choice of positive simple roots endows $\mathbb{A}$ with a coordinate system. Let $\mathbb{A}^{\circ}$ denote the open subset consisting of points whose coordinates are all nonzero.

Lemma 3.1.6. *[Vi1] The inverse image $\pi^{-1}(\mathbb{A}^{\circ})$ is isomorphic to the quotient $(G \times T)/Z$ of $G \times T$ by the antidiagonal action of the center $Z = Z(G)$. Moreover, this set forms the group of units of the semigroup $\mathbb{V}_G$.*

Remark 3.1.7. See [Br, Example 3.2.4] for the relation between the definition of the Vinberg semigroup presented in this section and Vinberg's original definition.

3.2 The wonderful compactification

The wonderful compactification of G^{ad} is a GIT quotient of the Vinberg semigroup by the action of a torus, as we now explain. Recall that $\Lambda_G = X^*(T)$ is the character lattice of the maximal torus T of G. Therefore, the algebra of functions $\mathcal{O}(T)$ on T is the precisely the algebra $\mathbb{C}[z^\lambda]$ considered above. The subalgebra generated by z^λ for λ in the root lattice is the algebra of functions $\mathcal{O}(T/Z)$ on the maximal torus T/Z of the adjoint group $G^{\mathrm{ad}} = G/Z$. Recall that the polynomial subalgebra of $\mathcal{O}(T/Z)$ generated z^{α_i} for $i \in \Delta$ is the algebra of functions on the affine space $\mathbb{A}$. Hence, $\mathbb{A}$ is a toric variety of T/Z. In particular, $\mathbb{A}$ has an action of T.

The Λ_G-grading on the Rees algebra $\mathcal{O}(\mathbb{V}_G)$ endows $\mathbb{V}_G$ with a T-action, and the map $\pi : \mathbb{V}_G \to \mathbb{A}$ is T-equivariant. Under the identification $\pi^{-1}(\mathbb{A}^\circ) = (G \times T)/Z$, the action of T is given via multiplication on the second factor.

Definition 3.2.1. Fix a regular dominant weight λ of G. The wonderful compactification of G^{ad} is defined as the GIT quotient of $\mathbb{V}_G$ by T along λ:

$$\overline{G^{\mathrm{ad}}} = \mathbb{V}_G /\!\!/_\lambda T.$$

The $G \times G$ action on $\mathbb{V}_G$ commutes with the action of T, and hence descends to a $G \times G$ action on $\overline{G^{\mathrm{ad}}}$. The GIT quotient of $\pi^{-1}(\mathbb{A}^\circ) = (G \times T)/Z$ by T forms a $G \times G$-stable open subset of $\overline{G^{\mathrm{ad}}}$. This subset is a copy of the adjoint group G^{ad}, and the $G \times G$ action restricts to the action by left and right multiplication on G^{ad}.

Proposition 3.2.2. *[EJ, Propositions 2.14, 3.1] The wonderful compactification $\overline{G^{\mathrm{ad}}}$ is a smooth projective variety. Up to isomorphism, it does not depend on the choice of regular dominant weight.*

The homogeneous coordinate ring of $\overline{G^{\mathrm{ad}}}$ is the graded ring $\bigoplus_{n\geq 0} \mathcal{O}(\mathbb{V}_G)_{n\lambda}$, where $\mathcal{O}(\mathbb{V}_G)_{n\lambda}$ denotes the $n\lambda$-weight-subspace. The wonderful compactification is the projective variety associated to this ring, and its category of quasicoherent sheaves can be described in terms of graded modules for the homogeneous coordinate ring (see Sections 1.2 and 2.5):

Lemma 3.2.3. *There is an equivalence of categories:*

$$\mathsf{QCoh}\left(\overline{G^{\mathrm{ad}}}\right) = \mathsf{Proj}\left(\bigoplus_{n\geq 0} \mathcal{O}(\mathbb{V}_G)_{n\lambda}\right).$$

We recall another construction of the wonderful compactification, given in [EJ]. Fix an irreducible representation $V = V_\lambda$ of G of regular highest weight λ, with action map $G \to \mathrm{GL}(V)$.

Lemma 3.2.4. *There is a map $\psi : G^{\mathrm{ad}} \to \mathbb{P}(\mathrm{End}(V)) \simeq \mathbb{CP}^{2\dim V - 1}$ making the following diagram commute:*

$$\begin{array}{ccccc} G & \longrightarrow & \mathrm{GL}(V) & \longrightarrow & \mathrm{End}(V)\setminus\{0\} \\ \downarrow & & & & \downarrow \\ G^{\mathrm{ad}} & & \xrightarrow{\quad\psi\quad} & & \mathbb{P}(\mathrm{End}(V)) \end{array}$$

Moreover, the map ψ is injective and equivariant for the action of $G \times G$.

The existence of ψ relies on the fact that the center $Z(G)$ acts on V by scalars, and the injectivity of ψ is seen by considering the weight spaces of V_λ. The following result follows from Theorem 5.3 of [MT]:

Theorem 3.2.5. *The closure $\overline{\psi(G^{\mathrm{ad}})}$ of the image of ψ in $\mathbb{P}(\mathrm{End}(V))$ is isomorphic to the wonderful compactification of G^{ad}.*

Corollary 3.2.6. *For any regular dominant weight λ, there is an ample line bundle $\mathcal{L}_\lambda$ on $\overline{G^{\mathrm{ad}}}$.*

3.3 Orbits on the wonderful compactification

In this section, we describe the $G \times G$ orbits on the wonderful compactification. Fix a subset $I \subseteq \Delta$ and consider the corresponding parabolic subgroup P_I, its opposite P_I^-, and its Levi L_I. There are projection maps $\pi^L : P_I \to L_I$ and $\pi^{L^-} : P_I^- \to L_I$, and each of these composes to a map valued in $L_I^{\mathrm{ad}} = L_I/Z(L_I)$.

Proposition 3.3.1. *[EJ] We have:*

1. *The $G \times G$ orbits on $\overline{G^{\mathrm{ad}}}$ are in bijection with subsets of Δ. Write Orb_I for the orbit corresponding to $I \subseteq \Delta$. For subsets I, J of Δ, the containment $\mathrm{Orb}_I \subseteq \overline{\mathrm{Orb}_J}$ holds if and only if $I \subseteq J$.*

2. *There is a point in Orb_I whose stabilizer is the subgroup*

$$H_I = P_I \times_{L^{ad}} P_I^- = \{(p, p^-) \in P_I \times P_I^- \mid \pi^L(p)\pi^{L^-}(p^-)^{-1} \in Z(L_I)\}.$$

3. *Let $\overline{L_I^{\mathrm{ad}}}$ denote the wonderful compactification of the adjoint group L_I^{ad} of L, and let $\overline{\mathrm{Orb}_I}$ denote the closure of Orb_I in $\overline{G^{\mathrm{ad}}}$. There are fibrations:*

$$\begin{array}{ccc} L_I^{ad} & \longrightarrow & \mathrm{Orb}_I \\ & & \downarrow \\ & & G/P_I \times P_I^- \backslash G \end{array} \qquad \begin{array}{ccc} \overline{L_I^{ad}} & \longrightarrow & \overline{\mathrm{Orb}_I} \\ & & \downarrow \\ & & G/P_I \times P_I^- \backslash G \end{array} .$$

Example 3.3.2. In the extreme cases, $\mathrm{Orb}_\Delta = G^{\mathrm{ad}}$ is the unique open orbit, and $\mathrm{Orb}_\emptyset = G/B \times B^- \backslash G$ is the unique closed orbit. When $G = \mathrm{SL}_2$, there are only two $G \times G$ orbits on $\overline{G^{\mathrm{ad}}} = \mathbb{P}^3$, and they are the extreme ones. In the SL_3 case, there are four orbits. The two nonextreme orbits each form PSL_2-bundles over $\mathbb{P}^2 \times \mathbb{P}^2$, and the closure of each forms a $\mathbb{P}^3$-bundle over $\mathbb{P}^2 \times \mathbb{P}^2$.

Given $I \subseteq \Delta$, let e_I be the point in $\mathbb{A}$ whose ith coordinate of e_I is zero if $i \notin I$ and 1 otherwise[1]. The T-orbit $T \cdot e_I$ of e_I in $\mathbb{A}$, consists of all elements of $\mathbb{A}$ whose ith coordinate is nonzero if and only if $i \in I$. Let $\Lambda_I = \mathbb{Z}\{\alpha_i \mid i \in I\} \subseteq \Lambda_R$ denote the sublattice of the root lattice spanned by the roots in I. The character lattice of the subtorus $Z(L_I)$ of T is the quotient Λ/Λ_I. Write $[\lambda]$ for the image of $\lambda \in \Lambda$ in Λ/Λ_I.

Proposition 3.3.3. *Fix a regular dominant weight $\lambda \in \Lambda_G^+$. There are isomorphisms:*

$$\mathrm{Orb}_I = \pi^{-1}(T \cdot e_I) /\!\!/_{\lambda} T = \pi^{-1}(e_I) /\!\!/_{[\lambda]} Z(L).$$

The homogeneous coordinate ring of Orb_I is the graded algebra $\bigoplus_{n \geq 0} \mathcal{O}(\pi^{-1}(e_I))_{[n\lambda]}$. The algebra $\mathcal{O}(\pi^{-1}(e_I))$ is a Λ/Λ_I-graded algebra that is the associated graded of $\mathcal{O}(G)$ with respect to a certain 'partial' Peter-Weyl filtration. We describe this filtration in later sections. The category of quasicoherent sheaves on Orb_I is equivalent to the Proj category for this algebra:

$$\mathsf{QCoh}(\mathrm{Orb}_I) = \mathsf{Proj}\left(\bigoplus_{n \geq 0} \mathcal{O}(\pi^{-1}(e_I))_{[n\lambda]}\right).$$

3.4 The case of SL_2

In this section, we describe in detail the constructions of the previous sections for the case of SL_2.

[1] Recall that the choice of positive simple roots gives $\mathbb{A}$ a coordinate system.

Identify the weight lattice Λ_{SL_2} for SL_2 with the integers $\mathbb{Z}$, the cone of dominant weights $\Lambda^+_{\mathrm{SL}_2}$ with the nonnegative integers $\mathbb{Z}_{\geq 0}$, and the root lattice Λ_R with the sublattice $2\mathbb{Z} \subseteq \mathbb{Z}$ generated by the positive simple root $\alpha_1 = 2$. A dominant weight n is regular if and only if $n \geq 1$. The partial order on $\Lambda^+ = \mathbb{Z}_{\geq 0}$ from Definition 2.2.1 reduces to

$$n \leq m \text{ if and only if } n - m \text{ is a nonnegative multiple of } 2.$$

For $n \geq 0$, write $V_n = \mathrm{Sym}^n(\mathbb{C}^2)$ for the irreducible representation of SL_2 of highest weight n. As a special case of Lemma 2.2.2, we have that, if V_k appears as an irreducible representation of the tensor product $V_n \otimes V_m$, then $k \leq n + m$.

The algebra of functions $\mathcal{O}(\mathrm{SL}_2)$ is given by $\mathcal{O}(\mathrm{SL}_2) = \mathbb{C}[a, b, c, d]/(ad - bc = 1)$. This is a Hopf algebra with

$$\Delta(a) = a \otimes a + b \otimes c \qquad \Delta(b) = a \otimes b + b \otimes d$$

$$\Delta(c) = a \otimes c + c \otimes d \qquad \Delta(c) = b \otimes c + d \otimes d$$

$$\epsilon(a) = \epsilon(d) = 1 \qquad \epsilon(b) = \epsilon(c) = 0$$

$$S(a) = d \qquad S(b) = -b \qquad S(c) = -c \qquad S(d) = a.$$

We will use the same notation for elements of $\mathbb{C}[a, b, c, d]$ and their images in $\mathcal{O}(\mathrm{SL}_2)$. The Peter-Weyl Theorem for SL_2 is:

Theorem 3.4.1. *The map of matrix coefficients gives an isomorphism of representations of* $\mathrm{SL}_2 \times \mathrm{SL}_2$*:*

$$\phi : \bigoplus_{n \geq 0} V_n \otimes V_n^* \xrightarrow{\sim} \mathcal{O}(SL_2).$$

The Peter-Weyl filtration on $\mathcal{O}(\mathrm{SL}_2)$ is given by $\mathcal{O}(\mathrm{SL}_2)_{\leq n} = \phi\left(\bigoplus_{m \leq n} V_m^* \otimes V_m\right)$ for $n \in \mathbb{Z}_{\geq 0}$. The proof of the following lemma is left as an exercise.

Lemma 3.4.2. *For $n \geq 0$, consider the subspace of $\mathbb{C}[a,b,c,d]$ spanned by monomials $a^{k_1}b^{k_2}c^{k_3}d^{k_4}$ with $k_1 + k_2 + k_3 + k_4 \leq n$ and $k_1 + k_2 + k_3 + k_4 \equiv n \mod 2$. The image of this subspace under the quotient map $\mathbb{C}[a,b,c,d] \to \mathcal{O}(\mathrm{SL}_2)$ is precisely $\mathcal{O}(\mathrm{SL}_2)_{\leq n}$.*

For example:

- $\mathcal{O}(\mathrm{SL}_2)_{\leq 0} = \phi(V_0 \otimes V_0^*) = \mathrm{Span}\{1\}$
- $\mathcal{O}(\mathrm{SL}_2)_{\leq 1} = \phi(V_1 \otimes V_1^*) = \mathrm{Span}\{a, b, c, d\}$
- $\mathcal{O}(\mathrm{SL}_2)_{\leq 2} = \phi(V_0 \otimes V_0^* \oplus V_2 \otimes V_2^*) = \mathrm{Span}\{1, a^2, b^2, c^2, d^2, ab, ac, \dots\}$
- $\mathcal{O}(\mathrm{SL}_2)_{\leq 3} = \phi(V_1 \otimes V_1^* \oplus V_3 \otimes V_3^*) = \mathrm{Span}\{\text{degree 3 and degree 1 monomials}\}$

Notation 3.4.3. Let A be a graded ring. We write $\mathrm{Proj}(A)$ for the projective scheme associated to A, and $\mathsf{Proj}(A)$ for the category of graded A-modules modulo graded torsion modules. In other words, roman font indicates a space, while sans serif font indicates a category.

Proposition 3.4.4. *The associated graded algebra* $\mathrm{gr}(\mathcal{O}(\mathrm{SL}_2))$ *is the homogeneous coordinate ring of* $\mathbb{P}^1 \times \mathbb{P}^1$. *Consequently, we have an isomorphism of varieties:*

$$\mathrm{Proj}(\mathrm{gr}(\mathcal{O}(\mathrm{SL}_2)) = \mathbb{P}^1 \times \mathbb{P}^1.$$

Proof. First, $\mathrm{gr}(\mathcal{O}(\mathrm{SL}_2)) \simeq \mathbb{C}[a,b,c,d]/(ad - bc)$. The latter is isomorphic as a graded algebra to $\bigoplus_{n \geq 0} \mathrm{Sym}^n(\mathbb{C}^2) \otimes \mathrm{Sym}^n(\mathbb{C}^2)$ via the map $a \mapsto x \otimes w$, $b \mapsto x \otimes z$, $c \mapsto y \otimes w$, and

$d \mapsto y \otimes z$, where x, y are coordinate functions in the first copy of $\mathrm{Sym}^\bullet(\mathbb{C}^2)$ and w, z are the coordinates in the second copy. Now, $\mathrm{Sym}^n(\mathbb{C}^2) \otimes \mathrm{Sym}^n(\mathbb{C}^2)$ is the space of global sections of the line bundle $\mathcal{O}(n,n) = \mathcal{O}(1,1)^{\otimes n}$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The line bundle $\mathcal{O}(1,1)$ is ample since it is the pullback of $\mathcal{O}_{\mathbb{P}^3}(1)$ under the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$. Therefore, there is an isomorphism $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} \mathrm{Proj}\left(\bigoplus_n \Gamma\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)^{\otimes n}\right)\right)$. $\square$

Proposition 3.4.5. *We have the following:*

1. *The Vinberg semigroup* $\mathbb{V}_{\mathrm{SL}_2}$ *is isomorphic to the semigroup* Mat_2 *of 2 by 2 matrices over* $\mathbb{C}$*, and the action of* $\mathbb{C}^\times$ *on* $\mathbb{V}_{\mathrm{SL}_2}$ *coincides with the scaling action on* Mat_2.

2. *The wonderful compactification of* PSL_2 *is* $\mathbb{P}^3 = \mathrm{Proj}(\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2}))$.

3. *The stratification of* $\overline{\mathrm{PSL}_2} = \mathbb{P}^3$ *into* $\mathrm{SL}_2 \times \mathrm{SL}_2$ *orbits is given by:*

$$\overline{\mathrm{PSL}_2} \;=\; \mathrm{PSL}_2 \coprod (\mathbb{P}^1 \times \mathbb{P}^1).$$

Proof. The algebra of functions on the Vinberg semigroup $\mathbb{V}_{\mathrm{SL}_2}$ is given by the Rees algebra $\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2}) = \bigoplus_{n \geq 0} \mathcal{O}(\mathrm{SL}_2)_{\leq n} z^n$. Lemma 3.4.2 and the relation $z^2 = (az)(dz) - (bz)(cz)$ imply that the subspace $\mathcal{O}(\mathrm{SL}_2)_{\leq n} z^n$ coincides with the span of monomials $(az)^{k'_1}(bz)^{k'_2}(cz)^{k'_3}(dz)^{k'_4}$ with $k'_1 + k'_2 + k'_3 + k'_4 \leq n$. Hence, $\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})$ is a commutative algebra on the four generators az, bz, cz, dz and no relations. It is straightforward to verify that the coproduct of $\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})$ coincides with that of $\mathcal{O}(\mathrm{Mat}_2)$. Since $\mathbb{V}_{\mathrm{SL}_2} = \mathrm{Spec}(\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2}))$, this proves the first statement.

Fix the regular dominant weight $n = 1$, regarded as a character of the maximal torus $T = \mathbb{C}^\times$ of SL_2. The second statement follows from the computation:

$$\overline{\mathrm{PSL}_2} = \mathbb{V}_{\mathrm{SL}_2} /\!\!/_1 \mathbb{C}^\times = \mathrm{Mat}_2 /\!\!/_1 \mathbb{C}^\times = (\mathrm{Mat}_2 \setminus \{0\})/\mathbb{C}^\times = \mathbb{P}^3.$$

To prove the last statement, first note that that $\mathrm{SL}_2 \times \mathrm{SL}_2$ orbits on $\mathrm{Mat}_2 \setminus \{0\}$ are precisely $\{\det^{-1}(d) \mid d \in \mathbb{C}\}$, where $\det : \mathrm{Mat}_2 \setminus \{0\} \to \mathbb{C}$ is the determinant map. If $d \neq 0$, then the $\mathbb{C}^\times$-action on $\mathrm{Mat}_2 \setminus \{0\}$ identifies $\det^{-1}(d)$ with $\det^{-1}(1) = \mathrm{SL}_2$. It follows that the disjoint union $\coprod_{d \in \mathbb{C}^\times} \det^{-1}(d)$ descends to a single $\mathrm{SL}_2 \times \mathrm{SL}_2$-orbit in $\overline{\mathrm{PSL}_2}$, and it is isomorphic to PSL_2. On the other hand, if $d = 0$, then we have:

$$\det{}^{-1}(0) = \{\text{matrices of rank } 1\} \simeq \frac{\mathbb{C}^2 \setminus \{0\} \times \mathbb{C}^2 \setminus \{0\}}{\mathbb{C}^\times}$$

The second identification is given by sending pair of nonzero vectors (x_1, x_2) and (y_1, y_2) to the matrix whose (i, j) entry is $x_i y_j$ for $i, j = 1, 2$. The $\mathbb{C}^\times$ action on $\mathrm{Mat}_2 \setminus \{0\}$ preserves this space, and the quotient is $\mathbb{P}^1 \times \mathbb{P}^1$. □

Remark 3.4.6. The orbit $\mathbb{P}^1 \times \mathbb{P}^1$ includes in $\overline{\mathrm{PSL}_2} = \mathbb{P}^3$ as the Segre embedding.

We give another perspective on the $\mathrm{SL}_2 \times \mathrm{SL}_2$ orbits on the wonderful compactification $\overline{\mathrm{PSL}_2} = \mathbb{P}^3$. First note that $\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})$ contains z^2, but does not contain z. On the level of points, i.e. set-theoretically, we have

$$\overline{\mathrm{PSL}_2} = \mathrm{Proj}(\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})) = \mathrm{Spec}\left(\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})[(z^2)^{-1}]^{\mathbb{C}^\times}\right) \coprod \mathrm{Proj}\left(\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})/\langle (z^2) \rangle\right). \tag{3.4.1}$$

It is straightforward to verify that $\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})[(z^2)^{-1}] = (\mathcal{O}(\mathrm{SL}_2)[z^{\pm 1}])^{\mathbb{Z}/2\mathbb{Z}} = \mathcal{O}(\mathrm{GL}_2)$. The action of $\mathbb{C}^\times$ on $\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})[(z^2)^{-1}]$ corresponds to the (free) action on GL_2 by its center. Thus,

$$\mathrm{Spec}\left(\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2})[(z^2)^{-1}]^{\mathbb{C}^\times}\right) = \mathrm{GL}_2/\mathbb{C}^\times = \mathrm{PSL}_2.$$

A standard argument shows that $\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2}))/(z^2) = \mathrm{gr}(\mathcal{O}(\mathrm{SL}_2))$. Therefore, an application of Proposition 3.4.4 shows that $\mathrm{Proj}\left(\mathcal{O}(\mathbb{V}_{\mathrm{SL}_2}))/(z^2)\right) = \mathbb{P}^1 \times \mathbb{P}^1$. We see that the decomposition above becomes precisely the decomposition of Proposition 3.4.5.3.

Chapter 4

The Proj category of a Λ-graded algebra

This chapter lays the groundwork for working with Proj categories for noncommutative rings graded by a lattice Λ. Most of the basic definitions appear in work of Artin and Zhang, of Ginzburg, and of Chan, and are ultimately inspired by results of Serre [AZ, C, Gi2, Se]. We present reformulations and refinements of constructions that appear in the aforementioned papers, and do not claim originality except for several elementary results. The Proj construction given in this chapter will reappear in our definition of the quantum wonderful compactification in Chapter 5.

Section 4.1 collects basic definitions related to rings graded by a lattice Λ, and modules for such rings. The notion of torsion modules is introduced in Section 4.2. We consider the case where Λ is the weight lattice in Section 4.3, and describe the special role of regular dominant weights. Section 4.4 recalls result that characterize Proj categories, and address to what extent a ring can be recovered from its Proj category. Finally, in Section 4.5, we describe an application of Proj categories that appears in work of Backelin and Kremnitzer on quantum flag varieties; this application admits a close connection to the quantum wonderful compactification, as we discuss in Section 5.2 of the next chapter.

4.1 Basic definitions

Let Λ be a lattice, that is, a finitely generated torsion-free abelian group. Let Λ^+ be a subsemigroup of Λ (i.e. a cone). We assume that Λ^+ has the following properties:

1. For any λ_1 and λ_2 in Λ, the intersection $(\lambda_1 + \Lambda^+) \cap (\lambda_2 + \Lambda^+)$ is nonempty.

2. $\Lambda^+ \cap (-\Lambda^+) = \{0\}$.

Define a partial order on Λ by declaring $\mu \succeq \lambda$ if $\mu - \lambda \in \Lambda^+$.

Example 4.1.1. The main example we will consider is when $\Lambda = X^*(T)$ is the weight lattice of G with respect to a maximal torus T, and Λ^+ is the cone of dominant weights[1].

Let k be a field.

Definition 4.1.2. A Λ-graded algebra is a Λ-graded vector space $R = \bigoplus_{\lambda \in \Lambda} R_\lambda$ over k equipped with an associative k-linear multiplication map that restricts to a map

$$m_{\lambda,\mu} : R_\lambda \otimes R_\mu \to R_{\lambda+\mu}$$

for any $\lambda, \mu \in \Lambda$. We say that R is locally finite if R_λ is finite-dimensional over k for every $\lambda \in \Lambda$. We say that R is Λ^+-graded if R is Λ-graded and $R_\lambda = 0$ unless $\lambda \in \Lambda^+$.

Remark 4.1.3. When $\Lambda = \mathbb{Z}$ and $\Lambda^+ = \mathbb{Z}_{\geq 0}$, we recover the usual notion of a graded algebra.

[1] Alternatively, one can choose Λ^+ to be the cone spanned by the positive roots, in which case one recovers the partial order that leads to a filtration on the algebra $\mathcal{O}(G)$ of functions on the group, as discussed in Chapter 3.

Remark 4.1.4. We will mostly be interested in the case where R is noetherian (or right noetherian) and locally finite. In applications, it will often have that $R_0 = k$.

Definition 4.1.5. Let R be a Λ-graded algebra.

1. A graded left R-module is a Λ-graded vector space $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ equipped with an action of R such the action map restricts to a map

$$R_\lambda \otimes M_\mu \to M_{\lambda+\mu}.$$

The category of graded left R-modules is denoted $\mathsf{Grmod}(R)$ and its objects will henceforth be referred to simply as R-modules.

2. An R-module M is finitely generated if there exist elements $m_1 \in M_{\nu_1}, \ldots, m_p \in M_{\nu_p}$, where $\nu_i \in \Lambda$, such that the map

$$\begin{aligned} \bigoplus R_{\lambda-\nu_i} &\to M_\lambda \\ (r_i) &\mapsto \sum_i r_i m_i \end{aligned}$$

is surjective for all $\lambda \in \Lambda$. The category of finitely generated graded left R-modules is denoted $\mathsf{grmod}(R)$, and it is a full subcategory of $\mathsf{Grmod}(R)$.

3. For an R-module M and $\mu \in \Lambda$, the μ-th shift of M is denoted $M[\mu]$ and defined by $M[\mu]_\lambda = M_{\mu+\lambda}$. The μ-th shift operation defines an autoequivalence

$$[\mu] : \mathsf{Grmod}(R) \to \mathsf{Grmod}(R)$$

that restricts to an autoequivalence of the subcategory $\mathsf{grmod}(R)$.

Remark 4.1.6. When R is noetherian, the category $\mathsf{grmod}(R)$ is abelian.

The following lemma is straightforward to verify.

Lemma 4.1.7. *Let $\omega_1, \ldots, \omega_r$ be a generating set for Λ^+ as a semigroup. Then R is generated by $R_{\omega_1}, \ldots, R_{\omega_r}$ if and only if the following two conditions are satisfied:*

1. *R is Λ^+-graded.*

2. *The multiplication map $R_\lambda \otimes R_\mu \to R_{\lambda+\mu}$ is surjective for all $\lambda, \mu \in \Lambda^+$.*

Definition 4.1.8. When R satisfies the equivalent conditions of Lemma 4.1.7 we say that R is generated in degree one.

4.2 Torsion modules

Let R be a Λ-graded algebra as above. In this section, we define the notion of a torsion graded R-module, and prove elementary properties of torsion modules.

Definition 4.2.1. An R-module M is called torsion if, for all m in M, there exists an $\lambda \in \Lambda^+$ such that R_μ acts by zero on m for any $\mu \in \lambda + \Lambda^+$. The full subcategory of torsion modules (resp. finitely generated torsion modules) is denoted $\mathsf{Tors}(R)$ (resp. $\mathsf{tors}(R)$).

Lemma 4.2.2. *If M is finitely generated, then M is torsion if and only if there exists $\lambda \in \Lambda$ such that $M_\mu = 0$ for $\mu \in \lambda + \Lambda^+$.*

Proof. Let $m_1 \in M_{\nu_1}, \ldots, m_p \in M_{\nu_p}$ be generators of M. Suppose first that M is torsion. Then for each m_i there exists $\lambda_i \in \Lambda^+$ such that R_μ acts by zero on m_i for $\mu \in \lambda_i + \Lambda^+$.

Choose $\lambda \in \bigcap_i(\nu_i + \lambda_i + \Lambda^+)$. Then, for every $\mu \in \lambda + \Lambda^+$, we have that $\mu \in \nu_i + \Lambda^+$ for all i, and so the map

$$\bigoplus_i R_{\mu-\nu_i} \to M_\mu$$
$$(r_i) \mapsto \sum_i r_i m_i$$

is surjective. On the other hand, $\mu - \nu_i \in \lambda - \nu_i + \Lambda^+ \subseteq \lambda_i + \Lambda^+$ for all i, so $r_i m_i = 0$ for all i. We conclude that $M_\mu = 0$ for all $\mu \in \lambda + \Lambda^+$.

Conversely, suppose that there exists μ such that $M_\mu = 0$ for $\mu \in \lambda + \Lambda^+$. Let $m \in M_\nu$. Let $\lambda' \in (\nu + \Lambda^+) \cap (\lambda + \Lambda^+) \subseteq \Lambda^+$. Then $R_\mu m = 0$ for all $\mu \in \lambda' - \nu + \Lambda^+$. $\square$

A dense subcategory of an abelian category is a full subcategory that is closed under extensions. See the appendix for more details.

Lemma 4.2.3. *Suppose R is noetherian. Then the full subcategory of torsion objects in either of* $\mathsf{grmod}(R)$ *or* $\mathsf{Grmod}(R)$ *is dense.*

Proof. Fix a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

in $\mathsf{Grmod}(R)$. If M is torsion, then it is clear that M' and M'' are both torsion. For the other direction, suppose M' and M'' are torsion and let m be a homogeneous element of M. It is enough to assume that m is of degree zero and M is generated by m, and thus we reduce to the setting of $\mathsf{grmod}(R)$. In this case, there are graded right ideals $I \subseteq J \subseteq R$ such that $M = R/I$, $M'' = R/J$, and $M' = J/I$. Since the image of m in M'' is torsion, there exists $\lambda_1 \in \Lambda^+$ such that $R_{\lambda_1} m \subseteq M'$. Since M' is a finitely generated torsion module, there exists λ_2 such that $M'_\mu = 0$ for $\mu \in \lambda_2 + \Lambda^+$. Let $\lambda \in (\lambda_1 + \Lambda^+) \cap (\lambda_2 + \Lambda^+)$. Then, for any $\mu \in \lambda + \Lambda^+$, $R_\mu m \subset M'_\mu = 0$. □

General results on the localization of abelian categories (outlined in Section 2.5) allow us to make the following definition.

Definition 4.2.4. Suppose R is noetherian. We form the quotient categories $\mathsf{Proj}(R) = \mathsf{Grmod}(R)/\mathsf{Tors}(R)$ and $\mathsf{proj}(R) = \mathsf{grmod}(R)/\mathsf{tors}(R)$.

Proposition 4.2.5. *We have:*

1. *The category* $\mathsf{Proj}(R)$ *is abelian.*

2. *Suppose* R *is noetherian. Then the category* $\mathsf{proj}(R)$ *is abelian and noetherian.*

3. *The functor* $\pi : \mathsf{Grmod}(R) \to \mathsf{Proj}(R)$ *has a right adjoint* $\omega : \mathsf{Proj}(R) \to \mathsf{Grmod}(R)$*, and the counit of the adjunction* $\pi\omega \to \mathrm{Id}$ *is a natural isomorphism.*

Proof. The first and third statements follow directly results described in [Pop]. The second statement is a straightforward verification. □

4.3 The case of the weight lattice

In this section, we specialize to the case where Λ is the weight lattice of a semisimple Lie algebra, and Λ^+ is the cone of dominant weights, which is the subsemigroup generated by the fundamental weights $\omega_1, \dots \omega_r$. Recall than an element $\lambda \in \Lambda^+$ is regular if it lies in the interior of Λ^+, that is, if λ is a non-negative linear combination of the fundamental weights.

Lemma 4.3.1. *Let $\lambda \in \Lambda^+$. Then λ is regular if and only if, for all $\nu \in \Lambda$, we have that $N\lambda \in \nu + \Lambda^+$ for $N >> 0$.*

Proof. Write $\lambda = \sum_{i=1}^r \lambda_i \omega_i$ for $\lambda_i \in \mathbb{Z}$. If λ is regular, then $\lambda_i > 0$ for all i. Choose $\nu = \sum_i \nu_i \omega_i \in \Lambda$ and let $N_0 = \max\left\{\frac{|\nu_i|}{\lambda_i}\right\}$. Then, if $N \geq N_0$, we have that

$$(N\lambda - \nu)_i = N\lambda_i - \nu_i \geq \frac{|\nu_i|}{\lambda_i}\lambda_i - \nu_i = |\nu_i| - \nu_i \geq 0.$$

It follows that $N\lambda \in \nu + \Lambda^+$. Conversely, take $\nu = \sum_i \omega_i$. By hypothesis, there is an N such that $N\lambda \in \nu + \Lambda^+$. Hence, $\lambda_i \geq \nu_i/N = 1/N > 0$ for all i, and so λ is regular. ☐

Lemma 4.3.2. *Suppose $R_\lambda \otimes R_\mu \to R_{\lambda+\mu}$ is surjective for all $\lambda, \mu \in \Lambda^+$. Let M be a finitely-generated graded R-module and $\lambda \in \Lambda^+$ a regular dominant weight. Then M is torsion if and only if $M_{N\lambda} = 0$ for $N >> 0$.*

Proof. If M is torsion, then there exists ν such that $M_\mu = 0$ for $\mu \in \nu + \Lambda^+$. Since λ is regular, we have that $N\lambda \in \nu + \Lambda^+$. The forward implication follows. Conversely, suppose that $M_{N\lambda} = 0$ for $N >> 0$. Let $m_1 \in M_{\nu_1}, \ldots, m_p \in M_{\nu_p}$ be generators of M. Let N_0 be such that, first, $M_{N\lambda} = 0$, and, second, $N\lambda \in \nu_i + \Lambda^+$ for all i (using Lemma 4.3.1). We claim that $M_\mu = 0$ for $\mu \in N\lambda + \Lambda^+$. Consider the following diagram

$$\begin{array}{ccc} \bigoplus_i R_{\mu-N\lambda} \otimes R_{N\lambda-\nu_i} & \longrightarrow & R_{\mu-N\lambda} \otimes M_{N\lambda} \, , \\ \downarrow & & \downarrow \\ \bigoplus_i R_{\mu-\nu_i} & \longrightarrow & M_\mu \end{array}$$

where the horizontal maps are the action maps evaluated at the elements m_i, and are surjective since the m_i generate the module M. The left vertical map is the multiplication map, and it is surjective by hypothesis. Therefore, the 'down then right' composition is surjective. On the other hand, since $M_{N\lambda} = 0$, the 'right then down' composition is zero. It follows that $M_\mu = 0$. □

Lemma 4.3.3. *Suppose R is locally finite and $R_\lambda \otimes R_\mu \to R_{\lambda+\mu}$ is surjective for all $\lambda, \mu \in \Lambda^+$. If λ is regular, then, for every object $\pi(M)$ of $\mathsf{proj}(R)$, there are integers $n_1, \ldots, n_p$ and a surjection*

$$\bigoplus_{i=1}^{p} \pi(R[-n_i\lambda]) \to \pi(M).$$

Proof. Since M is finitely generated, there is a surjection $\bigoplus_{j=1}^{s} R[-\nu_j] \to M$ in $\mathsf{grmod}(R)$. Applying the exact functor π to this morphism, we obtain a surjection $\bigoplus_{j=1}^{s} \pi(R[-\nu_j]) \to \pi(M)$. Therefore, it suffices to prove that, for every $\nu \in \Lambda$, there exist integers n and p and a homomorphism $R[-n\lambda]^{\oplus p} \to R[-\nu]$ with torsion cokernel. To this end, choose n large enough so that $n\lambda - \nu \in \Lambda^+$. Let $b_1, \ldots, b_p$ be a basis for $R_{n\lambda-\nu}$ as a vector space over k. For each $i = 1, \ldots, p$, define $f_i : R[-n\lambda] \to R[-\nu]$ by sending the generator $1[-n\lambda]$ of $R[-n\lambda]$ to b_i and extending linearly. Now define $f : R[-n\lambda]^{\oplus p} \to R[-\nu]$ to be the sum of the f_i. The hypothesis on the multiplication map for R implies that the image of f is contains $R[-\nu]_\mu$ for each $\mu \in n\lambda - \nu + \Lambda^+$. Therefore, the cokernel of f is torsion. □

Lemma 4.3.4. *Let $\lambda \in \Lambda^+$ be regular. Let M and N be finitely generated R-modules and write $\mathcal{M}$ and $\mathcal{N}$ for the images of M and N in $\mathsf{proj}(R)$.*

1. *Suppose $f : \mathcal{M} \to \mathcal{N}$ is a surjection in $\mathsf{proj}(R)$, and that $\omega(\mathcal{M})$ and $\omega(\mathcal{N})$ are finitely*

generated R-modules. Then the map

$$\mathrm{Hom}(\pi(R), \mathcal{M}[n\lambda]) \to \mathrm{Hom}(R, \mathcal{N}[n\lambda])$$

is surjective for $n >> 0$.

2. *Let $\mathcal{M}$ be an object in $\mathsf{proj}(R)$ with $\omega(\mathcal{M})$ finitely generated, and suppose $\mathrm{Hom}(R, \mathcal{M}[n\lambda]) = 0$ for $n >> 0$. Then $\mathcal{M} = 0$.*

Proof. For the first statement, observe that $\mathrm{Hom}(R, \mathcal{M}[n\lambda]) = \omega(\mathcal{M})_{n\lambda}$, and the map

$$\mathrm{Hom}(R, \mathcal{M}[n\lambda]) \to \mathrm{Hom}(R, \mathcal{N}[n\lambda])$$

is the $n\lambda$-th graded piece of the map $\omega(f) : \omega(\mathcal{M}) \to \omega(\mathcal{N})$. Since f is surjective, the map $\omega(f)$ has torsion cokernel. By Lemma 4.2.2, the $n\lambda$-th graded piece of $\omega(f)$ is surjective for $n >> 0$. For the second statement, we have that $\omega(\mathcal{M})_{n\lambda} = 0$ for $n >> 0$. Consider the short exact sequence

$$0 \to \tau(M) \to M \to \omega(\mathcal{M}).$$

We see that $M_{n\lambda} = \tau(M)_{n\lambda}$ for $n >> 0$. Since $\tau(X)$ is a finitely generated torsion module, $\tau(M)_{n\lambda} = 0$ for $n >> 0$. It follows that $\omega(\mathcal{M})$ is torsion, so $\mathcal{M} = \pi\omega(\mathcal{M}) = 0$. □

The assumptions about finite generation in the preceding lemma are crucial.

4.4 A characterization of Proj

In their seminal paper [AZ], Artin and Zhang give a characterization of categories that arise as proj categories of $\mathbb{Z}$-graded rings. They also describe to what extent a $\mathbb{Z}$-graded ring A can be recovered from the category $\mathsf{proj}(A)$. Backelin and Kremnitzer [BaKr] state a partial generalization of the results of Artin and Zhang for identifying proj categories of Λ-graded rings.

Let (Λ, Λ^+) be as above. Let $(\mathcal{C}, \mathscr{A}, S)$ be a triple consisting of a k-linear abelian category $\mathcal{C}$, a distinguished object $\mathscr{A}$ of $\mathcal{C}$, and a collection $S = \{s_1, \ldots, s_r\}$ of commuting autoequivalences[2] of $\mathcal{C}$. Given $\lambda = \sum_{i=1}^r \lambda_i \omega_i$, we write $X[\lambda]$ for the image of $X \in \mathcal{C}$ under the autoequivalence $s_1^{\lambda_1} \circ \cdots \circ s_r^{\lambda_r}$. Define a functor

$$\begin{aligned} \Gamma_* : \mathcal{C} &\to \mathsf{Vect}_k \\ X &\mapsto \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{C}}(\mathscr{A}, X[\lambda]). \end{aligned}$$

Set $A = \bigoplus_{\lambda \in \Lambda^+} \mathrm{Hom}_{\mathcal{C}}(\mathscr{A}, X[\lambda]) = \Gamma_*(\mathscr{A})_{\geq 0}$. We see that A has the natural structure of an associative, Λ-graded algebra, and the functor Γ_* factors through the category $\mathsf{Grmod}(A)$ of graded A-modules.

Definition 4.4.1. Let $(\mathcal{C}, \mathscr{A}, S)$ be as above. We say that S is ample if the following conditions are satisfied:

1. For every epimorphism $X \to Y$ in $\mathcal{C}$, there exists $\lambda \in \Lambda$ such the resulting map $\mathrm{Hom}_{\mathcal{C}}(\mathscr{A}, X[\mu]) \to \mathrm{Hom}_{\mathcal{C}}(\mathscr{A}, Y[\mu])$ is surjective for $\mu \in \lambda + \Lambda^+$.

[2] Alternatively, letting Λ be a lattice of rank r, one can formulate the notion of a categorical action of Λ on the category $\mathcal{C}$. The data of such an action is (in some sense) equivalent to the data of commuting autoequivalences $s_1, \ldots, s_r$.

2. For every object X of $\mathcal{C}$, there are elements $\nu_1, \ldots, \nu_p \in \Lambda$ and an epimorphism $\bigoplus_{i=1}^{p} \mathscr{A}[-\nu_i] \to X$.

The following theorem is stated in Backelin-Kremnitzer [BaKr], but is credited to Artin-Zhang [AZ, Theorem 4.5 and Corollary 4.6].

Theorem 4.4.2 ([BaKr, Proposition 2.1])**.** *Suppose the triple* $(\mathcal{C}, \mathscr{A}, s)$ *satisfies the following:*

- *(H1) The object* $\mathscr{A}$ *is noetherian.*
- *(H2) The set of autoequivalences* S *is ample.*
- *(H3)* $\mathrm{Hom}_{\mathcal{C}}(\mathscr{A}, X)$ *is a finite* A_0*-module for all objects* X *of* $\mathcal{C}$*.*

Let $A = \Gamma_*(\mathscr{A})_{\geq 0}$ *as above. Then the functor* Γ_* *takes values in* $\mathsf{grmod}(A)$*, and the composition* $\pi \circ \Gamma_* : \mathcal{C} \to \mathsf{proj}(A)$ *is an equivalence of categories. Moreover, this equivalence takes* $\mathscr{A}$ *to* $\pi(A)$ *and that intertwines the shift functors on* $\mathsf{proj}(A)$ *with the autoequivalences on* $\mathcal{C}$*. Finally,* A *is locally finite.*

In the case that $(\Lambda, \Lambda^+) = (\mathbb{Z}, \mathbb{Z}_{\geq 0})$, the preceding theorem is due to Artin and Zhang [AZ]. Artin and Zhang demonstrate that a $\mathbb{Z}$-graded algebra A cannot always be recovered from its proj category. However, they characterize those algebras that can be recovered via a condition on certain Ext groups.

Definition 4.4.3. Let A be a locally finite $\mathbb{Z}_{\geq}$-graded algebra. We say that A satisfies χ_1 if, for every finitely generated graded module M, the Ext group $\mathrm{Ext}^1(A_0, M[n])$ is nonzero for only finitely many n.

Remark 4.4.4. The condition χ_1 as stated above is actually the condition χ_1° of Artin-Zhang [AZ]. The two conditions coincide for locally finite algebras. We also note that any commutative graded $\mathbb{Z}$-graded algebra satisfies χ_1.

Theorem 4.4.5 ([AZ, Theorem 4.5 and Corollary 4.6])**.** *Let* $(\Lambda, \Lambda^+) = (\mathbb{Z}, \mathbb{Z}_{\geq 0})$.

1. *Suppose the triple* $(\mathcal{C}, \mathscr{A}, s)$ *satisfies the hypotheses (H1), (H2), and (H3) of Theorem 4.4.2. Then* $A = \Gamma_*(\mathscr{A})$ *satisfies* χ_1.

2. *Suppose that* B *is a noetherian locally finite* Λ*-graded algebra that satisfies* χ_1*. Then the triple* $(\mathsf{proj}(B), \pi(B), S)$ *satisfies the hypotheses (H1), (H2), and (H3) of Theorem 4.4.2. Let* $A = \Gamma_*(\pi(B))_{\geq 0}$*. The canonical homomorphism* $B \to A$ *has right bounded kernel and cokernel, and induces an equivalence of categories* $\mathsf{proj}(B) \to \mathsf{proj}(A)$ *that commutes with the shift functors.*

It remains an open question to generalize the notion of χ_1 to the multigraded setting in order to understand which algebras can be recovered from their proj categories. One major difficulty is the following. Suppose R is a Λ-graded algebra, for Λ of rank more than one. Then the R-module $R_{\succeq \lambda+\mu}/R_{\succeq \lambda}$ is in general unbounded for $\lambda, \mu \in \Lambda^+$. For this reason, Proposition 3.5(a) of [AZ] does not directly generalize to the multigraded setting.

Combining Corollary 3.2.6, Theorem 4.4.2, and Theorem 4.4.5 we have the following result:

Corollary 4.4.6. *Let* λ *be a regular dominant weight. Then there is an equivalence of categories:*

$$\mathsf{Proj}\,(\mathcal{O}(\mathbb{V}_G)) = \mathsf{Proj}\left(\bigoplus_{n\geq 0} \mathcal{O}(\mathbb{V}_G)_{n\lambda}\right)$$

for any regular dominant weight λ.

4.5 The quantum flag variety

In this section we recall and reinterpret some results from Backelin and Kremnitzer [BaKr]. The quantum coordinate algebra $\mathcal{O}_q(G)$ for G is a comodule for the quantum coordinate algebra $\mathcal{O}_q(B)$ of a Borel subgroup B of G, and the algebra structure on $\mathcal{O}_q(G)$ is compatible with the comodule structure. In other words, $\mathcal{O}_q(G)$ is an algebra object in the category of $\mathcal{O}_q(B)$-comodules. Hence we can consider the category $\mathcal{M}_{B_q}(G_q)$ of $\mathcal{O}_q(G)$-modules in the category of $\mathcal{O}_q(B)$-comodules. This category is a q-deformation of the category of quasicoherent sheaves on the flag variety G/B, and is known as the quantum flag variety.

We are interested in a doubled version of this category. Namely, let $\mathcal{M}_{B_q \times B_q^-}(G_q \times G_q)$ be the category of $\mathcal{O}_q(G \times G)$-modules within the category of $\mathcal{O}_q(B \times B^-)$-comodules. The following theorem is a consequence of [BaKr, Corollary 3.7].

Theorem 4.5.1. *Let λ be a regular dominant weight. There are equivalences of categories*

$$\mathcal{M}_{B_q \times B_q^-}(G_q \times G_q) \simeq \mathsf{Proj}(\mathrm{gr}(\mathcal{O}_q(G))).$$

Corollary 4.5.2. *Let $\mathcal{A}_q = \pi(\mathrm{gr}(\mathcal{O}_q(G)))$ as an object in $\mathsf{Proj}(\mathrm{gr}(\mathcal{O}_q(G)))$. Let λ be a regular dominant weight.*

1. *$\mathcal{A}_q$ is noetherian.*

2. *$\bigoplus_{n\geq 0} \mathrm{Hom}(\mathcal{A}_q, M)$ is a finitely generated module for $\bigoplus_{n\geq 0} \mathrm{End}(\mathcal{A}_q)$ for all M in $\mathsf{proj}(\mathrm{gr}(\mathcal{O}_q(G)))$.*

3. *Each object in* $\mathsf{proj}(\mathrm{gr}(\mathcal{O}_q(G)))$ *is a quotient of a direct sum of* $\mathcal{A}_q[n\lambda]$*'s.*

4. *For any surjection* $M \to M'$ *of objects in* $\mathsf{proj}(\mathrm{gr}(\mathcal{O}_q(G)))$*, there is a* λ *such that the induced map* $\mathrm{Hom}(\mathcal{A}_q, M[n\lambda]) \to \mathrm{Hom}(\mathcal{A}_q, M'[n\lambda]))$ *is a surjection for* $n >> 0$*.*

Theorems 4.4.2 and 4.4.5, together with the preceding corollary, imply that:

Corollary 4.5.3. *There is an equivalence of categories:*

$$\mathsf{Proj}(\mathrm{gr}(\mathcal{O}_q(G))) \xrightarrow{\sim} \mathsf{Proj}\left(\bigoplus_{n\geq 0} \mathrm{gr}(\mathcal{O}_q(G))_{n\lambda}\right).$$

Moreover, the algebra $\bigoplus_{n\geq 0} \mathrm{gr}(\mathcal{O}_q(G))_{n\lambda}$ *satisfies* χ_1*.*

Chapter 5

The wonderful compactification for quantum groups

This chapter is the heart of the thesis. In Section 5.1, we present two central constructions. First, the quantum coordinate algebra $\mathcal{O}_q(\mathbb{V}_G)$ of Vinberg semigroup is the Rees algebra for $\mathcal{O}_q(G)$ with a filtration by the weight lattice Λ. Second, the category of sheaves on the quantum wonderful compactification is the Proj category associated with $\mathcal{O}_q(\mathbb{V}_G)$. We briefly describe work-in-progress on different descriptions of the quantum wonderful compactification in Section 5.2, which is connected to work of Backelin and Kremnitzer on quantum flag varieties. We introduce filtrations on the quantum coordinate algebra $\mathcal{O}_q(G)$ in Section 5.3. We use these filtrations in Section 5.4 in order to describe the quantum $G \times G$ orbits in two different ways: first, as partial associated graded algebras of $\mathcal{O}_q(G)$, and, second, as invariants in $\mathcal{O}_q(G \times G)$ for a certain quantized enveloping algebra (regarded as a 'quantum stabilizer'). In Section 5.5, we define quantum differential operators on the wonderful compactification, and prove basic properties.

5.1 Main definitions

In this section, we define a quantum version of the Vinberg semigroup of a semisimple group G. Our starting point is the following result, which is a direct consequence of the definition of the quantum coordinate algebra $\mathcal{O}_q(G)$ (see Definition 2.3.7.):

Proposition 5.1.1. *There is an isomorphism of $U_q(\mathfrak{g})$-bimodules $\phi : \bigoplus_{\lambda \in \Lambda^+} V_\lambda \otimes V_\lambda^* \xrightarrow{\sim} \mathcal{O}_q(G)$, where the sum ranges over irreducible objects in $\mathcal{C}_q(\mathfrak{g})$ with highest weights in Λ_G^+.*

Recall from Definition 2.2.1 the partial order defined on Λ_G.

Definition 5.1.2. For $\lambda \in \Lambda$, define the following subspace of $\mathcal{O}_q(G)$:

$$\mathcal{O}_q(G)_{\leq \lambda} = \phi\left(\sum_{\mu \leq \lambda} V_\mu^* \otimes V_\mu\right).$$

Proposition 5.1.3. *The subspaces $\mathcal{O}_q(G)_{\leq \lambda}$ define a Λ_G-filtration on $\mathcal{O}_q(G)$. The associated graded algebra has λ-graded piece equal to $\phi(V_\lambda^* \otimes V_\lambda)$.*

Proof. General properties of matrix coefficients, together with Lemma 2.2.2 imply that the subspaces $\mathcal{O}_q(G)_{\leq \lambda}$ form a filtration. □

In Section 3.1, we considered the algebra $\mathbb{C}[z^\lambda] = \mathbb{C}[z^\lambda \mid \lambda \in \Lambda_G]$.

Definition 5.1.4. The Rees algebra for $\mathcal{O}_q(G)$ with the Peter-Weyl filtration is defined as the following Λ_G-graded subalgebra of $\mathcal{O}_q(G) \otimes \mathbb{C}[z^\lambda]$:

$$\mathrm{Rees}_q(G) = \bigoplus_{\lambda \in \Lambda} \mathcal{O}_q(G)_{\leq \lambda} z^\lambda.$$

Proposition 5.1.5. *The algebra $\mathrm{Rees}_q(G)$ has a natural bialgebra structure, and forms a flat deformation of the coordinate algebra $\mathcal{O}(\mathbb{V}_G)$ of the Vinberg semigroup.*

Proof. The first statement is a direct consequence of Proposition 5.1.3. The second statement follows from the fact that $\mathcal{O}_q(G)$ is a flat deformation of $\mathcal{O}(G)$. □

Definition 5.1.6. We make the following definitions:

1. The quantized coordinate algebra $\mathcal{O}_q(\mathbb{V}_G)$ of the Vinberg semigroup for G is defined as the Rees algebra for $\mathcal{O}_q(G)$:

$$\mathcal{O}_q(\mathbb{V}_G) = \bigoplus_{\lambda \in \Lambda} \mathcal{O}_q(G)_{\leq \lambda} z^{\lambda}.$$

2. The category of quasicoherent sheaves on the quantum wonderful compactification of G^{ad} is given by

$$\mathsf{QCoh}_q\left(\overline{G^{\mathrm{ad}}}\right) = \mathsf{Proj}\left(\mathcal{O}_q(\mathbb{V}_G)\right).$$

Remark 5.1.7. When $q = 1$, we recover from $\mathcal{O}_q(\mathbb{V}_G)$ the coordinate ring of the Vinberg semigroup [Br]. When $q = 1$, we recover from $\mathsf{QCoh}_q\left(\overline{G^{\mathrm{ad}}}\right)$ the category of quasicoherent sheaves on the wonderful compactification $\overline{G^{\mathrm{ad}}}$ [MT].

Recall from Section 3.1 that we denote by $\mathbb{A}$ the spectrum of the polynomial subalgebra $\mathbb{C}[z^{\alpha_i}]$ of $\mathbb{C}[z^{\lambda}]$ generated by the elements z^{α_i} for $i \in \Delta$.

Lemma 5.1.8. *For any $i \in \Delta$, the element z^{α_i} belongs to $\mathcal{O}_q(\mathbb{V}_G)$, and is central. Therefore, $\mathcal{O}_q(\mathbb{V}_G)$ forms a sheaf of algebras on $\mathbb{A}$.*

Proof. The assertion follows from the fact that $\alpha_i \geq 0$ for any positive simple root α_i. $\square$

We note that, for $\lambda \in \Lambda$, the monomial z^{λ} lies in $\mathcal{O}_q(\mathbb{V}_G)$ if and only if λ belongs to the cone spanned by the positive roots. The lattice Λ is naturally identified with the character lattice of the maximal torus T. Consequently, we regard $\mathbb{C}[z^{\lambda} \mid \lambda \in \Lambda]$ as the algebra of functions $\mathcal{O}(T)$ on the maximal torus T of G, and $\mathcal{O}_q(\mathbb{V}_G)$ as a subalgebra of $\mathcal{O}_q(G) \otimes \mathcal{O}(T)$. Moreover, the subalgebra generated by $z^{\pm\alpha_i}$ is identified with the algebra of functions $\mathcal{O}(T/Z(G))$ on the adjoint torus. Finally, the r-dimensional affine space $\mathbb{A}$ is naturally a partial compactification of $T/Z(G)$.

The choice of positive simple roots endows $\mathbb{A}$ with a coordinate system. The fiber of $\mathcal{O}_q(\mathbb{V}_G)$ over $\mathbb{A}$ is isomorphic to $\mathcal{O}_q(G)$ in the generic case when all coordinates are nonzero. At zero, the fiber is the associated graded algebra described in Proposition 5.1.3. The fibers at point in higher-dimensional root subspaces are certain 'partial' associated graded algebras for $\mathcal{O}_q(G)$. Sections 5.3 and 5.4 are devoted to a description of these algebras.

5.2 Independence of regular highest weight

In this section we discuss work-in-progress on the following conjecture:

Conjecture 5.2.1. *Let λ be a regular dominant weight. There is an equivalence of categories*

$$\mathsf{Proj}\left(\mathcal{O}_q(\mathbb{V}_G)\right) \xrightarrow{\sim} \mathsf{Proj}\left(\bigoplus_{n\geq 0} \mathcal{O}_q(\mathbb{V}_G)_{n\lambda}\right).$$

Fix a regular dominant weight λ. Abbreviate $\mathsf{proj}(\mathcal{O}_q(\mathbb{V}_G))$ by $\mathcal{C}$ and let $\mathcal{O}_q$ denote the object $\pi(\mathcal{O}_q(\mathbb{V}_G)))$ of $\mathcal{C}$. Consider the shift functor $s : \mathcal{C} \to \mathcal{C}$ given by shifting along the regular dominant weight λ. By Theorem 4.4.2, Conjecture 5.2.1 follows from the following conjecture:

Conjecture 5.2.2. *1. $\mathcal{O}_q$ is noetherian*

2. $\mathrm{Hom}(\mathcal{O}_q, \mathcal{M})$ is a finite-dimensional vector space for all noetherian objects $\mathcal{M}$ of $\mathcal{C}$.

3. Each object in $\mathcal{C}$ is a quotient of a direct sum of $\mathcal{O}_q[n\lambda]$'s.

4. *For any surjection* $\mathcal{M} \to \mathcal{N}$ *if noetherian objects in* $\mathcal{C}$*, the induced map*

$$\mathrm{Hom}(\mathcal{O}_q, \mathcal{M}[n\lambda]) \to \mathrm{Hom}(\mathcal{O}_q, \mathcal{N}[n\lambda])$$

is a surjection for $n >> 0$*.*

We discuss the parts of this conjecture in turn:

1. Part 1 is immediate.

2. Part 2 can presumably be shown using the localization functor $\mathcal{C} \to \mathsf{proj}(\mathcal{O}_q(\mathbb{V}_G)[\{(z^\alpha)^{-1}\}])$, which is expected to be faithful, and the fact that the target is equivalent to $\mathcal{O}_q(G)$-mod.

3. Part 3 is known to be true by straightforward arguments.

4. The major missing component of the conjecture is Part 4. One must show that the line bundle on the quantum wonderful compactification corresponding to a regular dominant weight λ is ample. This line bundle is ample in the commutative case. The noncommutative line bundle pulls back to an ample line bundle on quantum $G/B \times G/B$ by work of Backelin and Kremnitzer (see Corollary 4.5.2 above). While this strongly suggests that the original line bundle is ample, the technical argument needs an idea in the following spirit: if a sheaf on a stratified space is zero on the open stratum and zero on the unique closed stratum, then it is zero.

Finally, we note that Conjecture 5.2.1 is true for $q = 1$ by Corollary 4.4.6.

5.3 Filtrations on the quantum coordinate algebra

In this section, we consider certain filtrations on the quantized coordinate algebra $\mathcal{O}_q(G)$ of G, and describe the associated graded algebras. These associated graded algebras define the quantum orbits on the wonderful compactification.

Given a subset $I \subseteq \Delta$ of positive simple roots let $\Lambda_I = \mathbb{Z}\{\alpha_i \mid i \in I\} \subseteq \Lambda_R$ denote the sublattice of the root lattice spanned by the roots in I. Write $[\lambda]$ or $[\lambda]_I$ for the image of $\lambda \in \Lambda$ in Λ/Λ_I. Define a partial order on Λ/Λ_I by

$$[\mu]_I \leq [\lambda]_I \text{ whenever } \lambda - \mu = \sum_{i=1}^{r} n_i \alpha_i \text{ with } n_i \in \mathbb{Z} \text{ if } i \in I \text{ and } n_i \geq 0 \text{ if } i \notin I.$$

If $I = \emptyset$, we recover the partial order on Λ from Definition 2.2.1.

Definition 5.3.1. For $\lambda \in \Lambda$, define the following subspace of $\mathcal{O}_q(G)$:

$$\mathcal{O}_q(G)_{\leq[\lambda]_I} = \phi\left(\sum_{[\mu]_I \leq [\lambda]_I} V_\mu^* \otimes V_\mu\right).$$

If $I = \emptyset$, we write simply $\mathcal{O}_q(G)_{\leq\lambda}$ for $\mathcal{O}_q(G)_{\leq[\lambda]_\emptyset}$.

Proposition 5.3.2. *The subspaces $\mathcal{O}_q(G)_{\leq[\lambda]_I}$ define a filtration on $\mathcal{O}_q(G)$ by Λ/Λ_I. The associated graded algebra has $[\lambda]_I$-graded piece equal to*

$$\phi\left(\bigoplus_{\nu \in \Lambda_I} V_{\lambda+\nu}^* \otimes V_{\lambda+\nu}\right).$$

The coproduct Δ restricts to a map $\Delta : \mathcal{O}_q(G)_{\leq[\lambda]_I} \to \mathcal{O}_q(G)_{\leq[\lambda]_I} \otimes \mathcal{O}_q(G)_{\leq[\lambda]_I}$.

Proof. The fact that the subspaces $\mathcal{O}_q(G)_{\leq[\lambda]_I}$ form a filtration follows from general properties of matrix coefficients and from Lemma 2.2.2. For $\lambda \in \Lambda$, the $[\lambda]_I$-th graded piece of the associated graded algebra is given by

$$\frac{\mathcal{O}_q(G)_{\leq[\lambda]_I}}{\sum_{[\mu]\leq[\lambda],[\mu]\neq[\lambda]}\mathcal{O}_q(G)_{\leq[\mu]_I}} = \phi\left(\frac{\bigoplus_{[\mu]\leq[\lambda]}V_\mu^*\otimes V_\mu}{\bigoplus_{[\mu]\leq[\lambda],[\mu]\neq[\lambda]}V_\mu^*\otimes V_\mu}\right) = \phi\left(\bigoplus_{[\mu]=[\lambda]}V_\mu^*\otimes V_\mu\right).$$

The set of $\mu \in \Lambda$ with $[\mu] = [\lambda]$ is precisely $\{\lambda + \nu \mid \nu \in \Lambda_I\}$. The second claim is a consequence of Lemma 2.1.4.3. □

Definition 5.3.3. For $I \subseteq \Delta$, let $\mathrm{gr}_I(\mathcal{O}(G))$ denote the associated graded algebra of $\mathcal{O}(G)$ with the filtration of Definition 3.1.2.

Observe that $\mathcal{O}_q(G)$ and $\mathrm{gr}_I(\mathcal{O}_q(G))$ are isomorphic as $U_q(\mathfrak{g})$-$U_q(\mathfrak{g})$-bimodules.

Example 5.3.4. If $I = \emptyset$, we obtain the full associated graded algebra of $\mathcal{O}_q(G)$ discussed in the previous section. That is, $\mathrm{gr}_\emptyset(\mathcal{O}_q(G)) = \bigoplus_{\lambda\in\Lambda}\phi\left(V_\lambda^*\otimes V_\lambda\right)$. At the other extreme, if $I = \Delta$, then $\mathrm{gr}_\Delta(\mathcal{O}_q(G))$ is isomorphic as an algebra to $\mathcal{O}_q(G)$, and its grading coincides with the grading of $\mathcal{O}_q(G)$ by the finite group $\Lambda/\Lambda_R = \widehat{Z(G)}$:

$$\bigoplus_{[\lambda]\in\Lambda/\Lambda_R}\left(\sum_{\nu\in\Lambda_R}\mathcal{O}_q(G)_{\leq\lambda+\nu}\right).$$

The algebra $\mathrm{gr}_I(\mathcal{O}_q(G))$ is a 'partial' associated graded algebra, and its multiplication map can be described more explicitly as the composition of the ordinary multiplication map

$$\mathcal{O}_q(G)_{\leq\lambda}\otimes\mathcal{O}_q(G)_{\leq\mu}\to\mathcal{O}_q(G)_{\leq\lambda+\mu} = \phi\left(\bigoplus_{\nu\leq\lambda+\mu}V_\nu^*\otimes V_\nu\right)$$

with the projection onto the partial sum of the images of those $V_\nu\otimes V_\nu^*$ such that $\lambda+\mu-\nu$ is a linear combination of the α_i with $i \in I$.

Denote by e_I the point of $\mathbb{A}$ defined by the maximal ideal generated by $\{z^{\alpha_i}-1 \mid i \in I\}$ and $\{z^{\alpha_i} \mid i \notin I\}$. The following result follows from the discussion of this section:

Proposition 5.3.5. *The fiber of* $\mathcal{O}_q(\mathbb{V}_G)$ *over* e_I *is the associated graded algebra* $\mathrm{gr}_I(\mathcal{O}_q(G))$.

Definition 5.3.6. For $I \subseteq \Delta$, the category of quasicoherent sheaves on the quantum Orb_I is given by

$$\mathsf{QCoh}_q\left(\mathrm{Orb}_I\right) = \mathsf{Proj}\left(\mathrm{gr}_I(\mathcal{O}_q(G))\right).$$

5.4 The quantum orbits

This section is devoted to a different description of $\mathrm{gr}_I(\mathcal{O}_q(G))$, and hence of category of sheaves on the quantum orbits. Fix a subset $I \subseteq \Delta$. Throughout, we abbreviate $\mathfrak{l}_I$ by $\mathfrak{l}$ and $\mathfrak{u}_I$ by $\mathfrak{u}$. The map

$$U_q(\mathfrak{u} \times \mathfrak{l} \times \mathfrak{u}^-) = U_q(\mathfrak{u}) \otimes U_q(\mathfrak{l}) \otimes U_q(\mathfrak{u}^-) \to U_q(\mathfrak{g} \times \mathfrak{g})$$

$$x \otimes y \otimes z \to xy \otimes yz$$

is an injective morphism of algebras, and we henceforth identify $U_q(\mathfrak{u}\times\mathfrak{l}\times\mathfrak{u}^-)$ with its image in $U_q(\mathfrak{g} \times \mathfrak{g})$. We consider the action of $U_q(\mathfrak{g} \times \mathfrak{g}) = U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ on $\mathcal{O}_q(G \times G)$ given by

$$(x_1 \otimes x_2) \triangleright \phi(y_1, y_2) = \phi(y_1 x_1, S(x_2) y_2)$$

In particular, the action of $x_1 \otimes x_2$ takes a matrix coefficient $c_{f,v} \otimes c_{g,w}$ to the matrix coefficient $c_{f,x_1 \cdot v} \otimes c_{x_2 \cdot g,w}$. Consider the restriction of this action to the subalgebra $U_q(\mathfrak{u} \times \mathfrak{l} \times \mathfrak{u}^-)$ and the space of invariants $\mathcal{O}_q(G \times G)^{U_q(\mathfrak{u}\times\mathfrak{l}\times\mathfrak{u}^-)}$. This space of invariants carries an action of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ given by $(x_1 \otimes x_2) \triangleright \phi(y_1, y_2) = \phi(S(x_1) y_1, y_2 x_2)$.

Theorem 5.4.1. *For any* $I \subseteq \Delta$, *there is an isomorphism of algebras* $\mathrm{gr}_I(\mathcal{O}_q(G))) = \mathcal{O}_q(G \times G)^{U_q(\mathfrak{u}\times\mathfrak{l}\times\mathfrak{u}^-)}$. *Moreover, this isomorphism is* $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$*-equivariant.*

Remark 5.4.2. Bezrukavnikov and Kazhdan observe the classical version of the result of Theorem 5.4.1 in [BeKa, Remark 2.9].

The proof of this theorem requires some set-up.

Definition 5.4.3. Let V be an irreducible representation of $U_q(\mathfrak{g})$ with highest weight vector v_0. Let V_I denote the $U_q(\mathfrak{l})$-subrepresentation of V generated by v_0, that is, $V_I = \mathcal{U}_q(\mathfrak{l}) \cdot v_0$.

Lemma 5.4.4. *We collect the following facts:*

1. *The space V_I is the sum of weight subspaces of V with weights that differ from the highest weight by Λ_I.*

2. *The subspace V_I of V coincides with the $U_q(\mathfrak{u}_I)$-invariants: $V_I = V^{U_q(\mathfrak{u}_I)}$. Consequently, V_I is $U_q(\mathfrak{p}_I)$-stable.*

3. *The dual $(V_I)^*$ can be identified with the $U_q(\mathfrak{u}_I^-)$-invariants in V^*: $(V_I)^* = (V^*)^{U_q(\mathfrak{u}_I^-)}$*

4. *As representations of $U_q(\mathfrak{l})$, $(V_\lambda)_I$ and $(V_\mu)_I$ are isomorphic if and only if $\lambda = \mu$.*

Proof. By definition, the space V_I is obtained from v_0 by successive application of the operators F_i for $i \in I$. Thus, the weights spaces that appear in V_I are precisely those whose weights differ from the highest weight by elements of Λ_I, and the first claim is established. For the second claim, it is straightforward to show using the PBW theorem for $U_q(\mathfrak{g})$ that $V_I \subseteq V^{U_q(\mathfrak{u}_I)}$. The opposite inclusion follows from the description of V_I given in the first statement.

For the third claim, recall the fact that, if v_0 is a highest weight vector in V, then the dual vector v_0^* is a lowest weight vector in V^*. Now, $V_I^* = (\mathcal{U}_q(\mathfrak{l})v_0)^* = \mathcal{U}_q(\mathfrak{l})v_0^* = (V^*)^{U_q(\mathfrak{u}_I^-)}$.

Another way to prove the third claim is to use the second claim and match weight subspaces. For the last claim, one considers the action of the K_i in $U_q(\mathfrak{l})$ and the action of the quantized enveloping algebra $U_q(\mathfrak{l}/\mathfrak{z}(\mathfrak{l}))$ of the semisimple Lie algebra $\mathfrak{l}/\mathfrak{z}(\mathfrak{l})$, where $\mathfrak{z}(\mathfrak{l})$ denotes the center of $\mathfrak{l}$. $\square$

Remark 5.4.5. See Section 2.5 of [EJ] for analogous results in the classical case.

Write $V_\lambda \otimes V_\mu = \bigoplus_{\rho \leq \lambda+\mu} V_\rho^{\oplus N_{\lambda\mu}^\rho}$ for the decomposition of the tensor product $V_\lambda \otimes V_\mu$ into irreducibles. Note that $N_{\lambda\mu}^\rho$ is the dimension of $\mathrm{Hom}_{U_q(\mathfrak{g})}(V_\rho, V_\lambda \otimes V_\mu)$.

Lemma 5.4.6. *The $U_q(\mathfrak{g})$-subrepresentation of $V_\lambda \otimes V_\mu$ generated by $(V_\lambda)_I \otimes (V_\mu)_I$ coincides with the sum of $V_\rho^{\oplus N_{\lambda\mu}^\rho}$ for weights ρ that differ from $\lambda+\mu$ by Λ_I.*

Proof. Let W denote the $U_q(\mathfrak{g})$-subrepresentation of $V_\lambda \otimes V_\mu$ generated by $(V_\lambda)_I \otimes (V_\mu)_I$. Since $(V_\lambda)_I \otimes (V_\mu)_I$ is $U_q(\mathfrak{p}_I)$-stable, any element in W lying outside of $(V_\lambda)_I \otimes (V_\mu)_I$ is obtained by applying the action of F_i for $i \in \Delta \setminus I$ to elements in $(V_\lambda)_I \otimes (V_\mu)_I$. Hence, the resulting elements have lower weights than those in $(V_\lambda)_I \otimes (V_\mu)_I$. It follows that all highest weight vectors of W are be contained in $(V_\lambda)_I \otimes (V_\mu)_I$. Consequently, W is contained in the sum of $V_\rho^{\oplus N_{\lambda\mu}^\rho}$ for weights ρ that differ from $\lambda+\mu$ by Λ_I.

Next, we argue that the opposite inclusion holds. The sum $\bigoplus V_\rho^{\oplus N_{\lambda,\mu}^\rho}$ over ρ such that $\lambda+\mu-\rho \in \Lambda_I$ is the union of the images of all $U_q(\mathfrak{g})$-equivariant homomorphisms $V_\rho \to V_\lambda \otimes V_\mu$. Thus, it is enough to consider an arbitrary such nonzero homomorphism f and prove that its image lies in W. Let $v_0 \in V_\rho$ be a highest weight vector. Then $f(v_0) = \sum_j v_j \otimes w_j$ for some $v_j \in V_\lambda$ and $w_j \in V_\mu$. Without loss of generality, we can assume that v_j and w_j are weight vectors of weights $\mathrm{wt}(v_j)$ and $\mathrm{wt}(w_j)$. Since v_0 has weight ρ, we have that $\mathrm{wt}(v_j) + \mathrm{wt}(w_j) = \rho$ for all j.

We claim that $v_j \in (V_\lambda)_I$ and $w_j \in (V_\mu)_I$ for all j. To see this, observe that we can write $\lambda - \mathrm{wt}(v_j) = \sum_{i\in I} n_i\alpha_i$ and $\mu - \mathrm{wt}(w_j) = \sum_{i\in I} m_i\alpha_i$ for some $n_i, m_i \geq 0$. Now,

$$\sum_{i\in I}(n_i + m_i)\alpha_i = \lambda + \mu - (\mathrm{wt}(v_j) + \mathrm{wt}(w_j)) = \lambda + \mu - \rho \in \Lambda_I$$

Therefore, $n_i = m_i = 0$ for $i \notin I$, and so $v_j \in (V_\lambda)_I$ and $w_j \in (V_\mu)_I$ for all j.

Finally, the highest weight vector v_0 generates V_ρ, so the image of f will be contained in the subrepresentation of $V_\lambda \otimes V_\mu$ generated by $f(v_0)$. Since $f(v_0) \in (V_\lambda)_I \otimes (V_\mu)_I$, the image of f is contained in W. □

Proof of Theorem 5.4.1. The invariants in $\mathcal{O}_q(G)$ for the right (resp. left) action of $U_q(\mathfrak{u}_I)$ (resp. $U_q(\mathfrak{u}^-)$) can be expressed as:

$$\mathcal{O}_q(G)^{U_q(\mathfrak{u}_I)} = \bigoplus_{\lambda\in\Lambda^+} V_\lambda^* \otimes V_\lambda^{U_q(\mathfrak{u}_I)} = \bigoplus_{\lambda\in\Lambda^+} V_\lambda^* \otimes (V_\lambda)_I.$$

$${}^{U_q(\mathfrak{u}_I^-)}\mathcal{O}_q(G) = \bigoplus_{\lambda\in\Lambda^+} (V_\lambda^*)^{U_q(\mathfrak{u}_I^-)} \otimes V_\lambda = \bigoplus_{\lambda\in\Lambda^+} (V_\lambda)_I^* \otimes V_\lambda.$$

The residual right action of $U_q(\mathfrak{l})$ on $\mathcal{O}_q(G)^{U_q(\mathfrak{u}_I)}$ is on the second factor, and the residual left action of $U_q(\mathfrak{l})$ on ${}^{U_q(\mathfrak{u}_I^-)}\mathcal{O}_q(G)$ is on the first factor. Therefore, we have the following isomorphisms of $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$-modules:

$$\begin{aligned}
\mathcal{O}_q(G\times G)^{U_q(\mathfrak{u}\times\mathfrak{l}\times\mathfrak{u}^-)} &= \left[\mathcal{O}_q(G)^{U_q(\mathfrak{u}_I)} \otimes \mathcal{O}_q(G)^{U_q(\mathfrak{u}_I^-)}\right]^{U_q(\mathfrak{l})} \\
&= \left(\left[\bigoplus_{\lambda\in\Lambda^+} V_\lambda^* \otimes (V_\lambda)_I\right] \otimes \left[\bigoplus_{\mu\in\Lambda^+} (V_\mu)_I^* \otimes V_\mu\right]\right)^{U_q(\mathfrak{l})} \\
&= \bigoplus_{\lambda,\mu\in\Lambda^+} V_\lambda^* \otimes [(V_\lambda)_I \otimes (V_\mu)_I^*]^{\mathfrak{l}} \otimes V_\mu \\
&= \bigoplus_{\lambda\in\Lambda^+} V_\lambda^* \otimes [(V_\lambda)_I \otimes (V_\lambda)_I^*]^{U_q(\mathfrak{l})} \otimes V_\lambda
\end{aligned}$$

The last step follows from Lemma 5.4.4.4. The space $[(V_\lambda)_I \otimes (V_\lambda)_I^*]^{U_q(\mathfrak{l})}$ is one-dimensional.

Define a homomorphism

$$\Phi : \mathrm{gr}_I(\mathcal{O}_q(G)) \to \mathcal{O}_q(G \times G)^{U_q(\mathfrak{u} \times \mathfrak{l} \times \mathfrak{u}^-)}$$

$$c^{V_\lambda}_{f,v} \mapsto c^{V_\lambda}_{f,e_i} \otimes c^{V_\lambda}_{e^i,v},$$

where $v \in V_\lambda$, $f \in V_\lambda^*$, and $\{e_i\}$ and $\{e^i\}$ are dual bases of $(V_\lambda)_I$ and $(V_\lambda)_I^*$. (Here we adopt Einstein notation for summing over the index i.) Observe that, since V_λ is an irreducible representation of $U_q(\mathfrak{l})$, the space $[(V_\lambda)_I \times (V_\lambda)_I^*]^{U_q(\mathfrak{l})}$ is the span of $e_i \otimes e^i$. Hence, Φ is well-defined and does not depend on the choice of basis and dual basis. It is immediate that Φ is an isomorphism of Λ^+-graded $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$-modules.

We show that Φ is a homomorphism of algebras. To this end, recall from the proof of Lemma 5.4.6 the $U_q(\mathfrak{g})$-submodule $W \subseteq V_\lambda \otimes V_\mu$, defined as the $U_q(\mathfrak{g})$-submodule generated by the subspace $(V_\lambda)_I \otimes (V_\mu)_I$. (By Lemma 5.4.6, W coincides with the sum of $V_\rho^{\oplus N^\rho_{\lambda,\mu}}$ such that $\lambda + \mu - \rho \in \Lambda_I$.) Let $\mathrm{pr}_W : V_\lambda \otimes V_\mu \to W$ and $\mathrm{pr}_W^* : (V_\lambda \otimes V_\mu)^* \to W^*$ be the projections, which are $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$-equivariant maps. The algebra structure on $\mathrm{gr}_I(\mathcal{O}_q(G))$ can be described as

$$c^{V_\lambda}_{f,v} \cdot c^{V_\mu}_{g,z} = c^W_{\mathrm{pr}_W^*(f \otimes g), \mathrm{pr}_W(v \otimes z)}.$$

(See Section 5.3 above.) Let $\lambda, \mu \in \Lambda^+$, $v \in V_\lambda$, $f \in V_\lambda^*$, $z \in V_\mu$, $g \in V_\mu^*$. Further, let $\{e_i\}$ and $\{e_i^*\}$ be dual bases of $(V_\lambda)_I$ and $(V_\lambda)_I^*$, and let $\{\epsilon_i\}$ and $\{\epsilon_i^*\}$ be dual bases of $(V_\mu)_I$ and $(V_\mu)_I^*$. Then

$$\begin{aligned}\Phi\left(c^{V_\lambda}_{f,v} \cdot c^{V_\mu}_{g,z}\right) &= c^W_{\mathrm{pr}_W^*(f \otimes g), e_i \otimes \epsilon_i} \otimes c^W_{e^i \otimes \epsilon^j, \mathrm{pr}_W(v \otimes z)} = c^{V_\lambda \otimes V_\mu}_{f \otimes g, e_i \otimes \epsilon_i} \otimes c^{V_\lambda \otimes V_\mu}_{e^i \otimes \epsilon^j, v \otimes z} \\ &= \left(c^{V_\lambda}_{f,e_i} \otimes c^{V_\lambda}_{e^i,v}\right) \cdot \left(c^{V_\mu}_{g,\epsilon_i} \otimes c^{V_\mu}_{\epsilon^j,z}\right) = \Phi\left(c^{V_\lambda}_{f,v}\right) \cdot \Phi\left(c^{V_\mu}_{g,z}\right),\end{aligned}$$

where the multiplication in the last two expressions occurs in $\mathcal{O}_q(G \times G)^{U_q(\mathfrak{u}\times\mathfrak{l}\times\mathfrak{u}^-)}$ as a subalgebra of $\mathcal{O}_q(G \times G)$. □

Example 5.4.7. In the extreme cases we have $\mathrm{gr}_\emptyset(\mathcal{O}_q(G)) = \mathcal{O}_q\left(\frac{G/N\times N^-\backslash G}{T}\right)$ and $\mathrm{gr}_\Delta(\mathcal{O}_q(G)) = \mathcal{O}_q(G)$. The former is the quantized coordinate algebra of the asymptotic semigroup AsG of G; the asymptotic semigroup is considered in [P, Vi2].

5.5 Quantum differential operators

In this section, we fix $\kappa : U_q(\mathfrak{g}) \times \mathcal{O}_q(G) \to \mathbb{C}$ to be the perfect pairing given by evaluation and consider the right action of $U_q(\mathfrak{g})$ on $\mathcal{O}_q(G)$.

Lemma 5.5.1. *The action of $U_q(\mathfrak{g})$ on $\mathcal{O}_q(G)$ preserves the filtrations $\mathcal{O}_q(G)_{\leq[\lambda]_I}$, and hence descends to a well-defined action on $\mathrm{gr}_I(\mathcal{O}_q(G))$.*

Proof. Suppose $f \in \mathcal{O}_q(G)_{\leq[\lambda]_I}$. Then $f_{(1)} \in \mathcal{O}_q(G)_{\leq[\lambda]_I}$ by Proposition 5.1.3. Hence

$$x \triangleright f = f_{(1)}\kappa(h, f_{(2)}) \in \mathcal{O}_q(G)_{\leq[\lambda]_I}$$

for any $x \in U_q(\mathfrak{g})$. □

Definition 5.5.2. [BaKr, VV, Jo1] The algebra of quantum differential operators on G is defined as the smash product $\mathcal{O}_q(G) \star U_q(\mathfrak{g})$. Explicitly,

$$(a \otimes u)(b \otimes v) = a(u_{(1)} \triangleright b) \otimes u_{(2)}v = ab_{(1)}\kappa(u_{(1)}, b_{(2)}) \otimes u_{(2)}v.$$

For $\lambda \in \Lambda$ and a subset $I \subseteq \Delta$, define the following subspace of $\mathcal{D}_q(G)$:

$$\mathcal{D}_q(G)_{\leq[\lambda]_I} = \mathcal{O}_q(G)_{\leq[\lambda]_I} \otimes U_q(\mathfrak{g}).$$

If $I = \emptyset$, we write simply $\mathcal{D}_q(G)_{\leq\lambda}$ for $\mathcal{D}_q(G)_{\leq[\lambda]_\emptyset}$.

Remark 5.5.3. Jordan [Jo1] defines $\mathcal{D}_q(G)$ in terms of the reflection equation algebra (the ad-equivariant quantization of $\mathcal{O}(G)$). There is also a perspective involving Heisenberg doubles.

Proposition 5.5.4. *The subspaces $\mathcal{D}_q(G)_{\leq[\lambda]_I}$ define a filtration on $\mathcal{D}_q(G)$ by Λ/Λ_I. The associated graded algebra $\mathrm{gr}_I(\mathcal{D}_q(G))$ is identified with $\mathrm{gr}_I(\mathcal{O}_q(G)) \star U_q(\mathfrak{g})$.*

Proof. Fix $\lambda, \mu \in \Lambda$. Let e_i and e_i^* be dual bases for V_μ and V_μ^*. For $v \in V_\lambda$, $f \in V_\lambda^*$, $w \in V_\mu$, $g \in V_\mu^*$, and $x, y \in U_q(\mathfrak{g})$, we use Lemma 2.1.4.3 to compute

$$(c^{V_\lambda}_{f,v} \otimes x)(c^{V_\mu}_{g,w} \otimes y) = \sum_i c^{V_\lambda \otimes V_\mu}_{f\otimes g, v \otimes e_i} \kappa(x_{(1)}, c^{V_\mu}_{e_i^*, w}) \otimes x_{(2)} y$$

Observe that $c^{V_\lambda \otimes V_\mu}_{f\otimes g, v\otimes w} \in \mathcal{O}_q(G)_{\leq[\lambda+\mu]_I}$ and $\kappa(x_{(1)}, c^{V_\mu}_{e_i^*, w}) = e_i^*(x_{(1)} \triangleright w)$ is a scalar. It follows that the subspaces $\mathcal{D}_q(G)_{\leq[\lambda]_I}$ form a filtration. □

Example 5.5.5. If $I = \emptyset$, then $\mathrm{gr}_\emptyset(\mathcal{D}_q(G)) = \mathcal{O}_q(G \times G)^{U_q(\mathfrak{n}\times\mathfrak{t}\times\mathfrak{n}^-)} \star U_q(\mathfrak{g})$, and we regard this algebra as the algebra of quantum differential operators on the asymptotic cone $\frac{G/N \times N^- \backslash G}{T}$.

Example 5.5.6. At the other extreme, if $I = \Delta$, then $\mathrm{gr}_\Delta(\mathcal{D}_q(G))$ is isomorphic as an algebra to $\mathcal{D}_q(G)$, and its grading coincides with the grading of $\mathcal{D}_q(G)$ by the finite group $\Lambda/\Lambda_R = \widehat{Z(G)}$:

$$\bigoplus_{[\lambda]\in\Lambda/\Lambda_R} \left(\sum_{\nu \in \Lambda_R} \mathcal{O}_q(G)_{\leq \lambda+\nu} \star U_q(\mathfrak{g}) \right).$$

Consider the action of $U_q(\mathfrak{g})$ on $\mathcal{O}_q(\mathbb{V}_G)$ given by $x \triangleright (f z^n) = (x \triangleright f) z^n$, for $f \in \mathcal{O}_q(G)_{\leq n}$, where we use the action of $U_q(\mathfrak{g})$ on $\mathcal{O}_q(G)$ considered in Section 2.4.

Proposition 5.5.7. *There is an isomorphism of algebras* $\mathrm{Rees}(\mathcal{D}_q(G)) \xrightarrow{\sim} \mathcal{O}_q(\mathbb{V}_G) \star U_q(\mathfrak{g})$.

Proof. The map $\phi : \mathrm{Rees}(\mathcal{D}_q(G)) \to \mathcal{O}_q(\mathbb{V}_G) \star U_q(\mathfrak{g})$ given by $\phi((f \otimes x)z^\lambda) = (fz^\lambda) \otimes x$ for $f \in \mathcal{O}_q(G)_{\leq \lambda}$ is clearly a bijection. The fact that it is an algebra homomorphism is a computation:

$$\begin{aligned}\phi((f \otimes x)z^\lambda)\phi((g \otimes y)z^\mu) &= ((fz^\lambda) \otimes x)((gz^\mu) \otimes y)) = f(x_{(1)} \triangleright g)z^{\lambda+\mu} \otimes x_{(2)}y \\ &= \phi((f(x_{(1)} \triangleright g) \otimes x_{(2)}y)z^{\lambda+\mu}) = \phi(((f \otimes x)z^\lambda)((g \otimes y)z^\mu)).\end{aligned}$$

□

Remark 5.5.8. An possible definition for the algebra of differential operators on the Vinberg semigroup is as the smash product of $\mathcal{O}_q(\mathbb{V}_G)$ with $U_q(\mathfrak{g} \times \mathfrak{t})$. This smash product would contain the Rees algebra $\mathrm{Rees}(\mathcal{D}_q(G))$ as a proper subalgebra.

Definition 5.5.9 (Tentative)**.** Fix $I \subseteq \Delta$. The algebra $\mathcal{D}_q\left(\frac{G/U_I \times U_I^- \backslash G}{L}\right)$ of quantum differential operators on the orbit Orb_I is defined as the Λ/Λ_I-graded algebra $\mathrm{gr}_I(\mathcal{D}_q(G))$.

Let $\mathcal{D}_q(G)\text{-}\mathsf{mod}_I^{\mathrm{filt}}$ denote the category of $\mathcal{D}_q(G)$-modules that carry a filtration by Λ/Λ_I compatible with the corresponding filtration on $\mathcal{D}_q(G)$. Taking associated graded gives a functor from this category to the category of graded modules for the algebra $\mathcal{D}_q\left(\frac{G/U_I \times U_I^- \backslash G}{L}\right)$:

$$\mathcal{D}_q(G)\text{-}\mathsf{mod}_I^{\mathrm{filt}} \to \mathcal{D}_q\left(\frac{G/U_I \times U_I^- \setminus G}{L}\right)\text{-}\mathsf{grmod}.$$

This functor is a quantum version of the nearby cycles or Verdier specialization functor on the wonderful compactification.

Chapter 6

The case of SL_2

In this chapter, we describe explicitly the constructions of Chapter 5 in the case when $G = \mathrm{SL}_2$. The definitions of the Hopf algebras $\mathcal{O}_q(\mathrm{SL}_2)$ and $U_q(\mathfrak{sl}_2)$ appear in Section 6.1, followed by a description of the Peter-Weyl filtration for $\mathcal{O}_q(\mathrm{SL}_2)$ in Section 6.2. We state results on the quantum Vinberg semigroup and wonderful compactification for SL_2 in Section 6.3, and discuss the algebra $\mathcal{D}_q(\mathrm{SL}_2)$ of quantum differential operators on SL_2 in Section 6.4.

6.1 The algebras $\mathcal{O}_q(\mathrm{SL}_2)$ and $U_q(\mathfrak{sl}_2)$

Fix $q \in \mathbb{C}^\times$. The following discussion is adapted in part from [BG].

Definition 6.1.1. The quantum 2×2 matrix algebra is the bialgebra $\mathcal{O}_q(\mathrm{Mat}_2)$ generated by elements a, b, c, d with relations

$$ab = qba \qquad ac = qca \qquad bc = cb \qquad bd = qdb$$

$$cd = qdc \qquad ad - da = (q - q^{-1})bc,$$

and with coalgebra structure given by

$$\Delta(a) = a \otimes a + b \otimes c \qquad \Delta(b) = a \otimes b + b \otimes d$$

$$\Delta(c) = c \otimes a + d \otimes c \qquad \Delta(d) = c \otimes b + d \otimes d$$

$$\epsilon(a) = \epsilon(d) = 1 \qquad \epsilon(b) = \epsilon(c) = 0.$$

The quantum determinant is the (central) element $D_q := ad - qbc$ of $\mathcal{O}_q(\mathrm{Mat}_2)$.

Definition 6.1.2. The quantum coordinate algebra $\mathcal{O}_q(\mathrm{SL}_2)$ of SL_2 is the quotient of $\mathcal{O}_q(\mathrm{Mat}_2)$ by the ideal generated by the central element $D_q - 1$, and and quantum coordinate algebra $\mathcal{O}_q(\mathrm{GL}_2)$ of GL_2 is the localization of $\mathcal{O}_q(\mathrm{Mat}_2)$ at the central element D_q:

$$\mathcal{O}_q(\mathrm{SL}_2) = \mathcal{O}_q(\mathrm{Mat}_2)/\langle D_q - 1\rangle \qquad\qquad \mathcal{O}_q(\mathrm{GL}_2) = \mathcal{O}_q(\mathrm{Mat}_2)[D_q^{-1}].$$

We use the same notation for elements of $\mathcal{O}_q(\mathrm{Mat}_2)$ and their images in $\mathcal{O}_q(\mathrm{SL}_2)$ and $\mathcal{O}_q(\mathrm{GL}_2)$.

Lemma 6.1.3. *The bialgebra structure on $\mathcal{O}_q(\mathrm{Mat}_2)$ descends to a bialgebra structure on $\mathcal{O}_q(\mathrm{SL}_2)$ and $\mathcal{O}_q(\mathrm{GL}_2)$. Each of the latter bialgebras is a Hopf algebra with antipode is given by:*

$$S(a) = d \qquad S(b) = -q^{-1}b \qquad S(c) = -qc \qquad S(d) = a.$$

Lemma 6.1.4. *The algebra $\mathcal{O}_q(\mathrm{SL}_2)$ is a quantization of the Sklyanin Poisson bracket on $\mathcal{O}(\mathrm{SL}_2)$, namely,*

$$\{a,b\} = ab \qquad \{a,c\} = ac \qquad \{b,c\} = 0$$

$$\{b,d\} = bd \qquad \{c,d\} = cd \qquad \{a,d\} = 2bc.$$

Definition 6.1.5. The quantized enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is the Hopf algebra with generators $E, F, K^{\pm 1}$ subject to the relations

$$KE = q^2EK \qquad KF = q^{-2}FK \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}},$$

and Hopf structure given by:

$$\Delta(E) = E \otimes 1 + K \otimes E \qquad \Delta(F) = F \otimes K^{-1} + 1 \otimes F \qquad \Delta(K) = K \otimes K$$

$$\epsilon(E) = \epsilon(F) = 0 \qquad \epsilon(K) = 1$$

$$S(E) = -K^{-1}E \qquad S(F) = -FK \qquad S(K) = K^{-1}.$$

The finite dimensional representation theory of $\mathcal{U}_q(\mathfrak{sl}_2)$ is described in detail in [BG, Chapter I.4]; here we give a summary.

Definition 6.1.6. For a non-negative integer $n \geq 0$, define two $(n+1)$-dimensional modules $V(n,+)$ and $V(n,-)$ of $\mathcal{U}_q(\mathfrak{sl}_2)$ as follows. Let $v_0, v_1, \ldots, v_n$ be a basis of $\mathbb{C}^{n+1}$ and set

$$Kv_i = \pm q^{n-2i} v_i, \qquad Fv_i = \begin{cases} [i+1]_q v_{i+1} & \text{if } i < n \\ 0 & \text{if } i = n, \end{cases} \qquad Ev_i = \begin{cases} 0 & \text{if } i = 0 \\ \pm[n+1-i]_q v_{i-1} & \text{if } i > 0. \end{cases}$$

Theorem 6.1.7. *[BG, Theorems I.4.4 and I.4.5] Each of the modules $V(n,\pm)$ is irreducible, and any finite-dimensional irreducible module of $\mathcal{U}_q(\mathfrak{sl}_2)$ is isomorphic to $V(n,\pm)$ for some $n \geq 0$. Moreover, the category of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{sl}_2)$ is semisimple.*

A 'type 1' module for $\mathcal{U}_q(\mathfrak{sl}_2)$ is one whose irreducible constituents are all of the form $V(n,+)$. In what follows, we will consider only type 1 modules and abbreviate $V(n,+)$ by V_n. The full subcategory of $\mathcal{U}_q(\mathfrak{sl}_2)$-modules consisting of type 1 modules has strong parallels to the category of representations of the classical enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$. In particular:

Lemma 6.1.8. *Suppose $n \geq m$. Then the tensor product $V_n \otimes V_m$ of $\mathcal{U}_q(\mathfrak{sl}_2)$-modules decomposes as follows:*

$$V_n \otimes V_m = V_{n+m} \oplus V_{n+m-2} \oplus V_{n+m-4} \oplus \cdots \oplus V_{n-m}.$$

Consequently, if the $\mathcal{U}_q(\mathfrak{sl}_2)$-module V_k appears as an irreducible submodule of the tensor product $V_n \otimes V_m$, then $k \leq n+m$ for the partial order defined above.

6.2 The Peter-Weyl filtration on $\mathcal{O}_q(\mathrm{SL}_2)$

Theorem 6.2.1. *[BG, Theorem I.7.16] The sub-Hopf algebra of $\mathcal{U}_q(\mathfrak{sl}_2)^\circ$ generated by the matrix coefficients of the representations V_n is isomorphic to the Hopf algebra $\mathcal{O}_q(\mathrm{SL}_2)$ defined above.*

Corollary 6.2.2. *There is an isomorphism of $\mathcal{U}_q(\mathfrak{sl}_2) \times \mathcal{U}_q(\mathfrak{sl}_2)$-modules:*

$$\phi : \bigoplus_{n \geq 0} V_n \otimes V_n^* \xrightarrow{\sim} \mathcal{O}_q(SL_2).$$

Definition 6.2.3. Endow $\mathbb{Z}_{\geq 0}$ with the partial order of Definition 2.2.1. Define subspaces of $\mathcal{O}_q(\mathrm{SL}_2)$ by

$$\mathcal{O}_q(\mathrm{SL}_2)_{\leq n} = \phi \left(\bigoplus_{m \leq n} V_m \otimes V_m^* \right).$$

Thus, we have

$$\mathcal{O}_q(\mathrm{SL}_2)_{\leq 0} \subseteq \mathcal{O}_q(\mathrm{SL}_2)_{\leq 2} \subseteq \mathcal{O}_q(\mathrm{SL}_2)_{\leq 4} \subseteq \cdots$$

$$\mathcal{O}_q(\mathrm{SL}_2)_{\leq 1} \subseteq \mathcal{O}_q(\mathrm{SL}_2)_{\leq 3} \subseteq \mathcal{O}_q(\mathrm{SL}_2)_{\leq 5} \subseteq \cdots$$

and no inclusions between the two strings. The following lemmas are straightforward verifications; their proofs are parallel to their classical analogues.

Lemma 6.2.4. *The spaces of Definition 6.2.3 define a filtration on $\mathcal{O}_q(\mathrm{SL}_2)$:*

$$\mu : \mathcal{O}_q(\mathrm{SL}_2)_{\leq n} \otimes \mathcal{O}_q(\mathrm{SL}_2)_{\leq m} \to \mathcal{O}_q(\mathrm{SL}_2)_{\leq n+m}.$$

Lemma 6.2.5. *For $n \geq 0$, consider the subspace of the free algebra $\mathbb{C}\langle a, b, c, d\rangle$ spanned by monomials words of length k where $k \leq n$ and $k \equiv n \mod 2$. The image of this subspace under the quotient map $\mathbb{C}\langle a, b, c, d\rangle \to \mathcal{O}_q(\mathrm{SL}_2)$ is precisely $\mathcal{O}_q(\mathrm{SL}_2)_{\leq n}$.*

Lemma 6.2.6. *The coproduct Δ restricts to a map*

$$\Delta : \mathcal{O}_q(\mathrm{SL}_2)_{\leq n} \to \mathcal{O}_q(\mathrm{SL}_2)_{\leq n} \otimes \mathcal{O}_q(\mathrm{SL}_2)_{\leq n}.$$

Definition 6.2.7. We define the following algebras:

- Let $\mathrm{Sym}_q^k(\mathbb{C}^2)$ denote the kth graded piece of the algebra $\mathbb{C}\langle x, y\rangle/\langle xy - qyx\rangle$, and set

$$\mathbb{P}_q^1 \times \mathbb{P}_q^1 = \bigoplus_{k \geq 0} \mathrm{Sym}_q^k(\mathbb{C}^2) \otimes \mathrm{Sym}_q^k(\mathbb{C}^2).$$

- The associated graded algebra of $\mathcal{O}_q(\mathrm{SL}_2)$ is defined as the $\mathbb{Z}$-graded algebra

$$\mathrm{gr}(\mathcal{O}_q(\mathrm{SL}_2)) = \bigoplus_{n \geq 0} \mathcal{O}_q(\mathrm{SL}_2)_{\leq n}/\mathcal{O}_q(\mathrm{SL}_2)_{<n}.$$

Remark 6.2.8. Setting $q = 1$ in the definition of $\mathbb{P}_q^1 \times \mathbb{P}_q^1$, we obtain the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$. Note that $\mathbb{P}^1$ is the flag variety for SL_2.

Proposition 6.2.9. *The associated graded $\mathrm{gr}(\mathcal{O}_q(\mathrm{SL}_2))$ is isomorphic to $\mathbb{P}_q^1 \times \mathbb{P}_q^1$.*

Proof. Observe that there are isomorphisms

$$\mathrm{gr}(\mathcal{O}_q(\mathrm{SL}_2))) \simeq \mathcal{O}_q(\mathrm{Mat}_2)/(ad - qbc) \simeq \bigoplus_{k \geq 0} \mathrm{Sym}_q^k(\mathbb{C}^2) \otimes \mathrm{Sym}_q^k(\mathbb{C}^2),$$

where the second isomorphism is given by

$$a \mapsto x \otimes u \qquad b \mapsto x \otimes w \qquad c \mapsto y \otimes u \qquad d \mapsto y \otimes w.$$

Here (x, y) and (u, w) denote the coordinates on the first and second copies of $\mathbb{C}^2$, respectively.

□

6.3 The quantum Vinberg semigroup

For any filtered algebra, one associates a so-called Rees algebra to interpolate between the algebra and its associated graded. In this section, we consider the Rees algebra for $\mathcal{O}_q(\mathrm{SL}_2)$.

Definition 6.3.1. The quantum Vinberg semigroup $\mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2})$ for SL_2 is defined as the Rees algebra for $\mathcal{O}_q(\mathrm{SL}_2)$ with the Peter-Weyl filtration. Explicitly, letting z be a formal variable, $\mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2})$ is the following graded subalgebra of $\mathcal{O}_q(\mathrm{SL}_2)[z]$:

$$\mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2}) = \bigoplus_{n\geq 0} \mathcal{O}_q(\mathrm{SL}_2)_{\leq n} z^n.$$

Proposition 6.3.2. *There is a well-defined bialgebra structure on* $\mathcal{O}_q(\mathbb{V}_{SL_2})$ *given by*

$$\Delta : \mathcal{O}_q(\mathrm{SL}_2)_{\leq n} z^n \to \mathcal{O}_q(\mathrm{SL}_2)_{\leq n} z^n \otimes \mathcal{O}_q(\mathrm{SL}_2)_{\leq n} z^n$$

$$\Delta(f z^n) = \Delta_{\mathrm{SL}_2}(f) \cdot (z^n \otimes z^n),$$

and $\epsilon(f z^n) = \epsilon_{\mathrm{SL}_2}(f)$, *where* Δ_{SL_2} *and* ϵ_{SL_2} *denote the coproduct and counit on* $\mathcal{O}_q(\mathrm{SL}_2)$.

Proof. The proof is a routine computation. Note that $\mathcal{O}_q(\mathbb{V}_{SL_2})$ is generated as an algebra by $\mathcal{O}_q(\mathrm{SL}_2)_{\leq 0} = \mathbb{C} \cdot 1$ and $\mathcal{O}_q(\mathrm{SL}_2)_{\leq 1} z = \{az, bz, cz, dz\}$. We set $\Delta(az) = az \otimes az + bz \otimes cz$, $\Delta(bz) = az \otimes bz + bz \otimes dz$, etc. □

Remark 6.3.3. The algebra $\mathcal{O}_q(\mathbb{V}_{SL_2})$ does not have an antipode. If it did, the antipode $S(z)$ of z would satisfy $1 = \epsilon(z) = zS(z)$, but z is not invertible in $\mathcal{O}_q(\mathbb{V}_{SL_2})$.

Proposition 6.3.4. *There is an isomorphism of bialgebras* $\mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2}) \simeq \mathcal{O}_q(\mathrm{Mat}_2)$.

Proof. The proof is analogous to the proof in the classical case. By Lemma 6.2.5, an element of $\mathcal{O}_q(SL_2)_{\leq n}$ can be represented by a word in a, b, c, d of length k where $k \leq n$ and $k \equiv n \mod 2$. Similarly, an element of $\mathcal{O}_q(SL_2)_{\leq n} z^n$ can be represented by such a word together with a factor of z^n. Using the commutation relations between the generators of $\mathcal{O}_q(\mathrm{SL}_2)$ and the relation relation $z^2 = (az)(dz) - q(bz)(cz)$, one sees that such a word lies in the span of the words $(az)^{k_1'}(bz)^{k_2'}(cz)^{k_3'}(dz)^{k_4'}$ with $k_1' + k_2' + k_3' + k_4' \leq n$. Hence, $\mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2})$ is generated by the elements az, bz, cz, dz. The relations and coproduct on $\mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2})$ coincide with those of $\mathcal{O}_q(\mathrm{Mat}_2)$. □

Observe that $\mathcal{O}_q(\mathbb{V}_{SL_2})$ contains z^2, but does not contain z.

Corollary 6.3.5. *There are isomorphisms of algebras*

$$(\mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2})[(z^2)^{-1}])^{\mathbb{C}^\times} \simeq \mathcal{O}_q(\mathrm{SL}_2) \qquad \textit{and} \qquad \mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2})/(z^2) \simeq \mathbb{P}^1_q \times \mathbb{P}^1_q.$$

Proof. Under the isomorphism of Proposition 6.3.4, the element $z^2 \in \mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2})$ corresponds to the element $D_q \in \mathcal{O}_q(\mathrm{Mat}_2)$. Thus,

$$(\mathcal{O}_q(\mathbb{V}_{\mathrm{SL}_2})[(z^2)^{-1}])^{\mathbb{C}^\times} = (\mathcal{O}_q(\mathrm{Mat}_2)[D_q^{-1}])^{\mathbb{C}^\times} = \mathcal{O}_q(\mathrm{GL}_2)^{\mathbb{C}^\times} = \mathcal{O}_q(\mathrm{SL}_2).$$

It is straightforward to show that $\mathcal{O}_q(\mathbb{V}_{SL_2})/(z^2) = \mathrm{gr}(\mathcal{O}_q(\mathrm{SL}_2))$, and hence the second isomorphism is a consequence of Proposition 6.2.9. □

6.4 Quantum differential operators

There is a Hopf pairing $\kappa : \mathcal{O}_q(\mathrm{SL}_2) \times \mathcal{U}_q(\mathfrak{sl}_2) \to \mathbb{C}$ given by evaluation of matrix coefficients[1]. The table below lists the values of κ on the generators of $\mathcal{O}_q(\mathrm{SL}_2)$ and $\mathcal{U}_q(\mathfrak{sl}_2)$;

[1]See [BG, Section I.9.22] for the definition of a Hopf pairing.

these are obtained by considering the defining representation $V(1,+)$ of $\mathcal{U}_q(\mathfrak{sl}_2)$.

	E	F	K	K^{-1}
a	0	0	q	q^{-1}
b	1	0	0	0
c	0	1	0	0
d	0	0	q^{-1}	q

The pairing κ induces an action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on $\mathcal{O}_q(\mathrm{SL}_2)$ given by

$$x \triangleright f = f_{(1)}\kappa(f_{(2)}, x)$$

for $x \in \mathcal{U}_q(\mathfrak{sl}_2)$ and $f \in \mathcal{O}_q(\mathrm{SL}_2)$. Explicitly, we have

$$\begin{aligned} E \triangleright a &= 0 & F \triangleright a &= b & K \triangleright a &= qa \\ E \triangleright b &= a & F \triangleright b &= 0 & K \triangleright b &= q^{-1}b \\ E \triangleright c &= 0 & F \triangleright c &= d & K \triangleright c &= qc \\ E \triangleright d &= c & F \triangleright d &= 0 & K \triangleright d &= q^{-1}d. \end{aligned}$$

More precisely, these formulas define an action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on $\mathcal{O}_q(\mathrm{Mat}_2)$ that descends to $\mathcal{O}_q(\mathrm{SL}_2)$. The action of $\mathcal{U}_q(\mathfrak{sl}_2)$ preserves the filtration $\mathcal{O}_q(\mathrm{SL}_2)_{\leq n}$ and hence descends to a well-defined action on $\mathbb{P}^1_q \times \mathbb{P}^1_q$ that respects the grading.

Definition 6.4.1. [BaKr, VV, Jo1] The algebra of quantum differential operators on SL_2 is defined as the smash product $\mathcal{D}_q(\mathrm{SL}_2) = \mathcal{O}_q(\mathrm{SL}_2) \star \mathcal{U}_q(\mathfrak{sl}_2)$.

Explicitly, $\mathcal{D}_q(\mathrm{SL}_2)$ is generated by a, b, c, d and $E, F, K^{\pm 1}$ with relations as for $\mathcal{O}_q(\mathrm{SL}_2)$ and $\mathcal{U}_q(\mathfrak{sl}_2)$, and with cross relations given by

$$uf = (u_{(1)} \triangleright f)u_{(2)}$$

for $f \in \mathcal{O}_q(\mathrm{SL}_2)$ and $u \in \mathcal{U}_q(\mathfrak{sl}_2)$. The cross relations reduce to the following expressions:

$$
\begin{aligned}
&Ea = qaE && Fa = aF + bK^{-1} && Ka = qaK \\
&Eb = q^{-1}bE + a && Fb = bF && Kb = q^{-1}bK \\
&Ec = qcE && Fc = cF + dK^{-1} && Kc = qcK \\
&Ed = qdE + c && Fd = dF && Kd = q^{-1}dK
\end{aligned}
$$

Definition 6.4.2. For $n \geq 0$, define the following subspace of $\mathcal{D}_q(\mathrm{SL}_2)$:

$$\mathcal{D}_q(\mathrm{SL}_2)_{\leq n} = \mathcal{O}_q(\mathrm{SL}_2)_{\leq n} \otimes \mathcal{U}_q(\mathfrak{sl}_2).$$

Lemma 6.4.3. *The subspaces $\mathcal{D}_q(\mathrm{SL}_2)_{\leq n}$ define a filtration on $\mathcal{D}_q(\mathrm{SL}_2)$. The associated graded is $(\mathbb{P}^1_q \times \mathbb{P}^1_q) \star \mathcal{U}_q(\mathfrak{sl}_2)$.*

Proof. If $g \in \mathcal{O}_q(\mathrm{SL}_2)_{\leq m}$, then we have observed that $g_{(1)}$ and $g_{(2)}$ are also in $\mathcal{O}_q(\mathrm{SL}_2)_{\leq m}$. Therefore, for $f \in \mathcal{O}_q(\mathrm{SL}_2)_{\leq n}$, the product $fg_{(1)}$ is an element of $\mathcal{O}_q(\mathrm{SL}_2)_{\leq n+m}$. The lemma now follows by inspection of the cross relations in $\mathcal{D}_q(\mathrm{SL}_2)$, noting that $\kappa(u_{(1)}, g_{(2)})$ is a scalar. $\square$

Recall that the action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on $\mathcal{O}_q(\mathrm{SL}_2)$ considered in the previous section arises from an action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on $\mathcal{O}_q(\mathrm{Mat}_2)$.

Proposition 6.4.4. *There is an isomorphism $\mathrm{Rees}(\mathcal{D}_q(\mathrm{SL}_2)) \xrightarrow{\sim} \mathcal{O}_q(\mathrm{Mat}_2) \star \mathcal{U}_q(\mathfrak{sl}_2)$.*

Proof. The map $\phi : \mathrm{Rees}(\mathcal{D}_q(\mathrm{SL}_2)) \to \mathcal{O}_q(\mathbb{V}_{SL_2}) \star \mathcal{U}_q(\mathfrak{sl}_2)$ given by $\phi((f \otimes x)z^n) = (fz^n) \otimes x$ for $f \in \mathcal{O}_q(\mathrm{SL}_2)_{\leq n}$ is clearly a bijection. The fact that it is an algebra homomorphism is a computation:

$$\phi((f \otimes x)z^n)\phi((g \otimes y)z^m) = ((fz^n) \otimes x)((gz^m) \otimes y)) = f(x_{(1)} \triangleright g)z^{n+m} \otimes x_{(2)}y$$
$$= \phi((f(x_{(1)} \triangleright g) \otimes x_{(2)}y)z^{n+m}) = \phi(((f \otimes x)z^n)((g \otimes y)z^m)).$$

The claim now follows from Proposition 6.3.4. □

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