NN117

Problem

Consider the differential equations

$$\frac{dx_1}{dt} = - x_2 + a_0 x_1^3 + a_1 x_1^2 x_2 + a_2 x_1 x_2^2 + a_3 x_2^3$$

$$\frac{dx_2}{dt} = x_1 + b_0 x_2^3 + b_1 x_2^2 x_1 + b_2 x_2 x_1^2 + b_3 x_1^3$$

Show that the solution x

1

$= x$

2

$= 0$ is asymptotically stable (i.e., there exists a $\delta > 0$ such that for every solution $\underline{x} = \underline{x}(t)$ with $|\underline{x}(0)| < \delta$ we have

lim

$t\to\infty$

$\underline{x}(t) = \underline{o}$)

if $3a$

0

$+ 3b$

0

$+ a$

2

+ b

2

< 0

and completely unstable (i.e., there exists a $\delta > 0$ such that for every solution $\underline{x} = \underline{x}(t)$ with $\underline{x}(0) \neq \underline{o}$ there exists a $T \geq 0$ such that $|\underline{x}(t)| \geq \delta$ for $t \geq T$)

if 3a

0

+ 3b

0

+ a

2

+ b

2

> 0.

G.W. Veltkamp.

Solution

Consider the function

$$V(\underline{x}) = x_1^2 + x_2^2 + \frac{1}{2}(a_1 + b_3)x_1^4 - \frac{1}{2}(a_3 + b_1)x_2^4 - (a_0 - b_0)(x_1^3 x_2 + x_1 x_2^3) + \frac{1}{3}(a_2 + b_2)(x_1^3 x_2 - x_1 x_2^3).$$

Then along a trajectory

$$\dot{V}(t) = \frac{d}{dt} V(\underline{x}(t)) = \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} = [a_0 + b_0 + \frac{1}{3}(a_2 + b_2)](x_1^4 + x_2^4) + \text{terms of degree 6.}$$

It is obvious that V is a Lyapunov-function, i.e., V is continuously differentiable, $V(\underline{o}) = 0$, $V(\underline{x}) > 0$ in an open neighbourhood $\Omega(0 < |x| < \delta)$ of $\underline{x} = \underline{o}$ and

$\dot{V}(t) \leqq 0$ for $\underline{x}(t) \in \Omega$ if $a_0 + b_0 + \frac{1}{3}(a_2 + b_2) \leqq 0$.

The statements concerning stability and instability now follow from well-known theorems of Lyapunov (cf., e.g., La Salle and Lefschetz, *Stability by Lyapunov's Direct Method*, N.Y., 1961).

Note. The basic idea of the proof of Lyapunov's theorem is the following.

Let $0 < \delta_1$

< δ

2

, with sufficiently small δ

2

. Then the inequalities

δ

1

≤ $V(\underline{x})$ ≤ δ

2

define a closed annular subdomain Ω

1

of Ω which contains the origin inside its inner boundary.

If

.

V

< 0 for $\underline{x}(t) \in \Omega$ then

.

V

≤ -ε < 0 as long as $\underline{x}(t) \in \Omega$

1

.

Hence if $\underline{x}(0) \in \Omega$

1

then certainly $V(\underline{x}(t)) < \delta$

1

for $t > (a$

2

$-\delta$

1

$)/\varepsilon$.

Hence

lim

$t\to\infty$

$V(\underline{x}(a)) = 0$, which implies

lim

$t\to\infty$

$\underline{x}(t) = \underline{o}$. And similarly in the case of instability.

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transcribed by Corrado Cantelmi

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