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A Fragmentation Model for Sprays and L^2 Stability Estimates for Shocks Solutions of Scalar Conservation Laws Using the Relative Entropy Method

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Dedicated to my parents.

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A Fragmentation Model for Sprays and L^2 Stability Estimates for Shocks Solutions of Scalar Conservation Laws Using the Relative Entropy Method

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We present a mathematical study of two conservative systems in fluid mechanics. First, we study a fragmentation model for sprays. The model takes into account the break-up of spray droplets due to drag forces. In particular, we establish the existence of global weak solutions to a system of incompressible Navier-Stokes equations coupled with a Boltzmann-like kinetic equation. We assume the particles initially have bounded radii and bounded velocities relative to the gas, and we show that those bounds remain as the system evolves. One interesting feature of the model is the apparent accumulation of particles with arbitrarily small radii. As a result, there can be no nontrivial hydrodynamical equilibrium for this system.

Next, with an interest in understanding hydrodynamical limits in discontinuous regimes, we study a classical model for shock waves. Specifically, we consider scalar nonviscous conservation laws with strictly convex flux in one spatial dimension, and we investigate the behavior of bounded L^2 perturbations of shock wave solutions to the Riemann problem using the relative entropy method. We show that up to a time-dependent translation of the shock, the L^2 norm of a perturbed solution relative to the shock wave is bounded above by the L^2 norm of the initial perturbation.

Finally, we include some preliminary relative entropy estimates which are suitable for a study of shock wave solutions to $n \times n$ systems of conservation laws having a convex entropy.

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Chapter 1

Introduction

There are two parts. The first part of this thesis is dedicated to a problem in kinetic theory. Specifically, we study a fragmentation model for sprays. In Chapter 2, we discuss the general structure of kinetic models, and describe some of the qualitative asymptotic features that we expect from our model. For the sake of comparison, we also give an overview of the basic properties of the Boltzmann equation.

In Chapter 3, we introduce a generalized fragmentation model for sprays and prove our main results. A spray can be described simply as a collection of liquid droplets interacting with a gas. Our model takes into account the break-up of spray droplets due to drag forces. The model consists of the following coupled system of partial differential equations:

$$\begin{cases} \partial_t f + \nabla_x \cdot ((u+w)f) + \nabla_w \cdot (-\frac{w}{r^2} f) = \Gamma(f), \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} rw \cdot f \, dw \, dr, \\ \nabla \cdot u = 0, \\ f(0,x,w,r) = f^0(x,w,r), \\ u(0,x) = u^0(x). \end{cases} \tag{1.0.1}$$

The state of the gas is given by its velocity field $u(t,x) \in \mathbb{R}^3$ which verifies the incompressible Navier-Stokes equations. Under the assumption that the gradient of u is small and that the spray is dilute, the density of droplets, $f(t, x, w, r) \geq 0$, can be modeled by the Boltzmann-like kinetic equation appearing above. At time $t \geq 0$, the integral

$$\int_{\Omega} \int_{W} \int_{R} f(t, x, w, r)\, dr\, dv\, dx,$$

represents the expected number of particles contained in the set $\Omega \subset \mathbb{R}^3$, with relative velocity in $W \subset \mathbb{R}^3$, and with radius in $R \subset \mathbb{R}^+$. The effects of fragmentation are given by the operator

$$\begin{aligned} \Gamma(f)(t, x, w, r) &= -\nu(r, w) f(t, x, w, r) \\ &\quad + \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \nu(r^*, w^*) h(r, w, r^*, w^*) f(t, x, w^*, r^*)\, dw^*\, dr^*. \end{aligned} \tag{1.0.2}$$

The function $\nu(r, w)$ represents the break-up frequency, and $h(r, w, r^*, w^*)$ is the probability density describing the redistribution of particles after break-up. Finally, we observe that there is a coupling of the kinetic and fluid equations which corresponds to the interaction of the droplets with the surounding gas via drag forces.

Under this setup, the goal is to establish the existence of global weak solutions to the coupled system (1.0.1). The main mathematical difficulties are related to the lack of integrability of the factor $-\frac{w}{r^2}$ appearing in the kinetic equation. However, a priori estimates associated with the operator Γ are enough to ensure solutions in some suitably weak space. In our analysis, we assume the particles initially have bounded radii and bounded velocities relative to the gas, and we show that those bounds remain as the system evolves. Up to a few technical assumptions, our main result is the following.

Theorem 1.0.1. *Assume* $u^0 \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ *is divergence-free and* $f^0 \in L^1(\mathbb{R}^6 \times \mathbb{R}^+;\mathbb{R})$ *is bounded. Further assume* f^0 *has bounded support with respect to the variables* w *and* r. *Then for every* $T > 0$, *there exist bounded measurable functions* $f : [0,T] \to L^1(\mathbb{R}^6 \times \mathbb{R}^+;\mathbb{R})$ *and* $u : [0,T] \to L^2(\mathbb{R}^3;\mathbb{R}^3)$ *solving the initial value problem (1.0.1) in the sense of distributions.*

The second part of this thesis is dedicated to a problem in the classical theory of shock waves. The goal is to develop an L^2 stability theory for shocks using the relative entropy method. Our main results are included in Chapter 4. There we consider scalar conservation laws with strictly convex flux in one spatial dimension. Specifically, we study the initial value problem

$$\begin{cases} \partial_t U + \partial_x A(U) = 0; \\ U(x,0) = U^0(x), \end{cases} \tag{1.0.3}$$

in the case where the initial data U^0 is an L^2 perturbation the Riemann data

$$\phi(x) = \begin{cases} \phi_-, & \text{if } x < 0; \\ \phi_+ & \text{if } x > 0. \end{cases} \tag{1.0.4}$$

We assume $\phi_- > \phi_+$, so that for $\sigma \in \mathbb{R}$ given by the Rankine-Hugoniot relation, the function $\phi(x - \sigma t)$ forms an entropy admissible shock wave solution of (1.0.3). While L^2 stability, in the usual sense, fails for shocks, we show that it fails only up to a translation of the shock. Indeed, using techniques based on relative entropy, we establish the following global L^2 stability result.

Theorem 1.0.2. *Under the assumptions above, if* $U^0 \in L^\infty(\mathbb{R})$ *and* $U^0 - \phi \in L^2(\mathbb{R})$*, then there exists a Lipschitz continuous function* $\overline{x} : [0, \infty) \to \mathbb{R}$ *and a constant* $\lambda(\|U^0\|_{L^\infty}; \phi; A) > 0$ *such that*

$$\|U(\cdot, t) - \phi(\cdot - \sigma t - \overline{x}(t))\|_{L^2(\mathbb{R})} \leq \|U^0 - \phi\|_{L^2(\mathbb{R})}$$

and

$$|\overline{x}(t)| \leq \lambda \|U^0 - \phi\|_{L^2(\mathbb{R})} \sqrt{t}$$

for all $t \geq 0$*, where* $U(x, t)$ *is the unique entropy solution of (1.0.3) and* σ *is given by the Rankine-Hugoniot relation* $\sigma(\phi_- - \phi_+) = A(\phi_-) - A(\phi_+)$*.*

While our relative entropy techniques have been fully developed for scalar conservation laws, they do not easily extend to systems of conservation laws. A new idea is needed to obtain stability estimates for shocks in this case. However, given that the relative entropy method is generally applicable to any system endowed with a strictly convex entropy, the theory can be developed further and generalizations of our results are possible for those systems. As a first step in this direction, we establish some important structural lemmas concerning the variation of relative entropy along rarefaction curves and Hugoniot curves. These results are presented in Chapter 5.

Chapter 2

Background: Kinetic Theory

2.1 Transport Models

We begin with a brief introduction to some of the basic ideas of kinetic theory. The starting point of our discussion is the classical Liouville equation

$$\partial_t f + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f + \sum_{i=1}^{N} \frac{1}{m} F_i \cdot \nabla_{v_i} f = 0, \qquad (x_i, v_i) \in \mathbb{R}_x^3 \times \mathbb{R}_v^3 \qquad (2.1.1)$$

which governs the evolution of a joint position-velocity distribution function $f(t, x_1, v_1, x_2, v_2, ..., x_N, v_N) \geq 0$ for a system of N particles of mass m influenced by an external force $F_i = F(t, x_i, v_i) \in \mathbb{R}^3$. If we consider particle trajectories in phase space given by

$$\begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \frac{1}{m} F(t, x_i(t), v_i(t)), \end{cases}$$

then (2.1.1) derives formally from the assumption that f is constant along trajectories. That is,

$$\frac{d}{dt} f(t, x_1(t), v_1(t), x_2(t), v_2(t), ..., x_N(t), v_N(t)) = 0.$$

When the number of particles is very large, as in the case of a gas composed of billions of molecules, equation (2.1.1) is potentially difficult to manage computationally. In those cases it is often natural to impose two additional assumptions on the probability density f.

First, we assume that the particles are indistinguishable; that is, we assume that f is invariant under permutations of the pairs (x_i, v_i). This is a symmetry assumption that can be understood, alternatively, as the equivalence of the single-particle probability densities

$$f^i(t, x, v) = \int_{\mathbb{R}_x^{3(N-1)}} \int_{\mathbb{R}_x^{3(N-1)}} f \, dx_1 \, dv_1 ... \widehat{dx_i} \, \widehat{dv_i} ... \, dx_N \, dx_N.$$

On the other hand, we assume that the distributions f^i are independent in the sense that

$$f(t, x_1, v_1, x_2, v_2, ..., x_N, v_N) = \prod_{i=1}^{N} f^i(t, x_i, v_i). \tag{2.1.2}$$

This is usually called the molecular chaos assumption. We should point out that this is more than just the assertion that each particle behaves independently of the others. In particular, the initial probability density must factorize in this way.

Together these assumptions imply

$$f(t, x_1, v_1, x_2, v_2, ..., x_N, v_N) = \prod_{i=1}^{N} P(t, x_i, v_i). \tag{2.1.3}$$

We claim that the evolution of $f^i = P$ is governed by a one-particle Liouville equation. Indeed, let us assume that f vanishes at infinity and that $\mathrm{div}_v F$ is identically zero. Then integrating (2.1.1) over domains $\mathbb{R}_x^{3(N-1)} \times \mathbb{R}_v^{3(N-1)}$, we obtain formally the equation

$$\partial_t f^i + v \cdot \nabla_x f^i + \frac{1}{m} F \cdot \nabla_v f^i = 0. \tag{2.1.4}$$

It is easy to check that if the molecular chaos assumption holds at time $t = 0$, then (2.1.2) holds for all $t > 0$ with f^i given by (2.1.4).

In this way, we have reduced our problem to the study of a single-particle distribution P. In fact, scaling (2.1.4) by a factor of N, one usually considers the function $f^{\#} = NP$, so that, at time $t \geq 0$, the integral

$$\int_\Omega \int_V f^{\#}(t, x, v)\, dv\, dx$$

represents the *expected number* of particles contained in the set $\Omega \subset \mathbb{R}^3$, having velocity in the set $V \subset \mathbb{R}^3$. As an equation for $f^{\#}$, (2.1.4) is usually called the (homogeneous) Vlasov equation.

2.2 Redistribution Models

In the previous section, we assumed implicitly that there were no interactions between particles. Let us consider in this section kinetic models which take into account some kind of redistribution phenomenon. For simplicity, we will restrict our attention to spatially homogeneous models of the form

$$\partial_t f(t, v) = Q(f)(t, v). \tag{2.2.1}$$

We will call Q the redistribution operator. We are interested in the large-time asymptotic properties of (2.2.1). In particular, are there steady-state solutions, and what role does entropy play?

2.2.1 The Boltzmann Equation

The prototype for this kind of model is the spatially homogeneous Boltzmann equation,

$$\partial_t f(t, v) = Q(f, f)(t, v), \tag{2.2.2}$$

which takes into account the redistribution of particle velocities due to collisions. The Boltzmann collision operator is given by

$$Q(f, f) = \frac{1}{2} \int_{\mathbb{R}^n} \int_{S^{n-1}} \{f('v)f('v_*) - f(v)f(v_*)\} \, |(v - v_*) \cdot n| \, dn \, dv_*. \tag{2.2.3}$$

The symbols $'v$ and $'v_*$ denote the pre-collisional velocities and are defined by

$$\begin{aligned} 'v &= v - ((v - v_*) \cdot n)n, \\ 'v_* &= v_* + ((v - v_*) \cdot n)n. \end{aligned}$$

We would like to determine the functions f for which $Q(f, f) = 0$. These will be steady-state solutions of the Boltzmann equation. To begin with, we observe that the symmetry of (2.2.3) allows one to write

$$\begin{aligned} &\int_{\mathbb{R}^n} \phi(v) Q(f, f) \, dv \qquad (2.2.4) \\ &= \frac{1}{8} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} [\phi(v) + \phi(v_*) - \phi('v) - \phi('v_*)] \times \\ &\qquad \times [f('v)f('v_*) - f(v)f(v_*)] \, |(v - v_*) \cdot n| \, dn \, dv_* \, dv. \end{aligned}$$

We say that ϕ is a collision invariant if

$$\phi(v) + \phi(v_*) - \phi('v) - \phi('v_*) = 0$$

for all velocity pairs (v, v_*). One can check that functions of form

$$\phi(v) = A + B \cdot v + C|v|^2$$

are collision invariants. Indeed, this follows from the conservation of momentum and kinetic energy. In fact, it is a theorem of Boltzmann that says all collision invariants have this form. Now, if we consider again (2.2.4), this time with $\phi = \log f$, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} \log f \, Q(f,f) \, dv \qquad (2.2.5) \\ &\quad = \frac{1}{8} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} \log \left(\frac{f f_*}{'f' f_*} \right) ('f' f_* - f f_*) |(v - v_*) \cdot n| \, dn \, dv_* \, dv \leq 0, \end{aligned}$$

since $(x - y) \log (y/x) \leq 0$ for all $x, y \in \mathbb{R}^+$. Notice that we have equality precisely when $'f' f_* - f f_* = 0$, or equivalently when

$$\log f(v) + \log f(v_*) - \log f('v) - \log f('v_*) = 0.$$

In particular, if $\log f$ is a collision invariant, then $Q(f,f) = 0$. Therefore, the functions

$$f = \kappa e^{-\beta |v - v_0|^2}$$

form a class of steady-state solutions of the Boltzmann equation. These functions are called Maxwellians.

Finally we observe that (2.2.5) implies formally that

$$\partial_t \int_{\mathbb{R}^n} f \log f \, dv = \int_{\mathbb{R}^n} \partial_t f \log f \, dv + \int_{\mathbb{R}^n} \partial_t f \, dv$$
$$= \int_{\mathbb{R}^n} \log f \, Q(f,f) \, dv + \int_{\mathbb{R}^n} Q(f,f) \, dv \leq 0.$$

Note that we have used the fact that $\phi(v) = 1$ is a collision invariant. The estimate above shows that the entropy functional

$$\mathcal{H}(f)(t) = \int_{\mathbb{R}^n} f(t) \log f(t) \, dv$$

is non-increasing over time. Furthermore, if $f(t)$ is not a Maxwellian, then $\mathcal{H}(f)$ is strictly decreasing. This suggests that f approaches a Maxwellian distribution as $t \to \infty$. In fact, this can be made precise and proved rigorously, but it is outside the scope of our discussion. We simply wanted to highlight some of the qualitative features of the Boltzmann model which are important in the study of the large-time behavior of solutions. For further details we refer the reader to [15].

2.2.2 A Toy Model

While the Boltzmann equation enjoys a robust normalizing structure, other models may exhibit a concentration effect. In particular, some models may admit only dirac masses as time-asymptotic states. In those cases the large-time behavior of solutions may be difficult to describe quantitatively. As an illustration of this type of phenomenon, we consider the toy kinetic model

$$\partial_t f(t,z) = Q(f)(t,z) = -f(t,z) + \int_{\mathbb{R}^+} K(z,y) f(t,y)\, dy, \qquad z \in \mathbb{R}^+. \quad (2.2.6)$$

In this model, the redistribution of the kinetic function $f \geq 0$ is determined by the smooth kernel $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$. We assume that $\int_{\mathbb{R}^+} K(z,y)\, dz = 1$ for all $y \in \mathbb{R}^+$ so that the integral of f is conserved. Indeed, in that case we have

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^+} f(t,z)\, dz &= \int_{\mathbb{R}^+} \partial_t f(t,z)\, dz \\
&= -\int_{\mathbb{R}^+} f(t,z)\, dz + \int_{\mathbb{R}^+} f(t,y) \left\{ \int_{\mathbb{R}^+} K(z,y)\, dz \right\} dy \\
&= -\int_{\mathbb{R}^+} f(t,z)\, dz + \int_{\mathbb{R}^+} f(t,y)\, dy = 0.
\end{aligned}$$

Further, let us assume that $K(z,y) > 0$ if and only if $0 < z < y$. Under this hypothesis we will show that the mass of f accumulates near $z = 0$. More precisely, we will show that

$$\int_0^x f(t,z)\, dz \xrightarrow{t \to \infty} \int_0^{+\infty} f(0,z)\, dz \quad (2.2.7)$$

for all $x > 0$. In some sense this suggests that f approaches a dirac mass centered at $z = 0$. However, equation (2.2.6) does not admit generalized functions as solutions, so a delta function is technically not a steady-state for our model. This is part of the difficulty of characterizing the asymptotic properties of such models.

To prove our claim above, we integrate (2.2.6) over the interval $[0, x]$. Given our conditions on K we obtain

$$\begin{aligned}
\partial_t \int_0^x f(t,z)\,dz &= -\int_0^x f(t,z)\,dz + \int_0^\infty f(t,y)\left\{\int_0^x K(z,y)\,dz\right\}\,dy \\
&= -\int_0^x f(t,z)\,dz + \int_0^x f(t,y)\left\{\int_0^x K(z,y)\,dz\right\}\,dy \\
&\qquad + \int_x^\infty f(t,y)\left\{\int_0^x K(z,y)\,dz\right\}\,dy \\
&= \int_x^\infty f(t,y)\left\{\int_0^x K(z,y)\,dz\right\}\,dy \\
&= \int_x^\infty f(t,y)\phi_x(y)\,dy,
\end{aligned}$$

where $\phi_x(y) = \int_0^x K(z,y)\,dz$. Observe that $0 < \phi_x(y) \leq 1$ is continuous on $[x, \infty)$ and verifies $\phi_x(x) = 1$. Clearly $F(t,x) = \int_0^x f(t,z)\,dz$ is bounded above by the total mass of f which we denote by M. Also, thanks to the equation above $F(t,x)$ is non-decreasing as a function of t. Therefore, as $t \to \infty$, $F(t,x)$ converges pointwise to a function $F_\infty(x)$. We claim that this function is constant with value M.

First, we point out that F_∞ cannot be constant with value less than M since $F(t,x) \to M$ as $x \to \infty$. Now suppose $F_\infty(x_1) < F_\infty(x_2)$ for $0 < x_1 < x_2$. Then for T sufficiently large,

$$\int_{x_1}^{x_2} f(t,y)\,dy = F(t,x_2) - F(t,x_1) \geq \frac{1}{2}\left[F_\infty(x_2) - F_\infty(x_1)\right] \qquad (2.2.8)$$

for all $t \geq T$. Now, since ϕ_{x_1} is positive and continuous on the interval $[x_1, x_2]$, it is bounded below on that interval by some positive constant δ. Therefore,

$$\begin{aligned}\partial_t F(t, x_1) &= \int_{x_1}^{\infty} f(t,y)\phi_{x_1}(y)\,dy \\ &\geq \int_{x_1}^{x_2} f(t,y)\phi_{x_1}(y)\,dy \geq \frac{\delta}{2}\,[F_\infty(x_2) - F_\infty(x_1)] > 0,\end{aligned}$$

which implies that $F(t, x_1)$ is unbounded as $t \to \infty$. This is a contradiction. Hence, we have property (2.2.7).

2.3 Final Remarks

In the chapter to follow, we will consider a particle fragmentation model for which the redistribution operator has almost the same structure as (2.2.6). In place of the variable z considered in (2.2.6), we will introduce the new kinetic variable r which measures the radius of a particle. It is natural to expect an accumulation of small particles due to the aggregate effects of fragmentation. While the number of particles grows, the total mass $\int r^3 f(r)\,dr$ will be conserved, and it is the distribution of mass which behaves asymptotically like a delta function. Although we do not present a detailed study of the time-asymptotic properties of this model, we conjecture that the qualitative features are the same as those described in the previous section.

Chapter 3

Study of a Fragmentation Model for Sprays

3.1 Introduction

In this chapter, we study a fragmentation model for a general class of sprays. For our purposes, a spray can be described as an ensemble of liquid particles interacting with a gas. We describe the distribution of particles through a density function $f(t, x, v, r) \geq 0$, so that, at time $t \geq 0$, the integral

$$\int_\Omega \int_V \int_R f(t, x, v, r)\, dr\, dv\, dx,$$

represents the expected number of particles contained in the set $\Omega \subset \mathbb{R}^3$, with velocity in $V \subset \mathbb{R}^3$, and with radius in $R \subset \mathbb{R}^+$. The evolution of f is determined by the kinetic equation

$$\partial_t f + \nabla_x \cdot (vf) + \nabla_v \cdot (Af) = \Gamma(f),$$

where A represents the acceleration of a particle due to drag forces, and the fragmentation operator, Γ, is given by

$$\begin{aligned}\Gamma(f)(t, x, v, r) = &-\nu(r, v) f(t, x, v, r)\\ &+ \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \nu(r^*, v^*) h(r, v, r^*, v^*) f(t, x, v^*, r^*)\, dv^*\, dr^*.\end{aligned}$$

Here, $\nu(r, v)$ is the break-up frequency, and $h(r, v, r^*, v^*)$ is the probability density describing the redistribution of particles after break-up. More precisely, $\int_V \int_R h(r, v, r^*, v^*)\, dr\, dv$ is the probability that fragmentation of a particle with radius r^* and velocity v^* will produce a particle with radius $r \in R$ and velocity $v \in V$.

The fragmentation operator was introduced by Hylkema and Villedieu in [31] and has been previously studied by Dufour et al. [25] and Baranger [1], among others. In these contexts, ν and h were determined experimentally for a gas with constant velocity field u_g. In attempt to preserve the structure of Γ for general velocity fields, we introduce a change of variables. Namely, we express ν and h as functions of the relative velocity $w = v - u(t, x)$. More accurately, we assume the existence of ν and h which are independent of the velocity of the gas and depend only on the relative velocities of the particles and their radii. This is quite natural when one considers the direct dependence of fragmentation on drag forces. In these coordinates, we shall consider, instead, the density

$$g(t, x, w, r) = g(t, x, v - u(t, x), r) := f(t, x, v, r),$$

which satisfies the PDE

$$\begin{cases} \partial_t g + \nabla_x \cdot ((u + w)g) + \nabla_w \cdot (Ag) \\ \qquad + \nabla_w \cdot [(-\partial_t u - u \cdot \nabla_x u)g] + \nabla_w \cdot [(-w \cdot \nabla_x u)g] = \Gamma(g). \end{cases} \tag{3.1.1}$$

In order to simplify the mathematical analysis, we drop the last two terms on the left hand side of (3.1.1); that is, we assume the gradient of $u(t,x)$ is small, so that g is given, approximately, by

$$\begin{cases} \partial_t g + \nabla_x \cdot ((u+w)g) + \nabla_w \cdot (Ag) = -\nu(r,w)g(t,x,w,r) \\ \qquad + \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} \nu(r^*,w^*)h(r,w,r^*,w^*)g(t,x,w^*,r^*)\,dw^*\,dr^*. \end{cases} \tag{3.1.2}$$

As for the gas, we require that the velocity field $u(t,x) \in \mathbb{R}^3$ verifies the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = \mathbb{F}; \\ \nabla \cdot u = 0, \end{cases} \tag{3.1.3}$$

where $P(t,x), \mathbb{F}(t,x) \in \mathbb{R}^3$ represent the pressure and external force, respectively. We remind the reader that (3.1.3) represents a system of equations, where by definition

$$u \cdot \nabla u = \sum_{i=1}^{3} u_i \partial_{x_i} u.$$

Finally, equations (3.1.2) and (3.1.3) are coupled through drag forces. We will assume that the force on a particle of radius r moving at a velocity w relative to the fluid is approximated by Stokes' Law; that is, $F(r,w) = -rw$. (We set all constants equal to one.) Accordingly, we take the acceleration of a particle to be $A(r,w) = -\frac{w}{r^2}$. Integrating the density of drag forces exerted on the fluid, we obtain

$$\mathbb{F} = -\int_{\mathbb{R}^+}\int_{\mathbb{R}^3} Fg\,dw\,dr = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr. \tag{3.1.4}$$

Our goal in this chapter is to establish the existence of weak solutions to the coupled equations (3.1.2) and (3.1.3) when the initial distribution of particles, g^0, has bounded support with respect to the variables r and w. As our main result, we prove the following theorem.

Theorem 3.1.1. *Assume* $g^0 \in L^\infty \cap L^1(\mathbb{R}^6 \times \mathbb{R}^+)$. *Fix* R, $W > 0$ *and suppose* $\operatorname{supp}(g^0) \subset \mathbb{R}^3_x \times \Omega$ *where* $\Omega = \{(r, w) \in \mathbb{R}^+ \times \mathbb{R}^3 : 0 < r \leq R, 0 \leq |w| \leq W\}$. *Further suppose* $u^0 \in \mathbb{P}L^2(\mathbb{R}^3)$, *where* $\mathbb{P}$ *denotes the Leray projector. Then there exists a weak solution to the coupled system of initial value problems*

$$\begin{cases} \partial_t g + \nabla_x \cdot ((u + w)g) + \nabla_w \cdot (-\frac{w}{r^2} g) = \Gamma(g), \\ g(0, x, w, r) = g^0(x, w, r), \end{cases} \tag{3.1.5}$$

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} r w g \, dw \, dr, \\ \nabla \cdot u = 0, \\ u(x, 0) = u^0(x), \end{cases} \tag{3.1.6}$$

such that

$$\operatorname{supp}(g(t)) \subset \mathbb{R}^3_x \times \Omega, \text{ for a.e. } t \in [0, T], \tag{3.1.7}$$

$$g \in L^\infty(0, T; L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)) \cap L^\infty(0, T; L^1(\mathbb{R}^6 \times \mathbb{R}^+)), \tag{3.1.8}$$

with g *weakly continuous from* $[0, T]$ *into* $L^2_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)$, *and*

$$u \in L^2(0, T; H^1_0(\mathbb{R}^3)) \cap L^\infty(0, T; L^2(\mathbb{R}^3)), \tag{3.1.9}$$

with u *weakly continuous from* $[0, T]$ *into* $\mathbb{P}L^2(\mathbb{R}^3)$.

We say that g is a weak solution on $[0,T]$ if (3.1.5) holds in the sense of distributions; that is, for any $\varphi \in \mathcal{C}^\infty([0,T]\times\mathbb{R}^6\times\mathbb{R}^+)$ with $\varphi(\cdot,T)=0$ and $\varphi(\cdot,t)$ compactly supported for all $t\in[0,T]$, then

$$\int_0^T \int_{\mathbb{R}^6\times\mathbb{R}^+} g[\partial_t\varphi + (u+w)\cdot\nabla_x\varphi - \frac{w}{r^2}\cdot\nabla_w\varphi] + \varphi\Gamma(g)\,dr\,dw\,dx\,dt \\ + \int_{\mathbb{R}^6\times\mathbb{R}^+} g^0\varphi(0,x,w,r)\,dr\,dw\,dx = 0.$$

Fluid/particle models of this type have been well-studied in recent years, in part, due to a growing list of industrial applications ranging from sedimentation analysis (see Berres et al. [3] and Gidaspow [29]) to combustion theory (see Williams [45] and [46]). The simplest of such models, describing two-phase flow in one spatial dimension, was investigated by Domelevo and Roquejoffre in [24], where the authors prove the global existence and uniqueness of smooth solutions to a viscous Burgers equation coupled with a Vlasov equation. The existence and uniqueness of classical solutions to the IVP for a system of Vlasov-Fokker-Planck equations (which take into account the Brownian motion of particles) coupled with Poisson's equation was established by F. Bouchut in [9]. In a subsequent paper, [14], J. Carrillo studied the initial-boundary-value problems associated to the Poisson-Vlasov-Fokker-Planck system. The existence of global weak solutions for a system of compressible Navier-Stokes equations coupled with the Vlasov-Fokker-Planck equation is verified by Mellet and Vasseur in [40], and prior to that result, the existence of global weak solutions to the Vlasov-Stokes equations was established by

Hamdache in [30]. Finally, coupling of the Vlasov equation with the compressible Euler equations in the context of sprays is investigated by Baranger and Desvillettes in [2]. In this paper, the authors prove the existence and uniqueness of classical solutions for small time in the case of smooth initial data.

In addition to work related to well-posedness, there have been a number of recent results addressing the asymptotic behavior of kinetic models, including in many cases the characterization of steady-state solutions. An interesting feature of the present model is the apparent accumulation of mass near $w = 0$ and $r = 0$. Indeed, the only functions for which the fragmentation operator vanishes involve a dirac mass centered at $w = 0$. This type of phenomenon is seen, for example, in the inelastic Boltzmann models considered by Bobylev et al. [6–8], which admit (without additional forcing terms) only dirac masses as steady solutions. It is the goal of ongoing research to explore in more detail these asymptotic properties of the model.

Let us now present a brief outline of this chapter. In Section 3.2, we discuss some of the basic assumptions related to the operator Γ and prove that the kinetic equation is mass-preserving. Section 3.3 is devoted to the study of the kinetic equation. In particular, we show that for $u(t, x) \in L^2(0, T; W^{1,\infty}(\mathbb{R}^3))$, there exists a unique weak solution to the initial value problem (3.1.5). In Section 3.4, we couple the kinetic equation with the incompressible Navier-Stokes equations and prove Theorem 3.1.1 by means of a fixed point argument. In the final section, we include some additional comments on the redistribution kernel h.

3.2 The Fragmentation Operator: Preliminary Estimates

In this section we establish some preliminary estimates related to the fragmentation operator

$$\Gamma(g)(t,x,w,r) = -\nu(r,w)g(t,x,w,r) + \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} \nu(r^*,w^*)h(r,w,r^*,w^*)g(t,x,w^*,r^*)\,dw^*\,dr^*.$$

Let $\Phi = \mathbb{R}^+ \times (\mathbb{R}^3 \setminus \{0\})$, where $\mathbb{R}^+ = \{r \in \mathbb{R} : r > 0\}$. We will assume the following:

(A1) $\nu \in \mathcal{C}^1(\overline{\mathbb{R}^+ \times \mathbb{R}^3})$, $\nu \geq 0$.

(A2) $\nu(r,w) = 0$ if and only if $w = 0$.

(A3) $h \in \mathcal{C}^1(\overline{\Phi} \times \Phi)$, $h \geq 0$.

(A4) $h(r,w,r^*,w^*) = 0$ if $r \geq r^*$ or $|w| \geq |w^*|$.

(A5) $\exists\, C > 0 :\ \nu(r^*,w^*)h(r,w,r^*,w^*) \leq C$ for all $(r,w,r^*,w^*) \in \overline{\Phi} \times \Phi$.

(A6) $\displaystyle\int_a^b \int_{\mathbb{R}^3} h(r,w,r^*,w^*)\,dw\,dr = \int_{R(b)}^{R(a)} \int_{\mathbb{R}^3} h(r,w,r^*,w^*)\,dw\,dr$,
where $R(r) = \sqrt[3]{r^{*3} - r^3}$ and $0 \leq a \leq b \leq \frac{r^*}{\sqrt[3]{2}}$.

(A7) $\displaystyle\int_0^{\frac{r^*}{\sqrt[3]{2}}} \int_{\mathbb{R}^3} h(r,w,r^*,w^*)\,dw\,dr = \int_{\frac{r^*}{\sqrt[3]{2}}}^{r*} \int_{\mathbb{R}^3} h(r,w,r^*,w^*)\,dw\,dr = 1.$

First, let us point out that we assume in (A1) and (A3) that ν and h are continuous up to the boundary $\{r = 0\} \times \{w = 0\}$, while h may be unbounded near $\{r^* = 0\} \times \{w^* = 0\}$ (we will say more about this shortly). Furthermore, we will assume, as explained in Section 3.5, that the product νh which appears in the operator Γ verifies (A5). Condition (A2) expresses the fact that particles will break up if and only if the drag force is non-zero. Next, we assume that the size and relative velocity of a particle do not increase after break-up; this is condition (A4). It is also reasonable to assume that fragmentation of a particle produces exactly two particles with complementary radii ($r^3 + R^3 = r^{*3}$), and this is expressed in (A6). Additionally, the probability that fragmentation will produce a particle with volume less (or greater, respectively) than or equal to half the volume of the original particle should be equal to one.

It should be noted that hypotheses (A6) and (A7) are not typically found in the literature (cf. [25] and [1]). Instead, it is standard to assume that h verifies the relation: $\int_{\mathbb{R}^3} \int_{\mathbb{R}^+} r^3 h(r, w, r^*, w^*) \, dw \, dr = r^{*3}$, for all $r^* > 0$. This property is associated with mass conservation, and rightly so, however we prefer to include this as a lemma which follows from the fundamental assumptions above. In any case, taking this property into account, observe that (for fixed $r^* > 0$) as $w^* \to 0$, condition (A4) requires that the support of $r^3 h(r, w, r^*, w^*)$, with respect to r and w, has measure decreasing to zero. Therefore, if mass is conserved, h must blow-up. Further comments related to ν and h can be found in Section 3.5.

Lemma 3.2.1. *Consider a function h satisfying (A3)-(A7). Then h verifies*

$\int_{\mathbb{R}^+} \int_{\mathbb{R}^3} r^3 h(r,w,r^*,w^*)\,dw\,dr = r^{*3}$, *for every* $(r^*, w^*) \in (0,\infty) \times (\mathbb{R}^3 \setminus \{0\})$.

Proof. Fix $r^* > 0$ and $w^* \in \mathbb{R}^3 \setminus \{0\}$, and let $H(r) = \displaystyle\int_{\mathbb{R}^3} h(r,w,r^*,w^*)\,dw$. Also, as before, let $R(r) = (r^{*3} - r^3)^{\frac{1}{3}}$ be the complement of r. In terms of H, (A6) takes the form

$$\int_a^b H(r)\,dr = \int_{R(b)}^{R(a)} H(s)\,ds = \int_{R^{-1}(R(a))}^{R^{-1}(R(b))} H((r^{*3} - r^3)^{\frac{1}{3}}) \frac{r^2}{(r^{*3} - r^3)^{\frac{2}{3}}}\,dr,$$

where the last integral follows from the change of variables $s = R(r)$. Since this holds for all $0 \le a \le b \le \frac{r^*}{\sqrt[3]{2}}$, we conclude that

$$H(r) = \frac{r^2}{(r^{*3} - r^3)^{\frac{2}{3}}} H((r^{*3} - r^3)^{\frac{1}{3}}), \text{ a.e..}$$

Finally, we have

$$\begin{aligned}
\int_0^{\frac{r^*}{\sqrt[3]{2}}} r^3 H(r)\,dr &= \int_0^{\frac{r^*}{\sqrt[3]{2}}} r^3 \frac{r^2}{(r^{*3} - r^3)^{\frac{2}{3}}} H((r^{*3} - r^3)^{\frac{1}{3}})\,dr \\
&= \int_{R(0)}^{R(\frac{r^*}{\sqrt[3]{2}})} (r^{*3} - s^3) H(s)(-1)\,ds \\
&= \int_{\frac{r^*}{\sqrt[3]{2}}}^{r^*} (r^{*3} - s^3) H(s)\,ds = r^{*3} - \int_{\frac{r^*}{\sqrt[3]{2}}}^{r^*} s^3 H(s)\,ds,
\end{aligned}$$

which, together with (A4), ends the proof. Note that (A7) has been used in the last equality. ∎

Finally, thanks to the structure of Γ, we are able to show that the kinetic equation (3.1.5) is mass preserving.

Lemma 3.2.2 (Conservation of mass)**.** *Suppose u is smooth and bounded, and let g be a regular solution to (3.1.5) which vanishes at infinity. Further, suppose $r^3 g^0 \in L^1(\mathbb{R}^6 \times \mathbb{R}^+)$. Then, $\|r^3 g(t)\|_{L^1(\mathbb{R}^6\times\mathbb{R}^+)} = \|r^3 g^0\|_{L^1(\mathbb{R}^6\times\mathbb{R}^+)}$ for all $t > 0$.*

Proof. We multiply (3.1.2) by r^3 and integrate with respect to x, w, and r. Due to the divergence form of the equation we formally obtain

$$\frac{d}{dt}\int_{\mathbb{R}^6\times\mathbb{R}^+} r^3 g(t,x,w,r)\,dr\,dw\,dx = -\int_{\mathbb{R}^6\times\mathbb{R}^+} r^3 \nu(r,w) g(t,x,w,r)\,dr\,dw\,dx$$
$$+\int_{\mathbb{R}^6\times\mathbb{R}^+} \nu(r^*,w^*) g(t,x,w^*,r^*)\left\{\int_{\mathbb{R}^3\times\mathbb{R}^+} r^3 h(r,w,r^*,w^*)\,dr\,dw\right\} dr^*\,dw^*\,dx,$$

where we use Fubini's Theorem on the last term. By Lemma 3.2.1, we conclude

$$\frac{d}{dt}\|r^3 g(t)\|_{L^1(\mathbb{R}^6\times\mathbb{R}^+)} = 0.$$

∎

3.3 Study of the Kinetic Equation

The goal of this section is to prove the following proposition of existence and uniqueness of weak solutions to the initial value problem (3.1.5).

Proposition 3.3.1. *Assume $g^0 \in L^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^6 \times \mathbb{R}^+)$ and suppose* $\mathrm{supp}(g^0) \subset \mathbb{R}^3_x \times \Omega$*. Further suppose $u \in L^2(0,T; W^{1,\infty}(\mathbb{R}^3_x))$. Then there* *exists a unique weak solution to the initial value problem*

$$\begin{cases} \partial_t g + \nabla_x \cdot ((u+w)g) + \nabla_w \cdot (-\frac{w}{r^2} g) = \Gamma(g), \\ g(0,x,w,r) = g^0(x,w,r), \end{cases} \tag{3.3.1}$$

where g *has the following properties:*

(i) $g \in L^\infty(0,T; L^1(\mathbb{R}^6 \times \mathbb{R}^+))$.

(ii) $g \in L^\infty(0,T; L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$.

(iii) $g(t,x,w,r) \le C(\|g^0\|_{L^\infty}) e^{\frac{3t}{r^2}}$.

(iv) $g(t,x,w,r) \le C(\|g^0\|_{L^\infty}) \frac{1}{|w|^3}$.

(v) $g \in \mathcal{C}([0,T]; W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$, *for* $1 \le p < \infty$.

(vi) $\mathrm{supp}(g(t)) \subset \mathbb{R}^3_x \times \Omega$ *for a.e.* $t \in [0,T]$.

We begin by proving existence of solutions to (3.3.1) when g^0 and u are smooth and bounded, and when g^0 is compactly supported in r, w. The idea is to construct a sequence of solutions verifying

$$\begin{cases} \partial_t g_n + \nabla_x \cdot ((u+w)g_n) + \nabla_w \cdot (-\frac{w}{r^2} g_n) = -\nu(r,w) g_n(t,x,w,r) \\ \qquad + \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \nu(r^*,w^*) h(r,w,r^*,w^*) g_{n-1}(t,x,w^*,r^*)\, dw^*\, dr^*, \\ g_n(0,x,w,r) = g^0(x,w,r), \\ g_0(t,x,w,r) = 0. \end{cases} \tag{3.3.2}$$

Given g_{n-1} we can solve for g_n in (3.3.2) using the method of characteristics.

We then establish a priori bounds for the sequence g_n and show that $g = \lim_{n\to\infty} g_n$ solves the original PDE.

Consider the following trajectories (in phase space):

$$\begin{cases} \dot{x}(t) = u(t, x(t)) + w(t), \\ \dot{w}(t) = -\frac{1}{r^2} w(t), \\ x(t_0) = x_0, \\ w(t_0) = w_0. \end{cases}$$

If u is smooth and bounded, the Cauchy-Lipshitz theorem for ODEs ensures a smooth solution denoted $(x(t, t_0, x_0, w_0), w(t, t_0, w_0))$. Along trajectories (3.3.2) reduces to the ODE:

$$\begin{cases} \frac{d}{dt} g_n(t, x(t), w(t), r) = \left(-\nu(r, w(t)) + \frac{3}{r^2}\right) g_n(t, x(t), w(t), r) \\ \quad + \int_{\mathbb{R}^+} \int_{\mathbb{R}^3} \nu(r^*, w^*) h(r, w(t), r^*, w^*) g_{n-1}(t, x(t), w^*, r^*) \, dw^* \, dr^*, \\ g_n(0) = g^0. \end{cases} \tag{3.3.3}$$

Solving for g_n in (3.3.3) using an integrating factor and setting $t = t_0$, $x = x_0$ and $w = w_0$, we obtain

$$\begin{aligned} g_n(t, x, w, r) &= e^{-\int_0^t \left(\nu(r, w(s,t,w)) - \frac{3}{r^2}\right) ds} g^0(x(0, t, x, w), w(0, t, w), r) \\ &\quad + \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^+} e^{-\int_\tau^t \left(\nu(r, w(s,t,w)) - \frac{3}{r^2}\right) ds} \nu h(r, w(\tau, t, w), r^*, w^*) \times \\ &\quad \times g_{n-1}(\tau, x(\tau, t, x, w), w^*, r^*) \, dw^* \, dr^* \, d\tau. \end{aligned} \tag{3.3.4}$$

Note that since the trajectories are continuous with respect to the initial value parameters (again by Cauchy-Lipshitz), it is easy to show by induction that each g_n is continuous provided that g^0 is continuous.

Lemma 3.3.2. *Let* $\Omega = \{(r,w) \in \mathbb{R}^+ \times \mathbb{R}^3 : 0 < r \leq R, 0 \leq |w| \leq W\}$ *with* $R, W > 0$. *Suppose* $g^0 \in \mathcal{C}^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)$ *with supp*$(g^0) \subset \mathbb{R}^3 \times \Omega$. *Then supp*$(g_n(t)) \subset \mathbb{R}^3 \times \Omega$ *for all* $t \geq 0$ *and for all* n.

Proof. First, observe that the norm of the relative velocity decreases as we move forward in time along trajectories ($w(t) = e^{-\frac{(t-t_0)}{r^2}} w(t_0)$). Therefore we have $|w(\tau,t,w)| \geq |w|$ when $\tau \leq t$. Now we will prove the statement by induction. The previous observation implies

$$g_1(t,x,w,r) = e^{-\int_0^t \left(\nu(r,w(s,t,w)) - \frac{3}{r^2}\right) ds} g^0(x(0,t,x,w), w(0,t,w), r)$$

vanishes when $|w| \geq W$ or $r \geq R$ for all $t \geq 0$. Now assume the statement holds for $n-1$. Keeping in mind assumption (A4) from the previous section, we observe that for $|w| \geq W$, $h(r, w(\tau,t,w), r^*, w^*) = 0$ unless $|w^*| \geq W$. Since $g_{n-1}(\tau, x(\tau,t,x,w), w^*, r^*) = 0$ in that case, the integrand vanishes and as before we find $g_n(t,x,w,r) = 0$. A similar argument works for $r > R$. ∎

Remark. As we shall see, the previous lemma will allow us to consider only those measurable $g \geq 0$ such that *supp*$(g(t)) \subset \mathbb{R}^3 \times \Omega$. In that case, it is convenient to write

$$\Gamma(g)(t,x,w,r) = -\chi_\Omega \nu(r,w) g(t,x,w,r) + \int_{\Omega_r} \nu h(r,w,r^*,w^*) g(t,x,w^*,r^*)\, dw^*\, dr^*, \tag{3.3.5}$$

where

$$\Omega_r = \left\{ (r^*,w^*) \in \mathbb{R}^+ \times \mathbb{R}^3 : r \le r^* \le R, 0 \le |w^*| \le W \right\}. \tag{3.3.6}$$

Note that, by assumption, $h(r,w,r^*,w^*) = 0$ if $r \ge r^*$. Therefore, under the conditions above, it suffices to integrate over the compact set Ω_r. Also notice that the negative part of Γ is unchanged when multiplied by $\chi_\Omega(r,w)$.

In order to show that $g = \lim_{n\to\infty} g_n$ is a weak solution to (3.3.1), we need to verify that Γ is weakly continuous in following sense:

Lemma 3.3.3. *Consider the set*

$$K^p = \left\{ g \in L^p_{loc}([0,T] \times \mathbb{R}^6 \times \mathbb{R}^+) : supp(g) \subset [0,T] \times \mathbb{R}^3_x \times \Omega \right\},$$

where $1 \le p < \infty$. Suppose g_n is bounded in K^p. Then, $\Gamma(g_n)$ is bounded in K^p. Further, suppose that $g_n \xrightarrow{L^p_{loc}} g \in K^p$. Then, $\Gamma(g_n) \xrightarrow{L^p_{loc}} \Gamma(g) \in K^p$.

Proof. We must verify the statements above with respect to each compact subset of the domain $[0,T] \times \mathbb{R}^6 \times \mathbb{R}^+$. Keep in mind that $\{r = 0\}$ is excluded from the domain. This means that each compact subset is contained in a region $\{r \ge r_0\}$ with $r_0 > 0$.

We will first show that for $g \in K^p$ and $r_0, R_0 > 0$, there exists $C_{r_0} > 0$ such that

$$\|\Gamma(g)\|_{L^p([0,T]\times[-R_0,R_0]^3\times\Omega_{r_0})} \leq C_{r_0}\|g\|_{L^p([0,T]\times[-R_0,R_0]^3\times\Omega_{r_0})}. \tag{3.3.7}$$

Indeed, let $\tilde{\Omega} = [0,T] \times [-R_0, R_0]^3 \times \Omega_{r_0}$. Then,

$$\begin{aligned}
\|\Gamma(g)\|_{L^p(\tilde{\Omega})} &\leq \|\chi_\Omega \nu g\|_{L^p(\tilde{\Omega})} + \left\| \int_{\Omega_r} \nu h(r,w,r^*,w^*) g(t,x,w^*,r^*)\, dr^*\, dw^* \right\|_{L^p(\tilde{\Omega})} \\
&\leq \|\chi_\Omega \nu g\|_{L^p(\tilde{\Omega})} + \left\| \int\limits_{\Omega_{r_0}} \nu h(r,w,r^*,w^*) g(t,x,w^*,r^*)\, dr^*\, dw^* \right\|_{L^p(\tilde{\Omega})} \\
&\leq \|\nu\|_{L^\infty(\Omega)} \|g\|_{L^p(\tilde{\Omega})} + \left\| \|\nu h(r,w)\|_{L^{p'}(\Omega_{r_0})} \|g(t,x)\|_{L^p(\Omega_{r_0})} \right\|_{L^p(\tilde{\Omega})} \\
&\leq \|\nu\|_{L^\infty(\Omega)} \|g\|_{L^p(\tilde{\Omega})} + \|\nu h\|_{L^\infty(\Omega_{r_0}; L^{p'}(\Omega_{r_0}))} |\Omega_{r_0}|^{\frac{1}{p}} \|g\|_{L^p(\tilde{\Omega})} \\
&\leq \|\nu\|_{L^\infty(\Omega)} \|g\|_{L^p(\tilde{\Omega})} + \|\nu h\|_{L^\infty} |\Omega_{r_0}|^{\frac{1}{p'}} |\Omega_{r_0}|^{\frac{1}{p}} \|g\|_{L^p(\tilde{\Omega})} \\
&\leq \left(\|\nu\|_{L^\infty(\Omega)} + \|\nu h\|_{L^\infty} |\Omega_{r_0}| \right) \|g\|_{L^p(\tilde{\Omega})}.
\end{aligned}$$

Note that in the fifth inequality we used our assumption that νh is bounded. Also, note carefully that we have shown Γ is a bounded linear operator only when the domain is restricted to sets of the form $\tilde{\Omega}$ and when $g \in K^p$.

Now suppose g_n is bounded in K^p and consider $\Omega_0 \subset\subset [0,T] \times \mathbb{R}^6 \times \mathbb{R}^+$. We want to show that $\|\Gamma(g_n)\|_{L^p(\Omega_0)}$ is bounded uniformly for all g_n. Taking into account condition (A4) from Section 3.2, it is easy to verify that $supp(\Gamma(g_n)) \subset [0,T] \times \mathbb{R}^3_x \times \Omega$. Therefore, if we choose $r_0, R_0 > 0$ such that

$\Omega_0 \cap \{[0,T] \times \mathbb{R}^3_x \times \Omega\} \subset \tilde{\Omega} = [0,T] \times [-R_0, R_0]^3 \times \Omega_{r_0}$, we have

$$\|\Gamma(g_n)\|_{L^p(\Omega_0)} \leq \|\Gamma(g_n)\|_{L^p(\tilde{\Omega})} \leq C\|g_n\|_{L^p(\tilde{\Omega})}.$$

Since $\tilde{\Omega} \subset\subset [0,T] \times \mathbb{R}^6 \times \mathbb{R}^+$, the right hand side is bounded, and we conclude that $\Gamma(g_n)$ is bounded in K^p.

Next let us show that $\Gamma(g_n) \xrightarrow{L^p(\Omega_0)} \Gamma(g)$ for all compact subsets Ω_0. As before, we have $\Omega_0 \cap \{[0,T] \times \mathbb{R}^3_x \times \Omega\} \subset \tilde{\Omega}$. By assumption, $g_n \xrightarrow{L^p(\tilde{\Omega})} g$. Therefore using (3.3.7), we have $\Gamma(g_n) \xrightarrow{L^p(\tilde{\Omega})} \Gamma(g)$, which then gives us that $\Gamma(g_n) \rightharpoonup \Gamma(g)$ in $L^p(\Omega_0 \cap \{[0,T] \times \mathbb{R}^3_x \times \Omega\})$. Now, let $\varphi \in L^{p'}(\Omega_0)$. Then,

$$\int_{\Omega_0} \varphi(\Gamma(g_n) - \Gamma(g))\, dx\, dw\, dr\, dt = \int_{\Omega_0 \cap \{[0,T]\times\mathbb{R}^3_x\times\Omega\}} \varphi(\Gamma(g_n) - \Gamma(g))\, dx\, dw\, dr\, dt,$$

which goes to zero as $n \to \infty$ by the preceding observation. ∎

Next we establish two important bounds on the sequence g_n.

Lemma 3.3.4. *Assume $g^0 \in \mathcal{C}^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^\infty(\mathbb{R}^6 \times \mathbb{R}^+)$ with supp$(g^0) \subset \mathbb{R}^3 \times \Omega$. Then, the sequence g_n given by (3.3.3)-(3.3.4) is increasing. Furthermore, the sequences $|w|^3 g_n$ and $e^{\frac{-3t}{r^2}} g_n$ are uniformly bounded in $L^\infty(0,T; L^\infty(\mathbb{R}^6 \times \mathbb{R}^+))$. More precisely, there exists $C > 0$, depending only on Ω, such that*

(i) $\||w|^3 g_n(t)\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \leq \||w|^3 g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} + C\,\||w|^3 g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\, te^{Ct}$,

(ii) $\left\|e^{\frac{-3t}{r^2}} g_n(t)\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \leq \|g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} + C\,\|g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\, te^{Ct}$.

Proof. First we will show g_n increases pointwise with respect to n. By (3.3.4) $g_1 - g_0 = g_1 \geq 0$. Now assume $g_{n-1} - g_{n-2} \geq 0$. Then, due to Lemma 3.3.2

and Remark 3.3, (3.3.4) implies

$$g_n - g_{n-1} = \int_0^t \int_{\Omega_r} e^{-\int_\tau^t \left(\nu - \frac{3}{r^2}\right)\, ds} \nu h(g_{n-1} - g_{n-2})\, dw^*\, dr^*\, d\tau \geq 0.$$

So, by induction g_n is increasing. Now, using $|w(t)|^3 = e^{-3\frac{(t-t_0)}{r^2}}|w(t_0)|^3$ as an integrating factor in (3.3.3), we obtain

$$\begin{aligned}\frac{d}{dt}\left(|w(t)|^3 g_n(t, x(t), w(t), r)\right) &= -\chi_\Omega \nu(r, w(t))\left(|w(t)|^3 g_n(t, x(t), w(t), r)\right)\\ &+ \int_{\Omega_r} \nu h(r, w(t), r^*, w^*)|w(t)|^3 g_{n-1}(t, x(t), w^*, r^*)\, dw^*\, dr^*\\ &\leq \int_{\Omega_r} \nu h(r, w(t), r^*, w^*)|w(t)|^3 g_n(t, x(t), w^*, r^*)\, dw^*\, dr^*.\end{aligned}$$

Integrating in time and setting $t = t_0$, $x = x_0$ and $w = w_0$, we obtain

$$\begin{aligned}&|w|^3 g_n(t, x, w, r) \leq |w(0, t, w))|^3 g^0(x(0, t, x, w), w(0, t, w), r) \ + \qquad (3.3.8)\\ &\int_0^t \int_{\Omega_r} \nu h(r, w(\tau, t, w), r^*, w^*)|w(\tau, t, w)|^3 g_n(\tau, x(\tau, t, x, w), w^*, r^*)\, dw^*\, dr^*\, d\tau.\end{aligned}$$

Now, since $|w(\tau, t, w)| \leq |w^*|$ when $h(r, w(\tau, t, w), r^*, w^*)$ is non-zero, we have

$$\begin{aligned}&|w|^3 g_n(t, x, w, r) \leq |w(0, t, w))|^3 g^0(x(0, t, x, w), w(0, t, w), r)\\ &+ \int_0^t \int_{\Omega_r} \nu h(r, w(\tau, t, w), r^*, w^*)|w^*|^3 g_n(\tau, x(\tau, t, x, w), w^*, r^*)\, dw^*\, dr^*\, d\tau.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\||w|^3 g_n(t)\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} &\leq \left\||w|^3 g^0\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\\ &+ \|\nu h\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^+; L^1(\Omega_r))} \int_0^t \left\||w|^3 g_n(\tau)\right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\, d\tau.\end{aligned}$$

Since νh is bounded, we have

$$\|\nu h\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^+;L^1(\Omega_r))} \leq |\Omega_r| \|\nu h\|_{L^\infty} \equiv C.$$

Finally, by Gronwall's Lemma, we conclude

$$\left\| |w|^3 g_n(t) \right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \leq \left\| |w|^3 g^0 \right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} + C \left\| |w|^3 g^0 \right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} te^{Ct}.$$

On the other hand, since $|w(\tau,t,w)|^3 = e^{-3\frac{(\tau-t)}{r^2}}|w|^3$, we deduce from (3.3.8) that

$$\begin{aligned} &g_n(t,x,w,r) \leq e^{\frac{3t}{r^2}} g^0(x(0,t,x,w),w(0,t,w),r) \\ &+ \int_0^t \int_{\Omega_r} e^{\frac{3}{r^2}(t-\tau)} \nu h(r,w(\tau,t,w),r^*,w^*) g_n(\tau,x(\tau,t,x,w),w^*,r^*)\, dw^*\, dr^*\, d\tau. \end{aligned}$$

Now, since $e^{\frac{-3\tau}{r^2}} \leq e^{\frac{-3\tau}{r^{*2}}}$ for $r \leq r^*$, we have

$$\begin{aligned} &e^{\frac{-3t}{r^2}} g_n(t,x,w,r) \leq g^0(x(0,t,x,w),w(0,t,w),r) \\ &+ \int_0^t \int_{\Omega_r} \nu h(r,w(\tau,t,w),r^*,w^*) \left[e^{\frac{-3\tau}{r^{*2}}} g_n(\tau,x(\tau,t,x,w),w^*,r^*) \right] dw^*\, dr^*\, d\tau. \end{aligned}$$

Proceeding exactly as before, we conclude that

$$\left\| e^{\frac{-3t}{r^2}} g_n(t) \right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} \leq \left\| g^0 \right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} + C \left\| g^0 \right\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)} te^{Ct}.$$

∎

Proof of Proposition 5.2.3.

Step 1. First assume $g^0 \in \mathcal{C}^\infty(\mathbb{R}^6\times\mathbb{R}^+) \cap L^\infty(\mathbb{R}^6\times\mathbb{R}^+)$ with $\text{supp}(g^0) \subset \mathbb{R}^3\times\Omega$, and we assume that u is smooth and bounded. We claim that $g = \lim_{n\to\infty} g_n$ is a weak solution to the initial value problem (3.3.1) with properties (ii)-(vi).

Proof. According to Lemma 3.3.4, g_n (given by (3.3.4)) is an increasing sequence of measurable functions uniformly bounded above by $\phi_1(t,x,w,r) = C(\|g^0\|_{L^\infty})e^{\frac{3t}{r^2}} \in L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Therefore, by the monotone convergence theorem, $g(t,x,w,r) = \lim_{n\to\infty} g_n(t,x,w,r)$ is measurable, $g_n \to g$ in $L^p(0,T;L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$ for $1 \le p < \infty$, and (by the bound above) $g \in L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. This establishes properties (ii), (iii), and (vi). Also, since g_n is uniformly bounded by $\phi_2(t,x,w,r) = C(\||w|^3 g^0\|_{L^\infty})\frac{1}{|w|^3}$, we have property (iv). (Note that $\||w|^3 g^0\|_{L^\infty} \le W^3 \|g^0\|_{L^\infty}$ since $|w| \le W$ on the support of g^0.)

We now show that g satisfies (3.3.1) in the sense of distributions. We know that g_n given by the Duhamel formula (3.3.4), is a classical solution to (3.3.2). Therefore g_n satisfies

$$\partial_t g_n = -\nabla_x \cdot ((u+w)g_n) - \nabla_w \cdot \left(-\frac{w}{r^2} g_n\right) - \nu(g_n - g_{n-1}) + \Gamma(g_{n-1}). \quad (3.3.9)$$

Given that $g_n \xrightarrow{L^p(L^p_{loc})} g$ for $1 \le p < \infty$, it follows from Lemma 3.3.3 that $\Gamma(g_{n-1}) \xrightarrow{L^p(L^p_{loc})} \Gamma(g)$. Also, since multiplication by a smooth function and differentiation are continuous operations in $\mathcal{D}'((0,T) \times \mathbb{R}^6 \times \mathbb{R}^+)$, passing to the limit shows that g satisfies (3.3.1) in the sense of distributions.

Now it suffices to verify that $g(t) \rightharpoonup g^0$ in the sense of distributions as $t \to 0$. In fact, for $1 \le p < \infty$, we will show $g \in \mathcal{C}([0,T];W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$ so that $g(t) \to g^0$ strongly in $W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)$ as $t \to 0$.

First we will show that $\partial_t g_n$ is bounded in $L^p(0,T;W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Again, we examine the right-hand side of (3.3.9). Since $u + w$ and $-\frac{w}{r^2}$ are bounded on compact subsets of $[0,T]\times\mathbb{R}^6\times\mathbb{R}^+$, it is clear that $\nabla_x\cdot((u+w)g_n)$ and $\nabla_w\cdot(-\frac{w}{r^2}g_n)$ are bounded in $L^p(0,T;W^{-1,p}_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$. Thanks to Lemma 3.3.3, the remaining terms are bounded in $L^p(0,T;L^p_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$ which injects continuously into $L^p(0,T;W^{-1,p}_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$.

Now, since $L^p_{loc}(\mathbb{R}^6\times\mathbb{R}^+)$ is compactly embedded in $W^{-1,p}_{loc}(\mathbb{R}^6\times\mathbb{R}^+)$ by the Rellich-Kondrakov Theorem, we conclude, using the Aubin lemma (see Theorem 3.2.1 in [42]), that g_n is relatively compact in $\mathcal{C}([0,T];W^{-1,p}_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$. Therefore, there exists a convergent subsequence g_{n_k}, the limit of which must be equal to g for a.e. $t\in[0,T]$. (since $g_n\to g$ pointwise). This gives us property (v).

Finally, given $\Omega_0\subset\subset\mathbb{R}^6\times\mathbb{R}^+$, $\|g_{n_k}(t)-g(t)\|_{W^{-1,p}(\Omega_0)}\xrightarrow{n_k\to\infty}0$ uniformly in $t\in[0,T]$. It follows that

$$\|g(t)-g^0\|_{W^{-1,p}(\Omega_0)}\leq\|g(t)-g_{n_k}(t)\|_{W^{-1,p}(\Omega_0)}+\|g_{n_k}(t)-g^0\|_{W^{-1,p}(\Omega_0)}\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$

for n_k sufficiently large and for t sufficiently small. Therefore, $g(t)\to g^0$ strongly in $W^{-1,p}_{loc}(\mathbb{R}^6\times\mathbb{R}^+)$ as $t\to0$ and this completes the proof of the claim in Step 1. ∎

Step 2. Now suppose $g^0\in L^\infty(\mathbb{R}^6\times\mathbb{R}^+)$ with $\text{supp}(g^0)\subset\mathbb{R}^3\times\Omega$, and assume that u is smooth and bounded. Then there exists a weak solution to the initial value problem (3.3.1) with properties (ii)-(vi).

Proof. By mollification, we can construct $g^0_\delta\xrightarrow{\delta\to0}g^0$ in $L^\infty(\mathbb{R}^6\times\mathbb{R}^+)$ such that $\|g^0_\delta\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}\leq\|g^0\|_{L^\infty(\mathbb{R}^6\times\mathbb{R}^+)}$ for all $\delta>0$. By Step 1, there exists a sequence of regularized (weak) solutions, g_δ, satisfying

$$\begin{cases} \partial_t g_\delta + \nabla_x \cdot ((u+w)g_\delta) + \nabla_w \cdot (-\frac{w}{r^2} g_\delta) = \Gamma(g_\delta), \\ g_\delta(0,x,w,r) = g_\delta^0(x,w,r). \end{cases} \tag{3.3.10}$$

Since $g_\delta(t,x,w,r) \leq C(\|g_\delta^0\|_{L^\infty})e^{\frac{3t}{r^2}} \leq C(\|g^0\|_{L^\infty})e^{\frac{3t}{r^2}}$, we conclude that g_δ is bounded in $L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Also, as in the proof of Step 1, $\partial_t g_\delta$ is bounded in $L^p(0,T;W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$, for $1 \leq p < \infty$. Now, consider a sequence $\delta_n \to 0$. By the Aubin lemma, g_{δ_n} is relatively compact in $\mathcal{C}([0,T];W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Therefore, there exists a subsequence $g_{\delta_{n_k}}$ and a function g, such that

$$\begin{aligned} g_{\delta_{n_k}} &\rightharpoonup g \ \text{ weakly in } L^p(0,T;L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)), \\ g_{\delta_{n_k}} &\to g \ \text{ strongly in } \mathcal{C}([0,T];W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)). \end{aligned}$$

In particular, g has properties (iii) (which implies (ii)), (iv), (v) and (vi).

We claim that g is a weak solution to (3.3.1). Indeed, following the proof of Step 1, the former convergence implies that g satisfies (3.3.1) in the sense of distributions. It remains to show $g(t) \overset{\mathcal{D}'}{\rightharpoonup} g^0$ as $t \to 0$. As before, it is enough to have $g(t) \xrightarrow{W^{-1,p}_{loc}} g^0$. Given $\Omega_0 \subset\subset \mathbb{R}^6 \times \mathbb{R}^+$, the latter convergence above ensures $\|g_{\delta_{n_k}}(t) - g(t)\|_{W^{-1,p}(\Omega_0)} \xrightarrow{n_k \to \infty} 0$ uniformly in $t \in [0,T]$. Additionally, $\|g_{\delta_{n_k}}(t) - g^0_{\delta_{n_k}}\|_{W^{-1,p}(\Omega_0)} \xrightarrow{t \to 0} 0$. Therefore, given $\varepsilon > 0$, we can choose n_k sufficiently large and t sufficiently small so that

$$\begin{aligned} \|g(t) - g^0\|_{W^{-1,p}(\Omega_0)} &\leq \|g(t) - g_{\delta_{n_k}}(t)\|_{W^{-1,p}(\Omega_0)} + \|g_{\delta_{n_k}}(t) - g^0_{\delta_{n_k}}\|_{W^{-1,p}(\Omega_0)} \\ &\quad + \|g^0_{\delta_{n_k}} - g^0\|_{W^{-1,p}(\Omega_0)} \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

It follows that g is a weak solution to the initial value problem (3.3.1). ∎

Step 3. Finally, suppose $g^0 \in L^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^6 \times \mathbb{R}^+)$ with $\mathrm{supp}(g^0) \subset \mathbb{R}^3 \times \Omega$, and assume that $u \in L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))$. Then there exists a unique weak solution to the initial value problem (3.3.1) with properties (i)-(vi).

Proof. By mollification, we can construct $u^\varepsilon \to u$ in $L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))$ such that $\|u^\varepsilon\|_{L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))} \le \|u\|_{L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))}$ for all $0 < \varepsilon \le \varepsilon_0$. By Step 2, there exists a sequence of regularized (weak) solutions, g^ε, satisfying

$$\begin{cases} \partial_t g^\varepsilon + \nabla_x \cdot ((u^\varepsilon + w)g^\varepsilon) + \nabla_w \cdot (-\frac{w}{r^2} g^\varepsilon) = \Gamma(g^\varepsilon), \\ g^\varepsilon(0,x,w,r) = g^0(x,w,r). \end{cases} \tag{3.3.11}$$

As before, property (iii) implies g^ε is bounded in $L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Now we will show $\partial_t g^\varepsilon$ is bounded in $L^2(0,T;W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$ for $1 \le p < +\infty$. The main point is to show $\nabla_x \cdot (u^\varepsilon g^\varepsilon)$ is bounded in the aforementioned space. Given $\Omega_0 \subset\subset \mathbb{R}^6 \times \mathbb{R}^+$, we have

$$\begin{aligned} \|\nabla_x \cdot (u^\varepsilon g^\varepsilon)\|_{L^2(0,T;W^{-1,p}(\Omega_0))} &\le C\|u^\varepsilon g^\varepsilon\|_{L^2(0,T;L^p(\Omega_0))} \\ &\le C\|u^\varepsilon\|_{L^2(0,T;L^\infty(\Omega_0))}\|g^\varepsilon\|_{L^\infty(0,T;L^p(\Omega_0))} \\ &\le C\|u\|_{L^2(0,T;L^\infty(\mathbb{R}^3_x))}\|g^\varepsilon\|_{L^\infty(0,T;L^p(\Omega_0))} \\ &\le C_2\|u\|_{L^2(0,T;W^{1,\infty}(\mathbb{R}^3_x))}. \end{aligned}$$

The remaining terms in (3.3.11) are bounded in $L^2(0,T;W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$ for $1 \le p < +\infty$.

Finally, using the Aubin lemma, we extract a subsequence g^{ε_k} and pass to the limit exactly as before. We should just be careful to to check that

$\nabla_x \cdot (u^{\varepsilon_k} g^{\varepsilon_k}) \xrightarrow{\mathcal{D}'} \nabla_x \cdot (ug)$. Indeed, for $2 \le p < +\infty$, we have $u^{\varepsilon_k} \xrightarrow{L^2(L^\infty)} u$ and $g^{\varepsilon_k} \xrightarrow{L^p(L^p_{loc})} g$. Therefore $u^{\varepsilon_k} g^{\varepsilon_k} \rightharpoonup ug$ weakly in $L^{\frac{2p}{2+p}}(0, T; L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$, which implies convergence in $\mathcal{D}'$. Also, as in the previous steps, $g(t) \to g^0$ strongly in $W^{-1,p}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)$ as $t \to 0$. Therefore, g is a weak solution to (3.3.1) with properties (ii)-(vi).

Now, let us show that $g \in L^\infty(0, T; L^1(\mathbb{R}^6 \times \mathbb{R}^+))$. Integrating (3.3.1) with respect to x, w, and r, we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_{\Omega} g(t, x, w, r)\, dr\, dw\, dx = - \int_{\mathbb{R}^3} \int_{\Omega} \nu(r, w) g(t, x, w, r)\, dr\, dw\, dx$$

$$+ \int_{\mathbb{R}^3} \int_{\Omega_r} \nu(r^*, w^*) g(t, x, w^*, r^*) \left\{ \int_{\Omega} h(r, w, r^*, w^*)\, dr\, dw \right\} dr^*\, dw^*\, dx$$

$$= - \int_{\mathbb{R}^3} \int_{\Omega} \nu(r, w) g(t, x, w, r)\, dr\, dw\, dx$$

$$+ 2 \int_{\mathbb{R}^3} \int_{\Omega_r} \nu(r^*, w^*) g(t, x, w^*, r^*)\, dr^*\, dw^*\, dx$$

$$\le \int_{\mathbb{R}^3} \int_{\Omega} \nu(r, w) g(t, x, w, r)\, dr\, dw\, dx \le \|\nu\|_{L^\infty(\Omega)} \int_{\mathbb{R}^3} \int_{\Omega} g(t, x, w, r)\, dr\, dw\, dx.$$

Note that condition (A7) from Section 3.2 implies $\int_\Omega h(r, w, r^*, w^*)\, dr\, dw = 2$, since $(w^*, r^*) \in \Omega_r \subset \Omega$. The result follows from Gronwall's Lemma.

Finally, we prove that the weak solution g is unique. Suppose g_1 and g_2 are two weak solutions. Since (3.3.1) is a linear PDE, the difference $\overline{g} = g_1 - g_2$ solves (3.3.1) with initial condition $g^0 = 0$. Therefore, we have

$$\partial_t \overline{g} + \nabla_x \cdot ((u + w)\overline{g}) + \nabla_w \cdot (-\frac{w}{r^2}\overline{g}) = \Gamma(\overline{g}).$$

Multiplying by $r^3\,\mathrm{sgn}(\overline{g})$ and integrating with respect to x, w, and r, we find

$$\int_{\mathbb{R}^6\times\mathbb{R}^+} \partial_t(r^3|\overline{g}|)\,dr\,dw\,dx + \int_{\mathbb{R}^6\times\mathbb{R}^+} \nabla_x\cdot((w+u)r^3|\overline{g}|)\,dr\,dw\,dx$$
$$+ \int_{\mathbb{R}^6\times\mathbb{R}^+} \nabla_w\cdot(-wr|\overline{g}|)\,dr\,dw\,dx$$
$$= \int_{\mathbb{R}^6\times\mathbb{R}^+} -r^3\nu\,|\overline{g}|\;dr\,dw\,dx$$
$$+ \int_{\mathbb{R}^6\times\mathbb{R}^+} \left\{ r^3\,\mathrm{sgn}(\overline{g}) \int_{\mathbb{R}^6\times\mathbb{R}^+} \nu h\overline{g}\,dr^*\,dw^* \right\} dr\,dw\,dx.$$

We refer the reader to [36] (Lemma 2.3) for details of this type of analysis and in particular for a proof of the (formal) equality $\mathrm{sgn}(\overline{g})\nabla_x\cdot(u(\overline{g})) = \nabla_x\cdot(u|\overline{g}|)$ when $u\in L^2(0,T;W^{1,\alpha}(\mathbb{R}^3_x))$ and $\overline{g}\in L^\infty(0,T;L^\beta(\mathbb{R}^3_x))$ with $\frac{1}{\alpha}+\frac{1}{\beta}\le 1$. (We take $\alpha=\infty$ and $\beta=1$.) Now, integrating by parts on the left hand side yields

$$\frac{d}{dt}\int_{\mathbb{R}^6\times\mathbb{R}^+} r^3|\overline{g}|\,dr\,dw\,dx \le \int_{\mathbb{R}^6\times\mathbb{R}^+} -r^3\nu\,|\overline{g}|\;dr\,dw\,dx$$
$$+ \int_{\mathbb{R}^6\times\mathbb{R}^+} \left|r^3\,\mathrm{sgn}(\overline{g})\right| \left|\int \nu h\overline{g}\,dr^*\,dw^*\right| dr\,dw\,dx$$
$$\le \int_{\mathbb{R}^6\times\mathbb{R}^+} -r^3\nu\,|\overline{g}|\;dr\,dw\,dx + \int_{\mathbb{R}^6\times\mathbb{R}^+} r^3 \left\{ \int_{\mathbb{R}^6\times\mathbb{R}^+} \nu h|\overline{g}|\,dr^*\,dw^* \right\} dr\,dw\,dx$$
$$= \int_{\mathbb{R}^6\times\mathbb{R}^+} r^3\Gamma(|\overline{g}|)\,dr\,dw\,dx = 0.$$

Since $\overline{g}(0)=0$, we conclude that $\int_{\mathbb{R}^6\times\mathbb{R}^+} r^3|\overline{g}|\,dr\,dw\,dx = 0$, for almost every $t\in[0,T]$. Therefore $g_1 = g_2$ a.e.. This completes the proof. ∎

3.4 The Incompressible Navier-Stokes Equations and the Coupled Problem

The previous section established the existence and uniqueness of solutions to the kinetic equation (3.3.1) for a class of Lipschitz velocity fields $u(t,x) \in L^2(0,T;W^{1,\infty}(\mathbb{R}^3))$. Now, we turn our attention to the fluid equation. The goal of this section is to find a solution to the coupled problem by means of a fixed point argument. First, we need an existence and uniqueness result for the associated Galerkin solutions of the incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = \mathbb{F} \\ \nabla \cdot u = 0, \\ u(x,0) = u^0(x), \end{cases} \tag{3.4.1}$$

Specifically, following the framework detailed by Temam in [42], we consider the vector space $\mathcal{V} = \left\{ v \in \{\mathcal{D}(\mathbb{R}^3)\}^3,\ \mathrm{div}(v) = 0 \right\}$ and define the Hilbert spaces V and H to be the closure of $\mathcal{V}$ with respect to the $\{H_0^1(\mathbb{R}^3)\}^3$ and $\{L^2(\mathbb{R}^3)\}^3$ inner products, respectively. Assume $\{w_i\}_{i=1}^{\infty} \subset \mathcal{V}$ is an orthogonal basis of V and an orthonormal basis of H, and let $V_m = \mathrm{span}\{w_1, ..., w_m\} \subset V$ and $H_m = \mathrm{span}\{w_1, ..., w_m\} \subset H$. Finally, we denote the H_0^1 inner product by $((\cdot,\cdot))$ and the L^2 inner product by $(\cdot,\cdot)$, and consider the trilinear and continuous form on V defined by

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_{\mathbb{R}^3} u_i (D_i v_j) w_j.$$

Now, fix $m \in \mathbb{N}$. We say that $u_m \in L^2(0,T;V_m)$ is an approximate solution of (3.4.1), for a given $\mathbb{F} \in L^2(0,T;L^2(\mathbb{R}^3))$ and $u^0 \in H$, if for all $j = 1, ..., m$, u_m verifies

$$
\begin{aligned}
&- \int_0^T \left(u_m(t), \psi'(t) w_j\right) \, dt + \int_0^T \left(\left(u_m(t), \psi(t) w_j\right)\right) \, dt \\
&\quad + \int_0^T b\left(u_m(t), u_m(t), \psi(t) w_j\right) \, dt = \left(u^0, w_j\right) \psi(0) + \int_0^T \left(\mathbb{F}(t), \psi(t) w_j\right) \, dt,
\end{aligned} \tag{3.4.2}
$$

for all $\psi \in \mathcal{C}^1([0,T])$ with $\psi(T) = 0$.

Theorem 3.4.1. *Given $\mathbb{F} \in L^2(0,T;L^2(\mathbb{R}^3))$ and $u^0 \in H$, there exists a unique function u_m solving the variational problem (3.4.2) with the property*

$$u_m \in L^2(0,T;V_m) \cap L^\infty(0,T;H_m). \tag{3.4.3}$$

Moreover, by construction, we have

$$u_m \in \mathcal{C}([0,T];V_m), \tag{3.4.4}$$

and u_m has the following properties:

$$\|u_m\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \le C(\|u_0\|_{L^2(\mathbb{R}^3)}; \|\mathbb{F}\|_{L^2(0,T;L^2(\mathbb{R}^3))}), \tag{3.4.5}$$

$$\|u_m\|_{L^2(0,T;H_0^1(\mathbb{R}^3))} \le C(\|u_0\|_{L^2(\mathbb{R}^3)}; \|\mathbb{F}\|_{L^2(0,T;L^2(\mathbb{R}^3))}). \tag{3.4.6}$$

Proof. The existence of $u_m = \sum_{i=1}^m g_{im}(t) w_i$ verifying (3.4.2)-(3.4.6) is well-known and follows from Galerkin's method. We will sketch the proof of uniqueness for solutions with property (3.4.3). The argument follows closely the uniqueness proof for solutions of (3.4.1) in dimension two. First, note that (3.4.2) implies

$$\frac{d}{dt}\left(u_m(t),\cdot\right) = -\left(\left(u_m(t),\cdot\right)\right) - b\left(u_m(t), u_m(t),\cdot\right) + \left(\mathbb{F}(t),\cdot\right),$$

in the scalar distribution sense, as linear operators on V_m, and it can be shown that the right hand side represents an element of $L^2(0,T;V_m')$. In particular, this defines the weak time derivative $u_m' \in L^2(0,T;V_m')$. That is, we have the following equality in $\mathcal{D}'((0,T))$:

$$\langle u_m'(t),\cdot\rangle = -\left(\left(u_m(t),\cdot\right)\right) - b\left(u_m(t), u_m(t),\cdot\right) + \left(\mathbb{F}(t),\cdot\right).$$

Therefore, according to Lemma 3.1.2 in [42], u_m is almost everywhere equal to a function (absolutely) continuous from [0,T] to H_m and

$$\frac{d}{dt}|u_m|_{L^2}^2 = 2\langle u_m', u_m\rangle,$$

in $\mathcal{D}'((0,T))$. Assume $u_m^1, u_m^2 \in L^2(0,T;V_m)\cap L^\infty(0,T;H_m)$ are two functions verifying (3.4.2). Then, for a.e. $t\in[0,T]$, $v = u_m^1 - u_m^2$ verifies

$$\begin{aligned}\langle v'(t), v(t)\rangle + \left(\left(v(t), v(t)\right)\right) &= b\left(u_m^2(t), u_m^2(t), v(t)\right) - b\left(u_m^1(t), u_m^1(t), v(t)\right).\\ &= -b\left(v(t), u_m^2(t), v(t)\right)\end{aligned}$$

Therefore,

$$\frac{d}{dt}|v(t)|_{L^2}^2 + \|v(t)\|_{H_0^1}^2 \le \|\nabla u_m^2(t)\|_{L^\infty}|v(t)|_{L^2}^2.$$

Based on the continuity established above, we conclude using Gronwall's lemma, that $|v(t)|_{L^2}^2 \le C|v(0)|_{L^2}^2$.

It remains to show that $|v(0)|_{L^2}^2 = 0$. Since $t \mapsto (u_m(t), w_j)$ is absolutely continuous, a legitimate integration by parts applied to the first term in (3.4.2) yields

$$\int_0^T \frac{d}{dt}(u_m(t), w_j)\,\psi(t)\,dt + (u_m(0), w_j)\,\psi(0) + \int_0^T ((u_m(t), \psi(t)w_j))\ dt$$
$$+ \int_0^T b\,(u_m(t), u_m(t), \psi(t)w_j)\ dt = \left(u^0, w_j\right)\psi(0) + \int_0^T (\mathbb{F}(t), \psi(t)w_j)\ dt.$$

Now consider $\psi_\varepsilon \in \mathcal{C}^1([0,T])$ with $\text{supp}(\psi_\varepsilon) \subset [0,\varepsilon)$, $\psi_\varepsilon(t) \le 1$, and $\psi_\varepsilon(0) = 1$. Taking $\varepsilon \to 0$ in the equation above implies $(u_m(0), w_j) = (u^0, w_j)$ for all $j = 1, ..., m$. Therefore, for any approximate solution with property (3.4.3), $u_m(0)$ is the orthogonal projection in $L^2(\mathbb{R}^3)$ of u^0 onto H_m. Hence, $|u_m^1(0) - u_m^2(0)|_{L^2}^2 = 0$, which finishes the proof. ∎

We now prove that the force given by (3.1.4) is bounded in the appropriate space.

Lemma 3.4.2. *Suppose $u(t,x) \in L^2(0,T;W^{1,\infty}(\mathbb{R}^3))$. Let g be the unique weak solution to (3.3.1). Then, $\mathbb{F} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr \in L^\infty(0,T;L^2(\mathbb{R}^3_x))$. Furthermore, $\|\mathbb{F}\|_{L^\infty(0,T;L^2(\mathbb{R}^3_x))} \leq C(\|g^0\|_{L^\infty};\|g^0\|_{L^1})$.*

Proof. According to Proposition 5.2.3, $g \in L^\infty(0,T;L^1(\mathbb{R}^6 \times \mathbb{R}^+))$. Since $|rwg| \leq RWg$ on the support of g, it follows that $\mathbb{F}(t,x) = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr$ $\in L^\infty(0,T;L^1(\mathbb{R}^3_x))$. Also, by Proposition 5.2.3, $g(t,x,w,r) \leq \frac{C}{|w|^3}$. Therefore, $|\int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr| \leq \int_\Omega |rwg|\,dw\,dr \leq \int_\Omega R\frac{C}{|w|^2}\,dw\,dr \leq +\infty$ which implies that $\mathbb{F} \in L^\infty(0,T;L^\infty(\mathbb{R}^3_x))$. We conclude that $\mathbb{F} \in L^\infty(0,T;L^2(\mathbb{R}^3_x))$ and based on the properties of g given by Proposition 5.2.3, the norm of $\mathbb{F}$ is bounded by a constant depending only on $\|g^0\|_{L^\infty}$ and $\|g^0\|_{L^1}$. ∎

Lemma 3.4.3. *Assume $g^0 \in L^\infty(\mathbb{R}^6 \times \mathbb{R}^+) \cap L^1(\mathbb{R}^6 \times \mathbb{R}^+)$ and $u^0 \in H$. Then, the operator $\mathcal{T}_m : L^2(0,T;H_m) \to L^2(0,T;H_m)$ given by $u \mapsto g \mapsto u_m$, where g is the solution to (3.3.1) with velocity u, and u_m is the solution to (3.4.2) with $\mathbb{F} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr$, is well-defined and continuous.*

Proof. First we show that for $u \in L^2(0,T;H_m)$, we can apply Proposition 5.2.3 to find a unique $g \in L^\infty(0,T;L^1(\mathbb{R}^6 \times \mathbb{R}^+))$; that is, we must show $L^2(0,T;H_m) \subset L^2(0,T;W^{1,\infty})$. Assume, $u \in L^2(0,T;H_m)$. Since we can write $u = \sum_{i=1}^m d_i(t)w_i$, and $\{w_i\}_{i=1}^\infty$ is orthonormal in H, we have

$$\begin{aligned}\|u\|^2_{L^2(0,T;H_m)} &= \int_0^T \|u(t)\|^2_{L^2}\,dt = \int_0^T \|d_1(t)w_1 + ... + d_m(t)w_m\|^2_{L^2}\,dt \\ &= \int_0^T |d_1(t)|^2 + ... + |d_m(t)|^2\,dt \leq +\infty.\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|u\|^2_{L^2(0,T;W^{1,\infty})} &= \int_0^T \|u(t)\|^2_{W^{1,\infty}}\,dt = \int_0^T \|d_1(t)w_1 + ... + d_m(t)w_m\|^2_{W^{1,\infty}}\,dt \\
&\leq \int_0^T \left(\|d_1(t)w_1\|_{W^{1,\infty}} + ... + \|d_m(t)w_m\|_{W^{1,\infty}}\right)^2\,dt \\
&\leq \int_0^T \left(|d_1(t)|\|w_1\|_{W^{1,\infty}} + ... + |d_m(t)|\|w_m\|_{W^{1,\infty}}\right)^2\,dt \\
&\leq C^2 \int_0^T \left(|d_1(t)| + ... + |d_m(t)|\right)^2\,dt \\
&\leq C^2 m \int_0^T |d_1(t)|^2 + ... + |d_m(t)|^2\,dt \leq +\infty
\end{aligned}$$

where $C = \max\limits_{1\leq i\leq m} \|w_i\|_{W^{1,\infty}}$. Therefore $u \in L^2(0,T;W^{1,\infty})$ and Proposition 5.2.3 applies. This gives $g \in L^\infty(0,T;L^1(\mathbb{R}^6 \times \mathbb{R}^+))$, and according to Lemma 3.4.2, $\mathbb{F} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr \in L^\infty(0,T;L^2(\mathbb{R}^3_x))$. Applying Theorem 3.4.1 gives a unique $u_m \in L^2(0,T;V_m) \cap L^\infty(0,T;H_m) \subset L^2(0,T;H_m)$. This proves that $\mathcal{T}_m$ is well-defined.

Now we will show that the operator $\mathcal{T}_m$ is continuous with respect to the $L^2(0,T;L^2(\mathbb{R}^3))$ norm. Consider a sequence $u_n \xrightarrow{L^2_t(L^2_x)} u$. We want to show that $\tilde{u}_n = \mathcal{T}(u_n) \xrightarrow{L^2_t(L^2_x)} \mathcal{T}(u) = \tilde{u}$. It is equivalent to show that for every subsequence $\tilde{u}_{n_k}$ there exists a sub-subsequence $\tilde{u}_{n_{k_l}} \xrightarrow{L^2_t(L^2_x)} \tilde{u}$.

Let g_n be the unique solution to

$$\begin{cases} \partial_t g_n + \nabla_x \cdot ((u_n + w)g_n) + \nabla_w \cdot (-\frac{w}{r^2} g_n) = \Gamma(g_n), \\ g_n(0,x,w,r) = g^0(x,w,r). \end{cases} \tag{3.4.7}$$

By Proposition 5.2.3, g_{n_k} is bounded in $L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$ and it is easy to verify that $\partial_t g_{n_k}$ is bounded in $L^2(0,T;W^{-1,2}_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$. By the compactness lemma, there exists $g_{n_{k_l}}$ and g such that

$$\begin{aligned} g_{n_{k_l}} &\rightharpoonup g \;\text{ weakly in } L^p(0,T;L^p_{loc}(\mathbb{R}^6\times\mathbb{R}^+)),\\ g_{n_{k_l}} &\to g \;\text{ strongly in } \mathcal{C}([0,T];W^{-1,2}_{loc}(\mathbb{R}^6\times\mathbb{R}^+)). \end{aligned}$$

Following the proof of Proposition 5.2.3, it is easy to verify that g is the unique solution of (3.3.1) with velocity $u=\lim\limits_{n\to\infty}u_n$. (The main difference in Proposition 5.2.3 is that $u_n \xrightarrow{L^2_t(L^\infty_x)} u$. However, it is enough to have, as in this case, $u_n \xrightarrow{L^2_t(L^2_x)} u$). Also, we have

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg_{n_{k_l}}\,dw\,dr \rightharpoonup \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr \;\text{ weakly in } L^2(0,T;L^2_{loc}(\mathbb{R}^3)). \tag{3.4.8}$$

To see this, fix $\Omega_x \subset\subset \mathbb{R}^3$ and consider $\phi\in L^2(0,T;L^2(\Omega_x))$. Then,

$$\begin{aligned} &\int_0^T\int_{\Omega_x}\phi\left\{\int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rw(g_{n_{k_l}}-g)\,dw\,dr\right\}dx\,dt\\ &\qquad=\int_0^T\int_{\Omega_x}\phi\left\{\int_{\Omega} rw(g_{n_{k_l}}-g)\,dw\,dr\right\}dx\,dt\\ &\qquad=\int_0^T\int_{\Omega_x\times\Omega}\phi rw(g_{n_{k_l}}-g)\,dw\,dr\,dx\,dt\to 0, \end{aligned}$$

since $\phi rw\in L^2(0,T;L^2(\Omega_x\times\Omega))$ and $g_{n_{k_l}}\rightharpoonup g$ in $L^2(0,T;L^2_{loc}(\mathbb{R}^6\times\mathbb{R}^+))$.

Now, according to Theorem 3.4.1, there exists a unique function $\tilde{u}_n\in L^2(0,T;V_m)\cap L^\infty(0,T;H_m)$ verifying (3.4.2) with $\mathbb{F}=\mathbb{F}_n=\int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg_n\,dw\,dr$.

Since Lemma 3.4.2 implies $\|\mathbb{F}_n\|_{L^2(0,T;L^2(\mathbb{R}^3_x)} \leq C$, it follows from (3.4.5)-(3.4.6) that $\|\tilde{u}_n\|_{L^2(0,T;H^1_0(\mathbb{R}^3))} \leq C$ and $\|\tilde{u}_n\|_{L^\infty(0,T;L^2(\mathbb{R}^3))} \leq C$. Therefore, given any subsequence $\tilde{u}_{n_k}$ we can extract a subsequence $\tilde{u}_{n_{k_{l_p}}}$ such that (3.4.8) holds and

$$\tilde{u}_{n_{k_{l_p}}} \rightharpoonup u^* \text{ weakly in } L^2(0,T;V_m),$$
$$\tilde{u}_{n_{k_{l_p}}} \overset{*}{\rightharpoonup} u^* \text{ weak} - \text{star in } L^\infty(0,T;H_m).$$

Finally, using standard compactness results (for example, Theorem 3.2.2 in [42]), we can choose the subsequence above so that

$$\tilde{u}_{n_{k_{l_p}}} \to u^* \text{ strongly in } L^2(0,T;L^2(\mathbb{R}^3)).$$

Passing to the limit in (3.4.2) shows that u^* is the unique solution to (3.4.2) with $\mathbb{F} = \int_{\mathbb{R}^+}\int_{\mathbb{R}^3} rwg\,dw\,dr$. That is, $\tilde{u}_{n_{k_{l_p}}} \xrightarrow{L^2_t(L^2_x)} u^* = \tilde{u}$. Therefore, $\mathcal{T}_m$ is continuous with respect to the $L^2(0,T;L^2(\mathbb{R}^3))$ norm. ∎

We are looking for g and u which solve (3.1.5) and (3.1.6) simultaneously. The idea is to find approximate solutions g_m and u_m which correspond to a fixed point of $\mathcal{T}_m$, and then using previous estimates, extract a subsequence which converges to a solution of the original problem. With this goal in mind, we will now show that $\mathcal{T}_m$ has a fixed point.

Lemma 3.4.4. *$\mathcal{T}_m$ has a fixed point.*

Proof. It suffices to show, by the Shauder fixed point theorem, that the operator $\mathcal{T}_m$ maps some closed convex set compactly into itself. Indeed, according to Theorem 3.4.1,

$$\|\mathcal{T}_m(u)\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq \|\mathcal{T}_m(u)\|_{L^2(0,T;H_0^1(\mathbb{R}^3))} \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}; \|\mathbb{F}\|_{L^2(0,T;L^2(\mathbb{R}^3))})$$

where, by Lemma 3.4.2, $\|\mathbb{F}\|_{L^2(0,T;L^2(\mathbb{R}^3_x)} \leq C(\|g^0\|_{L^\infty}; \|g^0\|_{L^1})$. Therefore,

$$\|\mathcal{T}_m(u)\|_{L^2(0,T;L^2(\mathbb{R}^3))} \leq C(\|u_0\|_{L^2(\mathbb{R}^3)}; \|g^0\|_{L^\infty}; \|g^0\|_{L^1}). \tag{3.4.9}$$

Since $\mathcal{T}_m(u)$ is bounded in $L^2(0,T;L^2(\mathbb{R}^3))$ uniformly in u, we consider the convex closed set $B(0,R)$, where R is the constant in (3.4.9). Clearly $\mathcal{T}_m(B(0,R)) \subset B(0,R)$. The inclusion is also compact. Indeed, standard a priori estimates and compactness results (see [42]) ensure that solutions of (3.4.2) are contained in a subspace of $L^2(0,T;H_0^1(\mathbb{R}^3)) \cap L^\infty(0,T;L^2(\mathbb{R}^3))$ which embeds compactly into $L^2(0,T;L^2(\mathbb{R}^3))$. Thus, the Shauder fixed point theorem applies and we obtain a fixed point of $\mathcal{T}_m$. ∎

Proof of Theorem 3.1.1.

As a result of Lemma 3.4.4, there exist approximate solutions g_m and u_m verifying (3.3.1) and (3.4.2) simultaneously. By Proposition 5.2.3, the sequence g_m is bounded in $L^\infty(0,T;L^\infty_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. It is easy to check that $\partial_t g_m$ is bounded in $L^\infty(0,T;W^{-1,2}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$. Applying the Aubin lemma, there exists a subsequence g_{m_k} and a function g, such that

$$g_{m_k} \rightharpoonup g \text{ weakly in } L^p(0,T;L^p_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)),$$
$$g_{m_k} \to g \text{ strongly in } \mathcal{C}([0,T];W^{-1,2}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)).$$

Moreover, since $g \in L^\infty(0,T; L^2_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)) \cap \mathcal{C}([0,T]; W^{-1,2}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$, it follows from Lemma 3.1.4 in [42] that g is weakly continuous from $[0,T]$ into $L^2_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)$.

On the other hand, the bounds given by (3.4.5) and (3.4.6) allow us to apply compactness results, as in [42], and extract a subsequence $u_{m_{kl}}$ and a function u, such that

$$u_{m_{kl}} \rightharpoonup u \;\text{ weakly in } L^2(0,T; H^1_0(\mathbb{R}^3)),$$

$$u_{m_{kl}} \overset{*}{\rightharpoonup} u \;\text{ weak} - \text{star in } L^\infty(0,T; L^2(\mathbb{R}^3)),$$

$$u_{m_{kl}} \to u \;\text{ strongly in } L^2(0,T; L^2(\mathbb{R}^3)).$$

Also, according to (3.4.8), $\int\int rwg_{m_k}\, dw\, dr \xrightharpoonup{L^2(L^2_{loc})} \int\int rwg\, dw\, dr$.

Finally, given the information above, we claim that passing to the limit in (3.3.1) and (3.4.2) yields (3.1.5)-(3.1.6). Passing to the limit in the fluid equation is straightforward (cf. [42]) and we obtain (3.1.9). Next, for the kinetic equation, the former convergences imply, as in the proof of Proposition 5.2.3, that $g(t) \to g^0$ strongly in $W^{-1,2}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+)$ as $t \to 0$. It suffices to show (3.1.5) holds in the sense of distributions. Let us show that $\nabla_x \cdot (u_{m_{k_l}} g_{m_{k_l}}) \xrightharpoonup{\mathcal{D}'} \nabla_x \cdot (ug)$. For $2 \leq p < +\infty$, we have $u_{m_{k_l}} \xrightarrow{L^2(L^2)} u$ and $g_{m_{k_l}} \xrightharpoonup{L^p(L^p_{loc})} g$. Therefore, $u_{m_{k_l}} g_{m_{k_l}} \rightharpoonup ug$ weakly in $L^{\frac{2p}{2+p}}(0,T; L^{\frac{2p}{2+p}}_{loc}(\mathbb{R}^6 \times \mathbb{R}^+))$, which implies the statement above. It follows from previous arguments that the remaining terms converge in the sense of distributions to their respective terms in (3.3.1). Thus, g is a weak solution to (3.3.1) with velocity u. As before, integrating (3.3.1) shows $g \in L^\infty(0,T; L^1(\mathbb{R}^6 \times \mathbb{R}^+))$. This completes the proof. ∎

3.5 Comments on the Redistribution Density

We conclude with a few remarks about the redistribution density function, h. Specifically, we would like to describe one general case in which we can expect condition (A5) from Section 3.2. Recall that for non-constant flows, $u(x,t)$, we have chosen to express h as a function of the relative velocities of the particles and their radii. While a precise specification of h should depend, in some complex way, on the material properties of the particles as well as the gas, we will assume h has reasonably nice structure which is at least faithful to important qualitative features of the model such as mass conservation.

Let us assume first that

$$h(r, w, r^*, w^*) = H(r, r^*)G(w, w^*). \tag{3.5.1}$$

That is, we assume the size and relative velocity of derivative particles are determined independently according to the probabilities H and G. Surely this is a simplifying assumption, since one would expect some interdependence of those quantities, especially when taking momentum transfer into account. Next, it is reasonable to assume that both H and G satisfy a self-similarity (homogeneity) property, namely

$$H(r, r^*) = C(r^*)H\left(\frac{r}{r^*}, 1\right),$$
$$G(w, w^*) = C(|w^*|)G\left(\frac{w}{|w^*|}, \frac{w^*}{|w^*|}\right).$$

Additionally, G should be invariant under rotations of the pair (w, w^*).

Under these assumptions, it suffices to know the behavior of H and G for fixed values of the starred variables, e.g. $r^* = 1$ and $w^* = (1, 0, 0)$. In fact, let us assume that both $H(\cdot, 1)$ and $G(\cdot, \frac{w^*}{|w^*|})$ are bounded; that is, assume there exist $C_1, C_2 > 0$ such that

(i) $H(s, 1) \leq C_1$ for all $0 < s \leq 1$, and

(ii) $G(z, \eta) \leq C_2$ for all $|\eta| = 1$ and for all $z \in B(0, 1)$.

If this was not the case, then after mollification one could find suitable approximations which do have the indicated bounds.

Now, from the previous assumptions together with (A7) in Section 3.2, we deduce the following:

$$\begin{aligned}
2 &= \int_0^{r^*} \int_{B(0,|w^*|)} H(r, r^*) G(w, w^*) \, dw \, dr \\
&= C(r^*) C(|w^*|) \int_0^{r^*} \int_{B(0,|w^*|)} H\left(\frac{r}{r^*}, 1\right) G\left(\frac{w}{|w^*|}, \frac{w^*}{|w^*|}\right) dw \, dr \\
&= C(r^*) C(|w^*|) \int_0^1 \int_{B(0,1)} H(s, 1) \, G\left(z, \frac{w^*}{|w^*|}\right) r^* |w^*|^3 \, dz \, ds \\
&= r^* |w^*|^3 C(r^*) C(|w^*|) \int_0^1 \int_{B(0,1)} H(s, 1) \, G\left(z, \frac{w^*}{|w^*|}\right) dz \, ds \\
&= r^* |w^*|^3 C(r^*) C(|w^*|) \times 2.
\end{aligned}$$

Therefore, $C(r^*) = \frac{1}{r^*}$ and $C(|w^*|) = \frac{1}{|w^*|^3}$, and we deduce from (i)-(ii) that

$$h(r, w, r^*, w^*) = H(r, r^*) G(w, w^*) \leq \frac{C_1 C_2}{r^* |w^*|^3}.$$

Finally, we assume there exists $C > 0$ such that $\nu(r, w) \leq Cr|w|^3$. Notice, in particular, that we do not require $\nu(r, w) = 0$ for r and w sufficiently small (cf. [25]). Combining the previous two inequalities establishes the bound on νh required in Section 3.2.

Chapter 4

L^2 Stability Estimates for Shock Solutions of Scalar Conservation Laws Using the Relative Entropy Method

4.1 Introduction

For scalar nonviscous conservation laws with general flux, it is well-known from the theory of Kružkov [35] that the solution operator for the initial value problem

$$\begin{cases} \partial_t U + \partial_x A(U) = 0; \\ U(x,0) = U^0(x), \end{cases} \tag{4.1.1}$$

forms an L^1-contraction semigroup. As a result, the measure in $L^1(\mathbb{R})$ of the difference of any pair of entropy solutions is non-increasing over time. In particular, if $\phi(x - \sigma t)$ is a shock wave and $U^0 - \phi \in L^1(\mathbb{R})$, then

$$\|U(\cdot,t) - \phi(\cdot - \sigma t)\|_{L^1(\mathbb{R})} \leq \|U^0 - \phi\|_{L^1(\mathbb{R})} \tag{4.1.2}$$

for all $t \geq 0$.

While Kružkov's estimate is only valid for scalar conservation laws (cf. Temple [43]), global stability estimates for shocks with respect to the L^1 metric have also been obtained in the case of hyperbolic systems of conservation laws, at least for sufficiently small perturbations of suitably weak shock waves. One such result is established by Bressan et al. in [12], where the authors establish L^1 stability estimates corresponding to an "almost" contractive semigroup structure.

On the other hand, for systems of conservation laws with a convex extension, it is well-known that the relative entropy method developed by Dafermos [19] and DiPerna [23] provides L^2 stability estimates for solutions away from shocks. As illustrated in [21], one obtains stability estimates of the form

$$\|U(\cdot,t) - \overline{U}(\cdot,t)\|_{L^2([-R,R])} \leq ae^{bt}\|U^0 - \overline{U}^0\|_{L^2([-R-st,R+st])}, \tag{4.1.3}$$

where U and $\overline{U}$ are weak and strong solutions, respectively, and where a, b, and s are constants depending on the initial data U_0 and $\overline{U}_0$. However, simple examples show that this kind of result cannot hold when $\overline{U}$ is a shock wave or more generally when $\overline{U}$ is only a weak solution of the conservation law. Nonetheless, as we will prove, one can expect similar estimates to hold up to a suitable translation of the shock.

Consider the initial value problem (4.1.1) for a scalar conservation law in one space dimension. Our goal is to prove the the following global L^2 stability estimate for shocks.

Theorem 4.1.1. *Let $U^0 \in L^\infty(\mathbb{R})$ and assume $U^0 - \phi \in L^2(\mathbb{R})$ where*

$$\phi(x) = \begin{cases} C_L, & \text{if } x < 0; \\ C_R, & \text{if } x > 0, \end{cases} \tag{4.1.4}$$

with $C_L > C_R$. Further, assume U is the unique entropy solution of (4.1.1), for a smooth flux $A : \mathbb{R} \to \mathbb{R}$ verifying $A'' > 0$. Then there exists a Lipschitz continuous function $\overline{x} : [0, \infty) \to \mathbb{R}$ and a constant $\lambda(\|U^0\|_{L^\infty}; \phi; A) > 0$ such that

$$\|U(\cdot, t) - \phi(\cdot - \sigma t - \overline{x}(t))\|_{L^2(\mathbb{R})} \leq \|U^0 - \phi\|_{L^2(\mathbb{R})} \tag{4.1.5}$$

and

$$|\overline{x}(t)| \leq \lambda \|U^0 - \phi\|_{L^2(\mathbb{R})} \sqrt{t} \tag{4.1.6}$$

for all $t \geq 0$, where σ is given by the relation $\sigma(C_L - C_R) = A(C_L) - A(C_R)$.

In the same spirit as [12], our result can be characterized as an almost contractive variation of Kružkov's estimate (4.1.2). In fact, we will prove a slightly more general result (Theorem 4.3.1) in Section 4.3 which takes into account all strictly convex entropies associated with (4.1.1). However, while Theorem 4.3.1 is interesting in its own right, it is important to keep in mind that the estimates gained from the relative entropy method are purely of type L^2, regardless of the specific convex entropy used.

In a related result, Chen et al. [16] have used the relative entropy method to obtain stability estimates for shocks in the context of gas dynamics.

Specifically, the authors establish the time-asymptotic stability of Riemann solutions with arbitrarily large oscillation for the 3×3 system of Euler equations in one space dimension. The present work is another attempt at developing a stability theory for shocks using relative entropy techniques, beginning with a treatment of scalar equations, as proposed by Vasseur in [22]. For further applications of the relative entropy method in the context of fluid dynamics and kinetic theory, we refer the reader to the papers [4, 5, 10, 11, 34, 41, 47].

Let us clarify a few things regarding estimates (4.1.5) and (4.1.6). First, we are interested in controlling the relative entropy globally in time. Therefore, unlike the formulation of many large-time stability results (cf. [27, 32, 33, 37, 38]), we are forced to take a *time-dependent* translation of the shock. Also, we should point out that (4.1.6) is only intended to show that the shift $\overline{x}(t)$ has a bound proportional to the size of the initial perturbation. The estimate captures neither the Lipschitz continuity of $\overline{x}$ at $t = 0$, nor the expected large-time behavior of $\overline{x}$. Indeed, based on our techniques, it is reasonable to expect the time-asymptotic convergence of $\overline{x}(t)$ to the shift, x_0, considered in the large-time stability analysis of Il'in and Oleinik [32], at least when the initial perturbation is in $L^1(\mathbb{R})$.

The proof of Theorem 4.1.1 relies heavily on the work of Dafermos; in particular, on the theory of generalized characteristics and contingent equations considered in [18]. Roughly, the idea is to find curves $x_L(t)$ and $x_R(t)$, initially positioned at the origin, which preserve the quantities

$$\mathcal{E}_L(t) = \int_{-\infty}^{x_L(t)} |U(x,t) - C_L|^2 \, dx \quad \text{and} \quad \mathcal{E}_R(t) = \int_{x_R(t)}^{\infty} |U(x,t) - C_R|^2 \, dx.$$

Quite surprisingly, this condition leads to the constraint $x_R(t) \leq x_L(t)$ for all $t \geq 0$, so that (4.1.5) holds for all functions $\overline{x}(t)$ satisfying $x_R(t) - \sigma t \leq \overline{x}(t) \leq x_L(t) - \sigma t$. The bound on $\overline{x}$ then follows easily from the local L^1 stability of entropy solutions.

The paper is organized as follows. In Section 4.2, we introduce the entropy and relative entropy inequalities associated with the conservation law (4.1.1), and establish some preliminary estimates related to those quantities. In Section 4.3, we describe our method in detail and present the proof of the main theorem. Finally, we include in the appendix an existence result for differential inclusions arising in the context of conservation laws.

4.2 Relative Entropy Estimates

Consider the scalar conservation law

$$\partial_t U + \partial_x A(U) = 0, \tag{4.2.1}$$

with smooth flux $A : \mathbb{R} \to \mathbb{R}$. We say that $U \in L^\infty(\mathbb{R} \times (0,\infty))$ is an entropy solution of (4.2.1) if U satisfies

$$\partial_t \eta(U) + \partial_x G(U) \leq 0,$$

in the sense of measures, for all entropy/entropy-flux pairs $(\eta, G) \in [C^\infty(\mathbb{R})]^2$ verifying $\eta'' \geq 0$ and $G' = \eta' A'$.

If $\overline{U}$ solves (4.2.1) in the classical sense, and we consider, associated to each convex η, the relative entropy function

$$\eta(U \mid \overline{U}) = \eta(U) - \eta(\overline{U}) - \eta'(\overline{U}) \cdot (U - \overline{U}),$$

then an entropy solution U verifies additionally the inequality

$$\begin{aligned} \partial_t \left[\eta(U \mid \overline{U})\right] + \partial_x \left[G(U) - G(\overline{U}) - \eta'(\overline{U}) \cdot (A(U) - A(\overline{U}))\right] \\ \leq -\partial_t \left[\eta'(\overline{U})\right] \cdot (U - \overline{U}) - \partial_x \left[\eta'(\overline{U})\right] \cdot (A(U) - A(\overline{U})), \end{aligned} \tag{4.2.2}$$

for the same entropy/entropy-flux pairs (η, G).

The idea of the relative entropy method is to use (4.2.2) to estimate the quantity $\int \eta(U \mid \overline{U})\, dx$ in time. When $\eta(U) = U^2$, this corresponds directly to estimates in the L^2 metric. We are specifically interested in making comparisons with constant functions $\overline{U}(x,t) = C$, in which case (4.2.2) reduces to the inequality

$$\partial_t \eta(U \mid C) + \partial_x F(U, C) \leq 0, \tag{4.2.3}$$

where

$$\begin{aligned} F(U, C) &= G(U) - G(C) - \eta'(C) \cdot (A(U) - A(C)) \\ &= \int_C^U (\eta'(w) - \eta'(C)) A'(w)\, dw, \end{aligned} \tag{4.2.4}$$

represents the flux of relative entropy.

The goal of this section is to establish some preliminary estimates related to relative entropy inequality (4.2.3). Before presenting those results, let us remind the reader of the following facts.

Remark. Throughout the paper, we assume that U is an entropy solution of (4.2.1) with initial data $U^0 \in L^\infty(\mathbb{R})$. We assume also that the flux function, $A : \mathbb{R} \to \mathbb{R}$, is strictly convex and smooth, so that $A'' \circ U \geq \alpha > 0$. With these assumptions, Oleinik's estimate implies that $U(\cdot, t) \in BV_{loc}(\mathbb{R})$ for all $t > 0$. Therefore, the one-sided limits $U(x-, t)$ and $U(x+, t)$ exist and verify $U(x-, t) \geq U(x+, t)$ for all $x \in \mathbb{R}$ and for all $t > 0$. Furthermore, the trace theorem of Vasseur [44] shows that $U(\cdot, t)$ is continuous with values in $L^1_{loc}(\mathbb{R})$ up to $t = 0$.

We begin with the following estimate, which is an adaptation of a key lemma in [18] [Lemma 3.2] as it applies to (4.2.3).

Lemma 4.2.1. *Assume $\eta : \mathbb{R} \to \mathbb{R}$ is smooth and convex, and let $x_L : [t_0, T] \to \mathbb{R}$ and $x_R : [t_0, T] \to \mathbb{R}$ be Lipschitz continuous functions such that for all $t \in [t_0, T]$, $x_R(t) - x_L(t) \geq \delta$ for a fixed $\delta > 0$. Then for $C \in \mathbb{R}$ and for all a and b with $t_0 \leq a < b \leq T$,*

$$\begin{aligned}\int_{x_L(b)}^{x_R(b)} \eta(U(x, b) \mid C)\, dx - \int_{x_L(a)}^{x_R(a)} \eta(U(x, a) \mid C)\, dx \qquad (4.2.5)\\ \leq \int_a^b F(U(x_L(t)+, t), C) - \dot{x_L}(t)\eta(U(x_L(t)+, t) \mid C)\, dt\\ - \int_a^b F(U(x_R(t)-, t), C) - \dot{x_R}(t)\eta(U(x_R(t)-, t) \mid C)\, dt,\end{aligned}$$

for any entropy solution $U \in L^\infty(\mathbb{R} \times (0,\infty))$ of (4.2.1).

Proof. For $\varepsilon < \frac{\delta}{2}$, let $\psi_\varepsilon(\cdot,t)$ be an "inner" approximation to the characteristic function on $[x_L(t), x_R(t)]$ (instead of the "outer" approximation taken in [18]). Specifically, let

$$\psi_\varepsilon(x,t) = \begin{cases} 0, & \text{if } x < x_L(t); \\ \frac{1}{\varepsilon}(x - x_L(t)), & \text{if } x_L(t) \le x < x_L(t) + \varepsilon; \\ 1, & \text{if } x_L(t) + \varepsilon \le x \le x_R(t) - \varepsilon; \\ -\frac{1}{\varepsilon}(x - x_R(t)), & \text{if } x_R(t) - \varepsilon < x \le x_R(t); \\ 0, & \text{if } x_R(t) < x. \end{cases}$$

Also, as in [18] let

$$\chi_\varepsilon(t) = \begin{cases} 0, & \text{if } t < a; \\ \frac{1}{\varepsilon}(t - a), & \text{if } a \le t < a + \varepsilon; \\ 1, & \text{if } a + \varepsilon \le t \le b; \\ -\frac{1}{\varepsilon}(t - (b + \varepsilon)), & \text{if } b < t \le b + \varepsilon; \\ 0, & \text{if } b + \varepsilon < t. \end{cases}$$

Applying the non-negative test function $\varphi_\varepsilon(x,t) = \psi_\varepsilon(x,t)\chi_\varepsilon(t)$ to (4.2.3) and taking $\varepsilon \to 0$ yields (4.2.5). ∎

Now consider the Riemann data ϕ given by (4.1.4). Continuing in the spirit of [18] we have the following estimate.

Lemma 4.2.2. *Let ϕ be defined by (4.1.4) and suppose $U^0 \in L^\infty(\mathbb{R})$ satisfies $\|U^0 - \phi\|_{L^2(\mathbb{R})} < +\infty$. Also, assume $\eta : \mathbb{R} \to \mathbb{R}$ is smooth and convex, and let*

$x_L : [t_0, T] \to \mathbb{R}$ and $x_R : [t_0, T] \to \mathbb{R}$ be Lipschitz continuous functions. Then for all a and b with $t_0 \leq a < b \leq T$,

$$\begin{aligned}
&\left\{ \int_{-\infty}^{x_L(b)} \eta(U(x,b) \mid C_L)\, dx + \int_{x_R(b)}^{\infty} \eta(U(x,b) \mid C_R)\, dx \right\} \\
&- \left\{ \int_{-\infty}^{x_L(a)} \eta(U(x,a) \mid C_L)\, dx + \int_{x_R(a)}^{\infty} \eta(U(x,a) \mid C_R)\, dx \right\} \qquad (4.2.6) \\
&\leq - \int_a^b F(U(x_L(t)-,t), C_L) - \dot{x_L}(t)\eta(U(x_L(t)-,t) \mid C_L)\, dt \\
&\quad + \int_a^b F(U(x_R(t)+,t), C_R) - \dot{x_R}(t)\eta(U(x_R(t)+,t) \mid C_R)\, dt,
\end{aligned}$$

where U is the unique entropy solution of (4.2.1) with initial data U^0.

Proof. Choose R sufficiently large so that $-R < x_L(t) - \varepsilon$ and consider instead the test function

$$\psi_{\varepsilon,R}(x,t) = \begin{cases} 0, & \text{if } x < -2R; \\ \frac{1}{R}(x+2R), & \text{if } -2R \leq x < -R; \\ 1, & \text{if } -R \leq x \leq x_L(t) - \varepsilon; \\ -\frac{1}{\varepsilon}(x - x_L(t)), & \text{if } x_L(t) - \varepsilon < x \leq x_L(t); \\ 0, & \text{if } x_L(t) < x. \end{cases}$$

Testing (4.2.3) with $\varphi_{\varepsilon,R}(x,t) = \psi_{\varepsilon,R}(x,t)\chi_\varepsilon(t)$ and taking $\varepsilon \to 0$ and $R \to \infty$ we get

$$\begin{aligned}
&\int_{-\infty}^{x_L(b)} \eta(U(x,b) \mid C_L)\, dx - \int_{-\infty}^{x_L(a)} \eta(U(x,a) \mid C_L)\, dx. \\
&\qquad \leq \int_a^b \left\{ \lim_{R\to\infty} \frac{1}{R} \int_{-2R}^{-R} F(U(x,t), C_L)\, dx \right\} dt \\
&\qquad - \int_a^b F(U(x_L(t)-,t), C_L) - \dot{x_L}(t)\eta(U(x_L(t)-,t) \mid C_L)\, dt.
\end{aligned}$$

Since $\|U^0 - \phi\|_{L^2(\mathbb{R})} < +\infty$ and

$$|F(U(x,t), C_L)| \leq \frac{L_\eta}{2} \left[\sup_{|w| \leq \|U^0\|_{L^\infty} + |C_L|} |A'(w)| \right] |U(x,t) - C_L|^2$$

on account of (4.2.4), the first term on the right-hand side above vanishes. The relative entropy on the right is controlled by a similar argument. ∎

The following corollary is immediate.

Corollary 4.2.3. *Assume that the hypotheses of Lemma 4.2.2 are satisfied, and assume additionally that*

(i) $\dot{x_L}(t) \leq f(U(x_L(t)-, t), C_L)$ *for almost every* $t \in [t_0, T]$*, and*

(ii) $\dot{x_R}(t) \geq f(U(x_R(t)+, t), C_R)$ *for almost every* $t \in [t_0, T]$*,*

where $f(U, C) = \frac{F(U,C)}{\eta(U|C)}$*. Then, for all* a *and* b *with* $t_0 \leq a < b \leq T$*,*

$$\left\{ \int_{-\infty}^{x_L(b)} \eta(U(x,b) \mid C_L)\, dx + \int_{x_R(b)}^{\infty} \eta(U(x,b) \mid C_R)\, dx \right\}$$
$$- \left\{ \int_{-\infty}^{x_L(a)} \eta(U(x,a) \mid C_L)\, dx + \int_{x_R(a)}^{\infty} \eta(U(x,a) \mid C_R)\, dx \right\} \leq 0.$$

4.2.1 The Normalized Relative Entropy Flux

Consider the function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, which we call the normalized relative entropy flux, defined by

$$f(U, C) = \frac{F(U,C)}{\eta(U \mid C)} = \int_C^U \varphi(w, U, C) A'(w)\, dw, \tag{4.2.7}$$

where

$$\varphi(w, U, C) = \frac{\eta'(w) - \eta'(C)}{\eta(U \mid C)}.$$

The estimates obtained in Section 4.3 are based on the fact that for strictly convex functions A and η, the function defined by (4.2.7) is Lipschitz and increasing in the variables U and C, respectively. In order to prove this, let us first compute the first order partial derivatives of f. Assume $U \neq C$. Then,

$$\begin{aligned} \frac{\partial f}{\partial U}(U, C) &= \varphi(U, U, C)A'(U) + \int_C^U \frac{\partial \varphi}{\partial U}(w, U, C)A'(w)\, dw \\ &= \varphi(U, U, C)\left[A'(U) - f(U, C)\right] \end{aligned} \tag{4.2.8}$$

and

$$\begin{aligned} \frac{\partial f}{\partial C}(U, C) &= \int_C^U \frac{\partial \varphi}{\partial C}(w, U, C)A'(w)\, dw \\ &= \frac{-\eta''(C) \cdot (U - C)}{\eta(U \mid C)} \left[\frac{A(U) - A(C)}{U - C} - f(U, C)\right]. \end{aligned} \tag{4.2.9}$$

Using these formulas and taking suitable Taylor approximations, one can show that f and its gradient have a continuous extension to the line $U = C$. In particular, since

$$\int_C^U \varphi(w, U, C)\, dw = 1, \tag{4.2.10}$$

for all $U \neq C$, we define $f(C, C) = A'(C)$ for all $C \in \mathbb{R}$.

As suggested above, we would like to show that on bounded subsets of $\mathbb{R}^2$, (4.2.8) and (4.2.9) are bounded by positive constants from above and below, respectively. The latter estimate is delicate and relies on the following lemma.

Lemma 4.2.4. *Let $g : [0,1] \to \mathbb{R}$ and $h : [0,1] \to \mathbb{R}$ be continuously differentiable functions such that $g'(s) \geq \varepsilon_g > 0$ and $h'(s) \geq \varepsilon_h > 0$ for all $s \in (0,1)$. Further suppose $\int_0^1 g(s)\,ds = 1$. Then,*

$$\int_0^1 g(s)h(s)\,ds - \int_0^1 h(s)\,ds \geq \frac{\varepsilon_g \varepsilon_h}{12}.$$

Proof. Since g is increasing and $\int_0^1 g(s)\,ds = 1$, there exists a unique $s_0 \in (0,1)$ such that $g(s_0) = 1$. Therefore,

$$\begin{aligned}
\int_0^1 g(s)h(s)\,ds - \int_0^1 h(s)\,ds &= \int_0^1 (g(s) - g(s_0))h(s)\,ds \\
&= \int_0^1 (g(s) - g(s_0))(h(s) - h(s_0))\,ds \\
&= \int_0^1 \left[\int_{s_0}^s g'(\tau)\,d\tau\right]\left[\int_{s_0}^s h'(\tau)\,d\tau\right] ds \\
&\geq \varepsilon_g \varepsilon_h \int_0^1 (s - s_0)^2\,ds \geq \frac{\varepsilon_g \varepsilon_h}{12},
\end{aligned}$$

where we used the fact that the last term is minimized at $s_0 = \frac{1}{2}$. ∎

We can now prove our original claim.

Lemma 4.2.5. *Assume $A : \mathbb{R} \to \mathbb{R}$ and $\eta : \mathbb{R} \to \mathbb{R}$ are smooth, strictly convex functions, and let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by (4.2.7). Then, for bounded sets $\Omega \subset \mathbb{R} \times \mathbb{R}$, we have*

(i) $0 \le \left.\dfrac{\partial f}{\partial U}\right|_{\Omega} \le L_{\Omega}$, *and*

(ii) $0 < \varepsilon_{\Omega} \le \left.\dfrac{\partial f}{\partial C}\right|_{\Omega}$,

where ε_{Ω} and L_{Ω} are constants depending on Ω.

Proof. First, choose R sufficiently large so that $|(U, C)| \le R$ for all $(U, C) \in \Omega$, and let ε_A, ε_{η}, L_A, and L_{η} be positive constants such that we have the bounds

$$0 < \varepsilon_A \le A''(\xi) \le L_A; \tag{4.2.11}$$

$$0 < \varepsilon_{\eta} \le \eta''(\xi) \le L_{\eta}, \tag{4.2.12}$$

for all $|\xi| \le R$.

Taking into account (4.2.10) and the fact that A' and η' are increasing, the quantity computed in (4.2.8), for $U \ne C$, can be controlled in the following way.

$$\begin{aligned} 0 \le \frac{\partial f}{\partial U}(U, C) &\le \varphi(U, U, C)\,[A'(U) - A'(C)] \\ &= \frac{\int_C^U \eta''(\xi)\,d\xi}{\eta(U \mid C)} \left[\int_C^U A''(\xi)\,d\xi\right] \\ &\le \frac{L_{\eta} L_A (U - C)^2}{\frac{\varepsilon_{\eta}}{2}(U - C)^2} = \frac{2 L_{\eta} L_A}{\varepsilon_{\eta}}. \end{aligned}$$

Since the bound is independent of $|U - C|$, the estimate holds also for $U = C$. This proves part (i) of the lemma.

Next, assume $U \neq C$ and observe that, after a change of variables, equation (4.2.9) can be rewritten in the form

$$\frac{\partial f}{\partial C}(U, C) = \frac{-\eta''(C) \cdot |U - C|}{\eta(U \mid C)} \left[\int_0^1 h(s)\, ds - \int_0^1 g(s)h(s)\, ds \right], \qquad (4.2.13)$$

where h are g are defined respectively by

$$h(s) = sgn(U - C)A'(C + s(U - C))$$
$$g(s) = (U - C)\varphi(C + s(U - C), U, C).$$

Let us check that g and h satisfy the hypotheses of Lemma 4.2.4. First, according to (4.2.10), we have $\int_0^1 g(s)\, ds = 1$. Moreover, we compute

$$g'(s) = \frac{(U - C)^2 \eta''(C + s(U - C))}{\eta(U \mid C)} \geq \frac{\varepsilon_\eta (U - C)^2}{\frac{L_\eta}{2}(U - C)^2} = \frac{2\varepsilon_\eta}{L_\eta} > 0$$

and

$$h'(s) = |U - C| A''(C + s(U - C)) \geq |U - C| \varepsilon_A > 0.$$

Therefore, Lemma 4.2.4 applies and we deduce from (4.2.13) that

$$\frac{\partial f}{\partial C}(U, C) \geq \frac{-\eta''(C) \cdot |U - C|}{\eta(U \mid C)} \left[-\frac{\left(\frac{2\varepsilon_\eta}{L_\eta}\right) |U - C| \varepsilon_A}{12} \right]$$
$$\geq \frac{\varepsilon_A}{12} \left(\frac{2\varepsilon_\eta}{L_\eta} \right)^2 = \frac{\varepsilon_A {\varepsilon_\eta}^2}{3{L_\eta}^2} > 0.$$

Again, since this bound does not depend on $|U - C|$, the estimate extends to the line $U = C$. Therefore, we have (ii) and the proof of the lemma is complete. ∎

Note that the proof of Lemma 4.2.5 in the case $\eta(U) = U^2$ is much simpler. Indeed, a change of variables in (4.2.7) yields

$$f(U,C) = 2\int_0^1 sA'(C + (U - C)s)\, ds. \tag{4.2.14}$$

Therefore, differentiating (4.2.14) with respect to U and C and using (4.2.11) we get

$$0 < \frac{2}{3}\varepsilon_A \leq \left.\frac{\partial f}{\partial U}\right|_\Omega \leq \frac{2}{3}L_A$$

and

$$0 < \frac{1}{3}\varepsilon_A \leq \left.\frac{\partial f}{\partial C}\right|_\Omega \leq \frac{1}{3}L_A.$$

4.3 A Relative Entropy Technique for Shocks

It is well-known that when A is convex and when the initial data ϕ, given by (4.1.4), is non-increasing, the unique entropy solution of the Riemann problem is the traveling shock wave $\phi(x - \sigma t)$, where the shock speed σ is given by the Rankine-Hugoniot relation. The goal of this section is to show that traveling shock waves are stable in the sense of Theorem 4.1.1. In fact, we will show a slightly more general result.

Theorem 4.3.1. *Let $U^0 \in L^\infty(\mathbb{R})$ and assume $U^0 - \phi \in L^2(\mathbb{R})$ where ϕ is given by (4.1.4) with $C_L > C_R$. Further, assume U is the unique entropy solution of (4.1.1), for a smooth flux function $A : \mathbb{R} \to \mathbb{R}$ verifying $A'' > 0$. Then, for any smooth $\eta : \mathbb{R} \to \mathbb{R}$ verifying $\eta'' > 0$, there exists a Lipschitz continuous function $\overline{x} : [0, \infty) \to \mathbb{R}$ and a constant $\lambda(\|U^0\|_{L^\infty}; \phi; A; \eta) > 0$ such that*

$$\int_{-\infty}^{\infty} \eta(U(x,t) \mid \phi(x - \sigma t - \overline{x}(t)))\,dx \leq \int_{-\infty}^{\infty} \eta(U^0(x) \mid \phi(x))\,dx < \infty, \quad (4.3.1)$$

and

$$|\overline{x}(t)| \leq \lambda \|U^0 - \phi\|_{L^2(\mathbb{R})} \sqrt{t} \quad (4.3.2)$$

for all $t \geq 0$, where σ is given by the relation $\sigma(C_L - C_R) = A(C_L) - A(C_R)$.

In order to show (4.3.1) for a strictly convex entropy η, the idea is to construct curves $x_L : [0, \infty) \to \mathbb{R}$ and $x_R : [0, \infty) \to \mathbb{R}$, initialized with the data $x_L(0) = x_R(0) = 0$, for which the the total relative entropy

$$\mathcal{E}(t) = \int_{-\infty}^{x_L(t)} \eta(U(x,t) \mid C_L)\,dx + \int_{x_R(t)}^{\infty} \eta(U(x,t) \mid C_R)\,dx \quad (4.3.3)$$

is bounded above by $\mathcal{E}(0)$ for all $t \geq 0$. (Note that when $U^0 - \phi \in L^2(\mathbb{R})$ and $U^0 \in L^\infty(\mathbb{R})$, then $\mathcal{E}(0)$ is finite.) Due to the compressive nature of the solution $U(x,t)$, the construction will produce automatically the constraint

$x_R(t) \le x_L(t)$ for all $t \ge 0$. Therefore, (4.3.1) will follow for any function $\overline{x}(t)$ satisfying $x_R(t) \le \overline{x}(t) + \sigma t \le x_L(t)$. In particular, this includes the curves $\overline{x}(t) = x_L(t) - \sigma t$ and $\overline{x}(t) = x_R(t) - \sigma t$.

To control the total entropy, the idea is to exploit (4.2.6) by choosing $\dot{x_L}(t)$ and $\dot{x_R}(t)$ so that the right hand side vanishes. This makes sense at points of continuity of U; however, it turns out we do not have as much freedom at points where U is discontinuous. We borrow the following lemma from Dafermos [18].

Lemma 4.3.2. *Let* $x : [t_0, T] \to \mathbb{R}$, $0 \le t_0 < T < \infty$ *be a Lipschitz continuous function. Then for almost all* $t \in [t_0, T]$,

$$A(U(x(t)+,t)) - A(U(x(t)-,t)) - \dot{x}(t)\,[U(x(t)+,t) - U(x(t)-,t)] = 0.$$

The lemma simply asserts that if $x(t)$ moves along a discontinuity of U then its derivative must coincide with the shock speed given by the Rankine-Hugoniot condition.

Next, motivated by the idea of generalized characteristics, we consider a curve $x(t)$ solving in the sense of Filippov (see appendix), the differential inclusion

$$\dot{x}(t) \in [f(U(x(t)+,t),C), f(U(x(t)-,t),C)], \tag{4.3.4}$$

where $f(U,C) = \frac{F(U,C)}{\eta(U|C)}$. In view of Lemma 4.3.2, we find that (4.3.4) is actually quite restrictive.

Proposition 4.3.3. *Let $x : [t_0, T] \to \mathbb{R}$ be a Filippov solution of (4.3.4) on the interval $[t_0, T]$. Then for almost all $t \in [t_0, T]$,*

$$\dot{x}(t) = \begin{cases} f(U(x(t)\pm, t), C), & \text{if } U(x(t)-, t) = U(x(t)+, t); \\ \frac{A(U(x(t)+,t)) - A(U(x(t)-,t))}{U(x(t)+,t) - U(x(t)-,t)}, & \text{if } U(x(t)-, t) > U(x(t)+, t). \end{cases}$$

With these facts in mind, we consider functions $x_L : [0, \infty) \to \mathbb{R}$ and $x_R : [0, \infty) \to \mathbb{R}$, with initial values $x_L(0) = x_R(0) = 0$, solving

$$\begin{cases} \dot{x_L}(t) \in [f(U(x_L(t)+, t), C_L), f(U(x_L(t)-, t), C_L)], \\ \dot{x_R}(t) \in [f(U(x_R(t)+, t), C_R), f(U(x_R(t)-, t), C_R)], \end{cases} \tag{4.3.5}$$

in the sense of Filippov. Existence (and uniqueness beyond $t = 0$) is guaranteed by Proposition 4.4.1 (see appendix) and Lemma 4.2.5. Given (4.3.5), it follows immediately from Corollary 4.2.3 that the total entropy $\mathcal{E}(t)$ in (4.3.3) is bounded above by $\mathcal{E}(0)$. Moreover, since

$$\eta(U \mid C) = \int_C^U \int_C^w \eta''(\xi)\, d\xi\, dw$$

is non-negative, we easily deduce the following lemma.

Lemma 4.3.4. *Assume $x_L : [0, \infty) \to \mathbb{R}$ and $x_R : [0, \infty) \to \mathbb{R}$ verify (4.3.5) in the sense of Filippov. Further, assume $x_R(t) \le x_L(t)$ for all $t \ge 0$. Then for any function $x : [0, \infty) \to \mathbb{R}$ satisfying $x_R(t) \le x(t) \le x_L(t)$ for all $t \ge 0$, we have*

$$\int_{-\infty}^{x(t)} \eta(U(x,t) \mid C_L)\, dx + \int_{x(t)}^{\infty} \eta(U(x,t) \mid C_R)\, dx \leq \mathcal{E}(t) \leq \mathcal{E}(0),$$

for all $t \geq 0$*, where* $\mathcal{E}$ *is defined by (4.3.3).*

Given Lemma 4.3.4, it remains to show that when (4.3.5) holds and when $C_L > C_R$, then $x_R(t) \leq x_L(t)$ for all $t \geq 0$. Since x_L and x_R coincide at $t = 0$ and they move continuously, the idea is to show that x_R cannot pass x_L; that is, we would like to argue that when $x_R(t) = x_L(t)$, then in some sense $\dot{x_R}(t) \leq \dot{x_L}(t)$. This is not precise, of course, since the derivatives may be only measurable; however, note that when $x_R(t) = x_L(t) = \tilde{x}$ and $(\tilde{x}, t)$ is a point of continuity of U, then, formally, the monotonicity of f (Lemma 4.2.5) implies

$$\dot{x_L}(t) = f(U(\tilde{x}, t), C_L) > f(U(\tilde{x}, t), C_R) = \dot{x_R}(t).$$

This suggests that x_R may only pass x_L at a discontinuity of U; however, as we will show, Lemma 4.2.5 does not allow it. We will argue by contradiction, but first let us prove the following lemmas.

Lemma 4.3.5. *Assume* $x_L : [t_0, T] \to \mathbb{R}$ *and* $x_R : [t_0, T] \to \mathbb{R}$ *verify (4.3.5) in the sense of Filippov on the interval* $[t_0, T]$*, with* $C_L > C_R$*. Further, assume that* $x_R(t) - x_L(t) \geq \delta > 0$ *for all* $t \in [t_0, T]$*. Then for* $C \in (C_R, C_L)$ *there exists* $\lambda > 0$ *independent of* δ*, such that for all* a *and* b *with* $t_0 \leq a < b \leq T$*,*

$$\int_{x_L(b)}^{x_R(b)} \eta(U(x,b) \mid C)\, dx - \int_{x_L(a)}^{x_R(a)} \eta(U(x,a) \mid C)\, dx \leq -|S_{a,b}|\lambda \leq 0. \quad (4.3.6)$$

where $S_{a,b} = \{s \in [a,b] \mid \dot{x_R}(s) - \dot{x_L}(s) \geq 0\}$*.*

Proof. Using Lemma 4.2.1, for all a and b with $t_0 \leq a < b \leq T$,

$$\begin{aligned}
&\int_{x_L(b)}^{x_R(b)} \eta(U(x,b) \mid C)\, dx - \int_{x_L(a)}^{x_R(a)} \eta(U(x,a) \mid C)\, dx \\
&\leq \int_a^b \eta(U(x_L(t)+,t) \mid C) \left[f(U(x_L(t)+,t),C) - \dot{x_L}(t)\right] dt \\
&\quad - \int_a^b \eta(U(x_R(t)-,t) \mid C) \left[f(U(x_R(t)-,t),C) - \dot{x_R}(t)\right] dt \qquad (4.3.7)\\
&\leq \int_a^b \eta(U(x_L(t)+,t) \mid C) \left[f(U(x_L(t)+,t),C) - f(U(x_L(t)+,t),C_L)\right] dt \\
&\quad - \int_a^b \eta(U(x_R(t)-,t)|C) \left[f(U(x_R(t)-,t),C) - f(U(x_R(t)-,t),C_R)\right] dt,
\end{aligned}$$

where we used (4.3.5) to get the last inequality. Now, for $t \in S_{a,b}$

$$f(U(x_L(t)+,t),C_L) \leq \dot{x_L}(t) \leq \dot{x_R}(t) \leq f(U(x_R(t)-,t),C_R).$$

Therefore, since $U \in L^\infty$, we deduce using Lemma 4.2.5 that

$$\begin{aligned}
f(U(x_R(t)-,t),C_L) &- f(U(x_L(t)+,t),C_L) \\
&> f(U(x_R(t)-,t),C_L) - f(U(x_R(t)-,t),C_R) \\
&= \int_{C_R}^{C_L} \frac{\partial f}{\partial C}(U(x_R(t)-,t),z)\, dz \\
&> \varepsilon(C_L - C_R) > 0, \qquad (4.3.8)
\end{aligned}$$

for $t \in S_{a,b}$, where $\varepsilon > 0$ is a lower bound on $\frac{\partial f}{\partial C}$. We deduce further, using (4.3.8) and Lemma 4.2.5, that

$$U(x_R(t)-,t) - U(x_L(t)+,t) > \frac{\varepsilon}{L}(C_L - C_R) > 0,$$

for $t \in S_{a,b}$, where $L > 0$ is an upper bound on $\frac{\partial f}{\partial U}$. Thus, for $t \in S_{a,b}$, either

(i) $\eta(U(x_R(t)-,t) \mid C) \geq \kappa(U(x_R(t)-,t) - C)^2 \geq \kappa\left[\frac{\varepsilon}{2L}(C_L - C_R)\right]^2$, or

(ii) $\eta(U(x_L(t)+,t) \mid C) \geq \kappa(U(x_L(t)+,t) - C)^2 \geq \kappa\left[\frac{\varepsilon}{2L}(C_L - C_R)\right]^2$

where $\kappa > 0$ is a lower bound for $\frac{1}{2}\eta''(\cdot)$. Returning to (4.3.7), we get

$$\begin{aligned}
&\int_{x_L(b)}^{x_R(b)} \eta(U(x,b) \mid C)\, dx - \int_{x_L(a)}^{x_R(a)} \eta(U(x,a) \mid C)\, dx \\
&\leq \int_a^b \eta(U(x_L(t)+,t) \mid C) \left[-\int_C^{C_L} \frac{\partial f}{\partial C}(U(x_L(t)+,t),z)\, dz\right] dt \\
&\qquad - \int_a^b \eta(U(x_R(t)-,t) \mid C) \left[\int_{C_R}^{C} \frac{\partial f}{\partial C}(U(x_R(t)-,t),z)\, dz\right] dt \\
&\leq -\varepsilon K \int_a^b \left[\eta(U(x_L(t)+,t) \mid C) + \eta(U(x_R(t)-,t) \mid C)\right] dt \\
&\leq -\varepsilon K\kappa \left[\frac{\varepsilon}{2L}(C_L - C_R)\right]^2 |S_{a,b}|,
\end{aligned} \tag{4.3.9}$$

where $K = \min\{C_L - C, C - C_R\}$ (and we used the fact that $\eta(U \mid C)$ is non-negative). This proves (4.3.6) with $\lambda = \varepsilon K\kappa\left[\frac{\varepsilon}{2L}(C_L - C_R)\right]^2$. ∎

Lemma 4.3.6. *There exists $1 < \kappa < \infty$ such that for any $\delta > 0$, if $x_L : [t_0, T] \to \mathbb{R}$ and $x_R : [t_0, T] \to \mathbb{R}$ verify*

$$\begin{cases}
\dot{x_L}(t) \in [f(U(x_L(t)+,t),C_L), f(U(x_L(t)-,t),C_L)], \\
\dot{x_R}(t) \in [f(U(x_R(t)+,t),C_R), f(U(x_R(t)-,t),C_R)], \\
x_R(t_0) - x_L(t_0) = \delta,
\end{cases}$$

in the sense of Filippov, with $C_L > C_R$, and if $x_R(t) - x_L(t) \geq \delta$ for all $t \in [t_0, T]$, then $x_R(t) - x_L(t) \leq \kappa\delta$ for all $t \in [t_0, T]$.

Proof. Given $t \in (t_0, T]$, let $S_{t_0,t} = \{s \in [t_0, t] \mid \dot{x_R}(s) - \dot{x_L}(s) \geq 0\}$. Then,

$$\begin{aligned} x_R(t) - x_L(t) &= \delta + \int_{t_0}^{t} [\dot{x_R}(s) - \dot{x_L}(s)]\, ds \\ &\leq \delta + \int_{S_{t_0,t}} [\dot{x_R}(s) - \dot{x_L}(s)]\, ds. \end{aligned} \tag{4.3.10}$$

Since $U \in L^\infty$, Lemma 4.2.5 implies $|\dot{x_R}(s) - \dot{x_L}(s)|$ is bounded by some constant M_1. Furthermore, on account of Lemma 4.3.5, for $C \in (C_R, C_L)$ there exists $\lambda > 0$ such that

$$\begin{aligned} 0 &\leq \int_{x_L(t)}^{x_R(t)} \eta(U(x,t) \mid C)\, dx \leq \int_{x_L(t_0)}^{x_R(t_0)} \eta(U(x,t_0) \mid C)\, dx - |S_{t_0,t}|\lambda \\ &\leq \int_{x_L(t_0)}^{x_R(t_0)} M_2\, dx - |S_{t_0,t}|\lambda = M_2\delta - |S_{t_0,t}|\lambda, \end{aligned}$$

where again we used $U \in L^\infty$ to bound $\eta(U \mid C)$ by a constant M_2. Therefore, $|S_{t_0,t}| \leq \delta \left(\frac{M_2}{\lambda}\right)$, and we deduce using (4.3.10) that $x_R(t) - x_L(t) \leq \kappa\delta$, where $\kappa = 1 + \frac{M_1 M_2}{\lambda}$. ∎

We can now prove our previous claim.

Proposition 4.3.7. *Assume $x_L : [0,\infty) \to \mathbb{R}$ and $x_R : [0,\infty) \to \mathbb{R}$ verify (4.3.5) in the sense of Filippov with $C_L > C_R$. Then $x_R(t) \leq x_L(t)$ for all $t \geq 0$.*

Proof. We argue by contradiction. Let $d(t) = x_R(t) - x_L(t)$ and suppose $d(T) > 0$ for some $T > 0$. Then for $0 < \delta < d(T)$ define

$$d^{-1}(\delta) = \{0 \leq t \leq T \mid d(t) = \delta\}$$

and let $t_\delta = \sup_{t \in d^{-1}(\delta)} t$. Since d is continuous and $d(0) = 0$, $d^{-1}(\delta)$ is nonempty and $t_\delta < T$. Also, we must have $d(t) \geq \delta$ for $t \in [t_\delta, T]$, otherwise t_δ would be larger. Therefore Lemma 4.3.6 applies with $t_0 = t_\delta$ and we conclude that $d(t) \leq \kappa\delta$ for all $t \in [t_\delta, T]$. In particular, $d(T) \leq \kappa\delta$ for δ arbitrarily small. This is a contradiction. ∎

Proof of Theorems 4.1.1 and 4.3.1. Since U is the unique entropy solution of (4.1.1), the inequality (4.2.3) holds in the sense of measures for any strictly convex entropy η. Thus, given x_L and x_R defined by (4.3.5), the estimates in Section 4.2 are valid, and we deduce from Proposition 4.3.7 that there exists a Lipschitz continuous function $\overline{x} : [0, \infty) \to \mathbb{R}$ such that $x_R(t) \leq \overline{x}(t) + \sigma t \leq x_L(t)$ for all $t \geq 0$.

Also, since $U^0 - \phi \in L^2(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty} \eta(U^0(x) \mid \phi(x))\, dx \leq \frac{L_\eta}{2} \int_{-\infty}^{\infty} (U^0(x) - \phi(x))^2\, dx < \infty,$$

where $L_\eta > 0$ is an upper bound on η''. Therefore, $\mathcal{E}(0)$, defined by (4.3.3), is finite and Lemma 4.3.4 implies

$$\int_{-\infty}^{\infty} \eta(U(x,t) \mid \phi(x - \sigma t - \overline{x}(t)))\, dx \leq \mathcal{E}(t) \leq \mathcal{E}(0) = \int_{-\infty}^{\infty} \eta(U^0(x)|\phi(x))\, dx,$$

which completes the proof of estimate (4.3.1). Also, this gives (4.1.5), for Theorem 4.1.1, in the case $\eta(U) = U^2$.

Finally, let us show that the function $\overline{x}$ has a bound proportional to the size of $U^0 - \phi$ in $L^2(\mathbb{R})$. Recalling that $x_R(t) \leq \overline{x}(t) + \sigma t \leq x_L(t)$, we have

$$|\overline{x}(t) + \sigma t| \leq \max\{|x_R(t)|, |x_L(t)|\} \leq Lt,$$

since x_L and x_R are Lipschitz (Proposition 4.4.1) and $x_L(0) = x_R(0) = 0$. Note that L depends on $\|U^0\|_{L^\infty}$, ϕ, η and A, as the velocities $\dot{x_L}$ and $\dot{x_R}$ are given by (4.2.7). Next, observe that $\phi(x - \sigma t - \overline{x}(t)) - \phi(x - \sigma t)$ has support contained in the interval $[-(L + |\sigma|)t, (L + |\sigma|)t]$. Therefore,

$$\begin{aligned}(C_L - C_R)|\overline{x}(t)| &= \|\phi(\cdot - \sigma t - \overline{x}(t)) - \phi(\cdot - \sigma t)\|_{L^1(B_{(L+|\sigma|)t})} \\ &\leq \|\phi(\cdot - \sigma t - \overline{x}(t)) - U(\cdot, t)\|_{L^1(B_{(L+|\sigma|)t})} \\ &+ \|U(\cdot, t) - \phi(\cdot - \sigma t)\|_{L^1(B_{(L+|\sigma|)t})} \qquad (4.3.11)\end{aligned}$$

Then, by the L^1-stability theory of Kružkov, the last term is bounded by $\|U^0 - \phi\|_{L^1(B_{(M+L+|\sigma|)t})}$, where $M = \sup\{|A'(w)|; |w| \leq \|U^0\|_{L^\infty} + \|\phi\|_{L^\infty}\}$. Therefore, proceeding with the estimate (4.3.11), using Hölder's inequality, we get

$$\begin{aligned}(C_L - C_R)|\overline{x}(t)| \leq{}& \sqrt{2(L + |\sigma|)t} \cdot \|\phi(\cdot - \sigma t - \overline{x}(t)) - U(\cdot, t)\|_{L^2(\mathbb{R})} \\ &+ \sqrt{2(M + L + |\sigma|)t} \cdot \|U^0 - \phi\|_{L^2(\mathbb{R})} \qquad (4.3.12)\end{aligned}$$

Finally, since $\frac{\varepsilon_\eta}{2}(U - \phi)^2 \leq \eta(U \mid \phi) \leq \frac{L_\eta}{2}(U - \phi)^2$, using (4.3.1) we have

$$\|\phi(\cdot - \sigma t - \overline{x}(t)) - U(\cdot, t)\|_{L^2(\mathbb{R})} \leq \sqrt{\frac{L_\eta}{\varepsilon_\eta}} \cdot \|U^0 - \phi\|_{L^2(\mathbb{R})},$$

which together with (4.3.12) implies

$$|\overline{x}(t)| \leq \frac{1}{C_L - C_R}\left[\sqrt{\frac{2L_\eta}{\varepsilon_\eta}(L + |\sigma|)} + \sqrt{2(M + L + |\sigma|)}\right] \|U^0 - \phi\|_{L^2(\mathbb{R})}\sqrt{t}.$$

This completes the proof. Note that, by construction, $\overline{x}$ is actually Lipschitz even though this estimate does not show it.

4.4 Filippov Solutions and Conservation Laws

In this section we include an existence result for differential equations arising in the context of conservation laws. While this result is an easy application of the theory of Filippov [26], and by no means original, we have found no explicit statement of this kind in the literature.

Definition. A solution of (4.4.1) in the sense of Filippov on an interval $[t_0, T)$ is an absolutely continuous function $x(t)$ for which (4.4.1) holds for almost every $t \in [t_0, T)$.

Proposition 4.4.1. *Let U be the unique entropy solution of (4.1.1) with $A'' > 0$ and $U^0 \in L^\infty(\mathbb{R})$. Also, let $g : \mathbb{R} \to \mathbb{R}$ be continuous and non-decreasing. Then for $(x_0, t_0) \in \mathbb{R} \times [0, \infty)$ there exists a Lipschitz continuous function $x : [t_0, \infty) \to \mathbb{R}$, with initial value $x(t_0) = x_0$, solving*

$$\dot{x}(t) \in [g(U(x(t)+, t)), g(U(x(t)-, t))], \tag{4.4.1}$$

in the sense of Filippov (Definition 4.4). Furthermore, if $t_0 > 0$ and g is Lipschitz on bounded subsets of $\mathbb{R}$, then the solution x is unique.

Proof. The existence of solutions follows from [26] [Section 2.7, Theorem 1] provided the set valued function $G(x, t) = [g(U(x+, t)), g(U(x-, t))]$ is upper semicontinuous. Roughly this means that in the limit $(x', t') \to (x, t)$, the sets $G(x', t')$ will be contained in $G(x, t)$. In the present setting, upper semicontinuity is an immediate consequence of the continuity of g and the structure of solutions to (4.1.1) detailed in [18] (see also Remark 4.2). Also, since $g \circ U \in L^\infty$, the Filippov solutions are Lipschitz and defined for all $t \geq t_0$.

Now assume additionally that g is Lipschitz on bounded subsets of $\mathbb{R}$, and let us verify that solutions extend uniquely beyond $t = 0$. Applying [26] [Section 2.10, Theorem 1], it suffices to check that for any $T > 0$ there exists $\ell \in L^1([t_0, T])$ such that for any almost every (x, t) and (y, t) in $\mathbb{R} \times [t_0, T]$

$$(x - y) \cdot (g(U(x, t)) - g(U(y, t))) \leq \ell(t)|x - y|^2.$$

The assumptions on g together with Oleinik's well-known decay estimate easily imply the statement above. ∎

Existence results of this type have appeared implicitly in the work of Marson and Colombo (see [17] and [39]) on ODEs related to traffic modeling. In fact, the uniqueness argument above can be found directly in [17] for the case of concave flux functions. The study of contingent equations in the context of conservation laws can also be found in the papers [13, 18, 20].

Chapter 5

Relative Entropy Estimates for Systems of Conservation Laws

5.1 Preliminaries

In this final chapter we will lay the groundwork for a relative entropy study of shocks in the more general context of systems of conservation laws. Specifically, we will highlight some of the important structural properties of systems endowed with a convex entropy, particularly as that structure pertains to relative entropy estimates.

We will consider $n \times n$ systems of conservation laws in one spatial dimension of the form

$$\partial_t U_i + \partial_x A_i(U) = 0; \quad x \in \mathbb{R}, \quad i = 1, 2, ..., n. \tag{5.1.1}$$

Let us fix a state domain $\mathcal{D} \subset \mathbb{R}^n$, and assume for simplicity that the flux function $A : \mathcal{D} \to \mathbb{R}^n$ is smooth. We are interested in systems (5.1.1) which have a special structure; namely, those for which there exists a strictly convex function $\eta \in C^\infty(\mathcal{D})$ such that $\nabla\eta \cdot \nabla A$ is the gradient of a scalar function. Such an η is called an entropy of the system. We recall briefly the notion of entropy solutions.

Definition. A weak solution $U \in L^\infty$ of (5.1.1) is called an *entropy solution* if

$$\partial_t \eta(U) + \partial_x G(U) \leq 0, \tag{5.1.2}$$

in the sense of distributions, for all smooth entropy/entropy-flux pairs (η, G) with G verifying

$$\partial_i G = \nabla \eta \cdot \partial_i A. \tag{5.1.3}$$

Note that for smooth entropy solutions, (5.1.2) holds as an equality. Indeed, this follows directly from (5.1.3).

Recall also that, with respect to any fixed vector $C \in \mathcal{D}$, entropy solutions verify, in the sense of distributions, the additional inequality

$$\partial_t \eta(U \mid C) + \partial_x F(U, C) \leq 0, \tag{5.1.4}$$

where the relative entropy $\eta(U \mid C)$ and the relative entropy flux $F(U, C)$ are defined respectively by

$$\eta(U \mid C) = \eta(U) - \eta(C) - \nabla \eta(C) \cdot (U - C) \tag{5.1.5}$$

$$F(U, C) = G(U) - G(C) - \nabla \eta(C) \cdot (A(U) - A(C)) \tag{5.1.6}$$

This is an immediate consequence of (5.1.1) and (5.1.2), as explained in Section 4.2 for scalar conservation laws. The argument in the case of systems is exactly the same.

5.2 The Normalized Relative Entropy Flux

In the previous chapter, our stability analysis relied on monotonicity properties of the normalized relative entropy flux which we defined as

$$f(U, C) = \frac{F(U, C)}{\eta(U \mid C)}. \tag{5.2.1}$$

For scalar conservation laws with strictly convex flux, we showed, among other properties, that f is continuous and increasing in each variable. In the case of systems, neither property holds. In fact, the later property no longer makes sense over a multi-dimensional domain $\mathcal{D} \times \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^n$. On the other hand, continuity fails only on the set $U = C$, and we will show that there is a nice characterization of the behavior of f near such points. We will need the following lemma regarding positive matrices.

Lemma 5.2.1. *Let $A, B \in \mathcal{M}_{n\times n}$ such that A is symmetric and positive-definite ($A > 0$) and the product AB is symmetric. Then B is diagonalizable and*

$$\lambda_1 |x|^2 \leq ((A^{\frac{1}{2}})^{-1}x)^\top AB((A^{\frac{1}{2}})^{-1}x) \leq \lambda_n |x|^2 \tag{5.2.2}$$

for all $x \in \mathbb{R}^n$, where $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of B. In particular if $\lambda_1 \geq 0$, then

$$z^\top ABz \geq 0$$

for all $z \in \mathbb{R}^n$; that is, AB is positive-semidefinite.

Proof. Recall $A^{\frac{1}{2}} = OD^{\frac{1}{2}}O^\top$, where $D = O^\top AO$ is a diagonalization of A by an orthogonal matrix O. By hypothesis, $AB = (AB)^\top = B^\top A^\top = B^\top A$, therefore left and right multiplication by $(A^{\frac{1}{2}})^{-1}$ yields

$$A^{\frac{1}{2}}B(A^{\frac{1}{2}})^{-1} = (A^{\frac{1}{2}})^{-1}B^\top A^{\frac{1}{2}} = (A^{\frac{1}{2}}B(A^{\frac{1}{2}})^{-1})^\top,$$

where we used the fact that $A^{\frac{1}{2}}$ is symmetric in the last equality. Since B is conjugate to a symmetric matrix, it is diagonalizable. Furthermore, since $A^{\frac{1}{2}}B(A^{\frac{1}{2}})^{-1}$ is symmetric, we know

$$\lambda_1|x|^2 \leq x^\top A^{\frac{1}{2}}B(A^{\frac{1}{2}})^{-1}x \leq \lambda_n|x|^2, \tag{5.2.3}$$

for all $x \in \mathbb{R}^n$, where $\lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of $A^{\frac{1}{2}}B(A^{\frac{1}{2}})^{-1}$ (and hence also the eigenvalues of B). Rewriting (5.2.3) we arrive at (5.2.2). ∎

Proposition 5.2.2. *Let $A : \mathcal{D} \to \mathbb{R}^n$, $\eta : \mathcal{D} \to \mathbb{R}$, and $G : \mathcal{D} \to \mathbb{R}$ be smooth functions verifying (5.1.3). Further, assume η is strictly convex. Then, $\nabla A(U)$ is diagonalizable for all $U \in \mathcal{D}$, and*

$$\lambda_{min}(U)D^2\eta(U) \leq D^2\eta(U)\nabla A(U) \leq \lambda_{max}(U)D^2\eta(U) \tag{5.2.4}$$

where $\lambda_{min}(U)$ and $\lambda_{max}(U)$ are the smallest and largest eigenvalues, respectively, of $\nabla A(U)$.

Proof. Differentiating (5.1.3), we obtain the relation

$$\partial_i\partial_j G(U) = \partial_j\partial_k\eta(U)\partial_i A^k(U) + \partial_k\eta(U)\partial_i\partial_j A^k(U), \tag{5.2.5}$$

where, as usual, we sum over the upper and lower index k. Since the first and last terms are symmetric with respect to the indices i and j, the same is true for the second term. Hence, $D^2\eta(U)\nabla A(U)$ is a symmetric $n \times n$ matrix. Since η is strictly convex, $D^2\eta(U)$ is positive-definite, and we deduce from Lemma 5.2.1 that $\nabla A(U)$ is diagonalizable. (This is well-known, and was observed by Friedrichs and Lax in [28]).

Now let us show that (5.2.4) holds. We will show the first part of the inequality. The second part follows by a similar argument. The point is to show that

$$D^2\eta(U)\nabla A(U) - \lambda_{\min}(U)D^2\eta(U) = D^2\eta(U)(\nabla A(U) - \lambda_{\min} I)$$

is positive-semidefinite. Since the left-hand side is the difference of symmetric terms, the matrix product on the right-hand side is also symmetric. Moreover, $D^2\eta(U)$ is positive-definite and $(\nabla A(U) - \lambda_{\min} I)$ has non-negative eigenvalues. Therefore, Lemma 5.2.1 implies that the aforementioned product has the desired property. This completes the proof. ∎

Proposition 5.2.3. *Under the hypotheses of Proposition 5.2.2, we have the estimates*

$$[f(U,C) - \lambda_{min}(C)]_- = O(|U - C|); \tag{5.2.6}$$

$$[f(U,C) - \lambda_{max}(C)]_+ = O(|U - C|), \tag{5.2.7}$$

as $U \to C$.

Proof. First let us show that

$$F(U,C) = (U-C)^\top \cdot D^2\eta(C)\nabla A(C)\cdot(U-C) + O(|U-C|^3) \qquad (5.2.8)$$

as $U \to C$. Indeed, adding and subtracting $\nabla G(C)\cdot(U-C)$ from (5.1.6), we have

$$\begin{aligned}
F(U,C) &= G(U) - G(C) - \nabla\eta(C)\cdot(A(U)-A(C)) \\
&= \{G(U) - G(C) - \nabla G(C)\cdot(U-C)\} \\
&\qquad - \nabla\eta(C)\cdot\{A(U)-A(C)-\nabla A(C)\cdot(U-C)\} \\
&= G(U \mid C) - \nabla\eta(C)\cdot A(U\mid C) \\
&= (U-C)^\top \left\{D^2G(C) - \nabla\eta(C)\cdot D^2A(C)\right\}(U-C) + O(|U-C|^3) \\
&= (U-C)^\top \cdot D^2\eta(C)\nabla A(C)\cdot(U-C) + O(|U-C|^3)
\end{aligned}$$

as $U \to C$, where we used (5.2.5) in the last equality. On the other hand,

$$\eta(U \mid C) = (U-C)^\top \cdot D^2\eta(C)\cdot(U-C) + O(|U-C|^3) \quad \text{as } U \to C.$$

Therefore, we deduce from Lemma 5.2.2 that

$$[F(U,C) - \lambda_{\min}(C)\eta(U\mid C)]_- = O(|U-C|^3); \qquad (5.2.9)$$

$$[F(U,C) - \lambda_{\max}(C)\eta(U\mid C)]_+ = O(|U-C|^3), \qquad (5.2.10)$$

as $U \to C$. Finally, since η is strictly convex, there is a neighborhood of C on which $\eta(U \mid C)$ is bounded below by a positive multiple of $|U-C|^2$. To verify our claim, observe

$$\eta(U \mid C) = \int_0^1 \int_0^t (U-C)^\top D^2\eta(C+s(U-C))\,(U-C)\,ds\,dt$$
$$\geq \int_0^1 \int_0^t \delta|U-C|^2\,ds\,dt = \frac{\delta}{2}|U-C|^2$$

for a fixed $\delta > 0$ sufficiently small and for $|U-C|$ small enough. Therefore, dividing (5.2.9) and (5.2.10) by $\eta(U \mid C)$ we obtain the desired result. ∎

5.3 Estimates Along Rarefaction Curves

Proposition 5.2.3 states roughly that $\lambda_{min}(C) \leq f(U,C) \leq \lambda_{max}(C)$ in the limit $U \to C$. In fact, we will show next that f attains these extremal values if we take $U \to C$ along eigencurves corresponding to λ_{min} and λ_{max}. The interesting point is that, along these eigencurves, or as we say in the context of conservation laws, rarefaction curves, f enjoys the same averaging structure that it has in the scalar case (cf. equation (4.2.7)).

Let us fix C in the state domain, and assume for simplicity that $\nabla A(U)$ has n distinct real eigenvalues $\lambda_1(U) < \lambda_2(U) < \cdots < \lambda_n(U)$, in a neighborhood of C. For a smooth choice of eigenvectors $\{r_k(U)\}_{k=1}^n$, we define a family of eigencurves $w_k(s)$ solving

$$\begin{cases} \dot{w}_k(s) = r_k(w_k(s)); \\ w_k(0) = C. \end{cases} \tag{5.3.1}$$

Now, if we consider $U = w_k(s_0)$, $s_0 > 0$, we can compute the flux of relative entropy, $F(U,C)$, using a path integral along w_k. Specifically, we have

$$\begin{aligned}
F(U,C) &= G(U) - G(C) - \nabla\eta(C)\cdot(A(U)-A(C)) \\
&= \int_0^{s_0} \left[\nabla G(w_k(s))\cdot \dot{w}_k(s) - \nabla\eta(w_k(0))\cdot\nabla A(w_k(s))\cdot\dot{w}_k(s)\right] ds \\
&= \int_0^{s_0} \left[\nabla\eta(w_k(s)) - \nabla\eta(w_k(0))\right]\cdot\left[\nabla A(w_k(s))\cdot\dot{w}_k(s)\right] ds \\
&= \int_0^{s_0} \left[\nabla\eta(w_k(s)) - \nabla\eta(w_k(0))\right]\cdot\left[\lambda_k(w_k(s))\dot{w}_k(s)\right] ds \\
&= \int_0^{s_0} \varphi_k(s)\lambda_k(w_k(s))\, ds,
\end{aligned}$$

where $\varphi_k(s) = [\nabla\eta(w_k(s)) - \nabla\eta(w_k(0))]\cdot\dot{w}_k(s)$. On the other hand, a similar computation with respect to relative entropy yields

$$\begin{aligned}
\eta(U \mid C) &= \int_0^{s_0} \left[\nabla\eta(w_k(s))\cdot\dot{w}_k(s) - \nabla\eta(w_k(0))\cdot\dot{w}_k(s)\right] ds \\
&= \int_0^{s_0} \left[\nabla\eta(w_k(s)) - \nabla\eta(w_k(0))\right]\cdot\dot{w}_k(s)\, ds = \int_0^{s_0} \varphi_k(s)\, ds.
\end{aligned}$$

Notice also that

$$\varphi_k(s) = \dot{w}_k(s)\cdot\int_0^s \frac{d}{d\tau}\nabla\eta(w_k(\tau))\, d\tau = \int_0^s \dot{w}_k(s)^\top D^2\eta(w_k(\tau))\,\dot{w}_k(\tau)\, d\tau.$$

Thanks to the strict convexity of η and the continuity of $\dot{w}_k(s)$, we see that $\varphi_k(s)$ is non-negative for all $s \le s_0$ if s_0 is sufficiently small. In that case, $f(U,C) = f(w_k(s_0) \mid w_k(0))$ represents an average of $\lambda_k(w_k(s))$ over the interval $[0,s_0]$. Therefore, $f(w_k(s_0) \mid w_k(0)) \to \lambda_k(w_k(0))$ as $s_0 \to 0$.

5.4 Estimates Along Hugoniot Curves

It is also possible to check that f attains its extremal values along Hugoniot curves. We will discover, however, that the estimates we obtain in this case reveal a deeper structure. The information we obtain should play a key role in any extension of our relative entropy techniques to systems of conservation laws.

In this case, let us fix $U^- \in \mathcal{D}$ and assume, as in the previous section, that $\nabla A(U^-)$ has distinct eigenvalues. With respect to U^-, we consider the Hugoniot locus

$$\mathcal{H}(U^-) = \left\{ U \in \mathcal{D} \mid A(U) - A(U^-) = \sigma(U, U^-)(U - U^-) \right\},$$

which may be expressed, at least locally, as the union of n curves $\mathcal{H}_k(U^-)$, $k = 1, 2, ..., n$. Moreover, each Hugoniot curve $\mathcal{H}_k(U^-)$ shall be parametrized by a function $x_k(s)$ defined near $x_k(0) = U^-$, having the properties

$$\lim_{s \to 0} \sigma(x_k(s), U^-) = \lambda_k(U^-) \quad \text{and} \quad \lim_{s \to 0} \dot{x}_k(s) = r_k(U^-).$$

Note that the first property is independent of the parametrization x_k.

As in the previous section, we will compute the relative entropy flux using path integrals, this time for the paths x_k. To simplify notation we will write $\sigma_k(s) = \sigma(x_k(s), U^-)$. Then, for two states $U_1 = x_k(s_1)$ and $U_2 = x_k(s_2)$ belonging to $\mathcal{H}_k(U^-)$, we compute

$$
\begin{aligned}
F(U_2,U_1) &= \int_{s_1}^{s_2} \left[\nabla\eta(x_k(s)) - \nabla\eta(x_k(s_1))\right]\cdot\left[\nabla A(x_k(s))\cdot \dot{x}_k(s)\right]\, ds \\
&= \int_{s_1}^{s_2} \left[\nabla\eta(x_k(s)) - \nabla\eta(x_k(s_1))\right]\cdot \frac{d}{ds}A(x_k(s))\, ds \\
&= \int_{s_1}^{s_2} \left[\nabla\eta(x_k(s)) - \nabla\eta(x_k(s_1))\right]\cdot \frac{d}{ds}\left[\sigma_k(s)(x_k(s) - x_k(0))\right]\, ds \\
&= \int_{s_1}^{s_2} \left[\nabla\eta(x_k(s)) - \nabla\eta(x_k(s_1))\right]\cdot\left[\dot{\sigma}_k(s)(x_k(s) - x_k(s_1))\right]\, ds \\
&\qquad + \int_{s_1}^{s_2} \left[\nabla\eta(x_k(s)) - \nabla\eta(x_k(s_1))\right]\cdot\left[\dot{\sigma}_k(s)(x_k(s_1) - x_k(0))\right]\, ds \\
&\qquad\qquad + \int_{s_1}^{s_2} \left[\nabla\eta(x_k(s)) - \nabla\eta(x_k(s_1))\right]\cdot\left[\sigma_k(s)\dot{x}_k(s)\right]\, ds \\
&= \int_{s_1}^{s_2} \dot{\sigma}_k(s)\left[\eta(x_k(s)\mid x_k(s_1)) + \eta(x_k(s_1)\mid x_k(s))\right]\, ds \\
&\qquad + \int_{s_1}^{s_2} \dot{\sigma}_k(s)\left[\nabla\eta(x_k(s)) - \nabla\eta(x_k(s_1))\right]\cdot\left[x_k(s_1) - x_k(0)\right]\, ds \\
&\qquad\qquad + \int_{s_1}^{s_2} \sigma_k(s)\frac{d}{ds}\eta(x_k(s)\mid x_k(s_1))\, ds \\
&= \sigma_k(s_2)\,\eta(x_k(s_2)\mid x_k(s_1)) + \int_{s_1}^{s_2} \dot{\sigma}_k(s)\,\eta(x_k(s_1)\mid x_k(s))\, ds \\
&\qquad + \int_{s_1}^{s_2} \dot{\sigma}_k(s)\left[\nabla\eta(x_k(s)) - \nabla\eta(x_k(s_1))\right]\cdot\left[x_k(s_1) - x_k(0)\right]\, ds \\
&= \sigma_k(s_2)\,\eta(x_k(s_2)\mid x_k(s_1)) \\
&\qquad + \int_{s_1}^{s_2} \dot{\sigma}_k(s)\left[\eta(x_k(0)\mid x_k(s)) - \eta(x_k(0)\mid x_k(s_1))\right]\, ds \\
&= \sigma(U_2,U^-)\,\eta(U_2\,|\,U_1) + \int_{s_1}^{s_2} \dot{\sigma}_k(s)\left[\eta(U^-\,|\,x_k(s)) - \eta(U^-\,|\,U_1)\right]\, ds.
\end{aligned}
$$

The key step above is the application of the Rankine-Hugoniot relation in the third equality. Thanks to this relation, an interesting structure appears in the estimate. Among other features, we recover easily the properties of f studied in the previous sections. In fact, the behavior of $f(U, U^-)$ as $U \to U^-$ along Hugoniot curves is quite clear from our formula; we simply set $s_1 = 0$ and take the limit $s_2 \to 0$. Indeed, we have

$$\begin{aligned} F(x_k(s_2), U^-) &= \sigma(x_k(s_2), U^-)\, \eta(x_k(s_2) \,|\, U^-) + \int_0^{s_2} \dot{\sigma}_k(s) \eta(U^- \,|\, x_k(s))\, ds \\ &= \lambda_k(U^-)\, \eta(x_k(s_2) \,|\, U^-) + O(s_2) \qquad \text{as } s_2 \to 0. \end{aligned}$$

Note that the integral above has a sign when $\dot{\sigma}_k(s)$ has a sign. As we will see in the next section, if $U \to U^-$ along a shock curve, and we assume the parametrization is chosen so that $s \geq 0$ along this curve, then the entropy admissibility condition requires that $\dot{\sigma}_k(s) = \frac{d}{ds}\sigma(x_k(s), U^-) < 0$ for s sufficiently close to 0. In fact, for a number of important systems, including Euler's equation for compressible gas flow, this property holds globally along shock curves. In those cases, our estimate is especially useful.

5.5 Final Remarks

In the previous section we derived a formula for the flux of relative entropy associated to a pair of states (U_1, U_2) connected by a Hugoniot curve.

Dropping the index k, our formula reads

$$F(x(s_2), x(s_1)) = \sigma(s_2)\, \eta(x(s_2) \,|\, x(s_1)) + \int_{s_1}^{s_2} \dot{\sigma}(s) \left[\eta(x(0) \,|\, x(s)) - \eta(x(0) \,|\, x(s_1)) \right] ds. \tag{5.5.1}$$

It turns out that the derivatives of the left and right-hand sides of (5.5.1) with respect to s_2 agree even when $x(s_1)$ is replaced by a general state C. Therefore, up to a constant, (5.5.1) also holds in this generality. To determine the appropriate constant we set $s_2 = s_1$. In this way, we obtain the general estimate

$$F(x(s_2), C) - F(x(s_1), C) = \sigma(s_2)\, \eta(x(s_2) \,|\, C) - \sigma(s_1)\, \eta(x(s_1) \,|\, C) + \int_{s_1}^{s_2} \dot{\sigma}(s) \left[\eta(x(0) \,|\, x(s)) - \eta(x(0) \,|\, C) \right] ds. \tag{5.5.2}$$

There is a natural connection between this formula and the relative entropy inequality (5.1.4) introduced in Section 5.1. Let us consider a shock with end states $(U^-, U^+) = (x(0), x(s_2))$. In the same way that the endstates (U^-, U^+) verify a Hugoniot relation associated with (5.1.1), they also verify a weaker relation coming from (5.1.4); namely,

$$F(U^+, C) - F(U^-, C) \leq \sigma(U^+, U^-) \left[\eta(U^+ \,|\, C) - \eta(U^- \,|\, C) \right].$$

We now see that (5.5.2) tells us precisely the amount of relative entropy lost across a shock. Indeed, under the setup above, if we set $s_1 = 0$, then (5.5.2) takes the form

$$F(U^+, C) - F(U^-, C) = \sigma(U^+, U^-)\left[\eta(U^+ \,|\, C) - \eta(U^- \,|\, C)\right] + \int_0^{s_2} \dot{\sigma}(s)\eta(U^- \,|\, x(s))\, ds.$$

In particular, if $s_2 > 0$, then the remainder term has the appropriate sign when $\dot{\sigma}(s) < 0$, which can be verified directly for many special systems, including Euler's equations. We should also point out that, with respect to (5.1.2), the same remainder term appears if we compute the loss of entropy across a shock. That is, it follows directly from the previous equation that

$$G(U^+) - G(U^-) = \sigma(U^+, U^-)\left[\eta(U^+) - \eta(U^-)\right] + \int_0^{s_2} \dot{\sigma}(s)\eta(U^- \,|\, x(s))\, ds.$$

Note that we could have considered instead a parametrization $x(t)$ going from U^+ to U^-, in which case we would set $U^+ = x(s_1) = x(0)$ and $U^- = x(s_2)$. The implications are essentially the same as above. In either case, we may observe a particular asymmetry of f; namely,

$$f(U^+, U^-) \leq \sigma(U^+, U^-) = \sigma(U^-, U^+) \leq f(U^-, U^+).$$

On the other hand, if we consider systems for which the relative entropy, $\eta(x(0) \,|\, x(s))$, increases along shock curves (again, assume $s > 0$ corresponds to shocks), then reversing the role of s_1 and s_2 above leads to another interesting estimate. Specifically, applying (5.5.1) with respect to the parametrizations $\{U^+ = x(s_2) = x(0), U^- = x(s_1)\}$, and $\{U^- = x(s_2) = x(0), U^+ = x(s_1)\}$, we obtain, respectively, the relations

$$\lambda(U^+) \leq f(U^+, U^-) \quad \text{and} \quad f(U^-, U^+) \leq \lambda(U^-).$$

This concludes our study of relative entropy structures for systems.

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