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Behavior of Knot Floer Homology Under Conway and Genus Two Mutation

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Behavior of Knot Floer Homology Under Conway and Genus Two Mutation

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Dedicated to the memory of my grandparents.

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Behavior of Knot Floer Homology Under Conway and Genus Two Mutation

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In this dissertation we prove that if an n-stranded pretzel knot K has an essential Conway sphere, then there exists an Alexander grading s such that the rank of knot Floer homology in this grading, $\widehat{\mathrm{HFK}}(K, s)$, is at least two. As a consequence, we are able to easily classify pretzel knots admitting L-space surgeries. We conjecture that this phenomenon occurs more generally for any knot in S^3 with an essential Conway sphere. We also exhibit an infinite family of knots, each of which admits a nontrivial genus two mutant which shares the same total dimension of knot Floer homology, while being distinguished by knot Floer homology as a bigraded invariant. Additionally, the genus two mutation interchanges the δ-graded knot Floer homology groups in δ-gradings k and $-k$. This infinite family of examples supports a second conjecture, namely that the total rank of knot Floer homology is invariant under genus two mutation.

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Chapter 1

Introduction

Conway mutant knots (see Chapter 2.1) are popular examples in knot theory because they are difficult to distinguish, and are commonly used to determine how discriminating a knot invariant is. Classically, the Seifert genus was one of the few invariants robust enough to distinguish the famous Kinoshita-Terasaka/Conway mutant pair of knots [Gab86b]. More recently, the bigraded knot Floer homology invariant, whose definition is due to Ozsváth and Szabó [OS04b] and independently due to Rasmussen [Ras03], was also shown to distinguish this pair of knots [OS04c]. In the 1980s, the concept of mutation was generalized by Ruberman [Rub87] to a natural topological operation on a three-manifold M, in which M is cut along an embedded surface, and the two pieces are reglued by an involution (see Chapter 2.2). This dissertation is motivated by questions about how the knot Floer complex behaves with respect to such mutations.

The Heegaard Floer homology of three-manifolds, and its refinement for knots, knot Floer homology, have proved to be useful for answering Dehn surgery questions and understanding surfaces in three-manifolds. Knot Floer homology is a bigraded abelian group associated to a knot K in S^3

$$\widehat{HFK}(K) = \oplus_{m,s}\widehat{HFK}_m(K,s),$$

whose graded Euler characteristic is the symmetrized Alexander polynomial of K [OS04b],

$$\Delta_K(T) = \sum_s \chi(\widehat{\mathrm{HFK}}(K,s)) \cdot T^s.$$

For example, it is shown in [OS04a] that knot Floer homology detects the genus of a knot by

$$g(K) = \max\{s|\widehat{HFK}(K,s) \neq 0\},$$

and more generally, link Floer homology detects the Thurston norm [OS08]. Furthermore, work of Ghiggini and Ni shows that K is a fibered knot if and only if $\widehat{HFK}(K,g(K)) \cong \mathbb{Z}$ [Ghi08, Ni07]. In this dissertation, we conjecture that knot Floer homology is able to "see" a different type of essential surface, specifically essential Conway spheres.

Conjecture 1. *Let K be a knot in S^3. If K has an essential Conway sphere, then there exists an Alexander grading s such that*

$$\textit{rank } \widehat{HFK}(K,s) \geq 2.$$

We will establish Conjecture 1 for pretzel knots with any number of strands as evidence.

Theorem 2. *Let K be a pretzel knot which is neither $\pm P(-2,3,q)$ for odd $q \geq 1$ nor the torus knot $T(2,2n+1)$ for any n. Then, there exists $s \in \mathbb{Z}$ such that*

$$\textit{rank}\ \widehat{\mathrm{HFK}}(K,s) \geq 2$$

A pretzel knot (and more generally, a Montesinos knot) containing an essential Conway sphere in its complement must always have at least four tangles. Therefore we have the following.

Corollary 3. *Conjecture 1 holds for all pretzel knots.*

Note that if a knot is not fibered, then $\widehat{HFK}(K,g(K))$ has rank at least two, and Conjecture 1 is automatically true. Thus we will restrict our attention to fibered knots. Moreover, for any knot admitting an L-space surgery (see Chapter 3.1.2), $\widehat{HFK}(K,s) \cong 0$ or $\mathbb{Z}$ for all $s \in \mathbb{Z}$ [OS05, Theorem 1.2]. In light of Theorem 2 and what is known about the knot Floer homology of three-strand pretzel knots, we are able to easily classify the pretzel knots with positive L-space surgeries.

Corollary 4. *The knots $K = P(-2,3,q)$ for odd $q \geq 1$ and torus knots $T_{2,2n+1}$ for $n \geq 1$ are the only non-trivial pretzel knots with positive L-space surgeries.*

Proof. We first note that $P(-2,3,q)$ is the torus knot $T(2,5)$, $T(3,4)$, or $T(3,5)$ if $q = 1$, 3, or 5 respectively. These knots along with $T(2,2n+1)$ for $n > 0$ clearly admit positive L-space surgeries, since lens spaces are L-spaces. It is shown in [OS05] that $P(-2,3,q)$ admits a positive L-space surgery for odd

$q \geq 7$. Note that the remaining exceptional cases in Theorem 2 are mirrors of knots with positive L-space surgeries, and thus admit negative L-space surgeries. Finally, we recall that the only non-trivial knot with both positive and negative L-space surgeries is the unknot. This completes the list. $\square$

If Conjecture 1 is true, then the contrapositive states that L-space knots admit no non-trivial mutations. As an obvious and immediate consequence, we would be able to observe that for L-space knots, the total dimension of the knot Floer homology groups, i.e.

$$\sum_{m,s} \text{rank } \widehat{\text{HFK}}(K, s)$$

is invariant under Conway mutation. In fact, this is an observation that holds true more broadly, even when the bigraded knot Floer groups do change under mutation. Indeed, in all computed examples of the knot Floer groups of Conway mutant pairs K and K^τ (up to 12 crossings), the rank of $\widehat{\text{HFK}}(K)$ as an ungraded Abelian group is equal to that of $\widehat{\text{HFK}}(K^\tau)$ [BG12]. The question of whether the rank of knot Floer homology is unchanged under Conway mutation, or more generally, genus two mutation, remains an interesting open problem.

Question 5. *Is the total rank of* $\widehat{\text{HFK}}(S^3, K)$ *invariant under genus two mutation of* S^3*?*

Recently, Baldwin and Levine have conjectured [BL12] that the δ-

graded knot Floer homology groups

$$\widehat{\mathrm{HFK}}_\delta(L) = \bigoplus_{\delta=s-m} \widehat{\mathrm{HFK}}_m(L,s)$$

are unchanged by Conway mutation, which implies that their total ranks are preserved, amongst other things. In this dissertation, we offer an example of an infinite family of knots with isomorphic knot Floer homology, all of which admit a genus two mutant of the same total dimension in $\widehat{\mathrm{HFK}}$, although each pair is distinguished by $\widehat{\mathrm{HFK}}$ as a bigraded invariant.[1] Additionally, we show that the δ-graded $\widehat{\mathrm{HFK}}$ groups distinguish the genus two mutants pairs. Knot Floer homology computations are done with $\mathbb{F}_2$-coefficients.

Theorem 6. *There exists an infinite family of genus two mutant pairs* (K_n, K_n^τ), $n \in \mathbb{Z}^+$, *in which*

1. *each infinite family has isomorphic knot Floer homology groups,*

$$\begin{aligned} \widehat{\mathrm{HFK}}_m(K_n,s) &\cong \widehat{\mathrm{HFK}}_m(K_0,s), \text{ for all } m,s \\ \widehat{\mathrm{HFK}}_m(K_n^\tau,s) &\cong \widehat{\mathrm{HFK}}_m(K_0^\tau,s), \text{ for all } m,s, \end{aligned}$$

2. *each genus two mutant pair shares the same total dimension in* $\widehat{\mathrm{HFK}}$,

$$\sum_{m,s} \dim_{\mathbb{F}_2} \widehat{\mathrm{HFK}}_m(K_n,s) = \sum_{m,s} \dim_{\mathbb{F}_2} \widehat{\mathrm{HFK}}_m(K_n^\tau,s)$$

[1] Because we compute $\widehat{\mathrm{HFK}}$ as a graded vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, the theorem has been formulated in terms of dimension rather than rank.

3. *each genus two mutant pair is distinguished by* $\widehat{\mathrm{HFK}}$ *as bigraded groups, e.g.*

$$\widehat{\mathrm{HFK}}_m(K_n, s) \ncong \widehat{\mathrm{HFK}}_m(K_n^\tau, s) \text{ for some } m, s, \text{ and}$$

4. *each genus two mutant pair is distinguished by* δ*-graded* $\widehat{\mathrm{HFK}}$*, in particular,*

$$\begin{aligned} \widehat{\mathrm{HFK}}_\delta(K_n) &\cong \widehat{\mathrm{HFK}}_{-\delta}(K_n^\tau) \text{ for all } \delta, \text{ and} \\ \widehat{\mathrm{HFK}}_\delta(K_n) &\ncong \widehat{\mathrm{HFK}}_\delta(K_n^\tau) \text{ for all } \delta \neq 0. \end{aligned}$$

In [MS12], Theorem 6 appears along with a parallel theorem for Khovanov homology with $\mathbb{Q}$-coefficients. In particular, we prove an analogous result for parts (2), (3), and (4) of Theorem 6. There is a grading shift that makes an analogous statement to part (1) inapplicable in Khovanov homology. These theorems suggest that having invariant total dimension of knot Floer homology or Khovanov homology is a property shared not only by Conway mutants, but by genus two mutant knots as well, suggesting affirmative evidence towards Question 5.

Chapter 2

Mutation

Conjecture 1, Question 5, and Theorems 2 and 6 are unified by the goal of understanding how the knot Floer complex behaves under the different types of cutting and pasting operations known as mutation.

Definition 7. *Let F be a surface which is properly embedded in a three-manifold Y, meaning $F \cap \partial Y = \partial F$, or a surface which is contained in ∂Y. F is* compressible *is there is a disk $D \subset Y$ such that $D \cap F = \partial D$, where ∂D does not bound a disk in F. Otherwise, F is* incompressible.

We will be particularly interested in mutating along surfaces which are incompressible in the complement of knots in S^3.

2.1 Conway mutation

The original definition of mutation is due to Conway [Con70]. Let $S \subset S^3$ be a sphere which meets a knot K transversely in four points. S is called a *Conway sphere*. S bounds two balls, each ball containing a tangle whose union is K, i.e.

$$K = T_1 \cup_S T_2.$$

An *elementary Conway mutation* of K is performed by removing a ball containing a tangle, and replacing it by its image under a rotation by π about a coordinate axis, so that the four points of intersection with K are preserved. A knot is called a *Conway mutant of* K if it is obtained by a series of such Conway mutations. To obtain a Conway mutation diagrammatically, one can find a diagram of K in which the Conway sphere S projects to a disk, which can then be cut and rotated.

Conway mutants are indistinguishable by many knot invariants. Some examples of invariants which do not change under Conway mutation include skein-type polynomials, such as the HOMFLY-PT polynomial, the Jones polynomial, and the Alexander polynomial, and subsequently, the determinant of a knot. Moreover, we have the following observation due to Viro:

Theorem 8. *[Vir76]*
The branched double covers of Conway mutant knots in S^3 *are homeomorphic.*

By an *essential Conway sphere*, we mean a four-punctured Conway sphere $S - S \cap K$ which is incompressible in $S^3 - K$. We remark that mutation along an inessential Conway sphere results in a trivial mutation, i.e. a mutation which does not change the isotopy class of K. Essential Conway spheres lift to incompressible tori in $\Sigma_2(K)$, the branched double cover of S^3 over K. This follows from the $\mathbb{Z}_2$-equivariant version of the Loop Theorem (see for example [GL84]).

2.2 Genus two mutation

The concept of mutation was generalized to an operation on three-manifolds by Ruberman in [Rub87]. The original formulation of a mutation of a three-manifold along a surface F includes several types of surfaces; here we focus on genus two surfaces.

Definition 9. *Let F be an embedded, genus two surface in a compact, orientable three-manifold Y and let τ be the hyperelliptic involution. A genus two mutant of Y, denoted Y^τ, is obtained by cutting Y along F and regluing the two copies of F via τ.*

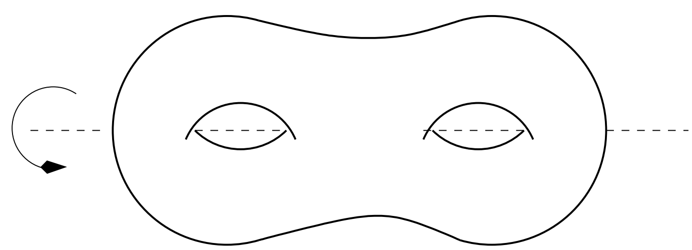


Figure 2.1: The genus two surface F and hyperelliptic involution τ.

The involution τ has the property that an unoriented simple closed curve γ on F is isotopic to its image $\tau(\gamma)$ and that $F/\tau \simeq S^2$ [Rub87, DGST10]. When $Y = S^3$, any closed surface $F \subset S^3$ is compressible, and so $(S^3)^\tau$ is homeomorphic to S^3 [DGST10]. In particular, mutation along F induces a homeomorphism of S^3. If S^3 contains a knot K which is disjoint from F, then denote by K^τ the image of K under this homeomorphism of S^3. The isotopy classes of K and K^τ are potentially different.

Definition 10. *Let* $F \subset S^3$ *be a genus two surface and let* $K \subset S^3, K \cap F = \emptyset$*. The knot that results from performing a genus two mutation of* S^3 *along* F *is the genus two mutant of* K*, denoted* K^τ*. Whenever the surface of mutation bounds a handlebody containing* K *in its interior, e.g.* $F = \partial H, K \subset H \subset S^3$ *with* $K \cap \partial H = \emptyset$*, the genus two mutation is called a* handlebody mutation*.*

In fact, a Conway mutation of a knot can be realized as a composition of genus two mutations. Since S separates K into two tangles, a genus two surface F is formed by taking S and tubing along either T_1 or T_2. The Conway mutation is then achieved by performing at most two such genus two mutations [DGST10]. Like Conway mutations, we are primarily interested in the case that F is incompressible in $S^3 - K$. Also like Conway mutants, genus two mutants are indistinguishable by many knot invariants [DGST10].

Theorem 11. *[MT88], [CL99]*
The Alexander polynomial and colored Jones polynomials for all colors of a knot in S^3 *are invariant under genus two mutation. Generalized* ω*-signature for links is invariant under genus two handlebody mutation.*

Above, generalized ω-signature refers to the signature of the Hermitian $(1-\omega)V + (1-\bar{\omega})V^T$, where $\omega \neq 1$ is a unit modulus complex number and V is a Seifert matrix of the link [CL99].

Theorem 12. *[Rub87, Theorem 1.3]*
Let K^τ *be a genus two mutation of the hyperbolic knot* K*. Then* K^τ *is also hyperbolic, and the volumes of their complements are the same.*

Theorem 12 is a special case of a more general theorem of [Rub87] which shows that the Gromov norm is preserved under mutation along any of several symmetric surfaces, including the genus two surface on which we are focused here. Ruberman also shows that cyclic branched coverings and Dehn surgeries along a Conway mutant knot pair yield manifolds of the same Gromov norm.

The manifold obtained by p/q-surgery along K^τ is a genus two mutant of p/q-surgery along K, i.e.

$$S^3_{p/q}(K^\tau) \cong [S^3_{p/q}(K)]^\tau.$$

However, these manifolds need not be homeomorphic. In [Kir89], Kirk gives an example of a pair K and F such that the genus two mutant manifolds $S^3_{1/n}(K)$ and $S^3_{1/n}(K^\tau)$ are not homeomorphic for some integer n.

2.3 Examples: 14^n_{22185} and 14^n_{22589}

In light of Theorem 8 and the discussion above, it is natural to ask whether $\Sigma_2(K)$ is homeomorphic to $\Sigma_2(K^\tau)$; however, this need not be true. We verify this by investigating the pair of genus two mutant knots in Figure 2.2, which we will denote by K_0 and K_0^τ and which are known as 14^n_{22185} and 14^n_{22589} in Knotscape notation. K_0 and K_0^τ appear along with several other examples of pairs of genus two mutant knots in [DGST10].

Proposition 13. *The branched double covers of K_0 and K_0^τ are not homeomorphic.*

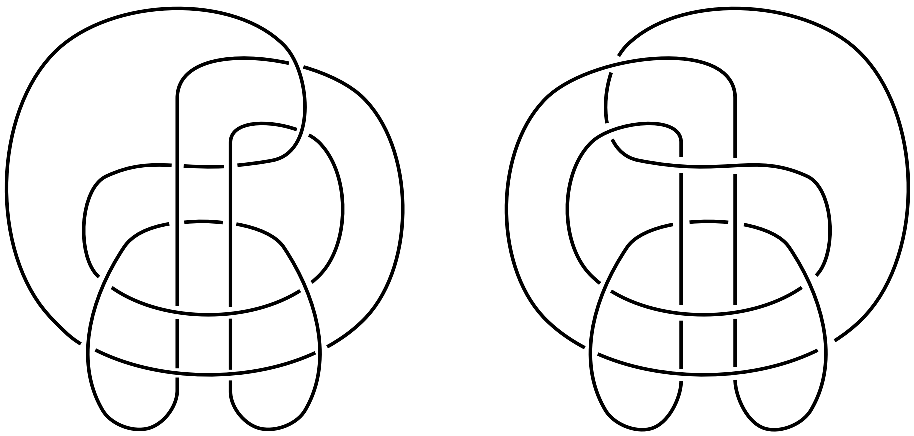


Figure 2.2: The genus two mutant pair $K_0 = 14^n_{22185}$ and $K_0^\tau = 14^n_{22589}$.

Proof. This is a fact which can be checked by computing the geodesic length spectra of $\Sigma_2(K_0)$ and $\Sigma_2(K_0^\tau)$ in SnapPy [CDW]. The code which performs this computation is included in the Appendix. The complex length spectrum of a compact hyperbolic three-orbifold Y is the collection of all complex lengths of closed geodesics in Y counted with their multiplicities (Chapter 12 of [MR03]). SnapPy demonstrates that the complex length spectra of $\Sigma_2(K)$ and $\Sigma_2(K^\tau)$ (computed to an upper bound of 1.5) are different, therefore these manifolds are not isospectral, and therefore not isometric. Mostow rigidity says that the geometry of a finite-volume hyperbolic three-manifold is unique, therefore $\Sigma_2(K)$ and $\Sigma_2(K^\tau)$ are not homeomorphic. □

Corollary 14. *The genus two mutant pair K_0 and K_0^τ are not Conway mutants.*

Proof. Since Conway mutants have homeomorphic branched double covers, this follows directly from Proposition 13. □

We will continue to explore the pair 14^n_{22185} and 14^n_{22589}. As genus two mutants, they share all of the properties mentioned in Theorems 11 and 12. Moreover, 14^n_{22185} and 14^n_{22589} are also shown in [DGST10] to have the same HOMFLY-PT and Kauffman polynomials, although in general these polynomials are known to distinguish larger examples of genus two mutant knots [DGST10]. Just as a subtler hyperbolic invariant was required to distinguish their branched double covers, we will require a subtler knot invariant to distinguish the knot pair. The categorified invariant $\widehat{\mathrm{HFK}}$ does the trick.

Theorem 15. *The genus two mutant knots K_0 and K_0^τ are distinguished by their bigraded knot Heegaard Floer homology groups, as well as by δ-graded knot Floer homology.*

Remark 16. *In [DGST10], it is shown using KhoHo [Shu] that K_0 and K_0^τ have different Khovanov homology with $\mathbb{Z}$ coefficients.*

The knot Floer homology and δ-graded knot Floer homology groups of K_0 and K_0^τ are displayed in Table A.1 and Table A.2 in the Appendix. Since $\widehat{\mathrm{HFK}}$ is known to detect Conway mutation [OS04c], it is not surprising that knot Floer homology can distinguish genus two mutant pairs. Indeed, we used the Python program of Droz [Dro] to compute the knot Floer groups $\widehat{\mathrm{HFK}}(K_0)$ and $\widehat{\mathrm{HFK}}(K_0^\tau)$. The key observation is that although knot Floer homology does distinguish the genus two mutants as bigraded vector spaces, the pairs are indistinguishable as ungraded objects. We will derive an infinite family of knots from the pair $K_0 = 14^n_{22185}$ and $K_0^\tau = 14^n_{22589}$. Each of these

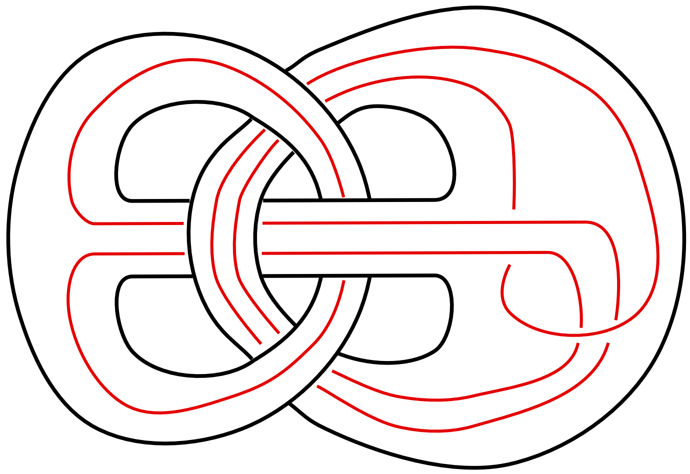


Figure 2.3: The surface of mutation for all K_n. Note the surface bounds a handlebody.

knots is formed as the band sum of a two-component unlink. By adding n half-twists with positive crossings to the bands of K_0 and K_0^τ, as in Figure 2.4, we obtain knots K_n and K_n^τ. It is visibly clear that K_n^τ is the genus two mutant of K_n by the same surface of mutation relating K_0 and K_0^τ, illustrated in Figure 2.3.

Observe that by resolving a crossing in the twisted band, K_n and K_{n-2} fit into an oriented skein triple (L_+, L_-, L_0) with L_0 equal to the two-component unlink $\mathcal{U}$ for all integers $n > 1$. K_n^τ, K_{n-2}^τ and $\mathcal{U}$ also fit into an oriented skein triple.

Lemma 17. *The Ozsváth and Szabó τ invariant invariant vanishes for all K_n and K_n^τ.*

Proof. The knots K_n and K_n^τ are formed from the band sum of a two-component

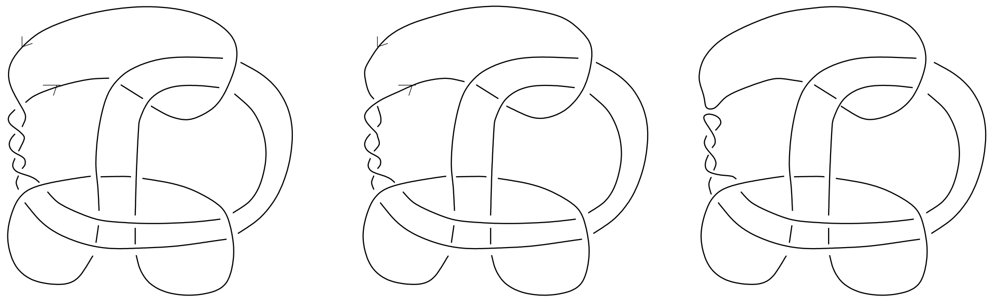


Figure 2.4: The oriented skein triple $(K_n, K_{n-2}, \mathcal{U})$.

unlink. In general, if K is any such knot, then K is smoothly slice. Recall that K is *smoothy slice* if K bounds a disk D which is smoothly embedded in B^4, with K contained in $S^3 = \partial B^4$. D is constructing by attaching a band along $K \times I$, and capping off the resulting pair of unknotted circles with disks. This is a standard fact (see for example [Lic97, p. 86]), and the slicing disk is illustrated in Figure 2.5. Ozsváth and Szabó define the smooth concordance invariant $\tau(K) \in \mathbb{Z}$ in [OS03b, Corollary 1.3] , and $\tau(K)$ provides a lower bound on the four-ball genus:

$$|\tau(K)| \leq g_*(K).$$

Since all of our knots are slice, we immediately obtain $\tau = 0$. □

We will utilize the fact that $\tau(K_n) = \tau(K_n^\tau) = 0$ in our proof of Theorem 6.

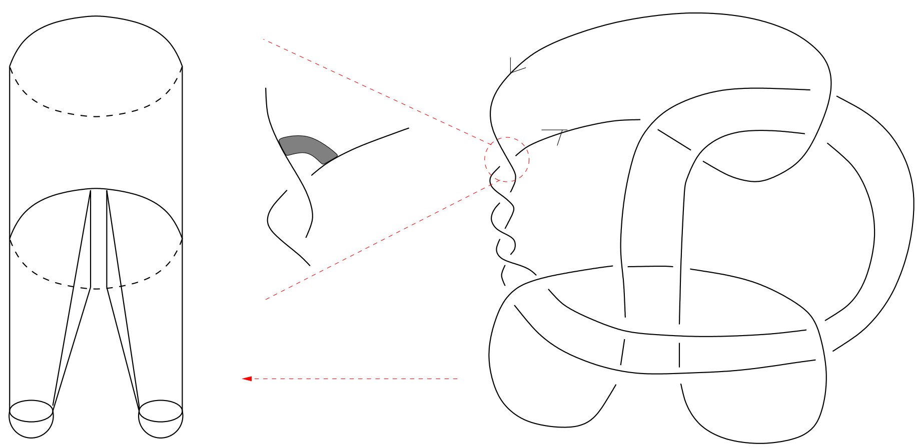


Figure 2.5: A smooth cobordism illustrating that K_n is slice.

Chapter 3

Knot Floer homology and Theorem 6

3.1 Preliminaries

Throughout, K (respectively L) will denote an oriented knot (respectively link) in S^3, unless otherwise stated. Y will denote a closed oriented three-manifold, and F will denote an embedded surface in Y.

3.1.1 Knot Floer homology

Knot Floer homology is a powerful invariant of oriented knots and links in an oriented three manifold Y, developed independently by Ozsváth and Szabó [OS04b] and Rasmussen [Ras03]. We tersely paraphrase Ozsváth and Szabó's construction of the invariant for knots from [OS04b], and refer the reader to [OS04b] for details of the construction.

To a knot $K \subset S^3$ is associated a doubly pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z, w)$. The data of the Heegaard diagram gives rise to a chain complex which comes in several varieties. Here, we are interested in the chain complexes $(\mathrm{CFK}^-(K), \partial^-)$ and $(\widehat{\mathrm{CFK}}(K), \widehat{\partial})$. These complexes come equipped with two integer-valued gradings (M, A), where M denotes Maslov grading and A denotes Alexander grading.[1] $\mathrm{CFK}^-(K)$ is an $\mathbb{F}_2[U]$ module, where the action of U reduces A by one and M by two. The differentials ∂^- and $\widehat{\partial}$ preserve A and reduce M by one. The homology groups

$$\widehat{\mathrm{HFK}}(K) = \bigoplus_{m,s} \widehat{\mathrm{HFK}}_m(K,s) \text{ and } \mathrm{HFK}^-(K) = \bigoplus_{m,s} \mathrm{HFK}^-{}_m(K,s)$$

are invariants of K. The graded Euler characteristic of $\widehat{\mathrm{HFK}}(S^3, K)$ recovers the symmetrized Alexander polynomial of K [OS04b],

$$\Delta_K(T) = \sum_s \chi(\widehat{\mathrm{HFK}}(K,s)) \cdot T^s.$$

We will require the use of several theorems from the corpus of Heegaard Floer homology, stated in the following subsections without proof.

3.1.2 L-spaces and L-space knots

Ozsváth and Szabó define a rational homology three-sphere Y to be an *L-space* if $|H_1(Y;\mathbb{Z})| = \operatorname{rank} \widehat{HF}(Y)$, where $\widehat{HF(Y)}$ is the Heegaard Floer homology group associated to a closed, oriented three-manifold [OS05]. The terminology L-space is derived from the notion that L-spaces generalize lens spaces; they are 'Heegaard Floer homology lens spaces'. A knot which admits an L-space surgery is called an *L-space knot*.

In the case of an L-space knot, there are strong restrictions on these groups. Namely, we have the following theorem of Ozsváth and Szabó.

[1] For a homogeneous element $\mathbf{x}$, such a bigrading is denoted $\mathbf{x}_{(m,s)}$, where $M(\mathbf{x}) = m$ and $A(\mathbf{x}) = s$

Theorem 18. *[OS05, Theorem 1.2]*

Suppose that $K \subset S^3$ is a knot for which there is a positive integer p for which $S^3_p(K)$ is an L-space. Then, there is an increasing sequence of integers

$$n_{-k} < ... < n_k$$

with the property that $n_i = -n_{-i}$, with the following significance. If for $-k \leq i \leq k$ we let

$$\delta_i = \begin{cases} 0 & \text{if } i = k \\ \delta_{i+1} - 2(n_{i+1} - n_i) + 1 & \text{if } k - i \text{ is odd} \\ \delta_{i+1} - 1 & \text{if } k - i > 0 \text{ is even,} \end{cases}$$

then $\widehat{\mathrm{HFK}}(K, j) = 0$ unless $j = n_i$ for some i, in which case $\widehat{\mathrm{HFK}}(K, j) \cong \mathbb{Z}$ and it is supported entirely in dimension δ_i.

In particular, Theorem 18 implies that the coefficients of the Alexander polynomial of an L-space knot are all either ± 1 or 0.

In [OS05], Ozsváth and Szabó show that if q is any an odd integer with $q \geq 7$, and p is any integer with $p \geq 2q + 4$, then the three-manifold $S^3_p(P(-2, 3, q))$ is an L-space. Moreover, they prove the following:

Theorem 19. *[OS05, Proposition 4.1]*

If K is an alternating knot with the property that all the coefficients a_s of its Alexander polynomial $\Delta_K(t)$ have $|a_s| \leq 1$, then K is the $(2, 2n + 1)$ torus knot.

Chapter 4 is dedicated to proving that $(2, 2n + 1)$ torus knots and pretzel knots of the form $(-2, 3, q)$ for odd $q > 0$ constitute the only instances of pretzel knots which admit L-space surgeries.

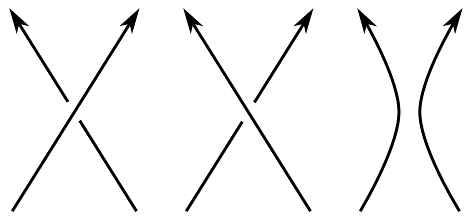


Figure 3.1: The skein triple (L_+, L_-, L_0).

3.1.3 The skein exact sequence of knot Floer homology

An oriented skein triple is a set of three oriented links (L_+, L_-, L_0) which admit diagrams which differ only in a neighborhood of a distinguished crossing, as in Figure 3.1. A key component in the construction of infinite families of genus two mutant knots (see Chapter 3.2) is a long exact sequence in homology which relates the knot Floer groups of an oriented skein triple. A skein exact sequence relating (L_+, L_-, L_0) first appeared in [OS04b], but here, we will be interested in the graded version appearing in [OS07]. The skein exact sequence referenced below is the specialization when L_+ and L_- are knots, and L_0 is a two-component link.

Theorem 20. *[OS07, Theorem 1.1]*

Let L_+, L_- and L_0 be three oriented links, which differ at a single crossing as indicated by the notation. Then, if L_+ and L_- are knots, there is a long exact sequence

$$\begin{aligned}\cdots \longrightarrow \mathrm{HFK}^-{}_m(L_+, s) \xrightarrow{f^-} \mathrm{HFK}^-{}_m(L_-, s) \xrightarrow{g^-} H_{m-1}\left(\frac{\mathrm{CFL}^-(L_0)}{U_1 - U_2}, s\right) \\ \xrightarrow{h^-} \mathrm{HFK}^-{}_{m-1}(L_+, s) \longrightarrow \cdots\end{aligned} \tag{3.1}$$

We remark that the skein exact sequence of Theorem 20 is derived from a mapping cone construction. Indeed, Ozsváth and Szabó show in [OS07, Theorem 3.1] that there is a chain map $f : \mathrm{CFK}^{-}(L_+) \to \mathrm{CFK}^{-}(L_-)$ whose mapping cone is quasi-isomorphic to the mapping cone of the chain map $U_1 - U_2 : \mathrm{CFL}^{-}(L_0) \to \mathrm{CFL}^{-}(L_0)$, which is in turn quasi-isomorphic to the complex $\mathrm{CFL}^{-}(L_0)/(U_1 - U_2)$. The maps in the diagram appearing in [OS07, Section 3.1] which determine the quasi-isomorphism from the cone of f to the cone of $U_1 - U_2$ are U-equivariant. The map f^- appearing in the sequence above is the map induced on homology by f. The maps g^- and h^- are induced by inclusions and projections of the mapping cone of f along with the quasi-isomorphism. Therefore the long exact sequence is U-equivariant.

3.2 Proof of Theorem 6

The main objective of this section is to show that each knot in the family $\{K_n\}$ has knot Floer homology isomorphic to $\widehat{\mathrm{HFK}}(K_0)$, and that each knot in the family $\{K_n^\tau\}$ has knot Floer homology isomorphic to $\widehat{\mathrm{HFK}}(K_0^\tau)$. Similar computations generating knots with isomorphic knot homologies occur in the work of Greene and Watson [GW11]. Theorem 22 is a special case of an observation originally due to Hedden. It will soon appear as part of a more general result of Hedden and Watson in [HW12]. Additionally, we will require the following fact:

Lemma 21. *Let $\mathcal{U}$ be the two-component unlink in S^3. $\mathcal{U}$ has the following* *knot Floer homology groups:*

$$\widehat{\mathrm{HFK}}(S^3,\mathcal{U}) \cong \mathbb{F}_2 {}_{\substack{m=0\\ s=0}} \oplus \mathbb{F}_2 {}_{\substack{m=-1\\ s=0}} \tag{3.2}$$

$$H_*\left(\frac{\mathrm{CFL}^-(\mathcal{U})}{U_1 - U_2}\right) \cong \widehat{\mathrm{HFK}}(S^3,\mathcal{U}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[U] \tag{3.3}$$

where in the module $\mathbb{F}_2[U]$*, the action of* U *drops the Maslov grading by two and the Alexander grading by one.*

Proof. The knot Floer homology groups of a two-component link $L \subset S^3$ are computed by first constructing an oriented knot $\widetilde{L}$ in $S^2 \times S^1$ which corresponds with L, called the *knotification* of L, and then computing the resulting knot Floer homology groups of $\widetilde{L} \subset S^2 \times S^1$. See [OS04b, Section 2.1] for an explanation of knotification. A Heegaard diagram for $\widetilde{\mathcal{U}} \subset S^2 \times S^1$ can be constructed by taking a genus one splitting of $S^2 \times S^1$ with two curves, α and β, intersecting in two points $\mathbf{x}$ and $\mathbf{y}$. Place basepoints z and w inside the annular region such that $\mathbf{x}$ is connected to $\mathbf{y}$ by two disks. Since it is a genus one splitting we count only ϕ corresponding to domains that are disks. As an application of the Riemann mapping theorem, $\#\widehat{\mathcal{M}}(\phi) = \pm 1$ for each such ϕ. Therefore the differential is zero in both $\widehat{\mathrm{CFK}}(S^2 \times S^1,\widetilde{\mathcal{U}})$ and $\mathrm{CFK}^-(S^2 \times S^1,\widetilde{\mathcal{U}})$. The relative grading difference is evident from the diagram and pinned down by the observation that the $\mathcal{U} \subset S^3$ fits into a skein exact sequence (Theorem 20) with the unknot. □

Theorem 22. *[HW12] Let* K *be a knot in* S^3 *formed from the band sum of a two-component unlink, and let* $\{K_n\}$ *denote the family of knots obtained by* *adding n half-twists with positive crossings to the band. For all $m, s \in \mathbb{Z}$ and $n \geq 2 \in \mathbb{Z}$, $\mathrm{HFK}^{-}{}_m(K_n, s) \cong \mathrm{HFK}^{-}{}_m(K_{n-2}, s)$.*

Proof. The proof is by induction on n. Just as with the specific families of knots described above, K_n fits into the skein triple $(K_n, K_{n-2}, \mathcal{U})$. Theorem 20 applied to the skein triple gives a long exact sequence

$$\begin{aligned} \cdots \to \mathrm{HFK}^{-}{}_m(K_n, s) \xrightarrow{f^-} \mathrm{HFK}^{-}{}_m(K_{n-2}, s) \xrightarrow{g^-} H_{m-1}\left(\frac{\mathrm{CFL}^{-}(\mathcal{U})}{U_1 - U_2}, s\right) \\ \xrightarrow{h^-} \mathrm{HFK}^{-}{}_{m-1}(K_n, s) \to \cdots . \end{aligned} \tag{3.4}$$

We will use this sequence in conjunction with information coming from the τ invariant. By Lemma 17, $\tau(K_n) = 0\ \forall n$. Because we are working with $\mathrm{HFK}^{-}(K)$, we will use the definition of τ appearing in [OST08, Appendix], where $m(K)$ denotes the mirror of K.

$$\begin{aligned} \tau(m(K)) = \max\{s \mid \exists\, \xi \in \mathrm{HFK}^{-}(K, s) \text{ such that } U^d \xi \neq 0 \\ \text{for all integers } d \geq 0\}. \end{aligned}$$

Moreoever, for a homogeneous element $\xi \in \mathrm{HFK}^{-}(K, \tau(m(K)))$ such that $U^d\xi \neq 0\ \forall d \geq 0$, the Maslov grading of ξ is given by $m = 2\tau(m(K))$. This fact can be verified by following the argument given in [OST08, Appendix], keeping careful track of the bigrading shifts at each step. Since $\tau(K_n) = 0$, we have the additional fact that $\tau(K_n) = \tau(m(K_n))$.

The non-torsion summand of $\mathrm{HFK}^{-}(K_n)$ is generated by an element ξ_n with maximal bigrading $(2\tau(m(K)), \tau(m(K)))$, which in this case is $(0, 0)$.

The third term $H_*\left(\frac{\mathrm{CFL}^-(L_0)}{U_1-U_2},0\right)$ of the skein triple corresponds with the two-component unlink and is freely generated over $\mathbb{F}_2[U]$ by elements z and z' in bigradings $(0,0)$ and $(-1,0)$. Since $\mathrm{HFK}^-(\mathcal{U})$ is supported entirely in bigradings $(-2d,-d)$ and $(-2d-1,-d)$ the long exact sequence immediately supplies isomorphisms $\mathrm{HFK}^-{}_m(K_n,s) \cong \mathrm{HFK}^-{}_m(K_{n-2},s)$ whenever $s=-d\leq 0$ and $|m-2s|>1$ or when $s>0$. The U-equivariant long exact sequence for the remaining case is displayed below, parameterized by $d \geq 0$.

$$0 \longrightarrow \mathrm{HFK}^-{}_{1-2d}(K_n,-d) \xrightarrow{f^-} \mathrm{HFK}^-{}_{1-2d}(K_{n-2},-d) \xrightarrow{g^-} \dots$$

$$\mathbb{F}_{2\{-2d,-d\}} \xrightarrow{h^-} \mathrm{HFK}^-{}_{-2d}(K_n,-d) \xrightarrow{i^-} \mathrm{HFK}^-{}_{-2d}(K_{n-2},-d) \xrightarrow{j^-} \mathbb{F}_{2\{-1-2d,-d\}}$$

$$\xrightarrow{k^-} \mathrm{HFK}^-{}_{-1-2d}(K_n,-d) \xrightarrow{\ell^-} \mathrm{HFK}^-{}_{-1-2d}(K_{n-2},-d) \longrightarrow 0$$

where

$$\begin{array}{ccccccc} \dots\mathbb{F}_{2\{-2d,-d\}} & \xrightarrow{h^-} & \mathrm{HFK}^-{}_{-2d}(K_n,-d) & \xrightarrow{i^-} & \mathrm{HFK}^-{}_{-2d}(K_{n-2},-d) & \xrightarrow{j^-} & \mathbb{F}_{2\{-1-2d,-d\}}\dots \\ \Cup & & \Cup & & \Cup & & \Cup \\ U^d\cdot z & \longmapsto & U^d\cdot\xi_n+\eta & & U^d\cdot\xi_{n-2} & \longmapsto & U^d\cdot z' \end{array}$$

In the diagram above, equivariance of the long exact sequence with respect to the action of U implies that $U^d\cdot z$ cannot be in the image of any $\mathbb{F}_2[U]$-torsion element. Since $\mathrm{HFK}^-{}_{1-2d}(K_{n-2},-d)$ is torsion, $U^d\cdot z$ is not in the image of g^-, and the map $g^-=0$. Exactness implies that f^- is an isomorphism, and also that h^- is an injection. Since the map h^- is degree preserving, $U^d \cdot z$ maps to a non-torsion element $U^d \cdot \xi_n + \eta \in \mathrm{HFK}^-{}_{-2d}(K, -d)$, where η is $\mathbb{F}_2[U]$-torsion. By exactness, $U^d \cdot \xi_n + \eta \in \mathrm{Ker}\, i^-$. Because the non-torsion summand gets mapped to zero by i^-, $U^d \cdot \xi_{n-2}$, which is also non-torsion, is not in the image of i^-. By exactness, $U^d \cdot \xi_{n-2} \notin \mathrm{Ker}\, j^-$ and $U^d \cdot \xi_{n-2}$ must map to $U^d \cdot z'$. Exactness implies that $k^- = 0$ and ℓ^- is an isomorphism. What remains is an isomorphism of torsion submodules at i^-. Hence, for all (m, s), $\mathrm{HFK}^-{}_m(K_n, s) \cong \mathrm{HFK}^-{}_m(K_{n-2}, s)$. □

Corollary 23. *Let $\{K_n\}$ and $\{K_n^\tau\}$ denote the infinite family of knots derived from 14^n_{22185} and 14^n_{22589}. Then*

$$\begin{aligned} \widehat{\mathrm{HFK}}_m(K_n, s) &\cong \widehat{\mathrm{HFK}}_m(K_0, s) \\ \widehat{\mathrm{HFK}}_m(K_n^\tau, s) &\cong \widehat{\mathrm{HFK}}_m(K_0^\tau, s). \end{aligned}$$

Proof. Once a suitable base case has been established, then the result follows from relating $\mathrm{HFK}^-(K_n)$, $\mathrm{HFK}^-(K_{n-2})$, $\widehat{\mathrm{HFK}}(K_n)$ and $\widehat{\mathrm{HFK}}(K_{n-2})$ by the five lemma.

$$\begin{array}{ccccccc}
\mathrm{HFK}^-{}_m(K_n, s) & \xrightarrow{U} & \mathrm{HFK}^-{}_{m-2}(K_n, s-1) & \xrightarrow{\pi} & \widehat{\mathrm{HFK}}_{m-2}(K_n, s-1) & \xrightarrow{\partial} & \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\mathrm{HFK}^-{}_m(K_{n-2}, s) & \xrightarrow{U} & \mathrm{HFK}^-{}_{m-2}(K_{n-2}, s-1) & \xrightarrow{\pi} & \widehat{\mathrm{HFK}}_{m-2}(K_{n-2}, s-1) & \xrightarrow{\partial} &
\end{array}$$

$$\begin{array}{ccccc}
\cdots & \xrightarrow{\partial} & \mathrm{HFK}^-{}_{m-1}(K_n, s) & \xrightarrow{U} & \mathrm{HFK}^-{}_{m-3}(K_n, s-1) \\
 & & \downarrow \cong & & \downarrow \cong \\
\cdots & \xrightarrow{\partial} & \mathrm{HFK}^-{}_{m-1}(K_{n-2}, s) & \xrightarrow{U} & \mathrm{HFK}^-{}_{m-3}(K_{n-2}, s-1)
\end{array}$$

There are four distinct families in our investigation, with base cases K_0, K_1, K_0^τ and K_1^τ, for even and odd values of n. The hat-version $\widehat{\text{HFK}}$ of each has been verified computationally with the program of Droz [Dro]. $\widehat{\text{HFK}}(K_1)$ and $\widehat{\text{HFK}}(K_1^\tau)$ have been found to be isomorphic with $\widehat{\text{HFK}}(K_0)$ and $\widehat{\text{HFK}}(K_0^\tau)$, respectively (see Table A.1). □

This verifies that $\{K_n\}$, $n \in \mathbb{Z}^+$, is an infinite family of knots each admitting a distinct genus two mutant of the same total dimension in knot Floer homology.

Chapter 4

Pretzel Knots and the proof of Theorem 2

4.1 Pretzel links

Throughout, K (resp. L) is an oriented knot (resp. link) in S^3. We write $L = (n_1, \ldots, n_k)$ if L is a pretzel link with k tangles, where the ith tangle consists of $n_i \in \mathbb{Z}$ half-twists. We will also use the integer n_i to refer to this specific tangle in the pretzel projection. Notice that tangles of length one can be permuted to any spot in a pretzel link by flype moves and that if there exist $n_i = +1$ and $n_j = -1$ in L, then n_i and n_j can be pairwise removed by flyping followed by an isotopy. We recall below the classification theorem for pretzel knots, due to Kawauchi [Kaw96]. In Kawauchi's notation $P(-\varepsilon b; n_1, n_2, \ldots, n_k)$ refers to a pretzel knot where the tangles of length one have been collected into the horizontal integer tangle b, $\varepsilon = \pm 1$, and $|n_i| > 1$ e.g.

$$K = (\underbrace{1, 1, \ldots, 1}_{b}, n_1, n_2, \ldots, n_k) = P(-b; n_1, n_2, \ldots, n_k)$$

In the special case that $b > 0$ and $n_i = \pm 2$, note that by flype moves and isotopy,

$$P(-\varepsilon b; n_1, \ldots, n_i, \ldots, n_k) \simeq P(-\varepsilon(b-1); n_1, \ldots, -n_i, \ldots, n_k)$$

Additionally, a pretzel link is a knot whenever either $k \geq 0$, all the n_i are odd, and $b + k$ are odd, or whenever $k \geq 1$ and exactly one n_i is even. These are called *odd* and *even* pretzels, respectively. Moreover, we have the following:

Theorem 24. *[Kaw96, Theorems 2.3.1 and 2.3.2]*

1. *A pretzel knot $P(-\varepsilon b; n_1, n_2 \ldots, n_k)$ is a 2-bridge knot (or possibly a trivial knot) if and only if $k \leq 2$ and $P(-b; n_1, n_2)$ has the same type as the 2-bridge knot $C(n_1, b, n_2)$.*

2. *Two pretzel knots $P(-b; n_1, n_2 \ldots, n_k)$ and $P(-c; n'_1, n'_2, \ldots, n'_\ell)$ which are neither 2-bridge nor trivial belong to the same type if and only if $k = \ell$, $b = c$ and one of the following conditions is satisfied:*

   (a) Both are odd pretzels and $(n'_1, n'_2, \ldots, n'_\ell)$ is a cyclic permutation of $(n_1, n_2, \ldots, n_k)$.

   (b) Both are even pretzels and $(n'_1, n'_2, \ldots, n'_\ell)$ is a cyclic permutation of $(n_1, n_2, \ldots, n_k)$ or $(n_k, \ldots, n_2, n_1)$.

3. *A pretzel knot $P(-\varepsilon b; n_1, n_2 \ldots, n_k)$ which is neither a 2-bridge knot nor a trivial knot is a torus knot if and only if $b = 0$, $k = 3$ and (n_1, n_2, n_3) is a cyclic permutation of $(3\varepsilon, 3\varepsilon, -2\varepsilon)$ or $(3\varepsilon, 5\varepsilon, -2\varepsilon)$, where $\varepsilon = \pm 1$.*

We will implicitly use this classification of pretzel knots throughout this chapter, though in general, we will not use Kawauchi's notation. Unless otherwise stated, we will assume a planar projection of any pretzel link L is in a standard pretzel form and that k is the minimal possible number of strands to present L as a pretzel projection. Note that this implies L does not have both $n_i = +1$ and $n_j = -1$. We will also assume that all $n_i \neq 0$, which is equivalent to the statement that K is prime. Throughout, we use $g(K)$ to denote the genus of K.

Since $\chi(\widehat{HFK}(K,s)) = a_s$, the coefficient of t^s in the symmetrized Alexander polynomial of K, this will give us an easy way to approach Conjecture 1 in many cases. We therefore establish the following lemma.

Lemma 25. *If $det(K) > 2g(K)+1$ then $\Delta_K(T)$ contains some coefficient a_s with $|a_s| > 1$.*

Proof. If the coefficients of $\Delta_K(T)$ are at most one in absolute value, then

$$\det(K) = |\Delta_K(-1)| \leq \sum_s |a_s| \leq 2g(K)+1,$$

which is a contradiction. □

Suppose that Y is a Seifert fibered rational homology sphere with base orbifold S^2 and Seifert invariants $(b; (a_1, b_1), \ldots (a_k, b_k))$. Then

$$|H_1(Y;\mathbb{Z})| = |a_1 \cdots a_k \cdot (b + \sum_{i=1}^{k} \frac{b_i}{a_i})|$$

(see for instance [Sav02]). The branched double covers of Montesinos knots (and subsequently, pretzel knots) are such Seifert fibered spaces. Since

$$det(K) = |H_1(\Sigma_2(K);\mathbb{Z})|,$$

then if K is of the form $(n_1, \cdots, n_r, \underbrace{1, \ldots, 1}_{d})$, where $|n_i| > 1$ for $1 \leq i \leq r$,

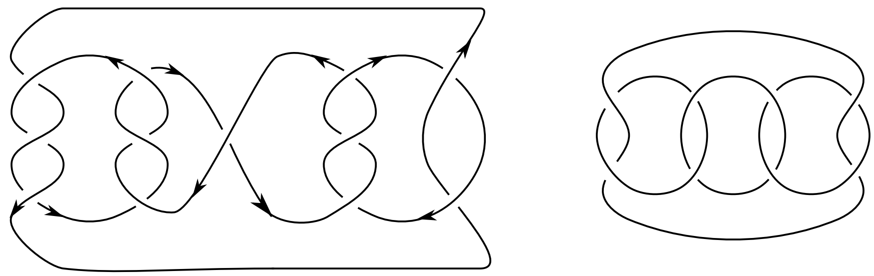
$$|H_1(\Sigma_2(K))| = det(K) = \left|n_1 \cdots n_r \cdot (d + \sum_{i=1}^{r} \frac{1}{n_i})\right|. \tag{4.1}$$

4.1.1 Fibered pretzel links

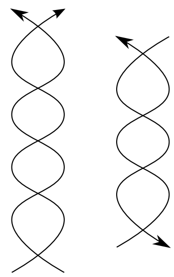
As mentioned earlier, $\widehat{\mathrm{HFK}}(K, g(K)) \cong \mathbb{Z}$ if and only if K is fibered [Ghi08, Ni07]. Since $\widehat{\mathrm{HFK}}(K, g(K))$ is always non-trivial [OS04a], Theorem 2 is automatic for any non-fibered knot. Thus for the proof of Theorem 2 (see Section 4) we will only be interested in fibered pretzel knots. In [Gab86a, Theorem 6.7], Gabai classified oriented fibered pretzel links together with their fibers; we recall this below for the case of links. We will deal with Theorem 2 for connected sums of pretzel knots in Section 4.6. Let m_i denote a tangle in an oriented pretzel diagram in which the two strands are oriented consistently (i.e. both up or both down) and let m_{ij} denote a tangle where the two strands are oriented inconsistently (i.e. one up and one down).[1] See Figure 4.1. Associated to a Type 2 or Type 3 link L (described below) will be an auxiliary oriented pretzel link L',

$$L' = \left(\frac{-2m_1}{|m_1|}, m_{11}, \ldots, m_{12}, \ldots, m_{1\ell_1}, \frac{-2m_2}{|m_2|}, m_{21}, \ldots, m_{2\ell_2}, \ldots \right.$$
$$\left. \ldots \frac{-2m_r}{|m_r|}, m_{r1}, \ldots, m_{r\ell_r}\right), \tag{4.2}$$

[1]At times, we use the notation m_i and m_{ij} to denote both an integer tangle as well as the numerical integer associated with such a tangle. The usage should be clear from the context.



(a) A pretzel knot K and its associated auxiliary link L'.



(b) The left tangle is $m_i = \pm 5$ and the right tangle is $m_{ij} = \pm 4$.

Figure 4.1: The notation m_i, m_{ij} and the auxiliary link L'.

where the term $\frac{-2m_i}{|m_i|}$ is omitted if $|m_i| = 1$. L' is oriented so that the surface obtained by applying the Seifert algorithm is of Type 1. The auxiliary link L' is derived from a procedure of Gabai in which a minimal genus Seifert surface is "desummed" to determine whether it fibers [Gab86a].

Theorem 26. *[Gab86a, Theorem 6.7]*
The algorithm which follows determines whether an oriented pretzel link fibers.[2]

Algorithm. *L falls into one of three types.*

Type 1: *Each tangle in L is an m_{ij}. The pretzel surface is the surface R obtained by applying the Seifert algorithm. L fibers if and only if L fibers*

[2]The original formulation of [Gab86a, Theorem 6.7] describes the fiber surfaces for all types; we include this information only when it is relevant to our calculations.

with fiber R. Moreover this happens if and only if one of the following holds:

1. *each* $n_i = \pm 1$ *or* ∓ 3 *and some* $n_i = \pm 1$.

2. $(n_1, \ldots, n_k) = \pm(2, -2, 2, -2, \ldots, 2, -2, n)$, $n \in \mathbb{Z}$ *(here, k is odd).*

3. $(n_1, \ldots, n_k) = \pm(2, -2, 2, -2, \ldots, -2, 2, -4)$ *(here, k is even).*

Type 2: *Both* m_i *and* m_{ij} *exist in* L.

Type 2A: *The number of positive and negative* m_i *differ by two.* L *fibers if and only if* $|m_{ij}| = 2$ *for all indices* ij *and* L *fibers if and only if* L *fibers with fiber* R*, where* R *is the surface obtained by applying the Seifert algorithm.*

Type 2B: *The number of positive and negative* m_i *in* L *are equal and* $L' \neq \pm(2, -2, \ldots, 2, -2)$. L *fibers if and only if* L' *fibers.*

Type 2C: *The number of positive and negative* m_i *are equal and* $L = \pm(2, -2, \ldots, 2, -2)$. *In this type,* L *can be isotoped to be in Type 3.*

Type 3: *These are Type 2 links where no* m_{ij} *exists. If either the numbers of positive and negative tangles are unequal or if* $L' \neq \pm(2, -2, \ldots, 2, -2)$*, then treat* L *as if it was Type 2A or 2B. Otherwise,* L *is fibered if and only if there is a unique* m_i *of minimal absolute value.*

Finally, if L *is Type 1, Type 2A, or the Type 2A subcase of Type 3, then the fiber surface is the surface obtained from the Seifert algorithm.*

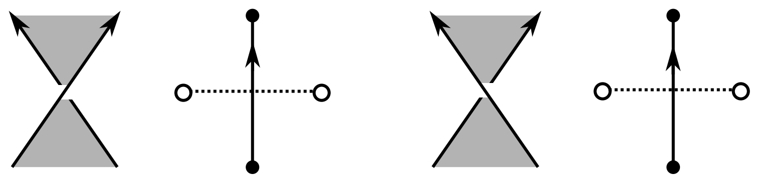
In our case analysis, will call the three subcases of Type 3 by Type 3-2A, Type 3-2B, and Type 3-min. Note that if a knot is Type 2, there is exactly one m_{ij}, which we denote by $\bar{m}$, and that this unique $\bar{m}$ must also be the unique even tangle. As permuting tangles corresponds with doing a series of Conway mutations, $\Delta_K(t)$, and consequently $det(K)$, are unchanged. Note that invariance of the determinant under permutations is also evident from Equation 4.1.

4.2 A state sum for the Alexander polynomial

The Alexander polynomial of K admits a state sum expression in terms of the set of Kauffman states $\mathcal{S}$ of a decorated projection of the knot [Kau83]. (In fact, we may also arrange that $\widehat{\mathrm{CFK}}(K)$ is freely generated by $\mathcal{S}$ [OS03a].) By a decorated knot projection we mean a knot projection with a distinguished edge. When using decorated knot projections, we will always choose the bottom-most edge in a standard projection of a pretzel knot to be the distinguished edge. Each state $\mathbf{x}$ is equipped with a bigrading $(A(\mathbf{x}), M(\mathbf{x}))$ such that the symmetrized Alexander polynomial of K is given by the state sum

$$\Delta_K(T) = \sum_{\mathbf{x}\in\mathcal{S}} (-1)^{M(\mathbf{x})} T^{A(\mathbf{x})}. \tag{4.3}$$

We follow the description of Kauffman states given in [OS03a]. Let G_B and G_W denote the black and white graphs associated with a checkerboard coloring of a decorated knot projection. $\mathcal{S}$ is in a one-to-one correspondence with the set of maximal trees in G_B. Each maximal tree $T \subset G_B$ uniquely corresponds with a dual maximal tree $T^* \subset G_W$; we frequently refer to the pair $T(\mathbf{x}) \cup T^*(\mathbf{x})$ as "the tree" corresponding to the state $\mathbf{x} \in \mathcal{S}$. For our purposes, it is more convenient to rephrase the state-sum formula in terms of maximal trees. We therefore describe $A(\mathbf{x})$ and $M(\mathbf{x})$ in this frameworks as follows. Label each



(above) $\eta(e) = -1,\ \eta(e^*) = 0$ (above) $\eta(e) = +1,\ \eta(e^*) = 0$

(below) $\eta(e) = 0,\ \eta(e^*) = +1$ (below) $\eta(e) = 0,\ \eta(e^*) = -1$

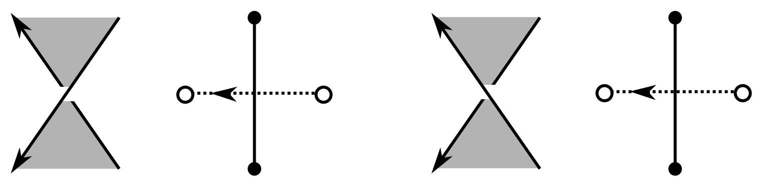


Figure 4.2: The labels $\eta(e)$ and $\eta(e^*)$ are defined for all edges $e \in G_B$ and $e^* \in G_W$. The edge orientations pictured are those induced by K on G_B or G_W.

edge e of $G_B(\mathbf{x})$ and $G_W(\mathbf{x})$ with $\eta(e) \in \{-1, 0, 1\}$ according to Figure 4.2. We describe two orientations on the edges of $T(\mathbf{x})$ and $T^*(\mathbf{x})$. The first is given by orienting the edges so that the root is the vertex corresponding to the region adjacent to the decorated edge and that each edge points away from the root [OS04c]. In this case, each edge of $T(\mathbf{x})$ and of $T^*(\mathbf{x})$ is oriented. The second orientation is induced by the orientation on the knot as in Figure 4.2; note that at each crossing exactly one of the edges of $T(\mathbf{x})$ or $T^*(\mathbf{x})$ is oriented.

Then, $A(\mathbf{x})$ is defined by

$$A(\mathbf{x}) = \frac{1}{2} \sum_{e \in T(\mathbf{x}) \cup T^*(\mathbf{x})} \sigma(e)\eta(e), \tag{4.4}$$

where

$$\sigma(e) = \begin{cases} +1 & \text{if the two induced orientations on } e \text{ agree} \\ -1 & \text{if the two induced orientations on } e \text{ disagree} \\ 0 & \text{otherwise.} \end{cases}$$

Here, σ is as in Figure 4.3 and η is as in Figure 4.2. Next, $M(\mathbf{x})$ is defined by summing only over edges on which the two orientations agree,

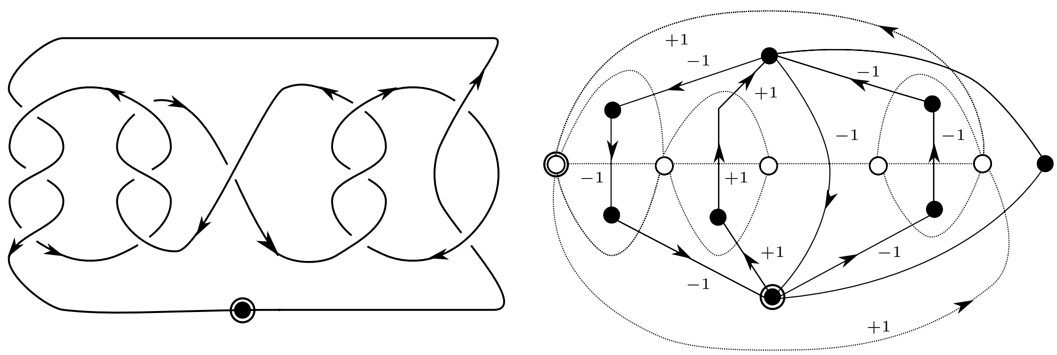
$$M(\mathbf{x}) = \sum_{\substack{e \in T(\mathbf{x}) \cup T^*(\mathbf{x}) \\ \sigma(e) = 1}} \eta(e) \tag{4.5}$$

An example is given in Figure 4.3.

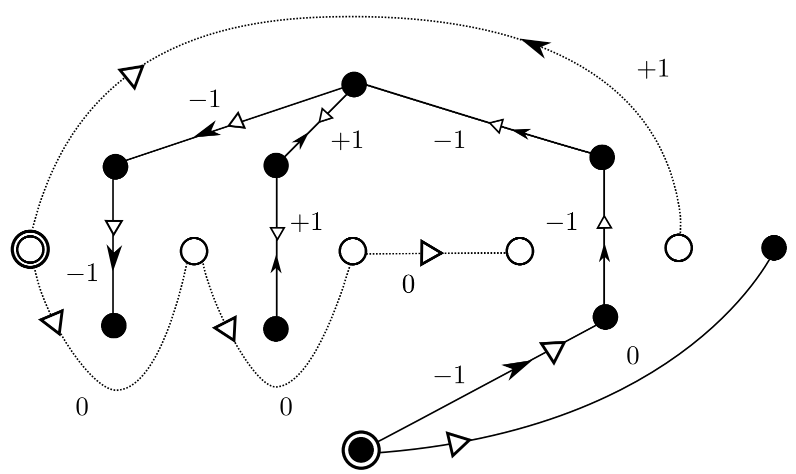
4.3 Counting lemmas

The state-sum formula (Chapter 4.2 above) provides an elementary way to determine the coefficients of the Alexander polynomial. Suppose that the state-sum decomposition of a diagram of K admits a unique state $\mathbf{x}_0$ with minimal A-grading $A(\mathbf{x}_0)$. When this happens, the class $[\mathbf{x}_0]$ generates the homology group $\widehat{\mathrm{HFK}}(K, A(\mathbf{x}_0))$. Moreover, since $\max\{s | \widehat{\mathrm{HFK}}(K, s) \neq 0\} = g(K)$ [OS04a], $A(\mathbf{x}_0) = -g(K)$. As discussed previously, this implies that K is fibered. When such a unique minimal element $\mathbf{x}_0$ exists, it is convenient to use $\mathbf{x}_0$ to count the states in A-grading $-g(K) + 1$.

Definition 27. *Let $K = (n_1, \cdots, n_k)$ be a pretzel knot with a standard decorated diagram and let $T(\mathbf{x}) \cup T^*(\mathbf{x})$ be the tree corresponding to some state*



(a) The Type 2A knot $K = (3, -3, 1, 3, 2)$ and the corresponding black and white graphs G_B and G_W. Edges without an arrow induced by the orientation on K are labelled $\eta(e) = 0$ or $\eta(e^*) = 0$. The decorated edge of K determines a decorated vertex (e.g. a root) in each of G_B and G_W.



(b) This is a state $\mathbf{x}$ of $K = (3, -3, 1, 3, 2)$ in bigrading $(A(\mathbf{x}) = -4, M(\mathbf{x}) = -5)$. The white arrows indicate the edge orientation which points away from the black and white roots and the black arrows indicate the orientation induced by K. When the two orientations agree, $\sigma(e) = 1$. When the two orientations disagree, $\sigma(e) = -1$. When the orientation induced by K does not exist, $\eta(e) = 0$.

Figure 4.3: An example to illustrate Formulas 4.4 and 4.5.

$\mathbf{x}$. *The* trunk *of* $T(\mathbf{x}) \cup T^*(\mathbf{x})$ *is the unique path in* $T(\mathbf{x})$ *which connects the bottom-most vertex of* G_B *(the* root of G_B*) to the top-most vertex of* G_B *(see Figure 4.4). Let* $T(n_i)$ *denote the path in* G_B *connecting the root to the top-most vertex, corresponding to the tangle* n_i*.*

Note that as long as n_i does not correspond with the trunk of $T(\mathbf{x}_0)$, then $T(\mathbf{x}_0) \cap T(n_i)$ is connected, and cannot have edges incident to both the top vertex and the root.

Definition 28. *Let* $K = (n_1, \cdots, n_k)$ *be a pretzel knot with a standard decorated diagram and suppose* $\mathbf{x}_0$ *is the unique state in A-grading* $A(\mathbf{x}_0) = -g(K)$*, should it exist. Fix a tangle* n_i *which does not correspond to the trunk. A* trade *is a state* $\mathbf{x}_T$ *whose corresponding tree is obtained by replacing the terminal edge of* $T(\mathbf{x}_0)$ *contained in* $T(n_i)$ *with the other edge in* $T(n_i)$ *adjacent to the black root or top vertex. See Figure 4.4.*

In a trade, $T(\mathbf{x}_T) \cup T^*(\mathbf{x}_T)$, along with its orientations and labels, differs from $T(\mathbf{x}_0) \cup T^*(\mathbf{x}_0)$ in exactly one edge of each and shares the same trunk.

Lemma 29. *Suppose that* K *and* $\mathbf{x}_0$ *are as in Definition 28 and that the tangle* n_j *corresponds with the trunk of* $T(\mathbf{x}_0)$*. Let* ℓ *be the number of tangles with* $n_i = \pm 1$*, excluding* n_j *if* $n_j = \pm 1$*. Then, there are* $k - \ell - 1$ *trades all of which are supported in bigrading* $(A(\mathbf{x}_0) + 1, M(\mathbf{x}_0) + 1)$*.*

Proof. Note that any branch of $T(\mathbf{x}_0)$ corresponding to a tangle of length one, except perhaps the trunk, would create a cycle. There are no edges of $T(\mathbf{x}_0)$

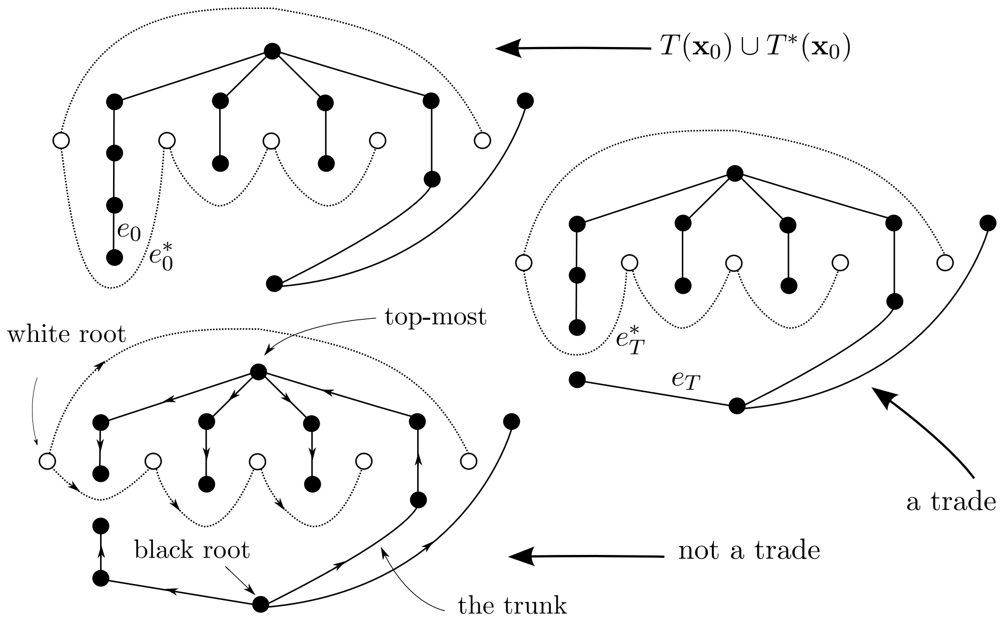


Figure 4.4: The first tree $T(\mathbf{x}_0) \cup T^*(\mathbf{x}_0)$ represents a state $\mathbf{x}_0$ with minimal A-grading. The tree to the right represents a trade $T(\mathbf{x}_T) \cup T^*(\mathbf{x}_T)$. The bottom tree represents a state which is not a trade. On the bottom tree, the orientation induced by the roots of the trees is indicated, as are the roots and the trunk.

corresponding with any tangle with $n_i = 1$, except possibly n_j. By definition, there is exactly one trade corresponding with each tangle of length greater than one (see Figure 4.4), and so there are $k - l - 1$ trades. Let $e_0 \in T(\mathbf{x}_0)$ and $e_T \in T(\mathbf{x}_T)$ (e_0^* and e_T^*, respectively) be the edges along which $T(\mathbf{x}_0)$ and $T(\mathbf{x}_T)$ ($T^*(\mathbf{x}_0)$ and $T^*(\mathbf{x}_T)$ respectively) differ. The edges e_0 and e_T are contained in the same branch of G_B, and therefore share the same value for η (see Figure 4.4). Assume first that $\eta(e_0) = \eta(e_T) = \pm 1$ and $\eta(e_0^*) = \eta(e_T^*) = 0$. Because $A(\mathbf{x}_0)$ is minimal, $\sigma(e_0)\eta(e_0) = -1$, or else $A(\mathbf{x}_T) < A(\mathbf{x}_0)$. This implies $\sigma(e_0) = 1$ and $\eta(e_0) = -1$ or $\sigma(e_0) = -1$ and $\eta(e_0) = 1$. In the trade, the orientation induced by the root on e_T is different than the orientation induced by the root on e_0, and so $\sigma(e_T) \neq \sigma(e_0)$, but the labels $\eta(e_T)$ and $\eta(e_0)$ remain the same. This implies $\sigma(e_T)\eta(e_T) = +1$, and therefore both $M(\mathbf{x}_T) = M(\mathbf{x}_0) + 1$ and $A(\mathbf{x}_T) = A(\mathbf{x}_0) + 1$. Assume next that $\eta(e_0^*) = \eta(e_T^*) = \pm 1$ and $\eta(e) = \eta(e_T) = 0$. The trade induces a change in $T^*(\mathbf{x}_0)$ wherein the edge e_0^* is replaced with an edge e_T^* which is vertically adjacent in G_W (see Figure 4.4). Similarly, since $A(\mathbf{x}_0)$ is minimal and $\sigma(e_0^*)\eta(e_0^*) = -1$, then again the induced orientation changes and σ changes but η does not. Both $M(\mathbf{x}_T) = M(\mathbf{x}_0) + 1$ and $A(\mathbf{x}_T) = A(\mathbf{x}_0) + 1$. □

4.4 Hyperbolic pretzel knots with four or more tangles

The goal of this section is prove

Proposition 30. *Let K be a fibered, hyperbolic pretzel knot of at least 4 strands which is not isotopic to the pretzel knot $(3, -5, 3, -2)$. Then, there exists a coefficient a_s of the Alexander polynomial such that $|a_s| \geq 2$.*

Since $\chi(\widehat{HFK}(K, s)) = a_s$, with the exception of a single knot, this will prove Theorem 2 in the case that K has at least four strands. Proposition 30 will be proved by the repeated use of two basic arguments: either analyzing a_{-g+1} with the state-sum formula or by using Lemma 25 to study the determinant of K. In the special case that K is isotopic to the knot $(3, -5, 3, -2)$, the Alexander polynomial of K is

$$\Delta_K(t) = t^{-3} - t^{-2} + 1 - t^2 + t^3.$$

For this knot, we compute its knot Floer homology groups to observe directly that there exist several bigradings (m, s) such that *rank* $\widehat{\mathrm{HFK}}_m(K, s) \geq 2$. We compute the knot Floer homology of $K \simeq (3, -5, 3, -2)$ by using the Python program for $\widehat{\mathrm{HFK}}$ with $\mathbb{F}_2$ coefficients by Droz [Dro]. The knot Floer groups are displayed in Table A.3 in the Appendix.

We proceed through the cases of Theorem 26.

4.4.1 Type 1 knots

In our case analysis, we discard Type 1 knots in the sub-cases (2) and (3) because these are links with at least two components.

Lemma 31. *If K is a fibered hyperbolic pretzel knot of Type 1, then* $\det(K) > 2g(K) + 1$.

Proof. Assume that K has r tangles with $n_i = \pm 3$ and $s > 0$ tangles with $n_i = \mp 1$. Note that when $r = 0$, K is a torus knot. If K is a hyperbolic, three strand pretzel knot, it has genus one. Therefore, if K is also a Type 1 fibered knot, it is the figure eight knot, which has $det(K) = 5$. Therefore, we assume that K has at least four strands (in fact five, since if K is a Type 1 knot, it must have an odd number of strands). More generally, the genus of the pretzel spanning surface (and in this case, the genus of K) is given by

$$g(K) = \frac{1}{2}(r + s - 1)$$

and so $2g(K)+1 = r+s$. By Equation 4.1,

$$det(K) = |3^r(-s + \sum_{i=1}^{r} \frac{1}{3})| = |3^{r-1}(r-3s)|.$$

We will verify the inequality in two cases, $r > 3s$ and $r < 3s$, where $s, r > 0$ and $r+s > 3$. (When $r = 3s$, $r+s$ is even and so K is a link.) If $r > 3s$, then

$$det(K) = |3^{r-1}(r-3s)| \geq |3^{r-1}| = 3^{r-1} > \frac{4r}{3} > r+s = 2g(K)+1.$$

If $r < 3s$, then

$$det(K) = |3^{r-1}(r-3s)| = 3^{r-1}(3s-r).$$

If $r < 3$, the inequality is easily checked by hand. If $3 \leq r < 3s$, we have

$$\begin{aligned} 3^{r-1} - 1 > 2r \quad &\Rightarrow \quad (3^{r-1}-1)(3s-r) > 2r - 2s \\ &\Rightarrow \quad (3^{r-1}-1)(3s-r) + (3s-r) > r+s \\ &\Rightarrow \quad 3^{r-1}(3s-r) > r+s. \end{aligned}$$

□

4.4.2 Type 2 knots

Note that if a knot K is Type 2, there is exactly one m_{ij}, which we denote by $\bar{m}$. Otherwise, K is a link with at least two components. Moreover, $\bar{m}$ must be the unique even tangle.

Type 2A. After mirroring, we may assume that a Type 2A knot K has $p+2$ positive odd tangles, p negative odd tangles, exactly one even tangle $\bar{m} = \pm 2$, and that $p > 0$.

Lemma 32. *For pretzel knots of Type 2A with four or more tangles, there exists a coefficient a_s of the Alexander polynomial such that $|a_s| \geq 2$.*

Proof. Note that the Type 2A condition is preserved under mutation; because these are fibered knots and the Alexander polynomial is preserved, we see that the genus is also preserved under mutation. Therefore, we may apply mutations to assume that n_i is negative when i is even, and positive when i is odd, except for $n_{2p+2} > 0$ and $n_{2p+3} = \bar{m} = \pm 2$. Thus for all edges $e \in T(n_i) \subset G_B$,

$$\eta(e) = \begin{cases} 0 & \text{if } i = 2p+3 \\ -1 & \text{if } i < 2p+3 \text{ is odd or } i = 2p+2 \\ +1 & \text{if } i \neq 2p+2 \text{ is even.} \end{cases}$$

Using the state sum expression for the Alexander polynomial in terms of maximal trees, we will first find a unique state in A-grading $-g(K)$, and then use this state to determine the the second coefficient $|a_{-g(K)+1}|$ of the Alexander polynomial.

Claim 33. *When K is oriented so that the first tangle points downward, K admits a unique state $\mathbf{x}_0$ with minimal A-grading $-g(K)$.*

Proof of Claim 33. Let $\mathbf{x}_0$ be the state which is represented by the tree $T(\mathbf{x}_0) \cup T^*(\mathbf{x}_0)$ as in Figure 4.5. In particular, the trunk of the tree must be along $T(n_{2p+2})$. The parts of the black tree corresponding with tangles $n_1, \ldots, n_{2p+1}$ are incident to the top vertex, and are not incident to the black root. In the white tree $T^*(\mathbf{x}_0)$ there is a single edge with a nonzero label (one of two edges depending on the sign of $\bar{m}$). Because the tangles were permuted to make the signs alternate, this means that for all $e \in T(\mathbf{x}_0)$,

$$\sigma(e) = \begin{cases} +1 & \text{if } e \in T(n_i), i \text{ odd or } i = 2p+2, i \neq 2p+3 \\ -1 & \text{if } e \in T(n_i), i \text{ even }, i \neq 2p+2 \\ 0 & \text{if } e \in T(n_{2p+3}) \end{cases}$$

As for edges in the white tree $T^*(\mathbf{x}_0)$, all are labelled $\eta(e) = \sigma(e) = 0$ except for one edge, where the configuration of the maximal tree is chosen so that $\sigma(e)\eta(e) = -1$. Thus, when $T(\mathbf{x}_0) \cup T^*(\mathbf{x}_0)$ is in the configuration described, every edge with $\eta(e) \neq 0$ contributes $\sigma(e)\eta(e) = -1$ to the sum for $A(\mathbf{x}_0)$ and hence $A(\mathbf{x}_0)$ is minimal. Moreover, it is not hard to see that in any other configuration of a maximal tree, there will be strictly fewer edges e contributing $\sigma(e)\eta(e) = -1$ to the A-grading sum. Hence $\mathbf{x}_0$ is unique. □

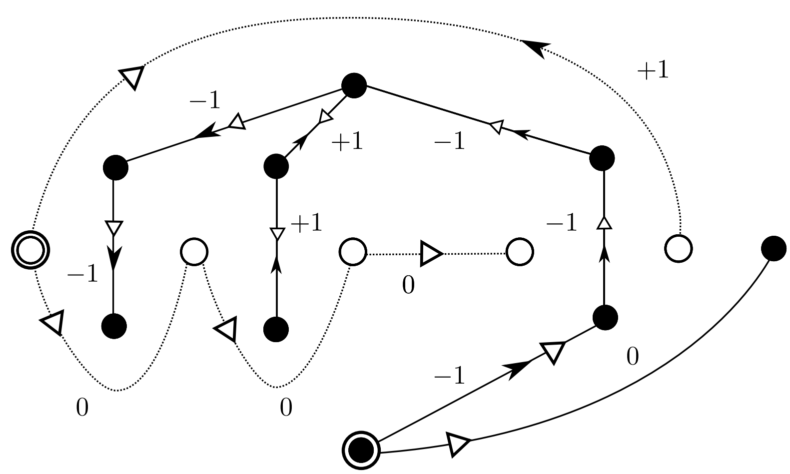


Figure 4.5: The tree $T(\mathbf{x}_0) \cup T^*(\mathbf{x}_0)$ corresponding with the unique minimal state $\mathbf{x}_0$ of the Type 2A knot $K = (3, -3, 1, 3, 2)$. The edges in the diagram are labeled by $\eta(e)$. The black and white arrows indicate the two orientations which define $\sigma(e)$.

By Lemma 29, there are $2p+2-\ell$ trades, all supported in bigradings $(-g(K)+1, M(\mathbf{x}_0)+1)$. To determine that $|a_{-g+1}| \geq 2$, we need to count all the other states in A-grading $-g(K)+1$, compute their M-gradings, and then use the Euler characteristic formula (Equation 4.3).

Because $\bar{m} = \pm 2$, all of the trees which share the same trunk as $T(\mathbf{x}_0)$ which are not trades represent states which have an A-grading greater than $-g(K)+1$. Thus, the remaining states in A-grading $-g(K)+1$ will appear as trees with different trunks. One of these states will be a tree which has its trunk along $T(n_{2p+3})$, and where $T(n_{2p+2})$ is no longer incident to the top vertex. This state is denoted $\mathbf{x}_{2p+3}$. If $\bar{m} = -2$, $\mathbf{x}_{2p+3}$ is supported in A-grading $-g(K)+1$ and M-grading $M(\mathbf{x}_{2p+3}) = M(\mathbf{x}_0)+2$. If $\bar{m} = +2$, $\mathbf{x}_{2p+3}$ is supported in A-grading $-g(K)+1$ and M-grading $M(\mathbf{x}_0)+1$. All of the remaining states in A-grading $-g(K)+1$ correspond with trees whose trunks are along tangles n_j of length one (where $n_j \neq n_{2p+2}$). Denote these states by $\mathbf{x}_j$. Such a tree $T(\mathbf{x}_j) \cup T^*(\mathbf{x}_j)$ has a trunk along $T(n_j)$ and the former trunk becomes a path which is no longer incident to the top vertex.[3] The trunk of $T(\mathbf{x}_j)$ must be length one, otherwise $A(\mathbf{x}_j) > -g(K)+1$ due to the contribution of edges labelled $\sigma(e)\eta(e) = 1$ in $T(n_j)$.

Claim 34. *Let $\mathbf{x}_j$ be a state in A-grading $-g(K)+1$ whose trunk is along the*

[3]We remark that a priori, in other diagrams of knots there may be states in A-grading $-g(K)+1$ which arise from configurations other than trades or trees with new trunks, but these do not occur in our calculations.

tangle n_j (which is necessarily length one). Then

$$M(\mathbf{x}_j) = \begin{cases} M(\mathbf{x}_0)+1 & \textit{if } j \textit{ odd and } j \neq 2p+3 \\ M(\mathbf{x}_0)+2 & \textit{if } j \textit{ even and } j \neq 2p+2. \end{cases}$$

Proof of Claim 34. In $T(\mathbf{x}_0)$ there are n_{2p+2} edges in the trunk, all labeled $\eta(e) = -1$ and $\sigma(e) = +1$. There is no edge along $n_j = \pm 1$ (otherwise $T(\mathbf{x}_0)$ contains a cycle). In $T(\mathbf{x}_j)$, the trunk becomes a path contributing $n_{2p+2} - 1$ edges with the same labels. Along the new trunk, if j is odd, $\sigma(n_j) = \eta(n_j) = -1$. If j is even, $\sigma(e) = \eta(e) = +1$. All other edges and labels of $T(\mathbf{x}_j)$ and $T(\mathbf{x}_0)$ agree and the change in $T^*(\mathbf{x})$ does not affect its contribution to M. The net change to the M-grading is $+1$ or $+2$, respectively. □

Since $\chi(\widehat{HFK}(K,s)) = a_s$, the coefficient $|a_{-g+1}|$ is given by difference of states in M-grading $M(\mathbf{x}_0)+1$ and states in M-grading $M(\mathbf{x}_0)+2$. Suppose first that $\bar{m} = 2$. If K contains any $n_j = -1$, then by flype moves, K is isotopic to a Type 3-2A knot addressed below (in particular, K can be isotoped so that the presentation has fewer strands). Hence when $\bar{m} = 2$, we may assume any tangle of length one is positive, and therefore all trades $\mathbf{x}_T$, states $\mathbf{x}_j$ and $\mathbf{x}_{2p+3}$ are supported in M-grading $M(\mathbf{x}_0)+1$, and so $|a_{g+1}| > 1$. Suppose now that $\bar{m} = -2$. Then $|a_{-g+1}|$ is given by difference in the number of trades $\mathbf{x}_T$ and $\mathbf{x}_{odd}$ states and the number of $\mathbf{x}_{even}$ states and $\mathbf{x}_{2p+3}$. Moreover, we may assume that no $n_j = +1$ and each length one tangle is negative, otherwise by flype moves, K is isotopic to a Type 3 knot (with fewer strands). Thus,

$$|a_{-g+1}| = (2p+2-\ell) - (\ell+1) = 2p - 2\ell + 1,$$

and so $|a_{-g+1}| > 1$ whenever $p > \ell$. Since there are at most p negative tangles of length one, we have reduced Lemma 32 to the case where every negative tangle which is not the trunk is length one. After isotopy, K is of the form

$$K \simeq (-2, \underbrace{-1, \ldots, -1}_{p}, n_1, n_2, \ldots, n_{p+2})$$

where here, we have reindexed the tangles so that $n_i \geq 3$ is odd for $1 \leq i \leq p+2$. We now apply Lemma 25. For Type 2A knots, the minimal genus Seifert surface and fiber for K is obtained by applying the Seifert algorithm to the standard projection [Gab86a], which gives

$$g(K) = \frac{1}{2}\Big(\sum_{i=1}^{p+2}(n_i - 1) + 2\Big).$$

Let $N = n_1 \ldots n_{p+2}$. By Equation 4.1, we have

$$\begin{aligned} det(K) &= \big| -2N\big(-p - \frac{1}{2} + \sum_{i=1}^{p+2} \frac{1}{n_i})\big| \\ &= \big|N + 2N(p - \sum_{i=1}^{p+2} \frac{1}{n_i})\big| \\ &\geq \big|N + 2N(\frac{2p-2}{3})\big| \end{aligned}$$

And since $p \geq 1$,

$$\begin{aligned} \left|N + 2N(\frac{2p-2}{3})\right| &\geq N \\ &> (\sum_{i=1}^{p+2} n_i) + 1 \\ &\geq \sum_{i=1}^{p+2} (n_i - 1) + 3 \\ &= 2g(K) + 1. \end{aligned}$$

Lemma 25 now completes the proof for Type 2A knots. □

Remark 35. *The L-space knots of the form $(-2, 3, q)$, where $q > 0$ is an odd integer are pretzel knots of Type 2A with three tangles ($p = 0$).*

Type 2B. In this case, K has exactly one even tangle $\bar{m}$ and an equal number of positive and negative odd tangles. The auxiliary link $L' \neq \pm(2, -2, \ldots, 2, -2)$, and K fibers if and only if L' fibers.

Lemma 36. *For pretzel knots of Type 2B, $|a_{-g+1}| \geq 2$.*

Proof. Suppose first K has $\ell > 0$ length one tangles. The existence of length one tangles impacts the auxiliary link L'. If $\ell \geq 3$, then L' cannot be of the required form for fibered Type 1 links. If $\ell = 2$, then L' may be of the correct form, but this implies $\bar{m} = \pm 2$ when the length one tangles are ∓ 1. But then by flype moves and isotopy, K is isotopic to a Type 3 knot (and it was not isotoped to have a minimal number of strands), which is addressed below. If $\ell = 1$, then L' must be of the form $\pm(2, -2, \ldots, 2, -4)$ and so

$$K \simeq \pm(1, m_1, \ldots, m_{2p-1}, -4)$$

where $m_i < -2$ for $1 \leq i \leq 2p-1$, i odd and $m_i > 2$ for $1 < i < 2p-1$, i even. For these knots, we will first perform an isotopy, and then make a counting argument similar to the one given for Type 2A knots. The isotopy to be performed on K is pictured in Figure 4.6. After isotopy, the knot admits

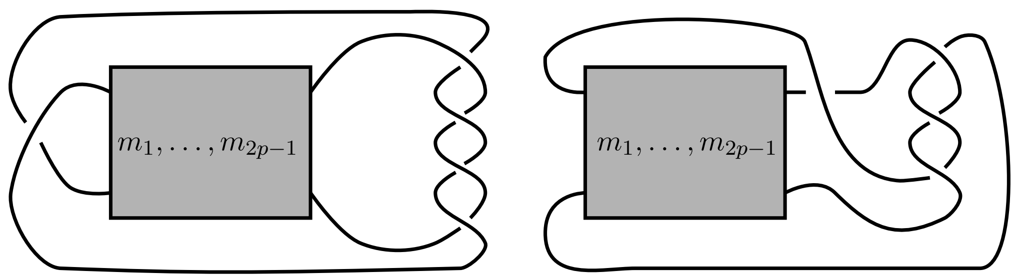


Figure 4.6: The isotopy performed on the Type 3 knot $K = (1, m_1, \ldots, m_{2p-1}, -4)$.

a black graph whose edges are all labelled ± 1, and a dual white graph where all of the edges are labelled 0. Thus we only need to consider maximal trees of the black graph.

Claim 37. *When K is oriented so that the first tangle (after isotopy) points upward, there is a unique state $\mathbf{x}_0$ in A-grading $-g(K)$.*

Proof of Claim 37. See Figure 4.7. Since $m_i < -2$ for i odd and $m_i > 2$ for i even, then for all edges $e \in T(m_i) \subset G_B$, $i = 1, \ldots 2p-1$,

$$\eta(e) = \begin{cases} +1 & \text{if } i \text{ is odd} \\ -1 & \text{if } i \text{ is even.} \end{cases}$$

There are four other edges in G_B, and each is labelled $\eta(e) = -1$, as in Figure 4.7. Let $\mathbf{x}_0$ be the state whose maximal tree $T(\mathbf{x}_0)$ has its trunk along the right-most upward oriented edge (the orientation induced by the knot), and

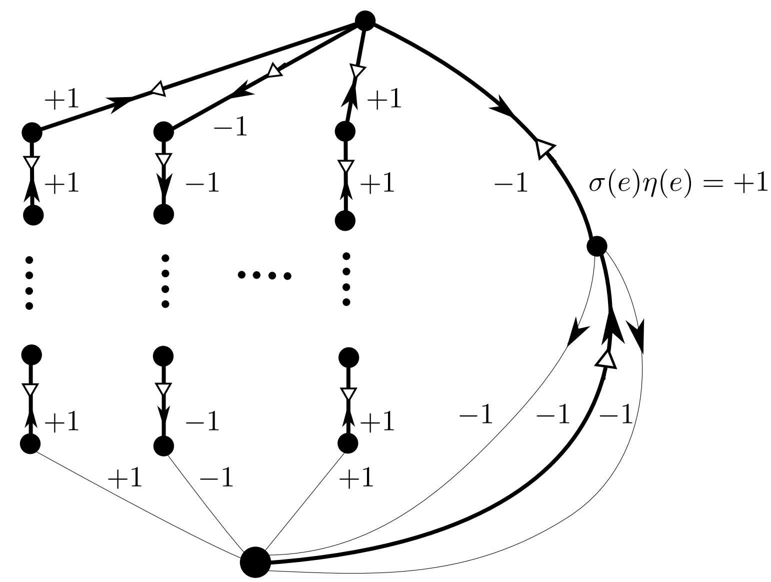


Figure 4.7: The black graph after isotoping the Type 2B knot $K \simeq \pm(1, m_1, \ldots, m_{2p-1}, -4)$. $T(\mathbf{x}_0)$ is pictured in bold. The black arrows indicate the orientation induced by the knot, and the white arrows indicate the orientation induced by the root. Labels $\eta(e)$ are given for each edge in G_B. There single edge for which $\sigma(e)\eta(e) = +1$ in $T(\mathbf{x}_0)$, also indicated.

with no other edges incident to the black root, as in Figure 4.7. Each edge $e \in T(\mathbf{x}_0)$ contributes $\sigma(e)\eta(e) = -1$ to the sum for $A(\mathbf{x}_0)$, with the exception of one edge, indicated in Figure 4.7. Because $|m_i| > 2$ for $i = 1, \ldots, 2p-1$ then in any other maximal tree configuration, the corresponding A-grading is strictly greater than $A(\mathbf{x}_0)$. Hence $\mathbf{x}_0$ is the unique state with minimal A-grading. □

There are exactly $2p+1$ states in A-grading $-g(K)+1$, and these are obtained by "trading" along the $m_1, \ldots, m_{2p-2}$ or by interchanging one of the remaining two edges not present in $T(\mathbf{x}_0)$. Each of these $2p+1$ states is supported in the same M-grading. Hence, $|a_{-g+1}| = 2p + 1$. □

Suppose that there are no tangles of length one. Since L' is a fibered Type 1 link, L' must be of the form

$$(n_1, \ldots, n_k) = \pm(2, -2, 2, -2, \ldots, 2, -2, n),\ n \in \mathbb{Z}.$$

This means that K is of the form

$$K \simeq (m_1, \ldots, m_{2p-1}, \bar{m}).$$

Up to mirroring and cyclic permutation, K admits a diagram where $m_i > 0$ when i is odd, $m_i < 0$ is negative when i is even, and $\bar{m}$ is both even and positioned to be the last tangle. When the pretzel diagram for K is oriented so that the first tangle points downward, K admits a unique state $\mathbf{x}_0$ in minimal A-grading $A(\mathbf{x}_0) = -g(K)$. This state is represented by a tree $T(\mathbf{x}_0) \cup T^*(\mathbf{x}_0)$ which has its trunk along $T(\bar{m})$, and with only one edge incident to the black root. See Figure 4.8. Because the tangles alternate sign, every edge of $T(\mathbf{x}_0) \cup T^*(\mathbf{x}_0)$ contributes $\sigma(e)\eta(e) = -1$ or 0 to $A(\mathbf{x}_0)$. Because there are no tangles of length one, any other maximal tree configuration will have a strictly greater A-grading. Hence $\mathbf{x}_0$ is unique and minimally A-graded. Moreover, every state supported in A-grading $-g(K) + 1$ is a trade because there is a unique $\bar{m}$ and there are no tangles of length one. By Lemma 29, there are $2p$ trades, each supported in M-grading $M(\mathbf{x}_0) + 1$, and hence $|a_{-g+1}| = 2p \geq 2$. (Note that this particular argument also applies to length three, Type 2B pretzels knots $K = (n_1, n_2, n_3)$ where $|n_i| > 1$, $i = 1, 2, 3$.)

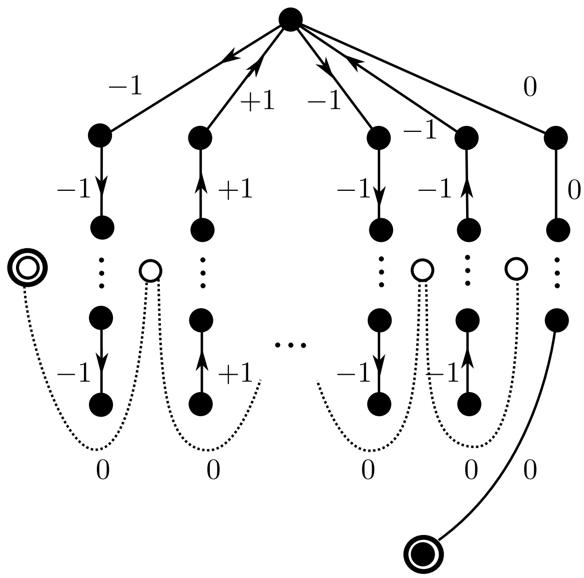


Figure 4.8: The unique minimal state for a Type 2B knot K with no tangles of length one. Labels $\eta(e)$ are indicated in the diagram.

Type 2C. Suppose K has exactly one even tangle $\bar{m}$ and an equal number of positive and negative odd tangles. By assumption,

$$L' = \pm(2, -2, \ldots, 2, -2).$$

Up to mirroring, this implies that $\bar{m} = -2$ and there is exactly one tangle with $n_i = +1$. Thus, we have

$$K = (n_1, \ldots, n_{2p-1}, 1, -2) \simeq (n_1, \ldots, n_{2p-1}, 2),$$

and K is exhibited as a Type 3 knot with fewer strands.

4.4.3 Type 3 knots

A tangle of the form m_i contributes nonzero terms only to G_B, whereas an m_{ij} contributes nonzero terms only to G_W. In particular, Type 3 knots contain no m_{ij} tangles, and hence their Alexander polynomials can be studied solely using the black graph G_B and black maximal trees $T(\mathbf{x})$.

Type 3-min. K has p positive tangles, and p negative tangles. Of these there is a unique tangle of minimal length and an even tangle, which are possibly the same tangle. By assumption, $L' = \pm(2, -2, \ldots, 2, -2)$ also has an even number of tangles, and thus by uniqueness of the minimal tangle, there are no tangles of length one.

Lemma 38. *For all pretzel knots of Type 3-min other than $K = (3, -5, 3, -2)$, there exists a coefficient of the Alexander polynomial such that $|a_s| \geq 2$.*

Proof. By the conditions on L', the tangles alternate sign. After mirroring and cyclic permutation, we may assume n_i is positive when i is odd, n_i is negative when i is even, and n_{2p} is minimal. For all $e \in T(n_i)$, $\eta(e)$ is labeled -1 when i is odd and $+1$ when i is even. When the pretzel diagram is oriented so that the first tangle points downward, K admits a unique state with $A(\mathbf{x}_0) = -g(K)$ represented by a tree $T(\mathbf{x}_0)$ with its trunk along the minimal tangle, as in the example in Figure 4.9. Because the tangles alternate sign, then each edge $e \in T(n_i)$ for $i = 1, \ldots, 2p-1$ contributes $\eta(e)\sigma(e) = -1$ to the A-grading sum. Along trunk, the edges contribute $\eta(e)\sigma(e) = +1$ to the sum, but since this is the unique minimal length tangle, $A(\mathbf{x}_0)$ is both minimal and unique. By Lemma 29, there are $2p-1$ trades in bigradings $(-g(K)+1, M(\mathbf{x}_0)+1)$. Since there is no m_{ij}, all states in A-grading $-g(K)+1$ which are not trades have a corresponding tree whose trunk is along a tangle n_j with $|n_j| = |n_{2p}|+1$.

Denote such a state by $\mathbf{x}_j$. Such trees $T(\mathbf{x}_j)$ must have trunks along tangles of length $|n_{2p}|+1$, otherwise $A(\mathbf{x}_j) > -g(K)+1$, as in Lemma 32.

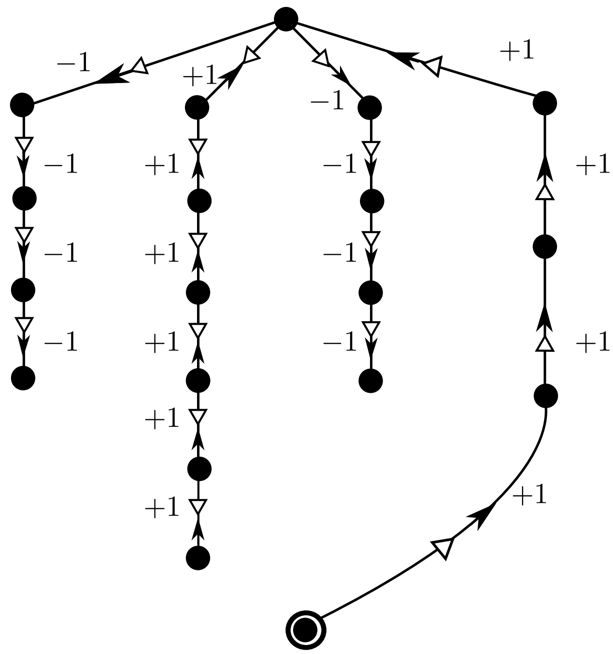


Figure 4.9: An example of the unique minimal state for a Type 3-min knot K, with trunk along the unique tangle of minimum length. We only need to consider the black graph for Type 3 knots.

Suppose n_{2p} is odd. Since there is exactly one even tangle, there is at most one state $\mathbf{x}_j$. Hence $|a_{-g+1}| = 2p$ or $2p-2$, and since $2p \geq 4$, we have $|a_{-g+1}| \geq 2$. Now suppose n_{2p} is even.

Claim 39. *Let $\mathbf{x}_j$ be a state in A-grading $-g(K)+1$ corresponding with a tangle n_j of length $|n_{2p}|+1$. Then*

$$M(\mathbf{x}_j) = \begin{cases} M(\mathbf{x}_0) & j \text{ odd} \\ M(\mathbf{x}_0)+1 & j \text{ even.} \end{cases}$$

Proof of Claim 39. Suppose j is odd. For all $e \in T(\mathbf{x}_0) \cap T(n_j)$, $\sigma(e) = 1$ and $\eta(e) = -1$, and for all $e \in T(\mathbf{x}_0) \cap T(n_{2p})$, $\sigma(e) = \eta(e) = 1$. In particular, the contribution to $M(\mathbf{x}_0)$ of the edges e from $T(\mathbf{x}_0) \cap T(n_{2p})$ and $T(\mathbf{x}_0) \cap T(n_j)$ cancel because $|n_j| = |n_{2p}| + 1$. In $T(\mathbf{x}_j)$, all labels and orientations remain the same, except for these same edges, where $\sigma(e)$ changes to -1. Thus the edges e from $T(\mathbf{x}_0) \cap T(n_{2p})$ and $T(\mathbf{x}_0) \cap T(n_j)$ do not contribute to $M(\mathbf{x}_j)$. The net change to the M-grading is zero. Suppose j is even. In $T(\mathbf{x}_j)$, the orientations along $T(n_j)$ and $T(n_{2p})$ change from -1 and 1 to 1 and -1, respectively. Similarly, the net change to the M-grading is $+1$. □

Now, $|a_{-g+1}|$ is equal the number of $\mathbf{x}_T$ states and $\mathbf{x}_{even}$ states minus the number of $\mathbf{x}_{odd}$ states. Therefore,

$$|a_{-g+1}| \geq \big|(2p-1) - (p)\big| = p - 1$$

and so $|a_{-g+1}| \geq 2$ whenever $p > 2$. Since $p > 1$, the case $p = 2$ remains. All of the above restrictions on K summarize to the following remaining case: K is now of the form

$$K \simeq (2n+1, -q, 2n+1, -2n) \tag{4.6}$$

where $n \geq 1$ and $q \geq 2n+3$ is odd. In this case, the minimal genus Seifert surface is obtained after performing a particular isotopy which is described in [Gab86a] and pictured in Figure 4.10. Applying the Seifert algorithm gives the genus of K,

$$g(K) = \frac{1}{2}(6n + q - 3).$$

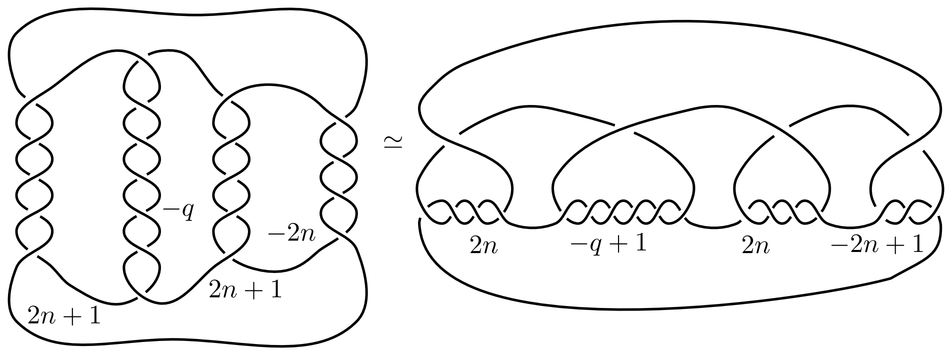


Figure 4.10: The isotopy performed on a Type 3-min knot K to obtain a minimal genus Seifert surface. This isotopy is described in [Gab86a].

By Equation 4.1,

$$det(K) = |4n(2n+1)q - (2n+1)^2 q - 2n(2n+1)^2|$$

Since $q \geq 2n+3$, then $det(K) > 2g(K)+1 \Longleftrightarrow$

$$\begin{aligned}
\Leftrightarrow \quad 4n(2n+1)q - (2n+1)^2 q - 2n(2n+1)^2 &> 6n + q - 2 \\
\Leftrightarrow \quad (2n^2-1)q &> 4n^3 + 4n^2 + 4n - 1 \\
\Leftrightarrow \quad (2n^2-1)(2n+3) &> 4n^3 + 4n^2 + 4n \\
\Leftrightarrow \quad n^2 &> 3n + 1
\end{aligned}$$

which is true for all $n > 3$. If $n = 3$, $n = 2$, or $n = 1$, then the inequality

$$(2n^2-1)q > 4n^3 + 4n^2 + 4n - 1$$

is satisfied if and only if $q \geq 11$, $q \geq 9$, or $q \geq 13$, respectively. The only pairs (n, q) not covered by this inequality are: $(3, 9), (2, 7), (1, 11), (1, 9), (1, 7)$, and $(1, 5)$. The knots and Alexander polynomials corresponding to the first five pairs are:

$$\Delta_{(7,-9,7,-6)} = t^{-5} - t^{-4} + 2t^{-2} - 3t^{-1} + 3 - 3t + 2t^2 - t^4 + t^5$$

$$\Delta_{(5,-7,5,-4)} = t^{-5} - t^{-4} + t^{-2} - 2t^{-1} + 3 - 2t + t^2 - t^4 + t^5$$

$$\Delta_{(3,-11,3,-2)} = t^{-6} - t^{-5} + 2t^{-3} - 3t^{-2} + 3t^{-1} - 3$$
$$+3t - 3t^2 + 2t^3 - t^5 + t^6$$

$$\Delta_{(3,-9,3,-2)} = t^{-7} - t^{-6} + t^{-4} - 2t^{-3} + 3t^{-2} - 4t^{-1} + 5$$
$$-4t + 3t^2 - 2t^3 + t^4 - t^6 + t^7$$

$$\Delta_{(3,-7,3,-2)} = t^{-4} - t^{-3} + 2t^{-1} - 3 + 2t - t^3 + t^4.$$

The last pair of integers corresponds to the special knot $(3, -5, 3, -2)$ mentioned above, whose knot Floer groups appear in Table A.3 in the Appendix. The Alexander polynomials were computed using the Mathematica package KnotTheory [BNMea]. □

Type 3-2A. After mirroring, we may assume that for pretzel knots of Type 3-2A, there are $p + 2$ positive tangles and p negative tangles, and that of these tangles, there is exactly one even tangle. There is no m_{ij}.

Lemma 40. *Up to mirroring, for all pretzel knots of Type 3-2A other than those with exactly p negative tangles of length one, $|a_{-g+1}| \geq 2$.*

Proof. Up to mutation, we may assume that n_i is positive when i is odd and that n_i is negative when i is even, except for the last tangle, where n_{2p+2} is positive. In G_B, $e \in T(n_i)$ is labeled $\eta(e) = -1$ if i odd or $i = 2p + 2$

and $\eta(e) = +1$ for $i \neq 2p+2$ even. When K is oriented so that the first tangle points downward, then there is a unique state $\mathbf{x}_0$ with $A(\mathbf{x}_0) = -g(K)$ represented by a tree $T(\mathbf{x}_0)$ with trunk corresponding with tangle n_{2p+2}, as in Lemma 32. In particular, for all $e \in T(\mathbf{x}_0)$, $\sigma(e) = +1$ if $e \in T(n_i)$ for i odd or $i = 2p+2$ and $\sigma(e) = -1$ if $i \neq 2p+2$ even. An example is given in Figure 4.11. Every edge contributes $\eta(e)\sigma(e) = -1$ to the sum for $A(\mathbf{x}_0)$, so $\mathbf{x}_0$ is clearly minimally graded. It is unique because in any other maximal tree configuration there will be an edge contributing $\sigma(e)\eta(e) = +1$ to the A-grading.

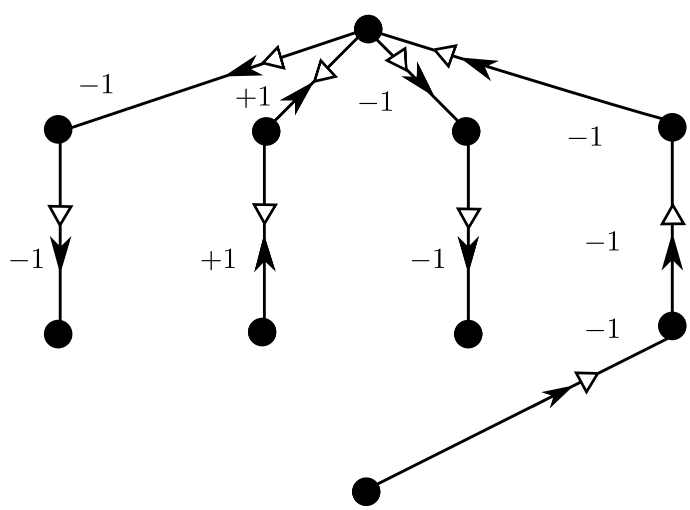


Figure 4.11: An example of the unique minimal state of a Type 3-2A knot.

There are $2p-\ell+1$ trades in A-grading $-g(K)+1$ by Lemma 29, and ℓ other states $\mathbf{x}_j$ in A-grading $-g(K)+1$, each of which corresponds to a tangle n_j of length one, as in Lemma 32.

Claim 41. *Let $\mathbf{x}_j$ be the state in A-grading $-g(K)+1$ corresponding with the* *tangle n_j of length one. Then*

$$M(\mathbf{x}_j) = \begin{cases} M(\mathbf{x}_0) + 1 & j \text{ is odd} \\ M(\mathbf{x}_0) + 2 & j \neq 2p + 2 \text{ is even.} \end{cases}$$

Proof of Claim 41. $T(n_j) \not\subset T(\mathbf{x}_0)$ and for all $e \in T(\mathbf{x}_0) \cap T(n_{2p})$, $\sigma(e) = 1$ and $\eta(e) = -1$. In $T(\mathbf{x}_j)$, all edges, labels, and orientations remain the same, except along $T(n_{2p})$, where one less edge is included, and along $T(n_j)$, where the single edge of $T(n_j)$ has $\sigma(e) = -1$ and $\eta(e) = -1$ when i is odd, or has $\sigma(e) = 1$ and $\eta(e) = 1$ when i is even. The net change to the M-grading is $+1$ and $+2$, respectively. □

If the length one tangles are positive, then

$$|a_{-g+1}| = (2p - \ell + 1) + \ell = 2p + 1 > 2,$$

and we are done. If the length one tangles are negative, then

$$|a_{-g+1}| = (2p - \ell + 1) - \ell > 1 \Longleftrightarrow \ell < p.$$

This verifies the statement of Lemma 40. □

To prove Proposition 30 for the remaining Type 3-2A pretzels, we must show that when K has exactly p negative length one tangles $det(K) > 2g(K)+1$. After reindexing the tangles, K is of the form

$$K \simeq (\underbrace{-1, \ldots, -1}_{p}, n_1, \cdots, n_{p+2})$$

where one of the $n_i = 2$ and the remaining $n_i \geq 3$ for $i = 1, \ldots p+2$. The genus of the surface obtained by applying the Seifert algorithm to the standard projection is

$$g(F) = \frac{1}{2}\big(\sum_{i=1}^{p+2}(n_i - 1) + 1\big),$$

and it does not depend on the order of the tangles. By Theorem 26, this is the fiber surface, and so $g(F) = g(K)$. Let $N = n_1 \cdots n_{p+2}$. Using Equation 4.1,

$$\begin{aligned} det(K) &= \big|N\big(-p + \sum_{i=1}^{p+2} \frac{1}{n_i}\big)\big| \\ &\geq \big|N(p - \sum_{i=1}^{p+1} \frac{1}{3} - \frac{1}{2})\big| \\ &\geq N \cdot \frac{4p-5}{6} \end{aligned}$$

and whenever $p \geq 2$, we have

$$det(K) > \sum_{i=1}^{p+2} n_i - p = 2g(K) + 1.$$

If $p = 1$ then K is of the form $(-1, n_1, n_2, n_3)$. Notice that the even tangle must be at least 4, otherwise K can be isotoped to be a three-stranded pretzel knot, which we address in Section 4.5. The determinant inequality is not satisfied unless one of the $n_i \geq 5$. So without loss of generality, assume $K =$

$(-1, n_1, n_2, n_3)$ where $n_1 \geq 3, n_2 \geq 5$, and $n_3 \geq 4$ is even. In this case,

$$\begin{aligned} det(K) &= \left|N\left(1 - \sum_{i=1}^{3} \frac{1}{n_i}\right)\right| \\ &\geq N \cdot \frac{13}{60} \\ &> \sum_{i=1}^{3} n_i - 1 \\ &= 2g(K) + 1. \end{aligned}$$

The only Type 3-2A pretzel knot with four or more strands which remains is $K = (-1, 3, 3, 4)$, which has Alexander polynomial

$$\Delta_{(-1,3,3,4)} = t^{-4} - t^{-3} + 2t^{-1} - 3 + 2t - t^3 + t^4.$$

This Alexander polynomial was computed using the Mathematica package KnotTheory [BNMea].

Type 3-2B. K is a Type 3 knot addressed via Type 2B. There are p positive tangles, and p negative tangles. By assumption the auxiliary link $L' \neq \pm(2, -2, \ldots, 2, -2)$, and K is fibered if and only if L' is fibered, where L' is Type 1. There are no terms of L' equal to ± 1, therefore L' cannot be of Type 1-(1). Since there is no $\bar{m}$, there are no terms equal to ± 4, and so we may also rule out Type 1-(3). This leaves $L' = \pm(2, -2, \ldots, 2, -2, n)$, where $n \in \mathbb{Z}$. This can only happen if $n = \pm 2$ and K contains a unique tangle of length one. Up to mirroring, $K \simeq (n_1 \ldots, n_{2p-1}, -1)$, where n_i is positive for i odd, and negative for i even. K can be oriented so that $\eta(e) = -1, \sigma(e) = 1$ when $e \in T(n_i)$ for i odd, and $\eta(e) = 1, \sigma(e) = -1$ when $e \in T(n_i)$ for i

even. As in the proof of Lemma 38, the minimal state $\mathbf{x}_0$ is represented by a tree with trunk along the unique tangle of minimal length, say n_{min}, and the only possible states which are not trades must occur along tangles n_j of length $|n_{min}| + 1$. Since there is a single even tangle, there is at most one such state $\mathbf{x}_j$. This implies that either $|a_{-g+1}| = 2p - 1$ or $|a_{-g+1}| = 2p - 2$. Either way, $|a_{-g+1}| \geq 2$.

4.5 Length three pretzel knots

Having established Proposition 30, we now complete the proof of Theorem 2 for pretzel knots. We will denote pretzel knots in this section by $P(n_1, \ldots, n_k)$ to avoid confusion with torus knots, denoted $T(m, n)$. Before proceeding, we point out that pretzel knots with one strand are unknotted and that $P(p, q) = T(2, p + q)$. As the only non-hyperbolic, pretzel knots are $\pm T(3, 4)$, $\pm T(3, 5)$, and $T(2, 2n + 1)$ for all $n \in \mathbb{Z}$, to prove Theorem 2 for pretzel knots, it suffices to establish the following.

Lemma 42. *If K is a hyperbolic pretzel knot with three strands which is not of the form $\pm P(-2, 3, q)$, where q is a positive odd integer, then there exists $s \in \mathbb{Z}$ such that rank $\widehat{\mathrm{HFK}}(K, s) \geq 2$.*

Proof. Let $K = P(p, q, r)$, where $p, q, r \in \mathbb{Z}$. By Theorem 26, there are three cases, up to mirroring, of length three pretzel knots to consider:

1. p, q, and r are all odd.

2. $p = 2k + 1 > 0, q = 2\ell + 1 > 0$, and $r = -2$, where $k, \ell \in \mathbb{Z}$.

3. $p = 2k + 1 > 0, q = 2\ell + 1 < 0$, and $r = 2m$, where $k, \ell, m \in \mathbb{Z}$.

The only cases we have omitted are alternating knots, in which case K is an L-space knot if and only if it is isotopic to a $T(2, 2n + 1)$ torus knot [OS05].

Case 1. These are Type 1 knots. This is handled in Lemma 31.

Case 2. These are Type 2A fibered knots. When $p = 1$, $P(-2, 1, q) \simeq P(2, q) \simeq T(2, 2+q)$. When $p = 3$ and q is a positive integer, this is the family exempted in the assumptions of the Lemma.

It follows from [Eft03, Theorem 1] that whenever $p > 3$ and $q > 3$, there exists a bigrading (m, s), with rank $\widehat{\mathrm{HFK}}_m(K, s) \geq 2$. This can also be verified with another simple determinant argument and by manually checking a few remaining cases. Without loss of generality, we may assume that $p \leq q$. The genus of the surface F obtained by applying the Seifert algorithm to the pretzel presentation for $K = P(p, q, -2)$ is $g(F) = \frac{1}{2}(p + q)$, which is equal to $g(K)$ by Theorem 26. Thus, whenever $p > 5$ and $q > 5$ or $q > 7$ and $p = 5$,

$$\begin{aligned} det(K) &= |2pq(\frac{1}{p} + \frac{1}{q} - \frac{1}{2})| \\ &= |2(p + q) - pq| \\ &\geq p + q + 1 \\ &= 2g(K) + 1. \end{aligned}$$

When $p = 5$ and $q = 5$ or 7, we obtain the desired result by computing the

Alexander polynomials:

$$\begin{aligned}\Delta_{P(-2,5,5)}(t) = \;& t^{-6} - t^{-5} + t^{-3} - 2t^{-2} + 3t^{-1} - 3 \\ & +3t - 2t^2 + t^3 - t^5 + t^6 \\ \Delta_{P(-2,5,7)}(t) = \;& t^{-5} - t^{-4} + t^{-2} - 2t^{-1} + 3 - 2t + t^2 - t^4 + t^5.\end{aligned}$$

The Alexander polynomials were computed using the Mathematica package KnotTheory [BNMea].

Case 3. These knots are Type 2B. As pointed out at the end of the proof of Lemma 36, the proof that $|a_{-g+1}| \geq 2$ applies for pretzel knots with three strands of Type 2B when neither p nor q is length one. Therefore, assume that one of p or q is length one. We consider the case where $q = -1$; the case that $p = 1$ is similar. If $K = P(-1, p, r)$, then $L' = P(-2, r)$. In this case, L' will fiber if and only if $r = 4$. Whenever $p > 3$, then

$$\begin{aligned} det(K) &= |4p(-1 + \frac{1}{p} + \frac{1}{4})| \\ &= |-3p + 4| \\ &> p + 2 \\ &= 2g(F) + 1 \\ &\geq 2g(K) + 1, \end{aligned}$$

where F is the surface obtained by applying the Seifert algorithm to the pretzel presentation of K. If $p = 3$, then K is the figure eight knot, which has $|a_0| = 3$. Finally, if $p = 1$, K is the unknot. □

4.6 Connected sums of pretzel knots

We now prove Theorem 2 and consequently Conjecture 1 for connected sums of pretzel knots.

Proof of Theorem 2 for composites. Suppose that K is a composite of fibered pretzel knots. Recall the Künneth formula for the knot Floer homology of a connected-sum $K = K_1 \# K_2$ [OS04b]:

$$\widehat{\mathrm{HFK}}(K, s) = \bigoplus_{s_1+s_2=s} \widehat{\mathrm{HFK}}(K_1, s_1) \otimes \widehat{\mathrm{HFK}}(K_2, s_2).$$

We begin with the case that $K = K_1 \# K_2$, where each K_i is a prime, fibered pretzel knot. We may assume that $\operatorname{rank} \widehat{\mathrm{HFK}}(K_i, s) < 2$ for all s, or else the result immediately follows from the Künneth formula, as we note that $\widehat{\mathrm{HFK}}(K)$ cannot be trivial. By the proof of Theorem 2 for prime knots, we can deduce that each K_i must be an L-space knot. For L-space knots, $\tau(K) = g(K)$ [OS05], and so $\widehat{\mathrm{HFK}}_0(K, g) \cong \mathbb{F}$. By work of Hedden and Watson [HW12], $\widehat{\mathrm{HFK}}_{-1}(K, g-1)$, a fact deduced from the spectral sequence relating $\widehat{\mathrm{HFK}}(K)$ to $\widehat{\mathrm{HF}}(S^3)$. In particular, the Alexander polynomial satisfies $|a_g| = |a_{g-1}| = 1$. This implies that the rank of $\widehat{\mathrm{HFK}}(K_1 \# K_2, g_1 + g_2 - 1)$ is at least two.

If K has at least three prime-summands, then we write $K = K_1 \# K_2$, where K_1 has exactly two prime-summands. The result now follows from combining the Künneth formula with the above discussion for pretzel knots with two prime-summands. □

Chapter 5

Observations

Here, we collect a few miscellaneous observations and remarks.

5.1 Conjecture 1 and Question 5 are related

In Theorem 6, we offer affirmative evidence towards the question of whether the total rank of knot Floer homology is preserved under Conway and genus two mutation. Let us suppose for a moment that the total rank is indeed preserved under mutation, and consider an arbitrary L-space knot K. Suppose that K^τ is a mutant of K, along some essential Conway sphere or genus two surface F. Since neither Conway nor genus two mutation affects the Alexander polynomial of K,

$$\Delta_K(t) = \Delta_{K^\tau}(t).$$

This fact, taken together with the restrictions on the knot Floer homology groups for L-space knots and the (temporary) assumption that the total rank of knot Floer homology is preserved under mutation, imply that for all Alexander gradings s,

$$\widehat{\mathrm{HFK}}(K^\tau, s) \cong \widehat{\mathrm{HFK}}(K, s),$$

modulo a possible shift in the Maslov gradings.

The existence of two distinct L-space knots which share the same knot Floer homology groups seems like a rare occurrence. An equally plausible scenario may be that K and K^τ have isomorphic knot Floer homology because K and K^τ are actually isotopic knots, meaning the mutation did not actually change the isotopy class of the knot. Thus, a natural question to ask after considering the possibility that the total rank of knot Floer homology is preserved under mutation is the following:

Question 43. *Do any L-space knots admit non-trivial mutations?*

Conjecture 1 addresses this question. Indeed, the contrapositive of Conjecture 1 states that a knot with *rank* $\widehat{\mathrm{HFK}}(K,s) \leq 1$ for all $s \in \mathbb{Z}$ (e.g. an L-space knot) contains no essential Conway spheres in its complement, thus admits only trivial mutations.

5.2 Observations following Theorem 2

Using Theorem 2 in conjunction with existing results of Mattman, Ozsváth and Szabó, we are able to recover the classification of pretzel knots which admit surgeries with finite fundamental group.

Corollary 44. *[IJ09, Ichihara-Jong]*
The only non-trivial pretzel knots which admit finite surgeries with positive surgery coefficients are $P(-2,3,7)$, $P(-2,3,9)$, $T(3,4)$, $T(3,5)$, *and* $T(2,2k+1)$ *for* $k>0$.

Proof. It remains to rule out the case of $P(-2,3,q)$ for odd $q \geq 11$. A theorem of Mattman shows that the only knots of the form $K = P(-2,3,q)$ with $q \neq 1,3,5$ which admit a finite surgery are $P(-2,3,7)$ and $P(-2,3,9)$ [Mat02]. This completes the proof. □

Remark 45. *Corollary 44 was shown to hold more generally for Montesinos knots by Ichihara and Jong [IJ09]. Their proof uses similar arguments to those in this paper, but first appeals to an analysis of essential laminations in the exteriors of Montesinos knots by Delman. This allows them to restrict their attention to a few specific families of pretzel knots.*

It is also interesting to compare Conjecture 1 with recent work of Hedden and Levine. They show that if Y is the splice of non-trivial knots in integer homology sphere L-spaces (such manifolds necessarily contain incompressible tori), then $\dim_{\mathbb{Z}/2} \widehat{HF}(Y;\mathbb{Z}/2) \geq 2$ [HL12]. It seems possible that using the machinery of bordered-sutured Floer homology [Zar11], one could use an argument similar to that of Hedden and Levine to prove Conjecture 1.

5.3 Observations following Theorem 6

Baldwin and Levine conjecture that δ-graded knot Floer homology is invariant under Conway mutation [BL12]. The examples in Theorem 6 demonstrate that this conjecture cannot be extended to include genus two mutations. If their conjecture is indeed true, it would be interesting to understand why there is an apparent difference in the behavior of the knot Floer complex under

Conway mutations and genus two mutations.

Genus two mutation provides a method for producing closely related knots and links, but more generally it is an operation on three manifolds. Ruberman proves [Rub99] that the instanton Floer homology and $\mathbb{Z}$-graded instanton homology for homology three-spheres is invariant under genus two mutation. Thus a natural question is whether this holds true in the Heegaard Floer homology context as well.

Question 46. *Let Y be a closed, oriented three-manifold with an embedded genus two surface F. If Y^τ is the corresponding genus two mutant of Y, is it true that*

$$\operatorname{rank}\widehat{\mathrm{HF}}(Y) = \operatorname{rank}\widehat{\mathrm{HF}}(Y^\tau)?$$

The families of knots which we have employed in this paper are all non-alternating slice knots, and in particular, are formed as the band sum of a two-component unlink. There are other infinite families of slice knots for which these computational techniques using skein exact sequences and concordance invariants work. For example, Hedden and Watson [HW12] prove that there are infinitely many knots with isomorphic Floer groups in any given concordance class, whereas Greene and Watson [GW11] have worked with the Kanenobu knots. Certain three-stranded pretzel knots also share this property. Nor is the non-alternating status of these knots a coincidence; in fact there can only be finitely many alternating knots of a given knot Heegaard Floer homology type.

Proposition 47. *Let K be an alternating knot. There are only finitely many other alternating knots with knot Floer homology isomorphic to $\widehat{\mathrm{HFK}}(K)$ as bigraded groups.*

Proof. Suppose to the contrary that K belongs to an infinite family $\{K_n\}_{n\in\mathbb{Z}}$ of alternating knots sharing the same knot Floer groups. Since $\widehat{\mathrm{HFK}}(K_n) \cong \widehat{\mathrm{HFK}}(K)$ and knot Floer homology categorifies the Alexander polynomial,

$$\det(K_n) = |\Delta_{K_n}(-1)| = |\Delta_K(-1)| = \det(K)$$

for all n. Each knot K_n admits a reduced alternating diagram D_n with crossing number $c(D_n)$. The Bankwitz Theorem implies that $c(K_n) \leq \det(K_n)$. However, there are only finitely many knots of a given crossing number, and in particular $c(K_n)$ grows arbitrarily large with n, which contradicts that $c(K_n) \leq \det(K)$. □

This fact leads to the interesting open question of whether there are infinitely many quasi-alternating knots of a given knot Floer type. As an even stronger statement, Greene conjectures that there exist only finitely-many quasi-alternating links with a given determinant [Gre10, Conjecture 3.1] and proves the cases where $\det(L) = 1, 2$ or 3.

Finally, one may also ask about the behavior of knot Floer homology under *skein equivalence*. This is an equivalence relation $\sim_S$ on the set of oriented links characterized by the following [Kaw96]:

1. If L is isotopic to L', then $L \sim_S L'$,

2. $L_+ \sim_S L'_+$ and $L_0 \sim_S L'_0$ imply $L_- \sim_S L'_-$, and

3. $L_- \sim_S L'_-$ and $L_0 \sim_S L'_0$ imply $L_+ \sim_S L'_+$,

for skein triples (L_+, L_-, L_0) and (L'_+, L'_-, L'_0). It is not hard to see that all pairs of Conway mutant knots are skein equivalent. Due to the recursive nature of the definition of skein equivalence and the difficulty in distinguishing mutants, pairs of knots which are skein equivalent but not Conway mutants are difficult to come by. One such pair are the knots 8_8 and 13_{6714}, each of which fits into a skein triple with 10_{129} and the two-component unlink. Interestingly, these knots all belong to the family of Kanenobu knots [Kan86]. The knot Floer homology groups of Kanenobu knots were computed in [GW11], and shown to form infinite families of knots with isomorphic $\widehat{\mathrm{HFK}}(K)$. In particular, the knot Floer homology groups of 8_8 and 13_{6714} are isomorphic, and consequently, have the same total rank. One example does not a conjecture make, but regardless of the outcome, it would be interesting to answer Question 5 for skein equivalent pairs of knots.

Appendices

Appendix A

Knot Floer homology tabular data

$\widehat{\mathrm{HFK}}(K_0)$					
	-2	-1	0	1	2
3					$\mathbb{F}$
2				$\mathbb{F}^2$	$\mathbb{F}$
1			$\mathbb{F}^2$	$\mathbb{F}^2$	
0		$\mathbb{F}^2$	$\mathbb{F}^3$		
-1	$\mathbb{F}$	$\mathbb{F}^2$			
-2	$\mathbb{F}$				
dim = 17					

$\widehat{\mathrm{HFK}}(K_0^\tau)$			
	-1	0	1
1			$\mathbb{F}^2$
0		$\mathbb{F}^5$	$\mathbb{F}^2$
-1	$\mathbb{F}^2$	$\mathbb{F}^4$	
-2	$\mathbb{F}^2$		
dim = 17			

Table A.1: Knot Floer groups are displayed with Maslov grading on the vertical axis and Alexander grading on the horizontal axis. Computation with the program of Droz [Dro] also confirms that $\widehat{\mathrm{HFK}}(K_0) \cong \widehat{\mathrm{HFK}}(K_1)$ and $\widehat{\mathrm{HFK}}(K_0^\tau) \cong \widehat{\mathrm{HFK}}(K_1^\tau)$.

$\delta -$ graded $\widehat{\text{HFK}}(K_0)$						
	-2	-1	0	1	2	dim
$a - m = -1$	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}^2$	$\mathbb{F}$	8
$a - m = 0$	$\mathbb{F}$	$\mathbb{F}^2$	$\mathbb{F}^3$	$\mathbb{F}^2$	$\mathbb{F}$	9
dim = 17						

$\delta -$ graded $\widehat{\text{HFK}}(K_0^\tau)$				
	-1	0	1	dim
$a - m = 0$	$\mathbb{F}^2$	$\mathbb{F}^5$	$\mathbb{F}^2$	9
$a - m = +1$	$\mathbb{F}^2$	$\mathbb{F}^4$	$\mathbb{F}^2$	8
dim = 17				

Table A.2: The δ-graded groups $\widehat{\text{HFK}}_\delta(K_0) \cong \widehat{\text{HFK}}_\delta(K_1)$ and $\widehat{\text{HFK}}_\delta(K_0^\tau) \cong \widehat{\text{HFK}}_\delta(K_1^\tau)$.

$\widehat{\text{HFK}}(K \simeq (3, -5, 3, -2))$							
	-3	-2	-1	0	1	2	3
4							$\mathbb{F}$
3						$\mathbb{F}^3$	
2					$\mathbb{F}^4$	$\mathbb{F}^2$	
1				$\mathbb{F}^3$	$\mathbb{F}^4$		
0			$\mathbb{F}^4$	$\mathbb{F}^4$			
-1		$\mathbb{F}^3$	$\mathbb{F}^4$				
-2	$\mathbb{F}$	$\mathbb{F}^2$					

Table A.3: The knot Floer groups of the knot $K \simeq (3, -5, 3, -2)$ are displayed with Maslov grading on the vertical axis and Alexander grading on the horizontal axis. There are clearly many summands with rank greater than one.

Appendix B

SnapPy code

```
>> M1=Manifold("14n22185.tri"); M2=Manifold("14n22589.tri")
>> M1.dehn_fill((2,0),0); M2.dehn_fill((2,0),0)
>> M1.covers(2,cover_type="cyclic"); M2.covers(2,cover_type="cyclic")

>> M1.length_spectrum(cutoff=1.5)
mult  length                                topology       parity
1     (0.618708509882-0.915396961493j)     mirrored arc   orientation-preserving
1     (1.02046533287-2.87373908997j)       mirrored arc   orientation-preserving
1     (1.19267652219-1.97573028631j)       circle         orientation-preserving
1     (1.2943687184-0.108601853389j)       mirrored arc   orientation-preserving
1     (1.4180061001+1.77458043688j)        circle         orientation-preserving

>> M2.length_spectrum(cutoff=1.5)
mult  length                                topology       parity
1     (0.61977975736+1.04574145952j)       mirrored arc   orientation-preserving
1     (0.946415249278+3.02707626124j)      mirrored arc   orientation-preserving
1     (1.07345426322+2.11448221051j)       circle         orientation-preserving
1     (1.2943687184-0.108601853389j)       mirrored arc   orientation-preserving
```

Figure B.1: The SnapPy [CDW] code used to compute the geodesic length spectra of $\Sigma_2(K_0)$ and $\Sigma_2(K_0^\tau)$.

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