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# Symmetries of knots, branched cyclic covers, and L-spaces

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# Symmetries of knots, branched cyclic covers, and L-spaces

**by**

**Hannah Kathryn Turner**

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**Symmetries of knots, branched cyclic covers, and L-spaces**

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This dissertation studies the L-space conjecture among manifolds which are branched cyclic covers of links. We present three main results. First, we construct new families of knots all of whose branched cyclic covers are L-spaces. Then, we give an almost complete characterization of which cyclic branched covers of double-twist knots have left-orderable fundamental groups. Finally, we relate the notion of visibility of certain symmetries of an alternating knot to the Heegaard Floer homology of its cyclic branched covers.

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# Introduction

## 1.1 Background

Heegaard Floer homology is a collection of invariants of 3- and 4-manifolds and their submanifolds introduced by Ozsváth-Szabó [40]. These invariants, and related gauge-theoretic invariants, have applications to 3- and 4-dimensional knot invariants, contact and symplectic structures, and have settled many outstanding topological questions in low-dimensional topology. Since these theories have proven very powerful, the search for other topological information they detect continues.

Let $\widehat{HF}(M)$ denote the Heegaard Floer homology of a closed, oriented 3-manifold $M$ with $\mathbb{Z}/2$ coefficients; we will only consider Heegaard Floor homology over $\mathbb{Z}/2$ throughout. $\widehat{HF}(M)$ is a finite dimensional vector space over $\mathbb{Z}/2$. Denote the order of $H_1(M;\mathbb{Z})$, as an abelian group, by $|H_1(M;\mathbb{Z})|$; if the first homology is infinite, then we say the order is 0.

**Definition 1.1.1.** *A closed, oriented 3-manifold $M$ is called an L-space if the rank of $\widehat{HF}(M)$ as a $\mathbb{Z}/2$-vector space satisfies* $\mathrm{rk}(\widehat{HF}(M)) = |H_1(M;\mathbb{Z})|$.

This is in fact the smallest possible rank $\widehat{HF}(M)$ could have. Thus, from the persepective of the rank of Heegaard Floer homology, an L-space is simple. A manifold $M$ which is *not* an L-space is therefore complex from the perspective of Heegaard Floer homology; it is an interesting question whether $M$ is complex in any geometric or topological sense. The following conjecture suggests several possibly equivalent measures of complexity which might be detected by Heegaard Floer homology.

**Conjecture 1.1.2** ([7, 26])**.** *Let $M$ be a closed, oriented, and prime 3-manifold. Then the following are equivalent:*

1. *$M$ is not an L-space*
2. *$\pi_1(M)$ is left-orderable*
3. *$M$ admits a co-oriented taut foliation*

A group is left-orderable if its fundamental group admits an order which is invariant under left-multiplication. A 3-manifold $M$ admits a taut foliation if it is filled out by surfaces in a particular geometric way. This conjecture is surprising since these three properties touch on very different aspects of 3-manifold theory. However, the conjecture has been confirmed for many classes of 3-manifolds, including all graph manifolds [19, 46, 6], and some relationships among the properties have been found; for example, it is known that an L-space cannot admit a taut foliation [39, 29, 5]. This conjecture has been well-studied for manifolds obtained by Dehn surgery on a knot. A knot $K$ is called an L-space knot if some non-trivial surgery on $K$ yields an L-space. Knots with this property and the corresponding slopes are topologically constrained [41, 38, 20]; see Section 3.1 for a discussion of L-space knots.

This thesis focuses on establishing analogous results for branched cyclic covers of knots in $S^3$. See Section 1.3 for background on branched cyclic covers of links.

## 1.2 Summary of results

**Definition 1.2.1.** *A knot in $S^3$ is called a branched L-space knot if all of its branched cyclic covers are L-spaces.*

Examples of such knots were first found among 2-bridge knots [45, 51]. Boileau-Boyer-Gordon studied whether quasipositive knots (see Section 3.3.3 for a definition) can be branched L-space knots. They obstructed this possibility unless $K$ is slice (bounds a properly embedded disk in $B^4$). They asked the following natural question.

**Question 1.2.2** ([4, Question 12.6])**.** *Does there exist a quasipositive branched L-space knot?*

In Chapter 3, we answer this question in joint work with Ahmad Issa. We produced examples of branched L-space knots which are quasipositive (and necessarily slice); these are the only known examples of branched L-space knots which are not 2-bridge, and in fact are non-alternating.

**Theorem A** ([25])**.** *There are infinitely many (prime) quasipositive branched L-space knots.* [1]

To generalize the notion of a branched L-space knot to links or knots outside of integer homology spheres, a choice of Seifert surface must be made so that the set of branched cyclic covers of the link is well-defined. For links in integer homology spheres this amounts to choosing an orientation. Thus, we will say an oriented link in an integer homology sphere is a *branched L-space link* if all of branched cyclic covers are L-spaces; a link in a general 3-manifold will be called a branched L-space link if for some fixed choice of Seifert surface, all of its associated branched cyclic covers are L-spaces.

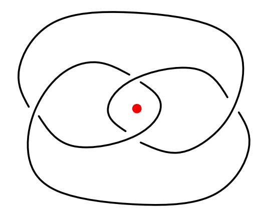

Figure 1.1: A 2-periodic knot rotationally symmetric about the central red point. This symmetry is "visible" (in this particular diagram).

Recent work of Hendricks, Lidman, and Lipshitz shows that such a knot cannot exist in a non-L-space [22]; see Theorem 3.4.4.

**Question 1.2.3.** *Which L-spaces contain a branched L-space knot?*

We show that some manifolds besides $S^3$ contain such knots in the following theorem.

[1]This theorem and all results in Sections 3.2 and 3.3 are joint with Ahmad Issa.

**Theorem B.** *Let $L$ be an alternating non-split link in $S^3$. Then the double-branched cover of $L$ contains a branched L-space knot.*

All of our examples of branched L-space knots are 2-periodic; see Chapter 2 for a discussion of 2-periodic symmetries. In particular we point out the following observation relating properties of branched cyclic covers of a knot and whether it admits a symmetry to the unknot, which is "visible" in an alternating diagram; see Chapter 2 for a definition of this notion.

**Proposition C.** *Let $L$ be an oriented link in $S^3$ with a 2-periodic symmetry to the unknot, whose symmetry is visible in an alternating diagram. Then $L$ is a branched L-space link.*

This provides an obstruction to many alternating links from admitting such a visible symmetry.

More generally, we are interested in which branched cyclic covers of links in $S^3$ are L-spaces. For an oriented link $L$ in $S^3$, define the set:

$$\mathcal{L}_{br}(L) := \{n \in \mathbb{N} \mid \text{ the n-fold branched cyclic cover } \Sigma_n(L) \text{ is an L-space}\}.$$

One can similarly define the set:

$$\mathcal{LO}_{br}(L) := \{n \in \mathbb{N} \mid \pi_1(\Sigma_n(L)) \text{ is left-orderable}\}.$$

If the L-space conjecture is true, then $\mathcal{L}_{br}(L) \sqcup \mathcal{LO}_{br}(L) = \mathbb{N}$. In particular, if $L$ is a branched L-space link, then we expect that $\mathcal{LO}_{br}(L) = \emptyset$. Little is known about what subsets of $\mathbb{N}$ the set $\mathcal{L}_{br}(L)$ or $\mathcal{LO}_{br}(L)$ can take.

**Question 1.2.4.** *Let $L$ be an oriented link in $S^3$.*

1. *Can $\mathcal{LO}_{br}(L)$ be a subset of $\mathbb{N}$ which is neither (i) $\emptyset$ nor (ii) $\{n \geq N\}$ for some positive integer $N$?*

2. *Similarly, can $\mathcal{L}_{br}(L)$ be a subset of $\mathbb{N}$ which is neither (i) all of $\mathbb{N}$ nor (ii) $\{1 \leq n \leq N\}$ for some positive integer $N$?*

In Chapter 4 we study $\mathcal{LO}_{br}(K)$ when $K$ is a double-twist knot; these knots are a family of two-bridge knots; an example is depicted in Figure 1.2. Unless the three-genus of such a knot is one, we obtain strong left-orderability results. With the following theorem, we answer a question of Boileau-Boyer-Gordon, who showed precisely for which indices the cyclic branched covers of a subfamily of these knots were L-spaces. Inspired by the L-space conjecture, they asked whether the corresponding results hold in the setting of left-orderability [4, Problem 12.11].

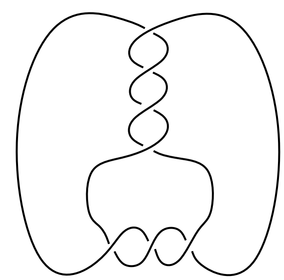

Figure 1.2: A double-twist knot.

**Theorem D** ([55])**.** *For double-twist knots with three-genus at least four, $\Sigma_n(K)$ is left-orderable if and only if $n \geq 3$. When the three-genus is two or three, then $\Sigma_n(K)$ is left-orderable if $n \geq 5$ and $n \geq 4$, respectively.* [2]

[2]This result was originally published in the Proceedings of the American Mathematical Society, Volume 149, Number 3, in 2021.

Theorem D confirms that part of the L-space conjecture ($M$ is a non-L-space if and only if $\pi_1(M)$ is left-orderable) holds for the branched cyclic covers of double-twists knots with genus at least four, and other subfamilies. The theorem also gives more evidence toward an affirmative answer to Question 1.2.4 (2).

## 1.3 Preliminaries on branched cyclic covers

Rolfsen gives a thorough introduction to branched cyclic covers of knots in $S^3$ [49, Chapters 5 and 9]; we follow his treatment closely, but discuss generalizations to null-homologous links in an arbitrary 3-manifold. Throughout, when we discuss branched cyclic covers of a link $L$, we will assume $L$ is null-homologous. In this case $L$ bounds a (not necessarily unique) orientable surface $F$ called a Seifert surface. We will show there is a unique 3-manifold $\Sigma_n(M, F, L)$ called the $n$-fold cyclic branched cover of $(M, L)$ induced by $F$. If the manifold $M$ and Seifert surface $F$ are clear from context, we will denote $\Sigma_n(M, F, L)$ simply by $\Sigma_n(L)$ and call it the $n$-fold branched cyclic cover of $L$.

The manifold $\Sigma_n(M, F, L)$ is constructed as follows. The exterior of the link $X(L) = M - \nu(L)$ is a manifold with torus boundary components. We first show that for fixed $n \in \mathbb{N}$ and choice of Seifert surface $F$, $X(L)$ has a unique cover $X_n(L)$ with deck group $\mathbb{Z}/n$ associated with $F$. Abusing notation, we will also denote the intersection $F \cap X(L)$ by $F$ throughout.

Any regular covering space of $X(L)$ with deck group $\mathbb{Z}/n$ has a homology class $F \in H_2(X(L), \partial(X(L)); \mathbb{Z})$ associated to it which determines its homeomorphism type; conversely any homology class $[F]$ determines a unique $n$-fold cylic cover of $X(L)$ as we now describe. Given a regular cover $X_n(L)$ of $X(L)$ with deck group $\mathbb{Z}/n$ we have the exact sequence

$$1 \to \pi_1(X_n(L)) \to \pi_1(X(L)) \xrightarrow{q} \mathbb{Z}/n \to 1.$$

The map $q$ must factor through the abelianization map. Thus we have an induced map $H_1(X(L)) \cong\to \mathbb{Z}/n$. This map must also factor through $\mathbb{Z}$. Thus we also have the induced map $H_1(X(L)) \xrightarrow{[F]} \mathbb{Z}$ which we have suggestively labelled $[F]$. Poincaré-Lefschetz duality and the universal coefficient theorem provide us with the isomorphism

$$\mathrm{Hom}(H_1(X(L);\mathbb{Z}),\mathbb{Z}) \cong H_2(X(L),\partial(X(L));\mathbb{Z}).$$

Thus, the map $[F]$ defines a class in relative second homology, and conversely a class in relative second homology defines a unique map $H_1(X(L)) \to \mathbb{Z}$. Let $\eta$ denote any surjection $\mathbb{Z} \xrightarrow{\eta} \mathbb{Z}/n$. Then the composite map

$$\pi_1(X(L)) \xrightarrow{ab} H_1(X(L)) \xrightarrow{[F]} \mathbb{Z} \xrightarrow{\eta} \mathbb{Z}/n \tag{1.1}$$

has kernel $G$ which does not depend on choice of $\eta$. This subgroup of $\pi_1(X(L))$ corresponds to a manifold $X_n(L)$ which covers $X(L)$ by a covering map $p$ and with deck group $\mathbb{Z}/n$. We have shown that for a fixed link $L$, a second homology class $[F]$ determines a unique $n$-fold cyclic cover, and conversely such a cover has unique homology class associated to it. This allows us to make the following definition.

**Definition 1.3.1.** *The $n$-fold cyclic unbranched cover of a knot $L$ with Seifert surface $F$ in a three-manifold $M$ is the covering space $X_n(L)$ corresponding to the subgroup $G$ which is the kernel of the map $\eta \circ [F] \circ ab$ defined in (1.1).*

*Remark* 1.3.2. Let $L = K_1 \cup K_2 \cup \cdots \cup K_m$ and denote by $\mu_i$ a meridian of the $i^{th}$ component of $L$ in $X(L)$. To check whether a given $X_n(L)$ corresponds to a homology class $S$, we need only check that $\widetilde{S} \cap_a [\widetilde{\mu_i}] = n$ for each $1 \leq i \leq m$.

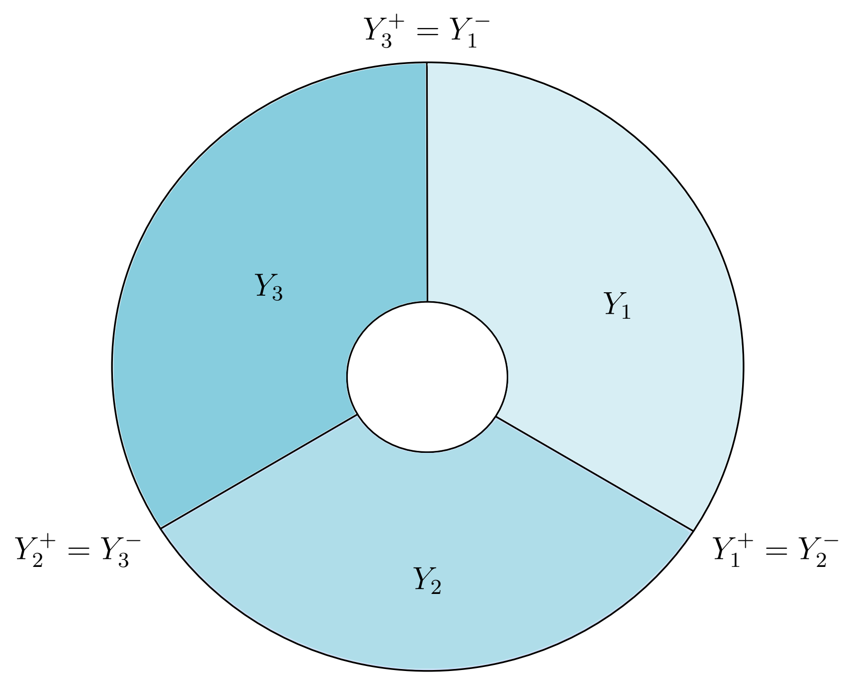


Figure 1.3: A schematic of the cyclic gluing procedure to obtain $Y_n(L)$. Here $n = 3$.

An explicit construction of $X_n(L)$ can be obtained as follows. Let $Y = X(L) - (\mathring{F} \times (-1, 1))$ be the manifold obtained by removing a tubular neighborhood of $\mathring{F}$ from $X(L)$. The boundary $\partial Y$ is decomposed into three pieces (i) $F^+ = F \times \{1\}$, (ii) $F^- = F \times \{-1\}$ and (iii) $\partial F \times [-1,1]$. We denote $F^{\pm} \subset \partial Y$ by $Y^{\pm}$. Take $n$ copies of $Y$ indexed by $0 \leq i \leq n-1$ glued cyclically: $Y_i^+$ is identified with $Y_{i+1}^-$ for all $0 \leq i \leq n-1$ with indices taken mod $n$; we call this identification space $Y_n(L)$; see Figure 1.3.

**Proposition 1.3.3.** *Suppose $L = K_1 \cup K_2 \cup \cdots \cup K_m$ has $m$ components. Denote by $T_i = \partial(\nu(K_i))$ the boundary component of $X(L)$ corresponding to $K_i$. Then the boundary of $Y_n(L)$ is the union $\widetilde{T_1} \cup \widetilde{T_2} \cup \cdots \cup \widetilde{T_m}$ of $m$ tori. Further, let $\lambda_i = \partial F \cap T_i$ and $\mu_i$ be a meridian of $K_i$. Then $p^{-1}(\mu_i)$ is connected and $p^{-1}(\lambda_i)$ is the disjoint union of $n$ parallel copies of one lift of $\lambda_i$.*

*Proof.* To see this, we recall that the boundary $\partial Y_i$ has a decomposition into three pieces; the surfaces $F^{\pm}$ are used in the identification of the $Y_i$ and their interiors are not in $\partial X_n(L)$ after the identification. What remains is a union of $n$ copies of $\partial F \times [-1,1]$, glued cyclically. This gives the $m$ tori $\widetilde{T_1} \cup \widetilde{T_2} \cup \cdots \cup \widetilde{T_m}$ one for each boundary component of $F$. In each of the $n$ identifications forming $\widetilde{T_i}$ we see a lift of $\lambda_i$. The lift of $\mu_i$ is a union of $n$ intervals glued cyclically.

□

It is easy to see that $Y_n(L)$ is a covering space of $X(L)$ with deck group $\mathbb{Z}/n$. In addition, the map $\pi_1(Y_n(L)) \to \mathbb{Z}/n$ determined by the deck group factors through the map $[F]$. There is a unique covering space of $X(L)$ with this property, and hence we identify $Y_n(L) \cong X_n(L)$.

We can now define the closed manifold $\Sigma_n(M, F, L)$.

**Definition 1.3.4.** *The n-fold (or index n) branched cyclic cover of* $(M, L)$ *induced by* $F$ *is the manifold* $\Sigma_n(M, F, L) \cong X_n(L) \cup_m S^1 \times D^2$ *where the gluing of the* $m$ *solid tori is determined by the condition that the preimage* $p^{-1}(\mu_i)$ *of a meridian* $\mu_i$ *of a component* $K_i$ *of* $L$ *bounds a disk in the filling.*

To see that this is in fact a branched covering, we need only extend the branched covering map across the solid tori. The map $p : S^1 \times D^2 \to \nu(K_i) \cong S^1 \times D^2$ maps via the identity on the first factor and the map $z \to \frac{z^n}{\|z^{n-1}\|}$ on the second. This gives a branched cover from the solid torus to $\nu(K_i)$ which matches $p$ on $\partial X_n(L)$. To see that this is the unique cyclic branched cover we note that Fox proved that a branched covering $N$ of $M$ branched along the branched set $B \subset M$ is determined by $M$, $B$, and an unbranched cover of $M - B$ [14]. Thus, since $X_n(L)$ was uniquely determined, so is $\Sigma_n(M, F, L)$.

We briefly consider certain abelian (non-cyclic) branched covers of links. Suppose that $L = L_1 \cup L_2 \cup \cdots \cup L_k$ where each $L_i$ is a null-homologous link. Label each $L_i$ with a positive integer $\ell_i$; this will be the branching index along each component. Choosing a Seifert surface $F_i$ for each $L_i$ determines a map $\pi_1(X(L)) \to \mathbb{Z}^k \to \mathbb{Z}/\ell_1 \oplus \mathbb{Z}/\ell_2 \oplus \cdots \oplus \mathbb{Z}/\ell_k$ following a similar argument in the case of a single branching index. This in turn, determines a unique associated unbranched covering. Again we fill all boundary components to obtain a manifold (unique up to homeomorphism) $\Sigma_{\ell_1,\ell_2,\ldots,\ell_k}(L_1, L_2, \ldots L_k)$ which is a branched covering of $S^3$ with branched set $L$.

*Remark* 1.3.5. Let $L = L_1 \cup L_2$ be a link in $M$ with a choice of Seifert surface $F_i$ for $L_i$ and choice of label $\ell_i$ for each $L_i$. Denote $M_{\ell_i} = \Sigma_{\ell_i}(L_i)$. Let $\widetilde{L_1}$ be the lift of $L_1$ in $M_{\ell_2}$ and similarly $\widetilde{L_2}$ be the lift of $L_2$ in $M_{\ell_1}$. The lift of each $F_1$ is a Seifert surface for $\widetilde{L_1}$ in $M_{\ell_2}$ and similarly for the lift of $F_2$. We then consider $M_{\ell_1,\ell_2} = \Sigma_{\ell_2}(M_{\ell_1}, \widetilde{L_2})$ and $M_{\ell_2,\ell_1} = \Sigma_{\ell_1}(M_{\ell_2}, \widetilde{L_1})$. It is not hard to show that both $M_{\ell_1,\ell_2}$ and $M_{\ell_2,\ell_1}$ are $\mathbb{Z}/\ell_1 \oplus \mathbb{Z}/\ell_2$ branched covers of $L_1 \cup L_2$ and that their quotient maps factor through the map determined by $[F_1 \cup F_2]$, and hence are homeomorphic.

Heuristically, this means that $\mathbb{Z}/\ell_1 \oplus \mathbb{Z}/\ell_2$ branched covers of a link $L_1 \cup L_2$ can be described as a sequence of $\ell_1-$fold and $\ell_2-$fold cyclic branched coverings, and that reordering the branching does not affect the homeomorphism type of the result.

# Chapter 2

# Periodic symmetries of knots

In this chapter we discuss links with periodic symmetries and the branched cyclic covers of such links.

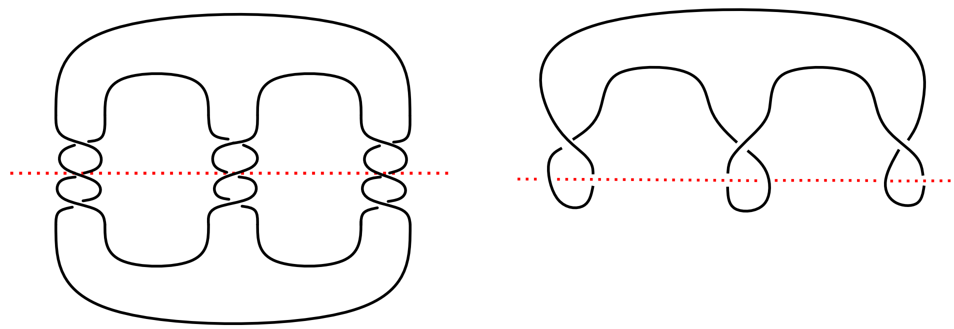

Figure 2.1: A 2-periodic knot rotationally symmetric about the red axis on the left. On the right the quotient of the knot with the quotient of the axis under the periodic symmetry is depicted.

**Definition 2.0.1.** *An order $m$ periodic symmetry of an (oriented) link $L$ is a rotation $h$ of angle $2\pi/m$ about a great circle $C$ in $S^3$ such that $h(L) = L$ as an oriented link and $C \cap L = \emptyset$. If $L$ admits such a rotation then $L$ is called periodic of order $m$.*

We emphasize that when we consider a periodic link we implicitly assume the ambient manifold is $S^3$.

## 2.1 Branched cyclic covers of periodic links

We now review a construction which has been widely used [15, 33, 56, 37] relating the cyclic branched covers of a link which is periodic of order $m$ to the $m$-fold branched cover of a related link $L_n$.

Let $L$ be periodic link of order $m$. Given such a link we also have a rotation of $h$ of order $m$ of $S^3$ which leaves $L$ invariant. We are interested in the quotient map determined by the action of $h$ which we denote $\rho(S^3, L, C) = (S^3, \overline{L}, \overline{C})$ pictured in Figure 2.1. In fact, $\rho$ defines a branched covering map from $S^3$ to itself where the branching set is $\overline{C}$ and the branching is of index $m$.

Then, a feature of m-periodic links $L$ with axis $C$, is that the $n$-fold branched cylic cover $\Sigma_n(L)$ can be expressed as the $\mathbb{Z}/n \times \mathbb{Z}/m$ branched cover of the quotient link $\overline{L} \cup \overline{C}$. The order of branching does not affect the homeomorphism type of the manifold $\Sigma_{n,m}(\overline{L} \cup \overline{C})$; see Remark 1.3.5. Thus one can consider first branching (of index $n$) over $\overline{L}$, and then branching (of index $m$) over the lift of $\overline{C}$ which we denote $L_n$; see Figure 2.2. From this commuting diagram, we see that if $L_n$ is the lift of $\overline{C}$ in $\Sigma_n(\overline{L})$, then $\Sigma_n(L) \cong \Sigma_{n,m}(\overline{L} \cup \overline{C}) \cong \Sigma_m(L_n)$ can be expressed both as an $n$-fold cyclic branched cover over $L$ and a $m$-fold branched cover over $L_n$. We have just proved the following.

**Proposition 2.1.1.** *Let $L$ be an $m$-periodic link. If the axis of the symmetry is $C$ and the quotient link is $\overline{L}$ we have that $\Sigma_n(L) \cong \Sigma_{n,m}(\overline{L} \cup \overline{C})$.*

$$
\begin{array}{ccc}
 & \Sigma_n(L) \cong \Sigma_{n,m}(\overline{L} \cup \overline{C}) \cong \Sigma_m(L_n) & \\
\swarrow & & \searrow \\
\Sigma_{1,m}(\overline{L} \cup \overline{C}) \cong S^3 & & \Sigma_{n,1}(\overline{L} \cup \overline{C}) \\
\searrow & & \swarrow \\
 & (S^3, \overline{L}, \overline{C}) &
\end{array}
$$

Figure 2.2: For an $m$-periodic link $L$, the $n$-fold branched cover can be expressed in two ways. The left-hand side of the diagram describes $\Sigma_{n,m}(\overline{L} \cup \overline{C})$ as first branching over $\overline{C}$, and then over $L$ (the lift of $\overline{L}$). On the right-hand side, we branch first over $\overline{L}$ and then over the lift of $\overline{C}$ which we denote $L_n$.

In the special case that $\overline{L} \cup \overline{C}$ is the union of two unknots, it is easy to understand the manifold $M = \Sigma_{n,1}(\overline{L} \cup \overline{C})$. In particular $M \cong S^3$ and it is possible to obtain a diagram for $(S^3, L_n)$ via the quotient diagram of $\overline{L} \cup \overline{C}$.

**Construction 2.1.2.** *Let $L$ be an $m$-periodic link with axis $C$; suppose that the quotient of $L$ under the symmetry $\overline{L}$ is unknotted. Then a diagram for $L_n$ where $\Sigma_m(L_n) \cong \Sigma_n(L)$ can be obtained as follows:*

1. *Note that $\overline{L} \cup \overline{C}$ is the union of unknots. Via isotopy, draw $\overline{L}$ as the standard unknot. Then $\overline{C}$ is a knot in the solid torus $S^3 \setminus \overline{L}$, called a pattern in a solid torus. Call this pattern $P$.*

2. *To move up the right-hand side of the diagram in Figure 2.2, $n$-fold branch along $\overline{L}$. Since $\overline{L}$ is unknotted, we obtain $S^3$, but more than that, we can describe the branched covering map. The solid torus $T = S^3 \setminus \overline{L}$*

*lifts to a solid torus $\widetilde{T}$; the branched covering map $\widetilde{T} \to T$ is the quotient of rotation along the core by angle $2\pi/n$. Then the lift of $P$, the link $L_n$ is represented as a pattern in $\widetilde{T}$ by cutting $P$ along a meridian of $T$, and cyclically gluing $n$ copies. See Figure 2.3.*

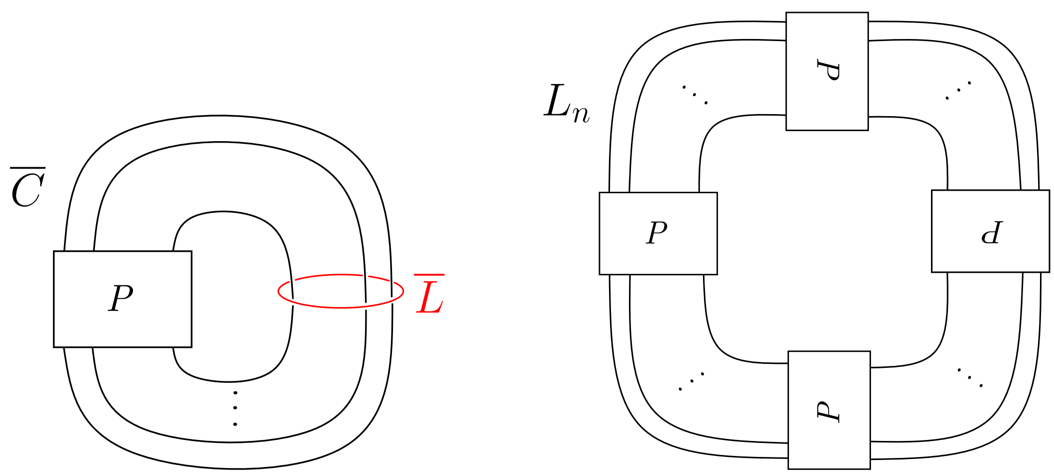


Figure 2.3: The quotient of the axis $\overline{C}$ as a pattern $P$ in the complement of $\overline{L}$, and the lift of $\overline{C} = L_n$ after taking the $n$-fold branched cover of $\overline{L}$. Here $n = 4$.

One reason Construction 2.1.2 is of interest in the case of a 2-periodic link whose quotient is unknotted, is the following. To study whether the cyclic branched covers of $L$ are L-spaces, we can instead study the double-branched covers of the related links $L_n$. The Heegaard Floer homology of double-branched covers are better understood than general branched cyclic covers. For instance, we have the following theorem.

**Theorem 2.1.3** ([42])**.** *The double-branched cover $\Sigma_2(L)$ of a non-split, alternating link $L$ is an L-space.*

Another reason, is that it produces examples of $(n, m) - twins$.

**Definition 2.1.4.** *A knot $K$ in $S^3$ is said to have an $(n, m)-$twin if there is a knot $K$ in $S^3$ such that $\Sigma_n(K) \cong \Sigma_m(K')$. Supppose $K$ has an $m$-periodic symmetry with axis $C$, and that quotient $\overline{K}$ is unknotted. In the case that the link $L_n$ described in Construction 2.1.2 is actually a knot, it is an $(n, m)-$twin of $K$. In this case we say $K$ has a periodic $(n, m)-$twin $L_n$.*

The $(2, 2)-$twins of $L$ are links with the same double-branched cover as $L$. It is still an open question to determine a set of diagrammatic moves relating such links (Problem 1.22 of the Kirby Problem List [31]). A variation of this problem is to ask which topological properties $(2, 2)-$twins must share. Greene conjectured that $(2, 2)-$twins must be either both alternating, or both non-alternating [18]. Greene's conjecture has a natural generalization.

**Question 2.1.5.** *Must $(n, m)-$twins be either both alternating or both non-alternating?*

We will give a partial solution to this question in Section 2.2.

## 2.2 Visible symmetries

We define what it means for a periodic symmetry to be *visible* in a particular diagram in the sense of [12]; this type of symmetry is also called *intravergent* [10].

**Definition 2.2.1.** *A periodic symmetry of a link $L$ in $S^3$ of order $m$ about an axis $C$ is visible in a diagram $D$ in $\mathbb{R}^2$ for $L$ if $D \cap C = x$ is a single point disjoint from the projection of $L$, and $D$ admits a rotation of order $m$ about the point $x$. In this situation we will say the diagram $D$ is $m$-periodic.*

For an example of a visible periodic symmetry, see Figure 1.1. For a non-example, see Figure 2.1; here the 2-periodic symmetry can be "seen" in the diagram, though it is not visible since the axis $C$ lies entirely in the diagram. Of course, this symmetry is visible in a different diagram $D'$ of the knot.

It is a classical question which knots have an $m$-periodic symmetry, and in which diagrams these symmetries are visible. Costa and Van Quach Hongler studied this question for alternating $m$-periodic knots, showing that for such a knot the symmetry is always visible in some alternating diagram when $m \geq 3$ [12]. This is a powerful detection result in the following sense. To check whether an alternating knot $K$ is $m$-periodic, one needs only determine whether any alternating diagram for $K$ is $m$-periodic. There are only finitely many alternating diagrams of a given knot one needs to check.

The failure of the result for $m = 2$ was known. For instance, for a 2-periodic alternating link the symmetry can only be visible in an alternating diagram if the crossing number of the link is even; many 2-periodic alternating knots with odd crossing number exist - for instance the torus knots $T(2, 2k+1)$. We will presently give more examples of 2-periodic alternating links whose symmetry is not visible in an alternating diagram. We remark that Boyle recently proved that all 2-periodic symmetries of alternating knots can be "seen" to some extent in an alternating diagram [10].

**Lemma 2.2.2.** *Let $L$ be a prime link which is 2-periodic, and suppose that the symmetry is visible in an alternating diagram. If in addition the quotient $\overline{L}$ is unknotted, then the pattern $P$ obtained in Construction 2.1.2 is an alternating pattern.*

*Proof.* By definition, there is an alternating diagram $D$ for $L$ where $C \cap D$ is a single point disjoint from the projection of $L$. Thus, the quotient link $\overline{L}$ has a diagram $\overline{D}$ which is again alternating and with $\overline{C} \cap \overline{D}$ a single point. If $L$ satisfies the conditions of Lemma 2.2.2, Paoluzzi shows that the quotient link $\overline{L} \cup \overline{C}$ is symmetric [44, Proposition 2]. That is, there is an isotopy of $S^3$ exchanging the components of the link.

Hence, there is another diagram $\overline{D'}$ for $\overline{L}$ in which $\overline{L}$ intersects the projection plane in a single point, and $\overline{C}$ is alternating. This diagram describes the pattern $P$. □

It is not difficult to see then that the link $L_n$ obtained from $L$ by Construction 2.1.2 is alternating for any $n \in \mathbb{N}$ when $P$ is alternating. $L_n$ is also non-split as it has a connected alternating diagram [35]. Combining this with Theorem 2.1.3, we conclude that $\Sigma_n(L) \cong \Sigma_2(L_n)$ is an L-space for all $n$; it is also true that $\pi_1(\Sigma_n(L)) \cong \pi_1(\Sigma_2(L_n))$ is non-left-orderable for all $n$ by a result of Boyer, Gordon and Watson [8]. This proves the following proposition.

**Proposition C.** *Let $L$ be an oriented link in $S^3$ with a 2-periodic symmetry to the unknot, whose symmetry is visible in an alternating diagram. Then $\mathcal{L}_{br}(L) = \mathbb{N}$, i.e. $L$ is a branched L-space link, while $\mathcal{LO}_{br}(L) = \emptyset$.*

Proposition C suggests that 2-periodic links whose symmetry is visible in an alternating diagram are rare, at least among links whose quotient is unknotted. A two-bridge knot which is not a branched L-space knot, or has a left-orderable branched cyclic cover for some index $n$ gives an example of a 2-periodic alternating knot whose symmetry is not visible in any alternating diagram. In particular, we obtain the following proposition which shows that having even crossing number is not a sufficient condition for a 2-periodic symmetry of an alternating knot to be visible in an alternating diagram.

**Proposition E.** *Let $J(r, s)$ be a double-twist knot with $r = 2k+1 \geq 3$ odd and $s = 2\ell \geq 4$ even. Then $J(2k+1, 2\ell)$ has even crossing number, is alternating and 2-periodic, yet this symmetry is not visible in an alternating diagram.*

*Proof.* For these double-twist knots, we prove in Theorem 4.1.3 that, with the stated assumptions on $r$ and $s$, there exists an $n \in \mathbb{N}$ so that $\Sigma_n(J(r, s))$ is left-orderable. Thus none of these knots satisfy the conclusion of Proposition C. To see that these knots have even crossing number note that the standard diagram for the double-twist knots $J(2k + 1, 2\ell)$ with $2(k + \ell) + 1$ crossings is not alternating, however an isotopy reduces the number of crossings by one and yields an alternating diagram. $\square$

We now give a partial solution to Question 2.1.5. Paoluzzi proved the same result when $n = m \geq 3$; our proof is a simple generalization of hers [44].

**Proposition F.** *Let $K$ be a hyperbolic knot in $S^3$ and suppose $n > 2$ and $m > 2$. If $K$ is alternating then so is any $(n,m)-$twin of $K$; similarly if $K$ is non-alternating, so is any $(n,m)-$twin of $K$.*

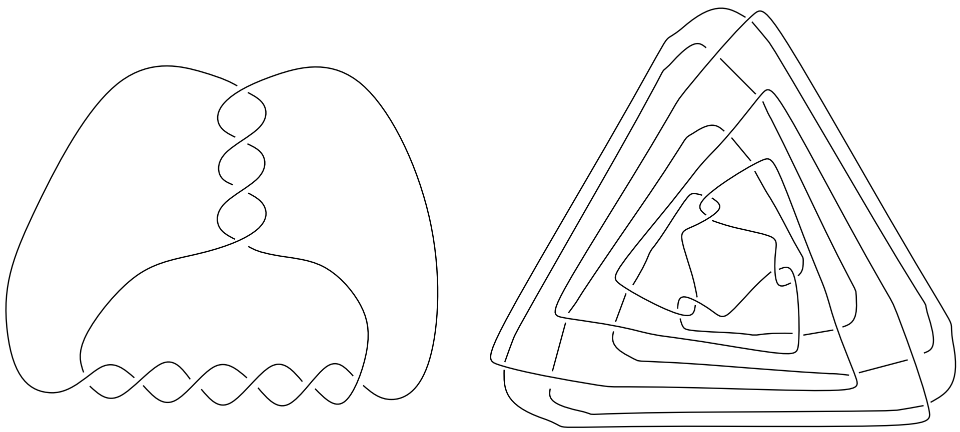

Figure 2.4: The non-alternating knot $L_{3,2,3}$ on the right has the property that $\Sigma_3(K_{2,3}) \cong \Sigma_2(L_{3,2,3})$ where $K_{2,3} = [4, -2, 2, -2, 2, -2]^+$ on the left.

*Proof.* Mecchia shows for $K$ hyperbolic and $n > 2$ and $m > 2$, an $(n,m)-$twin of $K$ is a *periodic* $(n,m)-$*twin* $L_n$ of $K$ [34]. That is, the link $L_n$ must be obtained from $K$ via applying Construction 2.1.2 to an $m$-periodic symmetry of $K$.

Since $m \geq 3$ the $m$-periodic symmetry is visible. By Lemma 2.2.2, the pattern $P$ obtained in Construction 2.1.2 is alternating. Its lift under the branching, $L_n$ is also alternating.

□

**Proposition G.** *Let $n \geq 3$. Then there exists a hyperbolic alternating knot with a non-alternating $(n, 2)-$twin.*

*Proof.* Let $K_{k,\ell} = [2k, -2, 2, -2, \ldots, 2, -2]^+$ where $2\ell \geq 6$ is the length of the sequence. These knots are hyperbolic and alternating. See Section 3.2 for an explanation of the notation and for a description of a 2-periodic symmetry of these knots. Applying Construction 2.1.2 to $K_{k,\ell}$ we obtain a pattern $P$ whose lift $L_{n,k,\ell}$ is an $(n, 2)-$twin of $K_{k,\ell}$; see Figure 2.4. That is, $\Sigma_n(K_{k,\ell}) \cong \Sigma_2(L_{n,k,\ell})$.

Boileau-Boyer-Gordon show that $\Sigma_n(K_{k,\ell})$ is not an L-space for $n \geq 3$ [4, Corollary 10.11]. Hence, $L_{n,k,\ell}$ cannot be alternating (since $L_{n,k,\ell}$ is non-split) for $n \geq 3$.

Technically, $L_{n,k,\ell}$ is an $(n, 2)-$twin of $K_{k,\ell}$ only if it is a knot. It can be checked that for instance, that if $\gcd(n, 2l - 1) = 1$ then $L_{n,k,\ell}$ is in fact a knot. □

We remark that the case $m = n = 2$ is a much more subtle problem since there are many ways to produce $(2, 2)-$twins besides periodic $(2, 2)-$twins; see e.g. [34]. Even restricting Question 2.1.5 to periodic $(2, 2)-$twins presents a challenge since we no longer have the result of Costa and Van Quach Hongler about the visibility of periodic symmetries [12] at our disposal.

# Chapter 3

# Branched L-space knots

In light of the L-space conjecture, it is an interesting problem to determine the following set for an oriented link $L$ in $S^3$:

$$\mathcal{L}_{br}(L) = \{n \in \mathbb{N} : \text{the } n\text{-fold cyclic branched cover } \Sigma_n(L) \text{ is an L-space}\}.$$ [1]

Evidence suggests the set $\mathcal{L}_{br}(L)$ always takes one of the two forms $\mathcal{L}_{br}(L) = \mathbb{N}$ or $\mathcal{L}_{br}(L) = \{1, \ldots, N\}$ for some $N$; see Question 1.2.4. In this section we will be particularly interested in the first case; oriented links with the property $\mathcal{L}_{br}(L) = \mathbb{N}$ will be called *branched L-space links.*

## 3.1 Introduction

Before discussing branched L-space knots in more detail, we survey some of the known results about L-space knots - an analogous notion in the setting of Dehn surgery. Denote $S^3_{p/q}(K)$ the manifold obtained by performing $p/q-$Dehn surgery along $K$ in $S^3$ where $p/q \in \mathbb{Q} \cup \{\infty\}$; see e.g. Rolfsen for a

[1] The work in Chapters 3.2 and 3.3 are joint with Ahmad Issa and appear in the following paper: Links all of whose branched cyclic covers are L-spaces, arXiv: 2008.06127. This paper has been submitted for publication. To this project I contributed mathematical ideas including all of those in Chapter 3.3.3 and to the writing of the paper.

definition of surgery and some properties [49, Chapter 9F]. It is interesting to fix a knot $K$ and vary the slope $p/q$ and investigate the manifolds obtained. The set

$$\mathcal{L}(K) := \{p/q \in \mathbb{Q}^{+} \cup \{\infty\} : S^3_{p/q}(K) \text{ is an L-space}\}$$

has been well-studied; we remark that since $\mathcal{L}(-K) = -\mathcal{L}(K)$ where $-K$ denotes the mirror of $K$, it is natural to restrict to non-negative slopes. This set takes one of two forms: Either $\mathcal{L}(K) = \{\infty\}$ or $\mathcal{L}(K) = [2g(K) - 1, \infty]$, where $g(K)$ denotes the Seifert genus [32, 43].

**Definition 3.1.1.** *A knot $K$ in $S^3$ for which $\mathcal{L}(K) = [2g(K) - 1, \infty]$ is called a (positive) L-space knot.*

L-space knots are a topologically restricted class of knots; they are known to be fibered [41, 38] and strongly quasipositive [20]; strong quasipositivity is a strengthening of the notion of quasipositivity defined in Section 3.3.3. One motivation for this section is to study whether similar restrictions apply to branched L-space knots in the setting of branched cyclic covers.

**Definition 3.1.2.** *An oriented link $L$ in $S^3$ for which $\mathcal{L}_{br}(L) = \mathbb{N}$ is called a branched L-space link.*

Some obstructions to being a branched L-space link are known. In the case of a branched L-space knot $K$, it has been shown that the $n$-fold cyclic branched cover of $K$ is not an L-space for $n$ sufficiently large, provided that $K$ is a non-slice quasipositive knot [4], or $K$ is fibered with non-zero fractional

Dehn twist coefficient [48, 28] see also [21]. This class includes, for example, all L-space knots.

Boileau-Boyer Gordon, curious whether the non-slice condition was a necessary assumption to guarantee that a quasipositive knot is *not* a branched L-space knot, asked the following question.

**Question 3.1.3.** *Does there exist a slice quasipositive branched L-space knot?*

In this Section 3.3 we present new examples of branched L-space knots and answer Question 3.1.3 in the affirmative in joint work with Issa [25]. Prior to our work the only known examples of branched L-space knots were an infinite family of two-bridge knots [45, 51].

See Sections 3.2 and 3.3 for the two-bridge and pretzel link notations used in the following theorems.

**Theorem H** ([25])**.** *Let $L$ be a two-bridge link with fraction of the form $[2a_1, 2a_2, \ldots, 2a_r]^+$ where $a_1, \ldots, a_r$ are all positive integers. In the case that $L$ is a two-component link, we orient the link as in Figure 3.2. Then $L$ is a branched L-space link.*

When $L$ is actually a knot this result was proved in work of Peters [45] and Teragaito [51]. We also prove the following theorem which has Theorem A as a corollary.

**Theorem I** ([25])**.** *The pretzel knots $K_k = P(3, -3, 2k+1)$ are branched L-space knots for any integer $k$. In addition, when $k$ is negative, the knot $K_k$ is quasipositive.*

The knots $K_k$ are classically known to be slice; hence Theorem I answers Question 3.1.3 affirmatively.

### 3.1.1 Preliminaries on double-branched covers and L-spaces

In this chapter we will focus on knots and links with a 2-periodic symmetry. While our primary goal is to understand when cyclic branched covers of any index are L-spaces, restricting ourselves to 2-periodic links allows us instead to solve questions about double-branched covers. In this section we summarize those results about double-branched covers of links and L-spaces which we will need in this chapter.

**Theorem 3.1.4** ([42])**.** *The double-branched cover $\Sigma_2(L)$ of a non-split alternating link $L$ is an L-space.*

In fact, Osváth-Szabó also showed that their result holds in the more general case that $L$ is quasi-alternating or reduced $\mathbb{Z}/2$ Khovanov-thin. One generalization of alternating which we will use is that of two-fold quasi-alternating links as defined by Scaduto-Stoffregen [50].

To define this notion we require the concept of a marking of link. A *marked link* is a pair $(L, \omega)$ where $L$ in $S^3$ is a link and $\omega$ is a marking of an element of $\mathbb{Z}/2$ to each component of $L$. If the number of components marked with $1 \in \mathbb{Z}/2$ is even (resp. odd) we say that $(L, \omega)$ is a *two-fold marked link* (resp. *odd marked link*). The trivial two-fold marking assigns $0 \in \mathbb{Z}/2$ to every component of $L$.

Let $D$ be a planar diagram representing a link $L$. An *arc* of $D$ is a strand of $D$ that descends to an edge of the 4-valent graph formed from $D$ upon forgetting its crossings. A *marking of* $D$ is an assignment $\tilde{\omega} : \Gamma(D) \to \mathbb{Z}/2$, where $\Gamma(D)$ is the set of arcs of $D$. We say $(D, \tilde{\omega})$ is *compatible with* or *represents* $(L, \omega)$ if

$$\sum_{\gamma \in \Gamma(K)} \tilde{\omega}(\gamma) = \omega(K) \pmod{2},$$

for each component $K$ of $L$. This data can be packaged diagrammatically by placing an odd number of dots on each arc $\gamma$ for which $\tilde{\omega}(\gamma) = 1$. Using this diagrammatic interpretation, when we smooth $D$ at a crossing (see Figure 3.1), we naturally obtain two marked diagrams $(D_0, \tilde{\omega}_0)$ and $(D_1, \tilde{\omega}_1)$ by counting dots mod 2.

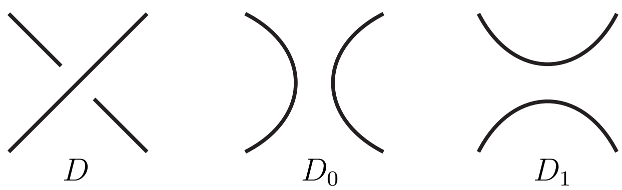


Figure 3.1: $D$ and its two smoothings $D_0$ and $D_1$ in a neighbourhood of a given crossing.

**Definition 3.1.5.** *The set of two-fold quasi-alternating (TQA) two-fold marked links, denoted $\mathcal{Q}$, is the smallest set of two-fold marked links satisfying the following:*

1. *The unknot with its unique trivial two-fold marking data is in $\mathcal{Q}$.*

2. *Any two-fold marked link that splits into two odd-marked links is in $\mathcal{Q}$.*

3. *Let* $(D, \tilde{\omega})$ *be a marked diagram representing* $(L, \omega)$ *such that the two smoothings* $(D_0, \tilde{\omega}_0)$ *and* $(D_1, \tilde{\omega}_1)$ *at a crossing represent marked links* $(L_0, \omega_0)$ *and* $(L_1, \omega_1)$*, respectively (see Figure 3.1). If*

   - *both smoothings* $(L_0, \omega_0)$ *and* $(L_1, \omega_1)$ *are in* $\mathcal{Q}$*, and*
   - $\det(L) = \det(L_0) + \det(L_1)$

   *then* $(L, \omega)$ *is in* $\mathcal{Q}$*.*

*We say that a link* $L$ *is two-fold quasi-alternating* (*TQA*) *if for the trivial marking* $\omega$*, we have* $(L, \omega) \in \mathcal{Q}$*.*

All non-split alternating links are TQA. Finally, the key fact about two-fold quasi-alternating links which we will use is the following.

**Theorem 3.1.6** ([50, Corollary 1])**.** *If a link* $L$ *is TQA, then* $\Sigma_2(L)$ *is an L-space.*

## 3.2 Two bridge links

Before proving Theorem H, we recall some background on two bridge links. The two bridge link associated with the fraction $\frac{p}{q} \in \mathbb{Q}$, where $p > q > 0$ are coprime is the unique link $L_{p/q}$ in $S^3$ with double branched cover the lens space $L(p, q)$. The links $L_{p/q}$ and $L_{p/(p-q)}$ are mirror images, and hence up to taking mirror images we may assume that precisely one of $p$ or $q$ is even. Following e.g. [27], we write $p/q$ as a (positive) even continued fraction expansion

$$p/q = [2a_1, 2a_2, \ldots, 2a_r]^+ := 2a_1 + \cfrac{1}{2a_2 + \cfrac{1}{\ddots + \cfrac{1}{2a_r}}},$$

where $a_1, \ldots, a_r$ are integers. Then $L_{p/q}$ is the link shown in Figure 3.2. If $r$ is even, $L_{p/q}$ is a knot, otherwise it is a two component link.

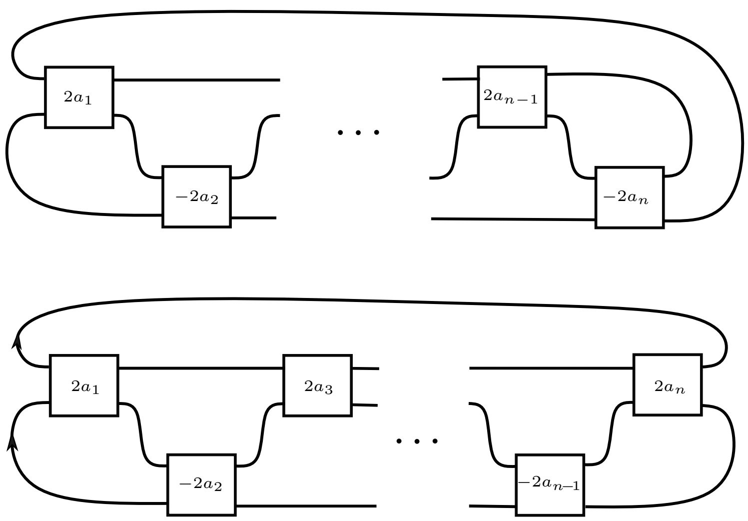


Figure 3.2: The top picture depicts a diagram for the two-bridge knot $K$ with fraction $[2a_1, \ldots, 2a_n]^+$ where $n$ is even. On bottom we have a diagram for the link $L$ with fraction $[2a_1, \ldots, 2a_n]^+$ where $n$ is odd, with a preferred orientation. In both diagrams the boxes should be replaced with the corresponding number of signed half-twists.

Two-bridge knots are all 2-periodic. Figure 3.3 shows a diagram for the link $L$, where $n$ is odd, which is rotationally symmetric about the origin (thinking of the diagram on the plane). An analogous process yields a symmetric diagram for the two bridge knot with fraction $[2a_1, \ldots, 2a_r]^+$ where $r$ is even.

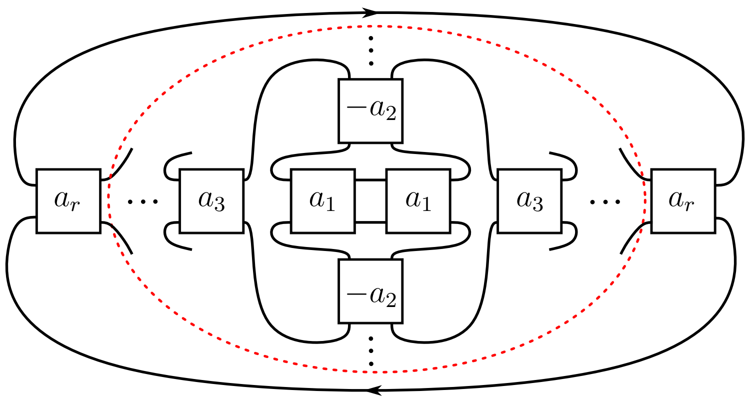


Figure 3.3: A symmetric diagram for the two bridge links with fraction $[2a_1, \ldots, 2a_r]^+$ where $r$ is odd. Imagining the diagram is placed on the plane, along the $x$-axis the twist boxes with coefficient $a_j$ for $j$ odd are placed at $(\pm j, 0)$. Along the $y$-axis the twist boxes with coefficient $-a_j$ for $j$ even are placed at $(0, \pm j)$. The boxes are connected so that for each box indexed with $a_j$ is connected to both boxes indexed with $a_j + 1$ in the manner depicted. The dotted circle indicates a Conway sphere that, when rotated through angle $\pi$, gives an isotopic link but transfers a half-twist from one box labelled $a_r$ to the other.

We sketch a proof that the diagrams in Figure 3.2 and Figure 3.3 are isotopic. Using the Conway sphere indicated in Figure 3.3 all of the half-twists of the left-hand twist box labelled $a_r$ can be untwisted at the expense of adding $a_r$ half twists to the box labelled $a_r$ on the right.

There is a corresponding Conway sphere $S_i$ for each $2 \leq i \leq r$ which contains all twist boxes labelled $a_j$ for $j < i$. Twisting along each $S_i$, either about its horizontal or vertical axis depending on the parity of $i$, in decreasing order from $S_r$ to $S_2$ yields the corresponding link in Figure 3.2.

**Theorem H** ([25])**.** *Let $L$ be a two-bridge link with fraction of the form $[2a_1, 2a_2, \ldots, 2a_r]^+$ where $a_1, \ldots, a_r$ are all positive integers. In the case that*

*$L$ is a two-component link, we orient the link as in Figure 3.2. Then $L$ is a branched L-space link.*

*Proof.* Since the diagram for $L = [2a_1, \ldots, 2a_r]^+$ in Figure 3.3 is symmetric about the origin, we can take the quotient of $(S^3, L \cup C)$ under the rotation about $C$ through angle $\pi$ where $C$ is the axis in $S^3$ perpendicular to the origin of the diagram, yielding $(S^3, \overline{L} \cup \overline{C})$; see Figure 3.4. It is not hard to see that $\overline{L}$ is unknotted. If we now assume that each $a_i > 0$ we also see that the quotient diagram of $\overline{L}$ is alternating and in fact the diagram can be modified by only Reidemeister 1 moves to obtain a diagram for $\overline{L}$ as the standard unknot. It follows that the link $\overline{L} \cup \overline{C}$ is symmetric, meaning that for any diagram there is an isotopy which exchanges the roles of $\overline{L}$ and $\overline{C}$, but otherwise leaves the picture unchanged.

Now via Construction 2.1.2, we see that we can now express $\Sigma_n(L)$ as $\Sigma_2(L_n)$ where $L_n$ is the lift of $\overline{C}$ under the $n$-fold branched cover of the unknotted $\overline{L}$. Since $\overline{C}$ has an alternating diagram as a pattern in the complement of $L$, its lift will also be alternating. Thus each $\Sigma_n(L)$ is homeomorphic to the double-branched cover of a non-split alternating link, and hence is an L-space. □

## 3.3 Pretzel knots

In this section we prove Theorem I which also proves Theorem A. Let the pretzel link $P(p_1, p_2, \ldots, p_n)$ be defined by replacing the boxes in Figure

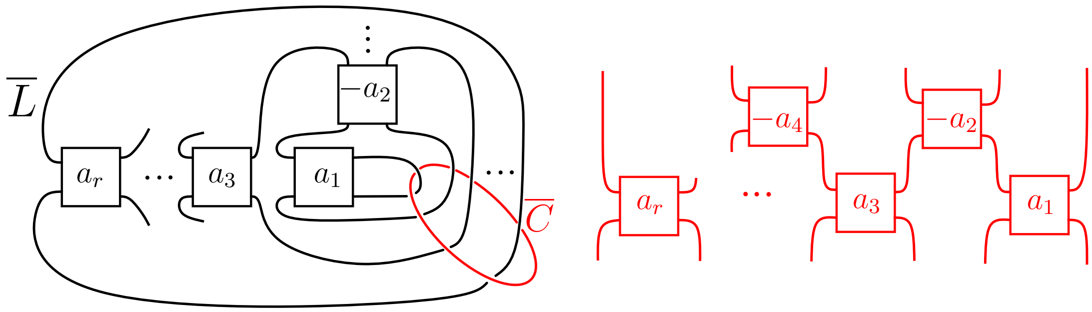


Figure 3.4: The link $\overline{L} \cup \overline{C}$ on the left. This link is symmetric; exchanging the roles of $\overline{L}$ and $\overline{C}$, and then cutting $\overline{C}$ along the obvious disk bounded by $\overline{L}$ yields the picture on the right.

3.5 with $|p_i|$ half-twists with sign determined by the sign of $p_i$. We remark that cyclic permutations of the parameters do not change the link isotopy class.

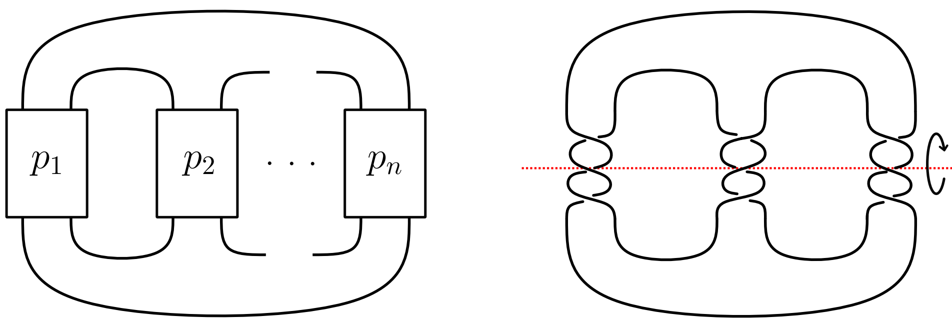


Figure 3.5: The pretzel link $P(p_1, p_2, \ldots p_r)$ and a symmetry of $P(-3, 3, -3)$.

### 3.3.1 Branched cyclic covers of $P(3, -3, 2k + 1)$

We now describe how to construct links $L_{n,k}$ such that

$$\Sigma_n(P(3, -3, 2k + 1)) \cong \Sigma_2(L_{n,k}).$$

The knot $K_k = P(3, -3, -2k - 1)$, and in fact any pretzel link $P(p_1, p_2, \ldots, p_r)$ with each $p_i$ odd, admits an involution whose axis is disjoint from the link which we will now describe. Let $C$ be the horizontal line which passes through each twist box in the standard diagram for $K_k$ through the central crossing of each twist box. Then $K_k$ is rotationally symmetric about $C$; see Figure 3.5. Rotating about this axis $C$ by an angle of $\pi$ defines an involution $\iota$ which preserves the knot setwise. The fixed set of this involution in $S^3$ is the axis $C$.

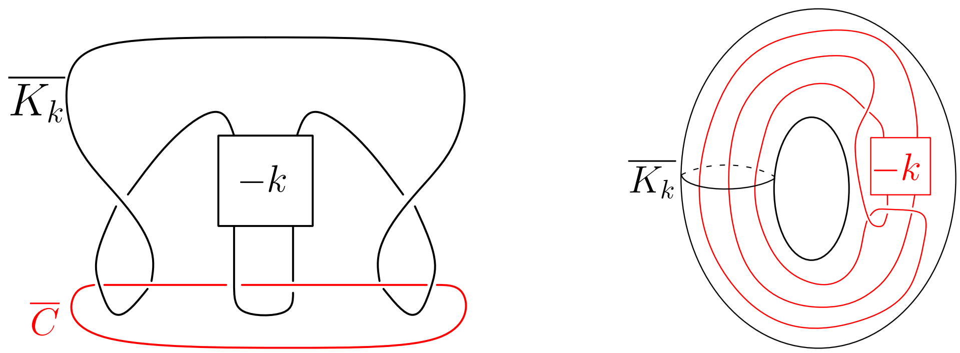


Figure 3.6: Let $K_k = P(-3, -2k-1, 3) \cong P(3, -3, -2k-1)$. There is an isotopy of $\overline{K_k} \cup \overline{C}$ to a diagram where $\overline{C}$ sits inside a solid torus with $\overline{K_k}$ its meridian.

**Proposition 3.3.1.** $\Sigma_n(P(3, -3, -2k-1)) \cong \Sigma_2(L_{n,k})$ *where* $L_{n,k}$ *is the link in Figure 3.7.*

*Proof.* We express $\overline{C}$ as a pattern in $S^3 - \overline{K_k}$; see Figure 3.6. We can now apply Construction 2.1.2 to obtain $L_{n,k}$ as the lift of $\overline{C}$. The link $L_{n,k}$ is depicted in Figure 3.7.

□

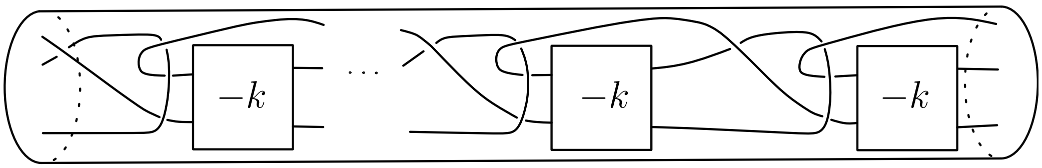


Figure 3.7: The link $L_{n,k}$ inside a torus (obtained by identifying the end disks of the cylinder by the identity map) sitting in $S^3$.

### 3.3.2 $L_{n,k}$ are two-fold quasi-alternating

In this section we show all the links $L_{n,k}$ in Figure 3.7 are two-fold quasi-alternating. We first describe a larger family of links which include the links $L_{n,k}$.

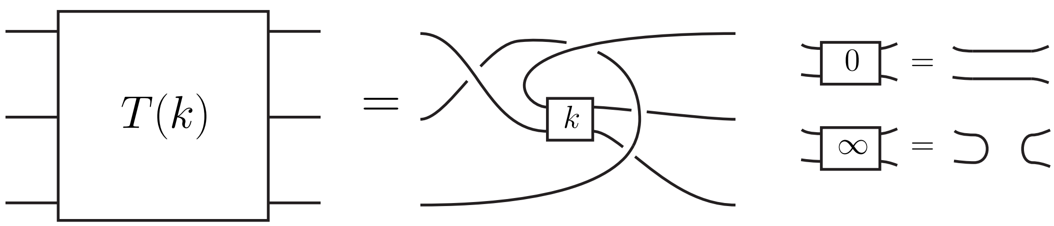


Figure 3.8: The tangle $T(k)$ where $k$ is the number of signed half-twists. See Figure 3.5 for an example of our signed twist conventions. We also allow the case $k = \infty$ which by convention is as shown.

**Definition 3.3.2.** *Let $T(k)$ denote the 3-tangle shown in Figure 3.8, where $k \in \mathbb{Z} \cup \{\infty\}$. If $T_1$ and $T_2$ are 3-tangles, denote by $T_1 \oplus T_2$ the 3-tangle obtained by "stacking" the two tangles, with $T_1$ to the left of $T_2$; see for example Figure 3.9. Let $L(k_1, \ldots, k_n) \subset S^3$ denote the link given by the closure of the tangle $T(k_1) \oplus T(k_2) \oplus \cdots \oplus T(k_n)$, where $k_i \in \mathbb{Z} \cup \{\infty\}$ for all $i$.*

The link $L_{n,k}$ in Figure 3.7 is the link $L(-k, -k, \ldots, -k)$. We will show that these links are all TQA.

**Lemma 3.3.3.** *The link $L(k_1, k_2, ..., k_n)$ is a knot so long as no $k_i = \infty$.*

*Proof.* The link $L = L(k_1, k_2, ..., k_n)$ is the closure of $T(k_1) \oplus \cdots \oplus T(k_n)$. To count the number of components of $L$, we only need to keep track of the way each tangle $T(k_i)$ connects the 6 endpoints of the tangle, which is determined by the parity of $k_i$. If $T$ and $T'$ are 3-tangles, write $T \sim T'$ if $T$ and $T'$ have the same connectivity of endpoints and number of connected components. One can check the four cases: $T(0) \oplus T(1) \sim T(1)$, $T(0) \oplus T(0) \sim T(0)$, $T(1) \oplus T(0) \sim T(0)$ and $T(1) \oplus T(1) \sim T(1)$. Thus, $T(k_1) \oplus \cdots \oplus T(k_n) \sim T(k)$, where $k \in \{0, 1\}$. Finally, the closure of both $T(0)$ and $T(1)$ are knots, hence $L$ is a knot. □

**Lemma 3.3.4.** *Let $L = L(k_1, \ldots, k_n)$ with $k_i \in \mathbb{Z} \cup \{\infty\}$ for all $i$. If $k_i = \infty$ for exactly one value of $i \in \{1, \ldots, n\}$, then $L$ is the two component unlink.*

*Proof.* Notice that the link $L = L(k_1, \ldots, k_n)$ is unchanged under any cyclic permutation of the parameters $(k_1, \ldots, k_n)$. Hence, we may assume that $k_n = \infty$. Figure 3.9 shows that $T(k) \oplus T(\infty)$ is isotopic to $T(\infty)$ as tangles, provided $k \neq \infty$. Applying this $n-1$ times, we see that $T(k_1) \oplus \cdots \oplus T(k_n)$ is the same as $T(\infty)$. Hence, $L$ is the closure of $T(\infty)$ which is the two component unlink. □

**Lemma 3.3.5.** *The links of the form $L(0, 0, \ldots, 0)$ are all non-split alternating knots.*

*Proof.* The link $L = L(0, 0, \ldots, 0)$ is the closure of $T(0) \oplus \cdots \oplus T(0)$. It is a knot by Lemma 3.3.3. If $T$ is a tangle conjugate to $T(0)$ then $L$ is also given

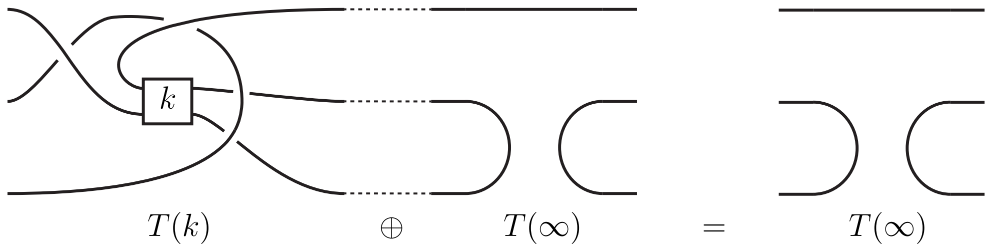


Figure 3.9: The identity $T(k) \oplus T(\infty) = T(\infty)$.

by the closure of $T \oplus \cdots \oplus T$. Figure 3.10 shows that we can conjugate $T(0)$ to an alternating tangle $T$, from which we see that the closure of $T \oplus \cdots \oplus T$ has a non-split alternating diagram. $\square$

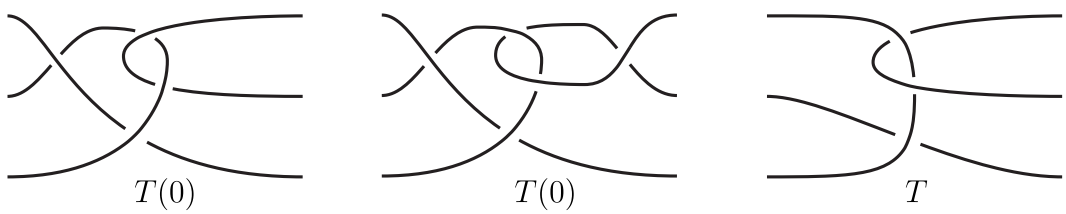


Figure 3.10: The alternating tangle $T$ is obtained from $T(0)$ by conjugating by a tangle interchanging the top two strands. The middle diagram of $T(0)$ is obtained from the one on the left by an isotopy.

**Theorem 3.3.6.** *The knot $L(k_1, \ldots, k_n)$, where $k_1, \ldots, k_n$ are integers is two-fold quasi-alternating.*

*Proof.* We prove this by induction on $N = |k_1| + \cdots + |k_n|$. Let $L = L(k_1, \ldots, k_n)$. If $N = 0$ then $k_i = 0$ for all $i$ and $L$ is a non-split alternating knot by Lemma 3.3.5, and hence is TQA. Assume that $N > 0$ and let $i \in \{1, \ldots, n\}$ be an index such that $|k_i| > 0$.

For convenience, let $\varepsilon = \text{sign}(k_i) \in \{\pm 1\}$. Pick any crossing in the twist box labeled $k_i$. Smoothing $L$ at the crossing results in two links: $L_0 = L(k_1, \ldots, k_{i-1}, \varepsilon(|k_i|-1), k_{i+1}, \ldots, k_n)$ and $L_1 = L(k_1, \ldots, k_{i-1}, \infty, k_{i+1}, \ldots, k_n)$. By Lemma 3.3.4, $L_1$ is the two component unlink. Let $D$ denote our diagram for $L$, and take any two arcs of $D$ which, after smoothing the crossing, belong to the two distinct components of $L_1$. Define a marking on $D$ which assigns $1 \in \mathbb{Z}/2$ (a single dot) to each of these two arcs, and 0 (no dots) to all other arcs. Since $L$ is a knot by Lemma 3.3.3, this marking represents the trivial marking of $L$ (the two dots cancel mod 2). Smoothing the crossing, with this choice of marking, we obtain marked diagrams representing:

- $L_0$ with its trivial marked diagram since $L_0$ is a knot by Lemma 3.3.3. This marked link is TQA by the induction hypothesis.
- The two component unlink $L_1$, where each component is assigned 1. This is TQA as it splits into two odd marked unknots.

We note that the determinant condition $\det(L) = \det(L_0) + \det(L_1)$ is automatically satisfied since $\det(L_1) = 0$. Hence, $L$ (with its trivial marking) is TQA.

□

### 3.3.3 Quasipositivity

In this section we show that the knots $P(3, -3, -2k-1)$ are quasipositive. The argument used also shows the well-known result that these pretzels knots are slice. Let $B_n$ denote the Artin braid group on $n$ strands and let $\sigma_i$ be the standard Artin generators for $1 \leq i \leq n$.

We now recall the definition of a quasipositive link. A braid $\beta \in B_n$ is called quasipositive if it can be written as a product of conjugates of the Artin generators: $\beta = \prod_k w_k \sigma_{i_k} w_k^{-1}$. A link in $L$ in $S^3$ is quasipositive if it is the closure of a quasipositive braid.

Quasipositivity is useful in determining properties of the branched cyclic covers of the knots in $S^3$ as the following theorem shows.

**Theorem 3.3.7** ([4])**.** *Suppose $K$ is a non-slice quasipositive knot. Then there is an $N = N(K)$ so that $\Sigma_n(K)$ is not an L-space for all $n \geq N$.*

The following proposition shows that the pretzel knots which were shown in Section 3.3.2 to have all L-space branched covers are also quasipositive. Note the knots in the proposition are slice; this is classical, but see Figure 3.13 for a ribbon diagram. This answers Question 3.1.3.

**Proposition 3.3.8.** *The knots $P(3, -3, -2k-1)$ are slice and quasipositive for $k \geq 1$.*

In order to prove this, we use the notion of a track knot defined by Baader [2].

**Definition 3.3.9.** *Let $C$ be the image of a generic immersion $i : [0, 1] \to \mathbb{R}^2$ with a labeling at each double point by a letter in $\{a, b, c, d\}$, and an angle in $\{0, \pi/2, \pi, 3\pi/2\}$; here we assume the diagram has been isotoped so that double*

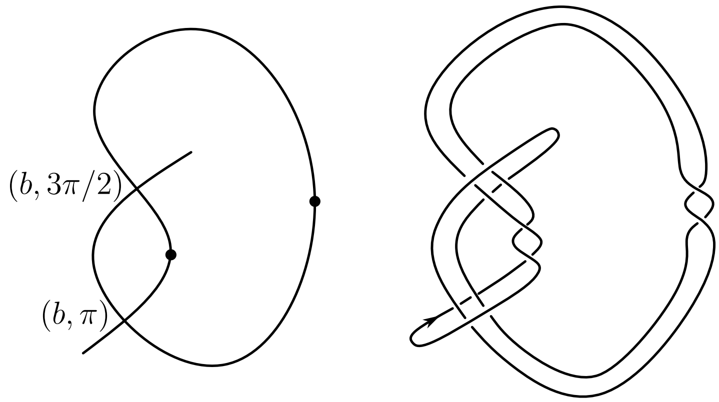


Figure 3.11: An immersed labeled interval and the corresponding track knot, which happens to be isotopic to $P(3, -3, -3)$.

*points locally look like* $\times$ *in the plane (not some arbitrary rotation of this picture). Finally, specify points* $p_1, p_2, \ldots, p_r$ *on the connected components of* $C - \{$*double points*$\}$ *such that each connected component of* $C - \{p_1, p_2, \ldots, p_r\}$ *is contractible. We will call* $C$ *a labeled immersed interval.*

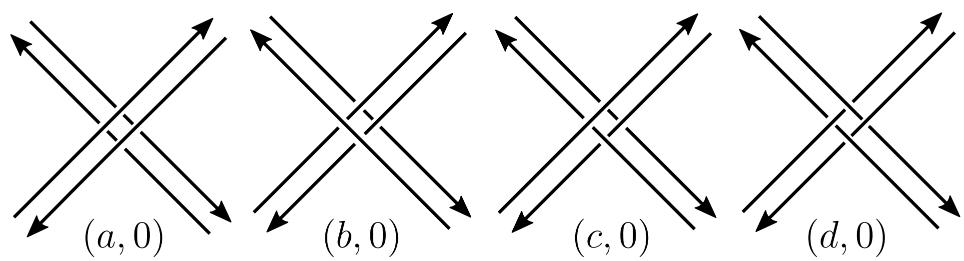


Figure 3.12: The process defining a knot from a labeled immersed curve replaces a double point labeled $(a, 0), (b, 0), (c, 0)$ and $(d, 0)$ with the corresponding crossing patterns above. If the angle at the crossing is not 0, rotate the corresponding diagram above by the angle specified in the counter-clockwise direction.

From such an interval one can associate a knot as follows (see Figure 3.11 for an example). Draw an immersed interval parallel to $C$ and join the two intervals by an arc at each end of $C$, this forms an immersed band following $C$ which we will call $B_C$. Orient $\partial B_C$ counter-clockwise. Each double point of $C$ corresponds to four self-intersections of $\partial B_C$, which we replace with over-and under-crossings according to the labeling and angle of $C$ at that double point as shown in Figure 3.12. Replace each point $p_i$ with a full-twist oriented to introduce positive crossings.

**Definition 3.3.10.** *A knot obtained from a labeled immersed interval by the above procedure is called a track knot.*

**Theorem 3.3.11** ([2])**.** *Track knots are quasipositive.*

*Remark* 3.3.12. Baader's proof that track knots are quasipositive describes an algorithm to obtain a quasipositive braid word for a track knot, though in general it will not be of minimal braid index.

Figure 3.13 shows that the knots $P(3, -3, -2k-1)$ are track knots and exhibits a slice disk for them. We then obtain the following proposition.

**Proposition 3.3.13.** *The pretzel knots* $P(3, -3, -2k-1)$ *are slice track knots for* $k \geq 1$*.*

Proposition 3.3.8 now follows from Theorem 3.3.11.

## 3.4 Knots in rational homology spheres

In this section we discuss branched L-space knots outside of $S^3$ and prove Theorem B. We also discuss obstructions to being a branched L-space

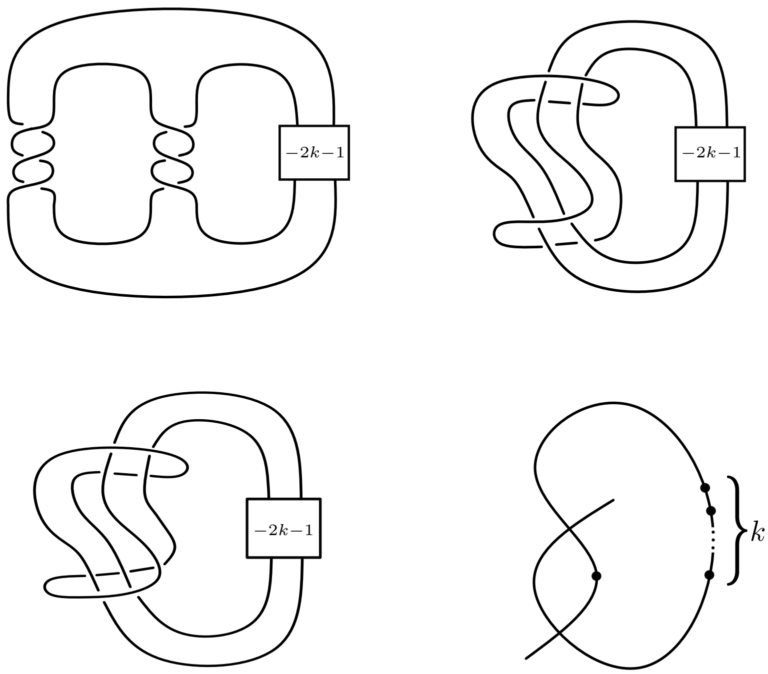


Figure 3.13: An isotopy of the pretzel knot $P(3, -3, -2k-1)$ which in particular exhibits ribbon disks for each knot. After redistributing a half twist from the twist box in the final pictured knot diagram by isotopy, we obtain the bottom right picture which gives a labeled immersed interval for $P(3, -3, -2k-1)$ realizing these knots as track knots.

link for periodic links.

**Lemma 3.4.1.** *Every alternating diagram for a link in $S^3$ can be related to a diagram of the form appearing on the right-hand side of Figure 3.14 by planar isotopy.*

*Proof.* Let $D$ be an alternating diagram for the link $L$. Then the projection of $L$ onto the projection sphere (thought of as a plane compactified by a point at infinity) of $D$ cuts the plane into regions, one of which contains the point at infinity. Then certainly, there is a sphere decomposing $D$ into two tangles, one of which is trivial. This is illustrated in the left-hand side of Figure 3.14. To obtain the tangle decomposition on the right-hand side of the figure, we choose a different sphere, the dotted red one illustrated in the middle of Figure 3.14. □

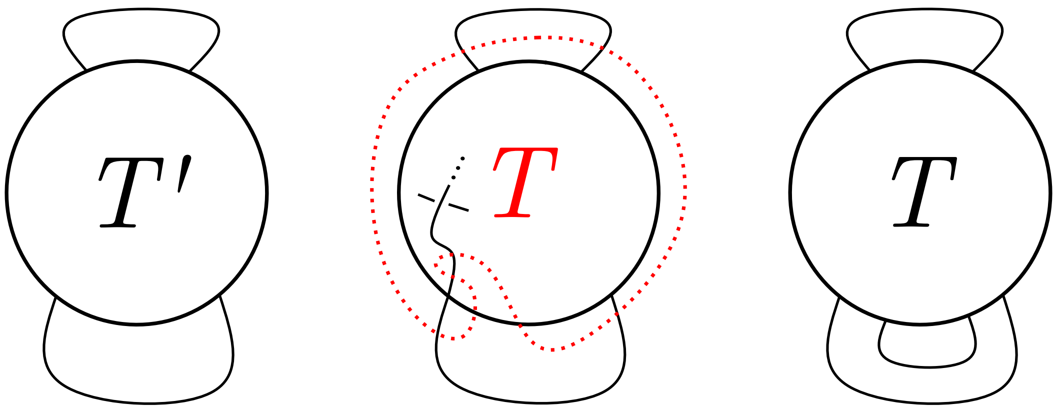


Figure 3.14: On the left, a sphere separating an alternating diagram for a link in $S^3$ into two tangles, one trivial and the other some 4-ended tangle $T'$. In the middle, a sphere for the same alternating diagram separating the link into two 6-ended tangles. Finally on the right, an isotopy of the middle tangle decomposition.

*Proof of Theorem B.* Let $L$ be an alternating link in $S^3$. We now construct diagrammatically an alternating periodic link $L_n$ with period $n$ whose quotient under this periodic symmetry is the original link $L$ for any $n \geq 1$.

Take $n$ copies of the tangle $T$ obtained in Lemma 3.4.1 and arrange them in a circle in the plane cyclically. Connect the endpoints of consecutive copies of $T$ as illustrated in Figure 3.15; there is a choice of sign of twisting

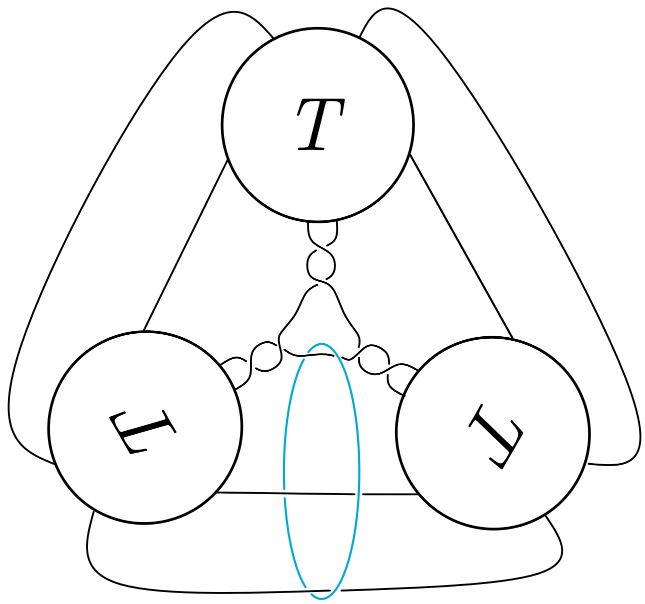


Figure 3.15: An alternating periodic knot $L_n$ (in this example $n = 3$) whose quotient is $L$. The blue curve is the axis of the symmetry $C$.

in the central region, we choose the sign which ensures the diagram in Figure 3.15 is alternating. This link is periodic of order $n$ and its quotient under the symmetry is our original link $L$ for each $n \geq 1$.

Let $C_n$ denote the axis of the symmetry of $L_n$ and $\overline{C_n}$ its quotient; for each $n$ the $\overline{C_n} \cup L$ are isotopic so we drop the subscript $n$. We denote the lift of $\overline{C}$ in the manifold $\Sigma_2(L)$ by $K$. We point out that $K$ is a knot since the linking number of $\overline{C}$ and $L$ is odd, so in particular is coprime to the index of the branching.

We now show that $K$ is a branched L-space knot in $\Sigma_2(L)$. First we remark that there is an induced Seifert surface for $K$; the knot $K$ is the lift of $\overline{C}$ which was unknotted and so bounds a disk $D$. The lift of $D$ in $\Sigma_2(L)$ is a Seifert surface for $K$. In this way, we choose a well-defined set of branched cyclic covers of $K$.

The branched cyclic cover $\Sigma_n(K) \cong \Sigma_2(L_n)$ for each $n$. Since $L_n$ was alternating for each $n$, the double-branched cover $\Sigma_2(L_n)$ is an L-space for each $n$. Hence $K$ is a branched L-space knot.

□

### 3.4.1 Obstructions for branched L-space knots

In this section we discuss how to obstruct a periodic link $L$ in $S^3$ from being a branched L-space link by studying its quotient link.

**Proposition J.** *Let $L$ be an oriented link in $S^3$. Suppose that $L$ is 2-periodic and the quotient of $L$ under this symmetry is $\overline{L}$. Then*

$$\mathcal{L}_{br}(L) \subset \mathcal{L}_{br}(\overline{L})$$

This gives an immediate obstruction to a 2-periodic link being a branched L-space link.

**Corollary 3.4.2.** *The quotient of a 2-periodic branched L-space link must also be a branched L-space link.*

As an example, we apply Proposition J to a family of pretzel links.

**Example 3.4.3.** *The family of pretzel knots $P(2k+1, 2m, 2k+1)$ is 2-periodic as illustrated in Figure 3.16. The quotient of the symmetry is the torus knot*

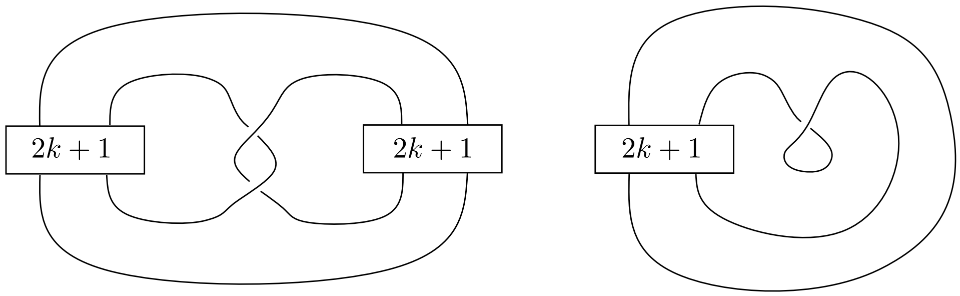


Figure 3.16: The pretzel knots $P(2k{+}1, 2m, 2k{+}1)$ have a 2-periodic symmetry whose quotient is the torus knot $T(2k+1,2)$. The symmetry is illustrated for the family $P(2k+1, 2, 2k+1)$; the diagram on the left is rotationally symmetric about its central point. The diagram on the right is the quotient of this symmetry.

$T(2k+1,2)$. *The set*

$$\mathcal{L}_{br}(T(2k+1,2)) = \begin{cases} \{1,2,3,4,5\} & \text{if } k=1 \\ \{1,2,3\} & \text{if } k=2 \\ \{1,2\} & \text{if } k \geq 3 \end{cases}$$

*Thus,*

$$\mathcal{L}_{br}(P(2k+1,2m,2k+1)) \subset \begin{cases} \{1,2,3,4,5\} & \text{if } k=1 \\ \{1,2,3\} & \text{if } k=2 \\ \{1,2\} & \text{if } k \geq 3 \end{cases}$$

A key tool in the proof of Proposition J is the following theorem of Hendricks-Lidman-Lipshitz.

**Theorem 3.4.4** ([23, Corollary 1.2])**.** *Suppose $L$ in $M$ is a link with a choice of Seifert surface. If $\Sigma_2(L)$ is an L-space, then $M$ is an L-space.*

We are now ready to prove Proposition J.

*Proof of Proposition J.* Let $C$ be the axis of the 2-periodic symmetry for $L$. Then there is a 2-fold branched cover map $S^3 \to S^3$ branched along $\overline{C}$ the quotient of $C$. Both $C$ and $\overline{C}$ are unknotted. Denote by $L_n$ the lift of $\overline{C}$ in $\Sigma_n(\overline{L})$; since $\overline{C}$ is unknotted it bounds a disk $D$; hence $L_n$ has a natural Seifert surface, the lift of $D$ in $\Sigma_n(\overline{L})$. By Proposition 2.1.1 all of the following manifolds are homeomorphic $\Sigma_n(L) \cong \Sigma_{n,2}(\overline{L} \cup \overline{C}) \cong \Sigma_2(L_n)$.

Now suppose that $\Sigma_n(\overline{L})$ is not an L-space. The link $L_n$ lies in $\Sigma_n(\overline{L})$; thus by Theorem 3.4.4, $\Sigma_2(L_n) \cong \Sigma_n(L)$ cannot be an L-space. Thus the set of indices $\mathcal{L}_{br}(\overline{L})$ for which $\Sigma_n(\overline{L})$ is an L-space constrains the same set $\mathcal{L}_{br}(L)$ for $L$.

□

# Chapter 4

# Left-orderability of cyclic branched covers

In this chapter we characterize, for certain classes of two-bridge knots, which branched cyclic covers have left-orderable fundamental groups, proving Theorem D. The techniques used are Riley polynomials of knots and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ representations of 3-manifold groups.[1]

## 4.1 Introduction

We recall the definition of a left-orderable group. In the setting of 3-manifold groups, we take as a convention that the trivial group is non-left-orderable.

**Definition 4.1.1.** *A non-trivial group is left-orderable if it admits a strict total order which is invariant under left-multiplication. We will say a 3-manifold $M$ is left-orderable if $\pi_1(M)$ is a left-orderable group.*

One method to prove a 3-manifold is left-orderable is via certain representations.

**Theorem 4.1.2** ([9])**.** *A compact, orientable, and irreducible 3-manifold group*

[1]The results in this chapter were originally published in the Proceedings of the American Mathematical Society, Volume 149, Number 3, in 2021.

*is left-orderable if it admits a non-trivial representation into a left-orderable group.*

The fact that $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is left-orderable [3] has been exploited to prove that certain 3-manifold groups are left-orderable. To prove theorem, we will construct non-trivial $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-representations for the manifolds in question.

In Theorem D, we study the set of left-orderable indices $\mathcal{LO}_{br}(K)$ for two families of two-bridge knots. First, we consider double twist knots with three-genus at least two. The double twist knots are a two-parameter family $J(r, s)$ as in Figure 4.1. Any double-twist knot with genus $g(J(r, s)) \geq 2$ is isomorphic to $J(2k+1, 2m)$ with $k \geq 1$ and $|m| \geq 2$; in fact, $g(J(2k+1, 2m)) = |m|$. See Section 4.1.2 for details.

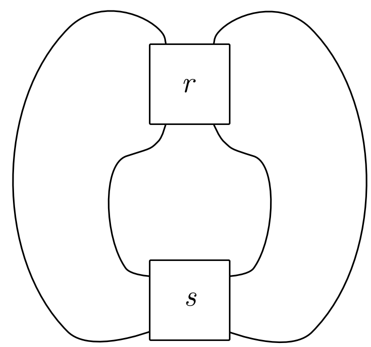

Figure 4.1: The knot $J(r, s)$ where $r$ and $s$ count the number of signed half twists in each box.

Tran determined an integer $N$ such that $\Sigma_n(J(2k+1, 2m))$ is left-orderable for $n \geq N$. The integer $N$ grows with $k$ and $|m|$ [54]. Improving on this result, for double twist knots with genus at least four, we completely classify the indices $n$ for which $\Sigma_n(K)$ is left-orderable. If the genus is two or three, we decide these indices with one or two exceptions. The following theorem is equivalent to Theorem D; see Section 4.1.1 for a discussion of the relationship between the genus of $K$ and $\mathcal{LO}_{br}(K)$.

**Theorem 4.1.3.** *Let $k \geq 1$ be fixed. Then $\Sigma_n(J(2k+1, 2m))$ is left-orderable in the following cases:*

1. *$n \geq 3$ when $m \leq -3$,*
2. *$n \geq 4$ when $m = -2$,*
3. *$n \geq 5$ when $m = 2$,*
4. *$n \geq 4$ when $m = 3$,*
5. *$n \geq 3$ when $m \geq 4$.*

*In cases* (1), (2) *and* (5), *$\Sigma_n(J(2k+1, 2m))$ is left-orderable if and only if the index $n$ satisfies the corresponding inequality.*

*Remark* 4.1.4. In light of Theorem 4.1.3, the only cyclic branched covers of double twist knots with genus at least two for which left-orderability remains unknown are $\Sigma_4(J(2k+1, 4))$ and $\Sigma_3(J(2k+1, 6))$. For the proof of this statement see Section 4.1.1. If $K$ is a double-twist knot with genus one, then in fact $K \cong J(2k, 2m)$ with $k$ positive, see Section 4.1.2. None of the branched covers of $K$ are left-orderable if $m$ is negative [13, Theorem 2(c)], while when $m$ is positve, Tran determines an integer $N$ which grows with $k$ and $|m|$, such that $\Sigma_n(K)$ is left-orderable for $n \geq N$ [54].

### 4.1.1 Left-orderable indices and genus

In Theorem 4.1.3 it is striking that $\mathcal{LO}_{br}(J(2k+1, 2m))$ is almost determined by the three-genus $|m|$, especially when $|m|$ is large. We point out that Teragaito gives examples two-bridge knots with arbitrarily large genus, all of whose cyclic-branched covers are L-spaces by showing these manifolds are the double-branched cover of non-split alternating links [52]. All of the cyclic branched covers of these knots are also not left-orderable [8, Theorem 8]. However, Theorem 4.1.3 gives evidence that there is a relationship between the properties of the L-space conjecture for $\Sigma_n(K)$ for a knot $K$ and its three-genus if $\Sigma_n(K)$ is left-orderable for some $n$. The following results of Ba also indicate that such a relationship might exist.

**Theorem 4.1.5** (Theorem 1.3 in [1]). *Let $K$ be a two-bridge knot with $g(K) = 2$. Then $\Sigma_3(K)$ is an L-space, and is not left-orderable.*

**Theorem 4.1.6** (Corollary 1.4 in [1]). *Let $K$ be a two-bridge knot with $g(K) = 1$. Then $\Sigma_n(K)$ is an L-space, and is not left-orderable for $n \leq 5$.*

Two-bridge knots have lens space branched double covers. These manifolds have finite fundamental groups and hence are not left-orderable. Together with Theorem 4.1.5, this allows us to conclude that the bounds we obtain in Theorem 4.1.3 are best possible in the cases that $m \leq -2$ or $m \geq 4$.

### 4.1.2 Notation

We now describe the family of two-bridge knots addressed in Theorem D. Two-bridge knots are the closures of rational tangles, and so have an (non-unique) associated fraction $p/q$ with $-p < q < p$, see [27, 11]. We will write $K(p, q)$ to denote the unique two-bridge knot associated to the fraction $p/q$.

The manifolds $\Sigma_n(K)$ and $\Sigma_n(-K)$ are orientation-reversing homeomorphic; we are interested in the fundamental groups of these manifolds so we need only consider one of $K$ or $-K$. The fractions $p/q$ and $p/q'$ correspond to the same knot if $q \equiv q' (\text{mod})\ p$. Finally, since $K(p, q) \cong -K(p, -q)$ we can assume that $p > q > 0$ with $q$ is odd since $K(p, 2n) \cong -K(p, p - 2n)$.

The double-twist knots are the two-bridge knots $J(r, s)$ pictured in Figure 4.1. We list some facts about $J(r, s)$:

1. If $rs$ is odd, then $J(r, s)$ is a link of two components.
2. If $rs = 0$ then $K$ is the unknot.
3. $J(-r, -s) \cong -J(r, s)$.
4. $J(r, s) \cong J(s, r)$.

Excluding the cases of the unknot and links of two components, we can consider without loss of generality knots of the form $J(r, 2m)$, with $|r|, |m| > 0$. As above, we need only consider one of $K$ or $-K$ to obtain our results, hence we can further assume that $r > 0$.

We can compute that the three-genus $g(J(2k+1,2m)) = |m|$ while $g(J(2k,2m)) = 1$. These families are disjoint except in the case that $|m| = 1$. The knots $J(2k+1,\pm 2)$ and $J(2k+1\mp 1,\mp 2)$ are isomorphic; thus all genus one double twist knots are addressed to some extent by Remark 4.1.4.

We present no new results in the case that $g(J(r,2m)) = 1$, so we exclude the case that $r$ is even, and the case that $|m| = 1$. Finally, if $k = 0$ then $K$ is a $(2, 2p+1)$-torus knot for some integer $p$. Gordon and Lidman completely determined the indices for which the branched covers of these knots are left-orderable [16, 17]. In summary, when $K$ is a double-twist knot, we will assume $K = J(2k+1, 2m)$ with $|m| \geq 2$ and $k \geq 1$.

## 4.2 Non-abelian representations and two-bridge knots

This section follows work of Hu [24] relating left-orderability of branched covers of two-bridge knots to finding roots of certain polynomials.

The following theorem of Hu allows us to prove left-orderability (and in fact construct$\widetilde{\mathrm{SL}(2,\mathbb{R})}$-reps for $\Sigma_n(K)$) by instead constructing constrained $\mathrm{SL}(2,\mathbb{R})$-reps on the knot complement.

**Theorem 4.2.1** (Theorem 3.1 of [24])**.** *Let $K$ be a prime knot in $S^3$ and $X_K$ denote its complement. Let $Z$ be a meridional element of $\pi_1(X_K)$. If there exists a non-abelian representation $\rho : \pi_1(X_K) \to SL(2,\mathbb{R})$ such that $\rho(Z^n) = \pm I$ then $\Sigma_n(K)$ is left-orderable.*

Let $K$ be a two-bridge knot for the remainder of the section. Then the knot group has a presentation of the form

$$\pi_1(X_K) = \langle a, b : va = bv \rangle \tag{4.1}$$

where $a$ and $b$ are meridians and $v$ is a word in $a$ and $b$, see eg. [27].

It follows from [47, Lemma 1] that a non-abelian representation $\rho : \pi_1(X_K) \to \mathrm{SL}(2, \mathbb{C})$ can be conjugated to be of the form:

$$\rho(a) = A = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \qquad \rho(b) = B = \begin{bmatrix} s & 0 \\ 2-y & s^{-1} \end{bmatrix} \tag{4.2}$$

where $s \in \mathbb{C} \setminus \{0\}$ and $y \in \mathbb{C}$ satisfying $VA - BV = 0$ where $V = \rho(v)$. A special case of (4.2) is

$$\rho(a) = A = \begin{bmatrix} e^{i\pi/n} & 1 \\ 0 & e^{-i\pi/n} \end{bmatrix} \qquad \rho(b) = B = \begin{bmatrix} e^{i\pi/n} & 0 \\ 2-y & e^{-i\pi/n} \end{bmatrix} \tag{4.3}$$

for fixed $n \in \mathbb{N}$ with $n \geq 2$. This map is closer to satisfying the conditions of Theorem 4.2.1 since it can be shown that $\rho(a^n) = -I$.

The map defined by (4.2) can be defined for any presentation of the form defined in (4.1), though it is not necessarily a homomorphism. To check that the map is in fact a representation for a given $(s, y) \in \mathbb{C}^* \times \mathbb{C}$, we need to see that $VA - BV = 0$ is satisfied. In Section 4.4 we compute the entries of $R = VA - BV$ explicitly in the case $K = J(2k+1, 2m)$ as in [54].

For two-bridge knots, work of Riley shows that determining when the map (4.2) is a representation reduces to determining when exactly one entry of the matrix $R$ is zero.

**Proposition 4.2.2** (Theorem 1 in [47])**.** *$R_{i,j} = 0$ for $1 \leq i, j \leq 2$ if and only if $R_{1,2} = 0$. In other words, if $R_{1,2} = 0$ then the map in (4.2) is a homomorphism.*

We see that $R_{1,2} = R_{1,2}(s, y)$ can be considered as a polynomial in $\mathbb{Z}[s^{\pm 1}, y]$.

**Proposition 4.2.3** (Proposition 1 in [47])**.** *We have that $R_{1,2}(s, y) = R_{1,2}(s^{-1}, y)$. Thus $R_{1,2}(s, y) = f(s + s^{-1}, y)$ where $f$ is a two-variable polynomial with coefficients in $\mathbb{Z}$.*

**Definition 4.2.4.** *Let $K$ be a two-bridge knot, and fix a presentation for $\pi_1(X_K)$. Let $x = s + s^{-1}$. Then we will call $\phi_K(x, y) := f(s + s^{-1}, y)$ a Riley polynomial of $K$.*

We note that the polynomial $\phi_K(x, y)$ is not an invariant of $K$, but depends on the choice of presentation for $\pi_1(X_K)$. The following statement should be compared to Hu's Proposition 4.1 and the proof of Theorem 4.3 [24].

**Theorem 4.2.5.** *Let $K$ be a two-bridge knot, and let $\phi_K(x, y)$ be a Riley polynomial of $K$. Fix $n \geq 2$. Suppose there exists $y_n > 2$ a real root of $\phi_K(2\cos(\pi/n), y)$. Then $\Sigma_n(K)$ is left-orderable.*

*Proof.* Since $\phi_K(2\cos(\pi/n), y_n) = 0$ it is clear that $R_{1,2}(e^{\pi i/n}, y_n) = 0$. Thus, setting $y = y_n$ in (4.3) defines a $\mathrm{SL}(2, \mathbb{C})$ representation of $\pi_1(X_K)$ by Proposition 4.2.2. In addition, $y_n > 2$ is real, so a result of Khoi tells us that (4.3) can be conjugated to a representation $\rho'$ into $\mathrm{SL}(2, \mathbb{R})$ [30, p. 786]. Since $\rho(a^n) = -I$ we also have that $\rho'(a^n) = -I$. Finally, two-bridge knots are prime; we can now see that $\rho'$ satisfies the conditions of Theorem 4.2.1, and we conclude that $\Sigma_n(K)$ is left orderable for that particular $n$. □

The following theorem therefore implies Theorem 4.1.3, and will be proved in Section 4.5.

**Theorem 4.5.5.** *Fix $n \geq 2$, and let $K = J(2k+1, 2m)$. Then there is a presentation of $\pi_1(X_K)$ with Riley polynomial $\phi_K(x, y)$ such that $\phi_K(2\cos(\pi/n), y)$ has a root $y_n > 2$ in the following cases:*

1. *$n \geq 3$ when $m \leq -3$,*
2. *$n \geq 4$ when $m = -2$,*
3. *$n \geq 5$ when $m = 2$,*
4. *$n \geq 4$ when $m = 3$,*
5. *$n \geq 3$ when $m \geq 4$.*

## 4.3 Chebyshev Polynomials

Let $S_n(z)$ be the sequence of Chebyshev polynomials defined by the recurrence relation $S_{n+1}(z) = zS_n(z) - S_{n-1}(z)$ with $S_0(z) = 1$ and $S_1(z) = z$. They allow simplifications of certain recurrences. For a well-chosen presentation of $\pi_1(X_K)$ for $K$ a double-twist knot, the Riley polynomial $\phi_K(x, y)$ can expressed in terms of these polynomials, and their properties allow us to understand the roots of $\phi_K(x, y)$.

**Lemma 4.3.1** (Lemma 3.2 in [53])**.** *If $a_n$ is a sequence of complex numbers satisfying $a_{n+1} = ca_n - a_{n-1}$ for some $c \in \mathbb{C}$, then $a_{n+1} = S_n(c)a_1 - S_{n-1}(c)a_0$.*

*Remark* 4.3.2. Calling them Chebyshev polynomials is apt since $S_n(2z) = U_n(z)$ where $U_n(z)$ are the Chebyshev polynomials of the second kind defined by $U_0(z) = 1$, $U_1(z) = 2z$ and $U_n(z) = 2zU_{n-1}(z) - U_{n-2}(z)$.

We will make use of properties of these Chebyshev polynomials in many arguments. One can allow $n$ to be negative and extend the recurrence; we do not need this generalization, so we will assume that $n \geq 0$ for the remainder of the section.

**Lemma 4.3.3.** *The Chebyshev polynomials $S_n(z)$ satisfy the following:*

1. *$S_n(2) = n + 1$ and $S_n(-2) = (-1)^n(n + 1)$.*

2. *The roots of $S_n(z)$ are $2\cos\left(\frac{k\pi}{n+1}\right)$ for $k = 1, 2, \ldots, n$.*

3. *$S_n(t) > 0$ when $t \geq 2$ for $t \in \mathbb{R}$.*

4. *The inequality* $S_{n+1}(t) > S_n(t)$ *holds when* $t \geq 2$ *for* $t \in \mathbb{R}$.

*Proof.*

1. This follows easily by induction.

2. Using the fact that the roots of $U_n(x)$ are $\cos\left(\frac{k\pi}{n+1}\right)$ for $k = 1, 2, \ldots, n$, the result follows from the fact that $S_n(2z) = U_n(z)$.

3. It is clear from the definition of Chebyshev polynomials that the leading coefficient is positive so that the end behavior as $t$ tends to infinity is positive. By (2) we have that all of the roots lie in the interval $(-2, 2)$. Thus, $S_N(t)$ is positive on $[2, \infty)$.

4. We proceed by induction. For $n = 0$ or 1 the statement is clear. Now suppose that the statement holds for all $0 \leq n < N$ and let $N > 1$.

   Let $t \geq 2$. We have by the induction hypothesis that $S_N(t) > S_{N-1}(t)$. Hence,

$$\begin{aligned} S_{N+1}(t) &= tS_N(t) - S_{N-1}(t) \geq 2S_N(t) - S_{N-1}(t) \\ &= S_N(t) + (S_N(t) - S_{N-1}(t)) > S_N(t). \end{aligned}$$

□

**Lemma 4.3.4.** *Let* $n \geq 1$. *Then* $(-1)^n S_n(t) < 0$ *on the interval*

$$\left(2\cos\left(\frac{n\pi}{n+1}\right), 2\cos\left(\frac{(n-1)\pi}{n+1}\right)\right).$$

*Proof.* We begin by noting that $r_1 = 2\cos\left(\frac{n\pi}{n+1}\right)$ is the smallest root of $S_n(t)$, and that $r_1$ and $r_2 = 2\cos\left(\frac{(n-1)\pi}{n+1}\right)$ are consecutive roots. By Lemma 4.3.3(2), all of the roots of $S_n(t)$ have multiplicity one. Thus the sign of $S_n(t)$ on $(r_1, r_2)$ is constant and opposite of the sign on $(-\infty, r_1)$ which is also constant. Since $-2 \in (-\infty, r_1)$ and $S_n(-2) = (-1)^n(n+1)$, the lemma follows. □

## 4.4 A formula for the Riley Polynomial

In this section we compute a formula for the Riley polynomial for $X_K$ in terms of Chebyshev polynomials. Let $K = J(2k+1, 2m)$. We fix a presentation for the knot group of $K$. For Sections 4.4 and 4.5 when we write $\phi_K(x, y)$ we mean the Riley polynomial of $K$ for presentation:

$$\pi_1(X_K) = \langle a, b \mid w^m a = b w^m \rangle \tag{4.4}$$

where $a$ and $b$ are meridians and $w = (ba^{-1})^k ba(b^{-1}a)^k$ [**?**, Prop 1].

An easy consequence of Lemma 4.3.1 gives a formula for powers of matrices in $\mathrm{SL}(2, \mathbb{C})$ in terms of Chebyshev polynomials.

**Lemma 4.4.1** (Lemma 2.2 of [36]). *Let $M \in SL(2, \mathbb{C})$. Then*

$$M^n = S_n(Tr(M))I - S_{n-1}(Tr(M))M^{-1}.$$

Let $\rho$ be the map in (4.2), and let $A = \rho(a)$, $B = \rho(b)$ and $W = \rho(w)$. Recall that $x = s + s^{-1} = \mathrm{Tr}(A) = \mathrm{Tr}(B)$ and note that $\mathrm{Tr}(BA^{-1}) =$

$\mathrm{Tr}(B^{-1}A) = y$. Then we can simplify the following expression involving these matrices using Chebyshev polynomials.

**Lemma 4.4.2** (Lemma 2.3 in [36])**.**

$$WA - BW = \begin{bmatrix} 0 & \alpha \\ (y-2)\alpha & 0 \end{bmatrix}$$

*where* $\alpha = \alpha_k(x,y) = 1 + (y+2-x^2)S_{k-1}(y)(S_k(y) - S_{k-1}(y))$.

Our goal is to compute $R = W^m A - BW^m$ in terms of Chebyshev polynomials since this matrix encodes the group relation. In order to use Lemma 4.4.1 we need to compute the trace of $W$.

**Lemma 4.4.3** (Lemma 2.4 in [36])**.** *The trace of* $W$ *is given by*

$$\lambda = \lambda_k(x,y) = Tr(W) = x^2 - y - (y-2)(y+2-x^2)S_k(y)S_{k-1}(y).$$

We now give an expression for the Riley polynomial in terms of Chebyshev polynomials. This is a mild reformulation of a proposition of Morifuji-Tran [36, Proposition 2.5].

**Proposition 4.4.4.** *If* $m \geq 1$ *then* $\phi_K(x,y) = S_{m-1}(\lambda)\alpha - S_{m-2}(\lambda)$. *If* $m \leq -1$ *then* $\phi_K(x,y) = S_{|m|}(\lambda) - S_{|m|-1}(\lambda)\alpha$.

*Proof.* Let $m \geq 1$. Since Lemma 4.4.3 computes the trace of $W$, Lemma 4.4.1 allows us to simplify $W^m$. Lemma 4.4.2 allows us to simplify further and conclude the following series of equalities.

$$\begin{aligned}
R = W^m A - BW^m &= S_{m-1}(\lambda)WA - S_{m-2}(\lambda)A - S_{m-1}(\lambda)BW + S_{m-2}(\lambda)B \\
&= S_{m-1}(\lambda)\left(WA - BW\right) - S_{m-2}(\lambda)\left(A - B\right) \\
&= S_{m-1}(\lambda)\begin{bmatrix} 0 & \alpha \\ (y-2)\alpha & 0 \end{bmatrix} - S_{m-2}(\lambda)\begin{bmatrix} 0 & 1 \\ (y-2) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & S_{m-1}(\lambda)\alpha - S_{m-2}(\lambda) \\ (y-2)(S_{m-1}(\lambda)\alpha - S_{m-2}(\lambda)) & 0 \end{bmatrix}
\end{aligned}$$

Now let $m \leq -1$ and note that $\mathrm{Tr}(W^{-1}) = \mathrm{Tr}(W)$. We have:

$$\begin{aligned}
R = W^m A - BW^m &= \left(W^{-1}\right)^{|m|} A - B\left(W^{-1}\right)^{|m|} \\
&= S_{|m|}(\lambda)A - S_{|m|-1}(\lambda)WA - S_{|m|}(\lambda)B + S_{|m|-1}(\lambda)BW \\
&= S_{|m|}(\lambda)(A - B) - S_{|m|-1}(\lambda)(WA - BW) \\
&= S_{|m|}(\lambda)\begin{bmatrix} 0 & 1 \\ (y-2) & 0 \end{bmatrix} - S_{|m|-1}(\lambda)\begin{bmatrix} 0 & \alpha \\ (y-2)\alpha & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & S_{|m|}(\lambda) - S_{|m|-1}(\lambda)\alpha \\ (y-2)(S_{|m|}(\lambda) - S_{|m|-1}(\lambda)\alpha) & 0 \end{bmatrix} \square
\end{aligned}$$

## 4.5 Roots of the Riley Polynomial and Double-Twist Knots

In light of Theorem 4.2.5, this section is devoted to finding roots $y > 2$ of $\phi_K(x, y)$. We now assume $y \in \mathbb{R}$. Some results of the section hold for any $x \in \mathbb{R}$; some only follow, or follow more easily in the case we take $x = x_n = e^{\pi i/n} + e^{-\pi i/n} = 2\cos(\pi/n)$. Our applications of the lemmas of the subsequent sections only require the statements in the case that $x_n = 2\cos(\pi/n)$.

**Lemma 4.5.1.** *For fixed $x \in \mathbb{R}$, we have that:*

$$\lim_{y\to\infty} \phi_K(x,y) = \begin{cases} \infty & \text{if } m \text{ odd and positive or } m \text{ even and negative} \\ -\infty & \text{if } m \text{ even and positive or } m \text{ odd and negative.} \end{cases}$$

*Proof.* Let $m \geq 1$ and $l, a, s_m$ denote the leading term of $\lambda(x,y), \alpha(x,y)$ and $S_m(y)$ respectively as polynomials in $y$. Then the leading term $p$ of $\phi_K(x,y) = S_{m-1}(\lambda)\alpha - S_{m-2}(\lambda)$ as a function of $y$ is $p = s_{m-1}(l)a$. It is not hard to see that $l = -y^2 s_k s_{k-1}$, $a = y s_k s_{k-1}$ and $s_m = y^m$. Thus the sign of $p$ depends only on the parity of $m$. In particular, the coefficient of $p$ is positive when $m$ is odd and negative when $m$ is even. A similar argument gives that the leading term of $\phi_K(x,y)$ is $-s_{|m|-1}(l)a$ when $m \leq -1$. □

Our goal in this section is to prove Theorem 4.5.5; the proof of the case $m = 2$ differs slightly from the general case. We prove this case first.

**Proposition 4.5.2.** *Let $K = J(2k+1, 4)$. Then $\phi_K(x_n, y)$ has a root $y_n > 2$ for $n \geq 5$.*

*Proof.* Here $m = 2$, so we have that $\phi_K(x,y) = S_1(\lambda)\alpha - S_0(\lambda) = \lambda(x,y)\alpha(x,y) - 1$ by Proposition 4.4.4. By Lemma 4.5.1, given any real $x$, there is a $y_- > 2$ so that $\phi_K(x, y_-) < 0$. For now we assume $n \geq 6$ so that $\sqrt{3} \leq x_n < 2$ to obtain the following inequality:

$$\phi_K(x_n, 2) = (x_n^2-2)(1+(4-x_n^2)k)-1 \geq (x_n^2-2)(1+(4-x_n^2))-1 > (x_n^2-2)-1 = x_n^2-3$$

so long as $k \geq 1$. Since $n \geq 6$ we have $x_n^2 = (2\cos(\pi/n))^2 \geq (2\cos(\pi/6))^2 = 3$. Hence $\phi_K(x_n, 2) > 0$ for $n \geq 6$. By the intermediate value theorem, there must be a root $y_n > 2$.

A direct computation for $n = 5$ shows we can do slightly better. Computing:

$$\begin{aligned}\phi_K(x_5, 2) &= ((2\cos(\pi/5))^2 - 2)(1 + (4 - (2\cos(\pi/5))^2)k) - 1 \\ &\geq ((2\cos(\pi/5))^2 - 2)(1 + (4 - (2\cos(\pi/5))^2)) - 1 \\ &= \left(\frac{-1+\sqrt{5}}{2}\right)\left(\frac{7-\sqrt{5}}{2}\right) - 1 \\ &= \left(\frac{8\sqrt{5}-12}{4}\right) - 1 > 0\end{aligned}$$

gives that the Riley polynomial is positive for $y_+ = 2$. Again we get a root $y_5 > 2$ by the intermediate value theorem. □

**Lemma 4.5.3.** *Let $x_n = 2\cos(\pi/n)$. For $y \geq 2$, we have that $\alpha = \alpha(x_n, y) > 1$ for all $n \geq 2$.*

*Proof.* Recall that $\alpha(x_n, y) = 1+(y+2-x_n^2)S_{k-1}(y)\,(S_k(y) - S_{k-1}(y))$. Lemma 4.3.3(3) gives that $S_{k-1}(y) > 0$ for all $y \geq 2$, and Lemma 4.3.3(4) gives that $S_k(y) - S_{k-1}(y) > 0$ for $y \geq 2$. Finally we have that $-2 < x_n < 2$ and in particular $x_n^2 < 4$ so that $(y + 2 - x_n^2) > 0$ for $y \geq 2$. Thus, $\alpha(x_n, y) - 1 = (y+2-x_n^2)S_{k-1}(y)\,(S_k(y) - S_{k-1}(y)) > 0$ for $y \geq 2$ as it is a product of positive functions. □

**Lemma 4.5.4.** *Fix $x \in \mathbb{R}$. For any $c \leq x^2 - 2$ there exists $y_c \geq 2$ such that $\lambda(x, y_c) = c$.*

*Proof.* Recall that $\lambda(x, y) = x^2 - y - (y - 2)(y + 2 - x^2)S_k(y)S_{k-1}(y)$. As in Lemma 4.5.3, we have that $S_k(y)$, $S_{k-1}(y)$ and $(y + 2 - x^2)$ are positive when

$y \geq 2$. Hence, $\lambda(x, y) - x^2 + y = -(y-2)(y+2-x^2)S_k(y)S_{k-1}(y) \leq 0$ for all $y \geq 2$.

Now $\lambda(x, y) \leq x^2 - y$, so letting $y \to \infty$ we see that $\lambda(x, y)$ tends to $-\infty$ as $y$ grows. We also have that $\lambda(x, 2) = x^2 - 2$. Since $\lambda$ is a continuous function, the lemma follows. □

**Theorem 4.5.5.** *Fix $n \geq 2$, and let $K = J(2k+1, 2m)$. Then there is a presentation of $\pi_1(X_K)$ with Riley polynomial $\phi_K(x, y)$ such that $\phi_K(2\cos(\pi/n), y)$ has a root $y_n > 2$ in the following cases:*

1. *$n \geq 3$ when $m \leq -3$,*
2. *$n \geq 4$ when $m = -2$,*
3. *$n \geq 5$ when $m = 2$,*
4. *$n \geq 4$ when $m = 3$,*
5. *$n \geq 3$ when $m \geq 4$.*

*Proof.* Again we choose the presentation of $\pi_1(X_K)$ as the one given in (4.4). We will argue carefully the case of $m$ positive; the case of $m$ negative is argued similarly. When $m = 2$ the result is proved by Proposition 4.5.2. We now assume that $m \geq 3$. In this case we have that $\phi_K(x_n, y) = S_{m-1}(\lambda)\alpha - S_{m-2}(\lambda)$.

We proceed by noting that by Lemma 4.5.1, there is $y_0 \geq 2$ such that $(-1)^m\phi_K(x_n, y_0) < 0$. Finding another $y_1 \geq 2$, with $(-1)^m\phi_K(x_n, y_1) > 0$, would give us a root larger than 2 by the intermediate value theorem.

Let $(r_1, r_2) = \left(2\cos\left(\frac{(m-1)\pi}{m}\right), 2\cos\left(\frac{(m-2)\pi}{m}\right)\right)$. Lemma 4.3.4 gives that $(-1)^{m-1}S_{m-1}(t) < 0$ on $(r_1, r_2)$. We also have that $c = 2\cos\left(\frac{(m-2)\pi}{m-1}\right)$ is a root of $S_{m-2}(t)$. Note that $c \in (r_1, r_2)$ for $m \geq 3$. By Lemma 4.5.4, if we assume $c \leq x_n^2 - 2$ then there is $y_c \geq 2$ so that $\lambda(x_n, y_c) = c$. Combined with the fact that $\alpha > 1$ by Lemma 4.5.3, we have that

$$\begin{aligned}(-1)^{m-1}\phi_K(x_n, y_c) &= (-1)^{m-1}(S_{m-1}(\lambda(x_n, y_c))\alpha(x_n, y_c) - S_{m-2}(\lambda(x_n, y_c))) \\ &= (-1)^{m-1}(S_{m-1}(c)\alpha - S_{m-2}(c)) \\ &= (-1)^{m-1}S_{m-1}(c)\alpha < 0.\end{aligned}$$

Thus, as long as $c \leq x_n^2 - 2$ we have that $y_c = y_1 \geq 2$ is the value we seek.

Before computing when $c \leq x_n^2 - 2$ holds, we highlight how the case of $m$ negative differs. The argument is similar; the differences come from the fact that in this case $\phi_K(x_n, y) = S_{|m|}(\lambda) - S_{|m|-1}(\lambda)\alpha$. Now, let $(r_1', r_2') = \left(2\cos\left(\frac{|m|\pi}{|m|+1}\right), 2\cos\left(\frac{(|m|-1)\pi}{|m|+1}\right)\right)$ and $c' = 2\cos\left(\frac{(|m|-1)\pi}{|m|}\right)$ and the we obtain a root of $\phi_K(x_n, y)$ so long as $c' \leq x_n^2 - 2$.

To conclude we need only determine when $c \leq x_n^2 - 2$ and when $c' \leq x_n^2 - 2$. Recall that $x_n = 2\cos(\pi/n)$. For $m = 3$ we see that $c = 0 \leq x_n^2 - 2 = 4\cos(\pi/n)^2 - 2$ so long as $n \geq 4$. Similarly if $m \geq 4$ then $c \leq -1 \leq 4\cos(\pi/n)^2 - 2$ for all $n \geq 3$. If $m = -2$ then $c' = 0 \leq x_n^2 - 2$ so long as $n \geq 4$. If $m \leq -3$ then $c' \leq -1 \leq x_n^2 - 2$ for $n \geq 3$. For these $n$, we can conclude that $\phi_K(x, y)$ has a root $y_n \in (2, \infty)$. □

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