The Dissertation Committee for Nicolas Z. Reyes
certifies that this is the approved version of the following dissertation:

The Moduli Space of Objects in Differential Graded Categories Glued along Bimodules and a Presentability Result in the Homotopy Theory of Commutative Differential Graded Algebras

Committee:

Andrew Blumberg, Supervisor

David Ben-Zvi

David Gepner

Sean Keel

The Moduli Space of Objects in Differential Graded Categories Glued along Bimodules and a Presentability Result in the Homotopy Theory of Commutative Differential Graded Algebras

by

Nicolas Z. Reyes

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

August 2019

This thesis is dedicated to the Students of the 2013-2015 Mashpee Wampanoag Native Tribal Scholars Program.

Acknowledgments

I am forever grateful to Andrew Blumberg for his guidance and support during my research process, and for being the best teacher I could hope for when it comes to the daunting subject of Homotopy Theory. I am also grateful to David Ben Zvi, Rok Gregoric, Sam Raskin, and John Berman for many illuminating discussions surrounding this work. Thank you to my family and friends for the love and kindness that fueled this mathematical journey.

The Moduli Space of Objects in Differential Graded Categories Glued along Bimodules and a Presentability Result in the Homotopy Theory of Commutative Differential Graded Algebras

Publication No. ______________

Nicolas Z. Reyes, Ph.D.
The University of Texas at Austin, 2019

Supervisor: Andrew Blumberg

The moduli space of objects of a dg-category, T, is a derived stack introduced in [31] that paramatrizes “pseudo-perfect T^{op}-modules.” This construction extends to a Morita invariant functor, $\mathcal{M}_{-} : Ho(dg{-}cat)^{op} \to Stacks$, which is right adjoint to the functor that assigns to a derived stack it’s dg-category of perfect complexes. In this thesis we are primarily concerned with the behavior of *semi-orthogonal decompositions* of dg categories under this functor. We show that when a dg category, $\mathcal{C}$ has a semi-orthogonal decomposition, $H^0(\mathcal{C}) =< H^0(\mathcal{C}_0), H^0(\mathcal{C}_1) >$, the moduli space of objects in $\mathcal{C}$ can be expressed as a certain pullback of stacks involving the moduli spaces of objects in $\mathcal{C}_0$ and $\mathcal{C}_1$. We also present a result on the cofibrant generation of a certain model category obtained as the total space of the Grothendieck

fibration associated to the “module category” functor mapping a derived ring to its model category of modules.

Table of Contents

Chapter 1

Introduction

1.1 Introduction

The first major result of this thesis centers around the operation of gluing categories. There are several incarnations of this type of gluing in the literature, primarily in the context of fibrations of simplicial sets as in [22], [18], [1], foundational work on dg-categories as in [29], log geometry in [3] and [26], and noncommutative geometry in [25], [7], and [21].

Perhaps the most tractable interpretation of this operation is as generalizing the construction of "lower triangular algebras," in the study of underived associative unital algebras. This classical operation takes as input data two algebras, A and B and an A-B-bimodule, M, and produces the algebra with elements

$$\{\begin{bmatrix} a & 0 \\ m & b \end{bmatrix} \mid a \in A, b \in B, m \in M\},$$

and obvious addition, multiplication, and units. These rings, including their module theory and ideal theory, are studied extensively in [15], [14], and elsewhere. We are, of course, working in the context of categories, but an interesting corollary of our work here is an interpretation of upper and lower triangular 2×2 matrix rings over an algebra, A, as the "algebra" $A \otimes \Delta^1$.

Joyal, who might make the first reference to this type of categorical join, uses the term collage ([18]), whereas in geometry, Orlov, Bondal and others refer to the dg-analogue as gluing because of geometric intuition ([25], [7], [21]). In those works, the authors construct semi-orthogonal decompositions of categories of sheaves on algebro-geometric spaces, implying that those categories can be glued from simpler categories which also arise geometrically. In still another context, that of [28], the term "cograph," is used, and this concept is indeed dual to the construction of graphs of functors in a precise way.

For example, in [1], the authors construct a category of correspondences of ∞-categories in which the join operation essentially produces morphism categories between objects. They show that this category classifies exponentiable fibrations of simplicial sets. Lurie hints at this interpretation frequently in *Higher Topos Theory*, for example in section 2.3.1 of that book, and later when formalizing the notion of adjunctions between ∞-categories.

In fact, this type of categorical join, or more accurately correspondences of ∞-categories, which are equivalent constructions (see section 2.3.1 of [22]) is the key ingredient for upgrading adjunctions to a homotopically meaningful relationship between functors. The idea there, is if that two functors between ∞-categories, $f : \mathscr{C} \to \mathscr{D}$ and $g : \mathscr{D} \to \mathscr{C}$ should be called *adjoint* if they define the same correspondence of ∞-categories.

Recently authors such as Scherotzke, Sibilla, and Taplo, have studied diagrams of glued categories with compatible structure maps and exhibited descriptions of limits of these diagrams as glued categories ([27]). Their results extend previous descent-type results by Bergh, and Schurer in [4], and these authors and others have applied their work to questions about log schemes and K theory ([3], [26]).

The other ingredient in this note are *moduli stacks of objects* in categories. There are several versions of this idea, the first published being Toen and Vaquie's moduli space of objects functor, as defined in [31]. There it is referred to as the "derived moduli stack of pseudo perfect modules." This functor assigns a stack to any dg-category, T, whose (connected components of) geometric points coincide with those perfect $T^{op}-$modules which take values in $k - Perf$, thus the term "pseudo-perfect T^{op}-modules."

Antieu and Gepner generalize this functor to the setting of spectral algebraic geometry in [12]. There they construct a derived stack on the category of $\mathbb{E}_\infty$ rings which parametrizes modules over a linear ∞-category.

Of course, regardless of context, the moduli stack of objects in a category should be a reasonably well behaved algebro-geometric object whose geometric points correspond to objects of that category, and whose homotopy theory reflects equivalences in that category (and equivalences of equivalences, etc.).

This type of moduli space has been studied in the underived setting, for example in [20], but in order to construct well behaved spaces, researchers have typically additional structure on the relevant categories, such as slope stability conditions or the more general Bridgeland stability conditions introduced in [8].

In these cases, which have found applications in physics ([5]), authors such as King have successfully constructed underived schemes parametrizing objects in better behaved subcategories. The broader approach to these moduli problems relevant in this note have the advantage of capturing all objects of the categories we're interested in and no additional categorical data and structure is necessary. The trade off, of course, is that the full machinery of homotopy theory becomes necessary, and we can generally not expect our moduli stacks to satisfy any kind of global compactness criteria. The stacks relevant to this note are always very large.

Nonetheless, derived moduli stacks of objects in homotopical categories have found utility in the literature. This type of stack is used by e.g. Blanc in [6] to define the topological K theory of noncommutative spaces. We will compute the moduli space of objects of the join of categories along a bimodule.

For the next primary result of this thesis we remain in the differential graded setting, but switch gears somewhat, to focus on the 1-category $\mathscr{M}od$, of pairs (A, U), with $A \in cdga_k^{\leq 0}$ a derived ring, and $U \in A - Mod$ an A

module. Arrows, $(A,U) \to (B,V)$, in this category are maps of rings $A \to B$ paired with maps of B-modules $U \otimes_A B \to V$.

This category arises naturally as the total space of the Grothendieck construction of the functor $cdga_k^{\leq 0} \to ModCat$ which maps a derived ring to it's model category of modules ($ModCat$ denotes the (2,1)-category of model categories with Quillen adjunctions and pseudo-natural isomorphisms of Quillen adjunctions). As such, Harpaz and Prasma show, in [16], that $\mathscr{M}od$ carries a natural model structure. We prove this model structure is cofibrantly generated.

1.1.1 Results

Now let $\mathcal{C}$ and $\mathcal{D}$ be dg-categories and let S be a $\mathcal{C}$-$\mathcal{D}$-bimodule. We will say S is *left-handed* if S takes the form $(c,d) \mapsto \mathcal{D}(fc,d)$ for some dg functor, $f : \mathcal{C} \to \mathcal{D}$, and S will be called *right-handed* if $S(c,d) = \mathcal{C}(c,gd)$ for a dg functor $g : \mathcal{D} \to \mathcal{C}$. Then the first primary result of this note is:

Theorem 1. *(Left-hand version) If S is left handed, then $\mathcal{C} \star_S \mathcal{D}$ is Morita equivalent to the homotopy colimit of the diagram:*

$$\begin{array}{ccc} \mathcal{C} \cong \mathcal{C} \otimes \{1\} & \longrightarrow & \mathcal{D} \\ \downarrow & & \\ \mathcal{C} \otimes \Delta^1 & & \end{array},$$

where the horizontal map is f and the vertical map comes from the inclusion $\{1\} \hookrightarrow \Delta^1$.

(Right-hand version) If S is right-handed, then $\mathcal{C} \star_S \mathcal{D}$ is Morita equivalent to the homotopy colimit of the diagram:

$$\begin{array}{ccc} \mathcal{D} \cong \mathcal{D} \otimes \{0\} & \rightarrow & \mathcal{C} \\ \downarrow & & \\ \mathcal{D} \otimes \Delta^1 & & \end{array},$$

where the horizontal map is g and the vertical map comes from the inclusion $\{0\} \hookrightarrow \Delta^1$.

This implies that in, say, the left-handed situation, the Moduli space of objects of $\mathcal{C} \star_S \mathcal{D}$ is given by the pullback stack of:

$$\begin{array}{ccc} & & \mathcal{M}_{\mathcal{C} \otimes \Delta^1} \\ & & \downarrow \\ \mathcal{M}_{\mathcal{D}} & \longrightarrow & \mathcal{M}_{\mathcal{C}} \end{array}.$$

We prove Theorem 1 in a decidedly unhomotopical manner, starting by using the Yoneda lemma to establish an equivalence between $\mathcal{C} \star_S \mathcal{D}$ and (the pre-triangulated hull of) an underived pushout. We then rely on a technical lemma of Holstein from [17] to show that this is a homotopy pushout under reasonable assumptions, and we then lift those assumptions by digging into the explicit construction of $\mathcal{C} \star_S \mathcal{D}$. However, we will also sketch an alternative proof of Theorem 1 which is entirely internal to homotopy theory in section 5.3.

The other primary theorem in this note regards the category $\mathscr{M}od$, defined in the previous subsection. We prove:

Theorem 2. *$\mathscr{M}od$ is cofibrantly generated.*

We prove this by explicitly listing the generating (acyclic) cofibrations, and the list will likely not surprise anybody familiar with differential graded algebras and their modules. We point out, however, that cofibrant generation does not seem to be guaranteed by any general theory (such as the results in [13]) and we rely critically on the well-behaved nature of tensor products to prove this result.

1.1.2 Applications

Theorem 1 has several applications in derived algebraic geometry, most of which are related to semi-orthogonal decompositions of categories of sheaves on geometric objects and will be presented in more detail in section 6 below. These include a description of the moduli stack of (derived) perfect sheaves on the projectivization of a finite dimensional vector bundle over an underived scheme, X, in terms of the moduli of perfect sheaves on X. Another example appears when studying the derived category of a regular scheme, X, blown up along a codimension 2 subscheme, Y. We show how to express this category (resp. its moduli of objects) as a pushout (resp. pullback) involving the derived categories of X and Y (resp. their moduli of objects).

1.1.3 Organization

The bulk of this article is devoted to Theorem 1 and it's application to moduli stacks of sheaves on geometric objects. After establishing the notational conventions that will pervade the paper in section 2, we devote section 3 to the model category theory of differential graded categories and Toen and Vaquie's construction of the moduli stack of pseudo-perfect modules. Most, but not all, of this material will be necessary in the proof of Theorem 1, and the rest is meant to put the theorem in context and help the reader apply the result to examples.

After that, in section 4, we quickly review the material of Harpaz and Prasma's [16], which establishes the model structure on $\mathscr{M}od$ that we will prove is cofibrantly generated. From there we move on to the proof of our two main results in section 5, devoting several subsections to the proof of Theorem 1. Section 6 provides a brief remark on the linear ∞-categorical analogue of theorem 1, and we conclude the paper with some examples of glued categories and apply Theorem 1 to those examples.

Chapter 2

Preliminaries

2.1 Notation and Conventions

Throughout this note we will work over a characteristic 0 field, k. We assume k is a field simply so that all dg-categories are h-projective in the sense of e.g. [21], and this assumption may likely be relaxed. The characteristic zero assumption is made to accommodate the symmetric monoidal Dold-Kan correspondence between simplicial commutative k-algebras and negatively graded commutative dg-algebras, however this assumption may be relaxed often, as it is only relevant in the sections pertaining to stacks.

All dg-categories will be cohomologically graded and we denote the dg-category of k-cochain complexes by $\underline{C(k)}$ and the 1-category of k-dg-categories with it's Dwyer-Kan, or "quasi-equivalence," model structure (see the background section below) by dg-cat. More accurately, we presume a pair of large universes, $\mathbb{U} \subset \mathbb{V}$, have been picked, and dg-cat will refer to $\mathbb{U}$-small dg categories.

The notation $\hat{\mathcal{C}}_{pe}$, for a dg-category, $\mathcal{C}$, will denote the category of perfect $\mathcal{C}^{op}$ modules, in agreement with Toen's notation (cf. [34]). Here "module" means a functor into $C(k)$.

The term $\mathcal{C}$-$\mathcal{D}$-module will refer to a dg-functor from $\mathcal{C}^{op} \otimes \mathcal{D}$ to $\underline{C(k)}$.

There are several types of morphism objects involved in this theory. We choose the notation $\mathcal{C}(-,-)$ for cochain complexes of morphisms between objects of a dg-category, $\mathcal{C}$, and use the notation $hom(-,-)$ when referring to the underived, $\mathbb{V}$-small morphism sets in the 1-category of dg-categories. The proof of our first main result in section 5 will involve a Yoneda morphism into the category of $\mathbb{V}$-small sets, and so it is critical that we haved picked universes, but we will otherwise not need to refer to set theory. $\underline{\mathbb{R}Hom}(-,-)$ will denote the dg-category valued derived mapping categories constructed in [34] and reconstructed in [9]. There is also an underived internal mapping category, $\underline{Hom}$, defined first by Keller in [19], and this functor makes the underived, "Hom-wise," tensor product into a closed symmetric monoidal product on dg-cat.

We will use the symbol Δ^1 to denote the dg category with two objects, 0 and 1, with $End(0) = End(1) = \Delta^1(0,1) = k[0]$ and $\Delta^1(1,0) = 0$. This dg-category accepts inclusions from two obvious full subcategories which we will denote 0 and 1.

We will sometimes refer to the composition in dg-categories as multiplication, and we will write it right to left, as one does when composing functions of sets.

2.2 Background on dg categories

2.2.1 dg-categories, the pretriangulated condition, and two model structures

The two model structures on the category of dg categories we are interested in are the Dwyer Kan model structure and the Morita model structure.

In the Dwyer Kan model structure, weak equivalences are the *quasiequivalences*, i.e. those dg-functors which induce quasi-isomorphisms on morphism complexes and are essentially surjective on homotopy categories. A fibration, $P : \mathcal{C} \to \mathcal{D}$, in this model structure is defined to be a dg functor which is degree-wise surjective on morphism complexes and has the additional property that for any isomorphism, $u' : x' \to y'$, in the homotopy category, $H^0(\mathcal{D})$, of $\mathcal{D}$, and any object, y, in $H^0(\mathcal{C})$ such that $P(y) = y'$, there is an isomorphism, $u : x \to y$, in $H^0(\mathcal{C})$ satisfying $H^0(P)(u) = u'$.

The Morita model structure on the category of dg-categories is the left Bousfield localization of the Dwyer-Kan model structure with respect to the *Morita equivalences*.

To define Morita equivalences, we begin by defining the $C(k)$-enriched category of modules over a dg-category, $\mathcal{C}$.

Definition 1. *$\hat{\mathcal{C}}$ is the category with dg-functors $\hat{\mathcal{C}} \to \underline{C(k)}$ as objects and closed, degree zero natural transformations as morphisms.*

$\hat{\mathcal{C}}$ carries a natural dg enhancement which we will abusively denote by the same symbol. Morphisms complexes in this category are given by the (enriched) *end* construction (c.f. e.g. [11]):

$$\hat{\mathcal{C}}(F, F') = \int_{\mathcal{C}} \underline{C(k)}(F(-), F'(-))$$

These categories also carry a combinatorial model structure with weak equivalences and fibrations defined pointwise in $C(k)$. (For more on $C(k)$ enriched model category theory, see e.g. [34]. In this model category, the representable functors are cofibrant, (trivially) fibrant, and compact objects.

Definition 2. *The category of* perfect $\mathcal{C}$ modules, $\hat{\mathcal{C}}_{pe}$, *is the smallest full dg subcategory of* $\hat{\mathcal{C}}$ *consisting of fibrant-cofibrant objects with the homotopy type of retracts of finite cell objects.*

Definition 3. *A morphism,* $\mathcal{C} \to \mathcal{D}$, *of dg categories is a* Morita Equivalence *if it induces a Dwyer-Kan equivalence,* $\hat{\mathcal{C}}_{pe} \to \hat{\mathcal{D}}_{pe}$ *on dg categories of perfect modules.*

Definition 4. $\mathcal{C}$ *is called* pretriangulated *if*

1. *There is an object,* $0 \in \mathcal{C}$, *such that* $\mathcal{C}(-, 0)$ *is weakly equivalent to the constant functor, 0.*

2. *Whenever a module,* M *has a pointwise degree -1 shift,* ΣM, *which is weakly equivalent to a representable* $\mathcal{C}$ *module,* M *itself is weakly equivalent to a representable* $\mathcal{C}$ *module.*

3. *Any map, $M \to N$ between $\mathcal{C}$ modules which are weakly equivalent to representables has a cofiber which is weakly equivalent to a representable $\mathcal{C}$ module.*

It turns out the pretriangulated dg categories are the fibrant objects in the Morita model structure and the dg Yoneda functors $\mathcal{T} \to \hat{\mathcal{T}}_{pe}$ are fibrant replacements. [[*citation: forget which tabuada article proves this]]

2.2.2 Mapping spaces, symmetric monoidal structure and internal Homs

We will quickly review a few of the essential points in [34], beginning with Definition 4.1 from that article.

Definition 5. *A $(\mathcal{T}, \mathcal{S})$-bimodule, $F : \mathcal{T}^{op} \otimes \mathcal{S} \to \underline{C(k)}$, is* right quasirepresentable *if for each $t \in \mathcal{T}$, the dg functor $F(t, -) : \mathcal{S} \to \underline{C(k)}$ it is weakly equivalent to a representable one.*

In the case that a dg category, $\mathcal{C}$, is cofibrant and $\mathcal{D}$ is arbitrary, Toen computes the simplicial mapping space $Map(\mathcal{C}, \mathcal{D})$ in $dg-cat$ in terms of these bimodules.

Fact 1. *(Theorem 4.2 in [34]) There is a weak equivalence of simplicial sets*

$$Map(\mathcal{C}, \mathcal{D}) \simeq N(M(\mathcal{C}, \mathcal{D}))$$

where $M(\mathcal{C}, \mathcal{D})$ is the 1-category with objects the right quasirepresentable $\mathcal{C}-\mathcal{D}$-bimodules, F, such that $F(x, -)$ is a cofibrant D^{op}-module for each $x \in \mathcal{C}$, and *morphisms the pointwise quasiequivalences of bimodules. N denotes the nerve functor.*

We can also derive the naive tensor product on dg categories and we record Theorem 6.1 from [34]:

Fact 2. *$(Ho(dg - cat), \otimes^{\mathbb{L}})$ forms a closed, $Ho(SSet)$-enriched, symmetric monoidal category, and for any two dg categories, $\mathcal{C}$ and $\mathcal{D}$, we may identify the internal mapping category, $\mathbb{R}\underline{Hom}(\mathcal{C}, \mathcal{D})$, with the full dg subcategory of cofibrant and right quasirepresentable $\mathcal{C} - \mathcal{D}$-bimodules.*

2.2.3 Gluing dg-categories

For the purposes of this note we choose to use the gluing of dg-categories defined in [25] as opposed to the definitions from, say, [21] or [29]. The so-called upper triangular dg-categories of [29] are defined by passing through a different category than dg-cat (namely the category of upper triangular dg-categories), and for the present work we select a construction internal to dg-cat for simplicity.

Meanwhile, the glued categories of [21] have a more complicated description than those of [25] with the upshot of remaining pre-triangulated when the input categories are. However. as Kuzntsov and Luntz point out in [21], the gluings of [21] and [25] have quasi-isomorphic pre-triangulated hulls and categories of perfect modules. As we are presently concerned with the Morita invariant moduli space of objects functor, we choose to use Orlov's construction for simplicity, and present this now.

Let $\mathcal{C}$ and $\mathcal{D}$ be dg-categories, and let S be a $\mathcal{C}$-$\mathcal{D}$-bimodule. We first introduce a dg category, $\mathcal{C}\tilde{\star}_S\mathcal{D}$, with objects $\mathrm{Ob}(\mathcal{C}) \coprod \mathrm{Ob}(\mathcal{D})$ and morphism complexes given below:

$$\mathcal{C}\tilde{\star}_S\mathcal{D}(x,y) := \begin{cases} \mathcal{C}(x,y) & x,y \in \mathcal{C} \\ \mathcal{D}(x,y) & x,y \in \mathcal{D} \\ S(x,y) & x \in \mathcal{C}, y \in \mathcal{D} \\ 0 & y \in \mathcal{C}, x \in \mathcal{D} \end{cases},$$

Now set $\mathcal{C} \star_S \mathcal{D} := \mathcal{C}\tilde{\star}_S\mathcal{D}_{pre-tr}$, the pre-triangulated hull of $\mathcal{C}\tilde{\star}_S\mathcal{D}$.

2.2.4 D^--stacks

Toen and Vezzosi introduced D^--stacks in [33] and the interested reader should refer to that document and it's sequel, [32], for a thorough treatment of the subject, but we will review basic aspects of the theory in this subsection.

Objects of the category, D^-Stk, of D^--stacks are, by definition, functors from the 1-category, $sk-CAlg$, of simplicial commutative k-algebras to the 1-category of simplicial sets. The minus sign in the terminology "D^--stack" comes from Dold and Kan's correspondence between simplicial commutative algebras and negatively graded commutative differential graded algebras (in characteristic 0).

The category, D^-Stk, is the homotopy category of the model category of simplicial presheaves Bousfield localized along Etale local equivalences.

2.2.5 Moduli of Objects

For a dg-category, $\mathcal{C}$, the *moduli space of objects*, $\mathcal{M}_{\mathcal{C}}$, of $\mathcal{C}$ is the derived stack given by:

$$A \mapsto Map(\mathcal{C}^{op}, \hat{A}_{pe})$$

for $A \in sk - Calg$. Here the mapping space is the simplicial mapping space of the model category dg-cat (with it's quasi-isomorphism model structure).

In particular, if we let $\mathbf{1}$ denote the dg-category with one object with endomorphism algebra k[0], then $\mathcal{M}_{\mathbf{1}}$ coincides with the stack $\mathbb{R}Perf$.

Toen and Vaquie show, in [31], that we have a pair of Ho(sSet) enriched adjoint functors:

$$L_{pe} : D^{-}Stk \rightleftarrows Ho(dg - cat)^{op} : \mathcal{M}_{-}$$

where the left category is Toen and Vezzozi's category of D^{-}-stacks and the functor L_{pe}, left adjoint to $\mathcal{M}_{-}$ assigns to a stack, T, the dg category of perfect complexes on T (this may be taken to be the definition of perfect complexes on T).

2.2.6 Tensor Products of dg-(Bi)Modules

The tensor product of dg-modules is a coend construction as in the usual tensor products of functors, though the linearity of dg categories accommodates a simpler formula recorded in e.g. [21] though likely due to Keller.

To start, let $\mathcal{C}$ be a dg-category, let M be a dg-$\mathcal{C}$-module, and let N be a dg-$\mathcal{C}^{op}$-module. We start by defining an element of $C(k)$, to be called $M \otimes_{\mathcal{C}} N$:

$$M \otimes_{\mathcal{C}} N := coker\Big(\Xi : \bigoplus_{A,B \in \mathcal{C}} M(B) \otimes_k \mathcal{C}(A,B) \otimes_k N(A) \to \bigoplus_{C \in \mathcal{C}} M(C) \otimes_k N(C)\Big), \tag{2.1}$$

where Ξ is defined as follows. If we let $v \in M(B)$ be homogeneous of degree, m, $h : A \to B$ homogeneous of degree n, and $u \in N(A)$, set

$$\Xi(v \otimes h \otimes u) := M(h)(v) \otimes u - (-1)^{mn} v \otimes N(h)(u) \in M(A) \otimes_k N(A) \oplus M(B) \otimes_k N(B). \tag{2.2}$$

Given three dg categories, $\mathcal{A}$, $\mathcal{C}$, and $\mathcal{D}$, the above definition generalizes naturally (by replacing chain complexes with functors valued in chain complexes) to the situation $M \in \mathcal{A} \otimes \mathcal{C} - Mod$ and $N \in \mathcal{C}^o \otimes \mathcal{D}$, in which case $M \otimes_{\mathcal{C}} N$ will belong to $\mathcal{A} \otimes \mathcal{D} - Mod$. This construction is dg-functorial in each variable, and of course it may be (left) derived as well.

2.3 Background on the model categorical Grothendieck construction

Our second primary result will rely on the foundational material in [16] on the model categorical Grothendieck construction.

Now let $ModCat$ denote (2,1)-category of model categories with Quillen adjunctions as arrows and pseudo-natural isomorphisms of Quillen adjunctions as 2-morphisms. We will not discuss this category in more detail here, as we will only be applying the results of [16] to a single example in this work, and when we come to that example we will be explicit about what those results imply.

If we let M denote a model category and are given a functor of underlying 1-categories:

$$\mathcal{F} : M \to ModCat,$$

we'll adopt the notation

$$h_! \dashv h^* : \mathcal{F}(A) \rightleftarrows \mathcal{F}(B)$$

for the adjunction $\mathcal{F}$ associates to a morphism, $h : A \to B$, in M. With that in mind, we define a category, $\int_M \mathcal{F}$, in the usual manner by letting objects be pairs (A, U) with $A \in Ob(M)$ and $U \in Ob(\mathcal{F}(A))$ and morphisms $(A, U) \to (B, V)$ be pairs (h, φ) with $h : A \to B$ an arrow in M and $\varphi : h_!U \to V$ an arrow in $\mathcal{F}(B)$.

Additionally, let Q symbolize (not necessarily functorial) cofibrant replacements in M. We now recall Harpaz an Prasma's Definition 3.0.4:

Definition 6. *(Integral model structure) A morphism,* $(h, \varphi) : (A, U) \to (B, V)$*, in* $\int_M \mathcal{F}$ *is*

- *a* weak equivalence *if h is a weak equivalence in M and the composite $h_!(QU) \to h_!(U) \xrightarrow{\varphi} V$ is a weak equivalence in $\mathcal{F}$;*

- *a* cofibration *if h is a cofibration in M and φ is a cofibration in $\mathcal{F}(B)$.*

In order for this to be a model structure, we need an additional pair of assumptions on $\mathcal{F}$ which we define below:

Definition 7. *(3.0.6. in [16]) $\mathcal{F}$ is* relative *if for every weak equivalence, $h : A \to B$, in M, the associated Quillen pair $h_! \dashv h^*$ is a Quillen equivalence.*

Definition 8. *(3.0.9. in [16]) $\mathcal{F}$ is*

- left proper *if whenever $h : A \to B$ is an acyclic cofibration in M, then $h_!$ preserves weak equivalences;*

- right proper *if whenever $h : A \to B$ is an acyclic fibration in M, then h^* preserves weak equivalences;*

- proper *if it is both right and left proper.*

With these in hand we may state Theorem 3.0.12. of [16]:

Fact 3. *If $\mathcal{F}$ is relative and proper, then $\int_M \mathcal{F}$ is, in fact, a model category with its integral model structure.*

We'd like to point out also that in this case a morphism $(h, \varphi) : (A, U) \to (B, V)$ in $\int_M \mathcal{F}$ will be a fibration if and only if h is a fibration in M and the map $U \to h^*V$ right adjoint to φ is a fibration in $\mathcal{F}(A)$. This fact may seem surprising at first, but it follows immediately from the definition of the integral model structure.

The last relevant piece of information we'd like to invoke from [16] follows from their Example 6.4:

Fact 4. *The natural functor* $cdga_k^{\leq 0} \to ModCat$ *associating a negatively graded commutative differential graded algebra to its category of modules with its projective model structure is relative and proper.*

Chapter 3

Results and Applications

3.1 Main Results

3.1.1 The Main Theorems

Throughout this section, let $\mathcal{C}$ and $\mathcal{D}$ be dg-categories and let S be the $\mathcal{C}$-$\mathcal{D}$-bimodule defined by

$$(c, d) \mapsto \mathcal{D}(fc, d)$$

for some dg-functor, $f : \mathcal{C} \to \mathcal{D}$. Let $\mathcal{A}$ be another dg-category.

Our first main lemma is similar to proposition 7.7 (ii) in [21].

Lemma 1. *Let $E(\mathcal{C}, \mathcal{D}, f, \mathcal{A})$ denote the set of triples, (F, G, η), where $F : \mathcal{C} \to \mathcal{A}$ and $G : \mathcal{D} \to \mathcal{A}$ are dg-functors and $\eta : F \to G \circ f$ is a closed, degree zero natural transformation. Then there is a bijection of sets,*

$$hom(\mathcal{C} \tilde{\star}_S \mathcal{D}, \mathcal{A}) \leftrightarrow E(\mathcal{C}, \mathcal{D}, f, \mathcal{A}). \tag{3.1}$$

Proof. First, let us define a map $\Phi : E(\mathcal{C}, \mathcal{D}, f, \mathcal{A}) \to hom(\mathcal{C} \tilde{\star}_S \mathcal{D}, \mathcal{A})$. Given $H = (F, G, \eta) \in E(\mathcal{C}, \mathcal{D}, f, \mathcal{A})$, the dg-functor, $\Phi(H)$ will act on objects and morphisms of $\mathcal{C}$ by F and similarly for $\mathcal{D}$ and G. If we let c and d be objects of $\mathcal{C}$ and $\mathcal{D}$ respectively, then the morphism of cochain complexes,

$$\Phi(H)_{c,d} : \mathcal{C}\tilde{\star}_S\mathcal{D}(c,d) = \mathcal{D}(fc,d) \to \mathcal{A}(Fc,Gd),$$

is defined by applying G then multiplying by η_c on the right (i.e. precomposing):

$$(t : fc \to d) \mapsto (Gt : Gfc \to Gd) \mapsto \Phi(H)_{c,d}(t) := (Gt \cdot \eta_c : Fc \to Gd).$$

It is not immediately clear that $\Phi(H)$ is a well-defined functor of dg-categories, but this is a straightforward check. The function, $\Psi : hom(\mathcal{C}\tilde{\star}_S\mathcal{D}, \mathcal{A}) \to E(\mathcal{C}, \mathcal{D}, f, \mathcal{A})$, which we will prove is inverse to Φ, is defined on a dg-functor, $J : \mathcal{C}\tilde{\star}_S\mathcal{D} \to \mathcal{A}$, by $\Psi(J) := (F', G', \eta')$ where F' and G' are the restrictions of J to $\mathcal{C}$ and $\mathcal{D}$ respectively, and $\eta'_c := J_{c,fc}(e_c)$ where for the remainder of this proof, e_c denotes the image of $1 \in k$ under the unit morphism $k \to \mathcal{D}(fc, fc) = \mathcal{C}\tilde{\star}_S\mathcal{D}(c, fc)$.

Again, it is a long but easy check that Ψ is well defined. We now wish to show Ψ and Φ are mutual inverses.

First, let $H = (F, G, \eta) \in E(\mathcal{C}, \mathcal{D}, f, \mathcal{A})$. The nontrivial aspect to checking $\Psi(\Phi(H)) = H$ is in proving agreement of the natural transformations "$\Psi(\Phi(\eta))$" and η, i.e. that the third component of $\Psi(\Phi(H))$ is η.

So let $c \in \mathcal{C}$ be an object, so $d = fc \in \mathcal{D}$. Applying Ψ to $\Phi(H)$ yields the natural transformation whose component at c is $\Phi(H)_{c,fc}(e_c)$. By construction, $\Phi(H)$ acts on $\mathcal{C}\tilde{\star}_S\mathcal{D}(c,d) = \mathcal{D}(fc, fc)$ by first applying G then precomposing with the degree zero cochain, η_c. But, as G is an (enriched) functor, $G(e_c) = unit_{Gfc}(1)$, the identity cochain of $G(fc)$. Therefore when we precompose this with η_c, η_c itself is returned. This proves Ψ is a left inverse of Φ.

Now let $J : \mathcal{C}\tilde{\star}_S\mathcal{D} \to \mathcal{A}$ be a dg-functor which restricts to functors F and G under the respective inclusions of full subcategories $\mathcal{C} \hookrightarrow \mathcal{C}\tilde{\star}_S\mathcal{D}$ and $\mathcal{D} \hookrightarrow \mathcal{C}\tilde{\star}_S\mathcal{D}$. $\Psi(J) = (F, G, \epsilon)$ for some closed degree zero natural transformation, $\epsilon : F \to G$. We want to understand ϵ in order to compute $\Phi(\Psi(J))$.

For $c \in \mathcal{C}$, we know $\epsilon_c = J_{c,fc}(e_c)$. Therefore, $\Phi(\Psi(J))$ acts on $\mathcal{C}\tilde{\star}_S\mathcal{D}(c, d)$ by first applying G to $\mathcal{D}(fc, d)$ then right multiplying by $J_{c,fc}(e_c)$.

Now, let t be an element of the $\mathbb{Z}$-graded vector space, $\mathcal{D}(fc, d)$ (which of course coincides with $\mathcal{C}\tilde{\star}_S\mathcal{D}(c, d)$, however the function Φ does not "see" this structure). We wish to compute $Gt \cdot J_{c,fc}(e_c)$. But by the the functoriality of J, and the fact that J agrees with G when restricted to $\mathcal{D}$, this is just

$$J_{fc,d}(t) \cdot J_{c,fc}(e_c) = J_{c,d}(t \cdot e_c) = J_{c,d}(t \cdot unit_{fc}(1)) = J_{c,d}(t),$$

and so we have recovered the functor J. This completes the proof of the lemma.

□

Now consider the dg-category, $\mathcal{B}$, defined to be the pushout in dg-cat of the diagram:

$$\begin{array}{ccc} \mathcal{C} \otimes \{1\} & \longrightarrow & \mathcal{D} \\ \downarrow & & \\ \mathcal{C} \otimes \Delta^1 & & \end{array},$$

where the horizontal arrow is f and the vertical arrow comes from the inclusion $\{1\} \hookrightarrow \Delta^1$.

$\mathcal{C}\tilde{\star}_S\mathcal{D}$ fits into a cocone under this diagram in a natural way. On one hand, there is an inclusion, $\mathcal{D} \hookrightarrow \mathcal{C}\tilde{\star}_S\mathcal{D}$. Via this inclusion, as well as $\mathcal{C} \hookrightarrow \mathcal{C}\tilde{\star}_S\mathcal{D}$, we will sometimes identify $\mathcal{C}$ and $\mathcal{D}$ with their images in $\mathcal{C}\tilde{\star}_S\mathcal{D}$. There is also a dg functor, $Y : \mathcal{C} \otimes \Delta^1 \to \mathcal{C}\tilde{\star}_S\mathcal{D}$ defined as follows. We use the notation c_i for $(c, i) \in \mathcal{C} \otimes \Delta^1$, $i = 0, 1$. We set $Y(c_1) = c \in \mathcal{C}\tilde{\star}_S\mathcal{D}$ and $Y(c_2) = fc \in \mathcal{C}\tilde{\star}_S\mathcal{D}$. The dg functor acts by the identity on $\mathcal{C} \otimes \Delta^1(c_0, c_0')$, and by f on $\mathcal{C} \otimes \Delta^1(c_1, c_1')$. On $\mathcal{C} \otimes \Delta^1(c_0, c_1') = \mathcal{C}(c, c')$, Y acts by f as a morphism of cochain complexes into $\mathcal{C}\tilde{\star}_S\mathcal{D}(Y(c_0), Y(c_1')) = \mathcal{C}\tilde{\star}_S\mathcal{D}(c, fc') = \mathcal{D}(fc, fc')$. It remains to check Y is well-defined, but this is straightforward. Also straightforward is a check that Y and $\mathcal{D} \to \mathcal{C}\tilde{\star}_S\mathcal{D}$ assemble into a cocone under our span with apex $\mathcal{C}\tilde{\star}_S\mathcal{D}$. As such, we get a morphism $Z : \mathcal{B} \to \mathcal{C}\tilde{\star}_S\mathcal{D}$.

With $\mathcal{A}$ as above, our first concern is understanding the pullback of Y under the functor $hom(-, \mathcal{A}) : dg-cat \to Set$. We will use the notation $[\epsilon : M_0 \to M_1]$ to denote elements of the set $hom(\Delta^1, \underline{Hom}(\mathcal{C}, \mathcal{A}))$, which paramatrizes ordered pairs, M_0, M_1 of dg-functors, $\mathcal{C} \to \mathcal{A}$, together with a closed degree zero natural transformation, ϵ, between them.

Lemma 2. *The function of sets,*

$$Y^* : hom(\mathcal{C}\tilde{\star}_S\mathcal{D}, \mathcal{A}) \to hom(\mathcal{C} \otimes \Delta^1, \mathcal{A}) = hom(\Delta^1, \underline{Hom}(\mathcal{C}, \mathcal{A})),$$

equals the function Ψ defined in the proof of Lemma 1, composed with the map

$$\chi : E(\mathcal{C}, \mathcal{D}, f, \mathcal{A}) \to hom(\Delta^1, \underline{Hom}(\mathcal{C}, \mathcal{A})),$$

$$(F, G, \eta) \mapsto [\eta : F \to G \circ f].$$

Proof. Let $J \in hom(\mathcal{C}\tilde{\star}_S\mathcal{D}, \mathcal{A})$ be a dg-functor, and consider the composite $J' = J \circ Y$. We would like to show $J' = \chi(\Psi(J))$. First, we compute J' on objects, $c_i = (c, i)$ of $\mathcal{C}\otimes\Delta^1$. For $i = 0$, Y maps c_0 to itself as an object of $\mathcal{C}\tilde{\star}_S\mathcal{D}$, and so J' will indeed carry c_0 to the image of c under the first "component" of $\Psi(J)$. Meanwhile, when $i = 1$, Y acts by f (followed by $\mathcal{D} \to \mathcal{C}\tilde{\star}_S\mathcal{D}$) on c_1, and then J acts by the second component of $\Psi(J)$, which we will call G. Hence J' sends c_1 to $G(f(c))$ as desired.

Next, let us calculate J' on morphism complexes. The nontrivial case to consider is how J' acts on $\mathcal{C} \otimes \Delta^1(c_0, c'_1) = \mathcal{C}(c, c')$, and we are particularly interested in the case when $c' = c$, as this will compute the natural transformation associated with J' when viewed as an element of $hom(\Delta^1, \underline{Hom}(\mathcal{C}, \mathcal{A}))$. By definition, Y acts on this cochain complex by f. Applying J afterwards, and tracing through the definition of Ψ, we see that the natural transformation associated with $J \circ Y$ is exactly the third component of $\Psi(J)$.

□

The main theorem of this work is:

Theorem 3. *$Z : \mathcal{B} \to \mathcal{C}\tilde{\star}_S\mathcal{D}$ is an equivalence.*

We emphasize that this a purely classical 1-categorical result. Upgrading our colimit to a homotopy colimit will be the focus of the next subsection.

We'd also like to remark that this theorem (and it's "adjoint" below) is a linear version of Example 1.8 in [24]. There, the analogous statement is not proven, however a proof can be extracted from the proof of 5.2.1.3, part (1), in *Higher Topos Theory*.

The argument contained there relies critically on the extensive machinery of ∞-categories developed in that book. While it may be possible to translate that argument into our context, we have not done that in the proof of our theorem below.

Proof. We will prove that Z is an equivalence using the Yoneda Lemma. So recall that $\mathcal{A}$ denotes an arbitrary dg-category in this section and consider the pullback of sets:

$$\begin{array}{ccc} hom(\mathcal{B},\mathcal{A}) & \longrightarrow & hom(\mathcal{D},\mathcal{A}) \\ \downarrow & & \downarrow \\ hom(\mathcal{C}\otimes\Delta^1,\mathcal{A}) & \longrightarrow & hom(\mathcal{C},\mathcal{A}) \end{array} .$$

Our first observation is that there is a canonical isomorphism,

$$hom(\mathcal{C}\otimes\Delta^1,\mathcal{A}) \cong hom(\Delta^1, \underline{Hom}(\mathcal{C},\mathcal{A})),$$

and we continue denoting elements of the latter set by $[\epsilon : M_0 \to M_1]$.

The bottom horizontal arrow maps $[\epsilon : M_0 \to M_1]$ to M_1 and the left vertical arrow maps $G : \mathcal{D} \to \mathcal{A}$ to $G \circ f$.

As pullbacks of sets are well understood, we conclude immediately that the pullback of

$$\begin{array}{ccc} & & hom(\mathcal{D}, \mathcal{A}) \\ & & \downarrow \\ hom(\mathcal{C} \otimes \Delta^1, \mathcal{A}) & \longrightarrow & hom(\mathcal{C}, \mathcal{A}) \end{array} \quad . \tag{3.2}$$

is in bijection with pairs of functors, $F : \mathcal{C} \to \mathcal{A}$ and $G : \mathcal{D} \to \mathcal{A}$, together with a closed, degree zero $\eta : F \to G \circ f$.

On the other hand, we computed $Y^* : hom(\mathcal{C}\tilde{\star}_S\mathcal{D}, \mathcal{A}) \to hom(\Delta^1, \underline{Hom}(\mathcal{C}, \mathcal{A}))$ in the previous lemma, and it follows immediately from that computation and the preceding observation that the natural cone over equation 4 with apex $hom(\mathcal{C}\tilde{\star}_S\mathcal{D}, \mathcal{A}) = \{\eta : F \to G \circ f\}$ is, in fact, a limit diagram. And, of course, by the definition of Z, the induced morphism into the pullback coincides with Z^*. It follows that $Z^* : hom(\mathcal{C}\tilde{\star}_S\mathcal{D}, \mathcal{A}) \to hom(\mathcal{B}, \mathcal{A})$ is an equivalence in *Set* for all dg categories, $\mathcal{A}$, so the natural transformation $Z^* : hom(\mathcal{C}\tilde{\star}_S\mathcal{D}, -) \to hom(\mathcal{B}, -)$ is an isomorphism of functors, and we conclude that Z is an isomorphism of dg categories.

□

Of course, an isomorphism of dg categories induces an equivalence of categories of perfect modules, and so we have an immediate corollary:

Corollary 1. *$\mathcal{B}$ is Morita equivalent to $\mathcal{C} \star_S \mathcal{D}$.*

We also provide "adjoints" to the preceding Lemma and Theorem, the proofs of which are nearly identical to their counterparts'.

Lemma 3. *Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{A}$ be as above, and let $g : \mathcal{D} \to \mathcal{C}$ be a dg-functor and let R be the $\mathcal{C}$-$\mathcal{D}$-bimodule $\mathcal{C}(-, g(-))$. Let $L(\mathcal{C}, \mathcal{D}, g, \mathcal{A})$ denote the set of triples, (F, G, ϵ) where $F : \mathcal{C} \to \mathcal{A}$ and $G : \mathcal{D} \to \mathcal{A}$ are dg-functors and $\epsilon : F \circ g \to G$ is a closed, degree zero natural transformation. Then there is a bijection of sets:*

$$hom(\mathcal{C} \tilde{\star}_R \mathcal{D}, \mathcal{A}) \leftrightarrow L(\mathcal{C}, \mathcal{D}, g, \mathcal{A}).$$

Theorem 4. *With the same notation as the above lemma, if we let $\mathcal{E}$ denote the pushout of the span*

$$\begin{array}{ccc} \mathcal{D} \otimes \{0\} & \longrightarrow & \mathcal{C} \\ \downarrow & & \\ \mathcal{D} \otimes \Delta^1 & & \end{array},$$

where the horizontal arrow is g, and the vertical is induced by $\{0\} \hookrightarrow \mathcal{D} \otimes \Delta^1$, then $\mathcal{E}$ is equivalent to $\mathcal{C} \tilde{\star}_R \mathcal{D}$.

3.1.2 Homotopy Coherence

There is a technical lemma of Holstein which ensures the pushouts of theorem 1 (resp. theorem 2) are, in fact, homotopy pushouts under the additional assumption that $\mathcal{C}$ (resp. $\mathcal{D}$) is cofibrant and f (resp. g) is a cofibration. Below we record Proposition 2.5 from [17].

Lemma 4. *(Proposition 2.5 of [17]) The category dg-cat with both it's Dwyer-Kan and it's Morita model structure is left proper.*

Corollary 2. *In the setting of Theorem 1 (resp. Theorem 2) if $\mathcal{C}$ is cofibrant and f is a cofibration (resp. $\mathcal{D}$ is cofibrant and g is a cofibration) then $\mathcal{C}\tilde{\star}_S\mathcal{D}$ is the homotopy pushout of the span $\mathcal{C}\otimes\Delta^1 \leftarrow \mathcal{C} \rightarrow \mathcal{D}$ (resp. $\mathcal{D}\otimes\Delta^1 \leftarrow \mathcal{D} \rightarrow \mathcal{C}$).*

However we may, in fact, lift our cofibrancy assumptions altogether. We will do this in the case S is defined by a functor $f : \mathcal{C} \to \mathcal{D}$, and of course when S is defined by $g : \mathcal{D} \to \mathcal{C}$ there is an "adjoint" proof of the analogous statement. In order to remove the assumption that $f : \mathcal{C} \to \mathcal{D}$ is a cofibration, first let us factor f as a cofibration followed by an acyclic fibration:

$$\mathcal{C} \xrightarrow{\overline{f}} \overline{\mathcal{D}} \xrightarrow{p} \mathcal{D}$$

Now consider the natural diagram:

$$\begin{array}{ccccc}
\mathcal{C}\otimes\{1\} & \xrightarrow{\overline{f}} & \overline{\mathcal{D}} & \xrightarrow{p} & \mathcal{D} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}\otimes\Delta^1 & \longrightarrow & \mathcal{C}\tilde{\star}_{\overline{S}}\overline{\mathcal{D}} & \xrightarrow{r} & \mathcal{C}\tilde{\star}_S\mathcal{D},
\end{array}$$

in which the outer rectangle and left square are pushouts, the bimodule $\overline{S}$ is defined by $(c, d) \mapsto \mathcal{D}(\overline{f}c, d)$, and the bottom right horizontal arrow, r, acts by the identity on $\mathcal{C}$, and by p on $\overline{\mathcal{D}}$ and the morphism complexes $\mathcal{C}\tilde{\star}_{\overline{S}}\overline{\mathcal{D}}(c, d) = \overline{\mathcal{D}}(\overline{f}c, d)$.

But, since p is a quasi-isomorphism, we see immediately that r is as well. Hence $\mathcal{C}\tilde{\star}_S\mathcal{D}$ is weakly equivalent to $\mathcal{C}\tilde{\star}_{\overline{S}}\overline{\mathcal{D}}$, which, in the case that $\mathcal{C}$ is cofibrant, is the homotopy pushout of

$$\begin{array}{ccc} \mathcal{C} \otimes \{1\} & \longrightarrow & \mathcal{D} \\ \downarrow & & \\ \mathcal{C} \otimes \Delta^1 & & \end{array} .$$

So now we wish to remove the assumption that $\mathcal{C}$ is cofibrant by arguing in a similar fashion. We denote the initial (empty) dg-category by $\emptyset$, and cofibrantly replace $\mathcal{C}$:

$$\emptyset \to \underline{\mathcal{C}} \xrightarrow{q} \mathcal{C}.$$

We will need the following fact about cofibrant replacement in *dgcat*, which is used frequently in the literature on dg categories and follows immediately from Tabuada's list of generating (acyclic) cofibrations of the Dwyer Kan model structure on *dgcat* in [29].

Fact 5. *Cofibrant replacements, $\underline{\mathcal{C}} \to \mathcal{C}$, in the Dwyer-Kan model structure on dgcat may be taken to be the identity on the level of objects.*

Now let $\underline{S}$ be the $\underline{\mathcal{C}} - \mathcal{D}$ bimodule $(c, d) \to \mathcal{D}(fqc, d)$ and consider the map

$$\underline{\mathcal{C}}\tilde{\star}_{\underline{S}}\mathcal{D} \to \mathcal{C}\tilde{\star}_S\mathcal{D}$$

which is the identity on objects as well as the subcategory isomorphic to $\mathcal{D}$, and acts by q on the subcategory isomorphic to $\underline{\mathcal{C}}$, and by the identity on the complexes

$$\underline{\mathcal{C}}\tilde{\star}_{\underline{S}}\mathcal{D}(c, d) = \mathcal{D}(fqc, d) = \mathcal{C}\tilde{\star}_S\mathcal{D}(qc, d).$$

This map is clearly a quasi-equivalence because q is.

This establishes the second necessary weak equivalence to show that, regardless of the cofibrancy of $\mathcal{C}, \mathcal{D}$, and f, we have a weak equivalence:

$$\mathcal{C}\tilde{\star}_S\mathcal{D} \simeq hocolim(\mathcal{C} \otimes \Delta^1 \hookleftarrow \mathcal{C} \otimes \{1\} \xrightarrow{f} \mathcal{D}).$$

Similarly, given arbitrary $\mathcal{C}, \mathcal{D}$, and $g : \mathcal{D} \to \mathcal{C}$, we have an equivalence:

$$\mathcal{C}\tilde{\star}_S\mathcal{D} \simeq hocolim(\mathcal{D} \otimes \Delta^1 \hookleftarrow \mathcal{D} \otimes \{0\} \xrightarrow{g} \mathcal{C}),$$

and we have established the homotopy coherence of our results.

3.1.3 Towards a proof entirely internal to homotopy theory

In this section we will attempt to argue in favor of Theorem 1 in a more homotopically meaningful manner, stopping short of carrying out the homological algebra which the preceeding sections have allowed us to sidestep.

Likely because fibrations are simpler than cofibrations in the Dwyer Kan model category of dg categories, researchers have been able to write down explicit formulas for certain homotopy limits of dg categories. For example, in section 4 of [2], the authors use Tabuada's explicit construction of path objects from [30] to write down a formula for homotopy pullbacks in this model category. As we will be using this formula for a (enriched, homotopical) Yoneda argument in the proof of the main results of this paper, we review it below.

So let $\mathcal{S}, \mathcal{T}$, and $\mathcal{R}$ be dg- categories equipped with functors as below

$$\begin{array}{ccc} & & \mathcal{S} \\ & & \downarrow{\scriptstyle s} \\ \mathcal{T} & \xrightarrow{r} & \mathcal{R} \end{array}$$

Bassat and Block construct the homotopy limit, $\mathcal{S} \times^h_{\mathcal{R}} \mathcal{T}$ as follows:

Objects in $\mathcal{S} \times^h_{\mathcal{R}} \mathcal{T}$ are tuples (X, Y, ϕ), where $X \in \mathcal{S}, Y \in \mathcal{T}, \phi \in \mathcal{R}(sX, tY)$, and ϕ is closed of degree zero and becomes invertible in $H^0(\mathcal{R})$. Morphism complexes are given by:

$$(\mathcal{S} \times^h_{\mathcal{R}} \mathcal{T})^i((X, Y, \phi), (X', Y', \phi')) = \mathcal{S}^i(X, X') \oplus \mathcal{T}^i(X', Y', \phi') \oplus \mathcal{R}^{i-1}(sX, tY'),$$

with differential

$$d(\mu, \nu, \tau) = (d\mu, d\nu, d\tau + \phi' s(\mu) - (-1)^i t(\nu)\phi).$$

So, informally, "arrows" in this category are comprised of morphisms, $\mu : X \to X'$ and $\nu : Y \to Y'$, together with a witness, τ, to the *(graded) homotopy commutativity* of the square:

$$\begin{array}{ccc} sX & \xrightarrow{s\mu} & sX' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ tY & \xrightarrow{t\nu} & tY'. \end{array}$$

Composition of morphisms is given on homogeneous elements by

$$(\mu, \nu, \tau)(\mu', \nu', \tau') = (\mu\mu', \nu\nu', \tau' s(\mu) + t(\nu')\tau).$$

Keeping the notation from the last three subsections (left-handed version), we turn our attention now to the homotopy pullback square:

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathbb{R}\underline{Hom}(\mathcal{D}, \mathcal{A}) \\ \downarrow & & \downarrow{\scriptstyle f^*} \\ \mathbb{R}\underline{Hom}(\Delta^1, \mathbb{R}\underline{Hom}(\mathcal{C}, \mathcal{A})) & \xrightarrow{pr_1} & \mathbb{R}\underline{Hom}(\mathcal{C}, \mathcal{A}). \end{array}$$

We will sometimes use exponential shorthand, such as $\mathcal{C}^{\mathcal{B}}$ for $\mathbb{R}\underline{Hom}(\mathcal{C}, \mathcal{A})$, in this section, but note that this notation might not be used the same way in all articles on dg categories.

Given a dg category, $\mathcal{T}$, we know from [34] that the simplicial mapping space $Map(\mathcal{T}, \mathcal{A})$ is equivalent to the mapping space $Map(\underline{C(k)}, \mathbb{R}\underline{Hom}(\mathcal{T}, \mathcal{A}))$, and the functor $Map(\underline{C(k)}, -)$ commutes up to homotopy with homotopy pullbacks, so we will abusively denote the homotopy pullback, P, by $\mathcal{A}^{\mathcal{B}}$, and we can prove our theorem by exhibiting a quasi-equivalence:

$$\mathbb{R}\underline{Hom}(\mathcal{C}\tilde{\star}_S\mathcal{D},\mathcal{A}) \to \mathcal{A}^{\mathcal{B}},$$

which is functorial in $\mathcal{A}$.

Now, we know that for any $\mathcal{T}$, $\mathbb{R}\underline{Hom}(\mathcal{T},\mathcal{A})$ is the full sub dg category of cofibrant and right quasi-representable objects in $\mathcal{T}\otimes^{\mathbb{L}}\mathcal{A}^{op}-Mod$. We also know that

$$\mathcal{T}\otimes^{\mathbb{L}}\mathcal{A} \simeq Q\mathcal{T}\otimes\mathcal{A} \simeq \mathcal{T}\otimes Q\mathcal{A},$$

where Q denotes a cofibrant replacement functor (which exists because the Dwyer Kan model structure on dg cat is cofibrantly generated). So throughout this subsection we will assume without loss of generality that $\mathcal{A}$ is cofibrant. As the dg category Δ^1 is cofibrant, we will need no more cofibrant replacements in this subsection.

Thanks to section 4 of [2], we have an explicit model for $\mathcal{A}^{\mathcal{B}}$. To begin unraveling this model, let us first describe the objects of

$$\mathbb{R}\underline{Hom}(\Delta^1,\mathbb{R}\underline{Hom}(\mathcal{C},\mathcal{A})) \simeq \Delta^1\otimes^{\mathbb{L}}\mathbb{R}\underline{Hom}(\mathcal{C},\mathcal{A})-Mod^{rqr,cof}.$$

Up to an isomorphism in H^0, an object in this category is determined by two objects, $M_0, M_1 \in \mathbb{R}\underline{Hom}(\mathcal{C},\mathcal{A})$, and an arrow, $\varphi : M_1 \to M_0$, in $H^0(\mathbb{R}\underline{Hom}(\mathcal{C},\mathcal{A}))$ (right quasi-representability is key here). We will denote these objects by $[M_0 \stackrel{\varphi}{\to} M_1]$.

So then objects of $\mathcal{A}^{\mathcal{B}}$ are triples, $([M_0 \xrightarrow{\varphi} M_1], N, \psi)$ with $N \in \mathbb{R}\underline{Hom}(\mathcal{D}, \mathcal{A})$ and $\phi : M_1 \to N$ a closed degree zero morphism which becomes an isomorphism in the homotopy category of quasi-functors $\mathcal{C} \to \mathcal{A}$.

On the level of objects we may then proceed in a similar manner to our proofs of the lemmas above, or even appeal to proposition 7.7 (ii) in [21], but at this stage we will not check the necessary weak equivalences of morphism complexes, only observe that based on the preceding work, they must exist, and we do not need to dig into the calculus of (derived) ends.

3.1.4 Second Primary Result

Relevant Category and Notation

Let $cdga_k^{\leq 0}$ denote the 1-category of nonpositively cohomologicaly graded commutative differential graded algebras over k with the model structure inherited from the projective model structure on nonpositively graded k cochain complexes. The words ring and module in this section will always refer to the differential graded versions of these objects.

For $n \in \mathbb{Z}$ we write S^n for the cochain complex of k vector spaces with exactly one copy of k in degree $-n$ and zeros elsewhere. D^{n+1} will denote the cochain complex with copies of k in degrees $-n-1$ and $-n$, the identity differential between them, and zeros elsewhere. Given a cdga, A, we will denote $A \otimes_k S^n$ by $S^n(A)$ and we'll similarly write $D^n(A)$.

When $n \geq 0$, we will also use the symbols $\mathbb{S}^n$ and $\mathbb{D}^{n+1}$ for the images of S^n and D^{n+1} under the graded symmetric algebra functor, $Vect_k \to cdga_k$.

It is well known (CITATION: SULLIVAN?) that the families of inclusions, $\mathbb{S}^n \hookrightarrow \mathbb{D}^{n+1}$ and $k \hookrightarrow \mathbb{D}^n$ respectively form classes of generating cofibrations and generating acyclic cofibrations in the model category $cdga_k^{\leq 0}$ (n ranges over nonpositive integers). Similarly, for $A \in cdga_k^{\leq 0}$, the inclusions $S^n(A) \hookrightarrow D^{n+1}(A)$ and $0 \hookrightarrow D^n(A)$ respectively form classes of generating cofibrations and generating acyclic cofibrations in the model category of A modules (here n runs through all integers).

Recall that we are interested in proving the following model category is cofibrantly generated:

$\mathscr{M}od$ denotes the category of pairs, (A, U) with $A \in cdga_k^{\leq 0}$ and $U \in A - mod$. Morphisms $(A, U) \to (B, V)$ are defined by pairs (f, ϕ), with $f : A \to B$ a ring map and $\phi : U \otimes_A B \to V$ a $B-$module homomorphism.

Definition 3.0.4, Theorem 3.0.12, and example 6.4 in Harpaz and Prasma ensure this category carries a model structure in which $(A, U) \to (B, V)$ is a weak equivalence when $A \to B$ is a weak equivalence of rings and $Q(U \otimes_A B) \to U \otimes_A B \to V$ (Q denotes cofibrant replacement) is a weak equivalence of B-modules, and a cofibration when $A \to B$ is a cofibration as well as $U \otimes_A B \to V$.

Pushouts in $\mathscr{M}od$

It is straightforward to see that the pushout (object) in $\mathscr{M}od$ of

$$(C, W) \leftarrow (A, U) \to (B, V)$$

is $(C \otimes_A B, (W \otimes_A B) \bigoplus_{C \otimes_A U \otimes_A B} (C \otimes_A V))$.

3.1.5 $\mathscr{M}od$ is cofibrantly generated

Define $I, J \subset Arr(\mathscr{M}od)$ by:

$$I := \{(\mathbb{S}^n, 0) \to (\mathbb{D}^{n+1}, 0)\}_{n\geq 0} \cup \{(k, S^n(k)) \to (k, D^{n+1}(k))\}_{n\in\mathbb{Z}}$$

$$J := \{(k, 0) \to (\mathbb{D}^n, 0)\}_{n\geq 0} \cup \{(k, 0) \to (k, D^{n+1}(k))\}_{n\in\mathbb{Z}}.$$

Any map, $(A, U) \to (B, V)$, in $\mathscr{M}od$ has an obvious factorization:

$$(A, U) \to (B, U \otimes_A B) \to (B, V).$$

This leads us to two claims:

Claim 1. *Given a cofibration (resp. trivial cofibration) of rings, $A \to B$, and any A module, U, the corresponding map of the form*

$$(A, U) \to (B, U \otimes_A B)$$

in $\mathscr{M}od$ is, up to finitely many retracts, a transfinite composition of I-cell (resp. J-cell) attachment maps in $\mathscr{M}od$.

Claim 2. *Given a cofibration (resp. trivial cofibration) of $B-$modules, $V \to W$, the corresponding map*

$$(B, V) \to (B, W)$$

in $\mathscr{M}od$ is, up to finitely many retracts, a transfinite composition of I-cell (resp. J-cell) attachment maps in $\mathscr{M}od$.

We prove claim 1 first. Under the premises of the claim, $A \to B$, is a finite retract of a transfinite composition of pushouts along coproducts of maps of the form $\mathbb{S}^n \hookrightarrow \mathbb{D}^{n+1}$ and $0 \hookrightarrow \mathbb{D}^n$. We can write this finite number of retracts as just one retract as, in any category, a retract of a retract is a retract.

We first assume there has been no retract, i.e. $A \to B$ is a transfinite composition of pushouts:

$$\begin{array}{ccc} \mathscr{S} & \longrightarrow & \mathscr{D} \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

where $\mathscr{S}$ is a coproduct of $\mathbb{S}^n$s and $\mathscr{D}$ is a coproduct of $\mathbb{D}^{n+1}$s indexed over the same set, and $B' \cong A' \otimes_{\mathscr{S}} \mathscr{D}$.

We then have, for any $U' \in A' - mod$, a pushout square in $\mathscr{M}od$:

$$\begin{array}{ccc} (\mathscr{S}, 0) & \longrightarrow & (\mathscr{D}, 0) \\ \downarrow & & \downarrow \\ (A', U') & \longrightarrow & (B', (U' \otimes_{\mathscr{S}} \mathscr{D}) \bigoplus_{A' \otimes_{\mathscr{S}} 0 \otimes_{\mathscr{S}} \mathscr{D}} (A \otimes_{\mathscr{S}} 0)) \end{array}$$

and we observe that

$$U' \otimes_{\mathscr{S}} \mathscr{D} \cong U' \otimes_{A'} A' \otimes_{\mathscr{S}} \mathscr{D} \cong U' \otimes_{A'} B'.$$

So the bottom right corner of the above pushout becomes $(B', U' \otimes_{A'} B')$. Now, by inducting over the natural numbers and writing B at the colimit of these B', we see that, for any $U \in A - mod$, the map

$$(A, U) \to (B, U \otimes_A B)$$

is a transfinite (really $\mathbb{N}$-indexed) composition of I-cell attachments. An identical argument carries through if $A \to B$ is assumed acyclic as well and we use J-cell attachment.

So then assume the cofibration $A \to B$ is a retract of a cellular attachment of rings. Say we have a retract diagram:

$$\begin{array}{ccccc} A & \longrightarrow & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & B' & \longrightarrow & B \end{array}$$

in $cdga_k^{\leq 0}$, where the middle vertical map is a cellular attachment. We then, given $U \in A - mod$, get a diagram in $\mathscr{M}od$:

$$\begin{array}{ccccc} (A, U) & \longrightarrow & (A', U \otimes_A A') & \longrightarrow & (A, U) \\ \downarrow & & \downarrow & & \downarrow \\ (B, U \otimes_A B) & \longrightarrow & (B', U \otimes_A A \otimes_{A'} B) & \longrightarrow & (B, U \otimes_A B) \end{array}$$

which is trivially seen to be a retract, and we know the middle vertical arrow is an I-cell attachment. Claim 1 follows.

The proof of Claim 2 mirrors the proof of Claim 1:

First assume we have a cofibration of B modules, $V \to W$, which is a transfinite composition of pushouts:

$$\begin{array}{ccc} \mathscr{S}(B) & \longrightarrow & \mathscr{D}(B) \\ \downarrow & & \downarrow \\ V' & \longrightarrow & W' \end{array}$$

where now we use the symbols $\mathscr{S}(B)$ and $\mathscr{D}(B)$ to denote coproducts of $S^n(B)$ and $D^{n+1}(B)$.

We have a pushout in $\mathscr{M}od$:

$$\begin{array}{ccc} (k, \mathscr{S}(k)) & \longrightarrow & (k, \mathscr{D}(k)) \\ \downarrow & & \downarrow \\ (B, V') & \longrightarrow & (B \otimes_k k, (V' \otimes_k k) \bigoplus_{B\otimes_k \mathscr{S}(k) \otimes_k k} (B \otimes_k \mathscr{D}(k))) \end{array}$$

And we compute the bottom right corner:

$$(B \otimes_k k, (V' \otimes_k k) \bigoplus_{B\otimes_k \mathscr{S}(k) \otimes_k k} (B \otimes_k \mathscr{D}(k))) \cong (B, V' \bigoplus_{\mathscr{S}(B)} \mathscr{D}(B)) \cong (B, W').$$

At this point the proof of Claim 2 follows the exact path of Claim 1.

And we have thus proven that cofibrations (resp. acyclic cofibrations) in $\mathscr{M}od$ may be written as retracts of transfinite compositions of $I-$cell (resp. J-cell) attachments.

It remains to be shown that (retracts of) transfinite compositions of $I-$cell (resp. J-cell) attachments are always cofibrations in $\mathscr{M}od$.

But this follows by transfinite induction from the fact that maps of the form $(\mathbb{S}^n, 0) \to (\mathbb{D}^{n+1}, 0)$ and $(k, S^n(k)) \to (k, D^{n+1}(k))$ commute in $\mathscr{M}od$.

Combining everything from this section, we conclude $\mathscr{M}od$ is cofibrantly generated. (It is a quick check also that I and J admit the small object argument).

3.2 A comment about the ∞-categorical analogue of Theorem 1

We would like to prove a similar result to our Theorem 1 in the setting of (stable, linear, presentable) ∞-categories, but our proof in the differential graded setting relies critically on the 1-category and model category theory specific to dg-categories. Results of Lee Cohn in [10], however, partially justify the definition we make at the end of this section, and a more thorough treatment of the ∞-categorical version of our work is left as a topic of future research.

As we treat this section as a remark, we will not recall the theory of ∞-categories and assume the reader has basic familiarity with the foundations, as well as working knowledge of (ring) spectra. R will denote an arbitrary fixed $\mathbb{E}_\infty$ spectrum throughout this section.

3.2.1 R-Linear ∞-categories and Comparison with Differential graded categories in [10]

A concise reference for this subsection is section 3 of [12], but a more thorough treatment appears in [23].

Let Mod_R denote the ∞-category of right R-modules. This category is stable and presentable, and the homotopy commutativity of R ensures this category has a tensor product, making it into an algebra object in Pr^L. We define the ∞-category of R-linear categories to be modules, in Pr^L, over this algebra object.

Definition 9. *$Cat_R := Mod_{Mod_R}(Pr^L)$ is the ∞-category of R-linear categories.*

We comment that categories linear over the sphere spectrum coincide with stable presentable ∞-categories and colimit preserving functors ([12], 17).

We also introduce the notation $Cat_{R,\omega} \subset Cat_R$ for the subcategory of *compactly generated* R-linear subcategories and functors which additionally preserve compact objects. Both of the categories we've defined carry natural symmetric monoidal structures with identity Mod_R. In [12], the authors construct derived moduli spaces of objects in these categories as in [31].

Now, let $Mor \subset Ar(dg - cat)$ denote the Morita equivalences of dg categories. Corollary 5.7 in [10] states we have the following equivalence of ∞-categories:

Fact 6. *There is an equivalence of ∞-categories:*

$$N(dg - cat)[Mor^{-1}] \simeq Cat_{R,\omega}.$$

3.2.2 A definition

Inspired by the concluding fact of the preceeding section and our work in section 5, we make the following definitions.

Definition 10. *Let Arr_R denote the idempotent completion of the R-linear category whose underlying spectral category has two objects, 0 and 1, with $Hom(0,0) = Hom(1,1) = Hom(0,1) = R$ and $Hom(1,0) = *$.*

There are obvious morphisms

$$Arr_R \overset{\iota_0}{\hookleftarrow} Mod_R \overset{\iota_1}{\hookrightarrow} Arr_R.$$

Definition 11. *Given a morphism $f : \mathscr{C} \to \mathscr{D}$ of R-linear categories, we say the* left handed join of $\mathscr{C}$ and $\mathscr{D}$ along f *is the pushout, in Cat_R, of the span*

$$\mathscr{C} \otimes Arr_R \xleftarrow{id \otimes \iota_1} \mathscr{C} \otimes Mod_R \xrightarrow{f} \mathscr{D}.$$

Given $g : \mathscr{D} \to \mathscr{C}$ we say the right handed join of $\mathscr{C}$ and $\mathscr{D}$ along g *is the pushout, in Cat_R, of the span*

$$\mathscr{D} \otimes Arr_R \xleftarrow{id \otimes \iota_0} \mathscr{D} \otimes Mod_R \xrightarrow{g} \mathscr{C}.$$

3.3 Examples of Glued Categories

3.3.1 Semi-orthogonal decompositions

The theory of gluing linear categories along a bimodule is equivalent to the theory of semi-orthogonal decompositions of linear categories in a sense to be made precise below. Our discussion of semi-orthogonal decompositions will be largely ported from [25], as that source collects and proves much of the relevant theory on the subject, but semi-orthogonal decompositions have a long history (appearing first in the context of triangulated categories most likely, although the author is not aware of the first article where the term appears) and appear frequently in nature. The next subsection contains examples.

Throughout this subsection, let N and T denote arbitrary triangulated categories (in practice, T will be the homotopy category of a pretriangulated dg category). Also let $\mathcal{C}$ denote a pretriangulated dg category. We will begin with some definitions.

Definition 12. *Given a full triangulated embedding $N \hookrightarrow T$, we say N is a* right admissible *(respectively* left admissible*)* subcategory *of T if there is a right (respectively left) adjoint functor $T \to N$. A subcategory which is both left and right admissible is called* admissible.

Definition 13. *A* semi-orthogonal decomposition *of T is a pair of admissible subcategories, $N, N' \subset T$ such that $Hom_T(x, y) = 0$ whenever $x \in N'$ and $y \in N$ and every object, $z \in T$ fits into a distinguished triangle of the form $x \to z \to y$ for some $x \in N'$ and $y \in N$. In this case we will write $T =< N, N' >$.*

Orlov proves the following lemmas, which make precise the relationship between semi-orthogonal decompositions and the gluing of dg-categories:

Proposition 1. (proposition 3.7 in [25])*Any time a dg category can be expressed as a gluing, $\mathcal{C} = \mathcal{C}_0 \star_S \mathcal{C}_1$ of two pretriangulated dg categories, $\mathcal{C}_0$ and $\mathcal{C}_1$, the dg embeddings, $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ and $\mathcal{C}_1 \hookrightarrow \mathcal{C}$ induce a semi-orthogonal decomposition, $H^0(\mathcal{C}) =< H^0(\mathcal{C}_0), H^0(\mathcal{C}_1) >$, of the triangulated category, T.*

Proposition 2. (proposition 3.8 in [25]) *Suppose we have a semi-orthogonal decomposition,* $H^0(\mathcal{C}) =< C_0, C_1 >$. *Then* $\mathcal{C}$ *is quasi-equivalent to the gluing,* $\mathcal{C}_0 \star_S \mathcal{C}_1$, *where* $\mathcal{C}_0, \mathcal{C}_1 \subset \mathcal{C}$ *are the full dg subcategories on the objects of* C_0 *and* C_1 *respectively, and the bimodule,* S, *is given by morphism complexes in* $\mathcal{C}$.

3.3.2 Examples of Semi-orthogonal decompositions

A first toy example is the case where $\mathcal{C} = \mathcal{D}$ is the one object category corresponding to a dg-k-algebra, A, and $S = A$ with it's natural A-bimodule structure. In this case, the category of $A \star_A A$ modules will be quasi-equivalent to the dg category of pairs of A modules together with a morphism between them.

Projectivizations of vector bundles

A more interesting example of a glued dg-category appears in [25] (example 3.9). To describe this, let $\mathcal{E}$ be a rank 2 vector bundle on an underived noetherian scheme, X. Let $p : \mathbb{P}(\mathcal{E}^\vee) \to X$ be the projectivization of $\mathcal{E}^\vee$. $\mathbb{P}(\mathcal{E}^\vee)$ carries a canonical line bundle, $\mathcal{O}(-1)$, with dual $\mathcal{O}(1)$, satisfying $Rp^*\mathcal{O}(1) \cong \mathcal{E}$. Also, p^* is quasi-fully-faithful, and the derived category of perfect complexes on $\mathbb{P}(\mathcal{E}^\vee)$ has a semi-orthogonal decomposition:

$$Perf - \mathbb{P}(\mathcal{E}^\vee) =< Rp^*Perf - X, Rp^*Perf - X \otimes \mathcal{O}(1) > \qquad (3.3)$$

We will abusively use the notation $Perf - X$ for its dg enhancement as well, and the above implies that the dg category $Perf - \mathbb{P}(\mathcal{E}^\vee)$ is quasi-isomorphic to $Perf - X$ glued to itself along some bimodule, and indeed, $Perf - \mathbb{P}(\mathcal{E}^\vee) \simeq Perf - X \star_{S_\mathcal{E}} Perf - X$ where

$$S_\mathcal{E}(V, W) = Perf - X(V, W \otimes \mathcal{E}). \tag{3.4}$$

The results of this thesis (Theorem 1, right-hand version) then imply we have an equivalence of derived stacks:

$$\mathcal{M}_{Perf-\mathbb{P}(\mathcal{E})} \simeq holim(\mathcal{M}_{\Delta^1 \otimes Perf-X} \xrightarrow{pr_0} \mathcal{M}_{Perf-X} \xleftarrow{(-\otimes \mathcal{E})^*} \mathcal{M}_{Perf-X}).$$

Blowups

For another example from [25] (example 3.9), let $\pi : \tilde{X} \to X$ be a blowup of a regular underived scheme, X, along a closed, regular, codimension 2 subscheme, Y. The functor $L\pi^*$ is fully faithful, and if we let $j : E \hookrightarrow \tilde{X}$ be the exceptional divisor, then the restricted projection $p : E \to Y$ is the projectivization of the normal bundle to Y in X. The functor Rj_*p^* is also fully faithful and we have a semi-orthogonal decomposition of the derived category

$$Perf - \tilde{X} = < L\pi^* Perf - X, Rj_*p^* Perf - Y > .$$

We then have a gluing of the respective dg enhancements:

$$Perf - \tilde{X} \simeq Perf - X \star_S Perf - Y,$$

where S takes the form

$$S(A, B) = Perf - Y(i^* A, B),$$

with $i : Y \hookrightarrow X$ denoting the inclusion. Therefore the left-hand version of Theorem 1 implies:

$$\mathcal{M}_{Perf-\mathbb{P}(\mathcal{E})} \simeq holim(\mathcal{M}_{\Delta^1 \otimes Perf-X} \xrightarrow{pr_1} \mathcal{M}_{Perf-X} \xleftarrow{(i^*)^*} \mathcal{M}_{Perf-Y}).$$

We remark in the case of either of the preceding examples that there is a natural interpretation of $\mathcal{M}_{\Delta^1 \otimes Perf-X}$ as the "moduli space of closed degree 0 arrows in $Perf - X$."

More examples

For a plethora of additional examples, see e.g. [21], where resolutions of singularities are discussed, or [3] and [26] where semi-orthogonal decompositions in log geometry and K theory are discussed.

Bibliography

[1] D Ayala and J Francis. Fibrations of ∞ - categories. *Preprint*, 2017.

[2] Oren Ben-Bassat and Jonathan Block. Milnor descent for cohesive dg-categories. *J. K-Theory*, 12(3):433–459, 2013.

[3] Daniel Bergh, Valery A Lunts, and Olaf M Schnu rer. Geometricity for derived categories of algebraic stacks. *Selecta Mathematica*, 22(4):2535–2568, 2016.

[4] Daniel Bergh and Olaf M Schnu rer. Conservative descent for semi-orthogonal decompositions. *Preprint, arXiv:1712.06845*, 2017.

[5] Aaron Bergman. Stability conditions and branes at singularities. *Journal of High Energy Physics*, 2008.10(07), 2008.

[6] Anthony Blanc. Topological k theory of noncommutative spaces. *Compos. Math.*, 152(3), 2016.

[7] A. Bondal and D. Orlov. Derived categories of coherent sheaves. *Proceedings of the InternationalCongress of Mathematicians, Vol. II (Beijing, 2002), 4756, Higher Ed. Press, Beijing, 2002.*

[8] Tom Bridgeland. Stability conditions on triangulated categories. *Ann. of Math*, (166):317–345, 2007.

[9] Alberto Canonaco and Paolo Stellari. Internal Homs via extensions of dg functors. *Adv. Math.*, 277:100–123, 2015.

[10] Lee Cohn. Differential graded categories are k-linear stable infinity categories. *Preprint*, 2013.

[11] Francesco Genovese. Adjunctions of quasi-functors between DG-categories. *Appl. Categ. Structures*, 25(4):625–657, 2017.

[12] David Gepner and Benjamin Antieau. Brauer groups and étale cohomology in derived algebraic geometry. *Geometry and Topology*, 18(2), 2012.

[13] David Gepner, Rune Haugseng, and Thomas Nikolaus. Lax colimits and free fibrations in ∞-categories. *Documenta Mathematica*, 22, 2017.

[14] E.L. Green. On the representation theory of rings in matrix form. *Pacific J. Math*, (100), 1982.

[15] A. Haghany and K. Varadarajan. Study of modules over formal matrix rings. *Journal of Pure and Applied Algebra*, (147), 2000.

[16] Yonatan Harpaz and Matan Prasma. The Grothendieck construction for model categories. *Adv. Math.*, 281:1306–1363, 2015.

[17] Julian Holstein. Morita cohomology. *PhD Thesis.*

[18] Andre Joyal. Notes on quasicategories. *preprint*, 2008.

[19] B. Keller. On differential graded categories. *International Congress of Mathematicians*, 2, 2006.

[20] A. D. King. Moduli of representations of finite-dimensional algebras. *Quart. J. Math. Oxford Ser. (2)*, 45(180):515–530, 1994.

[21] Alexander Kuznetsov and Valery A. Lunts. Categorical resolutions of irrational singularities. *Int. Math. Res. Not. IMRN*, (13):4536–4625, 2015.

[22] Jacob Lurie. *Higher Topos Theory.* Volume 170 of Annals of Mathematical Studies. Princeton University Press, 2009.

[23] Jacob Lurie. *Higher Algebra.* 2017.

[24] Aaron Mazel-Gee. All about the grothendeick construction. *Preprint, https://arxiv.org/abs/1510.03525*, 2015.

[25] D Orlov. Smooth and proper noncommutative schemes and glueings of dg categories. *Adv. Math.*, 302, 2016.

[26] Sarah Scherotzke, Nicol Sibilla, and Mattia Talpo. Parabolic semi-orthogonal decompositions and kummer flat invariants of log schemes, 2018.

[27] Sarah Scherotzke, Nicol Sibilla, and Mattia Talpo. Gluing semi-orthogonal decompositions, 2019.

[28] Ross Street. Correction to: "Fibrations in bicategories" [Cahiers Topologie Géom. Différentielle **21** (1980), no. 2, 111–160; MR0574662 (81f:18028)]. *Cahiers Topologie Géom. Différentielle Catég.*, 28(1):53–56, 1987.

[29] Goncalo Tabuada. Théorie homotopique des dg-catégories. *phd Thesis*, 2007.

[30] Goncalo Tabuada. Homotopy theory of dg categories via localizing pairs and drinfeld's dg quotient. *Homotopy, Homology, and Applications*, 12(1), 2010.

[31] B Toen and M Vaquie. Moduli of objects in dg categories. *Ann. Sci. E cole Norm. Sup. (4)*, 40(3), 2007.

[32] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. I. Topos theory. *Adv. Math.*, 193(2):257–372, 2005.

[33] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.

[34] Betrand Toen. Homotopy theory of dg-categories and derived morita theory. *Invent. Math*, 167(3), 2007.

Vita

Nicolas Reyes was born in 1991 in Austin, Texas. He received his Bachelor of Science in Mathematics from The University of Massachusetts, Amherst in 2013. While at the University of Texas, Austin he was a graduate research fellow of the National Science Foundation.

Permanent address: nicolaszreyes@gmail.com

This dissertation was typeset with LaTeX[†] by the author.

[†]LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.