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Hamiltonian and Action Principle formulations of plasma fluid models

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> “Begin at the beginning,” the King said gravely, “and go on till you come to the end: then stop.”
>
> - Lewis Carroll, *Alice’s Adventures in Wonderland*

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>
> - Christina Rossetti, *Goblin Market*

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> And thanks; and ever thanks"
> - William Shakespeare, *Twelfth Night, Or What You Will*

Hamiltonian and Action Principle formulations of plasma fluid models

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The Hamiltonian and Action Principle (HAP) formulations of plasmas and fluids are explored in a wide variety of contexts. The principles involved in the construction of Action Principles are presented, and the reduction procedure to obtain the associated noncanonical Hamiltonian formulation is delineated.

The HAP formulation is first applied to a 2D magnetohydrodynamics (MHD) model, and it is shown that one can include Finite Larmor Radius effects in a transparent manner. A simplified 2D limit of the famous Braginskii gyroviscous tensor is obtained, and the origins of a powerful tool - the gyromap - are traced to the presence of a gyroviscous term in the action. The noncanonical Hamiltonian formulation is used to extract the Casimirs of the model, and an Energy-Casimir method is used to derive the equilibria and stability; the former are shown to be generalizations of the Grad-Shafranov equation, and possess both flow and gyroviscous effects. The action principle of 2D MHD is generalized to encompass a wider class of gyroviscous fluids, and a suitable gyroviscous theory for liquid crystals is constructed.

The next part of the thesis is devoted to examining several aspects of extended MHD models. It is shown that one can recover many such models from a parent action, viz. the two-fluid model. By performing systematic orderings in the action, extended MHD, Hall MHD and electron MHD are recovered. In order to obtain these models, novel techniques, such as non-local Lagrange-Euler maps which enable a transition between the two fluid frameworks, are introduced. A variant of extended MHD, dubbed inertial MHD, is studied via the HAP approach in the 2D limit. The model is endowed with the effects of electron inertia, but is shown to possess a remarkably high degree of similarity with (inertialess) ideal MHD. A reduced version of inertial MHD is shown to yield the famous Ottaviani-Porcelli model of reconnection. Similarities in the mathematical structure of several extended MHD models are explored in the Hamiltonian framework, and it is hypothesized that these features emerge via a unifying action principle. Prospects for future work, reliant on the HAP formulation, are also presented.

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Chapter 1

The need for Hamiltonian and Lagrangian methods

"The time has come," the Walrus said,

"To talk of many things:

Of shoes – and ships – and sealing-wax –

Of cabbages – and kings –

And why the sea is boiling hot –

And whether pigs have wings."

- Lewis Carroll, *Through the Looking-Glass*

1.1 A brief historical overview

The history of physics is replete with models that have been derived from action principles (Lagrangians) or Hamiltonians. The importance of such methods in a plethora of fields, ranging from quantum physics [1] and general relativity [2] to condensed matter [3] and statistical physics [4] is well documented. Hence, it comes as no surprise that such methods have also been widely employed in the context of fluids and plasmas. In fact, the former field witnessed pioneering contributions from Lagrange himself in his remarkable treatise [5], which laid down the foundations for an action-principle based formulation of fluids. This was followed by a series of works in this area by several illustrious scientists in the 19th century [6, 7, 8, 9, 10, 11]. The 20th century witnessed a further slew of developments, which we shall not summarize here. Instead, the reader is referred to [12] which contains an excellent exposition of the developments in mathematical fluid mechanics up to the mid-20th century; see also [13, 14, 15] for associated expositions of these methods in the context of continuum classical models. A more comprehensive list of references and a broader historical discussion of Hamiltonian (and Lagrangian) methods can be found in the reviews of Salmon [16] and Morrison [17].

Although the action principle formulation of fluids existed since the pioneering work of Lagrange [5], the closely associated field of plasmas did not witness such an action principle formulation until nearly two centuries later. One such reason stemmed from the relatively recent rise of magnetohydrodynamics (MHD) in the mid-20th century. The corresponding MHD action was first derived by Newcomb in [18]; see also [19] for a near-contemporaneous action principle of MHD. It is unfortunate that Newcomb is a forgotten figure, as he also went on to derived extended fluid models in a series of prescient papers [20, 21, 22, 23] in the 1970s and 1980s. Although MHD possesses the Newcomb action principle formulation, it is *only* valid in the non-relativistic limit. The relativistic case still remains a work in progress, and the interested reader is referred to [24, 25, 26] for a summary of the recent progress in this arena.

But what of the Hamiltonian formulation of ideal hydrodynamics (HD) and MHD? As ideal, non-relativistic MHD possesses an action principle formulation, it would seem logical for it to also possess a Hamiltonian formulation. As we shall see in the subsequent sections, the great difficulty in constructing such a formulation stems from the fact that the MHD variables, such as the density, velocity, etc., are *noncanonical* in nature. The genesis of noncanonical Hamiltonian methods has been argued to have origins in the works of Sophus Lie in the 19th century, and several Russian and Polish physicists in the mid-20th century [27]. However, the modern noncanonical Hamiltonian formulation of HD and MHD was first set forth in the pioneering work of Morrison and Greene in 1980 [28]. The emergence of the noncanonical Hamiltonian formulation revolutionized progress in the fields of plasmas and fluids, and we shall explore some of these in further detail in the subsequent sections. For now, we observe that it has been successfully employed in geophysical fluid dynamics [16], guiding centre motion and the flow of magnetic field lines [29, 30], Maxwell-Vlasov dynamics [31, 32], nonlinear waves and plasma phenomena [33, 34], elasticity [35] and reduced plasma models [36, 37]. Moreover, the formalism has played a key role in influencing associated fields such as the Euler-Poincaré formulation of field theories [38, 39, 40], Clebsch variables and coadjoint orbits [41], perturbation theory [42] and dissipative dynamics [43, 44, 45, 46, 47, 48]. We emphasize that these selection of topics are not comprehensive; there are many other domains which have not been covered in this fleeting overview of the subject.

1.2 The motivation

It is clear that the preceding discussion has established the long and distinguished role of Hamiltonian and Action Principle (HAP) formulations in fluids and plasmas. The weight of historical progress and usage clearly serves as one tangible reason for the use of these methods, but we shall delineate other reasons in this section.

Firstly, let us recall some of the advantages of the action principle formalism. It is evident that one obvious use is its ability to extract information about the symmetries, and associated invariants, of the model via Noether's theorem, which was first proven by Emmy Noether in 1918; see [49] for an English translation of the original German work [50]. When applied correctly, Noether's theorem enables us to extract a great many invariants of the system, as shown in [51, 52, 53, 54]. Thus, instead of resorting to tedious manipulation of the equations, this represents an elegant method of recovering important invariants of HD and MHD, some of which are endowed with crucial topological properties. The latter includes the magnetic and fluid helicities, which share deep connections with knot theory [55, 56].

When one moves to the associated Hamiltonian formalism of fluids and plasmas, which is noncanonical in nature, the advantages are somewhat technical, albeit very powerful and general, in nature. Hence, we shall defer a full discussion until Section 2.6. For now, we note that the noncanonical formalism gives rise to a special class of invariants, the Casimirs, which foliate the phase space. They play a crucial role in determining the equilibria and stability of models via the Energy-Casimir method. A detailed exposition of this method can be found in several excellent reviews on the subject, see for e.g. [27, 17]. A second, and equally important, advantage of the Hamiltonian formalism is the presence of an 'energy', which is an invariant of the system. Through the proper construction of (noncanonical) Hamiltonian systems, it is easy to avoid the effects of 'spurious dissipation'.

It is well-known that virtually all real-world systems are endowed with dissipative effects, which raises the question of what 'spurious' dissipation refers to. We distinguish between 'real' dissipation, engendered by effects such as viscosity and resistivity, and 'spurious' dissipation which emerges when a system is claimed to be Hamiltonian, but actually fails to conserve the phase space or even the 'energy'. The latter can lead to several unwelcome instabilities such as the existence of false instabilities, which are caused primarily by the (mistaken) assumption that the system is Hamiltonian in nature. An excellent discussion of this issue in the context of extended MHD models is found in [57], and an associated discussion for hybrid fluid-kinetic models exists in [58].

This brings us to the third, and perhaps the most crucial, advantage of the HAP approach: how do we construct and design physical models? It is helpful to recall a general principle espoused by Popper in this regard:

> "Science may be described as the art of systematic over-simplification – the art of discerning what we may with advantage omit."

- Karl Popper, *The Open Universe: An Argument for Indeterminism*

If we follow this maxim, the HAP formalism is endowed with several advantages. In Chapter 2, we shall see that it entails the *a priori* inclusion of 'frozen-in' constraints, which possess an immediately intuitive and geometric meaning. Secondly, each term in the action (or the Hamiltonian) can be written down in a transparent manner, and one can use analogies with particle mechanics to justify their presence. As a result, this method enables a higher degree of physical clarity in determining what terms exist in our model, and the role that they play in determining the dynamics. Lastly, the process of obtaining dynamical equations via the HAP approach is a mathematically rigorous one, and ensures that effects such as 'spurious' dissipation do not creep in. Thus, we argue that the HAP formalism constitutes *both* a physically and mathematically clear approach to model-building, in contrast to the phenomenological, and sometimes *ad hoc*, approach used in designing fluid and plasma models.

The HAP approach also has a natural advantage when dealing with reduced models. Reduced models are obtained from a more complete, and complex, parent model by imposing a certain choice of ordering, typically entailing an expansion in small dimensionless parameter(s). However, the path to formulating reduced models is often a tricky one, as these manipulations are carried out on the level of the dynamical equations, which can result in a complex, and opaque, process. On the other hand, it is found that dropping one term in the action can amount to dropping multiple terms in the dynamical equations. Hence, the HAP approach represents an elegant, and simpler, means of obtaining reduced models through an ordering procedure on the level of the action (or the Hamiltonian).

Although we have advanced several reasons as to why the HAP formalism is beneficial, it is evident that most of them are of an abstract nature. In the subsequent sections, we shall address this issue in greater detail, and show how they can be gainfully employed in a wide variety of contexts, for *specific* plasma models.

1.3 Summary of the thesis

The aim of the thesis is to use the HAP formalism to tackle several existent, and new, plasma fluid models in the literature and employ the advantages of this approach in constructing and analysing them. In order to kick-start our analysis, we commence with an introduction to the action principle and noncanonical Hamiltonian formalisms in Chapter 2, which is applied to ideal MHD. This chapter primarily encapsulates results from the classic works of [18, 28, 17].

In Chapter 3, we shall introduce the concept of gyroviscosity, which is a crucial effect in plasmas. Gyroviscosity emerges from the simple notion of charged particles undergoing Larmor gyration in a magnetic field; such gyration-induced effects are known to give rise to momentum transport, and thereby to viscosity-like terms. However, gyroviscosity is an odd beast indeed, as it conserves momentum but it *also* conserves the energy, in stark contrast to conventional viscous effects, which are interpreted in a dissipative framework. We shall show the origins of gyroviscosity stem from the inclusion of a simple, yet unusual, term in the ideal MHD action. It is shown that the new term gives rise to terms that are identical to the famous Braginskii gyroviscous tensor [59] in a simplified limit. By moving to the associated noncanonical Hamiltonian through a systematic procedure, we investigate general properties of the equilibria and stability of this model. In particular, we derive generalizations of the well-known Grad-Shafranov equation [60] which includes the effects of flow and gyroviscosity.

After having studied the role of gyroviscosity in 2D ideal MHD in Chapter 3, we generalize the treatment to a very wide class of action principles for fluids and plasmas in Chapter 4. We begin by considering a generic Lagrangian, and obtain the corresponding dynamical equations for the model. Generalized criteria regarding the energy, momentum and angular momentum conservation laws for this general family are presented. As an illustration of the formalism, we construct fluids endowed with an intrinsic angular momentum, which can be viewed as a classical 'spin' variable, and derive a suitable gyroviscous model. We also highlight the similarity of this model with theories of nematic liquid crystals.

In Chapter 5, we begin our investigations of extended MHD models via an action principle formulation. In particular, we start with the two-fluid action principle for plasmas, and impose a series of orderings, which enable us to throw away some terms in this action. We re-express our model in terms of the one-fluid variables, but this necessitates some unique mathematical subtleties, which manifest in the form of complex Lagrange-Euler maps that need to be introduced. We introduce and physically motivate these maps, and demonstrate that a wide class of plasma models can be obtained from the two-fluid model via a rigorous variational procedure. We also invoke Noether's theorem to determine the conserved quantities corresponding to the Galilean symmetries of the models.

The HAP formalism can also be used to construct new models from scratch through the suitable imposition of *a priori* frozen-in constraints. We exploit this feature to our advantage in Chapter 6 by duly constructing a rarely studied model of extended MHD, which is dubbed inertial MHD, as it does not assume that the electrons are inertialess, i.e. the mass of the electrons is not entirely neglected. After presenting the equations for the model, and the noncanonical Hamiltonian formulation, we show that this model, in a highly simplified limit, reduces to the famous Ottaviani-Porcelli model [61] of reconnection, which has been widely used in fusion and astrophysical plasmas.

Each of the previous chapters increasingly point towards connections between several variants of extended MHD. In Chapter 7, we tackle this issue in greater detail and demonstrate that most extended MHD models possess a common mathematical structure, which is best understood by reverting to noncanonical Hamiltonian dynamics. Furthermore, we use the Lagrangian picture of the fluid, which models it as a continuum collection of particles, to physically motivate this common mathematical structure, thereby pointing towards the existence of a common action principle formulation for all extended MHD models.

Finally, we conclude in Chapter 8 with an overview of the work accomplished in the thesis, and some of the outstanding issues that remain currently unexplored. We also highlight avenues of interest which are well-suited to future investigations stemming from a HAP-based approach.

Although we present a fairly detailed discussion of the basics in Chapter 2, we shall endeavour to keep the subsequent chapters self-contained to a high degree. We believe that this will facilitate an easier, and more selective, reading experience although it comes at the price of repetitiveness.

Chapter 2

Hamiltonian and Action Principles: The basics

In this Chapter, we shall present an exposition of the principles Hamiltonian and Action Principle formalisms, and apply it to ideal MHD. We closely follow the approach and notation employed in [17, 28, 62, 63, 64]. [1]

2.1 Hamilton's Principle of Least Action

We commence with a brief discussion of the action principle in discrete classical mechanics. A more comprehensive discussion can be found in [15, 66].

The prescription outlined in most textbooks is the same in employing Hamilton's Principle of Least Action. We begin by identifying a configuration space and variables that describe the system in its entirety; these are the generalized coordinates $q^i(t)$, where $i = 1, 2, \ldots, N$ and N is the number of degrees of freedom of the system. The second step entails the construction of the Lagrangian, which is typically of the form $L := T - V$, and is obtained by identifying the kinetic energy T and potential energy V, yielding the action

[1] A part of this chapter was published in: "Hamiltonian and action formalisms for two-dimensional gyroviscous magnetohydrodynamics", P. J. Morrison, M. Lingam & R. Acevedo, 2014, *Phys. Plasmas*, **21**, 082102 [65]. The underlying physical principles were developed by P. J. Morrison and R. Acevedo, and extended by this author (M. Lingam).

functional,

$$S[q] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)\,. \tag{2.1}$$

Mathematically, the word "functional" refers to a quantity whose domain is comprised of functions and whose range is given by real numbers. Let us suppose that we are given path a $q(t)$, and the action functional $S[q]$ returns a real number upon substitution of this path into the above expression.

In Hamilton's principle, the initial and final limits of the path, $q(t_1)$ and $q(t_2)$, are fixed and the path that gives rise to the extremal value is sought. By "extremal", we imply that the functional derivative of the action vanishes, i.e. we require $\delta S[q]/\delta q^i = 0$, where the functional derivative is defined by

$$\begin{aligned} \delta S[q;\delta q] &= \left.\frac{dS[q+\epsilon\delta q]}{d\epsilon}\right|_{\epsilon=0} =: \left\langle \frac{\delta S[q]}{\delta q^i}, \delta q^i \right\rangle \\ &= \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} \right) \delta q^i\,. \end{aligned} \tag{2.2}$$

For an extended discussion of functional derivatives, the reader is referred to the excellent works by [67, 68]. In the above expression, $\delta q(t)$ represents the arbitrary perturbation of a path $q(t)$; as $\delta q(t)$ is specified to be entirely arbitrary, the only way for δS to vanish for all choices of $\delta q(t)$ is to have the quantity within the parentheses vanish, i.e.

$$\frac{\delta S[q]}{\delta q^i} = 0 \quad \Leftrightarrow \quad \frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0\,. \tag{2.3}$$

In other words, we see that the extremal path corresponds to the Euler-Lagrange equations of motion.

2.2 The Lagrangian and Eulerian viewpoints of fluids and magnetofluids

In this section we review the Lagrangian and Eulerian descriptions of a fluid and the relationship between them. The section is divided into two parts. Firstly, we describe the basic Lagrangian variable that describes the trajectory of a fluid element, and then present some useful algebraic identities and properties. The Lagrangian picture of a fluid is naturally endowed with a least principle, as it models the fluid as a (continuous) collection of particles. Next, we explore the relationship between the intrinsic properties of the fluid and their Eulerian counterparts. The two descriptions of the fluid are shown to be related through the Lagrange - Euler maps. For a more detailed exposition of the background material we suggest [12, 17, 18, 69].

Before proceeding further, some comments regarding the notation are in order. We shall adopt the notation employed in [17] where vectors and scalars possess a similar form - the former must be distinguished entirely by the context. For instance, the generalized coordinate $q(a,t)$, the velocity v, the magnetic field B, etc. are all vectors; quantities such as the density ρ and the entropy s serve as scalars.

2.2.1 The Lagrangian variable $q(a,t)$ and its properties

The *Lagrangian variable* $q(a,t)$ is a generalized coordinate that denotes the position of a particular fluid element, which is also referred to as the fluid particle or parcel, at a given time t. The coordinate, which indicates the position relative to the origin is denoted by $q = q(a,t) = (q^1, q^2, q^3)$; for the sake of simplicity Cartesian coordinates are used henceforth. The quantity $a = (a^1, a^2, a^3)$ denotes the *fluid element label* at time $t = 0$, which implies that $a = q(a, 0)$, but this means of labeling need not always be the case (cf. [70]). In general, the continuous label a is the continuum analog of the discrete index that enables us to track a given particle in a finite degree-of-freedom system. The map $q\colon D \to D$ is assumed to be one-to-one and onto at a given fixed time t, with D serving as the domain occupied by the fluid. We will further suppose that q is invertible and smooth and also impose any other "nice" properties that the problem necessitates. It is to be noted that these assumptions may not be entirely justified, and are primarily chosen to enable us to carry out the algebra on a formal level.

Given the Lagrangian coordinate q, we introduce two other related important quantities which play an important role: the deformation matrix, $\partial q^i/\partial a^j =: q^i_{\ ,j}$ and the corresponding determinant, the Jacobian, $\mathcal{J} := \det(q^i_{\ ,j})$; the latter, in three and two dimensions, has the form

$$\mathcal{J} = \frac{1}{6}\epsilon_{kjl}\epsilon^{imn} q^k_{\ ,i} q^j_{\ ,m} q^l_{\ ,n}\,, \tag{2.4}$$

$$= \frac{1}{2}\epsilon_{kj}\epsilon^{il} q^k_{\ ,i} q^j_{\ ,l}\,, \tag{2.5}$$

where $\epsilon_{ijk} = \epsilon^{ijk}$ and $\epsilon_{ij} = \epsilon^{ij}$ are the Levi-Civita tensors in the appropriate number of dimensions. By assuming that the label a specifies a unique trajectory, we conclude that $\mathcal{J} \neq 0$; this ensures the invertibility of $q = q(a,t)$, denoted by $a = a(q,t)$. Physically, we interpret the quantity $a(q,t)$ as the label of a fluid element that reaches the position q at time t. In general, for coordinate transformations we have

$$q^i_{,k} a^k_{,j} = a^i_{,k} q^k_{,j} = \delta^i_{,j}\,,$$

i.e. the the deformation matrix has an inverse given by $a^k_{,j} = \partial a^k/\partial q^j$. We also use the Einstein summation convention everywhere, unless explicitly specified. Using $q(a,t)$ or its inverse, we can express quantities such as $a^k_{,j}$ as functions of either q or a.

The volume element d^3a at time $t = 0$ maps into the volume element at time t according to

$$d^3q = \mathcal{J} d^3a\,, \tag{2.6}$$

and the components of an area element evolve as per

$$(d^2q)_i = \mathcal{J} a^j_{,i}\,(d^2a)_j\,, \tag{2.7}$$

where $\mathcal{J} a^j_{,i}$ is the transpose of the cofactor matrix of $q^j_{,i}$; it is given by

$$\mathcal{J} a^i_{,k} = \frac{1}{2}\epsilon_{kjl}\epsilon^{imn} q^j_{,m} q^l_{,n} \quad \text{or} \quad \mathcal{J} a^i_{,k} = \epsilon_{kj}\epsilon^{il} q^j_{,l}\,, \tag{2.8}$$

in three and two dimensions, respectively. A couple of other useful identities include

$$\frac{1}{\mathcal{J}}\frac{\partial \mathcal{J}}{\partial q^i_{,j}} = a^j_{,i}\,, \tag{2.9}$$

$$\frac{\partial(\mathcal{J} a^i_{,k})}{\partial a^i} = 0\,, \tag{2.10}$$

where (2.9), the standard rule for differentiation of determinants, follows from (2.4) or (2.5), and (2.10) follows from (2.8) by the antisymmetry of ϵ_{ijk} or ϵ_{ij}.

2.2.2 Attributes, observables, and the Lagrange to Euler map

Up to now, we have considered kinematical properties of the fluid, as described by the Lagrangian coordinate q. But, a fluid element is not solely characterized by its position q and its label a. In addition, we observe that the fluid 'particle' is endowed with several intrinsic properties, i.e. it may carry a certain density, or be endowed with some magnetic flux; the latter occurs when one considers magnetofluid theories such as magnetohydrodynamics (MHD). We shall now study these intrinsic properties in three spatial dimensions.

We will refer to quantities that the fluid element transports as the *attributes*, since they are intrinsic to the fluid under consideration. A fluid element that starts off at time $t = 0$ carries its attributes, which remain unchanged with time. Thus, by definition, we see that attributes are purely functions of the label a, and are Lagrangian variable constants of motion. We will use the subscript '0' to distinguish attributes from their Eulerian counterparts, discussed below.

Most of the times, in fluid theories, the Lagrangian variable description is not emphasized and, consequently, the attributes are usually not discussed. More typically, it is the Eulerian fields that are emphasized and observed. Before addressing the Eulerian fields, it is important to ask what the Eulerian picture of a fluid is.

In the Lagrangian picture, we have seen that the fluid 'particle' traverses through the domain, carrying with it several attributes along the way.

On the other hand, the Eulerian picture does not distinguish between fluid particles. Instead, imagine that we stick in a probe into the fluid at a certain point $r := (x, y, z) = (x^1, x^2, x^3)$ at a time t. We will find that the fluid has a certain density, velocity, etc. This entails the Eulerian picture of a fluid, wherein all fields are viewed as functions of r and t. We will refer to these fields as Eulerian observables, or just *observables* for short. Some of the most commonly used Eulerian observables include velocity field $v(r, t)$ and the mass density $\rho(r, t)$.

We reiterate that it is crucial to distinguish the Lagrangian coordinate q from the Eulerian observation point r. The latter is an independent variable that does not move with the fluid, although it is a point of D. The inability or unwillingness to distinguish between the two descriptions has led to confusion in the literature. As we have argued for the existence of two independent physical descriptions of the same system, it is quite natural to argue that the two must be connected somehow.

In other words, given $q(a, t)$ and the attributes, we require the observables to be uniquely determined based on the nature of the attributes, in particular, their tensorial properties. For example, consider the velocity field $v(r, t)$. If we were to insert a velocity probe into a fluid at (r, t), we would measure the velocity of the fluid element that happened to be at r at time t. Hence, $\dot{q}(a, t) = v(r, t)$, where the overdot indicates that the time derivative is obtained at fixed a. We are still left with the ambiguity of determining the label a, but the element at r is given by $r = q(a, t)$, whence $a = q^{-1}(r, t) =: a(r, t)$.

By combining all this information, we see that the Eulerian velocity field is given by

$$v(r,t) = \dot{q}(a,t)\big|_{a=a(r,t)} \ . \tag{2.11}$$

The above expression is an example of the Lagrange to Euler map that supplies a means of moving from one picture to the other.

Attributes, as part of their definition, possess rules for transformation to their corresponding Eulerian observables. The totality of these rules determines the set of observables. For a continuum system, in which mass is neither created nor destroyed, it is natural to attach a mass density, $\rho_0(a)$, to the element labelled by a. We note that the mass in a given volume is given by $\rho_0 d^3a$. By demanding that the mass by conserved, regardless of whether one uses the Eulerian or Lagrangian picture, we see that $\rho(r,t)d^3r = \rho_0 d^3a$. By using (2.6) we obtain $\rho_0 = \rho\mathcal{J}$. This defines the rule for transforming to the Eulerian description, and constitutes one example of a *Lagrange to Euler map.*

Next, let us consider the specific entropy (per unit particle). It is common to have the entropy conserved along the streamlines in the absence of heat flow or associated effects; in such a scenario, we require the entropy to be constant along a trajectory. Hence, we require the attribute $s_0(a)$ to equal the Eulerian observable $s(r,t)$. Thus, the Lagrange to Euler map for the specific entropy is $s_0 = s$. Similarly, we may attach a magnetic field $B_0(a)$ to a given fluid element, and define its transformation law by insisting on frozen-in flux. This yields $B \cdot d^2r = B_0 \cdot d^2a$, and from (2.7) we obtain $\mathcal{J}B^i = q^i_{\ ,j}\, B_0^j$.

However, there is still a missing link as the observables are functions of r and t, whilst the attributes and the trajectory q depend on a and t. Hence, we evaluate the expressions for the mass density, the specific entropy and the magnetic field at $a = q^{-1}(r,t) =: a(r,t)$, thereby yielding the Lagrange to Euler map for these quantities. In other words, given $q(a,t)$ and the attributes, the fields $\{\rho, s, v, B\}$, constituting the observables, are now defined.

Most of the time we will find it convenient to work with the alternative set of observables $\{\rho, \sigma, M, B\}$, where $M = \rho v$ is the kinetic momentum density and $\sigma = \rho s$ is the entropy density. This allows a convenient way to represent the Lagrange to Euler map in an integral form by using an appropriate Dirac delta function; the latter is used as a probe to 'pluck out' the fluid element that happens to be at the Eulerian observation point r at time t. As an example of this procedure, the mass density $\rho(r,t)$ is obtained by

$$\begin{aligned}\rho(r,t) &= \int_D d^3a\, \rho_0(a)\, \delta\left(r - q\left(a,t\right)\right) \\ &= \left.\frac{\rho_0}{\mathcal{J}}\right|_{a=a(r,t)} .\end{aligned} \tag{2.12}$$

The expression for σ is entirely akin to that of the density, and we do not present it separately, as we just replace $\rho \to \sigma$ in the above expression. We will introduce the canonical momentum density, $M^c = (M_1^c, M_2^c, M_3^c)$, which is related to the Lagrangian canonical momentum through the expression

$$\begin{aligned}M^c(r,t) &= \int_D d^3a\, \Pi(a,t)\, \delta\left(r - q(a,t)\right) \\ &= \left.\frac{\Pi(a,t)}{\mathcal{J}}\right|_{a=a(r,t)} .\end{aligned} \tag{2.13}$$

The superscript 'c' indicates that the momentum density constructed is the canonical one, as opposed to a different momentum density introduced in the next section. For most fluid theories, $\Pi(a,t) = (\Pi_1, \Pi_2, \Pi_3) = \rho_0 \dot{q}$. This in turn implies that $M^c = M = \rho v$. In general, note that $\Pi(a,t)$ can be found from the Lagrangian through $\Pi(a,t) = \delta L/\delta \dot{q}$ and is not always equal to $\rho_0 \dot{q}$. Lastly,

$$\begin{aligned} B^i(r,t) &= \int_D d^3a \, q^i_{\ ,j}(a,t) B_0^j(a) \, \delta\left(r - q(a,t)\right) \\ &= q^i_{\ ,j}(a,t) \frac{B_0^j(a)}{\mathcal{J}} \bigg|_{a=a(r,t)} , \end{aligned} \tag{2.14}$$

for the components of the magnetic field.

We round off this subsection with a couple of useful identities between the Eulerian and Lagrangian variables. In our subsequent calculations, we encounter Eulerian gradients quite often. The components of the gradient, in Eulerian form, can be mapped to the Lagrangian variables as follows

$$\frac{\partial}{\partial x^k} = a^i_{\ ,k} \frac{\partial}{\partial a^i} \bigg|_{a=a(r,t)} . \tag{2.15}$$

By using the condition that $r = q(a,t)$, the time derivative of any function $f(a,t) = \tilde{f}(r,t) = \tilde{f}(q(a,t),t)$ can be mapped to the corresponding Eulerian variables according to the expression

$$\begin{aligned} \dot{f}\Big|_{a=a(r,t)} &= \frac{\partial \tilde{f}}{\partial t} + \dot{q}^i(a,t) \frac{\partial \tilde{f}}{\partial x^i} \bigg|_{a=a(r,t)} \\ &= \frac{\partial \tilde{f}}{\partial t} + v \cdot \nabla \tilde{f}(r,t) \, . \end{aligned} \tag{2.16}$$

As stated earlier, we note that the overdot denotes the time derivative at constant a, $\partial/\partial t$ denotes the time derivative at constant r, and ∇ is the Eulerian derivative, i.e. $\partial/\partial r$ with components $\partial/\partial x^i$.

Lastly, we can obtain an evolution equation for the determinant $\mathcal{J}$ using Eq. (2.9)

$$\dot{\mathcal{J}} = \frac{\partial \mathcal{J}}{\partial q^i_{,j}} \dot{q}^i_{,j} = \mathcal{J} a^j_{,i} \dot{q}^i_{,j} , \tag{2.17}$$

which upon evaluation at $a = a(r,t)$ gives a formula due to Euler [12],

$$\frac{\partial \tilde{\mathcal{J}}}{\partial t} + v \cdot \nabla \tilde{\mathcal{J}} = \tilde{\mathcal{J}} \, \nabla \cdot v \, . \tag{2.18}$$

2.2.3 A note on Lie-dragged dynamical equations

As we have seen in the previous subsection, the Lagrange to Euler maps emerge via the natural imposition of conservation laws. In this section, we examine these conservation laws in further detail, and emphasize their geometric significance.

Let us first start with the specific entropy s. From the Lagrange-Euler map $s = s_0$ and (2.16), we can show that the evolution equation for s is

$$\frac{\partial s}{\partial t} + v \cdot \nabla s = 0, \tag{2.19}$$

and this lends itself to a twofold interpretation. We can view it as the Lie-dragging of a 0-form, or as the Lie-dragging of a scalar, i.e. the above expression can be written as

$$\left(\frac{\partial}{\partial t} + \mathfrak{L}_v \right) s = 0, \tag{2.20}$$

where $\mathfrak{L}$ denotes the Lie derivative with the velocity v serving as the flow field. For a discussion of Lie-dragging, and a geometric interpretation of invariants, in the context of fluid and plasma theories we refer the reader to [51, 54, 71].

Next, we can carry out the same procedure for the density by using $\rho_0 = \rho\mathcal{J}$, in conjunction with (2.16) and (2.18). We arrive at

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{2.21}$$

and it can be interpreted as the Lie-dragging of a scalar density of weight 1. Alternatively, one can rewrite the above expression as

$$\left(\frac{\partial}{\partial t} + \mathfrak{L}_v\right)\left(\rho\, d^3 r\right) = 0, \tag{2.22}$$

implying that the 3-form $\rho\, d^3 r$ is Lie-dragged.

Lastly, using the relation $\mathcal{J} B^i = q^i_{\ ,j}\, B_0^j$ for the magnetic field, along with (2.16) and (2.18), we obtain

$$\frac{\partial B}{\partial t} + B\,(\nabla \cdot v) - (B \cdot \nabla)\, v + (v \cdot \nabla)\, B = 0, \tag{2.23}$$

which can be cast into the more familiar form

$$\frac{\partial B}{\partial t} - \nabla \times (v \times B) = 0, \tag{2.24}$$

provided that $\nabla \cdot B = 0$. Notice that (2.24) is precisely the widely-used induction equation of ideal MHD [60]. In MHD, it is common to view flux-freezing as a consequence of the induction equation, but the Lagrangian viewpoint of MHD emphasizes the flux-freezing as a fundamental relation; the induction equation, then, becomes a consequence of flux-freezing. Following the same line of reasoning, it is possible to view (2.23) as the Lie-dragging of a vector density of weight 1. If the fluid were incompressible, the fourth term on the LHS would vanish yielding the Lie-dragging of one vector field (B) by another (v). We can also rewrite (2.23) to yield

$$\left(\frac{\partial}{\partial t} + \mathfrak{L}_v\right)\left(B \cdot d^2 r\right) = 0, \tag{2.25}$$

implying that the 2-form $B \cdot d^2r$, the magnetic flux, is Lie-dragged.

2.3 A general procedure for building an action principle for continuum models

In this section, we provide a brief summary of the general methodology advocated in [64, 65] for building action principles for continuum fluid models. As opposed to ordering in the equations of motion or *ad hoc* methods that are deployed in obtaining models from a basic set of equations, we can introduce each term in the action serially, and emphasize the physical relevance of each term to the model being built. This allows for an improved physical understanding and motivation as to why the different terms arise, and what roles they play in the model. In most cases, the terms in the action are the continuum analogs of their discrete counterparts, and the latter usually have clear-cut interpretations and uses.

The first step in constructing an action principle lies in choosing the domain D. For a fluid it would be either one, two, or three-dimensional, $D \subset$

$\mathbb{R}^{1,2,3}$. Furthermore, we suppose that there exists a Lagrangian (trajectory) variable $q\colon D \to D$. We also suppose that $q(a,t)$, where the label $a \in D$, is a well behaved function that is smooth, has an inverse, etc. as noted in Section 2.2.

The next step lies in choosing the sets of attributes and the corresponding observables, defined via a Lagrange to Euler map. There is some freedom in choosing the set of observables that interest us, as discussed in the previous section. It is important to recognize that the observables must be completely determined by the functions $q(a,t)$ and the attributes, but the converse statement is not a necessity.

From the analogy with Hamilton's action principle in mechanics, it is evident that the action will comprise of terms that involve the variable $q(a,t)$ and its derivatives with respect to both of its arguments. The last step of the method is to impose a most stringent requirement upon the terms in the action – viz. the existence of a *closure principle* which ultimately means that our theory must be 'Eulerianizable.' More precisely, we impose the condition that our action must be expressible entirely in terms of our set of observables. Such a requirement is well motivated, since it leads to energy-like quantities that are entirely expressible in terms of the desired Eulerian variables.

To illustrate the Eulerian closure principle, let us take the kinetic energy as an example, which satisfies

$$T[q] := \frac{1}{2}\int_D d^3a\, \rho_0(a)|\dot{q}|^2 = \frac{1}{2}\int_D d^3r\, \rho|v|^2\,, \tag{2.26}$$

where $|\dot{q}|^2 := \dot{q}^i g_{ij} \dot{q}^j = \dot{q}^i \dot{q}_i$ and for Cartesian coordinates, the metric $g_{ij} = \delta_{ij}$ is chosen. Thus, the Lagrangian variable description of the first equality can be written as the purely Eulerian description of the second. On the other hand, suppose we consider the term

$$E_M[q] := \frac{1}{2} \int_D d^3a \, |B_0(a)|^2, \tag{2.27}$$

we see that it resembles the magnetic energy density, but it *cannot* be written purely in terms of Eulerian variables. As a result, it is not a viable candidate for the magnetic energy density.

The imposition of the closure principle leads to important consequences: equations of motion that are purely expressible in terms of our observables, i.e. an Eulerian variable description, and an Eulerian Hamiltonian description in terms of noncanonical Poisson brackets, which are discussed in the subsequent sections. At this stage, we also reiterate that there is no evident *a priori* rationale that ensures that all (closed) fluid models possess both Eulerian and Lagrangian descriptions; there may be instances where only the former exist.

2.4 Ideal MHD: the Newcomb action

Hitherto, our discussion has veered towards the abstract, and we shall present a concrete model that embodies our methodology. We present the action originally derived by Newcomb [18], but we maintain consistency with our prior discussions and notation. We observe that associated discussions can also be found in [63, 64, 65].

The kinetic energy was already determined previously, and has the form

$$T[q] := \frac{1}{2}\int_D d^3a\,\rho_0(a)|\dot{q}|^2 = \frac{1}{2}\int_D d^3r\,\rho|v|^2\,, \tag{2.28}$$

The fluid is also endowed with an internal energy density U per unit mass. We assume that U is a function of the thermodynamical variables ρ and s. From the expression $dU = Tds - PdV$, and using the relation between V and ρ, we obtain the auxiliary relations

$$P = \rho^2\frac{\partial U}{\partial \rho}, \quad T = \frac{\partial U}{\partial s}. \tag{2.29}$$

The internal energy contribution to the action is

$$U[q] := \int_D d^3a\rho_0(a)\,U\left(\frac{\rho_0}{\mathcal{J}}, s_0\right) = \int_D d^3r\rho\,U\left(\rho, s\right), \tag{2.30}$$

and the first equality is chosen such that the second equality satisfies the Eulerian closure principle. Lastly, there is the magnetic energy density

$$E_M[q] := \frac{1}{2}\int_D d^3a\,\frac{q^i_{,j}q^i_{,k}B_0^jB_0^k}{\mathcal{J}} = \frac{1}{2}\int_D d^3r\,|B|^2, \tag{2.31}$$

and we observe that the second equality follows by applying the Lagrange-Euler maps to the first one. Once again, we see that this term is consistent with the tenets of the Eulerian closure principle. In the rest of the thesis, we operate in SI units with $\mu_0 = 1$, unless explicitly indicated otherwise.

The ideal MHD action is given by

$$S_{MHD}[q] := T[q] - U[q] - E_M[q], \tag{2.32}$$

where the RHS is determined via the equations (2.28), (2.30) and (2.31). We can now vary the above action with respect to q and use the identities presented in Section 2.2. We divide the ensuing result by $\mathcal{J}$ and apply the Lagrange-Euler maps throughout. Our final equation of motion is entirely Eulerian in nature, and is given by

$$\rho\left(\frac{\partial}{\partial t}+v\cdot\nabla\right)v=-\nabla\left(P+\frac{|B|^2}{2}\right)+B\cdot\nabla B, \tag{2.33}$$

where we have assumed $\nabla\cdot B=0$, and the pressure is defined in (2.29). We have already indicated, in Section 2.2, that the entropy, density and the magnetic field evolve as per (2.19), (2.21) and (2.23); the latter reduces to the familiar induction equation (2.24) of ideal MHD under the assumption $\nabla\cdot B=0$.

Finally, a comment regarding the internal energy U is in order. Although we have assumed it to be a function of ρ and s, we can extend this to include a $|B|$-dependence as well. In such an event, one finds that the pressure appearing in (2.33) becomes anisotropic in nature. In fact, it was shown in [57, 62] that a suitable choice of the $|B|$-dependent internal energy function U gave rise to the famous Chew-Goldberger-Low theory [72].

2.5 Ideal MHD: Reduction to noncanonical Hamiltonian dynamics

In the previous sections, we have presented an action principle (Lagrangian) formulation of fluid models, couched in the Lagrangian viewpoint of the fluid, where the variation of the action was solely undertaken with respect to q. However, the Eulerian viewpoint is the more commonly used one, and the Hamiltonian formalism is endowed with unique advantages of its own. Hence, we need to transition from a Lagrangian, action principle formulation to an Eulerian, Hamiltonian formulation. Formally, this procedure is termed reduction and involves mathematical intricacies that fall outside the scope of the thesis. The reduction procedure stems from the early works of [73, 74, 75], and was applied to fluid and kinetic models in [76, 77]. A geometric formulation of this procedure was presented in [32] and a summary can be found in [38, 66].

Instead, we adopt a different approach by focusing on a more direct approach, by using ideal MHD as a specific example. We shall work through the procedure in some detail, and other models in future chapters are also obtained through a similar approach.

2.5.1 Noncanonical Hamiltonian formulation of ideal MHD

We have specified the ideal MHD action, in Lagrangian variables, in (2.32). Now, we can compute the associated canonical momentum $\Pi = \delta L/\delta \dot{q}$, which yields $\Pi = \rho_0 \dot{q}$. The Hamiltonian functional is determined via a Legendre transform, and is given by

$$H[q,\Pi] = \int_D d^3a\, \dot{q} \cdot \Pi - L, \tag{2.34}$$

and the convexity property of ideal MHD allows us to write $\dot{q}$ in terms of Π and obtain the Hamiltonian. However, we observe that our action is in terms of q and Π, and we use the Lagrange-Euler maps to express the Hamiltonian in terms of Eulerian variables. Upon doing so, we find that it simplifies to

$$H_{MHD} = \int_D d^3r \left[\frac{\rho|v|^2}{2} + \rho U(\rho, s) + \frac{|B|^2}{2}\right], \tag{2.35}$$

and one can arrive at the same result through the use of Noether's theorem, and subsequent Eulerianization, i.e. expressing everything in terms of Eulerian variables.

In terms of Π and q, one can determine Hamilton's equations of motion through the canonical Poisson bracket in infinite dimensions,

$$\{\bar{F}, \bar{G}\} = \int_D d^3a \left(\frac{\delta\bar{F}}{\delta q}\cdot\frac{\delta\bar{G}}{\delta\Pi} - \frac{\delta\bar{G}}{\delta q}\cdot\frac{\delta\bar{F}}{\delta\Pi}\right), \tag{2.36}$$

where $\bar{F}$ and $\bar{G}$ are arbitrary functionals that depend on q and Π. In order to verify that the above bracket yields (2.33), we must calculate $\dot{q} = \{q, H\}$ and $\dot{\Pi} = \{\Pi, H\}$ and carry out the Eulerianization procedure; the Hamiltonian to be used is found from (2.34) by using the Lagrangian for ideal MHD.

As our system is endowed with a canonical Poisson bracket in *Lagrangian* variables that generates the correct dynamics, it is natural to look for a Poisson bracket that generates the same dynamics in *Eulerian* variables. However, it is crucial to recognize that Eulerian variables are *not* canonical in nature. As a result, the Poisson bracket in Eulerian variables does *not* possess the same form as (2.36). We will now outline the procedure by which the Eulerian noncanonical Poisson bracket is obtained from its Lagrangian and canonical counterpart.

The first step entails choosing our set of observables for ideal MHD. We shall work with $\{\rho, \sigma, M, B\}$, and for each of them, the Lagrange-Euler maps can be expressed in an integral form, as described in Section 2.2. Let us recall the relations once more:

$$\begin{aligned} \rho(r,t) &= \int_D d^3a\, \rho_0(a)\, \delta\left(r - q\left(a,t\right)\right) \\ &= \left.\frac{\rho_0}{\mathcal{J}}\right|_{a=a(r,t)}, \end{aligned} \tag{2.37}$$

$$\begin{aligned} \sigma(r,t) &= \int_D d^3a\, \sigma_0(a)\, \delta\left(r - q\left(a,t\right)\right) \\ &= \left.\frac{\sigma_0}{\mathcal{J}}\right|_{a=a(r,t)}, \end{aligned} \tag{2.38}$$

$$\begin{aligned} M^c(r,t) &= \int_D d^3a\, \Pi(a,t)\, \delta\left(r - q(a,t)\right) \\ &= \left.\frac{\Pi(a,t)}{\mathcal{J}}\right|_{a=a(r,t)}, \end{aligned} \tag{2.39}$$

$$\begin{aligned} B^i(r,t) &= \int_D d^3a\, q^i_{\ ,j}(a,t) B_0^j(a)\, \delta\left(r - q(a,t)\right) \\ &= q^i_{\ ,j}(a,t) \left.\frac{B_0^j(a)}{\mathcal{J}}\right|_{a=a(r,t)}, \end{aligned} \tag{2.40}$$

and ideal MHD yields $\Pi = \rho_0 \dot{q}$, implying that $M = \rho v = M^c$. As a result, we can replace the LHS of (2.39) with M instead.

In obtaining the noncanonical bracket, the cornerstone of our procedure stems, once again, from the fact that a fluid, or magnetofluid, theory must be equally describable by Lagrangian and Eulerian viewpoints. Hence, we demand that

$$\bar{F}[q,\Pi] = F[\rho,\sigma,M,B], \tag{2.41}$$

which implies

$$\begin{aligned} \delta\bar{F} &\equiv \int_D d^3a\, \frac{\delta\bar{F}}{\delta\Pi}\cdot\delta\Pi + \frac{\delta\bar{F}}{\delta q}\cdot\delta q \\ &= \delta F \equiv \int_D d^3r\, \frac{\delta F}{\delta\rho}\delta\rho + \frac{\delta F}{\delta\sigma}\delta\sigma + \frac{\delta F}{\delta M}\cdot\delta M + \frac{\delta F}{\delta B}\cdot\delta B. \end{aligned} \tag{2.42}$$

Now, let us consider (2.37), which yields

$$\delta\rho = -\int_D d^3a\,\rho_0(a)\,\nabla\delta\left(r-q\left(a,t\right)\right)\cdot\delta q, \tag{2.43}$$

and this can be substituted into (2.42), thereby yielding

$$\int_D d^3a\,\frac{\delta\bar{F}}{\delta\Pi}\cdot\delta\Pi + \frac{\delta\bar{F}}{\delta q}\cdot\delta q = -\int_D d^3r\,\frac{\delta F}{\delta\rho}\int_D d^3a\,\rho_0(a)\,\nabla\delta\left(r-q\left(a,t\right)\right)\cdot\delta q + \dots, \tag{2.44}$$

and the '. . . ' indicate that a similar procedure is carried out for (2.38), (2.39) and (2.40) as well. In the above expression, one can carry out an integration by parts and isolate the functional derivatives by eliminating $\int_D d^3a$ through equating the coefficients of δq and $\delta\Pi$. Thus, we arrive at

$$\frac{\delta\bar{F}}{\delta\Pi} = \int_D d^3r\,\frac{\delta F}{\delta M}\delta\left(r-q\left(a,t\right)\right), \tag{2.45}$$

and

$$\frac{\delta\bar{F}}{\delta q} = \mathcal{O}_\rho\frac{\delta F}{\delta\rho} + \mathcal{O}_\sigma\frac{\delta F}{\delta\sigma} + \mathcal{O}_M\frac{\delta F}{\delta M} + \mathcal{O}_B\frac{\delta F}{\delta B}, \tag{2.46}$$

and the $\mathcal{O}$'s are integral operators that involve factors of $\int_D d^3r$, Dirac delta functions, etc. The expression for $\mathcal{O}_\rho$ can be read off by inspecting (2.43); the others are found through similar means. Thus, in principle we have computed $\delta\bar{F}/\delta\Pi$ and $\delta\bar{F}/\delta q$, which can now be substituted into (2.36). After a fair amount of algebraic manipulations, we arrive at

$$
\begin{aligned}
\{F,G\}_{MHD} \;=\; -\int_D d^3r \Bigg[& M_i \left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta F}{\delta M_i} \right) \\
& +\rho \left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta F}{\delta \rho} \right) \\
& +\sigma \left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta F}{\delta \sigma} \right) \\
& +B_i \left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta G}{\delta B_i} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x_j}\frac{\delta F}{\delta B_i} \right) \\
& +B_i \left(\frac{\delta G}{\delta B_j}\frac{\partial}{\partial x_i}\frac{\delta F}{\delta M_j} - \frac{\delta F}{\delta B_j}\frac{\partial}{\partial x_i}\frac{\delta G}{\delta M_j} \right) \Bigg], \qquad (2.47)
\end{aligned}
$$

and this is clearly a very different beast when compared to (2.36) - it constitutes the noncanonical Poisson bracket of ideal MHD, which was first introduced in [28]. The dynamical evolution of ψ is found from $\dot{\psi} = \{\psi, H\}$ where the bracket is given by (2.47) and the Hamiltonian is (2.35); note that the latter must be re-expressed in terms of M and σ. Upon carefully working out the dynamical equations, we find that they are identical to the ones obtained via the Newcomb action in Section 2.4.

As a result, we conclude that (2.35) and (2.47) give rise to a fully (noncanonical) Hamiltonian formulation of ideal MHD dynamics that is entirely Eulerian in nature. The corresponding bracket and Hamiltonian for ideal hydrodynamics (HD) is obtained by simply dropping the B-dependent terms in (2.35) and (2.47).

Now, the bracket (2.47) is quite clearly antisymmetric and bilinear. Furthermore, it is also easy to verify that it satisfies the Leibnitz rule, and it serves as an example of a Lie-Poisson bracket [17, 62]. Yet, a simple inspection of (2.47) will *not* suffice to convince us that the Jacobi identity holds true, which is one of the requirements for ensuring that (2.47) is a valid Poisson bracket. However, as our starting point, the canonical bracket (2.36), did satisfy the Jacobi identity and the procedure employed above preserves this property; for more details and references, the reader is referred to [17, 78, 79].

2.6 Noncanonical Hamiltonian dynamics

Although we highlighted the reduction procedure and obtained a noncanonical Poisson bracket for ideal HD and MHD, there was no commentary offered on the usefulness and properties of noncanonical brackets. These topics shall form the subject of our discussion in this section. We shall confine ourselves primarily to a discussion of finite-dimensional noncanonical Hamiltonian systems, as the generalization to infinite dimensions can be undertaken, albeit with some subtleties.

Given a time-independent function $H(\{z\})$, a Hamiltonian system is of the form

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j}; \quad i,j = 1,2\ldots 2n, \tag{2.48}$$

and one defines the Poisson bracket via

$$[f,g] = \frac{\partial f}{\partial z^i}J^{ij}\frac{\partial g}{\partial z^j}; \quad i,j = 1,2\ldots 2n, \tag{2.49}$$

and J^{ij} must be bilinear, antisymmetric and satisfy the Jacobi identity [15]. Most systems familiar to the readers are *canonical*, i.e. J has the form:

$$J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \tag{2.50}$$

where 0_n is an $n \times n$ matrix with all zeroes and and I_n is the $n \times n$ identity matrix. It is easy to verify that this yields the familiar form of Hamilton's equations, upon identifying $\{z^1, z^2 \dots z^n\} \equiv \{q^1, q^2 \dots q^n\}$ and $\{z^{n+1}, z^{n+2} \dots z^{2n}\} \equiv \{p^1, p^2 \dots p^n\}$.

However, most real-world models do not possess a canonical structure, although they can still be described via (2.48), i.e. they are Hamiltonian. By a noncanonical structure, we simply mean that J is *not* of the form (2.50), although it is still bilinear, antisymmetric and satisfy the Jacobi identity. For instance, it is possible that J could depend on the z's themselves, which occurs for the rigid body dynamics [17]. When one has $\det J \neq 0$, it is possible to find a suitable coordinate transformation that maps J to (2.50) - this is the famous Darboux theorem of classical mechanics [15].

However, what happens when $\det J = 0$? The answer was given by Sophus Lie [80], who generalized the Darboux theorem, and demonstrated that one could find a coordinate transformation that maps J to

$$J = \begin{pmatrix} 0_m & I_m & 0 \\ -I_m & 0_m & 0 \\ 0 & 0 & 0_{n-2m} \end{pmatrix}, \tag{2.51}$$

and we see that this is identical to (2.50) if we identify $m \to n$, and recognize that we are endowed with $n - 2m$ extraneous coordinates. As $\det J = 0$ now, there exists a degeneracy in J whose rank is now given by $2m$. Hence, it is possible for $n-2m$ null eigenvectors of J to exist, and these constitute the Casimir invariants which satisfy

$$J^{ij}\frac{\partial C^I}{\partial z^j}=0; \quad I=1,2,\ldots n-2m, \tag{2.52}$$

which can also be represented as the condition $\left[f, C^I\right]=0$ for all arbitrary choices of f. Collectively, one can envision the n-dimensional phase space $\mathcal{Z}$ foliated by $2m$-dimensional symplectic leaves $\mathcal{P}$; the latter are found from the intersection of $\mathcal{Z}$ with the $(n-2m)$-dimensional surfaces determined via $C^I=$ const. A consequence of the degeneracy is that any dynamics that originates on $\mathcal{P}$ stays on $\mathcal{P}$ throughout. For more details, and a pictorial view of the dynamics, the reader is referred to [17].

Before proceeding further, we turn our attention to an important class of systems, which possess a special form - they satisfy $J^{ij}=c_k^{ij}z^k$, where the elements of c_k^{ij} are all constants. Such systems are said to be a Lie-Poisson form, since they serve as structure constants of a suitable Lie algebra. Infinite dimensional Hamiltonian systems can also possess analogous properties - it is known that the noncanonical bracket of ideal MHD, represented by (2.47) in Section 2.5, is one such example. Lie-Poisson brackets are endowed with an abundance of beautiful mathematical features, and the reader can find an exhaustive study of their properties in [81].

Now, let us suppose that we did not have a degeneracy and J was given by (2.50). Then, one could simply determine the equilibria via $\partial H/\partial z^i=0$, which is evident from (2.48) as $\det J \neq 0$. However, when we are confronted with $\det J = 0$, the Casimirs come into play in determining the equilibria of our model. They are obtained from $\partial F/\partial z^i = 0$, where $F = H + \sum_I \lambda_I C_I$. It is appropriate to think of the Casimirs as Lagrange multipliers allowing for the (constrained) determination of the equilibria.

The next issue that arises is the stability of these equilibria. Hamiltonian systems have close connections with stability, which goes back all the way to a theorem proven by Lagrange for Hamiltonians of the form $H = p^2/2m + V(q)$. It states that equilibria satisfying $p_e = 0$ and q_e serving as a local minimum of V are stable. However, it is evident that not all Hamiltonian systems possess such a nice, and separable, form. Fortunately, an old theorem by Dirichlet [82] states that one only needs to analyse $\partial^2 H/\partial z^i \partial z^j$ - the definiteness of this matrix constitutes a sufficient condition for stability, although it is *not* a necessary and sufficient condition. However, this statement is true when one has $\det J \neq 0$ alone. When we have degeneracy, one must replace H by F implying that the definiteness property of $\partial^2 F/\partial z^i \partial z^j$ will suffice for stability. In general, the field of Hamiltonian stability is a subtle one, and extended discussions of the same can be found in [17, 27].

From our preceding discussion, it is clear that Casimirs play a crucial role in determining the equilibria and stability of Hamiltonian systems; the latter is often referred to as the Energy-Casimir method of determining stability criteria. Another primary method of deducing stability, involving dynamically accessible variations, also relies heavily on the notions of noncanonical Hamiltonian systems and Casimir invariants [17]. Casimirs also play a crucial role in *dissipative* systems, where they serve as surrogates for the entropy, and preserving consistency with the laws of thermodynamics. We do not investigate such aspects in this dissertation, but succinct accounts can be found in [45, 83].

When we move to infinite-dimensional systems, one must replace the functions with functionals, and the partial derivatives with functional derivatives. The cosympletic form J is replaced by the corresponding cosymplectic operator $\mathcal{J}$. The Poisson bracket is now represented as:

$$\{F, G\} = \int_D d^n x \, \frac{\delta F}{\delta \psi^i} \mathcal{J}^{ij} \frac{\delta G}{\delta \psi^j}, \tag{2.53}$$

and $D \in \mathbb{R}^n$ while the ψ's constitute the dynamical variables of our system and are functions of $\{x^1, x^2, \ldots x^n\}$ and t. The above discussions pertaining to the Casimirs and their role in equilibria and stability can be formulated here in an analogous manner [17]. For instance, the Energy-Casimir method requires us to compute the Hessian $\delta^2 F/\delta\psi^a\psi^b$ where the ψ's are the Eulerian fields. However, we wish to emphasize that there are still several unresolved, or partially resolved, subtleties regarding Casimir invariants and their role in dynamical systems. The interested reader is referred to [84, 85] to explore these issues further.

Before closing this section, a few observations regarding noncanonical Hamiltonian dynamics of ideal MHD are in order. Firstly, we note that the magnetic helicity,

$$C_M = \int_D d^3r\, A \cdot B, \tag{2.54}$$

is a Casimir invariant of the system; this can be seen by computing $\{F, C_M\}$ from (2.47) and verifying that it is always zero. Secondly, through a similar procedure, one can verify that

$$C_0 = \int_D d^3r\, \rho \mathcal{F}(s), \tag{2.55}$$

where $\mathcal{F}$ is arbitrary is also a Casimir of ideal MHD. Now, suppose that we drop entropy from the theory - one finds that the cross helicity

$$C_H = \int_D d^3r\, \frac{M \cdot B}{\rho} \equiv \int_D d^3r\, v \cdot B, \tag{2.56}$$

is also a Casimir invariant. If we assume the presence of additional symmetry (axisymmetry, translational or helical), the class of Casimirs becomes much more richer. This was illustrated in [70, 86], where the equilibria of these models were also computed following the principles delineated above. A comprehensive stability analysis using three different methodologies, including the Energy-Casimir method, was carried out by the same authors in [87]. We observe that the Energy-Casimir method has been deployed in other plasma contexts to undertake stability analyses; see for e.g. [88, 89, 90, 91, 92].

Chapter 3

A two-dimensional MHD model with gyroviscosity

In this section, we shall construct a simple model for two-dimensional (2D) MHD which is endowed with Finite Larmor Radius (FLR) effects. We determine and analyse the equilibria, and offer a few comments on the stability. We also indicate how reduced fluid models with gyroviscosity can be constructed as limiting cases of our model. [1]

3.1 Finite Larmor Radius effects: A discussion

Before commencing our treatment, it is necessary to motivate the study of FLR effects. Firstly, we begin by noting that one of the crucial approximations in deriving ideal MHD is that $\epsilon \equiv \rho_i/a \ll 1$ [93] where a is the length scale of the plasma and $\rho_i = v_{Ti}/\omega_{ci}$; the expressions ρ_i, v_{Ti} and ω_{ci} represent the gyroradius, thermal velocity and cyclotron frequency of the ions respectively. It is, of course, quite manifest that that above assumption is but one

[1]This chapter is based on the publication: "Hamiltonian and action formalisms for two-dimensional gyroviscous magnetohydrodynamics", P. J. Morrison, M. Lingam & R. Acevedo, 2014, *Phys. Plasmas*, **21**, 082102 [65]. All the work presented in the chapter was undertaken by this author (M. Lingam) and supervised by P. J. Morrison, and built upon the earlier efforts by R. Acevedo.

of many that are employed in obtaining ideal MHD. Hence, it is possible that the condition $\epsilon \ll 1$ may break down; one such example is systems wherein the magnetic field is very weak. In such an event, one must include corrections that arise from the presence of a finite Larmor radius. MHD models with FLR corrections originated in the 1950s and 1960s [59, 94, 95, 96], of which the work by Braginskii is the best known [59]. We refer the reader to [97] for a more exhaustive list of references in this area.

However, these corrections are necessary, primarily in a mathematical sense, to ensure that the extended model captures FLR effects accurately. Beyond the mathematical aspects, FLR effects also possess a host of physical consequences; we shall not summarize the many fusion applications herein, and refer the reader instead to [97, 98]. One of the key consequences of FLR effects is their ability to alter momentum transport, which is of huge importance in fusion and astrophysical plasmas. In the latter, such effects are of tremendous import in accretion discs, where the transport of angular momentum plays a key role [99, 100]. Rather surprisingly, despite the ubiquitous nature of the magnetorotational instability (MRI) in astrophysics, FLR effects have been studied only for a handful of models [101, 102, 103]; this is likely since the MRI is operational only when the magnetic field is 'weak'. In addition, we observe that dilute and weakly magnetized plasmas are now being modelled with FLR effects, as evidenced from [104, 105, 106, 107]. It must be emphasized, however, that the works cited use *fluid* models, which do not possess the same level of complexity as models that are kinetic or gyrokinetic in nature; the latter in particular has proven to be very useful in a host of astrophysical and fusion contexts [108, 109].

Before moving on the HAP derivation of our simple 2D gyroviscous MHD model, we record some of the methodologies employed in constructing such models. Fluid models that include FLR effects are often constructed by incorporating kinetic effects, e.g., by moving from particle phase-space coordinates to guiding center coordinates [110, 111, 112, 113]. FLR models have also been extended to include contributions arising from Landau damping, anisotropic pressures and curvature [114, 115, 116, 117, 118, 119]. A second approach involves expansions in the smallness of the Larmor radius as compared to a characteristic length scale of the system and the imposition of closures for higher-order moments [120, 121, 122, 123]. A third method uses the Hamiltonian framework to construct full and reduced MHD models with FLR effects [36, 37, 111, 124, 125, 126, 127, 128]. Our approach is akin to the third path described, but its Hamiltonian nature actually emerges via a detailed consideration of its underlying action principle formulation.

3.2 Building an action principle for the 2D gyroviscous fluid model

Now we follow the method described in Sec. 2.3. First we introduce and motivate the set of observables, then we describe how their corresponding attributes are used to construct an action principle.

3.2.1 The observables of the 2D gyroviscous model

We start off by choosing the domain $D = \mathbb{R}^2$, with coordinates (x, y), since our theory is two-dimensional. Hence, our model is endowed with translational symmetry in the $\hat{z}$-direction. We can work with either the canonical momentum defined in (2.13) or the 'kinetic' momentum defined by

$$\begin{aligned} M(r,t) &= \int_D d^2a\, \rho_0(a)\dot{q}(a,t)\,\delta\left(r - q(a,t)\right) \\ &= \left.\frac{\rho_0 \dot{q}(a,t)}{\mathcal{J}}\right|_{a=a(r,t)} . \end{aligned} \tag{3.1}$$

The 2D version of the canonical momentum defined through (2.13) is given by

$$\begin{aligned} M^c(r,t) &= \int_D d^2a\, \Pi(a,t)\,\delta\left(r - q(a,t)\right) \\ &= \left.\frac{\Pi(a,t)}{\mathcal{J}}\right|_{a=a(r,t)} , \end{aligned} \tag{3.2}$$

where we suppose there is no momentum in the $\hat{z}$-direction. As we have emphasized, the kinetic and the canonical momenta are not always the same; in fact, their difference gives rise to the *gyromap*, one of the key results in this chapter. When deriving the equations of motion, we work with M, although we shall use M^c extensively in the Hamiltonian formalism for this model. Next consider the magnetic field, which also belongs to our set of observables. Since $\nabla \cdot B = 0$, we decompose it as follows:

$$B = B_z(x,y,t)\,\hat{z} + \hat{z} \times \nabla\psi(x,y,t), \tag{3.3}$$

which is a usual decomposition with ψ representing the parallel vector potential. Following the same line of reasoning of Sec. 2.2.2, the associated attribute takes on the form

$$B_0 = B_{0z}(a)\,\hat{z} + \hat{z} \times \nabla_a \psi_0(a)\,. \tag{3.4}$$

which satisfies the properti $\nabla_a \cdot B_0 = 0$ and the subscript a on the gradient indicates spatial derivatives obtained with respect to the a's. One can obtain the correspondence between these attributes and observables by using (2.14), which yields

$$\begin{aligned} B_z(r,t) &= \int_D d^2a\, B_{0z}(a)\, \delta\left(r - q\left(a,t\right)\right) \\ &= \left.\frac{B_{0z}}{\mathcal{J}}\right|_{a=a(r,t)}\,, \end{aligned} \tag{3.5}$$

$$\psi(r,t) = \left.\psi_0\right|_{a=a(r,t)}\,. \tag{3.6}$$

We know that ψ serves as a magnetic stream function, making it analogous to the velocity stream function. Since the latter is preserved along a fluid trajectory, we see that (3.6) is also consistent with this notion. We have argued in Section 2.2 that the magnetic field behaves as a vector density of weight 1; from (3.5), we conclude that B_z serves as a scalar density of weight 1 for our model. Our last observable is the density, which is given by the 2D version of (2.12)

$$\rho(r,t) = \int_D d^2a\, \rho_0(a)\, \delta\left(r - q\left(a,t\right)\right) = \left.\frac{\rho_0}{\mathcal{J}}\right|_{a=a(r,t)}\,. \tag{3.7}$$

Thus, our set of observables is now $\{\rho, M, B_z, \psi\}$. Notice that we have dropped entropy from our theory, which can be re-incorporated without much difficulty.

Up to now we have not specified anything about the internal energy per unit mass, which in general is $U := U(\rho, s)$. However, we restrict ourselves to the barotropic case, i.e., assume a thermodynamic energy that is independent of the entropy s. Thus, the pressure is obtained from $U := U(\rho)$ via $P = \rho^2 dU/d\rho = \kappa\rho^2$, with κ constant, and the last equality follows by choosing a specific ansatz for U which is linear in ρ. The Lagrange to Euler map between P and P_0 can be determined through the use of (3.7); it takes on the form

$$P = \left.\frac{P_0}{\mathcal{J}^2}\right|_{a=a(r,t)} . \tag{3.8}$$

We shall now introduce a new variable, the usage of which will seem somewhat *ad hoc* at the moment; however, its purpose will soon become evident. The new variable attribute-observable pair is the following:

$$\beta = \frac{P}{B_z} \quad \text{and} \quad \beta_0 = \frac{P_0}{B_{0z}} . \tag{3.9}$$

We use the Lagrange to Euler maps for the $\hat{z}$-component of the magnetic field and the pressure, respectively given by (3.5) and (3.8). Together, they enable us to conclude that

$$\beta = \left.\frac{\beta_0}{\mathcal{J}}\right|_{a=a(r,t)} , \tag{3.10}$$

which demonstrates that the above equation is similar to (3.7) and (3.5), implying that all three variables obey a similar dynamical equation of motion. From (3.9), we see that only 2 out of $\{P, B_z, \beta\}$ can be treated as independent functions. Thus, we proceed with the following set of variables $\{\rho, M, B_z, \psi, \beta\}$, although we shall introduce M^c in place of M and analyse the consequences later. We have assembled together all the requisite apparatus for building the action principle. We shall now proceed onwards to this task.

3.2.2 Constructing the gyroviscous action

Much of the discussion presented herein is a variant of the ideal MHD action, discussed in Section 2.4. However, for the sake of making the discussion self-contained, we recall the essential steps involved in the construction.

The kinetic energy is given by

$$T[q] := \frac{1}{2}\int_D d^2a\, \rho_0(a)|\dot{q}|^2 = \frac{1}{2}\int_D d^2r\, \rho|v|^2 \,. \tag{3.11}$$

Now, consider the multiple components that make up the potential energy of the Lagrangian. The first involves the internal energy of the fluid, which is given by the following functional:

$$U_{\rm int}[q] := \int_D d^2a\, \frac{B_{0z}\beta_0}{\mathcal{J}} = \int_D d^2r\, B_z\beta \,, \tag{3.12}$$

which, in light of (3.9), satisfies the closure principle. The expression for $U[q]$ follows from the specific ansatz chosen for $U(\rho)$ earlier, in conjunction with the definition of β from (3.9).

The next component of the internal energy is the magnetic field. The field energy density is $B^2/8\pi$; upon scaling away the factor of 4π we obtain

$$\begin{aligned} U_{\rm mag}[q] \;&:=\; \frac{1}{2}\int_D d^2a\, \left(\frac{|B_{0z}|^2}{\mathcal{J}} + \mathcal{J} g^{kl} a^i_{\ ,k}\, a^j_{\ ,l} \frac{\partial\psi_0}{\partial a^i}\frac{\partial\psi_0}{\partial a^j}\right) \\ &=\; \frac{1}{2}\int_D d^2r\, \left(|B_z|^2 + |\nabla\psi|^2\right) \,, \end{aligned} \tag{3.13}$$

an expression that by (3.3) satisfies the closure principle, while physically corresponding to the magnetic energy density. In our subsequent discussion, we work with Cartesian coordinates implying that $g^{kl} \equiv \delta^{kl}$.

Finally, we introduce a novel term that will be seen to account for gyroviscosity. Since gyroviscosity is ultimately gyroscopic in nature, this suggests a term of the following form:

$$G[q] := \int_D d^2a\, \Pi^\star \cdot \dot{q} = \int_D d^2r\, M^\star \cdot v\,, \tag{3.14}$$

which, unlike the other terms that are either independent of or quadratic in $\dot{q}$, is linear in $\dot{q}$. It remains to determine the form of $M^\star$ or its corresponding attribute $\Pi^\star$. We emphasized the importance of the closure principle in Section 2.4, and this ensures that the choices of $M^\star$ and $\Pi^\star$ are strongly constrained.

There are still an endless number of possibilities, but we shall assume that Π^* has the simple following form:

$$\Pi^{*i} = \frac{m}{2e}\mathcal{J}\epsilon^{ij} a^m_{,j}\, \frac{\partial}{\partial a^m}\left(\frac{\beta_0}{\mathcal{J}}\right)\,, \tag{3.15}$$

which is motivated in part by the knowledge that gyroviscous effects should be linear in the magnetic moment that scales as $\beta \sim P/B$. To be more precise, β is identical to the (magnitude of) magnetization inherent to the magnetofluid. From (3.14), we see that

$$M^\star = \left.\frac{\Pi^*}{\mathcal{J}}\right|_{a=a(r,t)}\,, \tag{3.16}$$

which can be used in conjunction with (3.15) to conclude that

$$M^\star = \frac{m}{2e}\nabla \times (\beta\hat{z})\,. \tag{3.17}$$

The $m/(2e)$ prefactor of (3.15) and (3.17) can be explained by introducing a new variable $L^\star$ - the intrinsic angular momentum - according to $M^\star =:$

$\nabla \times L^\star$. We shall identify $L^\star$ as an intrinsic angular momentum emerging from the inherent magnetic moment of the fluid particles due to gyro-effects. The magnetic moment and the angular momentum are related via the gyromagnetic ratio, which explains the presence of an m/e factor. Writing the magnetization in terms of the pressure, the magnetic moment can be identified and this leads to the factor of 2.

Let us now probe the definitions of $\Pi^\star$ and $M^\star$ further to gain a better feel for their physical existence. In the 1970s and 80s, Newcomb [20, 21, 22] developed a theory of incompressible gyrofluids. Later, it was shown in [64, 65] that the gyroviscous action that we define in (3.19) gives rise to a simplified 2D version of the Braginskii gyroviscous tensor [59, 120, 121], and serves as a compressible generalization of Newcomb's models. But, it is important to recognize that several of these reasons emerge only after *a posteriori* considerations.

Momentum transport by gyroviscosity arises from microscopic charged particle gyration [129, 130], and so it is natural to think that the mass and charge of the important species (ions for a single fluid model) would enter. Similarly, the presence of gyration is immediately suggestive, when visualized pictorially, of the presence of a curl. From (3.17), we do see that each of these properties are indeed satisfied by $M^\star$. If we were to add an additional component to the kinetic momentum M, such that the continuity equation remains unchanged, it is evident that the new momentum must be divergence free. In other words, it must be the curl of another quantity, which has the dimensions of angular momentum density. This provides a second reason for $M^\star$ involving a curl. Since M has already been "used" elsewhere, this leaves ρ, ψ, B_z and β to construct this curl. Using dimensional analysis, and the presence of m and e (outlined above), it is seen that (3.17) can also be justified on heuristic grounds.

A second way to visualize the gyroviscous term (3.14) is to interpret it as an additional contribution to the kinetic energy, which can be viewed as proportional to $M \cdot v$, with $M = \rho v$ denoting the momentum density. Now, in addition to the momentum density, suppose that we had an additional contribution that also couples to the velocity. Clearly, such a term could be visualized as another 'kinetic' energy of sorts. If we carefully inspect equation (30) of [121], we see that the magnetization current is defined as

$$J^\star = -enu^\star = \nabla \times \left(\frac{P_\perp}{|B|}\, \hat{b} \right). \tag{3.18}$$

Upon taking the 2D limit and assuming the simplified limit where $\hat{b} = \hat{z}$, we obtain a reduced expression for the magnetization velocity. By multiplying it with the density, the 'magnetization momentum density' can be obtained, which is analogous to M. Hence, we can interpret $M^\star$ as the momentum density arising from the magnetization, and (3.14) captures the effects of finite magnetization in plasmas.

In any event, all of the above interpretations really boil down to the presence of an intrinsic angular momentum. We shall see shortly hereafter that (3.14) gives rise to the gyroviscous tensor (3.25) and this remarkable tensor has been shown to originate in a variety of contexts, ranging from quantum Hall systems [131] to anomalous HD of vortex fluids [132], in addition to plasmas, where it was first emphasized. The commonality of these systems was elucidated in [133] by invoking the existence of an underlying internal angular momentum.

We end this subsection by presenting our action, which is obtained by combining equations (3.11), (3.12), (3.13) and (3.14) as follows:

$$S = \int_{t_1}^{t_2} dt \, \left(T[q] - U_{\rm int}[q] - U_{\rm mag}[q] + G[q] \right) \, , \tag{3.19}$$

and we are ready to explore its consequences.

3.2.3 The Eulerian equations and the gyromap

We begin by giving the Eulerian dynamical equations for the observables ρ, B_z, ψ and β. These are found from the expressions (3.7), (3.5), (3.6) and (3.10), respectively.

$$\frac{\partial \rho}{\partial t} = -\partial_s M_s \, , \tag{3.20}$$

$$\frac{\partial B_z}{\partial t} = -\partial_s \left(\frac{B_z M_s}{\rho} \right) \, , \tag{3.21}$$

$$\frac{\partial \psi}{\partial t} = -\frac{M_s}{\rho} \partial_s \psi \, , \tag{3.22}$$

$$\frac{\partial \beta}{\partial t} = -\partial_s \left(\frac{\beta M_s}{\rho} \right) \, . \tag{3.23}$$

The final Eulerian equation, which governs the evolution of momentum, is found from $\delta S = 0$. The computation is somewhat long and tedious, but straightforward. Hence, we shall only present the final result, and analyse the different terms. The momentum equation is given by

$$\begin{aligned} \dot{M}_s &= -\partial_l \left(M_s M_l/\rho\right) - \partial_s \left(P + |B|^2/2\right) \\ &\quad + B_l \partial_l B_s - \partial_l \pi_{ls}\,, \end{aligned} \tag{3.24}$$

where the pressure is given by (3.9) and the *gyroviscous tensor* π_{ls} is

$$\begin{aligned} \pi_{ls} &= N_{sjlk} \beta \partial_k \left(\frac{M_j}{\rho}\right) \\ N_{sjlk} &= \frac{m}{2e} \left(\delta_{sk}\epsilon_{jl} - \delta_{jl}\epsilon_{sk}\right) . \end{aligned} \tag{3.25}$$

Now consider the gyroviscous action given by (3.19). On varying the kinetic energy functional we obtain $\rho_0 \ddot{q}$, which yields the terms on either side of the equality sign in (3.24). The second term in the action, the internal energy, gives rise to the pressure gradient term. Similarly, the magnetic component of the internal energy, which comprises of two terms, as seen from (3.13), gives rise to the magnetic pressure and the penultimate term in (3.24).

Lastly, the gyroviscous part of the action gives rise to the gyroviscous tensor, as defined in (3.25), and constitutes the last term in (3.24). As mentioned earlier, the gyroviscous tensor is consistent with the results described in Braginskii [59, 120, 121], when the dissipative terms are neglected and restricted to a simplified 2D limit. Furthermore, a simpler version of the model was first constructed in [124]. The gyroviscous tensor has also been referred to as Hall viscosity or odd viscosity in other contexts [131, 132].

Finally, it is important to recognize that the action (3.19) has two different terms that involve $\dot{q}$, and hence the canonical momentum will *not* be the same as $\rho\dot{q}$. In fact, we find that

$$\Pi = \frac{\delta L}{\delta \dot{q}} = \rho\dot{q} + \Pi^{\star} \,. \tag{3.26}$$

Dividing throughout by $\mathcal{J}$ and evaluating the expression at $a = a(r,t)$, the Eulerian counterpart is obtained through the use of (3.1), (3.2), (3.16) and (3.17),

$$M^c = M + M^{\star} = M + \frac{m}{2e}\nabla \times (\beta\hat{z}) \ . \tag{3.27}$$

As we are dealing with a two-dimensional momentum vector, we can write the above equation as

$$M_s^c = M_s - \frac{m}{2e}\epsilon_{ls}\partial_l\beta \,, \tag{3.28}$$

which is the gyromap first introduced in [124], and subsequently employed in [37, 127]. As a result, we see that the true origins of the gyromap stem from the action principle formulation of gyroviscous MHD. However, the gyromap is more than just a relation between M^c and M - we shall explore its uses further in a Hamiltonian context.

3.3 The Hamiltonian description of gyroviscous MHD

Hitherto, our discussion has centred around an action principle formulation of gyroviscous MHD, expressed in Lagrangian variables. It is advantageous to transition to the Hamiltonian description in Eulerian variables. This is undertaken through a reduction procedure. We shall not summarize the details here, as the derivation closely mirrors that of ideal MHD in Section 2.5. We also refer the reader to Section 2.6 as we shall use the tools introduced in that section; it also summarizes the advantages inherent to the noncanonical Hamiltonian approach.

Now we study two different cases of the gyroviscous model and action developed in Sec. 3.2. First we set the attribute ψ_0 set to zero, and consequently ψ as well. Upon doing so, we obtain a simplified model, albeit one that possesses many features of the full model - this was first studied in [124]. The simpler model enables us to arrive at a better understanding of the role of the gyromap, the Hamiltonian and the bracket. Subsequently, we introduce ψ into our theory and analyse the equilibria and stability.

3.3.1 The $\psi \equiv 0$ model

For this reduced model we choose to work with M^c of (3.27), because it is directly obtained via the canonical momentum Π through (3.2). Thus, since $\psi \equiv 0$, only the $\hat{z}$-component of the magnetic field is present. The Hamiltonian, obtained through the Legendre transformation and Eulerianization, is

$$H = \int d^2r \left(\frac{1}{2\rho} \left| M^c - \frac{m}{2e} \nabla \times (\beta \hat{z}) \right|^2 + \beta B_z + \frac{B_z^2}{2} \right) , \tag{3.29}$$

which, when written in terms of the 'kinetic' momentum M takes on a more recognizable form

$$H = \int d^2r \left(\frac{|M|^2}{2\rho} + \beta B_z + \frac{B_z^2}{2} \right) . \tag{3.30}$$

In terms of M^c, the noncanonical bracket obtained by using the procedure of Sec. 2.5, is

$$\begin{aligned} \{F,G\}_c^0 \quad = \quad & \int d^2r \left[M_l^c \left(\frac{\delta G}{\delta M_k^c} \partial_k \frac{\delta F}{\delta M_l^c} - \frac{\delta F}{\delta M_k^c} \partial_k \frac{\delta G}{\delta M_l^c} \right) \right. \\ & + \rho \left(\frac{\delta G}{\delta M_k^c} \partial_k \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta M_k^c} \partial_k \frac{\delta G}{\delta \rho} \right) \\ & + B_z \left(\frac{\delta G}{\delta M_k^c} \partial_k \frac{\delta F}{\delta B_z} - \frac{\delta F}{\delta M_k^c} \partial_k \frac{\delta G}{\delta B_z} \right) \\ & \left. + \beta \left(\frac{\delta G}{\delta M_k^c} \partial_k \frac{\delta F}{\delta \beta} - \frac{\delta F}{\delta M_k^c} \partial_k \frac{\delta G}{\delta \beta} \right) \right] , \end{aligned} \tag{3.31}$$

and we see that this corresponds to Morrison-Greene bracket [64] restricted to $B = B_z \hat{z}$ with the momentum M^c replacing M. The index '0' indicates that we are considering $\psi = 0$, and 'c' is used to convey the information that the bracket is in terms of the canonical momentum M^c. Thus, we see that the effects of gyroviscosity are 'hidden' as we have used M^c. However, if we choose to write the bracket in terms of M, the gyroviscous contributions are manifest

$$\{F,G\}_G^0 = \{F,G\}^0 - \beta N_{ijsl} \left(\partial_s \frac{\delta G}{\delta M_i} \right) \left(\partial_l \frac{\delta F}{\delta M_j} \right) , \tag{3.32}$$

where we obtain a new term that produces the gyroviscous tensor and $\{F,G\}^0$ is the bracket of (3.31) with M_c replaced by M. This bracket is identical to that given in [124], which was obtained through a more *ad hoc* procedure. If we compare the Hamiltonian-bracket pair of Eqs. (3.29) and (3.31) with that of Eqs. (3.30) and (3.32), the significance of the gyromap becomes evident. We can choose to work with a system that possesses a relatively simple Hamiltonian with a more complex bracket, or vice versa, and it is the gyromap that allows us to move back and forth between these two versions. Both versions give the same equations of motion: those obtained in Sec. 3.2.3 with $\psi \equiv 0$.

We use the condition $\{F, C\} = 0$ for all F to find the Casimirs. Upon simplification, the only Casimirs that exist are independent of the gyro term, and in fact, are independent of the velocity of the fluid. We find the following infinite family of Casimirs:

$$C = \int d^2r\, \beta f \left(\frac{\rho}{\beta}, \frac{B_z}{\beta} \right) , \tag{3.33}$$

where f is an arbitrary function, a result that was first obtained in [124]. Because of its homogeneous form, the three variables of (3.33) are interchangeable, i.e., we can permute ρ, β and B_z without loss of generality.

Now that we have the Casimirs, we can employ energy-Casimir method, as described in Section 2.6, to determine the equilibria. They are found by demanding that $\delta F = 0$, where $F = H + C$. This yields two familiar conditions,

$$M = 0 \qquad \text{and} \qquad P + \frac{B_z^2}{2} = \text{const}\,, \tag{3.34}$$

which imply that there is zero equilibrium flow, and the existence of a total pressure balance. It is easy to verify that these are valid equilibria of the model from the equations of motion. However, it may well be possible for other equilibria to exist, not captured by the energy-Casimir method. This stems from the subtleties underlying Casimirs, which remain only partly understood [84, 85].

The astute reader will note that we have reverted back to M. Although this may appear confusing at first sight, there exists a well-defined general prescription: we first find the Casimirs by working in terms of M^c, and then apply the gyromap to express them in terms of M. Since the Hamiltonian is much simpler in terms of M, we can proceed to calculate δF, wherein M is the variable of choice, and *not* M^c. We shall use this procedure in the following sections as well to express our final results in terms of M.

Having determined the equilibria, we can carry out the stability analysis, as outlined in Section 2.6. Using the energy-Casimir methodology and computing the elements of the Hessian, we find that the equilibria satisfying (3.34) are always stable, regardless of the functional form of the Casimir. This is, of course, to be expected and serves as a sanity check.

3.3.2 The $\psi \not\equiv 0$ model

Now consider the full model with $\psi \not\equiv 0$ and magnetic field given by (3.3). We shall proceed to write down the Hamiltonian-bracket pair in terms of M^c, observing that the simplicity of the latter is obtained at the expense of the former. The Hamiltonian is

$$H = \int d^2r \left(\frac{1}{2\rho} \left| M^c - \frac{m}{2e} \nabla \times (\beta \hat{z}) \right|^2 + \beta B_z + \frac{B_z^2}{2} + \frac{|\nabla\psi|^2}{2} \right) , \tag{3.35}$$

which is equal to (3.29) plus the perpendicular magnetic energy, and the Poisson bracket is

$$\{F,G\}_c^{\psi} = \{F,G\}_c^0 + \int d^2r \, \nabla\psi \cdot \left(\frac{\delta F}{\delta M^c} \frac{\delta G}{\delta \psi} - \frac{\delta G}{\delta M^c} \frac{\delta F}{\delta \psi} \right) . \tag{3.36}$$

Thus, the noncanonical bracket (3.36) in conjunction with the Hamiltonian (3.35) successfully generates the equations of motion derived in Sec. 3.2.3.

The presence of a ψ leads to significant changes in the Casimirs obtained. Unlike the $\psi \equiv 0$ case, we do obtain Casimirs that depend on M^c, implying that they depend on the gyroviscous term inherent in M^c. We find that two broad Casimir families emerge, of which the first is independent of M^c, and has the form

$$C = \int d^2r \, \mathcal{C}\left(\rho, \beta, B_z\right) \mathcal{K}(\psi) \, , \tag{3.37}$$

where $\mathcal{C} = \beta f\left(\rho/\beta, B_z/\beta\right)$ or an equivalent function involving a permutation of ρ, β and B_z. The similarities with (3.33) are self-evident, as the two expressions only differ by $\mathcal{K}(\psi)$. If we choose $\mathcal{K} = \text{const}$, this eliminates ψ from (3.37), and renders it identical to (3.33). Thus, we can interpret (3.37) as a natural extension of (3.33).

Now, we turn our attention to the second Casimir family, which depends on M^c. From the condition $\{F, C\} = 0$, we arrive at

$$\partial_l \left(M_k^c \frac{\delta C}{\delta M_l^c} \right) + M_l^c \partial_k \left(\frac{\delta C}{\delta M_l^c} \right) + \ \rho \partial_k \left(\frac{\delta C}{\delta \rho} \right) + B_z \partial_k \left(\frac{\delta C}{\delta B_z} \right)$$
$$+\beta \partial_k \left(\frac{\delta C}{\delta \beta} \right) - \frac{\delta C}{\delta \psi} \partial_k \psi = 0 \,, \tag{3.38}$$
$$\partial_k \left(\frac{\delta C}{\delta M_k^c} \rho \right) = 0 \,, \qquad \partial_k \left(\frac{\delta C}{\delta M_k^c} \beta \right) = 0 \,, \tag{3.39}$$
$$\partial_k \left(\frac{\delta C}{\delta M_k^c} B_z \right) = 0 \,, \qquad \frac{\delta C}{\delta M_k^c} \partial_k \psi = 0 \,, \tag{3.40}$$

From the equation of (3.40) we obtain the candidate

$$C = \int d^2 r \, M^c \cdot (\hat{z} \times \nabla \psi) \, F \left(\rho, \beta, B_z, \psi \right) \,, \tag{3.41}$$

which when inserted in the first equation of (3.39) gives

$$C = \int d^2 r \, \frac{M^c \cdot (\hat{z} \times \nabla \psi)}{\rho} \mathcal{F} \left(\psi \right) \,, \tag{3.42}$$

while the remaining two equations of (3.39) and (3.40) imply

$$\left[\psi, \frac{B_z}{\rho} \right] = \left[\psi, \frac{\beta}{\rho} \right] = 0, \tag{3.43}$$

and we introduce the notation $[f, g] = f_x g_y - f_y g_x$. Equation (3.43) implies there are no velocity dependent Casimirs unless the model is reduced, which is well known for symmetric variants of MHD [70, 86]. The constraints of (3.43) are a consequence of over labeling [134], as it is impossible for the three advected labels of Eqs. (3.21), (3.22), and (3.23) to be independent. Thus, we assume B_z/ρ and β/ρ are functions of ψ, which effectively eliminates them from the dynamics. With this assumption (3.42) is a Casimir since it also satisfies (3.38). Upon collapsing (3.37), our general Casimir is then

$$C = \int d^2r \left(\frac{M^c \cdot (\hat{z} \times \nabla\psi)}{\rho} \mathcal{F}(\psi) + \rho \mathcal{J}(\psi) \right) . \tag{3.44}$$

Using $B^\perp = \hat{z} \times \nabla\psi$, $M \cdot B = M \cdot B^\perp$ and $M^c \cdot \hat{z} = 0$ (although parallel momentum could be included), and setting $\mathcal{F}$ = constant, the first term of (3.44) reduces to the well-known cross-helicity invariant of ideal MHD, exemplified by (2.56). However, there is a crucial difference since the velocity is now $v^c = M^c/\rho$ instead of v. Thus, in the absence of gyroviscosity, $v^c = v$ and the usual cross-helicity is recovered.

Given the Casimir invariants we can proceed to the variational equilibrium analysis, following the approach outlined in [70, 86]. As the analysis is a somewhat complicated one, we shall first consider the case with no flow as a trial run. For this case, the variational principle $\delta F = 0$ contains only the Casimir of (3.37), and gives rise to the following equilibria

$$M \equiv 0 \quad \text{and} \quad \Delta\psi = -P' - B_z B_z' , \tag{3.45}$$

where P and B_z are flux functions, and the prime denotes differentiation with respect to ψ. We obtain the Grad-Shafranov equation, which is to be expected.

Next consider the case with the Casimir of (3.44). As stated earlier, this involves a reduction owing the conditions imposed by (3.43). Hence, we introduce $B_z = \rho\,\varpi(\psi)$ and $\beta = \rho\,\varsigma(\psi)$ and the Hamiltonian becomes

$$H \;=\; \int d^2r \left(\frac{1}{2\rho} \left| M^c - M^\star \right|^2 + \;\rho^2 \left[\varsigma\varpi + \frac{\varpi^2}{2} \right] + \frac{|\nabla\psi|^2}{2} \right) . \tag{3.46}$$

The equilibrium conditions that follow from $\delta F = 0$, with (3.44) and (3.46), are

$$\begin{aligned}
\frac{\delta F}{\delta M^c} &= M^c - M^\star + (\hat{z} \times \nabla\psi)\, \mathcal{F} = 0, && (3.47)\\
\frac{\delta F}{\delta \rho} &= -\frac{1}{2\rho^2} \left| M^c - M^\star \right|^2 - \frac{M^c \cdot (\hat{z} \times \nabla\psi)}{\rho^2} \mathcal{F} \\
&\quad + \mathcal{J} + 2\rho \left[\varsigma\varpi + \frac{\varpi^2}{2} \right] = 0, && (3.48)\\
\frac{\delta F}{\delta \psi} &= -\Delta\psi + \rho^2 \left[\varsigma'\varpi + \varsigma\varpi' + \varpi\varpi_\psi \right] \\
&\quad + \mathcal{F} \nabla \cdot \left(\frac{\hat{z} \times M^c}{\rho} \right) + \rho \mathcal{J}' = 0. && (3.49)
\end{aligned}$$

By manipulating these equations further, we arrive at

$$\frac{1}{4} \left| \nabla\psi \right|^2 \left(\frac{\mathcal{F}}{\rho} \right)^2 + \frac{P_z}{\rho} + \frac{\mathcal{J}}{2} + \frac{m}{2e} \frac{\mathcal{F}}{2\rho^2} \nabla\beta \cdot \nabla\psi = 0\,, \qquad (3.50)$$

$$\nabla \cdot \left[\left(1 - \frac{\mathcal{F}^2}{\rho} \right) \nabla\psi \right] + \left| \nabla\psi \right|^2 \frac{\mathcal{F}\mathcal{F}'}{\rho} = \rho\mathcal{J}' - \rho^2 \left(\frac{P_z}{\rho^2} \right)' - \frac{m}{2e} \mathcal{F} \nabla \cdot \left(\frac{\nabla\beta}{\rho} \right) \qquad (3.51)$$

with $P_z := P + B_z^2/2 = \rho^2 \left(\varsigma\varpi + \varpi^2/2 \right)$ and recall that $\beta = \rho\,\varsigma(\psi)$ everywhere in the above expressions. Equations (3.50) and (3.51) compare with those of ordinary MHD as in [70], but with the addition of new gyro terms which are easily identified by the presence of factors of $m/(2e)$. In a manner analogous to ideal MHD, there are free functions of ψ that can be chosen to determine current and flow profiles. Equation (3.51) is a generalization of the Grad-Shafranov equation, but since the density is not a flux function it cannot be solved in isolation. Hence, we must use (3.50), the generalized Bernoulli equation, to close the system. We observe that the two expressions constitute generalizations with flow and gyroviscosity of the JOKF equation [135].

There are various ways of rewriting (3.50) and (3.51), one that brings out the Mach singularity is the following:

$$|\nabla\psi|^2\left[\frac{1}{4}\left(\frac{\mathcal{F}}{\rho}\right)^2+\frac{m}{2e}\frac{\mathcal{F}\varsigma'}{2\rho}\right]+\frac{P_z}{\rho}+\frac{\mathcal{J}}{2}+\ \frac{m}{2e}\frac{\mathcal{F}\varsigma}{2\rho^2}\nabla\rho\cdot\nabla\psi=0\,, \tag{3.52}$$

$$\nabla\cdot\left[\left(1-\frac{\mathcal{F}^2}{\rho}+\frac{m}{2e}\mathcal{F}\varsigma'\right)\nabla\psi\right]+\ |\nabla\psi|^2\left(\frac{\mathcal{F}\mathcal{F}'}{\rho}-\frac{m}{2e}\mathcal{F}'\varsigma'\right)+\frac{m}{2e}\frac{\mathcal{F}\varsigma'}{\rho}\nabla\rho\cdot\nabla\psi$$
$$+\frac{m}{2e}\mathcal{F}\varsigma\nabla\cdot\left(\frac{\nabla\rho}{\rho}\right)=\rho\mathcal{J}'-\rho^2\left(\frac{P_z}{\rho^2}\right)'\,. \tag{3.53}$$

An inspection of (3.52) and (3.53) makes it manifestly clear that they possess a rich structure. It is possible to undertake several analyses such as (i) their region of hyperbolicity, (ii) modification of the fast and slow magnetosonic waves due to the gyroviscous terms, etc. We shall not tackle these issues in this dissertation, but they constitute promising avenues for future work.

3.3.3 High-β gyro-RMHD

As noted in Sec. 3.1 there exists a large literature on reduced gyrofluid models that have been obtained by various means. In this subsection, we shall address the emergence of the three-field model given in Sec. IIIA of [37] via the HAP formalism.

From the action, we obtained, without approximation, the Poisson bracket of (3.36); we can also apply the gyromap on (3.36) to obtain $\{F,G\}^{\psi}_{G}$. Now, let us suppose that we assume $B_z\to B_0$ and $\rho\to\rho_0$ are constant.

This implies that $P \propto \beta$, which follows from (3.9) and the latter condition is consistent with incompressibility. Hence, we introduce the scalar vorticity $\Omega^c = \hat{z} \cdot \nabla \times M^c$ and $M^c = \nabla \varphi^c \times \hat{z}$, where φ^c is the stream function, up to the constant factor of ρ_0.

The subscript c is present everywhere to indicate that these include the gyroviscous terms. Following a similar line of analysis to that employed in [70], viz. chain rule relations of the form $\nabla^2 \delta F/\delta \Omega^c = \hat{z} \cdot \nabla \times \delta F/\delta M^c$, we reduce the bracket of (3.36) to the following:

$$\begin{aligned} \{F, G\} \;=\; & \int d^2 r \Big(\Omega^c \left[F_{\Omega^c}, G_{\Omega^c} \right] + \; \psi \left(\left[F_\psi, G_{\Omega^c} \right] - \left[G_\psi, F_{\Omega^c} \right] \right) \\ & + \beta \left(\left[F_\beta, G_{\Omega^c} \right] - \left[G_\beta, F_{\Omega^c} \right] \right) \Big) \,, \end{aligned} \tag{3.54}$$

which is precisely the high-β RMHD bracket first given in [36]. Because (3.54) is homogeneous of degree zero in β and ψ and of degree one in Ω^c, which means scaling Ω^c only scales time, these quantities can be identified with the corresponding quantities of [37]. Following a similar procedure, the reduced Hamiltonian is

$$H = \frac{1}{2} \int d^2 r \Big(|\nabla \varphi|^2 + |\nabla \psi|^2 \Big) \,, \tag{3.55}$$

and we have neglected the pressure term for simplicity; we have neglected the effect of toroidal curvature that usually occurs in high-β RMHD.

From (3.27), the gyromap relation between M^c and M, we obtain

$$\varphi^c = \varphi + \frac{m}{2e} \beta \,, \tag{3.56}$$

where $M = \nabla\varphi \times \hat{z}$. Equation (3.56) is precisely the gyromap used in [37]. Using (3.56) we can follow one of two paths: (i) eliminate φ from (3.55) and insert the resulting H into (3.54) to obtain the equations of motion in terms of Ω^c (ii) transform the bracket of (3.54) to one in terms of Ω and use the Hamiltonian of (3.55) in its current form. Both give gyrofluid evolution equations equivalent to the three-field model in Sec. IIIA of [37]. However, not all terms are recovered since our model lacks the effect of toroidal curvature and the Hall term. The latter is a consequence of extended MHD [136], which we shall tackle in subsequent chapters.

Thus, we have shown that our model of 2D gyroviscous MHD, constructed rigorously through a HAP formalism, yields a rich collection of equilibria. Moreover, we have shown that it can also be used to construct reduced gyrofluid models existent in the literature.

Chapter 4

Action principles for generalized fluid models with gyroviscosity

In Chapter 3, we considered a 2D model of gyroviscous MHD, and proceeded to analyse it in detail. Yet, it is evident that it constitutes but one example of gyroviscous models in general. In this context, we refer to any model that possesses a Lagrangian incorporating spatial derivatives of the velocity as being 'gyroviscous' in nature. This is quite different from our previous definition, and represents a generalization of the gyroviscous action introduced in Chapter 3. [1]

4.1 Action principle for the general gyroviscous fluid

In this section, we provide a brief summary of the general methodology advocated in [64, 65] for constructing action principles for fluid and magnetofluid models and obtain the gyroviscous fluid action. The advantages of this approach are manifold, and we shall refer the reader to Chapters 1 and

[1]This contents of this chapter were published in: "The action principle for generalized fluid motion including gyroviscosity", M. Lingam & P. J. Morrison, 2014, *Phys. Lett. A*, **378**, 3526-3532 [71]. The published work was entirely undertaken by this author (M. Lingam) under the supervision and guidance of P. J. Morrison.

2 for extended discussions of the same. Next, we describe how we build our model and obtain the corresponding equation of motion, thereby establishing the Eulerian closure principle along the way.

4.1.1 The general action

First, we choose the domain $D \subset \mathbb{R}^3$. We also assume the existence of the Lagrangian coordinate $q\colon D \to D$, which is a well behaved function that is sufficiently smooth, invertible, etc. Next, we specify our set of observables, which are fully determined by the attributes and q. In our case, the set corresponds to $\mathfrak{E} = \{v, \mathcal{S}^\alpha, \mathcal{P}^\beta, \mathcal{B}^\gamma\}$. Here v denotes the velocity, which is determined via $v(r,t) = \dot{q}(a,t)$, and the RHS is evaluated at $a = q^{-1}(r,t)$. The quantities $\mathcal{S}$, $\mathcal{P}$ and $\mathcal{B}$ are stand-ins for the entropy, density and magnetic field respectively. In other words, they are quantities that are Lie-dragged as 0-forms, 3-forms and 2-forms; alternatively, they can be viewed as Lie-dragged scalars, scalar and vector densities (of weight 1) respectively. This interpretation is a natural consequence of the geometric interpretation provided in Section 2.2. For instance, we see that the entropy density σ and the velocity stream function φ usually serve as examples of $\mathcal{P}$ and $\mathcal{S}$ respectively. A careful inspection reveals that there are no 1-forms in our theory. It is easy enough to incorporate them as well, and we only refrain from doing so as we are not aware of any fluid or magnetofluid theories that are endowed with such independent quantities; one may view the vector potential A as one such example, but it is clearly related to the magnetic field. In 3D, such quantities amount to the Hodge dual of the 2-forms and might lead to over specification [134], which we wish to avoid. The Greek indices α, β and γ are used to keep track of the number of fields that fall in each category.

The last step involves the imposition of a *closure principle*, which is necessary for our model to be 'Eulerianizable.' Mathematically, this principle is implemented by demanding the action to be expressible fully in terms of our Eulerian observables. In other words, we require our action to be expressible as follows:

$$S[q] := \int_D d^3a dt\, \mathcal{L}\left(q, \dot{q}, \partial q/\partial a\right) =: \bar{S}\left[\mathfrak{E}\right] . \tag{4.1}$$

Now, we shall make a simplification: $\bar{S} = \int_D d^3r dt\, \bar{\mathcal{L}}$, where $\bar{\mathcal{L}}$, can depend on the observables and their spatial and temporal derivatives of any order. However, for convenience we shall use the following ansatz for the Lagrangian density $\bar{\mathcal{L}}$:

$$\bar{S} = \int_D d^3r dt\, \bar{\mathcal{L}}\left(v, \mathcal{S}^\alpha, \mathcal{P}^\beta, \mathcal{B}^\gamma, \nabla v, \nabla\mathcal{S}^\alpha, \nabla\mathcal{P}^\beta, \nabla\mathcal{B}^\gamma\right) ; \tag{4.2}$$

i.e., we choose to work with actions that only involve the observables and their first-order spatial derivatives. Such a simplification is well-motivated since most of the widely used fluid and magnetofluid models possess this form. The generalization to higher derivatives is straightforward.

To sum up, there are two simplifications employed in this model. Firstly, we assumed that our model does not have observables that are akin to one-forms and, secondly, we chose the ansatz (4.2) for the action. In order to obtain the equation of motion, we must use Hamilton's principle to extremize the action (4.1). We shall instead show how we can extremize the action (4.2), and how it leads to equations of motion that are purely Eulerian.

As a result, for our family of models, this amounts to proving the Eulerian closure principle, which states that a completely Eulerianizable action yields Eulerian equations of motion. At this stage, an important caveat must be added: our discussion only holds true when there exist observables that exhibit Lie-dragging, thereby giving rise to the conservation laws discussed herein. In general, the observables, such as the magnetic flux or entropy, need not exhibit 'frozen-in' constraints such as the ones employed in the preceding analysis.

We shall drop the overbar in the action and the Lagrangian density described in (4.2), as it simplifies the notation. For the same reason, we also drop the Greek indices α, β and γ present in (4.2).

4.1.2 The Eulerian closure principle and equations of motion

The variation of the action (4.2) yields

$$\delta S \;=\; \int_D d^3r dt \left(\frac{\delta S}{\delta v^k}\delta v^k + \frac{\delta S}{\delta \mathcal{B}^k}\delta \mathcal{B}^k + \frac{\delta S}{\delta \mathcal{P}}\delta \mathcal{P} + \frac{\delta S}{\delta \mathcal{S}}\delta \mathcal{S}\right). \tag{4.3}$$

However, we need to express the quantities $\delta \mathcal{B}_k$, $\delta \mathcal{S}$, etc in terms of δq in order to derive the Euler-Lagrange equations of motion. We shall present this calculation in detail for $\delta \mathcal{P}$, since it is the most convenient for illustrating the procedure. From (2.12), we find that

$$\delta \mathcal{P} = -\int_D d^3a\, \mathcal{P}_0(a)\, \partial_k \delta\left(r - q\left(a,t\right)\right) \delta q^k, \tag{4.4}$$

where the partial derivative is now implemented with respect to r. Substituting this into the relevant component of (4.3), and integrating by parts yields the expression

$$\int_D d^3r dt\, \frac{\delta S}{\delta \mathcal{P}} \delta \mathcal{P} = \int_D d^3a dt \mathcal{P}_0 \left[\partial_k \frac{\delta S}{\delta \mathcal{P}} \right]_q \delta q^k, \tag{4.5}$$

where the notation $\left[\partial_k \frac{\delta S}{\delta \mathcal{P}}\right]_q$ is introduced as short-hand notation for

$$\left[\partial_k \frac{\delta S}{\delta \mathcal{P}} \right]_q = \int_D d^3r \delta \left(r - q\left(a, t\right)\right) \partial_k \frac{\delta S}{\delta \mathcal{P}}. \tag{4.6}$$

The above expression has a ready physical interpretation. In Section 2.2, we indicated that the observables can be determined from the corresponding attributes since the delta function allows us to 'pluck out' the appropriate fluid element. Here, the converse relation is true: given an Eulerian field (expressed in terms of the observables), the delta function allows us to pluck out the Lagrangian counterpart. As a result, the quantity (4.6) is fully Lagrangian, since the action is fully representable either in terms of q and its derivatives, or in terms of the Eulerian observables. Hence, the subscript q has been introduced to emphasize its Lagrangian nature.

Let us now return to (4.5) and extremize the action. This requires everything appearing in front of δq^k must vanish. The contribution from the $\mathcal{P}$ term is evidently

$$\mathcal{P}_0 \left[\partial_k \frac{\delta S}{\delta \mathcal{P}} \right]_q, \tag{4.7}$$

and since we know that the determinant $\mathcal{J} \neq 0$, we can divide throughout by $\mathcal{J}$. Next, evaluating this expression at the label $a = q^{-1}(r, t)$ and using (2.12) gives the following Eulerian contribution form the $\mathcal{P}$-term:

$$\mathcal{P}\partial_k \frac{\delta S}{\delta \mathcal{P}}\,, \tag{4.8}$$

where we have used the fact that $\left[\partial_k(\delta S/\delta\mathcal{P})\right]_q$ evaluated at $a = q^{-1}(r,t)$ yields $\partial_k(\delta S/\delta\mathcal{P})$. This effectively amounts to taking the quantity $\partial_k(\delta S/\delta\mathcal{P})$ and Lagrangianizing it - expressing it in terms of q, its derivatives and the attributes - and then re-Eulerianizing it again, i.e. re-expressing in terms of the Eulerian fields. We can also derive the same relation, by using the approach outlined in [137], which is often dubbed the Euler-Poincaré method [38], as it originated with the work of Poincaré in [138]. In terms of the notation employed in [87], the Lagrangian variation δq is denoted by ξ and the Eulerianized counterpart is denoted by η. The Euler-Poincaré method leads to an induced variation for $\mathcal{P}$ given by

$$\delta\mathcal{P} = -\partial_k\left(\mathcal{P}\eta^k\right). \tag{4.9}$$

Substituting this into (4.5), integrating by parts and separating out the algebraic expression in front of η^k, ultimately yields the same result as (4.8).

Consider now the $\mathcal{S}$-term. Since $\mathcal{S} = \mathcal{S}_0$, when the RHS is evaluated at $a = q^{-1}(r,t)$, the integral representation of this amounts to

$$\mathcal{S} = \int_D d^3a\,\mathcal{S}_0\mathcal{J}\delta\left(r - q\left(a,t\right)\right)\,. \tag{4.10}$$

With this term, we can either carry out the same approach outlined above for $\mathcal{P}$, or use the equivalent approach described in [137]. Substituting (4.10) into the appropriate term in (4.3), obtaining the Lagrangian expression, dividing throughout by $\mathcal{J}$, and Eulerianization gives

$$\mathcal{S}\partial_k \frac{\delta S}{\delta \mathcal{S}} - \partial_k \left(\mathcal{S} \frac{\delta S}{\delta \mathcal{S}} \right) . \tag{4.11}$$

Next, we consider the variable $\mathcal{B}$-term, which satisfies the relation

$$\mathcal{B}^j = \int_D d^3a \, q^j_{,i} \mathcal{B}^i_0 \delta \left(r - q \left(a, t \right) \right) . \tag{4.12}$$

We repeat the procedure for this term, and obtain

$$\mathcal{B}^j \partial_k \frac{\delta S}{\delta \mathcal{B}^j} - \partial_j \left(\mathcal{B}^j \frac{\delta S}{\delta \mathcal{B}^k} \right) . \tag{4.13}$$

Lastly, we note that the velocity is determined via

$$v^j = \int_D d^3a \, \dot{q}^j \mathcal{J} \delta \left(r - q \left(a, t \right) \right) , \tag{4.14}$$

and we can use this to determine δv in terms of δq and obtain the final Eulerian result. It is given by

$$\begin{aligned} v^j \partial_k \frac{\delta S}{\delta v^j} \quad &- \quad \partial_k \left(v^j \frac{\delta S}{\delta v^j} \right) - \partial_j \left(v^j \frac{\delta S}{\delta v^k} \right) \\ &- \quad \frac{\partial}{\partial t} \left(\frac{\delta S}{\delta v^k} \right) . \end{aligned} \tag{4.15}$$

Together, equations (4.8), (4.11), (4.13) and (4.15) constitute the pieces that make up the equation of motion. Putting them all together, we have

$$\begin{aligned} &\mathcal{P}\partial_k \frac{\delta S}{\delta \mathcal{P}} + \mathcal{S}\partial_k \frac{\delta S}{\delta \mathcal{S}} - \partial_k \left(\mathcal{S} \frac{\delta S}{\delta \mathcal{S}} \right) \\ &+\mathcal{B}^j \partial_k \frac{\delta S}{\delta \mathcal{B}^j} - \partial_j \left(\mathcal{B}^j \frac{\delta S}{\delta \mathcal{B}^k} \right) + v^j \partial_k \frac{\delta S}{\delta v^j} \\ &-\partial_k \left(v^j \frac{\delta S}{\delta v^j} \right) - \partial_j \left(v^j \frac{\delta S}{\delta v^k} \right) - \frac{\partial}{\partial t} \left(\frac{\delta S}{\delta v^k} \right) = 0. \end{aligned} \tag{4.16}$$

It is evident that (4.16) is fully Eulerian, since it does not contain any Lagrangian pieces. Earlier, we'd mentioned that two different assumptions were made in building our model. Of these, we have used only the absence of the 1-forms in proving that our equation of motion is Eulerian. This assumption can *also* be relaxed, and the ensuing result still remains the same.

Now, we shall make use of the second assumption, namely the ansatz from (4.2) to recast (4.16) into a more recognizable form. From the definition of the functional derivative, it can be shown that

$$\frac{\delta S}{\delta \Psi} = \frac{\partial \mathcal{L}}{\partial \Psi} - \partial_j \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_j \Psi\right)} \right), \tag{4.17}$$

where Ψ represents any of the observables. This follows from the fact that $\mathcal{L}$ only involves the observables and their first-order spatial derivatives. It is important to recognize that the LHS of (4.17) is primarily a compact way of representing the RHS of the same expression. Using this notation when necessary, one can rewrite (4.16) as

$$\begin{aligned} -\frac{\partial}{\partial t}\left(\frac{\delta S}{\delta v^k}\right) &+ \partial_j \left[\delta^j_k \left(\mathcal{P}\frac{\delta S}{\delta \mathcal{P}} + \mathcal{B}^j \frac{\delta S}{\delta \mathcal{B}^j} - \mathcal{L}\right)\right] \\ &+ \partial_j \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_j \mathcal{S}\right)}\left(\partial_k \mathcal{S}\right) + \frac{\partial \mathcal{L}}{\partial \left(\partial_j \mathcal{P}\right)}\left(\partial_k \mathcal{P}\right)\right] \\ &+ \partial_j \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_j \mathcal{B}^i\right)}\left(\partial_k \mathcal{B}^i\right) + \frac{\partial \mathcal{L}}{\partial \left(\partial_j v^i\right)}\left(\partial_k v^i\right)\right] \\ &- \partial_j \left[\mathcal{B}^j \frac{\delta S}{\delta \mathcal{B}^k} + v^j \frac{\delta S}{\delta v^k} + \dots\right] = 0\,. \end{aligned} \tag{4.18}$$

It is important to clarify the notation employed in the above equation. We commence with the observation that S and $\mathcal{L}$ are the Eulerian action and Lagrangian density respectively, since the overbars were dropped at the end of the previous subsection. The functional derivatives of S are just the shorthand notation for the RHS of (4.17). Hence, it must be noted that the final expression only involves the partial derivatives of $\mathcal{L}$ with respect to the observables, and with respect to the spatial gradients of the observables. Lastly, we note that the "..." indicate that higher order derivatives of the observables can be included in the action (4.2), which in turn induces higher order derivatives (and terms) in the above equation.

The equation of motion has been determined, and is given by (4.18). Now, let us evaluate the dynamical equations for the observables. From the Lagrange-Euler maps, one can use the procedure outlined in [64, 65] to obtain the corresponding dynamical equations. For $\mathcal{P}$, we find that

$$\frac{\partial \mathcal{P}}{\partial t} + \nabla \cdot (\mathcal{P} v) = 0. \tag{4.19}$$

The dynamical equation for $\mathcal{S}$ is found to be

$$\frac{\partial \mathcal{S}}{\partial t} + v \cdot \nabla \mathcal{S} = 0, \tag{4.20}$$

and lastly, the evolution equation for $\mathcal{B}$ is given by

$$\frac{\partial \mathcal{B}}{\partial t} + \mathcal{B}\left(\nabla \cdot v\right) - \left(\mathcal{B} \cdot \nabla\right) v + \left(v \cdot \nabla\right) \mathcal{B} = 0\,. \tag{4.21}$$

We observe that replacing the specific entropy $\mathcal{S}$ by the entropy density $\sigma = \mathcal{P}\mathcal{S}$ in (4.20) leads to a conservation law of σ.

Finally, we note that our Lagrangian density (typically) possesses an internal energy density U per unit mass/particle. We can close the fluid system by choosing U to be a suitable function of the thermodynamic variables.

By extending this framework to incorporate $|\mathcal{B}|$ as well, one can construct models with anisotropic pressure. We refer the reader to [62, 57] for a detailed account of the same. The Lagrangian counterpart of the internal energy is automatically achieved via the closure principle, i.e. we choose Lagrangian functionals such that they Eulerianize to their known (Eulerian) counterparts.

4.2 Analysis of fluids, magnetofluids, and gyrofluids

In this section, we use Noether's theorem in conjunction with (4.18) to make some general statements about fluids and magnetofluids. Then, we shall specialize to the case of the gyrofluid and discuss it in greater detail.

We work with the commonly used observables for fluid models, i.e., $\mathcal{S}$ is replaced by s, $\mathcal{B}$ by B and $\mathcal{P}$ by ρ, and we split the action into a part that depends on $\dot{q}$ and one that does not:

$$S[q] = \int dt\, \big(T[\dot{q}] - V[q]\big)\,. \tag{4.22}$$

It is important to note that there is no explicit q-dependence in our model. To see this, we must first recollect that none of our Lagrange-Euler maps involve q explicitly; instead, they involve only the derivatives of q with respect to t and a. Next, we notice that our Eulerian action is written solely in terms of the observables, and by the closure principle, it must have arisen from its Lagrangian counterpart. But, we see that none of the observables, on mapping back to their Lagrangian counterparts, involve q explicitly. As a result, we conclude that our Lagrangian action does not involve q explicitly. In general, let us suppose that we can write $T[\dot{q}]$ as

$$
\begin{aligned}
T[\dot{q}] &= \int_D d^3a\left(\mathcal{M}_{0i}\dot{q}^i + \wp_{0ij}\dot{q}^i\dot{q}^j + \mathcal{V}_{0ijk}\dot{q}^i\dot{q}^j\dot{q}^k + \dots\right) \\
&\quad + \int_D d^3r\left(\mathcal{M}_i v^i + \wp_{ij}v^i v^j + \mathcal{V}_{ijk}v^i v^j v^k + \dots\right) , \qquad (4.23)
\end{aligned}
$$

where we have used the fact that our action is fully Eulerianizable. In other words, we require $\mathcal{M}_i = \mathcal{M}_{0i}/\mathcal{J}$ and identical relations for $\wp$ and $\mathcal{V}$ in order to ensure this property. Note that the RHS of this relation is evaluated at $a = q^{-1}(r,t)$ as always. All of the expressions described above, such as $\mathcal{M}$, $\wp$, $\mathcal{V}$, etc. are purely functions of s, ρ and B, i.e. they are independent of v.

We have not yet specified anything about the tensors $\mathcal{M}$, $\wp$ and $\mathcal{V}$. At this stage, we only know that they are functions of the observables and their spatial derivatives, i.e., they must possess the same form as $\mathcal{L}$ from (4.2), minus the dependence of v. Let us postulate further that these tensors are fully symmetric under the exchange of any pair of indices - this is chosen purely for the sake of simplicity. Since we know that our action is independent of q, the corresponding canonical momentum must be conserved. The canonical momentum is given by

$$
\Pi_i = \mathcal{M}_{0i} + 2\wp_{0ij}\dot{q}^j + 3\mathcal{V}_{0ijk}\dot{q}^j\dot{q}^k + \dots, \qquad (4.24)
$$

since the tensors are symmetric. The Eulerian counterpart can be found from (2.13) by using the fact that $\mathcal{M}_i = \mathcal{M}_{0i}/\mathcal{J}$ (and the same for the rest). It turns out to be

$$
M_i^c = \mathcal{M}_i + 2\wp_{ij}v^j + 3\mathcal{V}_{ijk}v^j v^k + \dots. \qquad (4.25)
$$

This result can also be obtained from (4.18), thereby serving as a good consistency check. The first term in (4.18), which is given by $-\frac{\partial}{\partial t}\left(\frac{\delta S}{\delta v^k}\right)$, reduces to $\partial M_k^c/\partial t$. As a result, our equation of motion becomes

$$\frac{\partial M_k^c}{\partial t} + \partial_j T_k^j = 0, \tag{4.26}$$

which ensures that M^c is conserved. The conservation of angular momentum is a trickier business. The sufficient condition for angular momentum conservation is that T_k^j must be symmetric. Since we are dealing with a very general ansatz, it is not possible to determine *a priori* whether our classes of models will conserve angular momentum in general. The quantities $\mathcal{M}$, $\wp$, etc must be explicitly known in order to provide a definite answer. For the case of ideal hydrodynamics and magnetohydrodynamics, the tensor T_k^j is indeed symmetric.

Now, let us consider the simpler case wherein $\wp_{ij} = \frac{1}{2}\rho\delta_{ij}$. We define the kinetic momentum $M = \rho v$. We find that

$$M_i^c = \mathcal{M}_i + M_i + 3\frac{\mathcal{V}_{ijk}}{\rho^2} M^j M^k + \dots, \tag{4.27}$$

and we know that the LHS is conserved, i.e. $\frac{d}{dt}\left[\int_D d^3r M^c\right]$ is zero, provided we assume that all boundary terms that arise vanish. Let us now consider the constraints under which the conservation of M^c simplifies to the conservation of M. For starters, the first term on the RHS of (4.27) must be expressible as the divergence of a tensor. Upon integration by parts, it will then yield a boundary term which can be made to vanish. Hence, a sufficient condition for M to be conserved is to have $\mathcal{M}_i = \partial_j \mathfrak{L}_i^j$ and $\mathcal{V}_{ijk} = 0$.

We will now focus our attention on the model where the above constraints are satisfied. Let us choose to work with an action

$$S = S_{MHD} - \int_D d^3r dt\, \mathfrak{L}_i^j \partial_j v^i, \tag{4.28}$$

where our set of observables are now ρ, s, B and v. The quantity S_{MHD} represents the ideal MHD action, whose explicit expression is known (see, e.g., [18, 64]). The expression for the action is quite similar to the one studied in Chapter 3, and, in fact, constitutes the class of gyroviscous models.

From our preceding analysis, it is clear that both M^c and M are conserved for this model. It is also clear that this action satisfies the ansatz that we specified in (4.2). Furthermore, the ideal MHD action yields a symmetric momentum flux tensor, ensuring that T_k^j is symmetric. Hence, the first term in (4.28) also conserves angular momentum.

As a result, we only need to investigate $\mathfrak{L}$ and the constraints that must be imposed upon it to ensure that T_k^j is symmetric. Given that $\mathfrak{L}$ can only depend on B, s and ρ and their first order derivatives, there are still an infinite number of terms that can be generated. It is evident of course that this system is too elaborate to permit further analysis. Hence, to illustrate our mode of analysis, we shall work with a test case where $\mathfrak{L}_i^j$ is symmetric and has the form

$$\mathfrak{L}_i^j = \frac{1}{2}\left[\left(B^j B_i + B^i B_j\right)\alpha_I + \left(\delta_i^j + \delta_j^i\right)\alpha_{II}\right], \tag{4.29}$$

with $\alpha_{I,II}$ only depending on s, ρ and $|B|$. We use (4.28) and (4.29) in (4.18). Rather than use brute force, we shall use some of the inherent symmetries of

(4.18). Note that the first line in (4.18) contains terms that yield a symmetric contribution to T^j_k, since they are gradient terms, similar to the pressure. In the second line of (4.18), there are no contributions since there are no density and entropy gradients. The same is also true for the first term on the third line of (4.18), since (4.29) does not possess magnetic field gradients. As a result, we are left with only three terms of interest - the last three occurring in the LHS of (4.18). Upon evaluation, we find that the resulting tensor is not symmetric, and the ansatz (4.29) does not possess angular momentum conservation. Thus, from this test case, we see that one can analyse the model in considerable depth without explicitly working through all the intermediate details.

The existence of multiple observables, such as the magnetic field for instance, makes it difficult to address the issue of angular momentum conservation. The primary source of ambiguity emerges from the difficulty in imposing *a priori* constraints on the general *structure* that the Lagrangian should possess in order to exhibit rotational invariance. However, one may be able to generate an angular momentum, albeit not $r \times M$, that is conserved. We observe that the intricacies behind angular momentum conservation are quite interesting, and constitute a promising line of enquiry, both from a theoretical and applied standpoint; the latter is of interest as it affects angular momentum transport, which is widely studied in fusion and astrophysical plasmas.

Now, let us suppose that we consider the hydrodynamic case where B is absent. Let us further specialize to the case where gradients with respect to s and ρ are absent, and the gyroscopic term is of the form (4.28). In such a scenario, the condition for angular momentum conservation becomes particularly simple, since the tensor

$$\mathfrak{L}_i^j \left(\partial_k v^i\right) + v^j \partial_i \left(\mathfrak{L}_k^i\right) , \tag{4.30}$$

must be symmetric.

4.3 An illustration of the formalism

To demonstrate the utility of the formalism developed in this paper, we now consider a simple illustration that demonstrates how an additional attribute can be added to ideal HD. The attribute we add represents an internal degree of freedom, an intrinsic rotational (angular) velocity, attached to each fluid element. Given the new set of observables, we can immediately use (4.18) to compute the corresponding equation of motion.

There are many physical situations where an internal angular velocity or momentum is appropriate, because such microscopic behaviour influences the macroscopic dynamics. One example is the effect of finite Larmor radius gyration of charged particles in a magnetic field [129, 130] while another occurs in the theory of nematodynamics. We consider the latter which applies to liquid crystals, where the Lagrangian description of the fluid 'particles' models them as a collection of rods. These rods are endowed with a preferred direction, called the director, and an intrinsic angular momentum. The relevant dynamics for this system are described in [139]; see also [140] for an associated discussion. For an introduction to nematodynamics, we refer the reader to the classic works of [141, 142, 143] and the modern works of [144, 40, 145]. A simplified limit of this work where phenomenological dissipative relaxing is removed (their parameter $\gamma^{-1} \to 0$) results in a reduction to a single variable $\Omega_{\|}$, an angular velocity proportional to the intrinsic angular momentum parallel to the now constant director. The new variable $\Omega_{\|}$ is advected by the fluid velocity field, and thus behaves as a 0-form. We shall work with this subcase henceforth.

One must now construct an appropriate Lagrange-Euler map for our attribute-observable pair, denoted by $\Omega_{0\|}$ and $\Omega_{\|}$, respectively. Since we have noted that $\Omega_{\|}$ is advected, this corresponds to $\Omega_{\|} = \Omega_{0\|}$, with the RHS evaluated at $a = q^{-1}(r, t)$. The advection equation is given by

$$\frac{\partial \Omega_{\|}}{\partial t} + v^j \partial_j \Omega_{\|} = 0, \tag{4.31}$$

and the similarity to the entropy is self-evident. One can now construct an angular momentum density, $l^2 \omega_d := \rho\, l^2 \Omega_{\|}$, that behaves as a three-form. Here the quantity l represents the radius of gyration with l^2 being interpreted as the moment of inertia per unit mass. Its governing equation is

$$\frac{\partial \omega_d}{\partial t} + \partial_j \left(v^j \omega_d\right) = 0, \tag{4.32}$$

and the relationship between ω_d and its corresponding attribute ω_{0d} is $\omega_d = \omega_{0d}/\mathcal{J}$ with the RHS evaluated at $a = q^{-1}(r, t)$.

By analogy with classical (discrete) mechanics, we propose the continuum rotational kinetic energy functional,

$$K_{rot} := \int_D d^3a \, \frac{1}{2}\rho_0 l^2 \Omega_{0\|}^2 = \int_D d^3a \, l^2 \frac{\omega_{0d}^2}{2\rho_0} \,. \tag{4.33}$$

It is easily verified that the above functional satisfies the Eulerian closure principle, with its counterpart given by $l^2\omega_d^2/(2\rho)$. From (4.33), it is evident that the expression is entirely independent of q, implying that it serves as a Lagrangian invariant, and does not enter the equation of motion. This can also be verified by taking the Eulerian counterpart and substituting it into (4.18). Despite its absence in the momentum equation of motion, it is instructive to see how the rotational and translational kinetic energies stack up against each other. In order to compute the rotational energy, we assume that our fluid particles can be modelled as molecules. In such a scenario, we find that the ratio reduces to

$$\frac{l^2\Omega_\|^2}{v^2} \sim \frac{\Theta}{T}, \tag{4.34}$$

where Θ denotes the rotational temperature of the molecules [146]. We have assumed that v is characterized by the thermal velocity, and $l\Omega_\|$ by the rotational temperature. In general, it is evident that this ratio is extremely small for hot fluids, such as the ones observed in fusion reactors or in stars. However, there exist environments in nature, such as giant molecular clouds which possess temperatures of a few tens of Kelvin [147, 148]. They are comprised of molecular hydrogen, whose rotational temperature is known to be around 88 K [146]. As a result, we see that the two energies are comparable in this regime and there remains an outside possibility that such effects might be of importance. As stated earlier, (4.33) does not enter the equation of motion, but it does serve as a Casimir invariant, and is therefore of interest.

We have earlier mentioned that we study a subcase of [139] where coupling terms involving v and $\Omega_{\parallel}$ are non-existent. Now, let us add a simple term of the form $l^2\omega_d C_i^k \partial_k v^i$ to the action, where C_i^k is a tensor with constant coefficients. Then, the full action is given by

$$S := S_{HD} + \int_D d^3r dt\, l^2\omega_d C_i^k \partial_k v^i, \tag{4.35}$$

where S_{HD} represents the ideal HD action. The new term can be interpreted as follows. Firstly, observe that an integration by parts casts it in the form of $v\cdot\nabla\times L_{int}$, provided that we associate the tensor C_i^k with the three-dimensional Levi-Civita tensor - with one of the indices fixed to be $\hat{z}$, the director direction - and the term L_{int} with the intrinsic angular momentum density. By inspection, it is found that $L_{int} = (l^2\omega_d)\,\hat{z} = (\rho l^2)\,\Omega_{\parallel}\hat{z}$ and it is evidently the product of the moment of inertia (per unit volume) and the angular velocity. This term was not constructed at random - it corresponds to the analogue of 2D gyroviscous MHD studied in Chapter 3. In gyroviscous MHD, the particles undergo Larmor gyration as a result of the magnetic field, behaving as though they were indeed endowed with an intrinsic angular velocity (and angular momentum). The corresponding equation of motion is given by

$$\begin{aligned} \frac{\partial\left(\rho v_k\right)}{\partial t} &+ \partial_j\left[\left(C_k^j v^i - C_k^i v^j\right)\partial_i\left(l^2\omega_d\right)\right] \\ &+ \partial_j\left[\left(l^2\omega_d\right)\left(C_k^j \partial_i v^i - C_i^j \partial_k v^i\right)\right] + \cdots = 0, \end{aligned} \tag{4.36}$$

where the "..." indicates that the rest of the terms are identical to the ones present in the ideal HD momentum equation. The four additional terms involve gradients with respect to the velocity (or angular velocity), and they serve as the *de facto* viscosity tensor. If we assume that the fluid has the property that $\omega_d = \text{const}$, the two terms in the first line of (4.36) vanish identically. However, the next two terms are still present, which changes the ideal MHD momentum flux. With this special choice of ω_d, one notices a striking resemblance with the conventional viscous tensor - there are terms involving $\partial_k v^i$ and the divergence $\partial_i v^i$, and the coefficients in front of these terms correspond to the dynamic and bulk viscosities respectively. This explains the rationale behind treating it as a 'viscosity' despite the fact that it is dissipationless in general. Thus, we see that the angular momentum fluid, with some minor restrictions, mirrors the conventional viscous fluid. The viscous tensor that arises when we set $C^i_k \equiv \epsilon^i_k$ is a very special one - it corresponds to the 2D simplified limit of the Braginskii gyroviscous tensor, as noted in Chapter 3. We also remarked, in the same Chapter, that it arises in a wide range of contexts in condensed matter, and has been referred to as Hall viscosity. Thus, we see that there exists a natural commonality between plasmas and liquid crystals, which was further explored in [133].

In general, ω_d depends on time, and hence one can interpret (4.36) as comprising of time-dependent viscosities, thereby representing a theory of non-Newtonian fluids [149, 150]. The importance of such fluids in biological systems is well-documented [151, 152]. The action (4.35) conserves energy and linear momentum ρv, but it does *not* always conserve the angular momentum $r \times (\rho v)$. If we assume that the coupling tensor C^i_j is purely antisymmetric, the stress tensor in (4.36) becomes symmetric, thereby leading to angular momentum conservation. One such choice corresponds to the 2D Levi-Civita tensor, which was discussed in the previous paragraph.

In our discussion here, we have built a theory of fluids with intrinsic angular momentum by incorporating the rotational kinetic energy and gyroviscous terms. This illustrative model corresponds to a simplified version of [139] for nematic effects in liquid crystals, but with additional effects incorporated, and it serves as an illustration of building models from scratch. Clearly the nondissipative parts of more complete models can be built in this manner, and potential applications in a variety of fields, e.g., nematics, micromorphic systems [153], and plasma physics, come to mind.

Chapter 5

The derivation of extended MHD models via the two-fluid action

In this Chapter, we shall use the two-fluid action as our starting point, and highlight one of the chief advantages of the HAP formulation - the capacity to perform orderings in the action, as opposed to the equations of motion. We shall see that a variety of extended MHD models emerge as a consequence. Unlike the previous chapters, which relied on an action principle in purely Lagrangian variables, our treatment involves a mixed Eulerian-Lagrangian framework. Hence, we shall endeavour to keep the treatment as self-contained as possible. [1]

5.1 Review: Two-fluid model and action

In this section, we will briefly review the derivation of the non-dissipative two-fluid model equations of motion from the general two-fluid action. We shall

[1] This contents of this chapter were published in: "Action principles for extended magnetohydrodynamic models", I. Keramidas Charidakos, M. Lingam, P. J. Morrison, R. L. White & A. Wurm, 2014, *Phys. Plasmas*, **21**, 092118 [154]. The initial premise and results were contributed by A. Wurm and P. J. Morrison. The formalism and results were jointly derived by this author (M. Lingam), I. Keramidas Charidakos and R. L. White, with guidance and supervision from P. J. Morrison.

use the two-fluid action as our parent model in deriving reduced models via suitable orderings. We also use this section to briefly establish the notation employed.

To simplify our notation, we will avoid explicit vector notation, following the approach outlined in [17], and define the following: $q_s = q_s(a, t)$ is the position of a fluid element in a Cartesian coordinate system, where $a = (a_1, a_2, a_3)$ is any label identifying the fluid element and $q_s = (q_{s1}, q_{s2}, q_{s3})$. The label $s = (i, e)$ is used to denote the species under consideration. Here, we choose a to be the initial position of the fluid particle at $t = 0$, although other choices are possible [87]. The Lagrangian velocity will then be denoted by $\dot{q}_s$.

The Eulerian velocity field will be denoted by $v_s(r, t)$, where $v_s = (v_{s1}, v_{s2}, v_{s3})$ and $r = (x_1, x_2, x_3)$ denotes the position in the Eulerian framework. Similarly, we define the electric field vector $E(r, t)$, the magnetic field vector $B(r, t)$, and the vector and electrostatic potential $A(r, t)$ and $\phi(r, t)$. If we need to explicitly refer to components of these vectors, we will use subscripts (or superscripts) j and k. To simplify the equations, we will also often suppress the dependence on r, a, and t.

The action functional described below will include integrations over both the position space $\int d^3r$ and the label space $\int d^3a$. We will not explicitly specify the domains of integration, but we shall assume that our functions are well-defined on their respective domains, and that integrating them and taking functional derivatives is allowed. Moreover, we will assume that all variations on the boundaries of the domains and any surface terms (due to integration by parts) vanish. In the previous Chapters, we worked with in SI units with $\mu_0 = 1$, but we shall use CGS units for the duration of this Chapter.

5.1.1 Constructing the two-fluid action

The action functional of a general theory of a charged fluid interacting with an electromagnetic field should include the following components:

- The energy of the electromagnetic field, comprising of terms proportional to E^2 and B^2.
- The fluid-field interaction energy, expressed in terms of the electromagnetic potentials. There are two components of this term, one for each species.
- The fluid energy, comprising of the kinetic and internal energies for each species.

We will assume two independent fluids corresponding to two different species, chosen to be ions and electrons. The species are endowed with charge e_s, mass m_s and initial number density of $n_{s0}(a)$ and they interact with the electromagnetic field, but not directly with each other. Therefore, it is easy to see that the fluid-dependent parts of the action will naturally split into two parts, one for each species.

The complete action functional is given by

$$S[q_s, A, \phi] = \int_T dt\ L\,, \tag{5.1}$$

where T serves a finite time interval and the Lagrangian L is given by

$$L = \frac{1}{8\pi}\int d^3r \left[\left|-\frac{1}{c}\frac{\partial A(r,t)}{\partial t} - \nabla\phi(r,t)\right|^2 - |\nabla \times A(r,t)|^2\right] \tag{5.2}$$

$$+\sum_s \int d^3a\, n_{s0}(a) \int d^3r\ \delta\left(r - q_s(a,t)\right) \times \left[\frac{e_s}{c}\ \dot{q}_s \cdot A(r,t) - e_s\phi(r,t)\right] \tag{5.3}$$

$$+\sum_s \int d^3a\ n_{s0}(a)\Big[\frac{m_s}{2}\ |\dot{q}_s|^2 - m_s U_s\left(m_s n_{s0}(a)/\mathcal{J}_s, s_{s0}\right)\Big]\,. \tag{5.4}$$

The symbol $\mathcal{J}_s$ is the Jacobian of the map between Lagrangian positions and labels, $q(a,t)$, which shall be covered in more detail subsequently. We have expressed the electric and magnetic fields in terms of the vector and scalar potential, $E = -1/c\,(\partial A/\partial t) - \nabla\phi$ and $B = \nabla \times A$ everywhere. The first term (5.2) is the electromagnetic field energy, while the next expression (5.3) denotes the the coupling of the fluid to the electromagnetic field, which is achieved here by using the delta function. The delta function ensures that the fluid element passing by the Eulerian point r is 'plucked out'. The last line of the Lagrangian L (5.4) represents the kinetic and internal energies of the fluid. We see that the specific internal energy (energy per unit mass) of species s, U_s, depends both on the density as the entropy s_{s0} for each species. Also note, that the vector and scalar potentials are Eulerian variables. Thus, the Lagrangian, and thereby, the action, are mixed in nature, they comprise of both the Eulerian and Lagrangian variables.

5.1.2 Lagrange-Euler maps for the two-fluid model

In accordance with the Eulerian Closure Principle introduced in Chapter 2, we need to ensure that the action (5.1) can be completely expressed in terms of the desired set of Eulerian variables, which in turn ensures that the resulting equations of motion will also be completely Eulerian, hence representing a physically meaningful model. The connection between the Lagrangian and Eulerian pictures of fluids is achieved via the Lagrange-Euler maps. Before looking at the mathematical implementation of this map, let us recollect its physical meaning once more. As an example, consider the Eulerian velocity field $v(r,t)$ at a particular position r at time t. The velocity of the fluid at that point will be the velocity of the particular fluid element $\dot{q}(a,t)$ which has started out at position a at time $t=0$ and then arrived at point $r=q(a,t)$ at time t.

To implement this idea, we define the Eulerian number density $n_s(r,t)$ in terms of Lagrangian quantities as follows:

$$n_s(x,t) = \int d^3a \; n_{s0}(a) \; \delta\left(x - q_s(a,t)\right) \, . \tag{5.5}$$

Using the properties of the delta function, this relation can be rewritten as

$$n_s(x,t) = \left. \frac{n_{s0}(a)}{\mathcal{J}_s} \right|_{a=q_s^{-1}(x,t)} \, , \tag{5.6}$$

where, $\mathcal{J}_s = \det\left(\partial q_s/\partial a\right)$ is the Jacobian determinant. Note that (5.6) implies the continuity equation

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s v_s) = 0 \, , \tag{5.7}$$

automatically follows, which corresponds to local mass conservation if we define the mass density as $\rho_s = m_s n_s$. As per the interpretation discussed in Chapter 2, this arises as a consequence of the number density being Lie-dragged as a 3-form.

The corresponding relation for the Eulerian velocity is

$$v_s(r,t) = \dot{q}_s(a,t)\big|_{a=q_s^{-1}(r,t)}\,, \tag{5.8}$$

where the dot indicates differentiation with respect to time at fixed particle label a. This relation follows from integrating out the delta function in the Lagrange - Euler map for the (Eulerian) momentum density, $M_s := m_s n_s v_s$,

$$M_s(r,t) = \int d^3a\, n_{s0}(a,t)\delta\left(r - q_s(a,t)\right) m_s \dot{q}_s(a,t)\,. \tag{5.9}$$

Finally, our Eulerian entropy per unit mass, $s_s(r,t)$, is defined by

$$\rho_s s_s(r,t) = \int d^3a\, n_{s0}(a) s_{s0}(a) m_s\ \delta\left(r - q_s(a,t)\right)\,, \tag{5.10}$$

completing our set of fluid Eulerian variables for this theory, which is $\{n_s, s_s, M_s\}$. It is easy to check that the closure principle is satisfied by these variables.

For later use, we quote (without proof) some results from Chapter 2 involving the determinant and its derivatives

$$\frac{\partial q^k}{\partial a^j}\frac{A_k^i}{\mathcal{J}} = \delta_j^i\,, \tag{5.11}$$

where A_k^i is the cofactor of $\partial q^k/\partial a^i := q^k_{,i}$. A convenient expression for A_k^i is

$$A_k^i = \frac{1}{2}\epsilon_{kjl}\epsilon^{imn}\frac{\partial q^j}{\partial a^m}\frac{\partial q^l}{\partial a^n}\,, \tag{5.12}$$

where $\epsilon_{ijk}(= \epsilon^{ijk})$ is the Levi-Civita tensor. Using (5.11), one can show that

$$\frac{\partial \mathcal{J}}{\partial q^i_{,k}} = A^j_i \tag{5.13}$$

and using the chain rule

$$\frac{\partial}{\partial q^k} = \frac{1}{\mathcal{J}} A^i_k \frac{\partial}{\partial a^i} \, . \tag{5.14}$$

5.1.3 Varying the two-fluid action

The action of (5.1) depends on four dynamical variables: the scalar and vector potentials, ϕ and A, and the positions of the fluid elements q_s. Of these, the former duo are Eulerian in nature, whilst the latter are Lagrangian.

Varying with respect to ϕ yields Gauss's law

$$\partial_k \left(-\frac{1}{c}\frac{\partial A_k}{\partial t} - \partial_k \phi \right) = 4\pi e \left[\int d^3a \; n_{i0}(a) \, \delta \left(r - q_i \right) - \int d^3a \; n_{e0}(a) \, \delta \left(r - q_e \right) \right] ,$$

where $\partial_k := \partial / \partial x^k$, or in more familiar form

$$\nabla \cdot E = 4\pi e \left(n_i(x,t) - n_e(x,t) \right) \, . \tag{5.15}$$

Similarly, the variation with respect to A recovers the Maxwell-Ampere law

$$\frac{1}{4\pi} \left[-\nabla \times \nabla \times A + \frac{1}{c}\frac{\partial}{\partial t} \left(-\frac{1}{c}\frac{\partial A}{\partial t} - \nabla \phi \right) \right] - \frac{e}{c} \int d^3a \quad \left[\delta \left(r - q_e \right) n_{e0} \dot{q}_e + \delta \left(r - q_i \right) n_{i0} \dot{q}_i \right] = 0$$

which can be re-expressed to yield

$$\nabla \times B = \frac{4\pi J}{c} + \frac{1}{c}\frac{\partial E}{\partial t} \tag{5.16}$$

where the Eulerian current density J is defined as

$$J(r,t) = e\left(n_i v_i - n_e v_e\right) \ . \tag{5.17}$$

Recall that the other two Maxwell equations are contained in the definition of the potentials.

Variation with respect to the q_s's is slightly more complex, and we will illustrate a few intermediate steps. Varying the kinetic energy term is straight forward and yields

$$-n_{s0}(a) m_s \ddot{q}_s(a,t) \tag{5.18}$$

The j-th component of the interaction term results in

$$\begin{aligned} e_s n_{s0}(a) & \left[-\frac{1}{c}\frac{dA^j(q_s,t)}{dt} + \frac{1}{c}\dot{q}_s^k \frac{\partial A^k(q_s,t)}{\partial q_s^j} - \frac{\partial \phi(q_s,t)}{\partial q_s^j}\right] \\ = e_s n_{s0}(a) & \left[-\frac{1}{c}\frac{\partial A^j(q_s,t)}{\partial t} - \frac{1}{c}\dot{q}_s^k \frac{\partial A^j(q_s,t)}{\partial q_s^k} \right. \\ & \left. + \frac{1}{c}\dot{q}_s^k \frac{\partial A^k(q_s,t)}{\partial q_s^j} - \frac{\partial \phi(q_s,t)}{\partial q_s^j}\right] \\ = e_s n_{s0}(a) & \left[E(q_s,t) + \frac{1}{c}\dot{q}_s(a,t) \times \left(\nabla_{q_s} \times A\left(q_s,t\right)\right)\right]_j \end{aligned} \tag{5.19}$$

Notice that this expression is purely Lagrangian. The fields A and E are evaluated at the positions q_s of the fluid elements and the curl $\nabla_{q_s}\times$ is taken with respect to the Lagrangian position q_s. Also note, since $q_s = q_s(a,t)$, any total time derivative of, e.g., $A(q_s,t)$ will result in two terms.

Variation of the internal energy term yields

$$A_i^j \frac{\partial}{\partial a_j}\left(\frac{\rho_{s0}^2}{\mathcal{J}_s^2}\frac{\partial U\left(\frac{\rho_{s0}}{\mathcal{J}_s}, s_{s0}\right)}{\partial\left(\frac{\rho_{s0}}{\mathcal{J}_s}\right)}\right) \ . \tag{5.20}$$

Setting the sum of (5.18), (5.19) and (5.20) equal to zero and invoking the usual thermodynamic relations between internal energy and pressure and temperature,

$$p_s = (m_s n_s)^2 \frac{\partial U_s}{\partial (m_s n_s)} \quad \text{and} \quad T_s = \frac{\partial U_s}{\partial s_s} \tag{5.21}$$

results in the well-known (non-dissipative) two-fluid equations of motion

$$m_s n_s \left(\frac{\partial v_s}{\partial t} + v_s \cdot \nabla v_s \right) = e_s n_s \left(E + \frac{1}{c} v_s \times B \right) - \nabla p_s \tag{5.22}$$

Further analysis, see e.g. [60, 155], of these equations usually involves the addition and subtraction of the two-fluid equations and a change of variable transformation to

$$\begin{aligned} V &= \frac{1}{\rho_m} \, (m_i n_i v_i + m_e n_e v_e) \\ J &= e \, (n_i v_i - n_e v_e) \\ \rho_m &= m_i n_i + m_e n_e \\ \rho_q &= e \, (n_i - n_e) \ . \end{aligned} \tag{5.23}$$

The resulting equations are then simplified by making certain assumptions such as quasineutrality, $v << c$, etc. and suitable orderings to obtain two new *one*-fluid equations – one of these is commonly described as the *one-fluid momentum equation* and the other serves as the *generalized Ohm's law*.

5.2 The new one-fluid action

Before proceeding onwards to model-building, we must identify the variables of interest, i.e. the Eulerian observables that make up our new model.

Since we want to derive the two-fluid model of Lüst [155] and various reductions, our Eulerian observables are going to be the set $\{n, s, s_e, V, J, E, B\}$, where $s = (m_i s_i + s_e m_e)/m$, with $m = m_e + m_i$, and n is a single number density variable.

Next we have to define our Lagrangian variables and also construct the action. We shall implement all assumptions, and orderings, solely on the level of the action, which preserves the Hamiltonian nature of the underlying physical theories. Varying the new action will then result in equations of motion that, using properly defined Lagrange-Euler maps, will Eulerianize to, e.g., Lüst's equation of motion and the generalized Ohm's law.

5.2.1 New Lagrangian variables

We will start by defining a new set of Lagrangian variables, inspired by (5.23), as follows,

$$
\begin{aligned}
Q(a,t) &= \frac{1}{\rho_{m0}(a)}\left(m_i n_{i0}(a) q_i(a,t) + m_e n_{e0}(a) q_e(a,t)\right) \\
D(a,t) &= e\left(n_{i0}(a) q_i(a,t) - n_{e0}(a) q_e(a,t)\right) \\
\rho_{m0}(a) &= m_i n_{i0}(a) + m_e n_{e0}(a) \\
\rho_{q0}(a) &= e\left(n_{i0}(a) - n_{e0}(a)\right) \,.
\end{aligned}
\tag{5.24}
$$

Here $Q(a,t)$ can be interpreted as the centre-of-mass position variable and $D(a,t)$ as a local dipole moment variable, connecting an ion fluid element to an electron fluid element. Upon taking the partial time derivatives of Q and

D, we arrive at the center-of-mass velocity $\dot{Q}(a,t)$ and the Lagrangian current $\dot{D}(a,t)$, respectively. Using appropriately defined Lagrange-Euler maps, we can ensure that $\dot{Q}(a,t)$ will map to the Eulerian velocity $V(x,t)$ and $\dot{D}(a,t)$ to the Eulerian current $J(x,t)$ as defined by (5.23).

We will also need the inverse of this transformation,

$$
\begin{aligned}
q_i(a,t) &= \frac{\rho_{m0}(a)Q(a,t) + \frac{m_e}{e}D(a,t)}{\rho_{m0}(a) + \frac{m_e}{e}\rho_{q0}} \\
q_e(a,t) &= \frac{\rho_{m0}(a)Q(a,t) - \frac{m_i}{e}D(a,t)}{\rho_{m0}(a) - \frac{m_i}{e}\rho_{q0}} \\
n_{i0}(a) &= \frac{\rho_{m0}(a) + \frac{m_e}{e}\rho_{q0}(a)}{m} \\
n_{e0}(a) &= \frac{\rho_{m0}(a) - \frac{m_e}{e}\rho_{q0}(a)}{m} \,.
\end{aligned}
\tag{5.25}
$$

5.2.2 Ordering of fields and quasineutrality

Before commencing our ordering procedure, we emphasize that our method is rather novel, as most reductions are obtained by imposing an auxiliary ordering on the equations of motion. On the other hand, we perform orderings directly in the action, which has the chief advantage of preserving the variational formulation.

To construct the action, we will start with the two-fluid action of (5.1) and change variables to Q and D, we will first make two simplifying assumptions to recover models that are of interest to us. Firstly, we order the electromagnetic fields in the action so that the displacement current in (5.16) will vanish. Secondly, we shall enforce an ordering that automatically leads to quasineutrality. We shall now describe this field ordering in detail and discuss quasineutrality in the Lagrangian variable context, which as far as we know has not been done before.

We can neglect the displacement current in Ampére's law when the time scale of changes in the field configuration is much longer when compared to the time it takes for radiation to "communicate" these changes across the system [93, 156]. We transition to non-dimensional variables by introducing a characteristic scale B_0 for the magnetic field and a characteristic length scale ℓ for gradients. We normalize the $\dot{q}_s$'s by the Alfvén speed $v_A = B_0/\sqrt{4\pi\rho}$ and the times by the Alfvén time $t_A = \ell/v_A$, resulting in the following form for the sum of the field and interaction terms of the Lagrangian:

$$\frac{B_0^2}{8\pi}\int dt \int d^3\hat{r}\left[\left|-\frac{v_A}{c}\frac{\partial \hat{A}}{\partial \hat{t}} - \frac{\phi_0}{B_0\ell}\hat{\nabla}\hat{\phi}\right|^2 - \left|\hat{\nabla}\times\hat{A}\right|^2\right]$$
$$+\sum_s B_0^2\left[\int dt \int d^3\hat{a}\; n_0\hat{n}_{s0}(a)e_s \int d^3\hat{r}\;\delta\left(\hat{r}-\hat{q}_s\right)\times\left(\frac{v_A\ell}{B_0 c}\hat{\dot{q}}_s\cdot\hat{A} - \frac{\phi_o}{B_0^2}\hat{\phi}\right)\right],$$

where ϕ_0 and n_0 are yet to be specified scales for the electrostatic potential and the densities of both species, respectively. We also require that the two species' velocities are of the same scale.

Now, we demand that the two terms in the interaction play an equally important role, i.e. we require them of the same order. This automatically results in a scaling for ϕ; viz., $\phi_0 \equiv B_0\ell v_A/c$. Thus, both parts of the $|E|^2$ term are of order $\mathcal{O}\left(v_A/c\right)$. Neglecting this term and varying with respect to

$\hat{A}$ results in

$$\hat{\nabla} \times \hat{B} = \frac{4\pi e n_0 v_A}{c} \frac{\ell}{B_0} \left(\int d^3a \; \delta(\hat{r} - \hat{q}_i)\hat{n}_{i0}(a)\dot{\hat{q}}_i - \int d^3a \delta(\hat{r} - \hat{q}_e)\hat{n}_{e0}(a)\dot{\hat{q}}_e \right) ,$$

which can be written as

$$\frac{B_0}{\ell} \hat{\nabla} \times \hat{B} = \frac{4\pi j_0}{c} \hat{J} \, , \tag{5.26}$$

where $j_0 = e n_0 v_A$ is the appropriate scale for the current.

Varying the scaled action with respect to $\hat{\phi}$ yields

$$0 = \int d^3\hat{a} \delta(\hat{r} - \hat{q}_i)\hat{n}_i - \int d^3\hat{a} \delta(\hat{r} - \hat{q}_e)\hat{n}_e \equiv \Delta\hat{n} \tag{5.27}$$

The above equation states that the difference in the two densities is zero, i.e., the plasma is quasineutral, and this is an property that holds true locally, i.e. it amounts to $n_i(r,t) = n_e(r,t)$. Using (5.6), this statement would correspond to the following in the Lagrangian variable picture:

$$\left. \frac{n_{i0}(a)}{\mathcal{J}_i(a,t)} \right|_{a=q_i^{-1}(r,t)} = \left. \frac{n_{e0}(a)}{\mathcal{J}_e(a,t)} \right|_{a=q_e^{-1}(r,t)} . \tag{5.28}$$

In the Lagrangian picture we will make the additional assumption of homogeneity: $n_{i0}(a) = n_{e0}(a) =$ constant. This statement implies that, at $t = 0$, all fluid elements are identical in the amount of density that they carry. Therefore, we can replace n_{i0} and n_{e0} with a constant n_0. Equation (5.28) then reduces to a statement about the two Jacobians

$$\left. \mathcal{J}_i(a,t) \right|_{a=q_i^{-1}(r,t)} = \left. \mathcal{J}_e(a,t) \right|_{a=q_e^{-1}(r,t)} , \tag{5.29}$$

which will play a central role in our development below.

At first glimpse, it may appear as though we cannot describe plasmas with density gradients, as we have used the term 'homogeneity' in our description. But, it is important to recognize that this assumption is only on the Lagrangian level. In moving to the Eulerian picture, the presence of the Jacobians ensures that density gradients and inhomogeneity can exist, and evolve over time. Furthermore, we can select our labeling scheme, and hence the Jacobian, to suitably reflect the initial density gradient of the configuration. Thus, it is clear that there is still a considerable amount of freedom in this regard.

5.2.3 Action functional

We are now ready to implement the change of variables discussed in Section 5.2.1. Because of the homogeneity assumption $n_{i0}(a) = n_{e0}(a) = n_0$, the new variables of (5.24) reduce to

$$\begin{aligned} Q(a,t) &= \frac{m_i}{m} q_i(a,t) + \frac{m_e}{m} q_e(a,t) \\ D(a,t) &= e n_0 \left(q_i(a,t) - q_e(a,t) \right) \\ \rho_{m0}(a) &= m n_0 \\ \rho_{q0}(a) &= 0 \end{aligned} \tag{5.30}$$

and the inverse transformation of (5.25) to

$$\begin{aligned} q_i(Q,D) &:= q_i(a,t) = Q(a,t) + \frac{m_e}{m e n_0} D(a,t) \\ q_e(Q,D) &:= q_e(a,t) = Q(a,t) - \frac{m_i}{m e n_0} D(a,t) \,, \end{aligned} \tag{5.31}$$

where we choose the notation $q_s(Q, D)$ to emphasize that the Lagrangian variables q_s should not be thought of as ion/electron trajectories any more but as specific linear combinations of $Q(a, t)$ and $D(a, t)$. In addition, we will need the ion and electron Jacobians, $\mathcal{J}_i(Q, D)$ and $\mathcal{J}_e(Q, D)$, which must now expressed in terms of Q and D.

The resulting action functional has the form:

$$
\begin{aligned}
S = &-\frac{1}{8\pi}\int dt \int d^3r \; |\nabla \times A(r,t)|^2 \\
&+ \int dt \int d^3r \int d^3a \; n_0 \Big\{ \delta(x - q_i(Q,D)) \\
&\quad \times \left[\frac{e}{c} \left(\dot{Q}(a,t) + \frac{m_e}{men_0} \dot{D}(a,t) \right) \cdot A(x,t) - e\phi(x,t) \right] \Big\} \\
&+ \int dt \int d^3r \int d^3a \; n_0 \Big\{ \delta(x - q_e(Q,D)) \\
&\quad \times \left[-\frac{e}{c} \left(\dot{Q}(a,t) - \frac{m_i}{men_0} \dot{D}(a,t) \right) \cdot A(x,t) + e\phi(x,t) \right] \Big\} \\
&\quad + \frac{1}{2} \int dt \int d^3a \; n_0 \left[m_i |\dot{Q}|^2(a,t) + \frac{m_i m_e}{me^2 n_0^2} |\dot{D}|^2(a,t) \right] \\
&\quad - \int dt \int d^3a \; n_0 \Bigg[m_i U_i \left(\frac{m_i n_0}{\mathcal{J}_i(Q,D)}, (ms_0 - m_e s_{e0})/m_i \right) \\
&\qquad\qquad + m_e U_e \left(\frac{m_e n_0}{\mathcal{J}_e(Q,D)}, s_{e0} \right) \Bigg],
\end{aligned}
\tag{5.32}
$$

where we recall that $s_0 = (m_i s_{i0} + m_e s_{e0})/m$.

5.2.4 Nonlocal Lagrange-Euler maps

Hitherto, we have introduced Q and D and worked with them extensively, but we have not specified their Eulerian counterparts. Now we define the Lagrange-Euler maps that connect the Eulerian observables V and J to the new Lagrangian variables Q and D. Referring to Section 5.1.2, the Lagrange-Euler map was specified as a relationship between a Lagrangian quantity and some Eulerian observables, which holds only when it is evaluated on a trajectory $r = q_s(a,t)$. If we apply the inverse Lagrange-Euler maps from Eqs. (5.6) and (5.8) to (5.23) and assume quasineutrality, we get

$$
\begin{aligned}
V(r,t) &= \frac{m_i}{m}\,\dot{q}_i(a,t)\Big|_{a=q_i^{-1}(r,t)} + \frac{m_e}{m}\,\dot{q}_e(a,t)\Big|_{a=q_e^{-1}(r,t)} \\
J(r,t) &= e\left(\frac{n_0}{\mathcal{J}_i(a,t)}\dot{q}_i(a,t)\right)\Bigg|_{a=q_i^{-1}(r,t)} - e\left(\frac{n_0}{\mathcal{J}_e(a,t)}\dot{q}_e(a,t)\right)\Bigg|_{a=q_e^{-1}(r,t)} \\
n(r,t) &= \frac{m_i}{m}\left(\frac{n_0}{\mathcal{J}_i(a,t)}\right)\Bigg|_{a=q_i^{-1}(r,t)} + \frac{m_e}{m}\left(\frac{n_0}{\mathcal{J}_e(a,t)}\right)\Bigg|_{a=q_e^{-1}(r,t)} && (5.33) \\
s(x,t) &= \frac{m_i}{m}\,s_{i0}\Big|_{a=q_i^{-1}(r,t)} + \frac{m_e}{m}\,s_{e0}\Big|_{a=q_e^{-1}(r,t)} && (5.34) \\
s_e(x,t) &= s_{e0}\Big|_{a=q_e^{-1}(r,t)}\,. && (5.35)
\end{aligned}
$$

The definitions of $Q(a,t)$ and $D(a,t)$ in (5.30) are strongly indicative that their time-derivatives should be associated with V and J, respectively. However, it is important to recognize both $\dot{Q}$ and $\dot{D}$ are nonlocal objects, as they relate the velocities of electrons and ions which are located at different points in space. This implies that neither $\dot{Q}$ nor $\dot{D}$, when evaluated at the inverse maps for a, can Eulerianize to a purely local velocity or current, since, in general, $r = q_i(Q, D)$ and $r' = q_e(Q, D)$ with $r \neq r'$. In other words, $\dot{Q}$ and $\dot{D}$ are simultaneously evaluated at different trajectories. Therefore, we have two different inverse functions where the Lagrangian quantities are to be evaluated, namely, $a = q_i^{-1}(r, t)$ and $a = q_e^{-1}(r', t)$ which should be thought of as the inverse functions of $r = q_i(Q, D)$ and $r' = q_e(Q, D)$.

Thus, by taking these factors into account, we *define* our Lagrange-Euler maps, with $r = r'$ chosen to ensure locality, as follows

$$
\begin{aligned}
V(r,t) &= \frac{m_i}{m}\left(\dot{Q}(a,t) + \frac{m_e}{men_0}\dot{D}(a,t)\right)\bigg|_{a=q_i^{-1}(r,t)} \\
&+ \frac{m_e}{m}\left(\dot{Q}(a,t) - \frac{m_i}{men_0}\dot{D}(a,t)\right)\bigg|_{a=q_e^{-1}(r,t)} \\
J(r,t) &= \frac{en_0}{\mathcal{J}_i(a,t)}\left(\dot{Q}(a,t) + \frac{m_e}{men_0}\dot{D}(a,t)\right)\bigg|_{a=q_i^{-1}(r,t)} \\
&- \frac{en_0}{\mathcal{J}_e(a,t)}\left(\dot{Q}(a,t) - \frac{m_i}{men_0}\dot{D}(a,t)\right)\bigg|_{a=q_e^{-1}(r,t)} .
\end{aligned} \tag{5.36}
$$

Due to (5.29), the two Jacobian determinants are equal provided that they are evaluated at their respective inverse functions, and can be replaced by a common Jacobian determinant, $\mathcal{J}$.

The maps just defined are straight-forward to apply for mapping an Eulerian statement to a Lagrangian one, but for our purpose, we have to invert them. To keep careful track of the two inverse functions, we shall carry out an inversion of the intermediate quantities:

$$
\begin{aligned}
V(r,t) &+ \frac{m_e}{men(r,t)} J(r,t) \\
&= \left(\dot{Q}(a,t) + \frac{m_e}{men_0} \dot{D}(a,t) \right) \Bigg|_{a=q_i^{-1}(r,t)} \qquad (5.37) \\
V(r,t) &- \frac{m_i}{men(r,t)} J(r,t) \\
&= \left(\dot{Q}(a,t) - \frac{m_i}{men_0} \dot{D}(a,t) \right) \Bigg|_{a=q_e^{-1}(r,t)} , \qquad (5.38)
\end{aligned}
$$

where we have used (5.6). The inverse Lagrange-Euler maps are now given by

$$
\begin{aligned}
\dot{Q}(a,t) = {} & \frac{m_i}{m} \left(V(r,t) + \frac{m_e}{men(r,t)} J(r,t) \right) \Bigg|_{r=q_i(Q,D)} \\
& + \frac{m_e}{m} \left(V(r',t) - \frac{m_i}{men(r',t)} J(r',t) \right) \Bigg|_{r'=q_e(Q,D)} \\
\dot{D}(a,t) = {} & en_0 \left(V(r,t) + \frac{m_e}{men(r,t)} J(r,t) \right) \Bigg|_{r=q_i(Q,D)} \\
& - en_0 \left(V(r',t) - \frac{m_i}{men(r',t)} J(r',t) \right) \Bigg|_{r'=q_e(Q,D)} . \qquad (5.39)
\end{aligned}
$$

Note that the construction of the maps of Eqs. (5.36) and (5.39) could be undertaken with any invertible linear combination of the time derivatives of our Lagrangian variables. The only restriction, albeit a very stringent one, is that the final action should comply with the Eulerian Closure Principle, i.e., it should be expressible entirely in terms of the Eulerian observables. By using the above relations, it is easy to verify that such is indeed the case.

5.2.5 Lagrange-Euler maps without quasineutrality

Had we not assumed quasineutrality, we would have had to proceed in a completely different manner - (5.23) implies that the proper Lagrangian variables that would Eulerianize to the velocity and current would now be

$$\begin{aligned}
V(r,t) &= \frac{m_i\left(\frac{n_{i0}}{\mathcal{J}_i}\dot{q}_i(a,t)\right)\Big|_{a=q_i^{-1}(r,t)}}{m_i\left(\frac{n_{i0}}{\mathcal{J}_i}\right)\Big|_{a=q_i^{-1}(r,t)}+m_e\left(\frac{n_{e0}}{\mathcal{J}_e}\right)\Big|_{a=q_e^{-1}(r,t)}} \\
&+\frac{m_e\left(\frac{n_{e0}}{\mathcal{J}_e}\dot{q}_e(a,t)\right)\Big|_{a=q_e^{-1}(r,t)}}{m_i\left(\frac{n_{i0}}{\mathcal{J}_i}\right)\Big|_{a=q_i^{-1}(r,t)}+m_e\left(\frac{n_{e0}}{\mathcal{J}_e}\right)\Big|_{a=q_e^{-1}(r,t)}}\,, \\
J(x,t) &= e\left(\frac{n_{i0}}{\mathcal{J}_i}\dot{q}_i(a,t)\right)\Bigg|_{a=q_i^{-1}(x,t)} - e\left(\frac{n_{e0}}{\mathcal{J}_e}\dot{q}(a,t)\right)\Bigg|_{a=q_e^{-1}(x,t)}\,.
\end{aligned}$$

The above equations indicate that, in the absence of quasineutrality, the definitions for $\dot{Q}$, $\dot{D}$, etc. should be modified to the following:

$$\begin{aligned}
\dot{Q}(a,t) &= \frac{1}{\rho_{m0}(a)}\big(m_i\mathcal{J}_e n_{i0}(a)\dot{q}_i(a,t)+m_e\mathcal{J}_i n_{e0}(a)\dot{q}_e(a,t)\big) \\
\dot{D}(a,t) &= e\big(\mathcal{J}_e n_{i0}(a)\dot{q}_i(a,t)-\mathcal{J}_i n_{e0}(a)\dot{q}_e(a,t)\big) \\
\rho_{m0}(a) &= m_i\mathcal{J}_e n_{i0}(a)+m_e\mathcal{J}_i n_{e0}(a)
\end{aligned}$$

where $\dot{Q}/\left(\mathcal{J}_i\mathcal{J}_e\right)$ maps to $V(r,t)$ and $\dot{D}/\left(\mathcal{J}_i\mathcal{J}_e\right)$ to $J(r,t)$. In this case, however, both $\dot{Q}$ and $\dot{D}$ can only be implicitly defined, since the definitions of $\mathcal{J}_i$ and $\mathcal{J}_e$ cannot be simplified as before. This problem is absent when one is manipulating the Eulerian equations of motion. The difficulties encountered appear to suggest that a one-fluid description cannot emerge when there is *no* quasineutrality. Although this is only a hypothesis, it is borne out by the most general case derived by Lüst in [155]. The resulting equations of motion in V and J still contain terms explicitly referring to ion/electron quantities, e.g., n_i and n_e. It indicates that a complete one-fluid description of non-quasineutral extended MHD may not exist; if it does, it is likely that the corresponding action would be somewhat complicated. In constructing our action, and the Eulerian Closure Principle, we relied quite heavily on quasineutrality and it is not evident how this can be bypassed. In order to preserve the closure principle, one would need a means of distinguishing between integrations over ion and electron labels, to take into account the different factors of $\mathcal{J}_s$ floating around.

5.2.6 Derivation of the continuity and entropy equations

Before we derive the equations of motion for several different models in the next section, we derive here the continuity equation, which all of the models below have in common, and the entropy equations.

Due to the identity of the Jacobians from (5.29), the equation for n (5.33) reduces to

$$n(r,t) = \left(\frac{n_0}{\mathcal{J}_i(a,t)}\right)\Bigg|_{a=q_i^{-1}(r,t)} = \left(\frac{n_0}{\mathcal{J}_e(a,t)}\right)\Bigg|_{a=q_e^{-1}(r,t)}\,, \tag{5.40}$$

where the factors of q_s^{-1} are the inverse functions of $q_s(Q,D)$. Inverting the equation for the ions and taking the time derivative yields

$$\frac{dn}{dt}\bigg|_{x=q_i(Q,D)} = \frac{d}{dt}\frac{n_0}{\mathcal{J}_i(a,t)} = -\frac{n_0}{\mathcal{J}_i^2(a,t)}\frac{\partial \mathcal{J}_i}{\partial t}\,.$$

To Eulerianize the equation above, we use the well-known relations $d/dt = \partial/\partial t + v\cdot\nabla$ and $\partial\mathcal{J}/\partial t = \mathcal{J}\nabla\cdot v$. The key here is to use the correct Eulerian velocity, in this case the ion velocity, in terms of the one-fluid Eulerian variables V and J. The result is

$$\frac{\partial n}{\partial t} + \left(V + \frac{m_e}{men}J\right)\cdot\nabla n = -n\nabla\cdot\left(V + \frac{m_e}{men}J\right)$$

which can be further reduced to

$$\frac{\partial n}{\partial t} + \nabla \cdot (nV) + \frac{m_e}{me} \nabla \cdot J = 0\,.$$

However, we already know from (5.26) that $\nabla \cdot J = 0$. Therefore, no matter which equality we choose in (5.40), the same continuity equation will follow,

$$\frac{\partial n}{\partial t} + \nabla \cdot (nV) = 0 \tag{5.41}$$

Similarly, from (5.34), we obtain

$$\frac{\partial s}{\partial t} + V \cdot \nabla s = 0$$

and from (5.35)

$$\frac{\partial s_e}{\partial t} + \left(V - \frac{m_i}{men} J\right) \cdot \nabla s_e = 0\,,$$

which, to leading order in m_e/m_i, is

$$\frac{\partial s_e}{\partial t} + \left(V - \frac{1}{en} J\right) \cdot \nabla s_e = 0\,.$$

5.3 Derivation of reduced models

If we vary the action functional (5.32) with respect to Q and D and subsequently apply the Lagrange-Euler maps, we successfully obtain the momentum equation and generalized Ohm's law of Lüst[2] in the non-dissipative

[2]Note, there are typos in Eqs. (2.9) and (2.10) of Ref. [155] that prevent the term N_1 from vanishing when imposing quasineutrality.

limit:

$$nm\left(\frac{\partial V}{\partial t}+(V\cdot\nabla)V\right)-\nabla p+\frac{J\times B}{c}-\frac{m_i m_e}{me^2}(J\cdot\nabla)\left(\frac{J}{n}\right) \tag{5.42}$$

$$\begin{aligned} E+\frac{V\times B}{c} &= \frac{m_i m_e}{me^2 n}\left(\frac{\partial J}{\partial t}+(J\cdot\nabla)V+(\nabla\cdot V)\,J-(J\cdot\nabla)\left(\frac{J}{n}\right)\right) \\ &\quad +\frac{m_i m_e}{me^2}(V\cdot\nabla)\left(\frac{J}{n}\right)+\frac{(m_i-m_e)}{menc}(J\times B) \\ &\quad +\frac{m_i m_e}{mn^2e^2}J\,(V\cdot\nabla)\,n-\frac{m_i}{men}\nabla p_e+\frac{m_e}{men}\nabla p_i\,. \end{aligned} \tag{5.43}$$

The details behind this derivation have not been presented, since we show them for extended MHD, which entails one further ordering in (5.32); the terms that arise for both models and the steps involved are virtually identical.

5.3.1 Extended MHD

At this point we will make one more simplification - let us define the mass ratio $\mu = m_e/m_i$ and order the action functional by keeping terms up to first order in μ. The resultant model is referred to as extended MHD, and we observe that an associated action principle was studied via the Euler-Poincaré formalism in [157]. After making such a choice, we arrive at

$$\begin{aligned} q_i(Q,D) &= Q(a,t)+\frac{\mu}{en_0}D(a,t) \\ q_e(Q,D) &= Q(a,t)-\frac{1-\mu}{en_0}D(a,t) \end{aligned} \tag{5.44}$$

and the action takes on the form

$$
\begin{aligned}
S = &-\frac{1}{8\pi}\int dt \int d^3r \; |\nabla \times A(r,t)|^2 \\
&+ \int dt \int d^3r \int d^3a \; n_0 \Bigg\{ \delta(r - q_i(Q,D)) \\
&\quad \times \left[\frac{e}{c}\dot{Q}(a,t) + \frac{\mu}{cn_0}\dot{D}(a,t)\cdot A(r,t) - e\phi(r,t) \right] \Bigg\} \\
&+ \int dt \int d^3r \int d^3a \; n_0 \Bigg\{ \delta(r - q_e(Q,D)) \\
&\quad \times \left[-\frac{e}{c}\dot{Q}(a,t) + \frac{(1-\mu)}{cn_0}\dot{D}(a,t)\cdot A(r,t) + e\phi(r,t) \right] \Bigg\} \\
&+ \frac{1}{2}\int dt \int d^3a \, n_0 m_i \left((1+\mu)|\dot{Q}|^2(a,t) + \frac{\mu}{e^2 n_0^2}|\dot{D}|^2(a,t) \right) \\
&- \int dt \int d^3a \, n_0 \left(\mathfrak{U}_e\left(\frac{n_0}{\mathcal{J}_e(Q,D)}, s_{e0} \right) + \mathfrak{U}_i\left(\frac{n_0}{\mathcal{J}_i(Q,D)}, s_{i0} \right) \right). \qquad (5.45)
\end{aligned}
$$

where the internal energy per unit mass U_s has been duly replaced by $\mathfrak{U}_s$, the internal energy per particle. The pressure is obtained from the latter according to $p_s = n^2 \partial \mathfrak{U}_s / \partial n$.

Varying the action with respect to Q_k yields

$$
\begin{aligned}
0 = & -n_0 m_i(1+\mu)\ddot{Q}_k(a,t) - \partial_k p \\
& + n_0 \left[\frac{e}{c}\left(\dot{Q}_j(a,t) + \frac{\mu}{en_0}\dot{D}_j(a,t)\right)\frac{\partial A_j(r,t)}{\partial r^k} \right. \\
& \qquad \left. - e\partial_k \phi(r,t) - \frac{e}{c}\frac{d}{dt}A_k(r,t)\right]\Bigg|_{r=q_i(Q,D)} \\
& + n_0 \left[-\frac{e}{c}\left(\dot{Q}_j(a,t) - \frac{(1-\mu)}{en_0}\dot{D}_j(a,t)\right)\frac{\partial A_j(r,t)}{\partial r^k} \right. \\
& \qquad \left. + e\partial_k \phi(r,t) + \frac{e}{c}\frac{d}{dt}A_k(r,t)\right]\Bigg|_{r=q_e(Q,D)} . \qquad (5.46)
\end{aligned}
$$

The variation of the internal energy term proceeds by varying q_s through Eqs. (5.44), giving a contribution $\delta q_s = \delta Q$ (in addition to δD contributions as well) . We also make use of Eqs. (5.44) in the variation of the Jacobians $\mathcal{J}_s$. We have directly given the Eulerian result for the internal energies since the Lagrangian counterpart has two terms of the form of (5.20), and it is cumbersome to carry this through the rest of the calculation. For an alternative treatment on the Lagrangian level, we refer the reader to [57]. Consistent with Dalton's law, the total single fluid pressure is $p = p_i + p_e$ and both these pressures are present at the zeroth order of μ. It is important to recognize that the two time derivatives of A do not cancel, because they are advected by different flow velocities. In the the Lagrangian framework, it is equivalent to stating that they are evaluated at different arguments.

To find the Eulerian equations of motion, we start with (5.39), (up to first order in μ and impose locality, i.e., $r = r'$, such that $\dot{Q}$ maps to $V(r,t)$

and $\dot{D}$ to $J(r,t)$. However, the time derivatives of $\dot{Q}$ and $\dot{D}$ have to be treated with care as they each consist of two terms that are advected with different velocities. We will now provide the details of how the Eulerianization of the equations of motion is carried out.

The $\ddot{Q}$ in the first term of (5.46) can be re-written as

$$\begin{aligned}
\ddot{Q}(a,t) = {} & (1-\mu)\,\frac{d}{dt}\left(V(r,t) + \frac{\mu}{en(r,t)}J(r,t)\right)\bigg|_{r=q_i(Q,D)} \\
& + \mu\frac{d}{dt}\left(V(r,t) - \frac{(1-\mu)}{en(r,t)}J(r,t)\right)\bigg|_{r=q_e(Q,D)} \\
= {} & (1-\mu)\left(\frac{\partial V}{\partial t} + \frac{\partial q_i}{\partial t}\cdot\nabla V + \frac{\mu}{en}\frac{\partial}{\partial t}\left(\frac{J}{n}\right) + \frac{\mu}{e}\frac{\partial q_i}{\partial t}\cdot\nabla\left(\frac{J}{n}\right)\right) \\
& + \mu\left(\frac{\partial V}{\partial t} + \frac{\partial q_e}{\partial t}\cdot\nabla V - \frac{(1-\mu)}{en}\frac{\partial}{\partial t}\left(\frac{J}{n}\right) - \frac{(1-\mu)}{e}\frac{\partial q_e}{\partial t}\cdot\nabla\left(\frac{J}{n}\right)\right)\,. \qquad (5.47)
\end{aligned}$$

From Eqs. (5.44), we can compute the explicit expressions for the time derivatives of the $q_s(Q,D)$,

$$\frac{\partial q_i}{\partial t} = \dot{Q} + \frac{\mu}{en_0}\dot{D} \longrightarrow V + \frac{\mu}{en}J \qquad (5.48)$$

$$\frac{\partial q_e}{\partial t} = \dot{Q} - \frac{1-\mu}{en_0}\dot{D} \longrightarrow V - \frac{1-\mu}{en}J\,. \qquad (5.49)$$

Inserting these expression into (5.47), after some algebra, we arrive at

$$\ddot{Q}(a,t) \longrightarrow \frac{\partial V}{\partial t} + (V\cdot\nabla)V + \frac{\mu(1-\mu)}{ne^2}(J\cdot\nabla)\left(\frac{J}{n}\right)\,. \qquad (5.50)$$

Next we Eulerianize the interaction terms of (5.46) using (5.39), up to first order in μ, and (5.44). The result is

$$\frac{ne}{c}\left[\left(V_j+\frac{\mu}{en}J_j\right)\frac{\partial A_j}{\partial r^k}-c\partial_k\phi-\frac{\partial A_k}{\partial t}-\frac{\partial q_i}{\partial t}\cdot\nabla A_k\right]$$
$$+\frac{ne}{c}\left[\left(-V_j+\frac{(1-\mu)}{en}J_j\right)\frac{\partial A_j}{\partial r^k}+c\partial_k\phi+\frac{\partial A_k}{\partial t}\right.$$
$$\left.+\frac{\partial q_e}{\partial t}\cdot\nabla A_k\right], \tag{5.51}$$

which, after substitution using Eqs. (5.48) and (5.49), yields

$$\frac{1}{c}\left(J_j\frac{\partial A_j}{\partial r^k}-J_j\frac{\partial A_k}{\partial r^j}\right)=\frac{(J\times(\nabla\times A))_k}{c}\,. \tag{5.52}$$

The full Eulerian version of the equation of motion for the velocity of (5.46), also referred to as the *momentum equation* is

$$nm\left(\frac{\partial V}{\partial t}+(V\cdot\nabla)V\right) = -\nabla p+\frac{J\times B}{c}-\frac{m_e}{e^2}(J\cdot\nabla)\left(\frac{J}{n}\right)\,. \tag{5.53}$$

In Ref. [57], the last term on the RHS of (5.53) was shown to be absolutely necessary for energy conservation.

Next, varying the action with respect to D_k yields

$$0=\frac{m_i\mu}{n_0e^2}\ddot{D}_k(a,t)+\frac{(1-\mu)}{en}\partial_kp_e-\frac{\mu}{en}\partial_kp_i$$
$$+\mu\left[\left(-\frac{1}{c}\frac{d}{dt}A_k(r,t)-\partial_k\phi(r,t)+\frac{1}{c}\left(\dot{Q}_j(a,t)\right.\right.\right.$$
$$\left.\left.\left.+\frac{\mu}{en_0}\dot{D}_j(a,t)\right)\frac{\partial A_j(r,t)}{\partial r^k}\right)\right]\Bigg|_{r=q_i(Q,D)}$$
$$+(1-\mu)\left[\left(-\frac{1}{c}\frac{d}{dt}A_r(x,t)-\partial_k\phi(r,t)+\frac{1}{c}\left(\dot{Q}_j(a,t)\right.\right.\right.$$
$$\left.\left.\left.-\frac{(1-\mu)}{en_0}\dot{D}_j(a,t)\right)\frac{\partial A_j(r,t)}{\partial x^k}\right)\right]\Bigg|_{r=q_e(Q,D)}\,. \tag{5.54}$$

This time, the Jacobians of the internal energies are varied using the fact the variations contribute as follows: $\delta q_e \to -(1-\mu)\delta D/(en_0)$ and $\delta q_i \to \mu\delta D/(en_0)$, which again follow from Eqs. (5.44). It is precisely for this reason that only the electron pressure appears, to leading order, in the generalized Ohm's law for extended MHD.

Eulerianizing the $\ddot{D}$ term in (5.54) is carried out in an analogous manner to that of $\ddot{Q}$, and it yields

$$\frac{m_i\mu}{n_0 e^2}\ddot{D}(a,t) = \frac{m_i\mu}{e^2 n}\left(\frac{\partial J}{\partial t} + (J\cdot\nabla)V + (\nabla\cdot V)\,J - (J\cdot\nabla)\left(\frac{J}{n}\right)\right) + \frac{m_i\mu}{e^2}(V\cdot\nabla)\left(\frac{J}{n}\right) + \frac{m_i\mu}{e^2 n^2}J(V\cdot\nabla)n \tag{5.55}$$

where the continuity equation (5.41) was used to eliminate the time derivative of n; we also retained terms that were leading order in μ. The interaction terms in (5.54) reduce to

$$E + \frac{V\times(\nabla\times A)}{c} - \frac{(1-2\mu)}{enc}J\times(\nabla\times A)\,. \tag{5.56}$$

In Eqs. (5.55) and (5.56) we see the presence of some terms involving μ, in front of $\ddot{D}$ and $J\times B$, respectively. However, in the latter case it occurs only through the factor $(1-2\mu)$ and since our ordering is $\mu << 1$, we can drop the μ-dependence in Eq. (5.56), to lowest order. However, in Eq. (5.55), we cannot throw out all the terms that depend on μ to lowest order, as this would amount to a clear case of throwing out the baby along with the bath water, i.e. we lose a large number of crucial terms if we perform this operator. Moreover, the factor $\mu m_i/(ne^2)$ cannot be cast into a dimensionless form in a straightforward manner, and hence one cannot invoke the ordering $\mu <<$ 1 here. On the other hand, in $(1-2\mu)$, observe that the factor of '1' is already (trivially) in a dimensionless form, faciliating an easy comparison. Post-variation, the discrepancy in the order of the derived terms, i.e. the existence of these anomalous terms has also been observed in other contexts; see e.g. [158].

The Eulerian version of the equation of motion of the current (5.54) is obtained after retaining terms to lowest order in μ. It is commonly referred to as the *generalized Ohm's law*, and has the form

$$\begin{aligned} E+\frac{V\times B}{c} &= \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t}+(J\cdot\nabla)V+(\nabla\cdot V)\,J-(J\cdot\nabla)\left(\frac{J}{n}\right)\right) \\ &+\frac{(J\times B)}{enc}-\frac{\nabla p_e}{en}+\frac{m_e}{e^2}(V\cdot\nabla)\left(\frac{J}{n}\right)+\frac{m_e}{e^2n^2}J(V\cdot\nabla)n\,. \end{aligned} \tag{5.57}$$

The last two terms on the right hand side of (5.57) can be combined to yield a single term $(m_e/(e^2n))\,(V\cdot\nabla)J$. Since we also know that $\nabla\cdot J=0$, we can add a $V(\nabla\cdot J)$ term without changing the result, and combine most terms via the divergence of the tensor $VJ+JV$ to obtain the following equation:

$$\begin{aligned} E+\frac{V\times B}{c} &= \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t}+\nabla\cdot(VJ+JV)\right) \\ &-\frac{m_e}{e^2 n}(J\cdot\nabla)\left(\frac{J}{n}\right)+\frac{(J\times B)}{enc}-\frac{\nabla p_e}{en}\,. \end{aligned} \tag{5.58}$$

Equations (5.53) and (5.58) constitute the extended MHD model, which is in agreement with the results obtained in [57, 155].

5.3.2 Hall MHD

Hall MHD is a limiting case of extended MHD, and we observe that certain action principle formulations of Hall MHD already exist [159, 160]. Here, we obtain the actional functional by expanding it, and retaining only terms up to zeroth order in μ. In other words, it amounts to neglecting the electron inertia, i.e. we set ($m_e \to 0$), and we see that the action of (5.32) reduces to

$$
\begin{aligned}
S = &-\frac{1}{8\pi}\int dt \int d^3r \; |\nabla \times A(r,t)|^2 \\
&+ \int dt \int d^3a \; n_0 \left[\frac{1}{2} m|\dot{Q}|^2(a,t) - \mathfrak{U}_i\left(\frac{n_0}{\mathcal{J}_i(Q)}, s_{i0}\right) - \mathfrak{U}_e\left(\frac{n_0}{\mathcal{J}_e(Q,D)}, s_{e0}\right)\right] \\
&+ \int dt \int d^3r \int d^3a \; n_0 \left\{\delta\left(r - Q(a,t) + \frac{1}{en_0} D(a,t)\right)\right. \\
&\times \left.\left[-\frac{e}{c}\left(\dot{Q}(a,t) - \frac{1}{en_0}\dot{D}(a,t)\right)\cdot A(r,t) + e\phi(r,t)\right]\right\} \\
&+ \int dt \int d^3r \int d^3a \; n_0 \left\{\delta\left(r - Q(a,t)\right)\left[\frac{e}{c}\dot{Q}(a,t)\cdot A(r,t) - e\phi(r,t)\right]\right\}.
\end{aligned}
\tag{5.59}
$$

and Eqs. (5.44) simplify to

$$
\begin{aligned}
q_i(Q,D) &= Q(a,t) \\
q_e(Q,D) &= Q(a,t) - D(a,t)/(en_0)
\end{aligned}
\tag{5.60}
$$

Observe we have also replaced m_i by m in the kinetic energy term, which is correct to leading order in μ.

The inverse maps required for Eulerianizing the equations of motion are now given by

$$\dot{Q}(a,t) = V(r,t)\Big|_{r=q_i=Q}$$
$$\dot{D}(a,t) = en_0 V(r,t)\Big|_{r=q_i=Q} - en_0\left(V(r',t) - \frac{J(r',t)}{en(r',t)}\right)\Big|_{r'=q_e(Q,D)} .$$

Following the procedure outlined in the previous section for extended MHD, we arrive at the dynamical equations for the model commonly referred to as Hall MHD,

$$nm\left(\frac{\partial V}{\partial t} + (V\cdot\nabla)V\right) = -\nabla p + \frac{J\times B}{c} \tag{5.61}$$
$$E + \frac{V\times B}{c} = \frac{J\times B}{nec} - \frac{1}{ne}\nabla p_e\,, \tag{5.62}$$

and these constitute the usual forms of the momentum equation and Ohm's law for Hall MHD. Before moving ahead, it is worth mentioning that Hall MHD has proven to be very successful in modeling fusion and astrophysical plasmas in a wide range of contexts [161, 162, 163, 164].

5.3.3 Electron MHD

Electron MHD [59, 161, 157, 165] is another widely used model in both astrophysics and fusion. It represents another limiting case of our treatment, where we neglect the ion motion completely. Electron MHD is often used to model the motion over short time scales during which the ions are essentially assumed to be at rest. Since the ions are immobile, we require $\dot{q}_i = 0$ and $q_i = q_i(a)$. Also, we impose the condition that there be no electric field and, consequently, we neglect ϕ from the action. In this case, using the Q, D

formulation of the previous sections is redundant since there is only a single fluid. From $\dot{q}_i = 0$ we find $\dot{D} = -\left(en_0 m/m_e\right)\dot{Q}$. The same relation also holds between Q and D, up to an additive constant which represents the constant position of the ion. In addition, the Lagrange-Euler map takes on the simple form

$$v_e(r,t) = \left(1+\frac{1}{\mu}\right)\dot{Q}(a,t)\bigg|_{a=q_e^{-1}(r,t)}\,, \tag{5.63}$$

where

$$q_e(a,t) = (1+\frac{1}{\mu})Q(a,t)\,. \tag{5.64}$$

After using our ordering, the remaining terms in the action are given by

$$\begin{aligned} S = &-\frac{1}{8\pi}\int dt \int d^3r\; |\nabla\times A(x,t)|^2 \\ &+\int dt\int d^3a\; n_0\left[\frac{1}{2}m_e|\dot{q}_e|^2(a,t) - \mathfrak{U}_e\left(\frac{n_0}{\mathcal{J}_e(Q)}\right)\right] \\ &-\int dt\int d^3r\int d^3a\;\delta\left(r-q_e\right)\frac{en_0}{c}\dot{q}_e\cdot A(r,t)\,, \end{aligned} \tag{5.65}$$

which is essentially the same action as that of [157]. It is also straight-forward to express this action in terms of Q using Eqs. (5.63) and (5.64).

Upon varying the action, either in terms of q_e or Q, and Eulerianizing the resultant expressions, we arrive at the following dynamical equations:

$$\begin{aligned} m_e\left(\frac{\partial v_e}{\partial t} + v_e\cdot\nabla v_e\right) + \frac{e}{c}\frac{\partial A}{\partial t} =& \frac{e}{c}(v_e\times B) - \frac{\nabla p_e}{n} \\ \nabla\times B =& -\frac{4\pi}{c}env_e\,, \end{aligned}$$

which are the usual equations of electron MHD [165].

5.4 Noether's theorem

We shall end this Chapter by presenting a brief synopsis of the invariants of the quasineutral Lüst and the extended MHD actions, described by (5.32) and (5.45) respectively. We obtain these invariants by using Noether's theorem [49, 50].

Before proceeding further, let us recollect that the two actions can be expressed either in terms of (Q, D) or in terms of (q_i, q_e), which are related through a simple linear transformation; see for e.g. (5.30). Furthermore, both sets of variables obey the Eulerian Closure Principle. Hence, it is equivalent to work with an action expressed in terms of either set of variables. For convenience, we shall work with the latter set, as the Euler-Lagrange maps are easier to apply. Noether's theorem states that if an action is invariant under the transformations

$$q_s' = q_s + K_s\left(q_s, t\right); \quad t' = t + \tau\left(t\right), \tag{5.66}$$

i.e.,

$$S = \int_{t_1}^{t_2} dt \int d^3z\, \mathcal{L}\left(q_s, \dot{q}_s, z, t\right) = \int_{t_1'}^{t_2'} dt' \int d^3z'\, \mathcal{L}\left(q_s', \dot{q}_s', z', t'\right) ,$$

then there exist constants of motion given by

$$C = \int d^3z \left[\tau\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_s} \cdot \dot{q}_s - \mathcal{L}\right) - K_s \cdot \frac{\partial \mathcal{L}}{\partial \dot{q}_s}\right] , \tag{5.67}$$

and the variable z can denote a or x, as our actions are comprised of both Lagrangian and Eulerian components.

A Noetherian approach was also undertaken in [166] to determine invariants for reduced fluid models. The difference between the two is that we obtain invariants by investigating symmetries of the Lagrangian through suitable transformations of (Q, D), which serve as our fields; consequently, our approach is entirely Lagrangian in nature. The approach employed in [166] is complementary as it introduces variations induced by space-time translations, and investigates the ensuing symmetries; everything is undertaken on a Eulerian level.

1. Time translation

It is easy to verify that the action is invariant under time translation with

$$K_s = 0; \quad \tau = 1\,.$$

The corresponding constant of motion, the energy, is found to be

$$\mathcal{E} = \int d^3r \left[\frac{|\nabla \times A|^2}{8\pi} + \sum_s \int d^3a \left(\frac{1}{2} n_0 m_s |\dot{q}_s|^2 + n_0\, \mathfrak{U}_s \left(\frac{n_0}{\mathcal{J}_s}, s_{s0}\right)\right)\right]\,.$$

Using suitable Lagrange-Euler maps to express our answer in terms of the Eulerian variables $\{n, V, J\}$, we obtain

$$\mathcal{E} = \int d^3r \left[\frac{|B|^2}{8\pi} + n\mathfrak{U}_i + n\mathfrak{U}_e + mn\frac{|V|^2}{2} + \frac{m_e m_i}{mne^2}\frac{|J|^2}{2}\right] \tag{5.68}$$

for the quasineutral Lüst model and

$$\mathcal{E} = \int d^3r \left[\frac{|B|^2}{8\pi} + n\mathfrak{U}_i + n\mathfrak{U}_e + mn\frac{|V|^2}{2} + \frac{m_e}{ne^2}\frac{|J|^2}{2}\right] \tag{5.69}$$

for the extended MHD model. It is seen that the two energies are different since the extended MHD model includes an additional mass ratio ordering. In both models, the third term is very important, and is mostly neglected in extended MHD treatments; the reader is referred to [57] for a discussion of this issue.

2. Space translation

Space translations correspond to

$$K_s = k; \quad \tau = 0\,,$$

where k is an arbitrary constant vector. Under space translations, our actions are invariant, and the constant of motion is the momentum, which is found to be

$$\begin{aligned} P = k \cdot \int d^3a \left(n_0 m_i \dot{q}_i + n_0 m_e \dot{q}_e\right) \\ + k \cdot \int d^3r \, \frac{e}{c} A \left\{ \int n_0 \left[\delta\left(r - q_i\right) - \delta\left(r - q_e\right)\right] d^3a \right\} \,, \end{aligned}$$

Using the Lagrange-Euler maps, and the condition for quasineutrality, we find

$$P = k \cdot \int d^3r \; nmV$$

is the conserved quantity. Note that k is entirely arbitrary, and hence we see that the total momentum

$$P = \int d^3r \; \rho V \tag{5.70}$$

is conserved. This is also evident from the corresponding dynamical equation for V.

3. Rotations

The actions are also invariant under rotations which corresponds to

$$K_s = k \times q_s; \quad \tau = 0\,,$$

Following the same procedure as before, we have

$$\mathfrak{L} = k \cdot \int d^3r\; nm\, r \times V\,,$$

and since we know that k is arbitrary, we conclude that the angular momentum given by

$$\mathfrak{L} = \int d^3r\, \rho\, r \times V \tag{5.71}$$

is a constant of motion.

4. Galilean boosts

When discussing boosts, we have to consider that the action may remain invariant even when the following holds

$$\begin{aligned} S &= \int_{t_1}^{t_2} dt \int d^3z\; \mathcal{L}\left(q_s, \dot{q}_s, z, t\right) \\ &= \int_{t_1'}^{t_2'} dt' \int d^3z' \left(\mathcal{L}\left(q_s', \dot{q}_s', z', t'\right) + \partial_\mu \lambda^\mu\right)\,, \end{aligned}$$

because the second term, in the second equality, vanishes identically. In our previous investigations of the invariants of the models, the infinitesimal transformations did not involve time explicitly.

When we perform a boost on the other hand, it corresponds to

$$K_s = ut; \quad \tau = 0\,,$$

where u is an arbitrary constant velocity. For a Galilean boost in a one-fluid model, the corresponding invariant quantity is given by

$$\mathcal{B} = \int d^3a\; mn\,(q - \dot{q}t)\;,$$

and since we have two different species, the statement generalizes to

$$\mathcal{B} = \sum_s \int d^3a\; m_s n_s\,(q_s - \dot{q}_s t)\;.$$

Using the corresponding Lagrange-Euler maps, the Eulerianized expression is given by

$$\mathcal{B} = \int d^3r\; \rho\,(r - Vt)\;. \tag{5.72}$$

Chapter 6

Two-dimensional inertial magnetohydrodynamics

We shall work again with SI units for the rest of this Chapter. Our model constructed via the HAP formulation is 2D in nature with translational symmetry, i.e. z serves as an ignorable coordinate. One of the chief results presented herein is the emergence of the widely used Ottaviani-Porcelli model of reconnection [61], which is shown to be a limiting case of our model. The parent model, which we develop herein, takes the notion of flux-freezing from ideal MHD and generalizes it. [1]

6.1 Magnetic Reconnection: the need for extended MHD models

The process of reconnection is a ubiquitous one, and entails the modification of magnetic topology, i.e. we break the flux-freezing constraint of ideal MHD. By doing so, the stored magnetic energy is converted into other forms of energy, which could manifest as thermal or kinetic energies, for instance. Mag-

[1]The contents of this chapter were published in: "Inertial magnetohydrodynamics", M. Lingam, P. J. Morrison & E. Tassi, 2015, *Phys. Lett. A*, **379**, 570-576 [167]. The initial premise and results were set up by P. J. Morrison, while the subsequent calculations and results were jointly derived by this author (M. Lingam) and E. Tassi.

netic reconnection has been proposed as a solution for an enormously diverse range of issues, many of which have astrophysical connotations. The reader is referred to several excellent reviews on the subject [168, 169, 170, 171] for further information.

Early treatments of the subject relied on (ideal) MHD with resistivity added to the mix. However, it was shown that the reconnection rates thus obtained were much smaller than those observed/predicted in high-temperature plasma environments. As a result, extended MHD models began to play an increasingly important role as including effects such as the Hall term gave rise to models that were consistent with predictions. For an overview of Hall MHD reconnection, we refer the reader to [172, 173]; see also the works of [174, 175, 176, 177, 178, 179, 180].

However, Hall MHD still suffers from a crucial limitation, as it fails to take electron inertia into account. It was shown in several crucial works, such as [181, 182, 183, 184, 185], that electron inertia played in a key role in regulating the reconnection rates. Hence, one must take into account electron inertia effects when constructing a viable model of reconnection. At the same time, it is advantageous if our model is Hamiltonian in nature, as it avoids the possibility of producing spurious dissipation and/or instabilities; such effects can lead to potentially misleading results as observed in [57, 58].

Thus, we see that it is advantangeous to look for Hamiltonian models of reconnection. One of the first, and most influential, models in this area was developed by Ottaviani & Porcelli [61]. Hamiltonian models of reconnection include were also studied in [88, 186, 187, 188, 189, 190, 191, 192, 193, 194, 195]. Our investigations fall under this category, and we shall show that our model gives rise to a 6-field model that can be used to model fast reconnection.

6.2 The Inertial MHD action

In this section, we shall present a new dynamical variable, one that determines a frozen flux for our model. An action principle in terms of this new variable is developed, and the equations of motion are obtained and analysed.

6.2.1 The inertial magnetic field: a new dynamical variable

In Section 2.2.2, we discussed the implications of magnetic flux freezing in ideal MHD, which leads to the flux serving as a 2-form. Extended MHD lacks this feature, which implies that the magnetic flux can no longer be interpreted as a 2-form. From a purely geometric point-of-view, it would be logical to look for a new dynamical variable, but *not* $B \cdot dS$, which could play a similar role. We shall drop the dS henceforth, and refer to B as a 2-form, although this statement is not (strictly) true.

Hence, we introduce the variable B_e and its corresponding attribute B_{e0}. The relation between the two is akin to that obeyed by the magnetic field in ideal MHD, viz. $\mathcal{J} B_e^i = q^i_{\ ,j}\, B_{e0}^j$. Since we claim that our new theory is still a magnetofluid model, it is necessary for B_e to be a function of the MHD variables v, B, n and s. We shall work with the choice

$$B_e = B + \frac{m_e}{e^2}\nabla\times\left(\frac{J}{n}\right) = B + \frac{m_e}{\mu_0 e^2}\nabla\times\left(\frac{(\nabla\times B)}{n}\right). \tag{6.1}$$

In other words, this is also equivalent to stating that we replaced the vector potential A by A_e, the latter of which is given by

$$A_e = A + \frac{m_e}{e^2}\left(\frac{J}{n}\right) = A + \frac{m_e}{\mu_0 e^2}\left(\frac{\nabla\times B}{n}\right). \tag{6.2}$$

Although the expressions (6.1) and (6.2) may appear *ad hoc*, there are several good reasons that justify the choices of these expressions. The first stems from the inclusion of electron inertia, which is exemplified by the presence of an additional factor involving m_e and it also satisfies the consistency requirement, i.e. in the limit $m_e/m_i \to 0$, we recover the usual magnetic field and vector potential. A crucial reason arises from the fact that B_e serves as a natural dynamical variable in extended MHD theories; for instance, if one takes the curl of equation (20) in [57] and uses Faraday's law, we recover a dynamical equation for B_e. It is possible to carry out a similar procedure for the extended MHD models presented in [60, 196] and arrive at the same conclusion.

Lastly, the statement of flux freezing in ideal MHD is equivalent to stating that $\oint A\cdot dl$ serves an invariant, which is now altered in our model. To understand the alteration, consider the canonical momentum for the electrons, which is proportional to $A - (m_e v_e/e)$. Assuming $v_e \gg v_i$ permits the approximation $J \approx -env_e$ which is identical to the ordering used in electron MHD [165]. As a consequences, we see that the canonical momentum is (approximately) equal to A_e. If we let $m_e/m_i \to 0$, the canonical momentum reduces to A. This can be understood in a different manner as well. Note that this is a 2D theory, with z serving as the ignorable coordinate, which implies that the corresponding canonical momentum in the z-direction is conserved. This yields $\oint (A_e)_z \, dz$, which is akin to the condition $\oint A \cdot dl$ is conserved in ideal MHD. Later, we shall show that even better reasons can be advanced, albeit *a posteriori*, that further justify the choice of B_e.

Before we proceed to the next section, we introduce the nomenclature 'inertial magnetic field' to refer to B_e. The choice is natural since B_e plays the role of a magnetic field, whilst also incorporating the effects of electron inertia. Hence, we refer to this theory as inertial MHD (IMHD).

6.2.2 The IMHD action

We introduce the action for IMHD below, and then comment on its significance and interpretation. Our variables are chosen to be the density (3-form) ρ, the inertial magnetic field (2-form) B_e, the entropy (0-form) s and the velocity v. Observe that, as per the Lagrange-Euler maps, we are endowed with two components of the velocity v_x and v_y.

We present the Eulerianized version of the action; the Lagrangian version is obtained by noting that it must satisfy the closure principle and give rise to the following action:

$$S = \int\int \left[\frac{\rho v^2}{2} - \rho U\left(\rho, s\right) - \frac{B_e \cdot B}{2\mu_0}\right] d^2r \, dt. \tag{6.3}$$

The first term in (6.3) is the kinetic energy, which was already shown to obey the ECP in Chapter 2. The second term in (6.3) is the internal energy density, and is the product of density and the specific internal energy (per unit mass). This term generates the temperature and the pressure, given by $\partial U/\partial s$ and $\rho^2 \partial U/\partial \rho$ respectively. The third term in the above expression is the unusual part, as it deviates from the expression for the magnetic energy density of ideal MHD. In the limit where $m_e/m_i \to 0$ we have noted that $B_e \to B$, which in turn reduces the last term of (6.3) to the conventional magnetic energy density.

Although (6.3) is expressed in terms of the Eulerian variables, the ECP and the Euler-Lagrange maps, discussed in Chapter 2, allow us to express (6.3) purely in terms of the Lagrangian coordinate q and the attributes. In order to do so, we express the magnetic field B in terms of the inertial magnetic field B_e as follows

$$B\left(r,t\right) = \int K\left(r,t\,|\,r',t'\right) B_e\left(r',t\right)\, d^2r'\, dt', \tag{6.4}$$

where K is a complicated kernel. Using the kernel is quite complex, but we note that the self-adjoint property is preserved. Alternatively, one can use the Euler-Poincaré approach, a discussion of which can be found in [38, 71, 137]; see also Chapter 4 for additional details.

Before proceeding to the next section, a couple of remarks regarding (6.3) are in order. Firstly, the only term involving $\dot{q}$ is the kinetic energy term. Hence, one can perform a Legendre transformation, and recover the Hamiltonian (in Lagrangian variables). Upon Eulerianizing the Hamiltonian, we arrive at

$$H = \int \left[\frac{\rho v^2}{2} + \rho U(\rho, s) + \frac{B_e \cdot B}{2\mu_0} \right] d^2 r. \tag{6.5}$$

We can use the definition of B_e, given in (6.1), and simplify the above expression. The result is

$$H = \int \left[\frac{\rho v^2}{2} + \rho U(\rho, s) + \frac{B^2}{2\mu_0} + \frac{m_e}{ne^2} \frac{J^2}{2} \right] d^2 r. \tag{6.6}$$

The above expression is identical to equation (23) of [57]. We also note that the same expression was derived in Section 5.4. Furthermore, we see that (6.6) is identical to the MHD Hamiltonian, except for the last term. As a consistency check, we verify that the last term does vanish in the limit $m_e/m_i \to 0$. These facts lend further credence to our choice of B_e and the action (6.3).

6.2.3 The IMHD equations of motion

The Lagrange-Euler maps outlined in Section 2.2.2 permit us to obtain the corresponding dynamical equations for the observables by applying $\partial/\partial t$ on both sides of the map. We obtain

$$\frac{\partial s}{\partial t} + v \cdot \nabla s = 0, \tag{6.7}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \tag{6.8}$$

$$\frac{\partial B_e}{\partial t} + B_e (\nabla \cdot v) - (B_e \cdot \nabla) v + (v \cdot \nabla) B_e = 0. \tag{6.9}$$

The equations (6.7), (6.8) and (6.9) correspond to the Lie-dragging of zero, three and two forms respectively; see Section 2.2.2 for more details regarding the geometric interpretation of these dynamical equations. From the definition of (6.1), we see that $\nabla \cdot B_e = 0$, and this implies that one can rewrite (6.9) as follows

$$\begin{aligned} \frac{\partial B_e}{\partial t} &= \nabla \times (v \times B_e) \\ &= \nabla \times (v \times B) + \frac{m_e}{e^2} \nabla \times \left[v \times \left(\nabla \times \left(\frac{\nabla \times B}{n} \right) \right) \right]. \end{aligned} \tag{6.10}$$

The equation of motion is obtained by extremizing the action in Lagrangian variables, or by extremizing the Eulerian action via the Euler-Poincaré approach [137]. It is found to be

$$\rho \left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = -\nabla p + J \times B - \frac{m_e}{e^2} (J \cdot \nabla) \left(\frac{J}{n} \right). \tag{6.11}$$

Equations (6.10) and (6.11) constitute the heart of our model of inertial MHD. Let us first consider the latter expression. We see that it is nearly identical to the usual ideal MHD momentum equation, except for the presence of the last term, which can be neglected in the limit $m_e/m_i \to 0$. However, this term represents more than a correction - in the extended MHD models, which also possess electron inertia, this term is absolutely crucial for energy conservation, as pointed out in [57]. Secondly, we note that our equation of motion is exactly identical to equations (2) and (19) of [57], thereby lending further credence to our choice of the inertial magnetic field and action.

We turn our attention to (6.10), which represents the extended Ohm's law. It is instructive to compare this against the inertial Ohm's law of [57], represented by their equation (20). We find that our expression is exactly identical to equation (20) of [57], when the 2D limit of their model is considered and $B_z \to$ const (constant guide field) is assumed. Under these assumptions, the two results are exactly identical, irrespective of whether the fluid is compressible or incompressible. A few comments on the 3D generalization of this model are presented in Section 6.3.3.

To summarize thus far, we find that the momentum equations of our model and that of [57] are identical. The generalized Ohm's laws are also in perfect agreement with one another in the 2D, constant guide field limit. In addition, we have shown that both our model as well as [57] yields the same (conserved) energies and momenta. Collectively, it is self-evident that these represent ample grounds for justifying the form of the inertial magnetic field B_e and the IMHD action.

6.3 The Hamiltonian formulation of inertial MHD

In this section, we describe the methodology employed in recovering the (Eulerian) noncanonical Hamiltonian picture from the (Lagrangian) canonical action. After obtaining the bracket–Hamiltonian pair, we comment on potential extensions of this framework.

6.3.1 Derivation of the inertial MHD bracket

Our first step entails the determination of the Hamiltonian, which is done via a Legendre transformation and Eulerianizing the resultant expression. The exercise was already performed in Section 6.2.2, and the Hamiltonian is given by (6.5). An alternative route is to invoke Noether's theorem, which also leads to the same result.

Next, we need to obtain the noncanonical bracket. A detailed description of this procedure can be found in [64, 65] and in Chapter 2. Before proceeding on to the derivation, we reformulate our observables in the following manner. We replace the velocity v by the momentum M^c, and the entropy s by the entropy density $\sigma = \rho s$. The new set of observables result in a simpler and compact noncanonical Poisson bracket, which is of the Lie-Poisson form. Let us recall from Section 2.2.2 that the Lagrange-Euler maps can be represented in an integral form. We present them below

$$\rho = \int d^2a\, \delta\left(r - q(a,t)\right) \rho_0(a), \tag{6.12}$$

$$\sigma = \int d^2a\, \delta\left(r - q(a,t)\right) \sigma_0(a), \tag{6.13}$$

$$B_e^j = \int d^2a\, \delta\left(r - q(a,t)\right) q^j_{,k} B^k_{e0}(a), \tag{6.14}$$

$$M^c = \int d^2a\, \delta\left(r - q(a,t)\right) \Pi(a,t). \tag{6.15}$$

The last expression is also equivalent to $M^c = \rho v$, which can be found by computing Π from the Lagrangian, and then obtaining the Eulerian equivalent. We drop the subscript c henceforth, since the canonical momentum M^c is the same as the kinetic momentum $M = \rho v$. Although much of our analysis mirrors the derivation outlined in Section 2.5, we shall summarize the salient details to keep the discussion self-contained.

Next, we note that a given functional can be expressed either in terms of the canonical momenta and coordinates, Π and q, or in terms of the observables. Hence, we can denote the former by $\bar{F}$ and the latter by F, and note that $\bar{F} \equiv F$. As a result, we find that

$$\begin{aligned}&\int d^2a\, \frac{\delta \bar{F}}{\delta \Pi} \cdot \delta\Pi + \frac{\delta \bar{F}}{\delta q} \cdot \delta q \\ &= \int d^2r\, \frac{\delta F}{\delta M} \cdot \delta M + \frac{\delta F}{\delta B} \cdot \delta B + \frac{\delta F}{\delta \rho}\delta\rho + \frac{\delta F}{\delta \sigma}\delta\sigma.\end{aligned} \tag{6.16}$$

From (6.12), we can take the variation on the LHS and RHS, thereby obtaining

$$\delta\rho = -\int d^2a\, \rho_0 \nabla\delta\left(r - q(a,t)\right) \cdot \delta q. \tag{6.17}$$

A similar procedure can also be undertaken for (6.13), (6.14) and (6.15) as well. We substitute (6.17) into the second line of (6.16) and carry out an integration by parts, and a subsequent interchange of the order of integration. This process is repeated for the rest of the variables. By doing so, we can determine the functional derivatives $\delta\bar{F}/\delta q$ and $\delta\bar{F}/\delta\Pi$ in terms of the functional derivatives of the observables. Next, we note that the canonical bracket is given by

$$\{\bar{F}, \bar{G}\} = \int d^2a\, \left(\frac{\delta \bar{F}}{\delta q} \cdot \frac{\delta \bar{G}}{\delta \Pi} - \frac{\delta \bar{G}}{\delta q} \cdot \frac{\delta \bar{F}}{\delta \Pi}\right). \tag{6.18}$$

We can now substitute the expressions for $\delta\bar{F}/\delta q$ and $\delta\bar{F}/\delta\Pi$, obtained as per the procedure outlined above, into (6.18) and derive the noncanonical bracket.

It is found to be

$$\begin{aligned}\{F,G\} = -\int d^2r \bigg[& M_i \left(\frac{\delta F}{\delta M_j} \partial_j \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \partial_j \frac{\delta F}{\delta M_i} \right) \\ & + \rho \left(\frac{\delta F}{\delta M_j} \partial_j \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M_j} \partial_j \frac{\delta F}{\delta \rho} \right) \\ & + \sigma \left(\frac{\delta F}{\delta M_j} \partial_j \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M_j} \partial_j \frac{\delta F}{\delta \sigma} \right) \\ & + B_e^i \left(\frac{\delta F}{\delta M_j} \partial_j \frac{\delta G}{\delta B_e^i} - \frac{\delta G}{\delta M_j} \partial_j \frac{\delta F}{\delta B_e^i} \right) \\ & + B_e^i \left(\frac{\delta G}{\delta B_e^j} \partial_i \frac{\delta F}{\delta M_j} - \frac{\delta F}{\delta B_e^j} \partial_i \frac{\delta G}{\delta M_j} \right) \bigg]. \end{aligned} \tag{6.19}$$

The inertial MHD bracket, derived above, possesses a couple of remarkable features. Firstly, we note that the bracket is precisely identical to the ideal MHD bracket of [28], if we replace B_e in (6.19) with B everywhere. Secondly, it is interesting to note that one can replace M by M^c in the above expression. By doing so, we can obtain an expression for the gyroviscous inertial MHD bracket, yielding results identical to those of [64, 65].

We must reiterate the importance of the bracket, because it further highlights the merits of B_e as a dynamical variable. Our simple postulate in Section 6.2.1, that B_e behaves as a two form, ensures that inertial MHD and ideal MHD are identical to each other under the exchange $B_e \leftrightarrow B$. Not only does B_e yield equations of motion that are highly similar to those of extended MHD, but it also maintains a close connection with ideal MHD via its notion of flux freezing. We have established that the inertial and ideal MHD brackets are near-identical to each other, which ensures that an independent analysis of the former is not necessary; instead, one can simply migrate the results pertaining to the Casimirs, equilibria and stability of ideal MHD models, by replacing B by B_e in the suitable places. In particular, we note that the Casimir

$$\mathcal{C}_1 = \int d^3r\, \rho f(s), \tag{6.20}$$

still remains an invariant in inertial MHD. On the other hand, the counterpart of the magnetic helicity of ideal MHD is

$$\begin{aligned} \mathcal{C}_2 &= \int d^3r\, A_e \cdot B_e \qquad (6.21) \\ &= \int d^3r \left[A \cdot (\nabla \times A) + \frac{2m_e}{\mu_0 n e^2} B \cdot (\nabla \times B) + \frac{m_e^2}{e^4} \left(\frac{J}{n}\right) \cdot \left(\nabla \times \left(\frac{J}{n}\right)\right) \right], \end{aligned}$$

and it is seen that each of the three terms is of the form $W \cdot (\nabla \times W)$. Notice that the second and third terms in the second line vanish when $m_e/m_i \to 0$, thereby reducing to the ideal MHD magnetic helicity. The cross helicity of ideal MHD morphs into

$$\begin{aligned} \mathcal{C}_3 &= \int d^3r\, v \cdot B_e = \int d^3r\, v \cdot (\nabla \times A_e) \qquad (6.22) \\ &= \int d^3r \left[v \cdot B + \frac{m_e}{e^2} v \cdot \left(\nabla \times \left(\frac{J}{n}\right)\right) \right], \end{aligned}$$

and we see that it reduces to the ideal MHD cross helicity if we assume $m_e/m_i \to 0$. It is easily seen that the ideal and inertial MHD cross helicities are both expressible as $v \cdot (\nabla \times W)$.

6.3.2 The six-field model and its subcases

Although our model is 2D in nature, we have not fully exploited its nature - the choice was deliberate since the bracket and the equations of motion could be expressed in a relatively compact form. However, it comes at the cost of obtaining a narrower class of Casimirs, and an inability to clearly demarcate the behaviour of the different fields. We shall now exploit this 2D symmetry.

First, let us consider B_e, defined via (6.1). We see that it is divergence free. As z serves as our ignorable coordinate, we can immediately express it as

$$B_e = B_{ez}\hat{z} + \nabla\psi_e \times \hat{z}. \tag{6.23}$$

Next, we consider the momentum, recognizing that it involves two components. Hence, the most general possible representation is

$$M = \nabla\Gamma + \nabla\varphi \times \hat{z}. \tag{6.24}$$

We note that a similar analysis, albeit in terms of B instead of B_e, was carried out in [70, 86]. The advantage of inertial MHD is that the bracket is identical in structure to that of ideal MHD under the interchange $B_e \leftrightarrow B$. Upon substituting (6.23) and (6.24) into (6.19) and using the functional derivative chain rule, we obtain a bracket identical to that of equation (98) in [70], except for two differences. The bracket obtained involves an integration over d^2r, as opposed to d^3r in [70] since our model is 2D in nature. Secondly, we must impose $M_z = 0$ in equation (98) in [70] as our model lacks the z-component of the velocity.

In summary, we have a reduced model with scalar fields, and our observables given by $(\Gamma, \varphi, B_{ez}, \psi_e, \rho, \sigma)$. Although we have sacrificed the compactness, this comes at the advantage of writing out model solely in terms of scalar fields. We can whittle the model down to a 5-field model by assuming it to be isentropic, a common enough assumption, which eliminates σ. If we assume incompressibility, we eliminate ρ and Γ - the second relation follows from the condition $\nabla \cdot v = 0$. Lastly, we can eliminate the guide field B_{ez} by making it constant, and our resultant model now involves just two fields, i.e. φ and ψ_e. We introduce the notation $\omega = \Delta\varphi$, implying that the two functional derivatives are related via $\Delta F_\omega = -F_\varphi$, and the final bracket is given by

$$\{F, G\} = -\int d^2r \left[\omega[F_\omega, G_\omega] + \psi_e\Big([F_\omega, G_{\psi_e}] - [G_\omega, F_{\psi_e}]\Big)\right], \tag{6.25}$$

and the corresponding Hamiltonian takes on the form

$$H = \int d^2r \frac{1}{2}\left[\frac{d_e^2(\nabla^2\psi)^2}{\mu_0} + \frac{|\nabla\psi|^2}{\mu_0} + \frac{|\nabla\varphi|^2}{\rho}\right], \tag{6.26}$$

where $B = \nabla\psi \times \hat{z}$, and the relation between ψ_e and ψ is determined by using (6.1). We note that d_e represents the electron skin depth. The bracket and Hamiltonian, given by (6.25) and (6.26) are of high importance, as they give rise to the well known Ottaviani-Porcelli model [61], used in modelling collisionless magnetic reconnection. Thus, we see that a (highly) simplified limit of 2D inertial MHD gives rise to the Ottaviani-Porcelli model, and may point towards obtaining more complex models such as [197, 198].

6.3.3 Extensions of the inertial MHD bracket

In the preceding subsection, we have obtained the inertial MHD noncanonical bracket, with the corresponding expression given by (6.19). A crucial feature of inertial MHD was also identified, namely, the close affinity with the ideal MHD bracket, as one can be transformed into the other via $B_e \leftrightarrow B$.

The analogy between B and B_e also makes it possible to import the results of 2D gyroviscous MHD, and recast them in an inertial MHD framework. As noted in the previous subsection, the noncanonical brackets derived in [65] and Section 3 can be adapted for such a purpose. They are easily distinguishable from the non-gyroviscous brackets owing to the presence of the canonical momentum M^c in place of the kinetic momentum M. If the same methodology is employed herein, we can obtain a model for 2D gyroviscous inertial MHD [133, 187, 188]. It must be cautioned, however, that these methods are only applicable to the inertial and ideal brackets, as they are fully equivalent under $B_e \leftrightarrow B$. Modifying the Hamiltonians is a trickier task, as it requires us to explicitly use the relation (6.1) in determining the gyromap.

As we have stated thus far, our model of inertial MHD possesses an ignorable coordinate, thereby rendering it 2D. A natural generalization of the procedure is to undertake the same work in a 3D framework. Our central results thus far were the equation of motion (6.11) and the Ohm's law (6.10). We find that the former is unmodified, and is identical to that of [57]. However, in the 3D limit, we find that the Ohm's law of [57] and our model are *not* in agreement, although most of the terms are identical to one another. In fact, we find that the generalized Ohm's law of [57] reduces to our Ohm's law (6.10), when the flow is irrotational, or if the condition $J \parallel \omega$ is met.

Chapter 7

Connections between Hamiltonian extended MHD models

The work undertaken in Chapter 5 strongly indicates that most variants of extended MHD could be extracted via a common action principle. In [199], it was shown that these models possessed a unifying noncanonical Hamiltonian structure. In this Chapter, we explore the common properties of several extended MHD models in greater detail, and speculate on their connections. We also present a detailed proof of the Jacobi identity for the noncanonical bracket of Hall MHD. [1]

7.1 On the similarities and equivalences between extended MHD models

In this section, we analyse Hall MHD and demonstrate its equivalence with inertial MHD. We exploit this equivalence to determine the helicities, which are Casimir invariants, of these models in a straightforward manner.

[1]The results in this chapter are being prepared for a publication. The majority of the work was undertaken by this author (M. Lingam) under the supervision and guidance of P. J. Morrison. The proof of the Jacobi identity in Section 7.3 was jointly proven by G. Miloshevich and this author (M. Lingam).

7.1.1 Hall MHD: an analysis

Hall MHD represents one of the most widely used extended MHD models, and also ranks amongst the simplest. In Hall MHD, it is assumed that the two species drift with different velocities (as opposed to ideal MHD), but it is assumed that the electrons are inertialess (akin to ideal MHD). We commence our analysis with the Hall MHD bracket of [160, 199], expressed as

$$\begin{aligned}\{F,G\}^{HMHD} &= -\int_D d^3r \Bigg\{ [F_\rho \nabla\cdot G_V + F_V\cdot\nabla G_\rho] - \left[\frac{(\nabla\times V)}{\rho}\cdot(F_V\times G_V)\right] \\ &\quad - \left[\frac{B}{\rho}\cdot(F_V\times(\nabla\times G_B) - G_V\times(\nabla\times F_B))\right] \\ &\quad + d_i\left[\frac{B}{\rho}\cdot((\nabla\times F_B)\times(\nabla\times G_B))\right]\Bigg\}, \end{aligned} \tag{7.1}$$

where $d_i = c/(\omega_{pi}L)$ is the normalized ion skin depth and the likes of F_ρ, F_V, etc. represent the functional derivatives with respect to the corresponding variables. We can re-express (7.1) as

$$\{F,G\}^{HMHD} = \{F,G\}^{MHD} + \{F,G\}^{Hall}, \tag{7.2}$$

where $\{F,G\}^{MHD}$ is the ideal MHD bracket, first obtained in [28] and $\{F,G\}^{Hall}$ is the term in (7.1) that involves the ion skin depth d_i. As a consequence, we conclude that *any* Casimir of ideal MHD that is independent of B will automatically serve as a Casimir of Hall MHD. Next, observe that

$$\mathcal{C}_1 = \int_D d^3r\, A\cdot B, \tag{7.3}$$

is a Casimir of ideal MHD. Furthermore, it also satisfies $\{F,\mathcal{C}_1\}^{Hall} = 0$ as well. Together, they ensure that (7.3) is a Casimir of Hall MHD. Next, let us suppose that we introduce a new variable

$$\mathcal{B}_i = B + d_i \nabla \times V, \tag{7.4}$$

and re-express the bracket in terms of the new set of observables. We find that

$$\{F,G\}^{HMHD} \equiv \{F,G\}^{HMHD}[\mathcal{B}_i] = \{F,G\}^{MHD}[\mathcal{B}_i] - \{F,G\}^{Hall}[\mathcal{B}_i], \tag{7.5}$$

and the notation '$\mathcal{B}_i$' indicates that the respective components of (7.5) are the same as (7.2) except that B is replaced by $\mathcal{B}_i$. Thus, by following the same line of reasoning, we conclude that

$$\mathcal{C}_2 = \int_D d^3r\, \mathcal{A}_i \cdot \mathcal{B}_i = (A + d_i V) \cdot (B + d_i \nabla \times V), \tag{7.6}$$

is a Casimir of ideal MHD, with $B \to \mathcal{B}_i$ and it also satisfies $\{F, \mathcal{C}_1\}^{Hall}[\mathcal{B}_i] = 0$. Hence, we conclude that $\mathcal{C}_2$ is also a Casimir of Hall MHD.

The transformation $B \to \mathcal{B}_i$ exhibits two very special properties:

- We see that it preserves the form of the Hall MHD bracket, i.e. it is evident that (7.2) and (7.5) are identical to one another upon carrying out this transformation, apart from the change in sign. The latter can be absorbed simply via $d_i \to -d_i$ as well.

- It allows us to quickly determine the second Casimir of Hall MHD, without going through the conventional procedure of solving a set of constraint equations. In fact, we see that (7.3) and (7.6) possess the same form.

Thus, it is evident that such transformations play a crucial role, both in exposing the symmetries of the system and in determining the Casimirs. In Section 7.2, we shall explore this issue in greater detail.

7.1.2 Hall MHD and inertial MHD

Both ideal MHD and Hall MHD assume that the electrons are inertialess, i.e. this is undertaken by taking the limit $m_e/m_i \rightarrow 0$ everywhere. However, there are several regimes where electron inertia effects may be of considerable importance, such as reconnection [170, 171]. To address this issue, a new variant of MHD, dubbed inertial MHD, was studied in [57] and the Hamiltonian and Action Principle (HAP) formulation of two-dimensional inertial MHD was presented in [167].

We shall now turn our attention to inertial MHD, whose noncanonical bracket is given by

$$\{F,G\}^{IMHD} = \{F,G\}^{MHD}[B^\star] + d_e^2 \int_D d^3r \left[\frac{\nabla \times V}{\rho} \cdot \left((\nabla \times F_{B^\star}) \times (\nabla \times G_{B^\star})\right)\right], \tag{7.7}$$

and the bracket $\{F,G\}^{MHD}[B^\star]$ constitutes the ideal MHD bracket with $B \rightarrow B^\star$. The variable $B^\star$ is the 'inertial' magnetic field, and was first introduced in [167]. It is given by

$$B^\star = B + d_e^2 \nabla \times \left(\frac{\nabla \times B}{\rho}\right), \tag{7.8}$$

where $d_e = c/(\omega_{pe} L)$ represents the normalized electron skin depth. We shall now apply the transformation

$$\mathcal{B}_e = B^\star - d_e \nabla \times V, \tag{7.9}$$

and re-express our bracket in terms of the new set of observables. Upon doing so, we find that

$$\begin{aligned} \{F,G\}^{IMHD} &= \{F,G\}^{MHD}[\mathcal{B}_e] \\ &\quad -2d_e \int_D d^3r \left[\frac{\mathcal{B}_e}{\rho} \cdot \left((\nabla \times F_{\mathcal{B}_e}) \times (\nabla \times G_{\mathcal{B}_e})\right)\right]. \end{aligned} \tag{7.10}$$

The second term in the above expression can be compared against the last term in (7.1) - we see that the two are identical under $d_i \rightarrow 2d_e$ and $B \rightarrow \mathcal{B}_e$. Thus, we arrive at one of our central results:

$$\{F,G\}^{IMHD} \equiv \{F,G\}^{HMHD}[2d_e;\mathcal{B}_e]. \tag{7.11}$$

In other words, the inertial MHD bracket is equivalent to the Hall MHD bracket when the transformations $d_i \rightarrow 2d_e$ and $B \rightarrow \mathcal{B}_e$ are applied to the latter. As a result, we are led to a series of remarkable conclusions:

- As the inertial and Hall MHD brackets are identical under a change of variables (and constants), proving the Jacobi identity for one of them constitutes an automatic proof of the other.

- We can obtain the Casimirs of inertial MHD since the equivalent Casimirs were determined for Hall MHD. In particular, two helicities emerge:

$$\mathcal{C}_I = \int_D d^3r \, (A^\star - d_e V) \cdot (B^\star - d_e \nabla \times V), \tag{7.12}$$

$$\mathcal{C}_{II} = \int_D d^3r\ (A^\star + d_e V) \cdot (B^\star + d_e \nabla \times V)\,, \tag{7.13}$$

where $B^\star = \nabla \times A^\star$, and the LHS of this equation is determined via (7.8).

- By taking the difference of (7.13) and (7.12), we obtain a Casimir:

$$\mathcal{C}_{III} = \int_D d^3r\, V \cdot B^\star, \tag{7.14}$$

  which is identical to the cross-helicity invariant of ideal MHD, after performing the transformation $B \to B^\star$. The existence of this invariant has also been documented in [167].

We observe that (7.13) and (7.14) were obtained as the Casimirs for inertial MHD in [199], but the authors do not seem to have realized that inertial MHD has not one, but *two* Casimirs (helicities) of the form $\int_D d^3r\, P \cdot (\nabla \times P)$, as seen from (7.12) and (7.13). As a result, this allow us to emphasize a rather unique feature of inertial MHD:

- One can interpret inertial MHD as consisting of two helicities akin to the magnetic (or fluid) helicity, cementing its similarity to Hall MHD and the 2-fluid models [200].

- Alternatively, we can view inertial MHD as being endowed with one Casimir resembling the magnetic helicity and the other akin to the cross helicity. Such a feature renders it analogous to ideal MHD, which possesses similar features [28].

To summarize thus far, we have shown an unusual correspondence between Hall MHD (inertialess, finite Hall drift) and inertial MHD (finite electron inertia, no Hall drift) by showing that the two brackets are equivalent under a suitable set of transformations. We shall explore their origin in more depth in Section 7.2.

7.1.3 Comments on extended MHD

Hitherto, we have discussed models that incorporate the Hall drift and those that possess a finite electron inertia. Extended MHD clubs these effects together, giving rise to a more complete model. The noncanonical bracket for this model is

$$\{F, G\}^{ExMHD} = \{F, G\}^{IMHD} + \{F, G\}^{Hall}\left[B^{\star}\right], \tag{7.15}$$

and the second term on the RHS denotes the Hall term with $B \to B^{\star}$, and the latter is defined in (7.8).

It is evident that a clear pattern begins to emerge:

1. The Jacobi identity for the Hall bracket can be proven in a simple manner as it represents the sum of two components, one of which already satisfies the Jacobi identity (the ideal MHD component). The details are provided in Appendix 7.3.

2. The Jacobi identity for inertial MHD automatically follows as per the discussion in Section 7.1.2.

3. It is easy to see from (7.15) that the extended MHD bracket will then be composed of a component (inertial MHD) that already satisfies the Jacobi identity, apart from a second component that represents the Hall contribution. As a result, the calculation mirrors the proof of the Jacobi identity for Hall MHD, and the similarities are manifest upon inspecting (7.2) and (7.15).

As per the reasoning outlined above, we shall not delve too deeply into extended MHD, as it clearly shares close associations with the rest of the extended MHD models - Hall MHD and inertial MHD, which have been explored in detail in the previous sections.

Since we have argued that each of the extended MHD models shares a degree of commonality, it also follows that extended MHD must possess *two* helicities akin to the magnetic helicity (in form), and that they should involve the variables $B^\star$ and V. Thus, we postulate Casimirs of the form

$$\mathcal{C}_{ExMHD} = \int_D d^3r\ (V + \lambda A^\star) \cdot (\nabla \times V + \lambda B^\star)\,, \tag{7.16}$$

and solve for λ. A quadratic equation for λ emerges, given by

$$d_e^2 \lambda^2 + d_i \lambda - 1 = 0, \tag{7.17}$$

whose solutions are

$$\lambda = \frac{-d_i \pm \sqrt{d_i^2 + 4d_e^2}}{2d_e^2}, \tag{7.18}$$

and one of these solutions was obtained in [199]. However, it is important to recognize that there exist *two* helicities akin to the fluid (or magnetic) helicity, since this is a feature that the extended MHD models inherit from the parent two-fluid model.

In fact, we can recover these two helicities by following the same spirit of variable transformations introduced previously. Hence, we shall introduce the variable(s):

$$B_\lambda = B^\star + \lambda^{-1}\nabla \times V, \tag{7.19}$$

where λ satisfies (7.17). Upon doing so, we find that

$$\{F,G\}^{ExMHD} \equiv \{F,G\}^{HMHD}\left[d_i - 2\lambda^{-1}; B_\lambda\right], \tag{7.20}$$

where the RHS indicates that the extended MHD bracket is equivalent to the Hall MHD bracket, when the latter is subjected to $d_i \to d_i - 2\lambda^{-1}$ and $B \to B_\lambda$. It must be borne in mind that there are *two* such variable transformations since there are two choices for B_λ, which stem from (7.17) - the quadratic equation for λ. We find that these two variable transformations naturally allow us to determine the two helicities of the model. We recover (7.16) successfully, thereby confirming the power of variable transformations. Furthermore, we conclude from (7.20) that a proof of the Jacobi identity for Hall MHD automatically ensures that the extended MHD bracket also satisfies the same property.

In summary, we have established the remarkable result that a proof of the Jacobi identity for Hall bracket suffices to establish the validity of the inertial and extended MHD brackets as well.

7.2 The Lagrangian origin of the equivalence between the extended MHD models

In this section, we shall briefly explore the origin of the helicities derived in the previous sections, and comment on the equivalences between the various extended MHD models. In order to do so, we appeal to the Lagrangian picture of fluid models, which envisions the fluid as a continuum collection of particles. In this picture, laws are built in *a priori* through the imposition of suitable geometric constraints; we refer the reader to [18, 65, 154, 167] for further details.

In ideal MHD, we know that flux is frozen-in, and this translates into a local statement of flux conservation on the Lagrangian level. When one works out the algebra, it is shown that the magnetic induction equation of ideal MHD is just the Lie-dragging of a 2-form - the magnetic field $B \cdot dS$. Alternatively, one can interpret it, in 3D, as the Lie-dragging of a vector density [26, 71]. Now, let us take a step back and consider two-fluid theory, where one can define a canonical momentum $\mathcal{P} = m_s v_s + q_s A$ for each species. It is evident that A represents the electromagnetic component of the canonical momentum, whilst v_s gives rise to the kinetic component. Next, suppose that we consider a scenario where the kinetic momentum is much 'smaller' than its electromagnetic counterpart - this is achieved especially in the case of electrons, owing to their lower mass. In such an event, we see that the canonical momentum reduces to A (up to proportionality factors) and we can interpret B as a certain limit of $\nabla \times \mathcal{P}$. In ideal MHD, which is a pure one-fluid theory, it is easy to view B as being Lie-dragged by the center-of-mass velocity V.

Now, we shall proceed in the same heuristic manner, through the incorporation of two-fluid effects. Firstly, let us suppose that the electrons are inertialess, but *not* the ions. As a consequence, one finds that the center-of-mass velocity V and the ion velocity virtually coincide. The corresponding canonical momenta, after suitable normalization, reduce to B and $\mathcal{B}_i$ respectively, after rewriting them in terms of one-fluid variables. Following the analogy outlined above, we can choose to Lie-drag them as 2-forms, akin to the magnetic field in ideal MHD. Next, the question arises: by which velocity must we Lie-drag these variables? The answer is intuitive: we choose to Lie-drag them by their corresponding velocities. After some manipulation, it is easy to show that the resulting equations are equivalent to those of Hall MHD.

Next, suppose that we include the effects of electron inertia. The curls of the canonical momenta, viz. the canonical vorticities, when written in terms of the one-fluid variables, correspond to $B^\star \pm d_e \nabla \times V$, which are the variables that appear in (7.13) and (7.12) respectively. Following the same prescription, we can choose to Lie-drag these quantities. We choose to Lie-drag the canonical vorticities by suitable flow velocities, $V \pm d_e \nabla \times B/\rho$, which can be determined by an appropriate manipulation of the inertial MHD equations. It is found, after some algebraic simplification, that the resulting equations are equivalent to those of inertial MHD. The generalization to extended MHD is not entirely straightforward, but it can be done by using the variables from (7.16) as Lie-dragged 2-forms, and noting that each is Lie-dragged by the velocity of the corresponding species.

Thus, we see that our preceding analysis establishes two very important points. Firstly, the equations for extended MHD can be viewed as the natural manifestation of underlying (Lagrangian) geometric constraints. Secondly, we see that the variables $\mathcal{B}_i$, $\mathcal{B}_e$, etc. introduced earlier, and the helicities of the models, are also 'natural' - they emerge from the unified view that the canonical momenta are treated as Lie-dragged 2-forms. In both these aspects, we see that the Lagrangian picture of extended MHD presents a compelling argument as to why the variable transformations of Section 7.1 are *not* arbitrary, and, more importantly, it emphasizes the underlying geometric nature of the extended MHD models. The latter is all the more useful as it further serves to emphasize the existence of a unifying structure for the extended MHD models.

7.3 Jacobi identity for Hall MHD

In this section, we shall present a detailed proof of the Jacobi identity for the noncanonical Hall MHD bracket. The discussion in the preceding sections ensures that the proof of the Jacobi identity for other versions of extended MHD can also be established in an analogous manner. For this section *alone*, in our entire thesis, we shall introduce the boldface notation to distinguish vectors explicitly from their scalar counterparts.

In the absence of the Hall term, we see that (7.1) reduces to the ideal

MHD bracket, first derived in [28]:

$$
\begin{aligned}
\{F,G\}^{MHD} \;:=\; -\int_D d^3r \Big(& F_\rho \nabla \cdot G_{\mathbf{v}} - G_\rho \nabla \cdot F_{\mathbf{v}} + \frac{\nabla \times \mathbf{v}}{\rho} \cdot G_{\mathbf{v}} \times F_{\mathbf{v}} \\
& + \frac{\mathbf{B}}{\rho} \cdot \Big[F_{\mathbf{v}} \cdot \nabla G_{\mathbf{B}} - G_{\mathbf{v}} \cdot \nabla F_{\mathbf{B}} \Big] \\
& + \mathbf{B} \cdot \Big[\nabla \frac{F_{\mathbf{v}}}{\rho} \cdot G_{\mathbf{v}} - \nabla \frac{G_{\mathbf{v}}}{\rho} \cdot F_{\mathbf{v}} \Big] \Big),
\end{aligned}
\tag{7.21}
$$

which is known to satisfy Jacobi identity on its own [28, 62]. The convention that we will be using throughout is that ∇ operator acts only on the variable following it, and dyadics can be written in the coordinate form

$$
\mathbf{B} \cdot \nabla \frac{F_{\mathbf{v}}}{\rho} \cdot G_{\mathbf{v}} = B_i \partial_i \Big(\frac{F_v^j}{\rho} \Big) G_v^j. \tag{7.22}
$$

7.3.1 Hall - Hall Jacobi identity

The introduction of the Hall current leads to additional Hall bracket, identified previously in (7.2). We recollect that it is given by

$$
\{F,G\}^{Hall} := -d_i \int_D d^3r \frac{\mathbf{B}}{\rho} \cdot \Big[(\nabla \times F_{\mathbf{B}}) \times (\nabla \times G_{\mathbf{B}}) \Big], \tag{7.23}
$$

Demonstrating that Hall MHD bracket satisfies Jacobi is important since it is closely connected to the rest of the extended MHD models, as discussed previously. The Jacobi identity involves proving that cyclical permutations of any functionals F, G, H vanish, i.e. we require

$$
0 = \{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} \equiv \{\{F,G\},H\} + \circlearrowleft_{F,G,H} \tag{7.24}
$$

Here $\{,\} := \{,\}^{MHD} + \{,\}^{Hall}$. Because we already know that (7.21) satisfies Jacobi and according to the bilinearity of Poisson brackets, the general proof splits into two pieces

$$\{\{F,G\}^{MHD},H\}^{Hall}+\{\{F,G\}^{Hall},H\}^{MHD}+\circlearrowleft_{F,G,H}=0, \tag{7.25}$$

and

$$\{\{F,G\}^{Hall},H\}^{Hall}+\circlearrowleft_{F,G,H}=0. \tag{7.26}$$

This split occurs since (7.25) involves terms that are linear in d_i, whilst (7.26) is quadratic in d_i. We introduce the cosymplectic operator J which depends on the field variables u in general. It is known that Poisson brackets can be formally written in the form

$$\{F,G\}:=\left\langle\frac{\delta F}{\delta u}\middle|J\frac{\delta G}{\delta u}\right\rangle. \tag{7.27}$$

The outer brackets in both (7.25) and (7.26) require evaluation of the variational derivatives of the inner bracket with respect to the field variables:

$$\begin{aligned}\frac{d}{d\epsilon}\{F,G\}[u+\epsilon\delta u]\Big|_{\epsilon=0} &:= \left\langle\frac{\delta}{\delta u}\{F,G\}\middle|\delta u\right\rangle \\ &= \left\langle\frac{\delta^2 F}{\delta u\delta u}\delta u\middle|J\frac{\delta G}{\delta u}\right\rangle+\left\langle\frac{\delta F}{\delta u}\middle|J\frac{\delta^2 G}{\delta u\delta u}\delta u\right\rangle \\ &\quad+\left\langle\frac{\delta F}{\delta u}\middle|\frac{\delta J}{\delta u}(\delta u)\frac{\delta G}{\delta u}\right\rangle\end{aligned} \tag{7.28}$$

Proving the Jacobi identity for noncanonical Poisson brackets is aided by a theorem proven in [62], which states that the first two terms of the above expression vanish when plugged in the outer bracket, together with the other cyclic permutations. Thus, we can neglect second variations that appear throughout the following calculations. Since the outer Hall bracket involves variations with respect to $\mathbf{B}$, it is enough to consider

$$\frac{\delta}{\delta\mathbf{B}}\{F,G\}^{Hall}=-d_i\,(\nabla\times F_{\mathbf{B}})\times(\nabla\times G_{\mathbf{B}})+\cdots\equiv -d_i\,F_{\mathbf{A}}\times G_{\mathbf{A}}+\dots, \tag{7.29}$$

where the second variations that arise implicitly are suppressed because we have established that they will not contribute to the Jacobi identity. Hence, it suffices to compute the variations with respect to the field variables that enter the Poisson bracket explicitly. Note that the last relation in (7.29) arises due to $\mathbf{B} =: \nabla \times \mathbf{A}$. This is evident through

$$\begin{aligned} \delta F &= \int_D d^3r\, \frac{\delta F}{\delta \mathbf{B}} \cdot \delta \mathbf{B} = \int_D d^3r\, \frac{\delta F}{\delta \mathbf{B}} \cdot \nabla \times \delta \mathbf{A} \\ &= \int_D d^3r\, \nabla \times \frac{\delta F}{\delta \mathbf{B}} \cdot \delta \mathbf{A} = \int_D d^3r\, \frac{\delta F}{\delta \mathbf{A}} \cdot \delta \mathbf{A}. \end{aligned} \tag{7.30}$$

A corollary of the above relation is that $F_{\mathbf{A}} \equiv \frac{\delta F}{\delta \mathbf{A}}$ is divergence-free, i.e. $\nabla \cdot F_{\mathbf{A}} = 0$. Substituting (7.29) into the Hall-Hall part of the Jacobi relation (7.26), we obtain

$$d_i^2 \int_D d^3r\, \mathbf{B} \cdot \left(\nabla\Big(\frac{1}{2\rho^2}\Big) \times \Big[F_{\mathbf{A}} \times G_{\mathbf{A}}\Big] + \frac{1}{\rho^2} \nabla \times \Big[F_{\mathbf{A}} \times G_{\mathbf{A}}\Big] \right) \times H_{\mathbf{A}}. \tag{7.31}$$

This expression can be expanded using vector identities such as $\mathbf{X} \times (\mathbf{Y} \times \mathbf{Z}) = \mathbf{Y}\ (\mathbf{X} \cdot \mathbf{Z}) - \mathbf{Z}\ (\mathbf{X} \cdot \mathbf{Y})$ and $\nabla \times (\mathbf{X} \times \mathbf{Y}) = \nabla \cdot \big(\mathbf{Y}\, \mathbf{X}^T - \mathbf{X}\, \mathbf{Y}^T\big)$, which enables us to collect certain terms together. Since the Jacobi identity involves two additional cyclic permutations, we are allowed to carry out cyclic permutations of the above expression and collect similar terms together. Through a suitable permutation of the variables, and integrating by parts, we arrive at

$$\begin{aligned} \{\{F, G\}^{Hall}, H\}^{Hall} + {}_{F,\overset{\circlearrowleft}{G},H} &= d_i^2 \int_D d^3r\, \frac{1}{\rho^2} F_{\mathbf{A}} \times G_{\mathbf{A}} \cdot (H_{\mathbf{A}} \cdot \nabla)\mathbf{B} + {}_{F,\overset{\circlearrowleft}{G},H} \\ &= d_i^2 \int_D d^3r\, \frac{1}{\rho^2} \epsilon_{ijk} F_A^j\, G_A^k H_A^l \partial_l B^i + {}_{F,\overset{\circlearrowleft}{G},H} \\ &= d_i^2 \int_D d^3r\, \frac{F_{\mathbf{A}} \cdot G_{\mathbf{A}} \times H_{\mathbf{A}}}{\rho^2} \delta_i^l \partial_l B^i + {}_{F,\overset{\circlearrowleft}{G},H}, \end{aligned} \tag{7.32}$$

where the last step becomes apparent when we explicitly write down the other two permutations, and use the antisymmetry of Levi-Civita tensor ϵ_{ijk} in addition to the identity $\epsilon_{ijk}\epsilon^{ljk} = 2\delta^l_i$. Finally, upon invoking the identity $\nabla \cdot \mathbf{B} = 0$, we see that the Hall - Hall Jacobi identity is satisfied.

7.3.2 Hall - Ideal MHD Jacobi identity

We observe that this part is harder to tackle, owing to the greater complexity of the resultant expression. Let us first express the first term in (7.25). As described in the previous section, the outer Hall bracket (7.23) necessitates only the explicit variational derivatives with respect to $\mathbf{B}$. Hence, we only need to consider such variations of the inner MHD bracket (7.21):

$$\frac{\delta}{\delta \mathbf{B}}\{F,G\}^{MHD} = -\frac{F_{\mathbf{v}}}{\rho}\cdot\nabla G_{\mathbf{B}} + \frac{G_{\mathbf{v}}}{\rho}\cdot\nabla F_{\mathbf{B}} - \nabla\frac{F_{\mathbf{v}}}{\rho}\cdot G_{\mathbf{B}} + \nabla\frac{G_{\mathbf{v}}}{\rho}\cdot F_{\mathbf{B}} + \ldots, \quad (7.33)$$

and we have suppressed the implicit second-order variations, as they do not contribute to the Jacobi identity. After substitution into the outer Hall bracket, we get

$$\begin{aligned}\{\{F,G\}^{MHD},H\}^{Hall} &= -d_i\int_D d^3r\frac{\mathbf{B}}{\rho}\cdot\Bigg[\nabla\times\Big(\frac{F_{\mathbf{v}}}{\rho}\cdot\nabla G_{\mathbf{B}} - \frac{G_{\mathbf{v}}}{\rho}\cdot\nabla F_{\mathbf{B}} \\ &\quad +\nabla\frac{F_{\mathbf{v}}}{\rho}\cdot G_{\mathbf{B}} - \nabla\frac{G_{\mathbf{v}}}{\rho}\cdot F_{\mathbf{B}}\Big)\times\nabla\times H_{\mathbf{B}}\Bigg]\end{aligned} \quad (7.34)$$

We proceed to use the vector calculus identities $\nabla\times\nabla f = 0$ and $\mathbf{X}\times\nabla\times\mathbf{Y} = \nabla\mathbf{Y}\cdot\mathbf{X} - \mathbf{X}\cdot\nabla\mathbf{Y}$, which allows us to simplify the expression as follows

$$\{\{F,G\}^{MHD},H\}^{Hall} = -d_i\int_D d^3r\,\frac{\mathbf{B}}{\rho}\cdot\left(\nabla\times\frac{F_{\mathbf{v}}\times G_{\mathbf{A}} - G_{\mathbf{v}}\times F_{\mathbf{A}}}{\rho}\times H_{\mathbf{A}}\right). \tag{7.35}$$

In the second term of (7.25) the outer MHD bracket requires evaluation of variations with respect to both $\mathbf{B}$ and ρ. We already have the first one from (7.29), while the second yields

$$\frac{\delta}{\delta\rho}\{F,G\}^{Hall} = d_i\frac{\mathbf{B}}{\rho^2}\cdot F_{\mathbf{A}}\times G_{\mathbf{A}}. \tag{7.36}$$

Upon substituting them into the second term of (7.25), we end up with

$$-d_i\int_D d^3r\,\frac{\mathbf{B}}{\rho^2}\cdot(F_{\mathbf{A}}\times G_{\mathbf{A}})(\nabla\cdot H_{\mathbf{v}}) + \frac{\mathbf{B}}{\rho}\cdot\left(\nabla\times\frac{F_{\mathbf{A}}\times G_{\mathbf{A}}}{\rho}\times H_{\mathbf{v}}\right) + \circlearrowleft_{F,G,H} \tag{7.37}$$

Upon combining (7.35) and (7.37), we have

$$\begin{aligned}\mathcal{J} &= -d_i\int d^3r\left(\frac{\mathbf{B}}{\rho^2}\cdot F_{\mathbf{A}}\times G_{\mathbf{A}}\nabla\cdot H_{\mathbf{v}} + \frac{\mathbf{B}}{\rho}\cdot\left[\nabla\times\frac{F_{\mathbf{A}}\times G_{\mathbf{A}}}{\rho}\times H_{\mathbf{v}}\right]\right.\\ &\qquad\left.+\frac{\mathbf{B}}{\rho}\cdot\left[\nabla\times\frac{F_{\mathbf{v}}\times G_{\mathbf{A}} - G_{\mathbf{v}}\times F_{\mathbf{A}}}{\rho}\times H_{\mathbf{A}}\right]\right) + \circlearrowleft_{F,G,H}\\ &= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3,\end{aligned} \tag{7.38}$$

where $\mathcal{J}_i$'s represent the three contributions arising from (7.37) and (7.35) respectively. Applying the vector identities mentioned previously, and recollecting that variations with respect to $\mathbf{A}$ are divergence-free, the third term can be manipulated to yield

$$\begin{aligned}
\mathcal{J}_3 &= d_i \int_D d^3r\, \frac{\mathbf{B}}{\rho} \cdot H_{\mathbf{A}} \times \Bigg(G_{\mathbf{A}} \cdot \nabla \frac{F_{\mathbf{v}}}{\rho} - \nabla \cdot F_{\mathbf{v}} \frac{G_{\mathbf{A}}}{\rho} - F_{\mathbf{v}} \cdot \nabla \frac{G_{\mathbf{A}}}{\rho} - F_{\mathbf{A}} \cdot \nabla \frac{G_{\mathbf{v}}}{\rho} \\
&\qquad + \nabla \cdot G_{\mathbf{v}} \frac{F_{\mathbf{A}}}{\rho} + G_{\mathbf{v}} \cdot \nabla \frac{F_{\mathbf{A}}}{\rho} \Bigg) \\
&= -d_i \int_D d^3r \Bigg(- \frac{2\mathbf{B}}{\rho^2} \cdot (F_{\mathbf{A}} \times G_{\mathbf{A}}) (\nabla \cdot H_{\mathbf{v}}) \\
&\qquad - \mathbf{B} \cdot (F_{\mathbf{A}} \times G_{\mathbf{A}}) \left[H_{\mathbf{v}} \cdot \nabla \left(\frac{1}{\rho^2} \right) \right] - \mathbf{B} \cdot \left(\frac{H_{\mathbf{v}}}{\rho^2} \cdot \nabla \right) \left(F_{\mathbf{A}} \times G_{\mathbf{A}} \right) \\
&\qquad + \frac{\mathbf{B}}{\rho} \cdot \Big[F_{\mathbf{A}} \times (G_{\mathbf{A}} \cdot \nabla) - G_{\mathbf{A}} \times (F_{\mathbf{A}} \cdot \nabla) \Big] \frac{H_{\mathbf{v}}}{\rho} \Bigg) \qquad (7.39)
\end{aligned}$$

Here, we have used the freedom to permute F, G, H in a consistent manner. When combined with the first term $\mathcal{J}_1$, this results in

$$\begin{aligned}
\mathcal{J}_1 + \mathcal{J}_3 &= d_i \int_D d^3r \Bigg[\nabla \cdot \left[\frac{H_{\mathbf{v}}}{\rho} \frac{F_{\mathbf{A}} \times G_{\mathbf{A}}}{\rho} \right] \cdot \mathbf{B} \qquad (7.40) \\
&\qquad - \frac{\mathbf{B}}{\rho} \cdot \Big[F_{\mathbf{A}} \times (G_{\mathbf{A}} \cdot \nabla) - G_{\mathbf{A}} \times (F_{\mathbf{A}} \cdot \nabla) \Big] \frac{H_{\mathbf{v}}}{\rho} \Bigg].
\end{aligned}$$

The second term of (7.38) can be rewritten as

$$\mathcal{J}_2 = -d_i \int_D d^3r \left(\frac{H_{\mathbf{v}}}{\rho} \cdot \nabla \left(\frac{F_{\mathbf{A}} \times G_{\mathbf{A}}}{\rho} \right) \cdot \mathbf{B} - \mathbf{B} \cdot \nabla \left(\frac{F_{\mathbf{A}} \times G_{\mathbf{A}}}{\rho} \right) \cdot \frac{H_{\mathbf{v}}}{\rho} \right) \quad (7.41)$$

Upon using (7.40) and (7.41), we can condense (7.38) into

$$\begin{aligned}
\mathcal{J} &= d_i \int_D d^3r \Bigg(\mathbf{B} \cdot \left(\frac{F_{\mathbf{A}} \times G_{\mathbf{A}}}{\rho} \right) \left[\nabla \cdot \left(\frac{H_{\mathbf{v}}}{\rho} \right) \right] - \mathbf{B} \cdot \nabla \left(\frac{H_{\mathbf{v}}}{\rho} \right) \cdot \frac{F_{\mathbf{A}} \times G_{\mathbf{A}}}{\rho} \\
&\qquad - \frac{\mathbf{B}}{\rho} \cdot \Big[F_{\mathbf{A}} \times (G_{\mathbf{A}} \cdot \nabla) - G_{\mathbf{A}} \times (F_{\mathbf{A}} \cdot \nabla) \Big] \frac{H_{\mathbf{v}}}{\rho} \Bigg). \qquad (7.42)
\end{aligned}$$

The second term has been integrated by parts, by applying $\nabla\cdot\mathbf{B}=0$ to obtain this expression. We shall not use any further permutations of F, G and H, as one such permutation was used previously. It can be shown, in coordinates for instance, or using the vector identities introduced previously, that the first two and the last two terms collapse into

$$\begin{aligned}\mathcal{J} &= d_i\int_D d^3r\Bigg(\frac{\mathbf{B}}{\rho}\cdot\left[\Big((F_{\mathbf{A}}\times G_{\mathbf{A}})\times\nabla\Big)\times\frac{H_{\mathbf{v}}}{\rho}\right] \\ &\qquad -\frac{\mathbf{B}}{\rho}\cdot\left[\Big((F_{\mathbf{A}}\times G_{\mathbf{A}})\times\nabla\Big)\times\frac{H_{\mathbf{v}}}{\rho}\right]\Bigg)\equiv 0\end{aligned} \tag{7.43}$$

As a result, we see that the Hall - MHD Jacobi identity is satisfied.

Hence, from the results derived in Sections 7.3.1 and 7.3.2, we conclude that the Hall MHD bracket (7.1) satisfies the Jacobi identity, thereby rendering it a valid noncanonical Poisson bracket. In turn, this ensures the validity of the inertial MHD bracket, and by applying the same procedures, it is possible to show that the extended MHD bracket satisfies the Jacobi identity.

Chapter 8

Conclusions and outline for future work

"Our revels now are ended. These our actors,

As I foretold you, were all spirits and

Are melted into air, into thin air"

- William Shakespeare, *The Tempest*, Act 4, Scene 1

We have now reached the end of our journey, one which depicted the use of the HAP formulation of fluids and plasmas to tackle a diverse array of problems. At this stage, we recall the salient features of each chapter, and highlight the advantages endowed by the HAP formulation for the specific problem(s) that we tackled therein.

- In Chapter 2, the necessary mathematical preliminaries were developed. The advantages of both the action and the Hamiltonian formulations were highlighted, by using ideal MHD as a trial case.

- In Chapter 3, we emphasized the important of FLR effects. Through simple physical considerations, a new term was introduced in the action that was linear in the velocities. Amongst other things, we showed that it explained the origin of the gyromap. We moved to the associated

Hamiltonian model via reduction and used the Casimirs to derive the generalized Grad-Shafranov equations with flow and gyroviscosity.

- In Chapter 4, we showed that gyroviscous models could be analysed in a generic model-independent manner through the use of Noether's theorem. We also indicated that our formalism was powerful enough to derive generalized hydrodynamical models, which could describe liquid crystals and motivate the origins of gyroviscosity.

- In Chapter 5, we used the action principle to perform a set of rigorous orderings, and obtain a series of extended MHD models, and their conserved quantities, along the way. We also tackled, for the first time, the issues of non-local Lagrange-Euler maps inherent in multi-fluid models, and quasineutrality on a Lagrangian level.

- In Chapter 6, we illustrated the power of Lagrangian constraints to build a 2D MHD model with electron inertia that gave rise to a 6-field model, which included the famous Ottaviani-Porcelli model as a special subcase. By exploiting the HAP machinery, we also demonstrated the close connections between inertial and extended MHD.

- In Chapter 7, we established a unique commonality between the noncanonical brackets of Hall and inertial MHD, and their connections with extended MHD. Furthermore, we also speculations on the Lagrangian origins of these connections between several extended MHD models.

The reader will have observed that three of our chapters were centred around extended MHD models, and there are several promising avenues for future work. The first entails the use of the extended bracket from [199] to derive reduced fluid models of interest in a rigorous manner such as [37, 201, 202] whilst using the gyroviscous machinery from Chapters 3 and 4. In particular, the inclusion of gyroviscosity is likely to lead interesting modification of the equilibria, waves, etc. which have important secondary ramifications. We emphasize that the reduced models of extended MHD thus obtained incorporate electron inertia and FLR effects, which are of great importance in studying reconnection. Hence, we posit that such models could prove to be of considerable use in numerical simulations of magnetic reconnection.

On a more fundamental level, the actions introduced in Chapter 5 were comprised of both Lagrangian and Eulerian variables - a natural improvement of these actions is to seek ones that involve only the Lagrangian variables. The derivation of such a Lagrangian action is also likely to confirm, or disprove, the conjectures introduced in Chapter 7 to explain the surprising connections between the Hamiltonian formulations of Hall MHD and inertial MHD. Yet another fundamental use of the HAP formulation is to study the intriguing connections between several condensed matter systems and plasmas, as observed in [71, 133].

If we move beyond our (fairly vast) world of fluid models, unknown vistas open up to us. In [31], it was demonstrated that the Maxwell-Vlasov system possessed a noncanonical Hamiltonian structure. In a recent work

[203], it was shown that a wide spectrum of collisionless neutral kinetic models followed from a parent action, akin to the analysis undertaken in [154]. Hybrid models that combine fluid and kinetic effects have also been shown to possess a Hamiltonian structure [204], and such models are being increasingly used in modelling space and astrophysical plasmas; see e.g. the recent work by [205]. Owing to these similarities, we believe that subjecting the above models to a similar HAP-based analysis is likely to be very beneficial.

And lastly, we observe that our entire analysis has been Newtonian in nature. However, it is well-known that a strong understanding of relativistic MHD [206, 207], which is still riddled with ambiguities, would be highly beneficial, especially in astrophysical environments. The HAP approach, with its inherent rigour, constitutes a promising line of enquiry, and it is possible to use [26] as a starting point for future work(s) in this area.

To sum up, we have seen that the Hamiltonian and Action Principle formulation of fluid models is endowed with unique advantages such as a combination of simplicity and rigour, making it amenable both to interpreting old results and constructing (or deriving) new ones. Hence, these constitute strong and compelling reasons for applying the HAP approach to the issues outlined above. It seems not only possible, but quite probable, that interesting advances will emerge from future investigations of this kind.

Bibliography

[1] S. Weinberg. *The Quantum Theory of Fields*. Cambridge University Press, 1995.

[2] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W.H. Freeman and Co., 1973.

[3] A. Altland and B. D. Simons. *Condensed Matter Field Theory*. Cambridge University Press, 2010.

[4] M. Kardar. *Statistical Physics of Particles*. Cambridge University Press, 2007.

[5] J. L. Lagrange. *Mécanique Analytique*. Dordrecht Kluwer Academic, Boston, MA, 1997.

[6] R. F. A. Clebsch. *J. Reine Angew. Math.*, 54:293, 1857.

[7] H. von Helmholtz. *J. Reine Angew. Math.*, 55:25, 1858.

[8] R. F. A. Clebsch. *J. Reine Angew. Math.*, 56:1, 1859.

[9] H. Hanke. *Zur allgemeinen Theorie der Bewegung der Flussigkeiten*. Dieterichschen Univ.-Buchdruckerei, 1861.

[10] R. Kirchhoff. *Vorlesungenuber Mathematische Physik: Mechanik*. Teubner, Leipzig, 1876.

[11] U. Frisch and B. Villone. Cauchy's almost forgotten Lagrangian formulation of the Euler equation for 3D incompressible flow. *Eur. Phys. J. H*, 39:325, 2014.

[12] J. Serrin. Mathematical principles of classical fluid mechanics. In S. Flügge, editor, *Handbuch der Physik*, volume VIII, pages 125–263. Springer Verlag, Berlin, 1959.

[13] L. D. Landau and E. M. Lifshitz. *Fluid mechanics.* Pergamon Press, Oxford, 1959.

[14] G. K. Batchelor. *An Introduction to Fluid Dynamics.* Cambridge: Cambridge Univ. Press, 1967.

[15] V. I. Arnold. *Mathematical methods of classical mechanics.* Springer Verlag, Berlin, 1978.

[16] R. Salmon. Hamiltonian fluid mechanics. *Ann. Rev. Fluid Mech.*, 20:225–256, 1988.

[17] P. J. Morrison. Hamiltonian description of the ideal fluid. *Rev. Mod. Phys*, 70:467–521, 1998.

[18] W. A. Newcomb. *Nucl. Fusion Suppl.*, pt 2:451, 1962.

[19] M. G. Calkin. An action principle for magnetohydrodynamics. *Can. J. Phys.*, 41:2241, 1963.

[20] W. A. Newcomb. The perfect gyroscopic fluid. *Ann. Phys.*, 72:29–79, 1972.

[21] W. A. Newcomb. Gyroscopic-quasielastic fluid systems. *Ann. Phys.*, 81:231–331, 1973.

[22] W. A. Newcomb. The anisorrhopic guiding-center fluid. *Ann. Phys.*, 150:172–266, 1983.

[23] W. A. Newcomb. Guiding-center autonomy theorem. *Phys. Fluids*, 28:505–516, 1985.

[24] Z. Yoshida, Y. Kawazura, and T. Yokoyama. Relativistic helicity and link in Minkowski space-time. *J. Math. Phys.*, 55(4):043101, 2014.

[25] Y. Kawazura, Z. Yoshida, and Y. Fukumoto. Relabeling symmetry in relativistic fluids and plasmas. *J. Phys. A*, 47:5501, 2014.

[26] E. D'Avignon, P. Morrison, and F. Pegoraro. An Action Principle for Relativistic MHD. *Phys. Rev. D*, 91:084050, 2015.

[27] D. D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein. Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.*, 123:1–116, 1985.

[28] P. J. Morrison and J. M. Greene. Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics. *Phys. Rev. Lett.*, 45:790–794, 1980.

[29] R. G. Littlejohn. Hamiltonian formulation of guiding center motion. *Phys. Fluids*, 24:1730–1749, 1981.

[30] J. R. Cary and R. G. Littlejohn. Noncanonical Hamiltonian mechanics and its application to magnetic field line flow. *Ann. Phys.*, 151:1–34, 1983.

[31] P. J. Morrison. The Maxwell-Vlasov equations as a continuous hamiltonian system. *Phys. Lett. A*, 80:383–386, 1980.

[32] J. E. Marsden and A. Weinstein. The Hamiltonian structure of the Maxwell-Vlasov equations. *Physica D*, 4:394–406, 1982.

[33] V. E. Zakharov, S. L. Musher, and A. M. Rubenchik. Hamiltonian approach to the description of non-linear plasma phenomena. *Phys. Rep.*, 129:285–366, 1985.

[34] V. E. Zakharov and E. A. Kuznetsov. Hamiltonian formalism for nonlinear waves. *Phys. Usp.*, 40:1087–1116, 1997.

[35] D. D. Holm and B. A. Kupershmidt. Poisson brackets and clebsch representations for magnetohydrodynamics, multifluid plasmas, and elasticity. *Physica D*, 6:347–363, 1983.

[36] P. J. Morrison and R. D. Hazeltine. Hamiltonian formulation of reduced magnetohydrodynamics. *Phys. Fluids*, 27:886–897, 1984.

[37] R. D. Hazeltine, C. T. Hsu, and P. J. Morrison. Hamiltonian four-field model for nonlinear tokamak dynamics. *Phys. Fluids*, 30:3204–3211, 1987.

[38] D. D. Holm, J. E. Marsden, and T. Ratiu. The Euler-Poincaré Equations and Semidirect Products with Applications to Continuum Theories. *Adv. Math.*, 137:1–81, 1998.

[39] H. Cendra, D. D. Holm, M. J. W. Hoyle, and J. E. Marsden. The Maxwell-Vlasov equations in Euler-Poincaré form. *J. Math. Phys.*, 39:3138–3157, 1998.

[40] F. Gay-Balmaz and C. Tronci. Reduction theory for symmetry breaking with applications to nematic systems. *Physica D*, 239:1929–1947, 2010.

[41] J. Marsden and A. Weinstein. Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. *Physica D*, 7:305–323, 1983.

[42] R. G. Littlejohn. Hamiltonian perturbation theory in noncanonical coordinates. *J. Math. Phys.*, 23:742–747, 1982.

[43] A. N. Kaufman. Dissipative hamiltonian systems: A unifying principle. *Phys. Lett. A*, 100:419–422, 1984.

[44] P. J. Morrison. Bracket formulation for irreversible classical fields. *Phys. Lett. A*, 100:423–427, 1984.

[45] P. J. Morrison. A paradigm for joined Hamiltonian and dissipative systems. *Physica D*, 18:410–419, 1986.

[46] G. K. Vallis, G. F. Carnevale, and W. R. Young. Extremal energy properties and construction of stable solutions of the Euler equations. *J. Fluid Mech.*, 207:133–152, 1989.

[47] R. W. Brockett. Dynamical systems that sort lists, diagonalize matrices, and solve linear programming problems. *Linear Alg. Appl.*, 146:79–91, 1991.

[48] A. Bloch, P. S. Krishnaprasad, J. E. Marsden, and T. S. Ratiu. The Euler-Poincaré equations and double bracket dissipation. *Comm. Math. Phys.*, 175:1–42, 1996.

[49] E. Noether. Invariant variation problems. *Transp. Theory Stat. Phys.*, 1:186–207, 1971.

[50] E. Noether. Invariante Variations probleme. *Nachr. Knig. Gessell. Wissen. Gttingen, Mathphys. Kl.*, pages 235–257, 1918.

[51] A. V. Tur and V. V. Yanovsky. Invariants in dissipationless hydrodynamic media. *J. Fluid Mech.*, 248:67–106, 1993.

[52] N. Padhye and P. J. Morrison. Fluid element relabeling symmetry. *Phys. Lett. A*, 219:287–292, 1996.

[53] N. Padhye and P. J. Morrison. Relabeling symmetries in hydrodynamics and magnetohydrodynamics. *Plasma Phys. Rep.*, 22:869–877, 1996.

[54] G. M. Webb, B. Dasgupta, J. F. McKenzie, Q. Hu, and G. P. Zank. Local and nonlocal advected invariants and helicities in magnetohydrodynamics and gas dynamics: II. Noether's theorems and Casimirs. *J. Phys. A*, 47(9):095502, 2014.

[55] H. K. Moffatt. The degree of knottedness of tangled vortex lines. *J. Fluid Mech.*, 35:117–129, 1969.

[56] M. A. Berger and G. B. Field. The topological properties of magnetic helicity. *J. Fluid Mech.*, 147:133–148, 1984.

[57] K. Kimura and P. J. Morrison. On energy conservation in extended magnetohydrodynamics. *Phys. Plasmas*, 21(8):082101, 2014.

[58] C. Tronci, E. Tassi, E. Camporeale, and P. J. Morrison. Hybrid Vlasov-MHD models: Hamiltonian vs. non-Hamiltonian. *Plasma Phys. Cont. Fusion*, 56:095008, 2014.

[59] S. I. Braginskii. Transport Processes in a Plasma. *Rev. Plasma Phys.*, 1:205, 1965.

[60] J. P. H. Goedbloed and S. Poedts. *Principles of Magnetohydrodynamics.* Cambridge University Press, 2004.

[61] M. Ottaviani and F. Porcelli. Nonlinear collisionless magnetic reconnection. *Phys. Rev. Lett.*, 71:3802–3805, 1993.

[62] P. J. Morrison. Poisson Brackets for Fluids and Plasmas. In M. Tabor and Y. Treve, editors, *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, volume 88, pages 13–46. New York: Am. Inst. Phys., 1982.

[63] P. J. Morrison. Hamiltonian and action principle formulations of plasma physics. *Phys. Plasmas*, 12:058102, 2005.

[64] P. J. Morrison. On Hamiltonian and Action Principle Formulations of Plasma Dynamics. In B. Eliasson and P. K. Shukla, editors, *New Developments in Nonlinear Plasma Physics: Proceedings for the 2009 ICTP College on Plasma Physics*, volume 1188, pages 329–344. New York: Am. Inst. Phys., 2009.

[65] P. J. Morrison, M. Lingam, and R. Acevedo. Hamiltonian and action formalisms for two-dimensional gyroviscous magnetohydrodynamics. *Phys. Plasmas*, 21(8):082102, 2014.

[66] J. E. Marsden and T. Ratiu. *Introduction to Mechanics and Symmetry.* Springer, 2002.

[67] R. Courant and D. Hilbert. *Methods of mathematical physics - Vol.1.* New York: Interscience, 1953.

[68] I. M. Gelfand and S. V. Fomin. *Calculus of Variations.* Dover, 2000.

[69] N. G. van Kampen and B. U. Felderhof. *Theoretical methods in plasma physics*. Amsterdam: North Holland, 1967.

[70] T. Andreussi, P. J. Morrison, and F. Pegoraro. MHD equilibrium variational principles with symmetry. *Plasma Phys. Controlled Fusion*, 52(5):055001, 2010.

[71] M. Lingam and P. J. Morrison. The action principle for generalized fluid motion including gyroviscosity. *Phys. Lett. A*, 378:3526–3532, 2014.

[72] G. F. Chew, M. L. Goldberger, and F. E. Low. The Boltzmann Equation and the One-Fluid Hydromagnetic Equations in the Absence of Particle Collisions. *Proc. Roy. Soc. London, Ser. A*, 236:112–118, 1956.

[73] H. Bacry, H. Ruegg, and J.-M. Souriau. Dynamical groups and spherical potentials in Classical Mechanics. *Comm. Math. Phys.*, 3:323–333, 1966.

[74] J.-M. Souriau. *Structure des systemes dynamiques.* Dunod, Paris, 1970.

[75] J.-M. Souriau. Physique et Geometrie. *Preprint CTP-81/P.1298*, 1982.

[76] P. J. Morrison. Hamiltonian Field Description of the Two-Dimensional Vortex Fluids and Guiding Center Plasmas. *Princeton Plasma Physics Laboratory Report PPPL-1783*, 1981.

[77] P. J. Morrison. Hamiltonian Field Description of the One-Dimensional Poisson-Vlasov Equation. *Princeton Plasma Physics Laboratory Report PPPL-1788*, 1981.

[78] J. Batt, P. J. Morrison, and G. Rein. Linear stability of stationary solutions of the Vlasov-Poisson system in three dimensions. *Arch. Rational Mech. Anal.*, 130:163–182, 1995.

[79] Z. Yoshida, S. Ohsaki, A. Ito, and S. M. Mahajan. Stability of Beltrami flows. *J. Math. Phys.*, 44:2168–2178, 2003.

[80] R. G. Littlejohn. Singular Poisson tensors. In M. Tabor and Y. Treve, editors, *Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems*, volume 88, pages 47–66. New York: Am. Inst. Phys., 1982.

[81] J.-L. Thiffeault and P. J. Morrison. Classification and Casimir invariants of Lie-Poisson brackets. *Physica D*, 136:205–244, 2000.

[82] P. G. L. Dirichlet. *Crelle*, 32:3, 1846.

[83] P. J. Morrison. Thoughts on brackets and dissipation: Old and new. *J. Phys. Conf. Ser.*, 169(1):012006, 2009.

[84] Z. Yoshida, P. J. Morrison, and F. Dobarro. Singular Casimir Elements of the Euler Equation and Equilibrium Points. *J. Math. Fluid Mech.*, 16:41–57, 2014.

[85] Z. Yoshida and P. J. Morrison. A hierarchy of noncanonical Hamiltonian systems: circulation laws in an extended phase space. *Fluid Dyn. Res.*, 46(3):031412, 2014.

[86] T. Andreussi, P. J. Morrison, and F. Pegoraro. Hamiltonian magnetohydrodynamics: Helically symmetric formulation, Casimir invariants, and equilibrium variational principles. *Phys. Plasmas*, 19(5):052102, 2012.

[87] T. Andreussi, P. J. Morrison, and F. Pegoraro. Hamiltonian magnetohydrodynamics: Lagrangian, Eulerian, and dynamically accessible stability–Theory. *Phys. Plasmas*, 20(9):092104, 2013.

[88] E. Tassi, P. J. Morrison, F. L. Waelbroeck, and D. Grasso. Hamiltonian formulation and analysis of a collisionless fluid reconnection model. *Plasma Phys. Controlled Fusion*, 50(8):085014, 2008.

[89] E. Tassi and P. J. Morrison. Mode signature and stability for a Hamiltonian model of electron temperature gradient turbulence. *Phys. Plasmas*, 18(3):032115, 2011.

[90] P. J. Morrison, E. Tassi, and N. Tronko. Stability of compressible reduced magnetohydrodynamic equilibria–Analogy with magnetorotational instability. *Phys. Plasmas*, 20(4):042109, 2013.

[91] S. M. Moawad. Linear and nonlinear stability criteria for compressible MHD flows in a gravitational field. *J. Plasma Phys.*, 79:873–883, 2013.

[92] P. J. Morrison, E. Tassi, and C. Tronci. Energy stability analysis for a hybrid fluid-kinetic plasma model. In O. Kirillov and D. Pelinovsky, editors, *Nonlinear Physical Systems – Spectral Analysis, Stability and Bifurcations*, pages 311–329, 2014.

[93] J. P. Freidberg. Ideal magnetohydrodynamic theory of magnetic fusion systems. *Rev. Mod. Phys.*, 54:801–902, 1982.

[94] S. I. Braginskii. Transport Phenomena in a Completely Ionized Two-Temperature Plasma. *Sov. Phys. JETP*, 6:358, 1958.

[95] K. V. Roberts and J. B. Taylor. Magnetohydrodynamic Equations for Finite Larmor Radius. *Phys. Rev. Lett.*, 8:197–198, 1962.

[96] M. N. Rosenbluth and A. Simon. Finite Larmor Radius Equations with Nonuniform Electric Fields and Velocities. *Phys. Fluids*, 8:1300–1322, 1965.

[97] R. D. Hazeltine and J. D. Meiss. *Plasma Confinement*. Addison-Wesley, New York, 1992.

[98] F. L. Hinton and R. D. Hazeltine. Theory of plasma transport in toroidal confinement systems. *Rev. Mod. Phys.*, 48:239–308, 1976.

[99] S. A. Balbus and J. F. Hawley. Instability, turbulence, and enhanced transport in accretion disks. *Rev. Mod. Phys.*, 70:1–53, 1998.

[100] S. A. Balbus. Enhanced Angular Momentum Transport in Accretion Disks. *Annu. Rev. Astron. Astrophys.*, 41:555–597, 2003.

[101] N. M. Ferraro. Finite Larmor Radius Effects on the Magnetorotational Instability. *Astrophys. J.*, 662:512–516, 2007.

[102] E. Devlen and E. R. Pekünlü. Finite Larmor radius effects on weakly magnetized, dilute plasmas. *Mon. Not. Roy. Ast. Soc.*, 404:830–836, 2010.

[103] T. Heinemann and E. Quataert. Linear Vlasov Theory in the Shearing Sheet Approximation with Application to the Magneto-rotational Instability. *Astrophys. J.*, 792:70, 2014.

[104] S. A. Balbus. Viscous Shear Instability in Weakly Magnetized, Dilute Plasmas. *Astrophys. J.*, 616:857–864, 2004.

[105] T. Islam and S. Balbus. Dynamics of the Magnetoviscous Instability. *Astrophys. J.*, 633:328–333, 2005.

[106] R. Dong and J. M. Stone. Buoyant Bubbles in Intracluster Gas: Effects of Magnetic Fields and Anisotropic Viscosity. *Astrophys. J.*, 704:1309–1320, 2009.

[107] I. J. Parrish, M. McCourt, E. Quataert, and P. Sharma. The effects of anisotropic viscosity on turbulence and heat transport in the intracluster medium. *Mon. Not. Roy. Ast. Soc.*, 422:704–718, 2012.

[108] A. J. Brizard and T. S. Hahm. Foundations of nonlinear gyrokinetic theory. *Rev. Mod. Phys.*, 79:421–468, 2007.

[109] A. A. Schekochihin, S. C. Cowley, W. Dorland, G. W. Hammett, G. G. Howes, E. Quataert, and T. Tatsuno. Astrophysical Gyrokinetics: Ki-

netic and Fluid Turbulent Cascades in Magnetized Weakly Collisional Plasmas. *Astrophys. J.*, 182:310–377, 2009.

[110] A. Hasegawa and M. Wakatani. Finite-Larmor-radius magnetohydrodynamic equations for microturbulence. *Phys. Fluids*, 26:2770–2772, 1983.

[111] C. T. Hsu, R. D. Hazeltine, and P. J. Morrison. A generalized reduced fluid model with finite ion-gyroradius effects. *Phys. Fluids*, 29:1480–1487, 1986.

[112] A. Brizard. Nonlinear gyrofluid description of turbulent magnetized plasmas. *Phys. Fluids B*, 4:1213–1228, 1992.

[113] E. V. Belova. Nonlinear gyroviscous force in a collisionless plasma. *Phys. Plasmas*, 8:3936–3944, 2001.

[114] R. D. Hazeltine, M. Kotschenreuther, and P. J. Morrison. A four-field model for tokamak plasma dynamics. *Phys. Fluids*, 28:2466–2477, 1985.

[115] W. Dorland and G. W. Hammett. Gyrofluid turbulence models with kinetic effects. *Phys. Fluids B*, 5:812–835, 1993.

[116] M. A. Beer and G. W. Hammett. Toroidal gyrofluid equations for simulations of tokamak turbulence. *Phys. Plasmas*, 3:4046–4064, 1996.

[117] P. B. Snyder, G. W. Hammett, and W. Dorland. Landau fluid models of collisionless magnetohydrodynamics. *Phys. Plasmas*, 4:3974–3985, 1997.

[118] P. B. Snyder and G. W. Hammett. A Landau fluid model for electromagnetic plasma microturbulence. *Phys. Plasmas*, 8:3199–3216, 2001.

[119] G. M. Staebler, J. E. Kinsey, and R. E. Waltz. Gyro-Landau fluid equations for trapped and passing particles. *Phys. Plasmas*, 12(10):102508, 2005.

[120] J. J. Ramos. Fluid formalism for collisionless magnetized plasmas. *Phys. Plasmas*, 12(5):052102, 2005.

[121] J. J. Ramos. General expression of the gyroviscous force. *Phys. Plasmas*, 12(11):112301, 2005.

[122] J. J. Ramos. Fluid theory of magnetized plasma dynamics at low collisionality. *Phys. Plasmas*, 14(5):052506, 2007.

[123] T. Passot and P. L. Sulem. Collisionless magnetohydrodynamics with gyrokinetic effects. *Phys. Plasmas*, 14(8):082502, 2007.

[124] P. J. Morrison, I. L. Caldas, and H. Tasso. Hamiltonian Formulation of Two-Dimensional Gyroviscous MHD. *Z. Naturforsch. Teil A*, 39:1023, 1984.

[125] A. J. Brizard, R. E. Denton, B. Rogers, and W. Lotko. Nonlinear finite-Larmor-radius effects in reduced fluid models. *Phys. Plasmas*, 15(8):082302, 2008.

[126] F. L. Waelbroeck, R. D. Hazeltine, and P. J. Morrison. A Hamiltonian electromagnetic gyrofluid model. *Phys. Plasmas*, 16(3):032109, 2009.

[127] O. Izacard, C. Chandre, E. Tassi, and G. Ciraolo. Gyromap for a two-dimensional Hamiltonian fluid model derived from Braginskii's closure for magnetized plasmas. *Phys. Plasmas*, 18(6):062105, 2011.

[128] F. L. Waelbroeck and E. Tassi. A compressible Hamiltonian electromagnetic gyrofluid model. *Commun. Nonlinear Sci. Numer. Simulat.*, 17:2171–2178, 2012.

[129] S. Chapman and T. G. Cowling. *The mathematical theory of non-uniform gases. an account of the kinetic theory of viscosity, thermal conduction and diffusion in gases*. Cambridge University Press, 1953.

[130] A. N. Kaufman. Plasma Viscosity in a Magnetic Field. *Phys. Fluids*, 3:610–616, 1960.

[131] J. E. Avron, R. Seiler, and P. G. Zograf. Viscosity of Quantum Hall Fluids. *Phys. Rev. Lett.*, 75:697–700, 1995.

[132] P. Wiegmann and A. G. Abanov. Anomalous Hydrodynamics of Two-Dimensional Vortex Fluids. *Phys. Rev. Lett.*, 113(3):034501, 2014.

[133] M. Lingam. Hall viscosity: a link between quantum Hall systems, plasmas and liquid crystals. *Phys. Lett. A*, 379:1425–1430, 2015.

[134] J.-L. Thiffeault and P. J. Morrison. Invariants and Labels in Lie-Poisson Systems. *Ann. New York Acad. Sci.*, 867:109, 1998.

[135] J. L. Johnson, C. R. Oberman, R. M. Kulsrud, and E. A. Frieman. Some Stable Hydromagnetic Equilibria. *Phys. Fluids*, 1:281–296, 1958.

[136] N. A. Krall and A. W. Trivelpiece. *Principles of plasma physics*. McGraw Hill, New York, 1973.

[137] E. Frieman and M. Rotenberg. On Hydromagnetic Stability of Stationary Equilibria. *Rev. Mod. Phys*, 32:898–902, 1960.

[138] H. Poincaré. *C. R. Acad. Sci.*, 132:369–371, 1901.

[139] H. Stark and T. C. Lubensky. Poisson bracket approach to the dynamics of nematic liquid crystals: The role of spin angular momentum. *Phys. Rev. E*, 72(5):051714, 2005.

[140] H. Stark and T. C. Lubensky. Poisson-bracket approach to the dynamics of nematic liquid crystals. *Phys. Rev. E*, 67(6):061709, 2003.

[141] T. C. Lubensky. Molecular Description of Nematic Liquid Crystals. *Phys. Rev. A*, 2:2497–2514, 1970.

[142] P. C. Martin, O. Parodi, and P. S. Pershan. Unified Hydrodynamic Theory for Crystals, Liquid Crystals, and Normal Fluids. *Phys. Rev. A*, 6:2401–2420, 1972.

[143] M. J. Stephen and J. P. Straley. Physics of liquid crystals. *Rev. Mod. Phys.*, 46:617–704, 1974.

[144] J. L. Ericksen. Liquid crystals with variable degree of orientation. *Arch. Rational Mech. Anal.*, 113:97–120, 1991.

[145] F. Gay-Balmaz, T. S. Ratiu, and C. Tronci. Equivalent Theories of Liquid Crystal Dynamics. *Arch. Rational Mech. Anal.*, 210:773–811, 2013.

[146] L. E. Reichl. *A Modern Course in Statistical Physics, 2nd Edition.* Wiley-VCH, 1998.

[147] F. H. Shu, F. C. Adams, and S. Lizano. Star formation in molecular clouds - Observation and theory. *Annu. Rev. Astron. Astrophys.*, 25:23–81, 1987.

[148] C. F. McKee and E. C. Ostriker. Theory of Star Formation. *Annu. Rev. Astron. Astrophys.*, 45:565–687, 2007.

[149] C. Truesdell and W. Noll. *The Non-Linear Field Theories of Mechanics.* Springer Verlag, 1965.

[150] G. Astarita and G. Marrucci. *Principles of Non-Newtonian Fluid Mechanics.* McGraw-Hill Inc., 1974.

[151] S. Das and S. Chakraborty. Analytical solutions for velocity, temperature and concentration distribution in electroosmotic microchannel flows in a non-Newtonian bio-fluid. *Anal. Chim. Acta*, 559:15–24, 2006.

[152] C. L. A. Berli and M. L. Olivares. Electrokinetic flow of non-Newtonian fluids in microchannels. *J Colloid Interface Sci.*, 320:582–589, 2008.

[153] S. Forest. Micromorphic approach for gradient elasticity, viscoplasticity, and damage. *J. Eng. Mech.*, 135:117–131, 2009.

[154] I. Keramidas Charidakos, M. Lingam, P. J. Morrison, R. L. White, and A. Wurm. Action Principles for Extended MHD Models. *Phys. Plasmas*, 21(9):092118, 2014.

[155] R. Lüst. Über die Ausbreitung von Wellen in einem Plasma. *Fortschr. Phys.*, 7:503–558, 1959.

[156] P. H. Roberts. *An Introduction to Magnetohydrodynamics.* Longmans, Green and Co ltd, London, 1967.

[157] V. I. Ilgisonis and V. P. Lakhin. Lagrangean structure of hydrodynamic plasma models and conservation laws. *Plasma Phys. Rep.*, 25:58–69, 1999.

[158] R. G. Littlejohn. Variational principles of guiding centre motion. *J. Plasma Phys.*, 29:111–125, 1983.

[159] M. Hirota, Z. Yoshida, and E. Hameiri. Variational principle for linear stability of flowing plasmas in Hall magnetohydrodynamics. *Phys. Plasmas*, 13(2):022107, 2006.

[160] Z. Yoshida and E. Hameiri. Canonical Hamiltonian mechanics of Hall magnetohydrodynamics and its limit to ideal magnetohydrodynamics. *J. Phys. A*, 46:G5502.

[161] M. J. Lighthill. Studies on Magneto-Hydrodynamic Waves and other Anisotropic Wave Motions. *Proc. Roy. Soc. London, Ser. A*, 252:397–430, 1960.

[162] S. M. Mahajan and Z. Yoshida. Double Curl Beltrami Flow: Diamagnetic Structures. *Phys. Rev. Lett.*, 81:4863–4866, 1998.

[163] P. Goldreich and A. Reisenegger. Magnetic field decay in isolated neutron stars. *Astrophys. J.*, 395:250–258, 1992.

[164] M. Wardle. Magnetic fields in protoplanetary disks. *Astrophys. Space Sci.*, 311:35–45, 2007.

[165] A. V. Gordeev, A. S. Kingsep, and L. I. Rudakov. Electron magnetohydrodynamics. *Phys. Rep.*, 243:215–315, 1994.

[166] A. J. Brizard. Noether derivation of exact conservation laws for dissipationless reduced-fluid models. *Phys. Plasmas*, 17(11):112503, 2010.

[167] M. Lingam, P. J. Morrison, and E. Tassi. Inertial magnetohydrodynamics. *Phys. Lett. A*, 379:570–576, 2015.

[168] E. Priest and T. Forbes. *Magnetic reconnection: MHD theory and applications*. Cambridge University Press, 2000.

[169] R. M. Kulsrud. *Plasma physics for astrophysics.* Princeton University Press, 2005.

[170] E. G. Zweibel and M. Yamada. Magnetic Reconnection in Astrophysical and Laboratory Plasmas. *Annu. Rev. Astron. Astrophys.*, 47:291–332, 2009.

[171] M. Yamada, R. Kulsrud, and H. Ji. Magnetic reconnection. *Rev. Mod. Phys.*, 82:603–664, 2010.

[172] J. Birn, J. F. Drake, M. A. Shay, B. N. Rogers, R. E. Denton, M. Hesse, M. Kuznetsova, Z. W. Ma, A. Bhattacharjee, A. Otto, and P. L. Pritchett. Geospace Environmental Modeling (GEM) magnetic reconnection challenge. *J. Geophys. Res.*, 106:3715–3720, 2001.

[173] A. Bhattacharjee. Impulsive Magnetic Reconnection in the Earth's Magnetotail and the Solar Corona. *Annu. Rev. Astron. Astrophys.*, 42:365–384, 2004.

[174] D. Biskamp, E. Schwarz, and J. F. Drake. Ion-controlled collisionless magnetic reconnection. *Phys. Rev. Lett.*, 75:3850–3853, 1995.

[175] X. Wang, A. Bhattacharjee, and Z. W. Ma. Collisionless reconnection: Effects of Hall current and electron pressure gradient. *J. Geophys. Res.*, 105:27633–27648, 2000.

[176] M. A. Shay, J. F. Drake, B. N. Rogers, and R. E. Denton. Alfvénic collisionless magnetic reconnection and the Hall term. *J. Geophys. Res.*, 106:3759–3772, 2001.

[177] P. L. Pritchett. Collisionless magnetic reconnection in a three-dimensional open system. *J. Geophys. Res.*, 106:25961–25978, 2001.

[178] X. Wang, A. Bhattacharjee, and Z. W. Ma. Scaling of Collisionless Forced Reconnection. *Phys. Rev. Lett.*, 87(26):265003, 2001.

[179] J. F. Drake, M. A. Shay, and M. Swisdak. The Hall fields and fast magnetic reconnection. *Phys. Plasmas*, 15(4):042306, 2008.

[180] Y.-M. Huang, A. Bhattacharjee, and B. P. Sullivan. Onset of fast reconnection in Hall magnetohydrodynamics mediated by the plasmoid instability. *Phys. Plasmas*, 18(7):072109, 2011.

[181] D. Biskamp, E. Schwarz, and J. F. Drake. Two-fluid theory of collisionless magnetic reconnection. *Phys. Plasmas*, 4:1002–1009, 1997.

[182] M. A. Shay, J. F. Drake, R. E. Denton, and D. Biskamp. Structure of the dissipation region during collisionless magnetic reconnection. *J. Geophys. Res.*, 103:9165–9176, 1998.

[183] M. Hesse and D. Winske. Electron dissipation in collisionless magnetic reconnection. *J. Geophys. Res.*, 103:26479–26486, 1998.

[184] M. A. Shay, J. F. Drake, B. N. Rogers, and R. E. Denton. The scaling of collisionless, magnetic reconnection for large systems. *Geophys. Res. Lett.*, 26:2163–2166, 1999.

[185] M. A. Shay, J. F. Drake, M. Swisdak, and B. N. Rogers. The scaling of embedded collisionless reconnection. *Phys. Plasmas*, 11:2199–2213, 2004.

[186] B. N. Kuvshinov, F. Pegoraro, and T. J. Schep. Hamiltonian formulation of low-frequency, nonlinear plasma dynamics. *Phys. Lett. A*, 191:296–300, 1994.

[187] T. J. Schep, F. Pegoraro, and B. N. Kuvshinov. Generalized two-fluid theory of nonlinear magnetic structures. *Phys. Plasmas*, 1:2843–2852, 1994.

[188] E. Cafaro, D. Grasso, F. Pegoraro, F. Porcelli, and A. Saluzzi. Invariants and Geometric Structures in Nonlinear Hamiltonian Magnetic Reconnection. *Phys. Rev. Lett.*, 80:4430–4433, 1998.

[189] D. Grasso, F. Pegoraro, F. Porcelli, and F. Califano. Hamiltonian magnetic reconnection. *Plasma Phys. Controlled Fusion*, 41:1497–1515, 1999.

[190] E. Tassi, F. L. Waelbroeck, and D. Grasso. Gyrofluid simulations of collisionless reconnection in the presence of diamagnetic effects. *J. Phys. Conf. Ser.*, 260(1):012020, 2010.

[191] D. Grasso, E. Tassi, and F. L. Waelbroeck. Nonlinear gyrofluid simulations of collisionless reconnection. *Phys. Plasmas*, 17(8):082312, 2010.

[192] E. Tassi, P. J. Morrison, D. Grasso, and F. Pegoraro. Hamiltonian four-field model for magnetic reconnection: nonlinear dynamics and extension to three dimensions with externally applied fields. *Nuc. Fusion*, 50(3):034007, 2010.

[193] D. Grasso, F. L. Waelbroeck, and E. Tassi. Ion diamagnetic effects in gyrofluid collisionless magnetic reconnection. *J. Phys. Conf. Ser.*, 401(1):012008, 2012.

[194] L. Comisso, D. Grasso, F. L. Waelbroeck, and D. Borgogno. Gyro-induced acceleration of magnetic reconnection. *Phys. Plasmas*, 20(9):092118, 2013.

[195] E. Tassi. Hamiltonian derivation of a gyrofluid model for collisionless magnetic reconnection. *J. Phys. Conf. Ser.*, 561(1):012018, 2014.

[196] L. Spitzer. *Physics of Fully Ionized Gases*. Interscience, New York, 1956.

[197] R. Fitzpatrick and F. Porcelli. Collisionless magnetic reconnection with arbitrary guide field. *Phys. Plasmas*, 11:4713–4718, 2004.

[198] R. Fitzpatrick. Magnetic reconnection in weakly collisional highly magnetized electron-ion plasmas. *Phys. Plasmas*, 17(4):042101, 2010.

[199] H. M. Abdelhamid, Y. Kawazura, and Z. Yoshida. Hamiltonian Formalism of Extended Magnetohydrodynamics. *arXiv: 1412.6228*, 2014.

[200] R. G. Spencer and A. N. Kaufman. Hamiltonian structure of two-fluid plasma dynamics. *Phys. Rev. A*, 25:2437–2439, 1982.

[201] R. D. Hazeltine and J. D. Meiss. Shear-Alfvén dynamics of toroidally confined plasmas. *Phys. Rep.*, 121:1–164, 1985.

[202] M. Yagi and W. Horton. Reduced Braginskii equations. *Phys. Plasmas*, 1:2135–2139, 1994.

[203] C. Tronci and E. Camporeale. Neutral Vlasov kinetic theory of magnetized plasmas. *Phys. Plasmas*, 22(2):020704, 2015.

[204] C. Tronci. Hamiltonian approach to hybrid plasma models. *J. Phys. A*, 43:K5501, 2010.

[205] L. Wang, A. H. Hakim, A. Bhattacharjee, and K. Germaschewski. Comparison of multi-fluid moment models with particle-in-cell simulations of collisionless magnetic reconnection. *Phys. Plasmas*, 22(1):012108, 2015.

[206] A. Lichnerowicz. *Relativistic Hydrodynamics and Magnetohydrodynamics*. Benjamin, 1967.

[207] A. M. Anile. *Relativistic fluids and magneto-fluids : with applications in astrophysics and plasma physics*. Cambridge University Press, 1989.