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Splitting of Logarithmic Gromov-Witten Invariants Under Degeneration

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Splitting of Logarithmic Gromov-Witten Invariants Under Degeneration

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Splitting of Logarithmic Gromov-Witten Invariants Under Degeneration

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In this dissertation, I study the theory of punctured Gromov-Witten invariants built by Abramovich, Chen, Gross and Siebert. I prove a splitting formula that reconstructs the logarithmic Gromov-Witten invariants of simple normal crossing varieties from the punctured Gromov-Witten invariants of their irreducible components, under the assumption of the gluing strata being toric varieties.

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Chapter 1

Introduction

Relative Gromov-Witten invariants of a smooth projective variety Y and a smooth divisor D, developed in [15][12][16][17], has been one of the most important techniques to calculate Gromov-Witten invariants. For a degenerating family of projective schemes $X \to B$ with general fiber over $b \in B$ a smooth variety X_b and the central fiber X_0 the union of two smooth irreducible components Y_1, Y_2 meeting along a smooth divisor D, a degeneration formula is obtained to relate the Gromov-Witten invariants of X_b with the relative Gromov-Witten invariants of (Y_1, D) and (Y_2, D).

Recently, logarithmic Gromov-Witten theory developed in [11] [8] [2] has been proved to be a successful generalization of the relative Gromov-Witten theory to the case of D being a normal crossing divisor of Y. Especially, for a degenerating family with central fiber X_0 a normal crossing variety, a decomposition formula is obtained in [3] that relates the Gromov-Witten invariants of X_b with the logarithmic Gromov-Witten invariants on X_0 of rigid decorated *tropical types*. The rigid decorated *tropical types* $\boldsymbol{\tau}$ restrict the combinatorics of the maps, including the dual intersection graphs, the image cones of irreducible components, marked and nodal points, the contact orders and the curve classes.

To further decompose the logarithmic Gromov-Witten invariants of X_0 of type

$\boldsymbol{\tau}$ to the invariants of irreducible components of X_0, the theory of punctured Gromov-Witten invariants is built in [4]. Punctured Gromov-Witten theory studies logarithmic maps with domain being punctured logarithmic curves, which naturally occur after splitting log smooth curves along nodal points. The combinatorics of the split maps are encoded in tropical sub-types $\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_r$. There is a natural splitting morphism

$$\mathscr{M}(X/B, \boldsymbol{\tau}) \to \prod_{i=1}^{r} \mathscr{M}(X/B, \boldsymbol{\tau}_i).$$

In this paper, we prove an explicit formula (Theorem 1.1.5) presenting the virtual fundamental class of $\mathscr{M}(X/B, \boldsymbol{\tau})$ under splitting as the products of the strata of $\mathscr{M}(X/B, \boldsymbol{\tau}_i)$ associated to $\boldsymbol{\tau}_i$-marked tropical types, under the assumption that the gluing strata are toric varieties whose log stratifications are the same as the toric stratifications. A numerical splitting formula of logarithmic Gromov-Witten invariants (Corollary 1.1.6) is obtained as a direct corollary.

1.1 The main results

Let B be a log point $(\operatorname{Spec} \Bbbk, \mathcal{M}_B)$, whose log structure is determined by a chart $Q_B \to \Bbbk$ with Q_B a toric monoid. Let $X \to B$ be a projective log smooth morphism between log schemes with Zariski log structures. Let β be a curve class in X.

The moduli space $\mathscr{M}(X/B, \boldsymbol{\tau})$ of basic stable punctured maps marked by a global decorated type $\boldsymbol{\tau}$ is a logarithmic algebraic stack ([4, Thm A]). The tropicalization of $\mathscr{M}(X/B, \boldsymbol{\tau})$ is locally determined by the tropical types of the maps. For a geometric point in $\mathscr{M}(X/B, \boldsymbol{\tau})$ with tropical type $\boldsymbol{\omega}$, there is an *associated basic cone* $\overline{\boldsymbol{\omega}}$ (Definition 2.2.2) of $\boldsymbol{\omega}$ parametrizing the tropical maps of type $\boldsymbol{\omega}$. Supposing $\overline{x}'$ is a geometric point lying in the closure of $\overline{x}$, there is a canonical *contraction morphism* (Definition 2.2.7) from $\boldsymbol{\omega}'$ to $\boldsymbol{\omega}$, with $\boldsymbol{\omega}'$ the tropical type associated to $\overline{x}'$. The contraction morphism induces an inclusion of the associated basic cone $\overline{\boldsymbol{\omega}}$ as a face of $\overline{\boldsymbol{\omega}}'$. The tropicalization of $\mathscr{M}(X/B,\boldsymbol{\tau})$ is defined to be the colimit of the basic cones over the geometric points under the above maps.

In order to define logarithmic evaluation maps, we need to modify the log structure on $\mathscr{M}(X/B,\boldsymbol{\tau})$ based on the set $\mathbf{S}$ of nodal and punctured points where we evaluate at (Section 2.2). The tropicalizations of the modified moduli spaces are now determined by the *associated evaluation cones* $\widetilde{\boldsymbol{\omega}}_{\mathbf{S}}$ (Definition 2.2.2), parametrizing the tropical maps with type $\boldsymbol{\omega}$ together with a marking on each edge and leg corresponding to points in $\mathbf{S}$. There is a tropical evaluation map by taking the evaluations at the markings

$$\mathrm{evt}_{\boldsymbol{\omega}} : \widetilde{\boldsymbol{\omega}}_{\mathbf{S}} \to \prod_{p\in\mathbf{S}} \Sigma(X), \tag{1.1.1}$$

with $\Sigma(X)$ the tropicalization of X.

Splitting a logarithmic map along nodal points of the domain can be described easily using the tropical types. Fix a subset $\mathbf{S}$ of edges in the graph G of $\boldsymbol{\tau}$. Cutting along each edge $p \in \mathbf{S}$ results in a set of global decorated types $\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_r$ with $\mathbf{S}_1, ..., \mathbf{S}_r$ the set of additional half legs from the edges in each type. We use $\widetilde{\boldsymbol{\omega}} = \widetilde{\boldsymbol{\omega}}_{\mathbf{S}}$ and $\widetilde{\boldsymbol{\omega}}_i = \widetilde{\boldsymbol{\omega}}_{i,\mathbf{S}_i}$ to denote the associated evaluation cones of types $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_i$ marked by $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_i$.

Theorem 1.1.1. *[4, Thm C] There is a finite, representable morphism of moduli spaces of punctured log stable maps to X over B*

$$\delta : \mathscr{M}(X/B,\boldsymbol{\tau}) \to \prod_{i=1}^{r} \mathscr{M}(X/B,\boldsymbol{\tau}_i).$$

The reverse process of gluing punctured maps of type $\boldsymbol{\tau}_i$ requires both schematic and tropical matching conditions. Though in general complicated, under the case of the gluing strata being toric varieties, the gluings of the logarithmic maps are completely determined by the tropical information.

Assumption 1.1.2. Assume $X \to B$ is integral and $\overline{\mathcal{M}}_X$ is globally generated. Assume for each edge $p \in \mathbf{S}$, the strict closed subscheme $V_p := V_X(\boldsymbol{\sigma}(p))$ of the log scheme X associated to the cone $\boldsymbol{\sigma}(p)$ has the underlying scheme a toric variety and the log stratification of V_p is the same as the toric stratification.

Lemma 1.1.3. *(Theorem 4.1.1) There exists a toric variety X_p associated to the fan (Σ_p, N_p) with canonical toric log structure, such that V_p is isomorphic to a toric stratum of X_p.*

For curves of types $\boldsymbol{\omega}_i$, the tropical matching condition is a fiber diagram of cones

$$\begin{array}{ccc} \left(\prod_{i=1}^r \varepsilon_{\boldsymbol{\omega}_i}\right)^{-1}(0) & \longrightarrow & \prod_{i=1}^r \widetilde{\boldsymbol{\omega}}_i \\ \big\downarrow & & \big\downarrow {\scriptstyle \prod \varepsilon_{\boldsymbol{\omega}_i}} \\ 0 & \longrightarrow & \prod_{p\in \mathbf{S}} N_{p,\mathbb{R}}, \end{array}$$

with

$$\prod_{i=1}^r \varepsilon_{\boldsymbol{\omega}_i} : \prod_{i=1}^r \widetilde{\boldsymbol{\omega}}_i \xrightarrow{\prod \mathrm{evt}_{\boldsymbol{\omega}_i}} \prod_{p\in\mathbf{S}} N_{p,\mathbb{R}} \times N_{p,\mathbb{R}} \xrightarrow{\prod \operatorname{coker} \overline{\Delta}_p} \prod_{p\in\mathbf{S}} N_{p,\mathbb{R}}. \tag{1.1.2}$$

Here, the second map is the cokernel of the diagonal map. The map (1.1.2) tells the difference by evaluating at two half edges after splitting. Instead of requiring the evaluations to be matched along split edges, we introduce *generic displacement vectors* and require the maps to be matched after the perturbation along this vector.

The minimal types satisfying the new matching conditions are exactly the strata of $\prod_{i=1}^{r} \mathscr{M}(X/B, \boldsymbol{\tau})$ rationally equivalent to $\delta(\mathscr{M}(X/B, \boldsymbol{\tau}))$. The idea is inspired by the intersection theory of toric varieties in [9].

Definition 1.1.4. 1. A vector $\mathfrak{V} \in \prod_{p\in\mathbf{S}} N_p$ is *a displacement vector* if $\mathfrak{V}$ lies in the sublattice

$$N_{p_1} \times_{N_B} \dots \times_{N_B} N_{p_{|\mathbf{S}|}} \subseteq \prod_{p\in\mathbf{S}} N_p,$$

where the map $N_{p_i} \to N_B$ is induced by the tropicalization of the map $X_{p_i} \to B$.

2. For a displacement vector $\mathfrak{V}$, define $\Delta(\mathfrak{V})$ to be the set of types $[\boldsymbol{\rho}] = (\boldsymbol{\rho}_1, \dots, \boldsymbol{\rho}_r)$ such that

   (i) $\boldsymbol{\rho}_i$ admits a contraction morphism to $\boldsymbol{\tau}_i$, for $i = 1, \dots, n$,

   (ii) $\mathfrak{V} \in \mathrm{im}(\prod_{i=1}^{r} \varepsilon_{\boldsymbol{\rho}_i})$, for $\prod_{i=1}^{r} \varepsilon_{\boldsymbol{\rho}_i}$ defined in (1.1.2) and

   (iii)

$$\sum_{i=1}^{r} \dim \widetilde{\boldsymbol{\rho}}_i - \dim \widetilde{\boldsymbol{\tau}} = \sum_{p\in\mathbf{S}} \dim N_p - (|\mathbf{S}| - r + 1) \cdot \mathrm{rank}\, Q_B^{\mathrm{gp}}.$$

   By condition (i), the types in $\Delta(\mathfrak{V})$ determine strata in $\prod_{i=1}^{r} \mathscr{M}(X/B, \boldsymbol{\tau}_i)$. Condition (ii) requires the existence of tropical maps with type $(\boldsymbol{\rho}_1, \dots \boldsymbol{\rho}_r)$ that match along splitting edges after the perturbation along $\mathfrak{V}$. Condition (iii) requires the types to have expected virtual dimension.

3. A displacement vector $\mathfrak{V}$ is *generic* if for any type $[\boldsymbol{\rho}] \in \Delta(\mathfrak{V})$, the map $\prod_{i=1}^{r} \varepsilon_{\boldsymbol{\rho}_i}$ is injective and $\mathfrak{V}$ lies in the interior of the image cone $\mathrm{im}(\prod_{i=1}^{r} \varepsilon_{\boldsymbol{\rho}_i})$.

4. For each $[\boldsymbol{\rho}] \in \Delta(\mathfrak{V})$, we define the multiplicity

$$m_{[\boldsymbol{\rho}]} = \big[\,\mathrm{im}(\prod_{i=1}^{r} \overline{\varepsilon}_{\boldsymbol{\rho}_i})^{\mathrm{sat}} : \mathrm{im}(\prod_{i=1}^{r} \overline{\varepsilon}_{\boldsymbol{\rho}_i})\big],$$

where $\overline{\varepsilon}_{\boldsymbol{\rho}_i}$ is the lattice map associated to $\varepsilon_{\boldsymbol{\rho}_i}$ and $\mathrm{im}(\prod_{i=1}^{r} \overline{\varepsilon}_{\boldsymbol{\rho}_i})^{\mathrm{sat}}$ is the saturation of the sublattice $\mathrm{im}(\prod_{i=1}^{r} \overline{\varepsilon}_{\boldsymbol{\rho}_i})$ in $\prod_{p \in \mathbf{S}} N_p$.

Now, we are ready to state the main result:

Theorem 1.1.5. *Let X be a logarithmic smooth projective scheme over a log point $B = \mathrm{Spec}(Q_B \to \Bbbk)$, with Q_B a toric monoid. Let $\boldsymbol{\tau}$ be a decorated global tropical type. Fix a set of the splitting edges $\mathbf{S}$ and let $\boldsymbol{\tau}_1, ...\boldsymbol{\tau}_r$ be the decorated global types obtained after splitting.*

Suppose the Assumption 1.1.2 is satisfied. Let $\mathfrak{V}$ be a generic displacement vector defined in Definition 1.1.4. Then, for the finite, representable morphism of moduli spaces of punctured stable log maps

$$\delta : \mathscr{M}(X/B, \boldsymbol{\tau}) \to \prod_{i=1}^{r} \mathscr{M}(X/B, \boldsymbol{\tau}_i),$$

the following equation holds

$$\delta_*[\mathscr{M}(X/B, \boldsymbol{\tau})]^{\mathrm{virt}} = \sum_{[\boldsymbol{\rho}] \in \Delta(\mathfrak{V})} \prod_{i=1}^{r} \frac{m_{[\boldsymbol{\rho}]}}{|\,\mathrm{Aut}(\boldsymbol{\rho}_i/\boldsymbol{\tau}_i)|} \cdot j_{\boldsymbol{\rho}_i \boldsymbol{\tau}_i *}[\mathscr{M}(X/B, \boldsymbol{\tau}_i)]^{\mathrm{virt}}, \qquad (1.1.3)$$

with $j_{\boldsymbol{\rho}_i \boldsymbol{\tau}_i}$ the finite morphism from $\mathscr{M}(X/B, \boldsymbol{\rho}_i)$ to $\mathscr{M}(X/B, \boldsymbol{\tau}_i)$ associated to the contraction morphism $\boldsymbol{\rho}_i \to \boldsymbol{\tau}_i$, and $\mathrm{Aut}(\boldsymbol{\rho}_i/\boldsymbol{\tau}_i)$ the automorphism group of $\boldsymbol{\rho}_i$ relative to $\boldsymbol{\tau}_i$.

A special case of Theorem 1.1.5 is the splitting of $\boldsymbol{\tau}$ at all edges. Then each split type $\boldsymbol{\tau}_i$ consists of one vertex with a number of legs, with the associated image stratum strictly smaller than the full target. In this case, (1.1.3) expresses the punctured invariants of type $\boldsymbol{\tau}$ in terms of punctured invariants of these logarithmic strata. For example, in a degeneration situation as in [3], this expresses the Gromov-Witten invariants of a general fiber in terms of the punctured invariants of the strata of the central fiber. Such localization to the strata does not follow from the general gluing formulas in [22], [4, Thm C] and [23].

A direct corollary of the above theorem is a numerical formula of logarithmic Gromov-Witten invariants.

Corollary 1.1.6. *Follow the situation in Theorem 1.1.5. Let $\mathbf{P}$ be a subset of legs in $\boldsymbol{\tau}$ and $\mathbf{P}_i$ be the subset of $\mathbf{P}$ that lies in $\boldsymbol{\tau}_i$ after splitting, for $i = 1, ..., n$. There are evaluation maps $e : \underline{\mathscr{M}(X/B, \boldsymbol{\tau})} \to \underline{X}^{|\mathbf{P}|}$ along punctured points in $\mathbf{P}$ and $e_{\boldsymbol{\rho}_i} : \underline{\mathscr{M}_{\boldsymbol{\rho}_i}(X/B, \boldsymbol{\tau}_i)} \to \underline{X}^{|\mathbf{P}_i|}$ along punctured points $\mathbf{P}_i$, for $\boldsymbol{\rho}_i$ the $\boldsymbol{\tau}_i$-marked decorated types.*

Let $\beta \in H^*(X^{|\mathbf{P}|})$ be a cohomology class with a Künneth decomposition

$$\beta = \sum_{\mu} \alpha_\mu \cdot \beta_{\mu,1} \boxtimes ... \boxtimes \beta_{\mu,r},$$

where $\beta_{\mu,i} \in H^*(X^{|\mathbf{P}_i|})$, for $i = 1, ..., n$. Then,

$$\delta_* \Big[\int_{[\mathscr{M}(X/B,\boldsymbol{\tau})]^{\mathrm{virt}}} e^*(\beta) \Big] = \sum_{\mu} \sum_{[\boldsymbol{\rho}] \in \Delta(\mathfrak{V})} \alpha_\mu \cdot \prod_{i=1}^{r} \frac{m_{[\boldsymbol{\rho}]}}{|\operatorname{Aut}(\boldsymbol{\rho}_i/\boldsymbol{\tau}_i)|} \cdot \int_{[\mathscr{M}(X/B,\boldsymbol{\rho}_i)]^{\mathrm{virt}}} j^*_{\boldsymbol{\rho}_i \boldsymbol{\tau}_i} e^*_{\boldsymbol{\rho}_i}(\beta_{\mu,i}).$$

Proof. The claim is a direct result of Theorem 1.1.5, following the projection formula. ♠

1.2 Idea of the proof and structure of the paper

The foundation of the paper is based on the punctured Gromov-Witten invariants in [4]. In Chapter 2, we provide a brief review of punctured Gromov-Witten theory and the gluing formalism. We briefly cover the basic theory of the moduli spaces of punctured logarithmic maps and the virtual theory over the moduli of the maps to the relative Artin fans in Section 2.1. We study the evaluation log structures in Section 2.2. There are canonical *evaluation idealized structures* on the modified moduli spaces such that they are idealized log smooth (Proposition 2.2.5). In Section 2.3, we recall the gluing formalism studied in [4, §5.2]. It is shown in Proposition 2.3.2 that up to a reduction of the moduli spaces, it is sufficient to study the following the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}} & \xrightarrow{\delta^{\mathrm{ev}}_{\mathrm{red}}} & \prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}} \\ \Big\downarrow{\scriptstyle \mathrm{ev}} & & \Big\downarrow{\scriptstyle \prod \mathrm{ev}_{\boldsymbol{\tau}_i}} \\ X_{\boldsymbol{\tau}} & \xrightarrow{\Delta_X} & \prod_{i=1}^{r} X_{\boldsymbol{\tau}_i} \end{array} \tag{1.2.1}$$

following the fiber diagram (2.3.2).

Such diagram has nice properties. First, the moduli spaces $\prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}$ and the evaluation maps are both idealized log smooth (Prop 2.2.5, Cor 2.2.6). Hence, locally they admit charts of toric morphisms. Second, under Assumption 1.1.2, the gluing strata $X_{\boldsymbol{\tau}}$ and $X_{\boldsymbol{\tau}_i}$ have global toric structures. The global toric structures provide a canonical patching of the splitting formulas from the local charts.

Since the local gluings are toric, we review the intersection theory in toric varieties in Chapter 3 following [9]. We give the necessary generalization of the Fulton-Sturmfels formula to toric stacks in Corollary 3.0.6.

The local form of the splitting formula is explored in detail in the first two sections of Chapter 4. In Section 4.1, we study the structures of the gluing strata $X_{\boldsymbol{\tau}}$ and $X_{\boldsymbol{\tau}_i}$, based on the logarithmic fiber products of toric varieties studied in Appendix B. Each of them is a disjoint union of log schemes isomorphic with each other, denoted $Z_{\boldsymbol{\tau}}$ and $Z_{\boldsymbol{\tau}_i}$ correspondingly. In Section 4.2, we study the local chart of the gluing formalism (1.2.1). Étale locally, the moduli space $\prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}$ admits a smooth map to a quotient stack of a toric variety by an algebraic torus. The quotient stack $\mathcal{A}^{\mathrm{ev}}$ is an evaluation enhancement of the Artin cone defined in (4.2.5) and has a canonical evaluation map to the gluing strata $\prod_{i=1}^{r} X_{\boldsymbol{\tau}_i}$. By studying the gluing of $\mathcal{A}^{\mathrm{ev}}$ using the generalized Fulton-Sturmfels formula, we obtain the splitting formula for $\mathcal{A}^{\mathrm{ev}}$ in Lemma 4.2.2 and the local splitting formula in Proposition 4.2.1. In Section 4.3, we finish the proof of the global splitting formula 1.1.5 (Proof.4.2) by showing the splitting formula patches under a fixed generic displacement vector.

1.3 Other approaches

Relative Gromov-Witten invariants for smooth pairs (X, D), studied in [15][12][16][17], are defined through the moduli spaces of stable maps to expansions of X along D. The stable maps to the expansions are transverse, hence the degeneration formulas are obtained by gluing the underlying stable maps. Using the idea of expansion, the degeneration formulas for smooth pairs are studied using twisted stable maps in [5] and logarithmic stable maps in [13][7]. These different approaches are proved to be identical with logarithmic Gromov-Witten invariants for smooth pairs in [6]. In [14], Kim, Lho and Ruddat provided a proof of the gluing formula for logarithmic Gromov-Witten invariants for smooth pairs without expansions using logarithmic technique. Because of the transverse nature of the underlying tropical geometry, all these approaches come with splitting formulas according to strata similar to our Corollary 1.1.6.

Combining the idea of expanded degenerations and tropical geometry, Ranganathan showed a general gluing formula of log Gromov-Witten invariants in the normal crossing settings in [23]. The numerical degeneration formula there requires the knowledge of a Künneth decomposition of universal divisor expansions. We expect a similar splitting formula as we present can be obtained by proving an explicit Künneth formula for universal expansions of toric varieties.

The gluing and splitting formalism using punctured Gromov-Witten invariants has a symplectic parallel by the theory of exploded manifolds due to Brett Parker in [21] [22]. The concept of generic deformation vectors in [21] partially inspires our definition of the generic displacement vectors here. In a special case for rigid analytic Gromov-Witten invariants, a gluing formula has been proved by Yu [27].

1.4 Conventions

We follow the conventions in [3] and [4]. All logarithmic schemes and stacks are fine and defined of a field $\Bbbk$ over characteristic 0.

The affine log scheme with a global chart defined by a homomorphism $Q \to R$ from a monoid Q to a ring R is denoted $\operatorname{Spec}(Q \to R)$. For Q a toric monoid, we define $Q^\vee := \operatorname{Hom}(Q, \mathbb{N})$ and $Q^* := \operatorname{Hom}(Q, \mathbb{Z})$. We use S_Q to denote affine toric variety $\operatorname{Spec}(\Bbbk[Q])$ and T_Q to denote $\operatorname{Spec}(\Bbbk[Q^{\mathrm{gp}}])$. We define $\mathcal{A}_Q := [S_Q/T_Q]$ to be the

Artin cone of Q. Suppose $L \subseteq Q$ is an ideal of Q, then we use $S_{Q,K}$ to denote that subscheme of S_Q determined by the ideal generated by K. We use $\mathcal{A}_{Q,L}$ for the stack $[S_{Q,L}/T_Q]$.

For a toric variety X and a cone $\sigma \in \Sigma(X)$ in the fan of X, we use $V_X(\sigma)$ to denote the closed toric stratum associated σ and $\mathcal{O}_X(\sigma)$ for the open algebraic torus of $V_X(\sigma)$. For a Zariski log scheme X and a cone σ in the tropicalization of X, we use $V_X(\sigma)$ to denote the closed stratum whose dual cone of the stalk $\overline{\mathcal{M}}_X$ at the generic point of $V_X(\sigma)$ is σ. For a logarithmic stack X and a cone σ in the tropicalization of X, we use $V_X(\sigma)$ to denote the strict closed integral substack with pullback $V_W(\sigma)$ on each Zariski smooth chart $W \to X$.

For a proper, representable morphism between logarithmic stacks $f : X \to Y$, we use $f_*[X]$ to denote the pushforward class $\underline{f}_*[\underline{X}]$.

We use $|\mathbf{S}|$ to denote the cardinality of a finite set $\mathbf{S}$.

Chapter 2

Punctured invariants and the gluing formalism

In this section, we give a brief introduction to the punctured Gromov-Witten invariants and the gluing formalism studied in [4]. We show the gluing formalism admits a local model of fiber product of toric varieties in the category of fine, saturated log schemes.

2.1 Punctured Gromov-Witten invariants

Let X be a projective log smooth scheme over a log scheme B. A *punctured log curve* over a log scheme W is given by

$$(C^\circ \xrightarrow{p} C \xrightarrow{\pi} W, \mathbf{p} = (p_1, ..., p_n)),$$

where

1. $C \to W$ is a logarithmic curve with a set of disjoint sections $\{p_1, ..., p_n\}$.

2. C° is a logarithmic curve with the underlying curve $\underline{C}$ and log structure

$$\mathcal{M}_{C^\circ} \subset \mathcal{M}_C \oplus_{\mathcal{O}_C^\times} \mathcal{P}^{\mathrm{gp}}$$

for $\mathcal{P} \subset \mathcal{M}_C$ the divisorial log structure along sections $\mathbf{p}$, such that for any geometric point $\overline{x} \in \underline{C}$ and $s_{\overline{x}} \notin \mathcal{M}_{\overline{x}} \oplus_{\mathcal{O}_C^\times} \mathcal{P}_{\overline{x}}$, we have $\alpha_{C^\circ}(s_{\overline{x}}) = 0$.

We note that $\mathcal{M}_{C^\circ}$ is not necessarily saturated. Figure 1 in [4] provides a nice example. A *punctured log map to* $X \to B$ over $W \to B$ is a punctured log curve $(C^\circ \to C \to W, \mathbf{p})$ and a morphism $f : C^\circ \to X$ over B. It is *stable* if $\mathcal{M}_{C^\circ}$ is generated by $\mathcal{M}_C$ and $f^\flat(f^*(\mathcal{M}_X))$ and the underlying map $\underline{f}$ is stable in the usual sense.

The *contact orders* of a punctured map over a log point $W = \mathrm{Spec}(Q \to \Bbbk)$ at point $p \in \mathbf{p}$ is the composition

$$u_p : \overline{\mathcal{M}}_{X,\underline{f}(p)} \xrightarrow{f^\flat} \overline{\mathcal{M}}_{C,p} \to Q \oplus \mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z}.$$

The contact order is *negative* if the image of u_p is not contained in $\mathbb{N}$, which naturally occurs over the points p with $\mathcal{M}_{C,p}$ a strict submonoid of $\mathcal{M}_{C^\circ,p}$.

As in the theory of logarithmic Gromov-Witten, the moduli spaces of the stable punctured log maps to X are stratified by combinatorial data of *global types*.

Definition 2.1.1. [4, Def. 3.4] A *global type* τ of a family of tropical punctured maps is a tuple $(G, \mathbf{g}, \overline{\mathbf{u}}, \boldsymbol{\sigma})$ consisting of

1. A connected graph G with a set of vertices $V(G)$, a set of edges $E(G)$ and a set of legs $L(G)$.

2. A genus map $\mathbf{g} : V(G) \to \mathbb{N}$.

3. An image cone map $\boldsymbol{\sigma} : V(G) \cup E(G) \cup L(G) \to \Sigma(X)$.

4. A global contact order map $\overline{\mathbf{u}}$

$$\overline{\mathbf{u}} : E(G) \cup L(G) \to \bigsqcup_{\sigma \in \Sigma(X)} \mathfrak{C}_\sigma(X)$$

such that $\overline{\mathbf{u}}(x) \in \mathfrak{C}_{\boldsymbol{\sigma}(x)}(X)$, with $\boldsymbol{\sigma}(x)$ the image cone of any edge or leg x. Here, for any cone $\sigma \in \Sigma(X)$, we define

$$\mathfrak{C}_\sigma(X) := \operatorname{colim}^{\textbf{\textit{Sets}}}_{y \in V_X(\sigma)} N_{\sigma_y}.$$

for a point $y \in X$. By the cone σ_y we mean the dual cone $\overline{\mathcal{M}}^\vee_{X,y}$.

A *global decorated type* $\boldsymbol{\tau}$ is a tuple $(\tau, \mathbf{A})$ with τ a global type and $\mathbf{A}$ a function from $V(G)$ to a monoid of curve classes of X. We say a global type τ or a global decorated type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ is realizable if there exists a tropical map to $\Sigma(X)$ with associated global type τ.

A *marking by* $\boldsymbol{\tau}$ of a punctured map $(C^\circ/W, \mathbf{p}, f)$ is defined in [4, Def.3.7]. Roughly speaking, a map is marked by $\boldsymbol{\tau}$ if the genus decorated dual graph of the curve C admits a contraction to $(G_{\boldsymbol{\tau}}, \mathbf{g}_{\boldsymbol{\tau}})$, the image of each nodes and punctured points lies in the associated logarithmic strata of the cone $\boldsymbol{\sigma}$, both the contact orders of non-contracted edges and legs and the curve classes after contraction are determined by $\boldsymbol{\tau}$. The following theorem in [4] lays the foundation of the punctured Gromov-Witten theory.

Theorem 2.1.2. *[4, Thm A] Let $\boldsymbol{\tau}$ be a global decorated type. Then the moduli space $\mathscr{M}(X/B, \boldsymbol{\tau})$ of $\boldsymbol{\tau}$-marked basic stable punctured maps to $X \to B$ is a Deligne-Mumford logarithmic algebraic stack and is proper over B.*

The insights of Olsson's category of logarithmic schemes [20] lead to the concept of Artin fans. As defined in [3, §2.2], for a log Deligne-Mumford stack X, the Artin fan of X is the algebraic stack constructed by gluing toric quotient stacks, called Artin cones, of stalks of $\overline{\mathcal{M}}_X$. Let $\overline{x}$ be a geometric point on X and let $P_{\overline{x}}$ be $\overline{\mathcal{M}}_{X,\overline{x}}$. We define an Artin cone $\mathcal{A}_{\overline{x}} = [\operatorname{Spec} \Bbbk[P_{\overline{x}}] / \operatorname{Spec} \Bbbk[P_{\overline{x}}^{\mathrm{gp}}]]$. The generization of points results in open embeddings of Artin cones. The Artin fan $\mathcal{A}_X$ is the colimit of Artin cones along all points. Artin fans play an important role in the virtual theory and connect the tropical picture with the log picture.

Let $\mathcal{X} = \mathcal{A}_X \times_{\mathcal{A}_B} B$ be the relative Artin fan. The moduli space $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ of $\boldsymbol{\tau}$-marked basic stable punctured maps to $\mathcal{X} \to B$ is again an algebraic stack. For $\boldsymbol{\tau}$ realizable, the moduli space $\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ is pure dimensional([4, Prop.3.28]).

There is a natural evaluation map

$$\mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau}) \to \underline{\mathcal{X}} \times_{\underline{B}} \ldots \times_{\underline{B}} \underline{\mathcal{X}},$$

taken over all the edges and legs of type $\boldsymbol{\tau}$. We define

$$\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau}) = \mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau}) \times_{(\underline{\mathcal{X}} \times_{\underline{B}} \ldots \times_{\underline{B}} \underline{\mathcal{X}})} (\underline{X} \times_{\underline{B}} \ldots \times_{\underline{B}} \underline{X}). \tag{2.1.1}$$

Let $\mathbf{S}$ be a subset of edges of the graph G of $\boldsymbol{\tau}$. By splitting G along the edges in $\mathbf{S}$, we obtain a collection of types $\boldsymbol{\tau}_i$, $i = 1, \ldots, r$. As shown by the following theorem, the virtual theory of the splitting morphism of the moduli spaces of punctured maps to $X \to B$ is compatible with the splitting morphism of the moduli spaces of punctured maps to the relative Artin fans $\mathcal{X} \to B$.

Theorem 2.1.3. *[4, Thm C, Prop 5.15, Thm 5.17]*

There is a Cartesian diagram

$$\begin{array}{ccc} \mathscr{M}(X/B,\boldsymbol{\tau}) & \xrightarrow{\ \delta\ } & \prod_{i=1}^{r}\mathscr{M}(X/B,\boldsymbol{\tau}_i) \\ \Big\downarrow{\scriptstyle \hat{\varepsilon}} & & \Big\downarrow{\scriptstyle \varepsilon} \\ \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}) & \xrightarrow{\ \delta'\ } & \prod_{i=1}^{r}\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}_i), \end{array} \tag{2.1.2}$$

with horizontal splitting maps finite and representable, and vertical maps strict morphisms. There are obstruction theories

$$\mathbb{G} \to \mathbb{L}_{\mathscr{M}(X/B,\boldsymbol{\tau})/\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau})}$$

$$\mathbb{G}_{\mathrm{spl}} \to \mathbb{L}_{\prod_{i=1}^{r}\mathscr{M}(X/B,\boldsymbol{\tau}_i)/\prod_{i=1}^{r}\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}_i)},$$

such that the obstruction theory of the left vertical map is the pullback of the obstruction theory of the right vertical map. For $\alpha \in A_*(\mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B,\boldsymbol{\tau}))$, there is

$$\delta_*\hat{\varepsilon}^!(\alpha) = \varepsilon^!\delta'_*(\alpha),$$

where $\hat{\varepsilon}^!$ and $\varepsilon^!$ are the Manolache's virtual pullback defined using these two obstruction theories.

2.2 Logarithmic Evaluation Maps

Different from Jun Li's approach using expanded degenerations, we are not seeking for the transversality of the evaluation strata. Hence, gluing on the schematic level is no longer enough. In order to obtain a gluing formalism, we first need to fix the problem of not existing a logarithmic evaluation map from the moduli space $\mathfrak{M}(\mathcal{X}/B,\boldsymbol{\tau})$ to $\mathcal{X}$. It requires us to do a modification of the log structure on the moduli space.

For ease of notation, we use $\mathfrak{M}_{\boldsymbol{\tau}} := \mathfrak{M}(\mathcal{X}/B, \boldsymbol{\tau})$ and $\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}} := \mathfrak{M}^{\mathrm{ev}}(\mathcal{X}/B, \boldsymbol{\tau})$ for the rest of the paper. Let G be the graph associated to $\boldsymbol{\tau}$. For each element $p \in E(G) \cup L(G)$, let $\underline{s}_p : \underline{\mathfrak{M}_{\boldsymbol{\tau}}} \to \underline{\mathfrak{C}^{\circ}}$ be the universal section of the punctured or nodal point associated to p. Define $\widetilde{\mathfrak{M}}_p$ to be the logarithmic algebraic stack with the underlying stack $\underline{\mathfrak{M}_{\boldsymbol{\tau}}}$ and the log structure $\underline{s}_p^* \mathcal{M}_{\mathfrak{C}^{\circ}}$. With this log structure, there is a canonical evaluation map $\widetilde{\mathfrak{M}}_p \to \mathcal{X}$ on the section of p. Note that the log structure $\widetilde{\mathfrak{M}}_p$ is fine, but may not be saturated .

For a subset $\mathbf{S} \subseteq E(G) \cup L(G)$, let $\widetilde{\mathfrak{M}}_{\mathbf{S},\boldsymbol{\tau}}$ be the saturation of the fine fiber product

$$\widetilde{\mathfrak{M}}_{p_1} \times^{\mathrm{fine}}_{\mathfrak{M}_{\boldsymbol{\tau}}} \ldots \times^{\mathrm{fine}}_{\mathfrak{M}_{\boldsymbol{\tau}}} \widetilde{\mathfrak{M}}_{p_{|\mathbf{S}|}}, \quad p_i \in \mathbf{S} \tag{2.2.1}$$

in the category of fine log stacks. For the rest of the section, we fix a subset $\mathbf{S}$ and use $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ for $\widetilde{\mathfrak{M}}_{\mathbf{S},\boldsymbol{\tau}}$. Define $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} = \widetilde{\mathfrak{M}}_{\boldsymbol{\tau}} \times_{\mathfrak{M}_{\boldsymbol{\tau}}} \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}$.

Proposition 2.2.1. *[4, Prop.5.5] The canonical map $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} \to \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ is an isomorphism on the underlying stacks provided $\mathbf{S} \subseteq E(G)$, and generally induces an isomorphism on the reductions.*

There is a *canonical idealized structure* on $\mathfrak{M}_{\boldsymbol{\tau}}$, such that $\mathfrak{M}_{\boldsymbol{\tau}}$ is idealized log smooth ([4, Thm 3.24]). The idealized structure in [4, Def 3.22] comes from the fixed combinatorial conditions including dual graph G, image strata fixed by $\boldsymbol{\sigma}$, the contraction to the global type $\boldsymbol{\tau}$ and the puncturing ideal. We will construct an *evaluation idealized structure* on $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ following Construction 2.2.4, with which $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ is also idealized log smooth. Similar to the log structures on $\mathfrak{M}_{\boldsymbol{\tau}}$, both log structures and evaluation idealized structure on $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ are determined by the global type $\boldsymbol{\tau}$.

For a basic punctured log map of type $\boldsymbol{\tau}$ over a point w, the dual cone of the stalk $(\overline{\mathcal{M}}_{\mathfrak{M}_{\boldsymbol{\tau}},\overline{w}})^{\vee}_{\mathbb{R}}$, is called *associated basic cone* of $\boldsymbol{\tau}$. The associated basic cone parametrizes the tropical maps of type $\boldsymbol{\tau}$, which we describe concretely in the following Definition 2.2.2. Similarly, the dual cone of the stalk $(\overline{\mathcal{M}}_{\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}},\overline{w}})^{\vee}_{\mathbb{R}}$ also admits a simple description by *associated evaluation cone* of $\boldsymbol{\tau}$, which parametrizes the tropical maps of type $\boldsymbol{\tau}$ with an additional marking on each edge or leg in $\mathbf{S}$.

Definition 2.2.2. Let $\boldsymbol{\tau}$ be a realizable global decorated type. Define the *associated basic cone* $\overline{\boldsymbol{\tau}}$ *of* $\boldsymbol{\tau}$ the set of elements

$$((V_v)_{v\in V(G)}, (l_E)_{E\in E(G)}) \in \prod_{v\in V(G)} \boldsymbol{\sigma}(v) \times \prod_{E\in E(G)} \mathbb{R}_{\geq 0},$$

such that $V_{v_E} - V_{v'_E} = l_E \cdot \overline{\mathbf{u}}(E)$. Here v_E and v'_E are the vertices of the edge E, with order specified by $\overline{\mathbf{u}}(E)$. As V_{v_E} and $V_{v'_E}$ both lie in $\boldsymbol{\sigma}(E)$, the difference $V_{v_E} - V_{v'_E}$ is well-defined.

Define the *asscociated evaluation cone* $\widetilde{\boldsymbol{\tau}}_{\mathbf{S}}$ of $\boldsymbol{\tau}$ with respect to a set $\mathbf{S} \subseteq E(G) \cup L(G)$ to be the set of elements

$$((V_v)_{v\in V(G)}, (l_E)_{E\in E(G)}, (t_p)_{p\in\mathbf{S}}) \in \prod_{v\in V(G)} \boldsymbol{\sigma}(v) \times \prod_{E\in E(G)} \mathbb{R}_{\geq 0} \times \prod_{p\in\mathbf{S}} \mathbb{R}_{\geq 0},$$

such that $V_{v_E} - V_{v'_E} = l_E \cdot \overline{\mathbf{u}}(E)$, $V_{v_p} + t_p \cdot \overline{\mathbf{u}}(p) \in \boldsymbol{\sigma}(p)$ and $t_e \leq l_e$ for e in $E(G) \cap \mathbf{S}$. Here, if $p \in L(G)$, we define the vertex v_p to be the vertex of leg p; if p is an edge $E \in E(G)$, we define the vertex v_p to be the vertex v'_E with $V_{v_E} - V_{v'_E} = l_E \cdot \overline{\mathbf{u}}(E)$, specified by the orientation of the contact order. There is a *tropical evaluation map*

$$\mathrm{evt}_{\boldsymbol{\tau}} : \widetilde{\boldsymbol{\tau}}_{\mathbf{S}} \to \prod_{p\in\mathbf{S}} \boldsymbol{\sigma}(p),$$
$$((V_v)_{v\in V(G)}, (l_E)_{E\in E(G)}, (t_p)_{p\in\mathbf{S}}) \mapsto (V_{v_p} + t_p \cdot \overline{\mathbf{u}}(p))_{p\in\mathbf{S}}.$$

Under the case of $B = \mathrm{Spec}(Q_B \to \Bbbk)$, the tropical evaluation map $\mathrm{evt}_{\boldsymbol{\tau}}$ factors through the fiber product of cones $\boldsymbol{\sigma}(p)$ over $Q_B^\vee$. The map

$$\mathrm{ev}_{\boldsymbol{\tau}} : \widetilde{\boldsymbol{\tau}}_{\mathbf{S}} \to \boldsymbol{\sigma}(p_1) \times_{Q_B^\vee} \ldots \times_{Q_B^\vee} \boldsymbol{\sigma}(p_{|\mathbf{S}|}) \tag{2.2.2}$$

that $\mathrm{evt}_{\boldsymbol{\tau}}$ factors through is later used in Lemma 4.2.2.

Lemma 2.2.3. *Let $\boldsymbol{\tau}$ be the tropical type of the punctured map over a geometric point $\overline{w}$ on $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$. Then, there is an isomorphism between the dual cone $(\overline{\mathcal{M}}^\vee_{\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}},\overline{w}})_{\mathbb{R}}$ and $\widetilde{\boldsymbol{\tau}}_{\mathbf{S}}$.*

Proof. By the definition of $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ in (2.2.1), there is a projection from $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}} \to \widetilde{\mathfrak{M}}_p$, for every $p \in \mathbf{S}$. Let $\overline{w}_p$ be the geometric point in $\widetilde{\mathfrak{M}}_p$ under the projection. Let $\widetilde{Q}_p$ be the monoid $\overline{\mathcal{M}}_{\widetilde{\mathfrak{M}}_p,\overline{w}_p}$. From the tropical interpretation of the basic log structure in [4, §2.2], for $p \in \mathbf{S}$ and the associated punctured or nodal point $\underline{s}_p : \underline{\mathfrak{M}}_{\boldsymbol{\tau}} \to \underline{\mathfrak{C}}^\circ$, there are isomorphisms between $\widetilde{Q}^\vee_{p,\mathbb{R}} = \underline{s}_p^*(\overline{\mathcal{M}}_{\mathfrak{C}^\circ,\underline{s}_p(\overline{w})})$ and the cone

$$((V_v)_{v\in V(G)}, (l_E)_{E\in E(G)}, t_p) \in \prod_{v\in V(G)} \boldsymbol{\sigma}(v) \times \prod_{E\in E(G)} \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$$

with $V_{v_E} - V_{v'_E} = l_E \cdot \overline{\mathbf{u}}(E)$, $V_{v_p} + t_p \cdot \overline{\mathbf{u}}(p) \in \boldsymbol{\sigma}(p)$ and $t_p \leq l_p$ if $p \in E(G)$.

As $\widetilde{Q}$ is the saturation of $\widetilde{Q}_{p_1} \oplus_Q \ldots \oplus_Q \widetilde{Q}_{p_{|\mathbf{S}|}}$, the dual cone $\widetilde{Q}^\vee_{\mathbb{R}}$ is the fiber product of cones $\widetilde{Q}^\vee_{p_1,\mathbb{R}} \times_{Q^\vee_{\mathbb{R}}} \ldots \times_{Q^\vee_{\mathbb{R}}} \widetilde{Q}^\vee_{p_{|\mathbf{S}|},\mathbb{R}}$. Thus, there is an isomorphism of cones $\widetilde{Q}^\vee_{\mathbb{R}} \to \widetilde{\boldsymbol{\tau}}_{\mathbf{S}}$. ♠

Now we construct the idealized structure on $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$, with which $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ is idealized log smooth over B. It follows from the following general construction of an idealized log structure on a logarithmic stack M, given a strict closed embedding $(M, \mathcal{M}_M) \to (N, \mathcal{M}_N)$ and an idealized log structure $\mathcal{K}_N$ on N. The construction is the same as the log scheme case in [19, Prop III.1.3.4].

Construction 2.2.4. *Let $\mathcal{I}_N \subset \mathcal{O}_N$ be the ideal sheaf that defines M. Let $\mathcal{K}'$ be the preimage ideal sheaf in $\mathcal{M}_N$ under the structure morphism $\alpha_N : \mathcal{M}_N \to \mathcal{O}_N$. Define $\mathcal{K}_M$ to be the ideal sheaf of $\mathcal{M}_M$ generated by the pullback of $\mathcal{K}'$ and $\mathcal{K}_N$.*

It is easy to check that $\alpha_M(\mathcal{K}_M) = 0$, thus $\mathcal{K}_M$ is a well-defined idealized structure. The morphism $(M, \mathcal{M}_M, \mathcal{K}_M) \to (N, \mathcal{M}_N, \mathcal{K}_N)$ is idealized log smooth by [19, Variant IV.3.1.21].

Let $\mathcal{K}_{\mathfrak{M}_{\boldsymbol{\tau}}}$ be the *canonical idealized structure* on $\mathfrak{M}_{\boldsymbol{\tau}}$ defined in [4, Def.3.22]. For $p \in \mathbf{S}$, the map $e_p : \widetilde{\mathfrak{M}}_p \to \mathfrak{M}_{\boldsymbol{\tau}}$ is the composition of the strict section map $s_p : \widetilde{\mathfrak{M}}_p \to \mathfrak{C}^\circ$ and the universal curve $\mathfrak{C}^\circ \to \mathfrak{M}_{\boldsymbol{\tau}}$. Define an idealized log structure $\mathcal{K}_{\mathfrak{C}^\circ}$ on $\mathfrak{C}^\circ$ by the pullback of $\mathcal{K}_{\mathfrak{M}_{\boldsymbol{\tau}}}$ on $\mathfrak{M}_{\boldsymbol{\tau}}$. Define $\mathcal{K}_{\widetilde{\mathfrak{M}}_p}$ to be the canonical idealized structure on $\widetilde{\mathfrak{M}}_p$ associated to s_p constructed in Construction 2.2.4 and $\mathcal{K}_{\widetilde{\mathfrak{M}}_{\mathbf{S}}}$ to be the sheaf of ideals generated by the pullbacks of ideals $\mathcal{K}_{\widetilde{\mathfrak{M}}_p}$ under the projection maps $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}} \to \widetilde{\mathfrak{M}}_p$.

Proposition 2.2.5. *The logarithmic algebraic stack $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ with log ideal $\mathcal{K}_{\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}}$ is idealized log smooth over B.*

Proof. With the idealized structure $\mathcal{K}_{\mathfrak{C}^\circ}$ on $\mathfrak{C}^\circ$, the universal curve $\mathfrak{C}^\circ \to \mathfrak{M}_{\boldsymbol{\tau}}$ is ideally strict, that is, the idealized structure on $\mathfrak{C}^\circ$ is generated by the pullback of idealized structure on $\mathfrak{M}_{\boldsymbol{\tau}}$. Since $\mathfrak{C}^\circ \to \mathfrak{M}_{\boldsymbol{\tau}}$ is log smooth, it is idealized log smooth by [19, Variant IV.3.1.22]. By [19, Variant IV.3.1.21], the closed embedding $\widetilde{\mathfrak{M}}_p \to \mathfrak{C}^\circ$ is idealized log smooth. Hence, we obtain that $e_p : \widetilde{\mathfrak{M}}_p \xrightarrow{s_p} \mathfrak{C}^\circ \to \mathfrak{M}_{\boldsymbol{\tau}}$ is idealized log smooth.

Let $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{fine}}$ be the fiber product of fine logarithmic stacks

$$\widetilde{\mathfrak{M}}_{p_1} \times^{\mathrm{fine}}_{\mathfrak{M}_{\boldsymbol{\tau}}} \ldots \times^{\mathrm{fine}}_{\mathfrak{M}_{\boldsymbol{\tau}}} \widetilde{\mathfrak{M}}_{p_{|\mathbf{S}|}},$$

with p_i going over elements in $\mathbf{S}$. As the idealized log smoothness is stable under fine fiber products, with ideal sheaf $\mathcal{K}_{\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{fine}}}$ on $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{fine}}$ generated by the pullback of ideals $\mathcal{K}_{\widetilde{\mathfrak{M}}_p}$, the projection map $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{fine}} \to \mathfrak{M}_{\boldsymbol{\tau}}$ is idealized log smooth. By the idealized log smoothness of $\mathfrak{M}_{\boldsymbol{\tau}}$ over B, we obtain that $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{fine}}$ is idealized log smooth over B.

Let $g : \widetilde{\mathfrak{M}}_{\boldsymbol{\tau}} \to \widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{fine}}$ be the saturation morphism. By [19, §III.3.1.11], the saturation morphism g is log étale. As the projection maps $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}} \to \widetilde{\mathfrak{M}}_p$ factor through g, the ideal sheaf $\mathcal{K}_{\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}}$ is generated by $g^*(\mathcal{K}_{\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{fine}}})$. The morphism g is ideally strict, hence is idealized log smooth. The logarithmic algebraic stack $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ is idealized log smooth over B.

♠

Corollary 2.2.6. *Let $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau},\mathrm{red}}^{\mathrm{ev}}$ be the reduced induced logarithmic stack of $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{ev}}$. Let $\mathcal{K}_{\widetilde{\mathfrak{M}}_{\boldsymbol{\tau},\mathrm{red}}^{\mathrm{ev}}}$ be the idealized structure on $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau},\mathrm{red}}^{\mathrm{ev}}$ associated to the strict closed embedding to $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau},\mathrm{red}}^{\mathrm{ev}} \to \widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}^{\mathrm{ev}}$ constructed in Construction 2.2.4. Then the corresponding idealized log stack $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau},\mathrm{red}}^{\mathrm{ev}}$ is idealized log smooth over B.*

Proof. The statement follows from [19, Variant IV.3.1.21]. ♠

Following the idealized smoothness of $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$, we obtain that the stratification of $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ is encoded in the global types with *contraction morphisms* to $\boldsymbol{\tau}$, similar to [4, Rmk 3.29].

Definition 2.2.7. A *contraction morphism* of global decorated types $\boldsymbol{\omega} \to \boldsymbol{\tau}$ is a map of the graphs $G_\omega \to G_\tau$ contracting a subset of edges, such that, the following properties are satisfied:

1. the global contact order of $\boldsymbol{\tau}$ associated to an edge or a leg p is the same as the global contact order of the edge or leg in $\boldsymbol{\omega}$ surjective onto p,

2. the genus and the curve class of a vertex in $\boldsymbol{\tau}$ is the sum of those of the vertices in $\boldsymbol{\omega}$ mapped to v and

3. the cone of a vertex, edge or leg in $\boldsymbol{\tau}$ is a subcone of any vertices, edges or legs contained in the preimage.

Suppose $\boldsymbol{\omega} \to \boldsymbol{\tau}$ is a contraction of global decorated type. The preimage of elements in $\mathbf{S}$ form a subset of the edges and legs of $\boldsymbol{\omega}$, which we again denote $\mathbf{S}$. Then, by the definition of the associated evaluation cones in 2.2.2, there is a face inclusion $\widetilde{\tau}_{\mathbf{S}} \to \widetilde{\omega}_{\mathbf{S}}$ whose image is the locus corresponding to points with $l_E = 0$ for the contracted edges in the graph of G. The evaluation map $\mathrm{evt}_{\boldsymbol{\tau}}$ in Definition 2.2.2 is the restriction of $\mathrm{evt}_{\boldsymbol{\omega}}$ on $\widetilde{\boldsymbol{\tau}}_{\mathbf{S}}$.

Remark 2.2.8. Let us give the idealized structure on $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau},\mathrm{red}}$ a local description. Let $\boldsymbol{\omega}$ be a global decorated type that admits a contraction morphism to $\boldsymbol{\tau}$. Let $Q_{\boldsymbol{\omega}} = \mathrm{Hom}(\boldsymbol{\omega}_{\mathbb{Z}}, \mathbb{N})$ be the associated basic monoid of $\boldsymbol{\omega}$ as defined in [4, Def 2.38] and $Q_{\boldsymbol{\tau}} = \mathrm{Hom}(\boldsymbol{\tau}_{\mathbb{Z}}, \mathbb{N})$ be the associated basic monoid of $\boldsymbol{\tau}$. Let $L_{\boldsymbol{\omega}}$ be the stalk of the ideal sheaf $\overline{\mathcal{K}}_{\mathfrak{M}_{\boldsymbol{\tau}}}$ at a geometric point of type $\boldsymbol{\omega}$. Since $\boldsymbol{\tau}$ is realizable, by [4, Prop.3.23]), the ideal $L_{\boldsymbol{\omega}}$ is generated by the inverse image of $Q_{\boldsymbol{\tau}} \backslash \{0\}$ under the generization map $Q_{\boldsymbol{\omega}} \to Q_{\boldsymbol{\tau}}$.

Let $Q_{\boldsymbol{\omega},p} \subseteq Q_{\boldsymbol{\omega}} \oplus \mathbb{Z}$ be the stalk of $\overline{\mathcal{M}}_{\mathfrak{C}^\circ}$ at the punctured or nodal point associated to $p \in \mathbf{S}$ of a punctured map with type $\boldsymbol{\omega}$. Let $L_{\boldsymbol{\omega},p}$ be the ideal generated by the preimage of $L_{\boldsymbol{\omega}}$ under $Q_{\boldsymbol{\omega},p} \to Q_{\boldsymbol{\omega}}$ and the ideal $Q_{\boldsymbol{\omega},p} \cap (Q_{\boldsymbol{\omega}} \oplus \mathbb{Z}_{>0})$. It follows that $L_{\boldsymbol{\omega},p}$ is generated by the preimage of $L_{\boldsymbol{\tau},p}$ under the generization map $Q_{\boldsymbol{\omega},p} \to Q_{\boldsymbol{\tau},p}$, thus is generated by the preimage of $Q_{\boldsymbol{\tau},p} \backslash \{0\}$.

Let $\widetilde{Q}_{\boldsymbol{\omega}} = \mathrm{Hom}(\widetilde{\boldsymbol{\omega}}_{\mathbb{Z}}, \mathbb{N})$ and $\widetilde{L}_{\boldsymbol{\omega}}$ be the stalk of the ideal sheaf $\overline{\mathcal{K}}_{\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}}$ at the geometric point $\overline{x} \to \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau},\mathrm{red}} \to \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$. As the monoid $\widetilde{Q}_{\boldsymbol{\omega}}$ is the saturation of the fibered sum

$$Q_{\boldsymbol{\omega},p_1} \oplus_{Q_{\boldsymbol{\omega}}} \dots \oplus_{Q_{\boldsymbol{\omega}}} Q_{\boldsymbol{\omega},p_{|\mathbf{S}|}}$$

in the category of monoids, the ideal $\widetilde{L}_{\boldsymbol{\omega}}$ is generated by the image of $L_{\boldsymbol{\omega},p_i}$ together with the elements in $\widetilde{Q}_{\boldsymbol{\omega}}$ which are mapped to the nilpotent elements under the structure morphism. For type $\boldsymbol{\tau}$, the ideal $L_{\boldsymbol{\tau},p} = Q_{\boldsymbol{\tau},p} \backslash \{0\}$, hence $\widetilde{L}_{\boldsymbol{\tau}}$ is the prime ideal $\widetilde{Q}_{\boldsymbol{\tau}} \backslash \{0\}$. As $L_{\boldsymbol{\omega},p}$ is generated by the preimage of $L_{\boldsymbol{\tau},p}$, we obtain that $\widetilde{L}_{\boldsymbol{\omega}}$ is the preimage of $\widetilde{L}_{\boldsymbol{\tau}}$. The toric variety

$$\mathrm{Spec}\, \Bbbk[\widetilde{Q}_{\boldsymbol{\omega}}]/(\widetilde{L}_{\boldsymbol{\omega}}) = V_{\mathrm{Spec}\, \Bbbk[\widetilde{Q}_{\boldsymbol{\omega}}]}(\widetilde{\boldsymbol{\tau}})$$

is the toric strata associated to the subcone $\widetilde{\boldsymbol{\tau}}$ in $\widetilde{\boldsymbol{\omega}}$.

Corollary 2.2.9. *Let $\Sigma(\widetilde{\mathfrak{M}_{\boldsymbol{\tau}}})$ be the tropicalization of the Artin stack $\widetilde{\mathfrak{M}_{\boldsymbol{\tau}}}$ as mentioned in [3, § 2.1.4] and constructed in [1] and [25]. Then, the image of the finite morphism $\widetilde{\mathfrak{M}_{\boldsymbol{\omega}}} \to \widetilde{\mathfrak{M}_{\boldsymbol{\tau}}}$ is the substack $V_{\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}}(\widetilde{\boldsymbol{\omega}}_{\mathbf{S}})$ associated to the cone $\widetilde{\boldsymbol{\omega}}_{\mathbf{S}} \in \Sigma(\widetilde{\mathfrak{M}_{\boldsymbol{\tau}}})$.*

Proof. It follows from the idealized smoothness of $\widetilde{\mathfrak{M}_{\boldsymbol{\omega}}}$ and $\widetilde{\mathfrak{M}_{\boldsymbol{\tau}}}$ and the local description of the associated basic monoids and idealized structure in Remark 2.2.8. ♠

2.3 The Gluing Formalism

Fix a decorated global tropical type $\boldsymbol{\tau} = (\tau, \mathbf{A})$ with τ realizable and $\mathbf{S} \subseteq E(G)$ a subset of edges of the graph G of $\boldsymbol{\tau}$. By splitting along edges in $\mathbf{S}$, we obtain sub-types $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, ...\boldsymbol{\tau}_r$. For $i = 1, 2, ..., r$, let $\mathbf{S}_i$ be the subset of legs of the graph in $\boldsymbol{\tau}_i$, obtained from the splitting edges.

In the rest of the section, we use $\widetilde{\boldsymbol{\tau}}$ and $\widetilde{\boldsymbol{\tau}}_i$ to denote the evaluation cones $\widetilde{\boldsymbol{\tau}}_{\mathbf{S}}$ and $\widetilde{\boldsymbol{\tau}}_{i,\mathbf{S}_i}$. For a global decorated type $\boldsymbol{\omega}$ that admits a contraction to $\boldsymbol{\tau}$, the set $\mathbf{S}$ is a subset of edges of $\boldsymbol{\omega}$, we use $\widetilde{\boldsymbol{\omega}}$ to denote the evaluation cone $\widetilde{\boldsymbol{\omega}}_{\mathbf{S}}$. Similarly, we use $\widetilde{\boldsymbol{\omega}}_i$ to denote the evaluation cone $\widetilde{\boldsymbol{\omega}}_{i,\mathbf{S}_i}$ for $\boldsymbol{\omega}_i$ that admits a contraction to $\boldsymbol{\tau}_i$.

In the previous section, we constructed the logarithmic evaluation map $\mathrm{ev}_p : \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} \to X$ for each $p \in \mathbf{S}$. The global type restricts the reduction of the image strata of ev_p to $V_X(\boldsymbol{\sigma}(p))$. We use V_p to denote $V_X(\boldsymbol{\sigma}(p))$. Define

$$X_{\boldsymbol{\tau}} := V_{p_1} \times^{\mathrm{fs}}_B ... \times^{\mathrm{fs}}_B V_{p_{|\mathbf{S}|}}, \quad p_j \in \mathbf{S} \tag{2.3.1}$$

As $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ is reduced by [4, Prop.3.28], we obtain an evaluation map $\mathrm{ev}_{\boldsymbol{\tau}}$ from $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ to $X_{\boldsymbol{\tau}}$. Similarly, let

$$X_{\boldsymbol{\tau}_i} := V_{p_1} \times^{\mathrm{fs}}_B ... \times^{\mathrm{fs}}_B V_{p_{|\mathbf{S}_i|}}, \quad p_j \in \mathbf{S}_i.$$

and $\mathrm{ev}_{\boldsymbol{\tau}_i}$ be the corresponding evaluation map $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i} \to X_{\boldsymbol{\tau}_i}$.

Define $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}$ to be the following fiber product in the category of fine, saturated logarithmic stacks

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}} & \xrightarrow{\delta^{\mathrm{ev}}} & \prod_{i=1}^r \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i} \\ \downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \prod \mathrm{ev}_{\tau_i}} \\ X_{\boldsymbol{\tau}} & \xrightarrow{\Delta_X} & \prod_{i=1}^r X_{\boldsymbol{\tau}_i}. \end{array} \tag{2.3.2}$$

The gluing formalism of [4, Cor.5.13] relates the fiber product $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}$ with $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$. By the reducedness of $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ in [4, Prop.3.28], we obtain the following Lemma.

Lemma 2.3.1. *Let $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$ be the reduction of logarithmic algebraic stack $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}$. Then, the morphism from $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ to $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}$ induced by the fiber diagram factors through the map $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}} \to \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}$. Furthermore, it induces an isomorphism between $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$ and $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$.*

Before we prove Lemma 2.3.1, let us first use it to show the main result of this section Proposition 2.3.2, which implies that in order to study the pushforward of the virtual fundamental class under the splitting morphism (1.1.1), it is enough to study the map δ^{ev} in diagram (2.3.2).

Proposition 2.3.2. *Let $\gamma_{\boldsymbol{\tau}} : \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} \to \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ and $\gamma_i : \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i} \to \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$ be the canonical maps from moduli spaces with evaluation logarithmic structures to basic log structures. Let $\beta_i : \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}} \to \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$ be the canonical maps from the reduced induced stack $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}$ to $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$. Then, for the splitting morphism $\delta' : \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}} \to \prod_{i=1}^{r} \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$ defined in (2.1.2), in the Chow group of $\prod_{i=1}^{r} \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$, the following equation holds*

$$\delta'_*[\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}] = (\prod_{i=1}^{r} \gamma_i \circ \beta_i)_* \delta^{\mathrm{ev}}_{\mathrm{red}*}[\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}].$$

Here $\delta^{\mathrm{ev}}_{\mathrm{red}} : \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}} \to \prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}$ is the map induced from $\delta^{\mathrm{ev}} : \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}} \to \prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$ in diagram (2.3.2) by taking the reduction.

Proof. By [4, Prop.5.5], which we recalled in Proposition 2.2.1, the underlying stack morphism of $\gamma_{\boldsymbol{\tau}} : \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} \to \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ is an isomorphism. Then, by Lemma 2.3.1, the following diagram is commutative

$$\begin{array}{ccc}
\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}} = \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} & \xrightarrow{\gamma_{\boldsymbol{\tau}}} & \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}} \\
\Big\downarrow \delta^{\mathrm{ev}}_{\mathrm{red}} & & \Big\downarrow \delta' \\
\prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}} & \xrightarrow{\prod \gamma_i \circ \beta_i} & \prod_{i=1}^{r} \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}.
\end{array}$$

Therefore,

$$\begin{aligned}
\delta'_*[\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}] &= \delta'_*\gamma_{\boldsymbol{\tau}*}[\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}] \\
&= (\prod_{i=1}^{r} \gamma_i \circ \beta_i)_* \delta^{\mathrm{ev}}_{\mathrm{red}*}[\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}].
\end{aligned}$$

♠

In order to show Lemma 2.3.1, we need punctured maps *weakly marked* by a global type $\boldsymbol{\tau}$ defined in [4, Def.3.7], and the moduli space of basic log punctured maps of weak marking by $\boldsymbol{\tau}$, which is denoted $\widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}}$. In contrast to the moduli spaces of punctured maps marked by $\boldsymbol{\tau}$, it carries an extra non-reducedness obtained from the infinitesimal deformation along $\boldsymbol{\tau}$, which naturally occurs in the gluing process. See [4, §3.5.6] for a more detailed discussion of moduli spaces of maps of weak marking. Here, we use this as a bridge between $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}}$ and $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}$.

Theorem 2.3.3. *[4, Cor.5.13] There is a fine, saturated fiber product of logarithmic stacks*

$$\begin{array}{ccc}
\widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}} & \longrightarrow & \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_1} \times_B \ldots \times_B \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_r} \\
\downarrow & & \downarrow \\
X_{\boldsymbol{\tau}} & \xrightarrow{\Delta} & X_{\boldsymbol{\tau}_1} \times_B \ldots \times_B X_{\boldsymbol{\tau}_r}.
\end{array}$$

Following the fiber diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_1} \times_B \ldots \times_B \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_r} & \longrightarrow & \prod_{i=1}^r \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_i} \\ \downarrow & & \downarrow \\ X_{\boldsymbol{\tau}_1} \times_B \ldots \times_B X_{\boldsymbol{\tau}_r} & \longrightarrow & \prod_{i=1}^r X_{\boldsymbol{\tau}_i}, \end{array}$$

with horizontal maps induced from the universal property of the fiber products, we have

$$\widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}} = \prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_i} \times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}} X_{\boldsymbol{\tau}}. \tag{2.3.3}$$

Proof of Lemma 2.3.1. It is shown in [4, Prop 3.28] that $\mathfrak{M}_{\boldsymbol{\tau}}$ is reduced. The smoothness of the underlying stacks morphism of $\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}} \to \mathfrak{M}_{\boldsymbol{\tau}}$ induces that $\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ is reduced. As $\mathbf{S}$ is a subset of edges, by Proposition 2.2.1, the canonical map of moduli spaces $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} \to \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ is an isomorphism on the underlying stacks. Therefore, the moduli space $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ is reduced.

By [4, Prop.3.31], there are closed embeddings $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} \to \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}}$ and $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i} \to \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_i}$ defined by nilpotent ideals. Hence,

$$\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} = \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau},\mathrm{red}}, \quad \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}} = \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_i,\mathrm{red}}.$$

Then, by Theorem 2.3.3 and equation (2.3.3), we obtain that

$$\begin{aligned} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}} = \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau},\mathrm{red}} &= (\prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}'}_{\boldsymbol{\tau}_i} \times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}} X_{\boldsymbol{\tau}})_{\mathrm{red}} \\ &= (\prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i} \times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}} X_{\boldsymbol{\tau}})_{\mathrm{red}} = \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}. \end{aligned}$$

♠

Chapter 3

Generalization of Fulton-Sturmfels formula

The idealized log smoothness of the evaluation maps provides us with local toric models, where the local splitting maps can be seen as a toric morphism of toric stacks. We defer the discussion of the local toric models to the next Chapter. In this chapter, we study the pushforward of fundamental class under the morphisms of toric stacks. It is a generalization of the classical result of Fulton and Sturmfels on the intersection product of toric varieties. This chapter serves as a technical foundation for the splitting formula. The readers can feel free to skip the chapter first and check back later.

Let X be a toric variety associated to a fan $(\Sigma(X), N(X))$. Let $N(Y) \subseteq N(X)$ be a saturated sublattice defining a subtorus $T_Y \subseteq T_X$. Define the scheme Y to be the closure of T_Y in X.

Definition 3.0.1. A vector $v \in N(X)$ is *generic with respect to pairs* (X, Y) if for any cone $\delta \in \Sigma(X)$ with dimension $\dim X - \dim Y$, the affine space $N(Y)_{\mathbb{R}} + v$ intersects δ at at most one point, and if they intersect, the intersection point lies in the interior of δ.

Define $\Delta^0(v)$ to be the set of cones

$$\Delta^0(v) := \{\delta \in \Sigma(X) \mid \dim \delta = \dim X - \dim Y, (N(Y)_{\mathbb{R}} + v) \cap \delta \neq \varnothing\}.$$

The Chow groups of a toric variety are generated by its toric strata. It is proved in [9, Lemma 4.4] that the subvariety Y in X is rationally equivalent to a linear combination of the toric strata determined by cones in $\Delta^0(v)$, for any generic displacement vector v with respect to (X, Y). Here, we provide a slightly different proof using the $\mathbb{G}_m$-orbit of Y under the torus action associated to a generic vector v.

Lemma 3.0.2. *Let $v \in N(X)$ be a generic vector with respect to pairs (X, Y). Then, in the Chow group $A_{\dim Y}(X)$,*

$$[Y] = \sum_{\delta \in \Delta^0(v)} m(\delta) \cdot [V_X(\delta)]. \tag{3.0.1}$$

Here $m(\delta) = [N(X) : N_\delta + N(Y)]$, with N_δ the sublattice of $N(X)$ generated by the cone δ. The subscheme $V_X(\delta)$ is the closed subvariety associated to cone δ.

Proof. Let L be the toric variety $X \times \mathbb{P}^1$ with the product fan structure. Let $\alpha : L \to \mathbb{P}^1$ be the projection and $\overline{\alpha} : N(L) \to \mathbb{Z}$ be the associated projection of lattices.

We first construct the $\mathbb{G}_m$-orbit of Y under the torus action of v as a subvariety of $X \times \mathbb{P}^1$. Define $N(K)_\mathbb{R} := \{(x + tv, t) \mid x \in N(Y)_\mathbb{R} \text{ and } t \in \mathbb{R}\}$ and $N(K)$ the saturated integral lattice $N(K)_\mathbb{R} \cap N(L)$. Let $T_K \subseteq T_L$ be the corresponding subtorus. The closure of T_K defines a subvariety K of L. By toric geometry, the preimage subscheme $\alpha|_{T_K}^{-1}(1)$ is isomorphic to T_Y and $\alpha|_K^{-1}(1)$ is isomorphic to Y.

Let $\Sigma(K)$ be the fan with lattice $N(K)$ and cones $\delta \cap N(K)_\mathbb{R}$ for $\delta \in \Sigma(L)$. Let $\widetilde{K}$ be the toric variety associated to $(\Sigma(K), N(K))$. Then K is the image of $\widetilde{K}$ under the proper toric morphism associated to the lattice morphism $\beta_N : N(K) \hookrightarrow N(L)$. Let $\alpha' : \widetilde{K} \xrightarrow{\beta} L \xrightarrow{\alpha} \mathbb{P}^1$ be the induced projection to $\mathbb{P}^1$.

By [3, Prop 3.1], the subscheme $(\alpha')^{-1}(0)$ satisfies the equation

$$[(\alpha')^{-1}(0)] = \sum_{\tau} m_\tau \cdot [V_{\widetilde{K}}(\tau)],$$

where τ goes over the rays in $\Sigma(K)$, whose image under the $\mathbb{R}$-linear map

$$\overline{\alpha}'_{\mathbb{R}} : N(K)_{\mathbb{R}} \xrightarrow{\beta_{N,\mathbb{R}}} N(L)_{\mathbb{R}} \xrightarrow{\overline{\alpha}_{\mathbb{R}}} \mathbb{R}$$

is $\mathbb{R}_{\geq 0}$. The multiplicity m_τ is given by the image of the primitive generator of τ under $\overline{\alpha}'_{\mathbb{R}}$. Since α is a flat dominant morphism, by the alternative definition of rational equivalence in [10, §1.6], we obtain a rational equivalence relation in the Chow group of X:

$$\begin{aligned} [Y] = [\alpha|_K^{-1}(1)] \sim [\alpha|_K^{-1}(0)] &= \beta'_*[(\alpha')^{-1}(0)] \\ &= \sum_{\tau} m_\tau \cdot \beta'_*[V_{\widetilde{K}}(\tau)]. \end{aligned} \tag{3.0.2}$$

Here $\beta' : \widetilde{K} \to X$ is composition of the map $\beta : \widetilde{K} \to L$ with the projection $L \to X$. It is sufficient to show that the above equation (3.0.2) is the same as equation (3.0.1).

First, there is an one-to-one correspondence between rays τ and cones in $\Delta^0(v)$. Note each ray $\tau \in \Sigma(K)$ is the intersection of $\delta \times \mathbb{R}_{\geq 0}$ and $N(K)_{\mathbb{R}}$ for a cone δ in $\Sigma(X)$. The preimage $\overline{\alpha}_{\mathbb{R}}^{-1}(1) \cap \tau$ is a point $(x + v, 1)$ in $\delta_{\mathbb{R}} \times \mathbb{R}_{\geq 0}$ for some $x \in N(Y)_{\mathbb{R}}$. Hence $\{(N(Y)_{\mathbb{R}} + v) \cap \delta\}$ is non-empty. By the genericity of v, there is only one intersection point. It follows that the cone δ lies in $\Delta^0(v)$. On the other hand, for every δ in $\Delta^0(v)$, the intersection of $\delta \times \mathbb{R}_{\geq 0}$ and $N(K)_{\mathbb{R}}$ is a ray with image $\mathbb{R}_{\geq 0}$ under $\overline{\alpha}'_{\mathbb{R}}$.

Next, we need to show the corresponding multiplicities are the same. Under the proper morphism $\beta : \widetilde{K} \to K$, the image of $V_{\widetilde{K}}(\tau)$ is $V_L(\delta \times \mathbb{R}_{\geq 0})$ up to a multiplicity.

The multiplicity is decided by the degree of the finite map of the open toric strata from $O_{\widetilde{K}}(\tau)$ to $O_L(\delta \times \mathbb{R}_{\geq 0})$, that is, the lattice index

$$
\begin{aligned}
&[N(L)/(N_\delta \times \mathbb{Z}) : N(K)/N_\tau] \\
=&[N(L)/(N_\delta \times \mathbb{Z}) : N(K)/((N_\delta \times \mathbb{Z}) \cap N(K))] \\
=&[N(L)/(N_\delta \times \mathbb{Z}) : (N_\delta \times \mathbb{Z} + N(K))/(N_\delta \times \mathbb{Z})] \\
=&[N(L) : N_\delta \times \mathbb{Z} + N(K)].
\end{aligned}
$$

Thus we have

$$
\beta'_*[V_{\widetilde{K}}(\tau)] = [N(L) : N_\delta \times \mathbb{Z} + N(K)] \cdot [V_X(\delta)]. \tag{3.0.3}
$$

The sublattice

$$
\begin{aligned}
N_\delta \times \mathbb{Z} + N(K) &= N_\delta \times \mathbb{Z} + N(Y) \times \{0\} + \mathbb{Z} \cdot (v, 1) \\
&= N_\delta \times \mathbb{Z} + N(Y) \times \{0\} + \mathbb{Z} \cdot (v, 0) \\
&= (N_\delta + N(Y)) \times \mathbb{Z} + \mathbb{Z} \cdot (v, 0),
\end{aligned}
$$

with the second and the third equality following from the fact that $(0, 1) \in N_\delta \times \mathbb{Z}$.

Write $v = \frac{a_1}{a_2} \cdot x + \frac{b_1}{b_2} \cdot y$, where $x \in N_\delta$, $y \in N(Y)$ and integers pairs (a_1, a_2), (b_1, b_2) are coprime. As N_δ and $N(Y)$ has complementary dimension in $N(X)$, such presentation of v is unique. The lattice index

$$
\begin{aligned}
&[(N_\delta \times \mathbb{Z} + N(K)) : (N_\delta + N(Y)) \times \mathbb{Z}] \\
=&[(N_\delta + N(Y)) \times \mathbb{Z} + (\mathbb{Z} \cdot v, 0) : (N_\delta + N(Y)) \times \mathbb{Z}] \\
=&[N_\delta + N(Y) + (\mathbb{Z} \cdot v, 0) : N_\delta + N(Y)] = \mathrm{lcm}(a_2, b_2),
\end{aligned} \tag{3.0.4}
$$

where $\mathrm{lcm}(a_2, b_2)$ is the least common multiple of integers a_2 and b_2. The integral generator v_τ of the ray $N(K)_\mathbb{R} \cap \delta$ has form

$$\begin{aligned} v_\tau &= n \cdot (v, 1) + (y', 0) \\ &= \Big(\frac{n \cdot a_1}{a_2} \cdot x + \frac{n \cdot b_1}{b_2} \cdot y + y', n \Big) \end{aligned}$$

for $y' \in N(Y)$ and $n \in \mathbb{Z}$. Since v_τ is integral, then n is a multiple of $\operatorname{lcm}(a_2, b_2)$. As n is the smallest integer such that v_τ is integral, then $n = \operatorname{lcm}(a_2, b_2)$. Therefore

$$m_\tau = \operatorname{lcm}(a_2, b_2) = [N_\delta \times \mathbb{Z} + N(K) : (N_\delta + N(Y)) \times \mathbb{Z}].$$

The multiplicity

$$\begin{aligned} & m_\tau \cdot [N(L) : (N_\delta \times \mathbb{Z} + N(K))] \\ =& [N_\delta \times \mathbb{Z} + N(K) : (N_\delta + N(Y)) \times \mathbb{Z}] \cdot [N(L) : N_\delta \times \mathbb{Z} + N(K)] \\ =& [N(L) : (N_\delta + N(Y)) \times \mathbb{Z}] \\ =& [N(X) : N_\delta + N(Y)] = m_\delta. \end{aligned}$$

Then by (3.0.2) and (3.0.3), we obtain

$$[Y] = \sum_\tau m_\tau \cdot \beta'_*[V_{\widetilde{K}}(\tau)] = \sum_{\delta \in \Delta^0(v)} m_\delta \cdot [V_X(\delta)].$$

♠

Example 3.0.3. Let m be a non-negative integer. Let X be the Hirzebruch surface F_m, whose fan Σ_X in $\mathbb{Z}^2$ contains four rays $r_1, ..., r_4$ with directions $(1,0), (0,1), (-1,m)$ and $(0,-1)$. Let $N(Y)$ be one dimensional sublattice generated by $v_Y = (1,1)$.

Suppose the generic displacement vector $v = (1,0)$, then $\Delta^0(v)$ contains rays r_1 and r_4. Since the lattice generated by v_Y and v_1, v_2 are both $\mathbb{Z}^2$, the multiplicities for both rays are 1. Suppose we take the generic displacement vector $v = (-1,0)$, then

$\Delta^0(v)$ contains rays r_2 and r_3. The multiplicity for r_2 is 1 and the multiplicity for r_4 is $m+1$. We obtain that in $A_1(X)$,

$$[Y] = [V_X(\delta_1)] + [V_X(\delta_4)] = [V_X(\delta_2)] + (m+1) \cdot [V_X(\delta_3)].$$

We generalize Lemma 3.0.2 to morphisms of toric strata. Let $\widetilde{f} : Y \to X$ be a proper morphism of toric varieties associated to an injective map of lattices $f_N : N(Y) \to N(X)$. Let τ be a cone in $\Sigma(Y)$ and let τ' be the smallest cone in $\Sigma(X)$ that contains the image of $\tau \in \Sigma(Y)$. Let $f : V_Y(\tau) \to V_X(\tau')$ be the restriction of $\widetilde{f}$ on $V_Y(\tau)$. We wish to study $f_*[V_Y(\tau)]$ using the same idea.

Definition 3.0.4. A vector $v \in N(X)$ is *generic with respect to* $(X, Y, V_Y(\tau))$ if its image under the quotient map $q_X : N(X) \to N(X)/N_{\tau'}$ is generic with respect to the pair $(V_X(\tau'), f(V_Y(\tau)))$ as defined in Definition 3.0.1.

Similarly, we define $\Delta^\tau(v)$ to be the collection of cones δ in $\Sigma(X)$ satisfying that

1. $\tau' \subseteq \delta$,
2. $\dim N_\delta = \dim N(X) - \dim f_N(N(Y)) + \dim(f_N(N(Y)) \cap N_{\tau'})$,
3. $(f_N(N(Y))_{\mathbb{R}} + v) \cap \delta$ is not empty.

Proposition 3.0.5. *In the Chow group $A_l(V_X(\tau'))$:*

$$f_*[V_Y(\tau)] = \sum_{\delta \in \Delta^\tau(v)} m(\delta) \cdot [V_X(\delta)], \tag{3.0.5}$$

where $l = \dim N(Y) - \dim \tau$ and $m(\delta) = [N(X) : f_N(N(Y)) + N_\delta]$.

Proof. Let $q_X : N(X) \to N(X)/N_{\tau'}$ be the quotient of the lattice. Let N' be the saturation of the image $q_X(f_N(N(Y)))$ in $N(X)$. Then the image of $V_Y(\tau)$ under f is the closure of $T_{N'} \subseteq T_{N(X)/N_{\tau'}}$ inside $V_X(\tau')$. The degree of the map is the degree of the cover of torus induced from the saturation $q_X(f_N(N(Y))) \to N'$. Therefore,

$$f_*[V_Y(\tau)] = [N' : q_X(f_N(N(Y)))] \cdot [f(V_Y(\tau))]. \tag{3.0.6}$$

We first apply Lemma 3.0.2 to study $[f(V_Y(\tau))]$. By definition, the vector $q_X(v)$ is generic with respect to $(V_X(\tau'), V_Y(\tau))$. Hence, in the Chow group of $V_X(\tau')$,

$$[f(V_Y(\tau))] = \sum_{\omega \in \Delta^0(q_X(v))} [N(X)/N_{\tau'} : N' + N_\omega] \cdot [V_{V_X(\tau')}(\omega)]. \tag{3.0.7}$$

Let us first show that a cone $\omega \in \Delta^0(q_X(v))$ if and only if $\delta \in \Delta^\tau(v)$ for δ the unique cone containing τ' and $q_X(\delta) = \omega$. Note that the intersection $\omega \cap (N'_\mathbb{R} + q_X(v))$ is not empty if and only if the preimage of it under q_X is not empty. That is, the intersection of $\delta + N_{\tau',\mathbb{R}}$ with $f_N(N(Y))_\mathbb{R} + N_{\tau',\mathbb{R}} + v$ is not empty. It is equivalent to that the vector v lies in $\delta + f_N(N(Y))_\mathbb{R} + N_{\tau',\mathbb{R}}$. Note that

$$\delta + f_N(N(Y))_\mathbb{R} + N_{\tau',\mathbb{R}} = \delta + f_N(N(Y))_\mathbb{R},$$

as any element in $N_{\boldsymbol{\tau}',\mathbb{R}}$ can be written as a linear combination $a \cdot w + b \cdot w'$, for a $w \in f_N(N(Y))$ in the interior of τ' and for some $w' \in \tau' \subseteq \delta$ and non-negative a, b. Thus the condition (3) of Definition 3.0.4 that $(f_N(N(Y))_\mathbb{R} + v) \cap \delta$ is not empty is equivalent to $\omega \cap (N'_\mathbb{R} + q_X(v))$ being not empty.

For the dimension condition, as $\tau' \subseteq \delta$,

$$\dim \omega = \dim q_X(N_\delta) = \dim N_\delta - \dim N_{\tau'}.$$

Then

$$\dim \omega = \dim q_X(N(X)) - \dim q_X(f_N(N(Y)))$$

if and only if

$$\begin{aligned}\dim N_\delta &= \dim N_{\tau'} + \dim q_X(N(X)) - \dim q_X(f_N(N(Y))) \\ &= \dim N(X) - \dim q_X(f_N(N(Y))) \\ &= \dim N(X) - \dim f_N(N(Y)) + \dim(f_N(N(Y)) \cap N_{\tau'}).\end{aligned}$$

Hence $\omega \in \Delta^0(q_X(v))$ if and only if $\delta \in \Delta^\tau(v)$.

Note that $V_X(\delta) = V_{V_X(\tau')}(\omega)$ by definition. The equation (3.0.7) is then equivalent to

$$[f(V_Y(\tau))] = \sum_{\delta \in \Delta^\tau(v)} [N(X)/N_{\tau'} : N' + q_X(N_\delta)] \cdot [V_X(\delta)]. \tag{3.0.8}$$

Together with the equation (3.0.6), we obtain that

$$f_*[V_Y(\tau)] = \sum_{\delta \in \Delta^\tau(v)} [N' : q_X(f_N(N(Y)))][N(X)/N_{\tau'} : N' + q_X(N_\delta))] \cdot [V_X(\delta)]. \tag{3.0.9}$$

Since N' and $q_X(N_\delta)$ have complementary dimensions in the lattice $N(X)/N_{\tau'}$, and the intersection of $q_X(f_N(N(Y)))$ and $q_X(N_\delta)$ is zero dimensional. Thus,

$$[N' : q_X(f_N(N(Y)))] = [N' + q_X(N_\delta) : q_X(f_N(N(Y))) + q_X(N_\delta)]. \tag{3.0.10}$$

Since the quotient lattice

$$\begin{aligned}q_X(f_N(N(Y))) + q_X(N_\delta) &= q_X(f_N(N(Y)) + N_\delta) \\ &= (f_N(N(Y)) + N_\delta)/N_{\tau'},\end{aligned}$$

the lattice index in the equation (3.0.9) satisfies that

$$\begin{aligned}&[N' : q_X(f_N(N(Y)))] \cdot [N(X)/N_{\tau'} : N' + q_X(N_\delta))] \\ &\overset{(3.0.10)}{=} [N(X)/N_{\tau'} : q_X(f_N(N(Y))) + q_X(N_\delta)] \\ &= [N(X)/N_{\tau'} : (f_N(N(Y)) + N_\delta)/N_{\tau'}] \\ &= [N(X) : f_N(N(Y)) + N_\delta].\end{aligned}$$

We now finish the proof of the equation (3.0.5).

♠

Corollary 3.0.6. *With the same assumption in Proposition 3.0.5, let N_Q be a sublattice of $N(Y)$. The subtorus $T_Q \subseteq T_Y$ induces a T_Q-action on $V_Y(\tau)$ and $V_X(\tau')$. The morphism $f : V_Y(\tau) \to V_X(\tau')$ is T_Q-equivariant.*

Let $f_Q : [V_Y(\tau)/T_Q] \to [V_X(\tau')/T_Q]$ be the induced map on the quotient stacks. Let v be a generic displacement vector with respect to $(X, Y, V_Y(\tau))$ as defined in Definition 3.0.4. Then there is a closed substack of $[V_X(\tau')/T_Q] \times \mathbb{P}^1$ which induces the rational equivalence of $[f_Q([V_Y(\tau)/T_Q])]$ and

$$\sum_{\delta \in \Delta^\tau(v)} \frac{m(\delta)}{[\mathrm{im}(q_X \circ f_N)^{\mathrm{sat}} : \mathrm{im}(q_X \circ f_N)]} \cdot [V_X(\delta)/T_Q],$$

where $\mathrm{im}(q_X \circ f_N)^{\mathrm{sat}}$ is the saturation of sublattice $\mathrm{im}(q_X \circ f_N)$ in $N(X)/N_{\tau'}$ and

$$m(\delta) = [N(X) : f_N(N(Y)) + N_\delta].$$

In the Chow group $A_l([V_X(\tau')/T_Q])$,

$$f_{Q*}[V_Y(\tau)/T_Q] = \sum_{\delta \in \Delta^\tau(v)} m(\delta) \cdot [V_X(\delta)/T_Q], \tag{3.0.11}$$

where $l = \dim Y - \dim \tau - \dim N_Q$.

Proof. Let N' be the saturation of $f_N(N(Y))$ in $N(X)$. Then $q_X(N')$ is saturated in $N(X)/N_{\tau'}$. We first show that

$$[f_Q([V_Y(\tau)/T_Q])] = \sum_{\delta\in\Delta^\tau(v)} [N(X)/N_{\tau'} : q_X(N') + q_X(N_\delta)] \cdot [V_X(\delta)/T_Q] \tag{3.0.12}$$

similar to the equation (3.0.8) in the toric variety case.

In the toric subvariety $V_X(\tau')$, the closure of the torus associated to N' is $f(V_Y(\tau))$. Let v' be the vector $q_X(v)$ in $N(X)/N_{\tau'}$. By Lemma 3.0.2, there is a closed subvariety K in $V_X(\tau') \times \mathbb{P}^1$ defined to be the closure of the torus associated to the subspace

$$\{(x + t\cdot v', t) \mid x \in N(X)_\mathbb{R}/N_{\tau',\mathbb{R}} \text{ and } t \in \mathbb{R}\},$$

such that the projection map $\alpha : K \to \mathbb{P}^1$ induces the rational equivalence of $[f(V_Y(\tau))]$ and

$$\sum_{\delta\in\Delta^\tau(v)} [N(X)/N_{\tau'} : q_X(N') + q_X(N_\delta)] \cdot [V_X(\delta)].$$

The inclusion of lattices

$$N_Q \times \{0\} \hookrightarrow N_Q \times \mathbb{Z} \hookrightarrow N(Y)\times\mathbb{Z} \xrightarrow{f_N\times \mathrm{id}} N(X)\times\mathbb{Z} \xrightarrow{q_X\times\mathrm{id}} N(X)/N_{\tau'}\times\mathbb{Z}$$

induces a T_Q-action on $V_X(\tau')\times\mathbb{P}^1$. An easy lattice computation tells us that K is invariant under the T_Q-action and each fiber of α is T_Q-invariant. Therefore, the closed substack $[K/T_Q]$ together with the dominant morphism $\alpha' : [K/T_Q] \to \mathbb{P}^1$ satisfies the equations

$$[\alpha'^{-1}(1)] = [f(V_Y(\tau))/T_Q] = [f_Q([V_Y(\tau)/T_Q])],$$

$$[\alpha'^{-1}(0)] = \sum_{\delta\in\Delta^\tau(v)} [N(X)/N_{\tau'} : q_X(N') + q_X(N_\delta)] \cdot [V_X(\delta)/T_Q].$$

Hence this induces the rational equivalence of $[f(V_Y(\tau))/T_Q]$ and

$$\sum_{\delta \in \Delta^\tau(v)} [N(X)/N_{\tau'} : q_X(N') + q_X(N_\delta)] \cdot [V_X(\delta)/T_Q].$$

In Proposition 3.0.5, we showed that the equation (3.0.8) induces that

$$f_*[V_Y(\tau)] = \sum_{\delta \in \Delta^\tau(v)} m(\delta) \cdot [V_X(\delta)].$$

With the same argument, the equation (3.0.12) induces that

$$\begin{aligned} f_{Q*}[V_Y(\tau)/T_Q] &= [N' : f_N(N(Y))] \cdot [f_Q([V_Y(\tau)/T_Q])] \\ &= \sum_{\delta \in \Delta^\tau(v)} [N(X) : f_N(N(Y)) + N_\delta] \cdot [V_X(\delta)/T_Q]. \end{aligned}$$

♠

Chapter 4

Proof of the splitting formula

From now on, let us assume the Assumption 1.1.2 is satisfied.

Assumption 1.1.2. *Assume $X \to B$ is integral and $\overline{\mathcal{M}}_X$ is globally generated. Assume for each edge $p \in \mathbf{S}$, the strict closed subscheme $V_p := V_X(\boldsymbol{\sigma}(p))$ of the log scheme X associated to the cone $\boldsymbol{\sigma}(p)$ has the underlying scheme a toric variety and the log stratification of V_p is the same as the toric stratification.*

4.1 Toric Strata Assumption

In the following theorem, we show that the toric strata assumption is satisfied as long as logarithmic strata match toric strata. Note that for any $p \in \mathbf{S}$, we can associate an idealized structure on log subscheme V_p as in Construction 2.2.4 and in [19, Prop III.1.3.4], such that by [19, Variant IV.3.1.21], the idealized log scheme V_p is idealized log étale over log scheme X, hence is idealized log smooth over B.

Theorem 4.1.1. *Let $B = (\operatorname{Spec} \Bbbk, Q_B)$ be a log point. Suppose X is a scheme over B with Zariski log structure $\mathcal{M}_X$ and idealized structure $\mathcal{K}_X$, such that $X \to B$ is integral and idealized log smooth. Assume $\overline{\mathcal{M}}_X$ is globally generated. Suppose further that X satisfies the following conditions:*

1. The underlying scheme $\underline{X}$ is a toric variety.

2. The log stratification of X is the same as the toric stratification of X. In other words, for each point x of X, let $\widetilde{\sigma}$ be the dual cone of $\overline{\mathcal{M}}_{X,x}$, the underlying scheme of the logarithmic stratum $\underline{V_X(\widetilde{\sigma})}$ is the smallest closed toric stratum of $\underline{X}$ containing x.

Then there is a toric variety Y and a cone $\widetilde{\sigma}_0 \in \Sigma_Y$, such that X is isomorphic to $V_Y(\widetilde{\sigma}_0)$ as a log scheme.

Proof. Let $Q_0 = \overline{\mathcal{M}}_X(T)$ for T the maximal torus of X. Let N be the cocharacter lattice and M be the character lattice of $\underline{X}$. We first construct the fan of Y in the lattice $\widetilde{N} = N \times Q_0^*$ by constructing a cone $\widetilde{\sigma}$ in $\widetilde{N}$ for each cone σ in fan Σ of X.

Since $\mathcal{M}_X|_T$ is a constant sheaf by assumption (2), there is an isomorphism $s : \mathcal{M}_X(T) \to \Bbbk^\times \oplus M \oplus Q_0$. For each cone $\sigma \in \Sigma$, the restriction map $\chi_\sigma : \mathcal{M}_X(U(\sigma)) \to \mathcal{M}_X(T)$ determines a map

$$\phi_\sigma : \mathcal{M}_X(U(\sigma)) \xrightarrow{\chi_\sigma} \mathcal{M}_X(T) \xrightarrow{s} \Bbbk^\times \oplus M \oplus Q_0 \xrightarrow{\mathrm{pr}} M \oplus Q_0,$$

where $U(\sigma) = \operatorname{Spec} \Bbbk[\sigma^\vee \cap M]$. Define $\widetilde{\sigma}$ in $\widetilde{N}$ to be the dual cone of the image monoid $\mathrm{im}(\phi_\sigma)$. For the zero cone $0 \in \Sigma$, $\widetilde{\sigma}_0$ is simply $\{0\} \times Q_0^\vee$. Although the isomorphism s is not canonical, different s differs by a morphism $Q_0 \to M$, which results in a linear transformation of $N \times Q_0^*$ of determinant 1.

We now show that the collection of cones $\widetilde{\sigma}$ forms a fan in $\widetilde{N}$. Since $\overline{\mathcal{M}}_X$ is globally generated, the map $(\operatorname{im} \phi_\sigma) \hookrightarrow M \oplus Q_0 \to \Bbbk[M]$ is a chart of $\mathcal{M}_X|_T$, with the second map the composition of the map to $\Bbbk^\times \oplus M \oplus Q_0$, s^{-1} and structure morphism $\mathcal{M}_X(T) \to \mathcal{O}_X(T)$. Hence $(\operatorname{im}\phi_\sigma) \to Q_0$ is the quotient of $(\operatorname{im}\phi_\sigma)$ by $(\operatorname{im}\phi_\sigma) \cap (M \oplus \{0\})$. The dual map of $(\operatorname{im}\phi_\sigma) \hookrightarrow M \oplus Q_0$ then induces a face inclusion $\widetilde{\sigma}_0 \to \widetilde{\sigma}$. Similarly, for each $\tau \subseteq \sigma \in \Sigma$, the restriction map $\mathcal{M}_X(U(\sigma)) \to \mathcal{M}_X(U(\tau))$ induces an inclusion of submonoids $(\operatorname{im}\phi_\sigma) \hookrightarrow (\operatorname{im}\phi_\tau)$, with the induced map $(\operatorname{im}\phi_\sigma) \to Q_\tau$ the quotient of $(\operatorname{im}\phi_\sigma)$ by $(\operatorname{im}\phi_\sigma) \cap (P_\tau^\times \oplus \{0\})$. Here $P_\tau^\times$ is the subgroup of units in $\tau^\vee \cap M$. We thus obtain the face inclusions of the dual cones $\widetilde{\tau} \hookrightarrow \widetilde{\sigma}$. The collection of cones forms a fan $\widetilde{\Sigma}$ in $\widetilde{N}$.

Let Y be the toric variety of $(\widetilde{\Sigma}, \widetilde{N})$. Note in the quotient fan in $\widetilde{N}/(\{0\} \times Q_0^*)$, the affine subvariety associated to $(\operatorname{im}\phi_\sigma)^\vee$ is determined by the image of monoid

$$(\operatorname{im}\phi_\sigma) \to M \oplus Q_0 \xrightarrow{p} M,$$

where p sends $(m, 0)$ to m and (m, q) to 0 for $q \neq 0$. Let F_σ be the face $(\operatorname{im}\phi_\sigma) \cap (M \oplus \{0\})$. We claim that $p(F_\sigma)$ is isomorphic to $P_\sigma = \sigma^\vee \cap M$. As a consequence, the underlying variety $\underline{V_Y(\widetilde{\sigma}_0)}$ is isomorphic to $\underline{X}$. We leave the proof of this claim to the last paragraph.

For each affine variety $\underline{U(\sigma)}$, the log structure obtained from $\underline{U(\sigma)} \hookrightarrow V_Y(\widetilde{\sigma}_0)$ has chart $(\operatorname{im}\phi_\sigma) \to F_\sigma \to \Bbbk[F_\sigma] = \Bbbk[P_\sigma]$, where the first map sends elements in F_σ to themselves and others to 0. The log structure obtained from X is $\mathcal{M}_X(U(\sigma)) \xrightarrow{\alpha_\sigma} \Bbbk[P_\sigma]$. Both are induced by the following map f_σ in (4.1.1), whose image lies in $\Bbbk[P_\sigma]$. Therefore, the log schemes $V_Y(\widetilde{\sigma}_0)$ and X are isomorphic.

We now prove that F_σ is isomorphic to P_σ under $p : M \oplus Q_0 \to M$. It is sufficient to show that under

$$f_\sigma : \mathcal{M}_X(U(\sigma)) \xrightarrow{\chi_\sigma} \mathcal{M}_X(T) \xrightarrow{\alpha_T} \Bbbk[M], \tag{4.1.1}$$

the image of f_σ is exactly the monoid of monomials $\Bbbk^\times \oplus P_\sigma$. First, as there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_X(U(\sigma)) & \xrightarrow{\alpha_\sigma} & \Bbbk[P_\sigma] \\ \downarrow{\scriptstyle \chi_\sigma} & & \downarrow \\ \mathcal{M}_X(T) & \xrightarrow{\alpha_T} & \Bbbk[M], \end{array}$$

the image of f_σ lies in $\Bbbk[P_\sigma]$. Since the image of α_T in $\Bbbk[M]$ are monomials, the image of f_σ lies in $\Bbbk^\times \oplus P_\sigma$. As each toric stratum of $U(\sigma)$ determined by a generator $p \in P_\sigma/P_\sigma^\times$ is a logarithmic stratum, there is a minimal positive integer n with the monomial $z^{n\cdot p}$ lies in the image of f_σ. On the other hand, since $U(\sigma)$ is idealized log smooth over B, the sheaf of relative differential forms $\Omega_{U(\sigma)/B}$ is locally free. As $U(\sigma)$ is affine and $\mathcal{M}_{U(\sigma)}$ is globally generated, the global sections of

$$\Omega_{U(\sigma)/B} = \Omega_{\underline{U(\sigma)}} \oplus (\mathcal{O}_{\underline{U(\sigma)}} \otimes_{\mathbb{Z}} \mathcal{M}^{\text{gp}}_{U(\sigma)})/\sim$$

has generators $(dz^p, 0)$ for p the generators of P_σ and $(0, 1 \otimes D(q))$ for $q \in \mathcal{M}_X(U(\sigma))^{\text{gp}}$, with relations

$$(dz^{np}, 0) = (0, z^{np} \otimes D(q)),$$

$$\Rightarrow (n \cdot z^{p(n-1)} dz^p, 0) = (0, z^{np} \otimes D(q)),$$

for $f_\sigma(q) = z^{np}$. Let p be in the minimal generator of P_σ, then both $(dz^p, 0)$ and $(0, 1 \otimes D(q))$ are in a set of minimal generators of $\Gamma(\Omega_{U(\sigma)/B})$. The relation induces that relative differentials can not be locally free unless $n = 1$. We then obtain that $(p, 0)$ is in $\text{im}(\phi_\sigma)$ and

$$\text{im}(\phi_\sigma) \cap (M \oplus \{0\}) = P_\sigma \oplus \{0\}.$$

♠

With the assumption of the above theorem, the log map $X \to B$ induces a lattice map $N \times Q_0^* \to Q_B^*$ by taking the dual of

$$Q_B^{\mathrm{gp}} = \overline{\mathcal{M}}_B^{\mathrm{gp}}(B) \to \overline{\mathcal{M}}_X^{\mathrm{gp}}(T) = Q_0^{\mathrm{gp}} \hookrightarrow M \oplus Q_0^{\mathrm{gp}}.$$

It induces a toric morphism $Y \to \operatorname{Spec} \Bbbk[Q_B]$. The map $X \to B$ is the restriction of $Y \to \operatorname{Spec} \Bbbk[Q_B]$ on X. In the gluing situation, since V_p is a logarithmic stratum of a log scheme smooth over a log point B, by theorem 4.1.1, there exists a toric variety X_p such that V_p is a strict toric stratum of X_p. Let (Σ_p, N_p) be the fan of X_p and δ_p be the cone with $V_p = V_{X_p}(\delta_p)$. There is a toric morphism $X_p \to S_B = \operatorname{Spec} \Bbbk[Q_B]$, whose restriction on V_p is the map $V_p \to B \hookrightarrow S_B$.

Recall the definitions of $X_{\boldsymbol{\tau}}$ and $X_{\boldsymbol{\tau}_i}$ in (2.3.1)

$$X_{\boldsymbol{\tau}} := V_{p_1} \times_B^{\mathrm{fs}} \dots \times_B^{\mathrm{fs}} V_{p_{|\mathbf{S}|}}, \quad p_j \in \mathbf{S}$$

$$X_{\boldsymbol{\tau}_i} := V_{p_1} \times_B^{\mathrm{fs}} \dots \times_B^{\mathrm{fs}} V_{p_{|\mathbf{S}_i|}}, \quad p_j \in \mathbf{S}_i.$$

The next proposition studies the structure of $X_{\boldsymbol{\tau}}$ and $X_{\boldsymbol{\tau}_i}$. By [18], the fine, saturated fiber product of toric varieties is determined by the fiber product of fans, which is defined in Appendix B.0.1. As $X_{\boldsymbol{\tau}}$ and $X_{\boldsymbol{\tau}_i}$ are fiber products of toric strata, they are the subschemes of the fiber product of toric varieties. Though the ideal determining $X_{\boldsymbol{\tau}}$ and $X_{\boldsymbol{\tau}_i}$, which is generated by the pullback ideals from V_p, might not be radical, the reduction of them are well understood in terms of toric strata.

Proposition 4.1.2. *The fiber product $X_{\boldsymbol{\tau}}$ is a disjoint union of log schemes and each of them is isomorphic to an irreducible, but possibly non-reduced subscheme $Z_{\boldsymbol{\tau}}$ of the* *toric variety* $X'_{\boldsymbol{\tau}}$ *with fan*

$$\Sigma_{\boldsymbol{\tau}} = \Sigma_{p_1} \times_{\Sigma(B)} \ldots \times_{\Sigma(B)} \Sigma_{p_{|\mathbf{S}|}}, \quad p_j \in \mathbf{S},$$

and with the toric log structure. The reduction of $Z_{\boldsymbol{\tau}}$ *is* $V_{X'_{\boldsymbol{\tau}}}(\delta)$*, with* δ *the fiber product of cones* δ_p *for* $p \in \mathbf{S}$ *over* δ_B*.*

Similarly, for each $i = 1, 2, \ldots, r$*, the fiber product* $X_{\boldsymbol{\tau}_i}$ *is a disjoint union of log schemes and each of them is isomorphic to an irreducible, but possibly non-reduced subscheme* $Z_{\boldsymbol{\tau}_i}$ *of the toric variety* $X'_{\boldsymbol{\tau}_i}$ *with fan*

$$\Sigma_{\boldsymbol{\tau}_i} = \Sigma_{p_1} \times_{\Sigma(B)} \ldots \times_{\Sigma(B)} \Sigma_{p_{|\mathbf{S}_i|}}, \quad p_j \in \mathbf{S}_i,$$

with the toric log structures. The reduction of $Z_{\boldsymbol{\tau}_i}$ *is* $V_{X'_{\boldsymbol{\tau}_i}}(\delta_i)$*, with* δ_i *the fiber product of* δ_p *for* $p \in \mathbf{S}_i$ *over* δ_B*.*

The fiber product $X'_{\boldsymbol{\tau}} \times_{\prod_{i=1}^r X'_{\boldsymbol{\tau}_i}} \prod_{i=1}^r Z_{\boldsymbol{\tau}_i}$ *is* $Z_{\boldsymbol{\tau}}$*. With* $Z_{\boldsymbol{\tau}_i} \to X_{\boldsymbol{\tau}_i}$ *being the embedding of one component, the fiber product* $X_{\boldsymbol{\tau}} \times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}} \prod_{i=1}^r Z_{\boldsymbol{\tau}_i}$ *is a disjoint union of* $\mathcal{N}$ *schemes, each of which is isomorphic to* $Z_{\boldsymbol{\tau}}$*. Here,*

$$\mathcal{N} = [L^{\mathrm{sat}} : L], \quad L = \mathrm{im}(\overline{\Delta}_p) + \prod_{i=1}^{r} \mathrm{im}(\overline{\Delta}_{\boldsymbol{\tau}_i}),$$

where $\overline{\Delta}_p$ *is the diagonal map of* $\prod_{p \in \mathbf{S}} N_p \to \prod_{p \in \mathbf{S}} N_p \times N_p$ *and* $\overline{\Delta}_{\boldsymbol{\tau}_i}$ *is the lattice projection from* $\prod_{i=1}^r \Sigma_{\boldsymbol{\tau}_i}$ *to* $\prod_{p \in \mathbf{S}} N_p \times N_p$*.*

Proof. By Lemma B.0.2, the fine, saturated log fiber product

$$X_{p_1} \times^{\mathrm{fs}}_{S_B} \ldots \times^{\mathrm{fs}}_{S_B} X_{p_{|\mathbf{S}|}}, \quad p_j \in \mathbf{S} \tag{4.1.2}$$

is a disjoint union of log schemes, each of which is isomorphic to the toric variety $X'_{\boldsymbol{\tau}}$ of the fiber product of fans $(\Sigma_{\boldsymbol{\tau}}, N_{\boldsymbol{\tau}})$, with its toric log structure. Let I_{p_i} be the ideal sheaf of X_{p_i} that defines V_{p_i}. The scheme $X_{\boldsymbol{\tau}}$ is then the subscheme of (4.1.2) generated by the pullback of I_{p_i}. For toric morphisms $X'_{\boldsymbol{\tau}} \to X_p$ and a cone $\delta \in \Sigma(X')$, the image of $V_{X'_{\boldsymbol{\tau}}}(\delta)$ is contained in V_p if and only if the image of δ under the fan map $N_{\boldsymbol{\tau}} \to N_p$ intersects with the interior of δ_p. Hence, the reduction of $Z_{\boldsymbol{\tau}}$ is determined by the minimal cones δ with image intersecting with the interior of δ_p. Let $\delta = \delta_{p_1} \times_{\delta_B} ... \times_{\delta_B} \delta_{p_{|\mathbf{S}|}}$. The maps $\delta_p \to \delta_B$ are surjective, following the integrality of X/B. Thus δ is mapped to the interior of δ_p under the projection map, and is the minimal cone satisfying the conditions. Therefore, the ideal I defines an irreducible subscheme $Z_{\boldsymbol{\tau}}$ whose reduction is the toric strata $V_{X'_{\boldsymbol{\tau}}}(\delta)$. The proof works the same for $\boldsymbol{\tau}_i$.

The subscheme $Z_{\boldsymbol{\tau}}$ and the fiber product $X'_{\boldsymbol{\tau}} \times_{\prod_{i=1}^r X'_{\boldsymbol{\tau}_i}} \prod_{i=1}^r Z_{\boldsymbol{\tau}_i}$ are both the subschemes of $X'_{\boldsymbol{\tau}}$ determined by the pullback of ideals I_p under $X'_{\boldsymbol{\tau}} \to \prod_p X_p$. Hence they are the same. For the last statement, by the following lemma 4.1.3, the equation

$$X_{\boldsymbol{\tau}} \times_{\left(\prod_{i=1}^r X_{\boldsymbol{\tau}_i}\right)} \left(\prod_{i=1}^r Z_{\boldsymbol{\tau}_i}\right) = \prod_{p \in \mathbf{S}} X_p \times_{\left(\prod_{p\in\mathbf{S}} X_p \times X_p\right)} \left(\prod_{i=1}^r Z_{\boldsymbol{\tau}_i}\right)$$

holds. By Lemma B.0.2, the right side is the union of $Z_{\boldsymbol{\tau}}$ with the number of the components $\mathcal{N}$ being the lattice index

$$\mathcal{N} = [(\mathrm{im}(\overline{\Delta}_p) + \prod_{i=1}^r \mathrm{im}(\overline{\Delta}_{\boldsymbol{\tau}_i}))^{\mathrm{sat}} : \mathrm{im}(\overline{\Delta}_p) + \prod_{i=1}^r \mathrm{im}(\overline{\Delta}_{\boldsymbol{\tau}_i})].$$

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Lemma 4.1.3. *Assume the graph G of $\boldsymbol{\tau}$ is connected. There is a Cartesian diagram in the category of fine, saturated logarithmic schemes*

$$\begin{array}{ccc} X_{\boldsymbol{\tau}} & \longrightarrow & \prod_{i=1}^{r} X_{\boldsymbol{\tau}_i} \\ \downarrow & & \downarrow \\ \prod_{p\in\mathbf{S}} X_p & \xrightarrow{\Delta_p} & \prod_{p\in\mathbf{S}} X_p \times X_p, \end{array}$$

with horizontal maps the diagonal maps and the vertical maps the composition of the projections

$$g_{\boldsymbol{\tau}} : X_{\boldsymbol{\tau}} \to \prod_{p\in\mathbf{S}} V_p, \quad g_{\boldsymbol{\tau}_i} : X_{\boldsymbol{\tau}_i} \to \prod_{p\in\mathbf{S}_i} V_p$$

with the closed embeddings $V_p \hookrightarrow X_p$.

Proof. As $V_p \hookrightarrow X_p$ is a strict closed embedding, it is sufficient to show that the following diagram is Cartesian in the category of fine, saturated log schemes

$$\begin{array}{ccc} X_{\boldsymbol{\tau}} & \xrightarrow{\Delta_X} & \prod_{i=1}^{r} X_{\boldsymbol{\tau}_i} \\ \Big\downarrow{\scriptstyle g_{\boldsymbol{\tau}}} & & \Big\downarrow{\scriptstyle \prod g_{\boldsymbol{\tau}_i}} \\ \prod_{p\in\mathbf{S}} V_p & \xrightarrow{\Delta_p} & \prod_{p\in\mathbf{S}} V_p \times V_p. \end{array} \tag{4.1.3}$$

Let Z be a fine, saturated log scheme with $\alpha : Z \to \prod_{i=1}^{r} X_{\boldsymbol{\tau}_i}$ and $\beta : Z \to \prod_{p\in\mathbf{S}} V_p$, such that $\prod_{i=1}^{r} g_{\boldsymbol{\tau}_i} \circ \alpha = \Delta_p \circ \beta$ as logarithmic maps. Then, there is a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\alpha} & \prod_{i=1}^{r} X_{\boldsymbol{\tau}_i} & \longrightarrow & \prod_{i=1}^{r} B \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\prod g_{\boldsymbol{\tau}_i}} & & \Big\downarrow \\ \prod_{p\in\mathbf{S}} V_p & \xrightarrow{\Delta_p} & \prod_{p\in\mathbf{S}} V_p \times V_p & \searrow & \\ \downarrow & & & & \\ \prod_{p\in\mathbf{S}} B & & \longrightarrow & & \prod_{p\in\mathbf{S}} B \times B. \end{array}$$

By the universal property of the fiber products, there is a morphism

$$Z \to \Big(\prod_{i=1}^{r} B\Big) \times_{\left(\prod_{p\in\mathbf{S}} B\times B\right)} \Big(\prod_{p\in\mathbf{S}} B\Big).$$

As the dual graph G of $\boldsymbol{\tau}$ is connected, the pullback of $\prod_{i=1}^{r} B$ along the diagonal map identifies the base of each $X_{\boldsymbol{\tau}_i}$. Thus $\left(\prod_{i=1}^{r} B\right) \times_{\left(\prod_{p\in\mathbf{S}} B\times B\right)} \left(\prod_{p\in\mathbf{S}} B\right) = B$, with the maps from B to each factor being diagonal maps. For each $p \in \mathbf{S}$, the projection $Z \to V_p \to B$ is the same as the map

$$Z \to B \xrightarrow{\Delta_B} \prod_{p\in\mathbf{S}} B \xrightarrow{\mathrm{pr}_p} B.$$

By the universal property of the logarithmic fiber products, there is a unique morphism

$$\psi : Z \to X_{\boldsymbol{\tau}} = V_{p_1} \times_B \ldots \times_B V_{p_{|\mathbf{S}|}},$$

with $g_{\boldsymbol{\tau}} \circ \psi = \beta$. On the other hand, both α and $\Delta_X \circ \psi$ are the unique morphism induced from the universal property of the fiber product

$$\prod_{i=1}^{r} X_{\boldsymbol{\tau}_i} = \big(\prod_{i=1}^{r} B\big) \times_{\left(\prod_{p\in\mathbf{S}} B\times B\right)} \big(\prod_{p\in\mathbf{S}} V_p \times V_p\big).$$

Hence $\Delta_X \circ \psi = \alpha$ and we finish the proof of the diagram (4.1.3) being Cartesian in the category of log fine, saturated schemes.

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4.2 Local Toric Models of the Gluing Formalism

We are now ready to study the gluing formalism under the Assumption 1.1.2. We first study the local structure of the splitting morphism

$$\delta^{\mathrm{ev}}_{\mathrm{red}} : \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}} \to \prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}. \tag{4.2.1}$$

The main result of this section is Proposition 4.2.1, which provides a local splitting equation ($\star$) of a geometric point after a base change to an étale neighborhood. The idea is to analyze $\delta^{\mathrm{ev}}_{\mathrm{red}}$ under the following commutative diagram obtained from the fiber diagram (2.3.2)

$$\begin{array}{ccc} \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}} & \xrightarrow{\delta^{\mathrm{ev}}_{\mathrm{red}}} & \prod_{i=1}^r \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}} \\ \Big\downarrow{\scriptstyle \mathrm{ev}} & & \Big\downarrow{\scriptstyle \prod \mathrm{ev}_{\tau_i}} \\ X_{\boldsymbol{\tau}} & \xrightarrow{\Delta_X} & \prod_{i=1}^r X_{\boldsymbol{\tau}_i}. \end{array}$$

Let us first construct the étale base change for a geometric point $\overline{x}$ on $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$. Let

$$Q_{\overline{x}} = \overline{\mathcal{M}}_{\prod \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}},\delta^{\mathrm{ev}}_{\mathrm{red}}(\overline{x})}, \quad L_{\overline{x}} = \overline{\mathcal{K}}_{\prod \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}},\delta^{\mathrm{ev}}_{\mathrm{red}}(\overline{x})}.$$

By [4, Appendix B.2], there is a connected strict étale neighborhood $U_{\overline{x}}$ of $\delta^{\mathrm{ev}}_{\mathrm{red}}(\overline{x})$ in $\prod_{i=1}^r \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}$ such that there is a commutative diagram

$$\begin{array}{ccccc} \prod_{i=1}^r \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}} & \longleftarrow & U_{\overline{x}} & \longrightarrow & \mathcal{A}_{Q_{\overline{x}},L_{\overline{x}}} \\ \downarrow{\scriptstyle \prod \mathrm{ev}_{\boldsymbol{\tau}_i}} & & \downarrow{\scriptstyle \mathrm{ev}_U} & & \downarrow \\ \prod_{i=1}^r X_{\boldsymbol{\tau}_i} & \longleftarrow & \prod_{i=1}^r Z_{\boldsymbol{\tau}_i} & \rightarrow & \prod_{i=1}^r \mathcal{A}_{\boldsymbol{\tau}_i}, \end{array} \tag{4.2.2}$$

with $\mathcal{A}_{\tau_i}$ the Artin fan of the toric variety $X'_{\boldsymbol{\tau}_i}$. Define

$$U^{\mathrm{gl}}_{\overline{x}} := U_{\overline{x}} \times_{\prod_{i=1}^r \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}} \widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}. \tag{4.2.3}$$

We wish to study the diagonal map $\delta^{\mathrm{ev}}_U : U^{\mathrm{gl}}_{\overline{x}} \to U_{\overline{x}}$.

Proposition 4.2.1. *Let $\mathfrak{V}$, $\Delta(\mathfrak{V})$ and $m_{[\boldsymbol{\rho}]}$ be the global gluing data associated to the splitting morphism $\delta : \mathscr{M}(X/B,\boldsymbol{\tau}) \to \prod_{i=1}^r \mathscr{M}(X/B,\boldsymbol{\tau}_i)$ defined in Definition 1.1.4. Let $\delta^{\mathrm{ev}}_U : U^{\mathrm{gl}}_{\overline{x}} \to U_{\overline{x}}$ be an étale local model of (4.2.1) at a geometric point $\overline{x}$ of $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$, defined by the above diagrams (4.2.2) and (4.2.3). Let*

$$U^{\boldsymbol{\rho}}_{\overline{x}} = U_{\overline{x}} \times_{\prod_{i=1}^r \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}} \prod_{i=1}^r \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i,\mathrm{red}},$$

with $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i,\mathrm{red}}$ the image substack of the finite morphism $\widetilde{j}_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i}:\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\rho}_i,\mathrm{red}}\to\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}$.

Then, in the Chow group of $U_{\overline{x}}$,

$$\delta^{\mathrm{ev}}_{U*}([U^{\mathrm{gl}}_{\overline{x}}])=\sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})}m_{[\boldsymbol{\rho}]}\cdot[U^{\boldsymbol{\rho}}_{\overline{x}}].\qquad(\star)$$

Proof. Note that $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$ is the reduction of $\prod_{i=1}^r\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}\times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}}X_{\boldsymbol{\tau}}$, so $U^{\mathrm{gl}}_{\overline{x}}$ is the reduction of $U_{\overline{x}}\times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}}X_{\boldsymbol{\tau}}$. As $U_{\overline{x}}$ is connected, the evaluation map ev_U factors through one component of $\prod_{i=1}^r Z_{\boldsymbol{\tau}_i}$. Hence,

$$U^{\mathrm{gl}}_{\overline{x}}=[U_{\overline{x}}\times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}}X_{\boldsymbol{\tau}}]_{\mathrm{red}}=[U_{\overline{x}}\times_{\prod_{i=1}^r Z_{\boldsymbol{\tau}_i}}(\prod_{i=1}^r Z_{\boldsymbol{\tau}_i}\times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}}X_{\boldsymbol{\tau}})]_{\mathrm{red}}.$$

By Proposition 4.1.2, there is a lattice index $\mathcal{N}$, such that $\prod_{i=1}^r Z_{\boldsymbol{\tau}_i}\times_{\prod_{i=1}^r X_{\boldsymbol{\tau}_i}}X_{\boldsymbol{\tau}}$ is a disjoint union of $\mathcal{N}$ schemes, each of which is isomorphic to $Z_{\boldsymbol{\tau}}$. Hence, $U^{\mathrm{gl}}_{\overline{x}}$ is the disjoint union of $\mathcal{N}$ schemes, each of which is isomorphic to the reduction of

$$U^{\mathrm{gl,ir}}_{\overline{x}}:=U_{\overline{x}}\times_{\prod_{i=1}^r Z_{\boldsymbol{\tau}_i}}Z_{\boldsymbol{\tau}}=U_{\overline{x}}\times_{\prod_{i=1}^r X'_{\boldsymbol{\tau}_i}}X'_{\boldsymbol{\tau}},\qquad(4.2.4)$$

where the equality follows from Proposition 4.1.2. Hence, it is sufficient to study the diagonal map $\delta'_U:U^{\mathrm{gl,ir}}_{\overline{x}}\to U_{\overline{x}}$.

First, we observe that the evaluation map $U_{\overline{x}}\to\prod_{i=1}^r X'_{\boldsymbol{\tau}_i}$ is idealized log smooth. It is sufficient to show that $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}\to X'_{\boldsymbol{\tau}_i}$ is idealized log smooth. By Corollary 2.2.6, $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}_i,\mathrm{red}}$ is idealized log smooth over B. Locally, the map $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}_i,\mathrm{red}}\to B$ factors through

$$\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}_i,\mathrm{red}}\xrightarrow{\mathrm{ev}_{\boldsymbol{\tau}_i}}\mathcal{Z}_{\boldsymbol{\tau}_i}\hookrightarrow\mathcal{X}'_{\boldsymbol{\tau}_i}\to B,$$

with $\mathcal{Z}_{\boldsymbol{\tau}_i}$ being the relative Artin fan $\mathcal{A}_{Z_{\boldsymbol{\tau}_i}}\times_{\mathcal{A}_B}B$ and $\mathcal{X}'_{\boldsymbol{\tau}_i}$ being $\mathcal{A}_{\boldsymbol{\tau}_i}\times_{\mathcal{A}_B}B$. Note $\mathcal{X}'_{\boldsymbol{\tau}_i}$ is logarithmically étale over B, hence the evaluation map $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau}_i,\mathrm{red}}\to\mathcal{X}'_{\boldsymbol{\tau}_i}$ is idealized log smooth. The map from the evaluation enhancement

$$\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}} = \widetilde{\mathfrak{M}}_{\boldsymbol{\tau}_i,\mathrm{red}} \times_{\mathcal{X}_{\boldsymbol{\tau}_i}} X_{\boldsymbol{\tau}_i} = \widetilde{\mathfrak{M}}_{\boldsymbol{\tau}_i,\mathrm{red}} \times_{\mathcal{Z}_{\boldsymbol{\tau}_i}} Z_{\boldsymbol{\tau}_i} = \widetilde{\mathfrak{M}}_{\boldsymbol{\tau}_i,\mathrm{red}} \times_{\mathcal{X}'_{\boldsymbol{\tau}_i}} X'_{\boldsymbol{\tau}_i}$$

to $X'_{\boldsymbol{\tau}_i}$ is then idealized log smooth.

Next, we study the local splitting morphism using the toric local model of idealized log smooth morphisms. Following (4.2.2), let us define

$$\mathcal{A}^{\mathrm{ev}}_{\overline{x}} = \mathcal{A}_{Q_{\overline{x}},L_{\overline{x}}} \times_{\prod_{i=1}^r \mathcal{A}_{\boldsymbol{\tau}_i}} \prod_{i=1}^r X'_{\boldsymbol{\tau}_i} \quad \text{and} \quad \mathcal{A}^{\mathrm{gl,ev}}_{\overline{x}} = \mathcal{A}^{\mathrm{ev}}_{\overline{x}} \times_{\prod_{i=1}^r X'_{\boldsymbol{\tau}_i}} X'_{\boldsymbol{\tau}}. \tag{4.2.5}$$

In the fine, saturated Cartesian diagram

$$\begin{array}{ccccc} U^{\mathrm{gl,ir}}_{\overline{x}} & \xrightarrow{\phi} & \mathcal{A}^{\mathrm{gl,ev}}_{\overline{x}} & \longrightarrow & X'_{\boldsymbol{\tau}} \\ \downarrow{\scriptstyle \delta'_U} & & \downarrow{\scriptstyle \delta^{\mathrm{ev}}_{\mathcal{A}}} & & \downarrow \\ U_{\overline{x}} & \xrightarrow{\iota} & \mathcal{A}^{\mathrm{ev}}_{\overline{x}} & \longrightarrow & \prod_{i=1}^r X'_{\boldsymbol{\tau}_i} \\ & & \downarrow & & \downarrow \\ & & \mathcal{A}_{Q_{\overline{x}},L_{\overline{x}}} & \longrightarrow & \prod_{i=1}^r \mathcal{A}_{\boldsymbol{\tau}_i}, \end{array} \tag{4.2.6}$$

the map $\iota : U_{\overline{x}} \to \mathcal{A}^{\mathrm{ev}}_{\overline{x}}$ is induced by the universal property of the fiber product. Since the evaluation map $U_{\overline{x}} \to \prod_{i=1}^r X'_{\boldsymbol{\tau}_i}$ is idealized log smooth, by [4, Appendix B.4], the map ι is smooth. As the diagonal map $X'_{\boldsymbol{\tau}} \to \prod_{i=1}^r X'_{\boldsymbol{\tau}_i}$ is proper, both vertical maps δ'_U and $\delta^{\mathrm{ev}}_{\mathcal{A}}$ in diagram (4.2.6) are proper. We obtain that

$$\delta'_{U*}[U^{\mathrm{gl,ir}}_{\overline{x}}] = \delta'_{U*}\phi^*[\mathcal{A}^{\mathrm{gl,ev}}_{\overline{x}}] = \iota^*\delta^{\mathrm{ev}}_{\mathcal{A}*}[\mathcal{A}^{\mathrm{gl,ev}}_{\overline{x}}] \tag{4.2.7}$$

for $\phi : U^{\mathrm{gl,ir}}_{\overline{x}} \to \mathcal{A}^{\mathrm{gl,ev}}_{\overline{x}}$ in the diagram (4.2.6).

It is sufficient to study $\delta^{\mathrm{ev}}_{\mathcal{A}*}[\mathcal{A}^{\mathrm{gl,ev}}_{\overline{x}}]$ using the concrete toric stack description of $\mathcal{A}^{\mathrm{gl,ev}}_{\overline{x}}$ and $\mathcal{A}^{\mathrm{ev}}_{\overline{x}}$. Let $[\boldsymbol{\omega}] = (\boldsymbol{\omega}_1, ..., \boldsymbol{\omega}_r)$ be the global type of $\delta^{\mathrm{ev}}_{\mathrm{red}}(\overline{x})$. For $[\boldsymbol{\rho}] = (\boldsymbol{\rho}_1, ..., \boldsymbol{\rho}_r)$

a global type that admits a contraction to $[\boldsymbol{\tau}] = (\boldsymbol{\tau}_1, ..., \boldsymbol{\tau}_r)$, we define

$$\mathcal{A}_{\overline{x}}^{\mathrm{ev},\boldsymbol{\rho}} := \begin{cases} V_{\mathcal{A}_{\overline{x}}^{\mathrm{ev}}}(\widetilde{\boldsymbol{\rho}}), & \text{if } [\boldsymbol{\rho}] \text{ is a contraction of } [\boldsymbol{\omega}], \\ \varnothing, & \text{otherwise,} \end{cases}$$

where $\widetilde{\boldsymbol{\rho}} = \prod_{i=1}^{r} \widetilde{\boldsymbol{\rho}}_i$ is the evaluation cone associated to $[\boldsymbol{\rho}]$ defined in Definition 2.2.2. By Lemma 2.2.3 and Corollary 2.2.9, the evaluation cone $\widetilde{\boldsymbol{\rho}}$ is a subcone of $\widetilde{\boldsymbol{\omega}} = Q_{\overline{x}}^{\vee}$, hence the stratum $V_{\mathcal{A}_{\overline{x}}^{\mathrm{ev}}}(\widetilde{\boldsymbol{\rho}})$ is well-defined. The following lemma answers the question of $\delta_{\mathcal{A}*}^{\mathrm{ev}}[\mathcal{A}_{\overline{x}}^{\mathrm{gl,ev}}]$, whose proof we defer to later.

Lemma 4.2.2. *Let $\mathfrak{V}$ be a generic displacement as defined in Definition 1.1.4. Then, in the Chow group of $\mathcal{A}_{\overline{x}}^{\mathrm{ev}}$,*

$$\delta_{\mathcal{A}*}^{\mathrm{ev}}[\mathcal{A}_{\overline{x}}^{\mathrm{gl,ev}}] = \sum_{\boldsymbol{\rho} \in \Delta(\mathfrak{V})} m'_{[\boldsymbol{\rho}]} \cdot [\mathcal{A}_{\overline{x}}^{\mathrm{ev},\boldsymbol{\rho}}], \tag{4.2.8}$$

where

$$m'_{[\boldsymbol{\rho}]} = [\prod_{i=1}^{r} N_{\boldsymbol{\tau}_i} : \mathrm{im}(\overline{\Delta}) + \prod_{i=1}^{r} \mathrm{im}(\overline{\mathrm{ev}}_{\boldsymbol{\rho}_i})],$$

Here $\overline{\Delta} : N_{\boldsymbol{\tau}} \to \prod_{i=1}^{r} N_{\boldsymbol{\tau}_i}$ is the lattice diagonal map of $X'_{\boldsymbol{\tau}} \to \prod_{i=1}^{r} X'_{\boldsymbol{\tau}_i}$. The map $\overline{\mathrm{ev}}_{\boldsymbol{\rho}_i} : N_{\widetilde{\boldsymbol{\rho}}_i} \to N_{\boldsymbol{\tau}_i}$ is defined by the factorization of the tropical evaluation maps discussed in (2.2.2)

$$\overline{\mathrm{evt}}_{\boldsymbol{\rho}_i} : N_{\widetilde{\boldsymbol{\rho}}_i} \xrightarrow{\overline{\mathrm{ev}}_{\boldsymbol{\rho}_i}} N_{\boldsymbol{\tau}_i} \to \prod_{p \in \mathbf{S}_i} N_p.$$

At last, we are ready to finish the proof by the following arguments. By Lemma 4.2.2 and equation (4.2.7), in the Chow group of $U_{\overline{x}}$,

$$\delta'_{U*}[U_{\overline{x}}^{\mathrm{gl,ir}}] = \sum_{[\boldsymbol{\rho}] \in \Delta(\mathfrak{V})} m'_{[\boldsymbol{\rho}]} \cdot [\iota^{-1}(\mathcal{A}_{\overline{x}}^{\mathrm{ev},\boldsymbol{\rho}})] = \sum_{[\boldsymbol{\rho}] \in \Delta(\mathfrak{V})} m'_{[\boldsymbol{\rho}]} \cdot [U_{\overline{x}}^{\boldsymbol{\rho}}].$$

The second equality follows from Corollary 2.2.9. By (4.2.7),

$$\delta^{\mathrm{ev}}_{U*}([U^{\mathrm{gl}}_{\overline{x}}]) = \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} \mathcal{N}\cdot m'_{[\boldsymbol{\rho}]}\cdot [U^{\boldsymbol{\rho}}_{\overline{x}}].$$

It is now sufficient to show $\mathcal{N}\cdot m'_{[\boldsymbol{\rho}]} = m_{[\boldsymbol{\rho}]}$.

All the involved lattices can be fit into a commutative diagram

$$\begin{array}{ccccc} N_{\widetilde{\boldsymbol{\rho}}} \times_{\prod_{i=1}^r N_{\boldsymbol{\tau}_i}} N_{\boldsymbol{\tau}} & \longrightarrow & N_{\boldsymbol{\tau}} & \longrightarrow & \prod_{p\in\mathbf{S}} N_p \\ \big\downarrow & & \big\downarrow{\scriptstyle\overline{\Delta}} & & \big\downarrow{\scriptstyle\overline{\Delta}_p} \\ N_{\widetilde{\boldsymbol{\rho}}} = \prod_{i=1}^r N_{\widetilde{\boldsymbol{\rho}}_i} & \xrightarrow{\prod \overline{\mathrm{ev}}_{\boldsymbol{\rho}_i}} & \prod_{i=1}^r N_{\boldsymbol{\tau}_i} & \xrightarrow{\prod_{i=1}^r \overline{\Delta}_{\boldsymbol{\tau}_i}} & \prod_{p\in\mathbf{S}} N_p \times N_p. \end{array} \tag{4.2.9}$$

Note that $\mathcal{N} = [N^{\mathrm{sat}}_{\mathcal{N}} : N_{\mathcal{N}}]$ and $m'_{[\boldsymbol{\rho}]} = [\prod_{i=1}^r N_{\boldsymbol{\tau}_i} : N_m]$, where

$$N_{\mathcal{N}} = \mathrm{im}(\overline{\Delta}_p) + \prod_{i=1}^r \mathrm{im}(\overline{\Delta}_{\boldsymbol{\tau}_i}), \quad N_m = \mathrm{im}(\overline{\Delta}) + \prod_{i=1}^r \mathrm{im}(\overline{\mathrm{ev}}_{\boldsymbol{\rho}_i}).$$

We have $\mathcal{N}\cdot m'_{[\boldsymbol{\rho}]} = [K^{\mathrm{sat}} : K]$ for $K = \mathrm{im}(\overline{\Delta}_p) + \prod_{i=1}^r \mathrm{im}(\overline{\Delta}_{\boldsymbol{\tau}_i}\circ \overline{\mathrm{ev}}_{\boldsymbol{\rho}_i})$. We then obtain that

$$\begin{aligned} [K^{\mathrm{sat}} : K] &= [\mathrm{coker}\, \overline{\Delta}_p(K)^{\mathrm{sat}} : \mathrm{coker}\, \overline{\Delta}_p(K)] \\ &= \big[\mathrm{im}(\prod_{i=1}^r \overline{\varepsilon}_{\boldsymbol{\rho}_i})^{\mathrm{sat}} : \mathrm{im}(\prod_{i=1}^r \overline{\varepsilon}_{\boldsymbol{\rho}_i})\big] = m_{[\boldsymbol{\rho}]}, \end{aligned}$$

with $\overline{\varepsilon}_{\boldsymbol{\rho}_i}$ defined in equation (1.1.2) and Definition 1.1.4. ♠

Proof of Lemma 4.2.2. First, we show that $\delta^{\mathrm{ev}}_{\mathcal{A}}$ is the map of quotient stacks induced from a $T_{Q_{\overline{x}}}$-equivariant map of toric varieties. By Proposition A.0.2,

$$\mathcal{A}^{\mathrm{ev}}_{\overline{x}} = \mathcal{A}_{Q_{\overline{x}},L_{\overline{x}}} \times_{\prod_{i=1}^r \mathcal{A}_{\boldsymbol{\tau}_i}} \prod_{i=1}^r X'_{\boldsymbol{\tau}_i} = [S_{Q_{\overline{x}},L_{\overline{x}}} \times \prod_{i=1}^r T_{\boldsymbol{\tau}_i}/T_{Q_{\overline{x}}}].$$

Let Y be $S_{Q_{\overline{x}},L_{\overline{x}}}\times \prod_{i=1}^r T_{\boldsymbol{\tau}_i}$ and let $Y^{\mathrm{gl}} := Y\times_{\prod_{i=1}^r X'_{\boldsymbol{\tau}_i}} X'_{\boldsymbol{\tau}}$, where the map $Y\to \prod_{i=1}^r X'_{\boldsymbol{\tau}_i}$ is the composition of the quotient map $Y\to \mathcal{A}^{\mathrm{ev}}_{\overline{x}}$ and projection $\mathcal{A}^{\mathrm{ev}}_{\overline{x}} \to \prod_{i=1}^r X_{\boldsymbol{\tau}_i}$.

By Lemma A.0.1, the torus action of $T_{Q_{\overline{x}}}$ on Y induces an action of $T_{Q_{\overline{x}}}$ on Y^{gl}, such that the quotient stack $[Y^{\text{gl}}/T_{Q_{\overline{x}}}]$ is isomorphic to $\mathcal{A}_{\overline{x}}^{\text{gl,ev}}$. The diagonal map $\delta_{\mathcal{A}}^{\text{ev}} : \mathcal{A}_{\overline{x}}^{\text{gl,ev}} \to \mathcal{A}_{\overline{x}}^{\text{ev}}$ is then induced from the quotient of the $T_{Q_{\overline{x}}}$-equivariant map $Y^{\text{gl}} \to Y$. In order to study $\delta_{\mathcal{A}*}^{\text{ev}}[\mathcal{A}_{\overline{x}}^{\text{gl,ev}}]$, it is sufficient to study its reduction $\delta_{\mathcal{A}*}^{\text{ev}}[\mathcal{A}_{\overline{x},\text{red}}^{\text{gl,ev}}]$, hence it is enough to study $Y_{\text{red}}^{\text{gl}} \to Y$.

Next, we show that $Y_{\text{red}}^{\text{gl}} \to Y$ is the restriction of a toric morphism on a toric stratum. Such description allows us to use the generalized Fulton-Sturmfels formula in Proposition 3.0.6 to obtain Lemma 4.2.2. Let $\mathcal{Y}$ be $S_{Q_{\overline{x}}} \times \prod_{i=1}^{r} T_{\boldsymbol{\tau}_i}$. The tropicalization of $\mathcal{A}_{\overline{x}}^{\text{ev}} \to \prod_{i=1}^{r} X'_{\boldsymbol{\tau}_i}$ induces a toric morphism $\text{ev}_{\mathcal{Y}} : \mathcal{Y} \to \prod_{i=1}^{r} X'_{\boldsymbol{\tau}_i}$. By Lemma B.0.2, the fine, saturated fiber product $\mathcal{Y} \times_{\prod_{i=1}^{r} X'_{\boldsymbol{\tau}_i}} X'_{\boldsymbol{\tau}}$ is the disjoint union of n varieties, each of which is isomorphic to a toric variety $\mathcal{Y}^{\text{gl}}$, with

$$n = [\prod_{i=1}^{r} N_{\boldsymbol{\tau}_i} : \overline{\Delta}(N_{\boldsymbol{\tau}}) + \overline{\text{ev}}_{\mathcal{Y}}(N(\mathcal{Y}))], \tag{4.2.10}$$

for $\overline{\Delta}$ defined in (4.2.9). By Corollary A.0.3, the lattice map $\overline{\text{ev}}_{\mathcal{Y}}$ is defined as

$$\overline{\text{ev}}_{\mathcal{Y}} : N(\mathcal{Y}) = N_{Q_{\overline{x}}} \times \prod_{i=1}^{r} N_{\boldsymbol{\tau}_i} \to \prod_{i=1}^{r} N_{\boldsymbol{\tau}_i}, \quad (a, b) \mapsto e(a) - b, \tag{4.2.11}$$

with e the lattice map of the evaluation map $\mathcal{A}_{Q_{\overline{x}}, L_{\overline{x}}} \to \prod_{i=1}^{r} \mathcal{A}_{\boldsymbol{\tau}_i}$. The map $\overline{\text{ev}}_{\mathcal{Y}}$ is surjective, hence $n = 1$.

The scheme

$$Y^{\text{gl}} = Y \times_{\prod_{i=1}^{r} X'_{\boldsymbol{\tau}_i}} X'_{\boldsymbol{\tau}} = Y \times_{\mathcal{Y}} \mathcal{Y}^{\text{gl}}$$

is the subscheme of $\mathcal{Y}^{\text{gl}}$ determined by the pullback of ideal generated by $L_{\overline{x}}$ in $S_{Q_{\overline{x}}}$. In particular, we claim that the subscheme $Y_{\text{red}}^{\text{gl}}$ is the toric stratum $V_{\mathcal{Y}^{\text{gl}}}(\sigma^{\text{gl}})$, for $\sigma^{\text{gl}} = \widetilde{\boldsymbol{\tau}} \times \{0\}$. By the idealized structure on $\widetilde{\mathfrak{M}}_{\boldsymbol{\tau},\text{red}}^{\text{ev}}$ in Remark 2.2.8, $S_{Q_{\overline{x}}, L_{\overline{x}}}$ is $V_{S_{Q_{\overline{x}}}}(\prod_{i=1}^{r} \widetilde{\boldsymbol{\tau}}_i)$.

Therefore Y is the toric strata $V_{\mathcal{Y}}(\prod_{i=1}^{r} \widetilde{\boldsymbol{\tau}}_i \times \{0\})$. We use σ_Y to denote $\prod_{i=1}^{r} \widetilde{\boldsymbol{\tau}}_i \times \{0\}$. Since $\boldsymbol{\tau}$ is realizable and the types $\boldsymbol{\tau}_i$ are obtained by splitting $\boldsymbol{\tau}$, by the construction of the evaluation cones in Definition 2.2.2, $\sigma^{\mathrm{gl}} = \widetilde{\boldsymbol{\tau}} \times \{0\}$ is exactly the fiber product of cones

$$\sigma_Y \times_{\prod_{i=1}^{r} \Sigma(X'_{\boldsymbol{\tau}_i})} \Sigma(X'_{\boldsymbol{\tau}}),$$

from the fiber product of toric varieties $\mathcal{Y} \times_{\prod_{i=1}^{r} X'_{\boldsymbol{\tau}_i}} X'_{\boldsymbol{\tau}}$. Then, the cone σ^{gl} is the unique minimal cone in $\Sigma(\mathcal{Y}^{\mathrm{gl}})$ whose image in $\Sigma(\mathcal{Y})$ intersects the interior of σ_Y. The reduction $Y^{\mathrm{gl}}_{\mathrm{red}}$ is the toric stratum $V_{\mathcal{Y}^{\mathrm{gl}}}(\sigma^{\mathrm{gl}})$.

Now, we are ready to use the generalized Fulton-Sturmfels formula to study the diagonal map $\delta^{\mathrm{ev}}_{\mathcal{A}} : \mathcal{A}^{\mathrm{gl,ev}}_{\overline{x},\mathrm{red}} \to \mathcal{A}^{\mathrm{ev}}_{\overline{x}}$, which is the toric morphism of toric stacks

$$\left[V_{\mathcal{Y}^{\mathrm{gl}}}(\sigma^{\mathrm{gl}})/T_{Q_{\overline{x}}}\right] \to \left[V_{\mathcal{Y}}(\sigma_Y)/T_{Q_{\overline{x}}}\right].$$

Let $\mathfrak{V} \in \prod_{p \in \mathbf{S}} N_p$ be a generic displacement vector defined in Definition 1.1.4. Let ψ be the map

$$\psi : \prod_{i=1}^{r} N_{\boldsymbol{\tau}_i} \to \prod_{p \in \mathbf{S}} N_p \times N_p \to \prod_{p \in \mathbf{S}} N_p,$$

whose first map is the projections of fiber products $N_{\boldsymbol{\tau}_i}$ to N_p and the second map is the cokernel of the diagonal map of N_p. We first show that there exists an element $v \in N(\mathcal{Y})$ such that $\psi \circ \overline{\mathrm{ev}}_{\mathcal{Y}}(v) = \mathfrak{V}$. Let N' be the sublattice of $\prod_{p \in \mathbf{S}} N_p$, whose image in N_B under maps $\prod_{p \in \mathbf{S}} N_p \to N_p \to N_B$ are the same for any $p \in \mathbf{S}$. By definition, vector $\mathfrak{V}$ lies in N'. Note that

$$\psi(N_{\boldsymbol{\tau}_1} \times_{N_B} \ldots \times_{N_B} N_{\boldsymbol{\tau}_r}) = N'.$$

Hence $\psi^{-1}(\mathfrak{V})$ is non-empty. Since $\overline{\mathrm{ev}}_{\mathcal{Y}}$ is surjective, we obtain that there exists v in $N(\mathcal{Y})$ such that $\psi \circ \overline{\mathrm{ev}}_{\mathcal{Y}}(v) = \mathfrak{V}$. Next, we want to show that v is a generic displacement vector associated to $(\mathcal{Y}, \mathcal{Y}^{\mathrm{gl}}, Y^{\mathrm{gl}}_{\mathrm{red}})$ as defined in Definition 3.0.4. The equation (4.2.8) can be obtained using Fulton-Sturmfels formula in Corollary 3.0.6 associated to v.

1. *The vector v is generic with respect to* $(\mathcal{Y}, \mathcal{Y}^{\mathrm{gl}}, Y^{\mathrm{gl}}_{\mathrm{red}})$. Let $f : N(\mathcal{Y}^{\mathrm{gl}}) \to N(\mathcal{Y})$ be the lattice map and let $q_{\mathcal{Y}} : N(\mathcal{Y}) \to N(\mathcal{Y})/N_{\sigma_Y}$ be the lattice quotient map. We need to show for any cone $\omega \in \Sigma(\mathcal{Y})$ satisfying the conditions $(1), (2), (3)$ in Definition 3.0.4, the intersection $q_{\mathcal{Y},\mathbb{R}}((\mathrm{im}(f) + v) \cap \omega)$ lies in the interior of the cone $q_{\mathcal{Y},\mathbb{R}}(\omega)$. Following conditions $(1), (3)$, ω determines a unique type $[\boldsymbol{\rho}] = (\boldsymbol{\rho}_1, ..., \boldsymbol{\rho}_r)$ with property $(i), (ii)$ in Definition 1.1.4 (2). Furthermore, the dimension of ω satisfies the dimension condition in property (iii) since

$$\begin{aligned}\sum_{i=1}^{r} \dim \widetilde{\boldsymbol{\rho}}_i - \dim \widetilde{\boldsymbol{\tau}} &= \dim \omega - \dim \sigma_Y^{\mathrm{gl}} \\ &\overset{Def 3.0.4}{=} \dim N(\mathcal{Y}) - \dim \mathrm{im}(f) + \dim(\mathrm{im}(f) \cap \sigma_Y) - \dim \sigma_Y^{\mathrm{gl}} \\ &\overset{(1)}{=} \dim N(\mathcal{Y}) - \dim \mathrm{im}(f) = \sum_{i=1}^{r} N_{\boldsymbol{\tau}_i} - \dim N_{\boldsymbol{\tau}} \\ &= \sum_{p \in \mathbf{S}} \dim N_p - (|\mathbf{S}| - r + 1) \cdot \mathrm{rank}\, Q_B^{\mathrm{gp}}.\end{aligned}$$

Here (1) is true since f is injective and $\mathrm{im}(f) \cap \sigma_Y = \mathrm{im}(\sigma_Y^{\mathrm{gl}})$.

Recall that $\varepsilon_{\boldsymbol{\rho}_i}$ is defined in (1.1.2)

$$\prod_{i=1}^{r} \varepsilon_{\boldsymbol{\rho}_i} : \prod_{i=1}^{r} \widetilde{\boldsymbol{\rho}}_i \xrightarrow{\prod \mathrm{evt}_{\boldsymbol{\rho}_i}} \prod_{p \in \mathbf{S}} N_{p,\mathbb{R}} \times N_{p,\mathbb{R}} \xrightarrow{\prod \mathrm{coker}\, \overline{\Delta}_p} \prod_{p \in \mathbf{S}} N_{p,\mathbb{R}}.$$

As $\mathfrak{V}$ is generic, the map $\prod_{i=1}^{r} \varepsilon_{\boldsymbol{\rho}_i}$ is injective and $\mathfrak{V}$ lies in the interior of its image. It is equivalent to say that $(\prod_{i=1}^{r} \varepsilon_{\boldsymbol{\rho}_i})^{-1}(\mathfrak{V})$ intersects with the interior of

ω', with $\omega = \omega' \times \{0\}$. Hence, $(\overline{\mathrm{ev}}_{\mathcal{Y}})^{-1}\psi^{-1}(\mathfrak{V})$ intersects with the interior of cone ω. As $\mathrm{im}(f)$ contains the kernel of $\psi(\overline{\mathrm{ev}}_{\mathcal{Y}})$, the set $(\overline{\mathrm{ev}}_{\mathcal{Y}})^{-1}\psi^{-1}(\mathfrak{V})$ is contained in $\mathrm{im}(f) + v$. Hence, the intersection of $\mathrm{im}(f) + v$ with ω is not empty and is in the interior of ω. It follows that the intersection $q_{\mathcal{Y},\mathbb{R}}((\mathrm{im}(f) + v) \cap \omega)$ lies in the interior of the cone $q_{\mathcal{Y},\mathbb{R}}(\omega)$.

2. *Lemma (4.2.2) follows from Fulton-Sturmfels Formula in Corollary 3.0.6 on v.* By Corollary 3.0.6, we have

$$\delta^{\mathrm{ev}}_{\mathcal{A}*}\Big[V_{\mathcal{Y}^{\mathrm{gl}}}(\sigma^{\mathrm{gl}})/T_{Q_{\overline{x}}}\Big] = \sum_{\rho' \in \Delta^{\sigma^{\mathrm{gl}}}(v)} [N(\mathcal{Y}) : \mathrm{im}(f) + N_{\rho'}] \cdot [V_{\mathcal{Y}}(\rho')/T_{Q_{\overline{x}}}],$$

with $\Delta^{\sigma^{\mathrm{gl}}}(v)$ the set of cones in $N(\mathcal{Y})$ defined in Definition 3.0.4. There is a bijection between types in $\Delta(\mathfrak{V})$ and cones in $\Delta^{\sigma^{\mathrm{gl}}}(v)$, by taking a type $[\boldsymbol{\rho}]$ to $\prod_{i=1}^{r} \widetilde{\boldsymbol{\rho}}_i \times \{0\}$. The substack $\mathcal{A}^{\mathrm{ev},\boldsymbol{\rho}}_{\overline{x}}$ is the same as $[V_{\mathcal{Y}}(\rho')/T_{Q_{\overline{x}}}]$ following the stratification of the moduli space in Corollary 2.2.9. As for multiplicities, since $\overline{\mathrm{ev}}_{\mathcal{Y}}$ is surjective and the kernel is contained in $\mathrm{im}(f)$, by taking the quotient of $\ker(\overline{\mathrm{ev}}_{\mathcal{Y}})$, we get

$$\begin{aligned}[N(\mathcal{Y}) : \mathrm{im}(f) + N_{\rho'}] &= [\prod_{i=1}^{r} N_{\boldsymbol{\tau}_i} : \overline{\mathrm{ev}}_{\mathcal{Y}}(\mathrm{im}(f) + N_{\rho'})] \\ &= [\prod_{i=1}^{r} N_{\boldsymbol{\tau}_i} : \mathrm{im}(\overline{\Delta}) + \prod_{i=1}^{r} \mathrm{im}(\overline{\mathrm{ev}}_{\boldsymbol{\rho}_i})] = m'_{[\boldsymbol{\rho}]}.\end{aligned}$$

♠

Remark 4.2.3. With the same assumption in Lemma 4.2.2, by Corollary 3.0.6, there is a closed substack $K_{\mathcal{A}} \subseteq \mathcal{A}^{\mathrm{ev}}_{\overline{x}} \times \mathbb{P}^1$ and a projection map $\alpha_{\mathcal{A}} : K_{\mathcal{A}} \to \mathbb{P}^1$ such that as algebraic cycles in $\mathcal{A}^{\mathrm{ev}}_{\overline{x}}$,

$$[\alpha_{\mathcal{A}}^{-1}(1)] = [\delta_{\mathcal{A}}^{\mathrm{ev}}(\mathcal{A}^{\mathrm{gl,ev}}_{\overline{x},\mathrm{red}})], \quad [\alpha_{\mathcal{A}}^{-1}(0)] = \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} \frac{m'_{[\boldsymbol{\rho}]}}{\mathfrak{I}_{\overline{x}}} \cdot [\mathcal{A}^{\mathrm{ev},\boldsymbol{\rho}}_{\overline{x}}],$$

with $\mathfrak{I}_{\overline{x}} = [\mathrm{im}(q_{\mathcal{Y}} \circ f)^{\mathrm{sat}} : \mathrm{im}(q_{\mathcal{Y}} \circ f)]$ for $N(\mathcal{Y}^{\mathrm{gl}}) \xrightarrow{f} N(\mathcal{Y}) \xrightarrow{q_{\mathcal{Y}}} N(\mathcal{Y})/N_{\sigma_Y}$ defined as above. The index $\mathfrak{I}_{\overline{x}}$ is the degree of map $\delta^{\mathrm{ev}}_{\mathcal{A}}$.

Similarly, take $K_{U_{\overline{x}}}$ to be the preimage of $K_{\mathcal{A}}$ under the smooth map $\iota \times \mathrm{id}$ from $U_{\overline{x}} \times \mathbb{P}^1$ to $\mathcal{A}^{\mathrm{ev}}_{\overline{x}} \times \mathbb{P}^1$. Let $\alpha_U : K_{U_{\overline{x}}} \to \mathbb{P}^1$ be the projection map. As the diagram (4.2.6) is fine, saturated Cartesian, as algebraic cycles in $U_{\overline{x}}$,

$$[\alpha_U^{-1}(1)] = [\delta_U^{\mathrm{ev}}(U^{\mathrm{gl}}_{\overline{x}})], \quad [\alpha_U^{-1}(0)] = \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} \frac{m_{[\boldsymbol{\rho}]}}{\mathfrak{I}_{\overline{x}}} \cdot [U^{\boldsymbol{\rho}}_{\overline{x}}]. \tag{4.2.12}$$

The closed substack $K_{U_{\overline{x}}}$ induces the local splitting equation ($\star$) in Proposition 4.2.1.

4.3 Gluing of the Local Models

Now, we are ready to prove the main theorem.

Proof of Theorem 1.1.5. For $\overline{x}$ a geometric point on $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$, by Section 4.2, there is an étale neighborhood $U_{\overline{x}}$ of $\delta^{\mathrm{ev}}_{\mathrm{red}}(\overline{x})$ and $U^{\mathrm{gl}}_{\overline{x}}$ such that the Theorem 4.2.1 is satisfied. As $\overline{x}$ goes over the geometric points on $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$, we obtain an étale cover $\bigsqcup_{\overline{x}} U_{\overline{x}}$ of $\prod_{i=1}^{r} \widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}$.

Let $\mathfrak{V}$ be a generic displacement vector defined in Definition 1.1.4. By Proposition 4.2.1 and Remark 4.2.3, there are closed substacks $K_{U_{\overline{x}}}$ of $U_{\overline{x}} \times \mathbb{P}^1$ that induces the rational equivalence condition (4.2.12). We first show that for the geometric points $\overline{x}$ and $\overline{x}'$ in the same connected component of $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$, the indices $\mathfrak{I}_{\overline{x}}$ in (4.2.12) are the same. It is sufficient to show $\mathfrak{I}_{\overline{x}} = \mathfrak{I}_{\overline{x}'}$ supposing $\overline{x}$ is a generization of $\overline{x}'$. As the idealized structure on $\widetilde{\mathfrak{M}}^{\text{gl,ev}}_{\text{red}}$ is coherent, by [19, Prop II.2.6.1], the ideal $L_{\overline{x}}$ is generated by $L_{\overline{x}'}$ under the map $Q_{\overline{x}'} \to Q_{\overline{x}}$. We then obtain an open embedding of stacks $\mathcal{A}_{Q_{\overline{x}},L_{\overline{x}}} \to \mathcal{A}_{Q_{\overline{x}'},L_{\overline{x}'}}$. Then there is a fiber diagram

$$\begin{array}{ccccc} \mathcal{A}^{\text{gl,ev}}_{\overline{x}} & \hookrightarrow & \mathcal{A}^{\text{gl,ev}}_{\overline{x}'} & \longrightarrow & X'_{\boldsymbol{\tau}} \\ \downarrow{\scriptstyle \delta^{\text{ev}}_{\mathcal{A},\overline{x}'}} & & \downarrow{\scriptstyle \delta^{\text{ev}}_{\mathcal{A},\overline{x}}} & & \downarrow \\ \mathcal{A}^{\text{ev}}_{\overline{x}} & \hookrightarrow & \mathcal{A}^{\text{ev}}_{\overline{x}'} & \longrightarrow & \prod_{i=1}^r X'_{\boldsymbol{\tau}_i} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{A}_{Q_{\overline{x}},L_{\overline{x}}} & \hookrightarrow & \mathcal{A}_{Q_{\overline{x}'},L_{\overline{x}'}} & \longrightarrow & \prod_{i=1}^r \mathcal{A}_{\boldsymbol{\tau}_i}. \end{array}$$

As the indices $\mathfrak{I}_{\overline{x}}$ and $\mathfrak{I}_{\overline{x}'}$ are the degrees of the morphisms $\delta^{\text{ev}}_{\mathcal{A},\overline{x}}$ and $\delta^{\text{ev}}_{\mathcal{A},\overline{x}'}$, it follows from the diagram that they are the same.

Assume that $\widetilde{\mathfrak{M}}^{\text{gl,ev}}_{\text{red}}$ has one connected component and let $\mathfrak{I}$ be the index $\mathfrak{I}_{\overline{x}}$ for any geometric point $\overline{x}$. Let $K_{\boldsymbol{\tau}}$ to be the closure of the image of $\bigsqcup K_{U_{\overline{x}}}$ in $\prod_{i=1}^r \widetilde{\mathfrak{M}}^{\text{ev}}_{\boldsymbol{\tau}_i,\text{red}} \times \mathbb{P}^1$. Let α be the projection map $\alpha : K_{\boldsymbol{\tau}} \to \mathbb{P}^1$. Then, as algebraic cycles in $\prod_{i=1}^r \widetilde{\mathfrak{M}}^{\text{ev}}_{\boldsymbol{\tau}_i,\text{red}}$,

$$\begin{aligned} [\alpha^{-1}(1)] &= \big[\bigcup_{\overline{x}} \psi_{\overline{x}} \circ \delta^{\text{ev}}_U(U^{\text{gl}}_{\overline{x}})\big] = \big[\delta^{\text{ev}}_{\text{red}}(\widetilde{\mathfrak{M}}^{\text{gl,ev}}_{\text{red}})\big] \\ [\alpha^{-1}(0)] &= \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} \frac{m_{[\boldsymbol{\rho}]}}{\mathfrak{I}} \cdot \big[\bigcup_{\overline{x}} \psi_{\overline{x}}(U^{\boldsymbol{\rho}}_{\overline{x}})\big] \\ &= \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} \frac{m_{[\boldsymbol{\rho}]}}{\mathfrak{I}} \cdot \big[\prod_{i=1}^r \widetilde{j}_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i}(\widetilde{\mathfrak{M}}^{\text{ev}}_{\boldsymbol{\rho}_i,\text{red}})\big], \end{aligned}$$

where $\psi_{\overline{x}}$ is the étale map from $U_{\overline{x}}$ to $\prod_{i=1}^r \widetilde{\mathfrak{M}}^{\text{ev}}_{\boldsymbol{\tau}_i,\text{red}}$ and $\widetilde{j}_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i} : \widetilde{\mathfrak{M}}^{\text{ev}}_{\boldsymbol{\rho}_i,\text{red}} \to \widetilde{\mathfrak{M}}^{\text{ev}}_{\boldsymbol{\tau}_i,\text{red}}$ is the finite map induced from the contraction morphism from $\boldsymbol{\rho}_i$ to $\boldsymbol{\tau}_i$ for each $i = 1, ..., r$. As the degree of $\delta^{\text{ev}}_{\text{red}}$ is the same as δ^{ev}_U, we obtain the equation

$$\delta^{\text{ev}}_{\text{red}*}[\widetilde{\mathfrak{M}}^{\text{gl,ev}}_{\text{red}}] = \mathfrak{I} \cdot [\delta^{\text{ev}}_{\text{red}}(\widetilde{\mathfrak{M}}^{\text{gl,ev}}_{\text{red}})] = \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} m_{[\boldsymbol{\rho}]} \cdot \prod_{i=1}^r [\widetilde{j}_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i}(\widetilde{\mathfrak{M}}^{\text{ev}}_{\boldsymbol{\rho}_i,\text{red}})]. \tag{4.3.1}$$

Following the notations in Proposition 2.3.2, we have γ_i from $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$ to $\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$ and β_i from the reduced induced stack $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i,\mathrm{red}}$ to $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$, and

$$\begin{aligned}
\delta'_*[\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}] &= (\prod_{i=1}^{r} \gamma_i \circ \beta_i)_* \delta^{\mathrm{ev}}_{\mathrm{red}*}[\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}] \\
&\overset{(4.3.1)}{=} \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} m_{[\boldsymbol{\rho}]} \cdot \prod_{i=1}^{r} (\gamma_i \circ \beta_i)_* [\widetilde{j}_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i}(\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\rho}_i,\mathrm{red}})] \\
&\overset{(1)}{=} \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} m_{[\boldsymbol{\rho}]} \cdot \prod_{i=1}^{r} [j_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i}(\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\rho}_i})] \\
&= \sum_{[\boldsymbol{\rho}]\in\Delta(\mathfrak{V})} \frac{m_{[\boldsymbol{\rho}]}}{|\operatorname{Aut}(\boldsymbol{\rho}_i/\boldsymbol{\tau}_i)|} \cdot \prod_{i=1}^{r} j_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i *}[\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\rho}_i}],
\end{aligned}$$

with $j_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i} : \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\rho}_i} \to \mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}_i}$ from the contraction morphism $\boldsymbol{\rho}_i \to \boldsymbol{\tau}_i$. Since $\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\rho}_i}$ is reduced over B as shown in [4, Prop 3.28],

$$j_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i}(\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\rho}_i}) = \gamma_i \circ \beta_i \circ \widetilde{j}_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i}(\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\rho}_i,\mathrm{red}}).$$

By [4, Prop 5.5], γ_i induces an isomorphism on reductions. Hence $\gamma_i \circ \beta_i$ has degree one and we get equality (1). The last equality follows from the fact that $j_{\boldsymbol{\rho}_i,\boldsymbol{\tau}_i}$ is finite of degree $|\operatorname{Aut}(\boldsymbol{\rho}_i/\boldsymbol{\tau}_i)|$.

Now suppose $\widetilde{\mathfrak{M}}^{\mathrm{gl,ev}}_{\mathrm{red}}$ has more than one component, then by Lemma 2.3.1 and Proposition 2.2.1, moduli spaces $\widetilde{\mathfrak{M}}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ and $\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}$ have more than one component. For each component, we have equation (4.3.1). As $m_{[\boldsymbol{\rho}]}$ is independent of the geometric point $\overline{x}$ from the component, the above equation holds for general $\mathfrak{M}^{\mathrm{ev}}_{\boldsymbol{\tau}}$. Then, following Theorem 2.1.3, we finish the proof of Theorem 1.1.5.

♠

Appendices

Appendix A

Lift of Artin Cones

Recall that for an algebraic group G and a scheme X with a G-action, the quotient stack $[X/G]$ is the groupoid fibered over the category of schemes, such that

1. An object over a scheme B is a diagram $\left\{ \begin{array}{ccc} E & \xrightarrow{h} & X \\ \downarrow{\scriptstyle \pi} & & \\ B, & & \end{array} \right\}$ where E is a principal G-bundle over B and $h : E \to X$ is a G-equivariant map.
2. A morphism from an object $\left\{ \begin{array}{ccc} E' & \xrightarrow{h'} & X \\ \downarrow{\scriptstyle \pi} & & \\ B', & & \end{array} \right\}$ over B' to an object $\left\{ \begin{array}{ccc} E & \xrightarrow{h} & X \\ \downarrow{\scriptstyle \pi} & & \\ B, & & \end{array} \right\}$ over B is a pair (g, g') with $g : B' \to B$ and $g' : E' \to E \times_B B'$ such that g' is an isomorphism and $h' = h \circ g'$.

Let us first show a general lemma regarding quotient stacks.

Lemma A.0.1. *Let G be an algebraic group and X be a scheme with a G-action $\gamma : G \times X \to X$. Let Y and Z be algebraic stacks.*

Assume there is a G-invariant morphism $f' : X \to Y$, hence a map f from the quotient stack $[X/G]$ to Y. Let $g : Z \to Y$ be a representable morphism of algebraic stacks. There is a G-action on scheme $X \times_Y Z$ induced by its action on X. Then, *there is a 2-isomorphism*

$$[X \times_Y Z/G] \to [X/G] \times_Y Z.$$

Proof. By [26, Lemma 2.3.2], there is a 2-Cartesian diagram

$$\begin{array}{ccc} [X \times_Y Z/G] & \longrightarrow & [Z/G] \\ \downarrow & & \downarrow \\ [X/G] & \longrightarrow & [Y/G], \end{array}$$

where $[Z/G]$ and $[Y/G]$ are the quotient stacks induced by the trivial G-actions. As $[Z/G]$ is 2-isomorphic to the product stack $Z \times BG$ and $[Y/G]$ is 2-isomorphic to $Y \times BG$, we obtain that

$$[Z/G] = [Y/G] \times_Y Z.$$

Hence

$$[X \times_Y Z/G] = [X/G] \times_{[Y/G]} [Z/G] = [X/G] \times_Y Z.$$

♠

Let P, Q be toric monoids. For a monoid morphism $m : P \to Q$, we let $f : S_Q \to S_P$ be the associated toric morphism, $f_T : T_Q \to T_P$ be the algebraic torus morphism and $f_{\mathcal{A}} : \mathcal{A}_Q \to \mathcal{A}_P$ the morphism of Artin cones.

Proposition A.0.2. *Let $[S_Q \times T_P/T_Q]$ be the toric stack obtained by the torus action of T_Q on $S_Q \times T_P$ associated to the monoid morphism*

$$Q \oplus P^{\mathrm{gp}} \to Q^{\mathrm{gp}}, \quad (q, p) \mapsto q + m^{\mathrm{gp}}(p)$$

Then there is a Cartesian diagram of Artin stacks

$$\begin{array}{ccc} [S_Q \times T_P/T_Q] & \xrightarrow{g} & S_P \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \chi} \\ \mathcal{A}_Q & \xrightarrow{f_{\mathcal{A}}} & \mathcal{A}_P, \end{array} \tag{A.0.1}$$

where $\chi : S_P \to \mathcal{A}_P$ *is the quotient map, the morphism* η *is induced from the* T_Q*-equivariant map* $S_Q \times T_P$ *to* S_Q *by taking the projection and* g *is induced from the* $\widetilde{\eta} : S_Q \times T_P \to S_P$ *associated to the monoid morphism*

$$P \to Q \oplus P^{\mathrm{gp}}, \quad p \mapsto (m(p), -p), \tag{A.0.2}$$

invariant under the T_Q*-action on* $S_Q \times T_P$*.*

Proof. With trivial T_Q-action on $\mathcal{A}_P$, the map $S_Q \to S_P \to \mathcal{A}_P$ is T_Q-invariant. Since $\chi : S_P \to \mathcal{A}_P$ is representable, by Lemma A.0.1,

$$\mathcal{A}_Q \times_{\mathcal{A}_P} S_P = [S_Q \times_{\mathcal{A}_P} S_P/T_Q],$$

with T_Q acting on the fiber product by acting on S_Q. On the other hand, there is a commutative diagram

$$\begin{array}{ccccc} S_Q \times T_P & \xrightarrow{f \times \mathrm{id}} & S_P \times T_P & \xrightarrow{a} & S_P \\ \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \chi} \\ S_Q & \xrightarrow{f} & S_P & \xrightarrow{\chi} & \mathcal{A}_P, \end{array}$$

where a is the group action. By [24, Lemma 77.22.2], the commutative diagram on the right hand side is 2-Cartesian. We then obtain that $S_Q \times_{\mathcal{A}_P} S_P = S_Q \times T_P$ as both squares are fiber diagrams. The induced T_Q-action on $S_Q \times T_P$ is given by the monoid morphism in (A.0.2), which is the unique action that makes the above fiber diagram T_Q-equivariant. ♠

Proposition A.0.2 can be generalized to the toric strata of affine toric varieties.

Corollary A.0.3. *Let $m : P \to Q$ be a morphism of toric monoids. Let K be an ideal of P and L be an ideal of Q such that $m(K) \subseteq L$. Then there is a Cartesian diagram of idealized log stacks*

$$\begin{array}{ccc} [S_{Q,L} \times T_P/T_Q] & \xrightarrow{g} & S_{P,K} \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \chi} \\ \mathcal{A}_{Q,L} & \xrightarrow{f_m} & \mathcal{A}_{P,K}, \end{array}$$

where χ is the canonical quotient map and f_m is the map of toric stacks associated to the monoid morphism m. The morphism η is induced from the T_Q-equivariant map $S_{Q,L}\times T_P$ to $S_{Q,L}$ by taking the projection and g is induced from the $\widetilde{\eta} : S_{Q,L}\times T_P \to S_{P,K}$ associated to the monoid morphism

$$P \to Q \oplus P^{\mathrm{gp}}, \quad p \mapsto (m(p), -p),$$

invariant under the T_Q-action on $S_{Q,L} \times T_P$.

Proof. Since $S_{P,K} = \mathcal{A}_{P,K} \times_{\mathcal{A}_P} S_P$, we obtain that

$$\begin{aligned} \mathcal{A}_{Q,L} \times_{\mathcal{A}_{P,K}} S_{P,K} &= \mathcal{A}_{Q,L} \times_{\mathcal{A}_{P,K}} (\mathcal{A}_{P,K} \times_{\mathcal{A}_P} S_P) \\ &= \mathcal{A}_{Q,L} \times_{\mathcal{A}_P} S_P \\ &= \mathcal{A}_{Q,L} \times_{\mathcal{A}_Q} (\mathcal{A}_Q \times_{\mathcal{A}_P} S_P) \\ &= \mathcal{A}_{Q,L} \times_{\mathcal{A}_Q} [S_Q \times T_P/T_Q]. \end{aligned}$$

The map $[S_Q \times T_P/T_Q] \to \mathcal{A}_Q$ is induced by the T_Q-invariant map $S_Q \times T_P \xrightarrow{\mathrm{pr}_1} S_Q \to \mathcal{A}_Q$. By Lemma A.0.1, we then obtain that

$$\begin{aligned} \mathcal{A}_{Q,L} \times_{\mathcal{A}_Q} [S_Q \times T_P/T_Q] &= [\mathcal{A}_{Q,L} \times_{\mathcal{A}_Q} (S_Q \times T_P)/T_Q] \\ &= [(\mathcal{A}_{Q,L} \times_{\mathcal{A}_Q} S_Q) \times T_P/T_Q] = [S_{Q,L} \times T_P/T_Q]. \end{aligned}$$

♠

Appendix B

Logarithmic Fiber Product of Toric Varieties

In this section, we study the logarithmic fine, saturated fiber products of toric varieties. Unlike the fiber product in the category of schemes, the log fine, saturated fiber products of toric varieties are totally determined by the fiber product of the fans.

Definition B.0.1. Let $\Sigma(X) \to \Sigma(Y)$ and $\Sigma(Z) \to \Sigma(Y)$ be morphisms of fans. Define the *fiber product of fans* $\Sigma(X) \times_{\Sigma(Y)} \Sigma(Z)$ to be the fan $(\widetilde{\Sigma}, \widetilde{N})$ with

1. $\widetilde{N}$ being the fiber product of lattices $N(X) \times_{N(Y)} N(Z)$,
2. $\widetilde{\Sigma}$ consisting of the cones $\sigma_X \times_{\sigma_Y} \sigma_Z$ with $\sigma_X \in \Sigma(X)$, $\sigma_Y \in \Sigma(Y)$ and $\sigma_Z \in \Sigma(Z)$.

Lemma B.0.2. *Let $f : X \to Y$ and $g : Z \to Y$ be toric morphisms of toric varieties. Then, the fine, saturated fiber product $X \times_Y^{\mathrm{fs}} Z$ is a disjoint union of toric varieties and each of them is isomorphic to the toric variety of the fiber product of fans $\Sigma(X) \times_{\Sigma(Y)} \Sigma(Z)$. The number of the components is the lattice index $[(\operatorname{im} \alpha)^{\mathrm{sat}} : \operatorname{im} \alpha]$ under the map*

$$\alpha : N(X) \times N(Z) \to N(Y), \quad (x, z) \mapsto f_N(x) - g_N(z) \tag{B.0.1}$$

with f_N and g_N the lattice maps associated to the toric morphisms f and g.

In particular, if the lattice map α has full dimensional image in $N(Y)$, then the lattice index is $[N(Y) : \operatorname{im} \alpha]$.

Proof. The fine, saturated logarithmic fiber product of the toric varieties is discussed in [18, Rmk 2.2.5] in detail. By [18], the fine, saturated log fiber product $X \times^{\mathrm{fs}}_Y Z$ is a disjoint union of schemes, each of which is isomorphic to the toric variety of the fiber product of fans $\Sigma(X) \times_{\Sigma(Y)} \Sigma(Z)$. The reason of the fiber products containing several components is due to the fact that the torsion subgroup Tor of the fibered sum of monoids $M(X) \oplus_{M(Y)} M(Z)$ is nontrivial, with $M(X)$, $M(Y)$ and $M(Z)$ the character lattices. As we are working over a field of characteristic 0, the number of the components is the order of the group Tor.

Note that we have an exact sequence of monoids

$$M(Y) \xrightarrow{\phi} M(X) \oplus M(Z) \xrightarrow{\psi'} M(X) \oplus_{M(Y)} M(Z) \to 0, \tag{B.0.2}$$

with $\phi = (f_N^*, -g_N^*)$ and ψ' the cokernel map of ϕ. Let

$$\psi : M(X) \oplus M(Z) \xrightarrow{\psi'} M(X) \oplus_{M(Y)} M(Z) \xrightarrow{q} M(X) \oplus_{M(Y)} M(Z)/\operatorname{Tor}.$$

Then, the torsion group

$$\operatorname{Tor} = \ker q = \psi'(\ker \psi) \cong \ker \psi / (\ker \psi \cap \ker \psi') = \ker \psi / \operatorname{im} \phi.$$

As $\operatorname{im} \phi$ is a full dimensional sublattice of $\ker \psi$, by lattice geometry, the order of the torsion group Tor is the same as the lattice index $[(\operatorname{im} \phi)^* : (\ker \psi)^*]$.

We finish the proof by showing that under the inclusion of the lattice

$$\alpha' : (\operatorname{im} \phi)^* \to N(Y),$$

the image of $(\operatorname{im} \phi)^*$ is $(\operatorname{im} \alpha)^{\mathrm{sat}}$, and the image of $(\ker \psi)^* \to (\operatorname{im} \phi)^*$ is $(\operatorname{im} \alpha)$, with α defined in (B.0.1).

The cokernel of the map α' is isomorphic to $(\ker\phi)^*$, following the dual of the short exact sequence

$$0 \to \ker\phi \to M(Y) \to \operatorname{im}\phi \to 0.$$

Hence, $\operatorname{coker}\alpha'$ is torsion free. Therefore, the image of $(\operatorname{im}\phi)^*$ under α' is a saturated sublattice of $N(Y)$ containing $(\operatorname{im}\alpha)$.

Note that α is the dual map of ϕ in (B.0.2). Taking the dual of the exact sequence (B.0.2), we obtain an exact sequence of the lattices

$$0 \to N(X) \times_{N(Y)} N(Z) \xrightarrow{\psi^*} N(X) \times N(Z) \xrightarrow{\alpha} N(Y).$$

Therefore $\operatorname{im}\alpha \cong \operatorname{coker}\psi^* = (\ker\psi)^*$, with the second equality following the dual of the short exact sequence

$$0 \to \ker(\psi) \to M(Y) \xrightarrow{\psi} \operatorname{im}(\psi) \to 0.$$

The lattice quotient

$$[(\operatorname{im}\phi)^* : (\ker\psi)^*] = [(\operatorname{im}\alpha)^{\text{sat}} : \operatorname{im}\alpha],$$

which equals to the order of the torsion group Tor.

♠

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