Quantum Field Theory for Homological Algebraists

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Abstract

The BV formalism in quantum field theory provides a homological theory of integration that can be used to compute path integrals. The present note is an overview of the BV formalism. Starting with a discussion of the combinatorial algorithms used by physicists to compute correlators, we motivate the structures involved in studying observables in BV theories via the finite dimensional case. In particular, we show that the quantum BV formalism recovers the same algorithms from structures manifestly arising from integration. We then discuss the infinite dimensional case.

1 Introduction

The path integral approach to quantum field theory leads to several beautiful results in both physics and mathematics. According to this approach, numbers of physical interest arise as as expectation values for certain probability measures. However, these integrals often end up being mere heuristic notions: the "space of fields" over which we wish to integrate is infinite dimensional, yielding ill-behaved measure theory.

Despite its mathematical shortcomings, this perspective has been used by physicists to develop various algorithms that have proven to be successful in predicting outcomes of experiments. Thus, the mathematician is left with the task of developing a rigorous formalism that reproduces such algorithms. Since the "space of fields" is defined via some algebraic input, a key desiradatum of such a formalism would be a recasting of integration theory that is suited for such inputs. Developed to address questions in supergravity [2], the BV (Batalin-Vilkovisky) formalism provides an answer to this question via a homological version of integration theory. The present note aims to give an impressionistic account of this formalism.

The note is broken down as follows:

Section 1 is the current introduction

Section 2 is a brief overview of the path integral approach to QFT. Our main goal is to discuss the diagrammatics by which physicists compute path integrals.

Section 3 introduces the BV formalism in the simpler setting of 0-dimensional QFT. We define P_0 and BD algebras, which are the key algebraic structures involved in the classical and quantum settings respectively. In particular, we show that computing expectation values via the quantum BV formalism leads to the same diagrammatics as before.

Section 4 combines the algebraic machinery established in the previous section with the language of elliptic L_∞-algebras (used to describe moduli of solutions to systems of elliptic PDE) to describe the BV formalism in the general case.

1.1 Remarks

All the presented material is well-known to experts, and the author makes no claims of originality. The author would like to warmly thank Dr. David Ben-Zvi for his continued guidance over the past few years.

2 Path Integrals in Quantum Field Theory

In this section, we begin with an informal discussion of the general framework of field theory following [8], [9]. The main goal is to illustrate how perturbative computations in QFT are simplified via the use of Feynman diagrams. We focus our efforts on the example of a 0-dimensional QFT, where the infinite dimensional Gaussian integrals of the more general setting are replaced by finite dimensional ones. We then discuss Wick's Lemma, which gives a combinatorial expression for evaluating the relevant Gaussian integrals.

A (Wick-rotated) field theory on a Riemannian n-manifold M is determined by the data of a *Lagrangian density*, which is a nonlinear functional defined on a space of fields. The fields are usually taken to be smooth sections of some bundle over M, and we write the Lagrangian as

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + I(\phi).$$

Here:

- $\frac{1}{2}(\partial\phi)^2 = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ is the *kinetic term*. Here, the metric $g^{\mu\nu}$ has Euclidean signature as our theory is Wick-rotated.
- $\frac{m^2}{2}\phi^2$ is the *mass term*. Here, m is a positive real number.
- $I(\phi)$ is the *interaction term* and consists of a polynomial in ϕ of degree at least 3.

By the *principal of least action*, the classical fields of the theory are the critical points of the *action functional* $S[\phi]$, which is given by integrating the Lagrangian density over the spacetime manifold M. An *observable* of a scalar field theory is a functional $\mathcal{O}(\phi)$ on the configuration space of classical fields and can be thought of roughly as a measurement that can be performed.

The physics of the theory is entirely contained in the *expectation values* of the observables, also referred to as *correlation functions*, which are defined by the heuristic expressions

$$\langle\mathcal{O}(\phi)\rangle = \int \mathcal{D}[\phi]e^{-S[\phi]}\mathcal{O}(\phi).$$

We also define the *partition function* $Z = \int \mathcal{D}[\phi] e^{-S[\phi]}$ (the expectation value of the constant observable 1); we sometimes wish to normalize the correlation functions as $\frac{1}{Z}\langle \mathcal{O}(\phi)\rangle$. The integration here is done with respect to the mythical path-integral measure on the configuration space of fields, so the expression is ill-defined mathematically. However, we can regard this expression as shorthand for a certain asymptotic expansion, so it is still useful.

2.1 Feynman diagrams

For concreteness, we consider the case of a 0-dimensional theory of a single real scalar field X. That is, M is just a point, and our field $X \in C^\infty(M, \mathbb{R})$ is just a real variable. We can therefore regard the action $S[X]$ as a real valued function of X.

Consider the toy model with action given by $S[X] = \frac{m}{2}X^2 + \frac{\lambda}{4!}X^4$, where λ is a coupling constant. We denote the partition function $Z(m, \lambda)$ as it depends on two parameters. We now wish to compute the partition function. For $\lambda = 0$, the partition function is $\int_{\mathbb{R}} e^{-\frac{m}{2}X^2}$, which is just an ordinary finite dimensional Gaussian integral with value $\sqrt{\frac{2\pi}{m}}$ via the usual trick. The idea will be to treat the interaction term as a perturbation of an ordinary Gaussian integral. Expanding the partition function $Z(m, \lambda)$ about $\lambda = 0$ gives

$$\int_{\mathbb{R}} \sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \frac{X^{4n}}{n!} e^{-\frac{m}{2}X^2}.$$

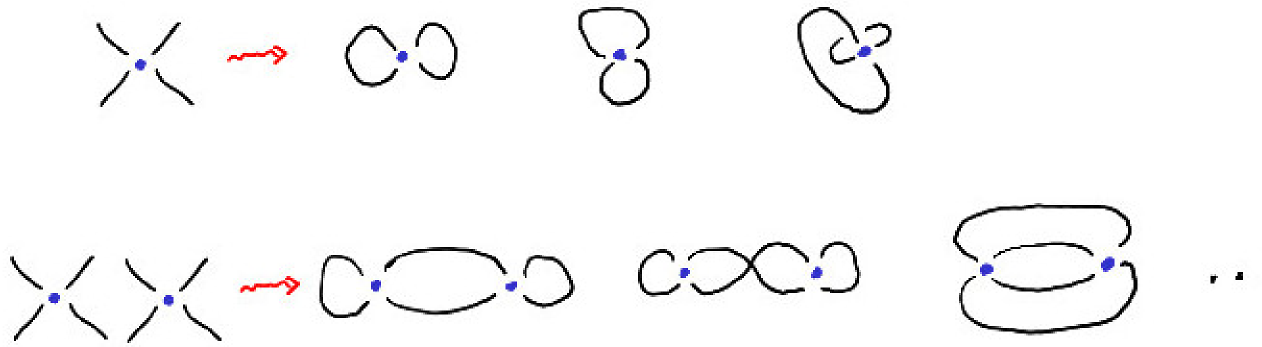
We define the *perturbation series expansion* to be the series obtained by commuting the sum and the integral:

$$\sum_{n=0}^{\infty} \left(\frac{-\lambda}{4!}\right)^n \frac{1}{n!} \int_{\mathbb{R}} X^{4n} e^{-\frac{m}{2}X^2}.$$

Note that this series diverges for all λ. Indeed, note that $Z(m, \lambda)$ can be analytically continued to $\mathbb{C}^\times$, but with an essential singularity at the origin. Such a function can not have a convergent series expansion. However, the series is an *asymptotic series* for $Z(m, \lambda)$ as $\lambda \to 0^+$, so the perturbation series expansion still tells us useful information about $Z(m, \lambda)$.

We now wish to compute the coefficients in the perturbation series. For motivation, consider the correlator $\langle X^{2k}\rangle = \int_{\mathbb{R}} X^{2k} e^{-\frac{m}{2}X^2}$. Note that this integral is gotten by applying $(-2\frac{d}{dm})^k$ to the ordinary Gaussian integral, which is equal to $\sqrt{\frac{2\pi}{m}}$. The derivative works out to $\sqrt{\frac{2\pi}{m}} \frac{1}{m^k} \frac{(2k)!}{k!2^k}$. Now note that the factor of $\frac{(2k)!}{k!2^k}$ is the number of ways of pairing $2k$ objects. This motivates the following interpretation of the terms in the perturbation series. Each power of $-\lambda$ in the series comes with 4 objects, and we wish to consider the possible ways of pairing them up.

We introduce certain diagrammatic gadgets in order to better manage these combinatorial considerations. Each power of $-\lambda$ will be denoted by a vertex, and each object will be a half-edge incident on this vertex. We wish to count all the ways of pairing up these half-edges, that is, the number of *Wick contractions*. The diagrams gotten after performing these Wick contractions are called *Feynman diagrams*. Below are the Feynman diagrams for the first and second order terms in the perturbation series expansion:



The contribution of each Feynman diagram to the term in the pertubration series should be weighted by the number of ways the diagram can be formed via Wick contraction. Let D_n denote the set of all n-vertex diagrams. This set is naturally acted on by the group $(S_4)^n \rtimes S_n$, where the S_n permutes the vertices and each S_4 permutes the half-edges incident on a given vertex. We compute that $|(S_4)^n \rtimes S_n| = n!(4!)^n$, and we may write the perturbation series expansion as

$$\sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \frac{1}{m^{2n}} (-\lambda)^n \frac{|D_n|}{|(S_4)^n \rtimes S_n|}.$$

Now, let O_n denote the set of orbits in D_n. By the orbit stabilizer theorem, the ratio $\frac{|D_n|}{|(S_4)^n \rtimes S_n|}$ is the same as the sum $\sum_{\Gamma \in O_n} \frac{1}{|\operatorname{Aut} \Gamma|}$. Substituting this above, we see that

$$\begin{aligned} Z &= \sqrt{\frac{2\pi}{m}} \sum_{n=0}^{\infty} \frac{1}{m^{2n}} (-\lambda)^n \sum_{\Gamma \in O_n} \frac{1}{|\operatorname{Aut} \Gamma|} \\ &= \sqrt{\frac{2\pi}{m}} \sum_{\Gamma} \frac{(-\lambda)^{|v(\Gamma)|}}{m^{|e(\Gamma)|}} \frac{1}{|\operatorname{Aut} \Gamma|}. \end{aligned}$$

Thus, $\frac{Z}{Z(\lambda=0)} = \sum_{\Gamma} \operatorname{ev}(\Gamma)$, where $\operatorname{ev}(\Gamma)$ is built from factors of $-\lambda$ for each vertex of Γ, $\frac{1}{m}$ for each edge (*propogators* in the physics literature), and $\frac{1}{|\operatorname{Aut} \Gamma|}$.//

The usual approach to QFT then defines infinite dimensional Gaussian integrals via analogous combinatorial expressions. The BV approach, which we introduce in the next section, is to give a homological repharsing of the familiar approach to integration over finite dimensional manifolds. In the finite dimensional case, we will see that this recovers the same combinatorial considerations. Furthermore, this homological perspective lets us define integration rigorously in the inifinite dimensional setting, without any need for path-integral measures.

3 The BV formalism in finite dimensions

We now introduce the BV formalism via the same toy model of a 0-dimensional QFT, where we may regard our space of fields as a finite dimensional manifold. The material in this section is drawn directly from [6]. There are three main ingredients to discuss:

- The *classical BV formalism* provides a framework for studying the critical locus of an action functional via derived geometry.
- The *quantum BV formalism* sets up a homological version of integration theory suited for adaptation to the infinite dimensional case.
- A *quantization procedure* describes how one moves between the classical and quantum problems.

We begin by discussing the classical case.

3.1 The classical BV formalism

Let M be a finite dimensional manifold, and let $S : M \to \mathbb{C}$ be smooth. We want to study the critical locus of S. Note that $\mathrm{crit}(S) = \mathrm{graph}(dS) \underset{T^*M}{\times} M$, the intersection of the graph of dS and the zero section inside the cotangent bundle to M. As the intersection of two Lagrangians in the cotangent bundle, this intersection may fail to be transverse for certain values of S.

Following the general yoga of derived geometry, in order to keep track of exactly how the intersection fails to be transverse, we replace the fiber product above with a derived fiber product. The functions on the derived critical locus will then be given by the derived tensor product $\mathcal{O}(\mathrm{dcrit}(S)) = \mathcal{O}(\mathrm{graph}(dS)) \underset{\mathcal{O}(T^*M)}{\otimes}{}^{\mathbb{L}} \mathcal{O}(M)$.

We wish to analyze the structure of $\mathcal{O}(\mathrm{dcrit}(S))$. To do so, it is desirable to have an explicit model for it as a commutative dg algebra. To this end, we choose the following resolution of $\mathcal{O}(\mathrm{graph}(dS))$ over $\mathcal{O}(T^*M)$:

$$K^* = 0 \to \mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} \wedge^n TM \to \cdots \to \mathcal{O}(T^*M) \otimes_{\mathcal{O}(M)} TM \to \mathcal{O}(M).$$

Viewing $\mathcal{O}(T^*M)$ as $\mathrm{Sym}_{\mathcal{O}}(M)(TM)$, we define the differential on K^* in degree -1 by $1 \otimes X \mapsto X - dS(X)$. The differential on all of K^* is then defined by extending this differential to the left as a Koszul complex. Thus, after tensoring, we get the following explicit presentation of $\mathcal{O}(\mathrm{dcrit}(S))$ as a commutative dg algebra:

$$K^* \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M) = 0 \to \wedge^n TM \to \cdots \to TM \to \mathcal{O}(M)$$

with the differential $-\iota_{dS}$ defined on vector fields by $X \mapsto -X(S)$. That is,

$$K^* \otimes_{\mathcal{O}(T^*M)} \mathcal{O}(M) = (\mathrm{Sym}_{\mathcal{O}(M)}(T[1]M), -\iota_{dS}),$$

which is the graded algebra of *polyvector fields*, equipped with a nontrivial differential. We may think of this as functions on the shifted cotangent bundle $T^*[-1]M$.

Polyvector fields carry a natural operation called the *Schouten bracket*, defined by extending the lie bracket on vector fields and the lie derivative on functions (both of which are degree 1) to all polyvector fields in a graded symmetric way. We denote this bracket by $\{-,-\}$ and note that it is a Poisson bracket of cohomological degree 1. We remark that for other choices of resolutions, we may not obtain a strict Poisson bracket; however, in general, we will get a homotopy Poisson bracket.

The Schouten bracket gives us an alternative description of our differential $-\iota_{dS}$.

Lemma 1. The operators $\{S,-\}$ and $-\iota_{dS}$ are equal as operators.

Proof. It suffices to verify the statement for elements in cohomological degree -1, as both operators were defined by extending an action on vector fields in a graded-symmetric way. For X a vector field, we have that

$$\{S,X\} = -\{X,S\} = -\mathcal{L}_X S = -X(S) = -\iota_{dS}(X).$$

□

Thus, we have the following structure on $\mathcal{O}(\mathrm{dcrit}(S))$:

Definition 1. A P_0*-algebra* $(A, d, \{-,-\})$ is a commutative dg algebra with a Poisson bracket $\{-,-\}$ of cohomological degree 1.

Our main example of such a gadget will be the graded algebra of polyvector fields equipped with the Schouten bracket.

3.2 The quantum BV formalism

The goal of the quantum BV formalism is to give a homological rephrasing of the path integral approach to QFT, in which one defines a probabilistic system starting with a space of fields equipped with an action functional. This is achieved by recasting the usual structures encoding integration on finite dimensional manifolds, namely the deRham complex.

To this end, let M be a closed, oriented, n-manifold and let $\Omega^n(M)$ denote the algebra of top forms. Elements of $\Omega^n(M)$ are smooth measures, and we have a natural linear map $\int_M : \Omega^n(M) \to \mathbb{R}$ given by integration. By Stoke's theorem, we have that $\int_M \mu = 0$ if and only if μ is exact. Therefore, $\int_M$ descends to a map $\Omega^n(M)/d\Omega^{n-1}(M) = H^n_{dR}(M) \to \mathbb{R}$. Adopting the homological point of view, we want to think of the n-th deRham cohomology as a space of "integrals" and ask for a resolution.

The above discussion reveals that the top forms are central to integration, so we may ask for a rephrasing of the deRham complex that focuses on the role of top forms. We may rewrite the exterior derivative $\Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M)$ as an operator $TM \otimes \Omega^n(M) \to \Omega^n(M)$ since contraction gives an isomorphism $TM\otimes\Omega^n(M)\cong\Omega^{n-1}(M)$. Similarly, via the isomorphisms $\wedge^k TM\otimes\Omega^n(M)\cong\Omega^{n-k}(M)$, the deRham complex is isomorphic to the complex

$$\wedge^n TM\otimes\Omega^n(M)\to\cdots\to TM\otimes\Omega^n(M)\to\Omega^n(M)$$

describing the natural action of polyvector fields on top forms.

This rephrasing suggests the following maneuver. As before, let M be a closed, oriented n-manifold. Fix a top form μ on M, which we think of as defining a probability density on M (which we think of as $e^{-S/\hbar}\mathcal{D}[\phi]$). We have a map $\mathcal{O}(M)\to\Omega^n(M)$ given by $f\mapsto f\mu$. Let $[\mu]$ denote the class of μ in $H^n_{dR}(M)$, and let $\langle f\rangle_\mu$ denote the expectation value of f with respect to the probability density μ. That is $\langle f\rangle_\mu=\frac{\int_M f\mu}{\int_M\mu}$. In terms of cohomology, this is exactly $\frac{[f\mu]}{[\mu]}$. Thus, we have a purely cohomological way of computing the expectation value of f with respect to μ. Our main goal will be to axiomatize this process.

Observe that the choice of a measure μ gives us a map $\wedge^k TM\to\Omega^{n-k}(M)$ given by contracting with μ. If the measure is nonvanishing, we can invert the contraction map and transfer the exterior derivative to polyvector fields. Thus, we define an operator $\Delta_\mu=m_\mu^{-1}\circ d\circ m_\mu$ on polyvector fields.

Definition 2. We call the operator Δ_μ a *BV Laplacian*. We call the chain complex $(\mathrm{Sym}_{\mathcal{O}(M)}(T[1]M),\Delta_\mu)$ the *quantum BV complex*. It is isomorphic to the deRham complex.

We remark that on TM, Δ_μ coincides with the familiar divergence operator. That is, $\Delta_\mu X=\mathrm{div}_\mu X\in\mathcal{O}(M)$, where $\mathrm{div}_\mu(X)$ is such that $\mathcal{L}_X\mu=(\mathrm{div}_\mu X)\mu$.

Lemma 2. Given f a function on M, the cohomology class $[f]_{BV}$ in $H^0(\mathrm{Sym}(T[1]M),\Delta_\mu)$ satisfies $[f]_{BV}=\langle f\rangle[1]_{BV}$.

We wish to examine properties of Δ_μ that can be made into an axiomatic definition. We begin with a description of Δ_μ in local coordinates. Let $M=\mathbb{R}^n$, and let $\mu_{Leb}=dx_1\wedge\cdots\wedge dx_n$ denote the Lebesgue measure on $\mathbb{R}^n$. Let Δ_{Leb} denote the corresponding BV laplacian. Using the correspondence between $\wedge^k TM$ and $\Omega^{n-k}(M)$ given by contraction, we see that

$$\Delta_{Leb}(f\partial_1\wedge\cdots\wedge\partial_n)=\sum_i(-1)^{i-1}(\partial_i f)\partial_1\wedge\cdots\wedge\hat{\partial}_i\wedge\cdots\wedge\partial_n.$$

Thus, the Koszul rule of signs gives us that in local coordinates, $\Delta_{Leb}=\sum_i\frac{\partial}{\partial x_i}\frac{\partial}{\partial(\partial_i)}$. Now, let μ be an arbitrary density on $\mathbb{R}^n$, and note that we can write it in the form $e^{-S(x)}dx_1\wedge\cdots\wedge dx_n$. An explicit computation yields that $\Delta_\mu=\Delta_{Leb}-\sum_i\frac{\partial S}{\partial x_i}\frac{\partial}{\partial(\partial_i)}=-\iota_{dS}+\Delta_{Leb}$.

Thus, we see that Δ_μ enjoys the following properties:

- Δ_μ is a second order differential operator on $\mathrm{Sym}(T[1]M)$.
- $\Delta_\mu^2=0$.

- For all polyvector fields X, Y, we have that

$$\Delta_\mu(XY) = (\Delta_\mu X)Y + (-1)^{|X|}X(\Delta_\mu Y) + \{X, Y\}.$$

That is, the failure of Δ_μ to be a derivation on polyvector fields is measured exactly by the Schouten bracket.

We abstract these into the following definition:

Definition 3. A *Beilinson-Drinfeld (BD) algebra* $(A, d, \{-,-\})$ is a graded commutative algebra A, flat as a module over $\mathbb{R}[[\hbar]]$, equipped with a degree 1 Poisson bracket such that

$$d(ab) = (da)b + (-1)^{|a|}(db) + \hbar\{a, b\}.$$

The above discussion suggests that the structure of a BD algebra encodes information about expectation values of observables in its zeroth cohomology. We can extend the relation between path integrals and the quantum BV formalism by showing that the problem of describing boundaries in zeroth cohomology of a BD algebra recovers the Feynman diagrams from before. The more formally-minded reader may safely absorb the upshot and skip to the next section.

Recall our basic problem: consider $V = \mathbb{R}^N$ with the Gaussian probability measure

$$\mu_{Gauss} = \frac{(2\pi\hbar)^{N/2}}{\sqrt{\det A}} e^{-\langle x, Ax\rangle/2\hbar} dx^1 \cdots dx^N,$$

where $A = (a_{ij})$ is a positive definite, symmetric, real, $N \times N$ matrix and $\hbar > 0$. We wish to compute expectation values with respect to this measure $\langle f\rangle_{Gauss} = \int_{\mathbb{R}^N} f\mu_{Gauss}$.

We will treat $\hbar$ as a formal parameter and give expressions for $\langle f\rangle_{Gauss}$ of any formal power series f in $\hbar^{N/2}\mathbb{C}[[\hbar]]$. Following the previous discussion, we rephrase the problem homologically. Namely, we want to work with the deRham complex of $\mathbb{R}^N$ and identify the cohomology class $[f\mu]_{Gauss}$ in $H^N_{dR}(\mathbb{R}^N)$. However, $H^N_{dR}(\mathbb{R}^N) = 0$, so this fails immediately. This has to do with the fact that smooth top forms are not integrable on a vector space. The standard fix is to work with compactly supported deRham cohomology, but alas, the measure μ_{Gauss} is not compactly supported. We present two successive idealizations of the situation that allow us to apply the BV formalism.

We attempt to remedy the above by working with the algebra $\mathcal{S} = \mathcal{S}(\mathbb{R}^N)$ of Schwartz functions on $\mathbb{R}^N$. Consider the Schwartz-deRham complex which consists of deRham forms with coefficients in $\mathcal{S}$:

$$\Omega^\bullet_{\mathcal{S}}(\mathbb{R}^N) = \mathcal{S} \to \oplus_{i=1}^N \mathcal{S}dx_i \to \cdots \to \mathcal{S}dx_1 \wedge \cdots \wedge \mathcal{S}dx_N.$$

Like the compactly supported deRham complex, this complex is supported in degree N. As before, given $f \in \mathcal{S}$, define $\langle f\rangle_{Gauss}$ to be such that $[f\mu_{Gauss}] = \langle f\rangle_{Gauss}[\mu_{Gauss}] \in H^N_{\mathcal{S}}(\mathbb{R}^N)$. Translating between the deRham complex and polyvector fields as before, we get a quantum BV complex consisting with underlying graded algebra the algebra of Schwartz polyvector fields and a BV laplacian given by $\Delta_{Gauss} = -\frac{1}{\hbar}\sum_{i,j} x_i \frac{\partial}{\partial \xi_j} + \Delta_{Leb}$. We may then use the BV formalism to study expectation values as before.

Consider the following further algebraic idealization: replace Δ_{Gauss} with $\hbar\Delta_{Gauss}$ so that the $\hbar$ factor appears in the Lebesgue BV Laplacian term. Furthermore, replace $\mathcal{S}$ with formal power series on $V = \mathbb{R}^N$. Our problem is then rephrased as follows: consider the algebra of formal power series

$$\begin{aligned} A(V) &= \widehat{\mathrm{Sym}}_{\mathbb{C}[[\hbar]]}(V^\vee \oplus V[1]) \\ &= \mathbb{C}[[x_1, \cdots, x_N, \xi_1, \cdots, \xi_N, \hbar]]. \end{aligned}$$

Here, the x_i have cohomological degree 0 (think of these as coordinates on V), ξ_i have cohomological degree -1 (think of these as vector fields on V), and $\hbar$ has cohomological degree 0. We topologize $A(V)$ as a power series algebra and equip it with the BV Laplacian

$$\begin{aligned} \Delta &= -\sum_{i,j=1}^N a_{ij} x_i \frac{\partial}{\partial \xi_j} + \hbar \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\frac{\partial}{\partial(\partial_i)} \\ &= \hbar\Delta_{Leb} - \sum_{i,j=1}^N a_{ij} x_i \frac{\partial}{\partial \xi_j}, \end{aligned}$$

where $A = (a_{ij})$ as before. We have an associated degree 1 Poisson bracket $\{-,-\}$ such that $\{x_i, x_j\} = \{\xi_i, \xi_j\} = 0$ and $\{x_i, \xi_j\} = \delta_{ij}$. We are then left with the following two problems

1. Compute the cohomology of $(A(V), \Delta)$.
2. Compute the cohomology of $(A(V), \Delta + \{I, -\})$, for an interaction $I \in \mathbb{C}[[x_1, \cdots, x_N]]$ having cubic and higher terms.

That is, we wish to completely describe the class $[f]_I$ of any $f \in \mathbb{C}[[\hbar, x_1, \cdots, x_N]]$. Note that the element 1 is not a boundary; choosing $[1]_I$ as a basis for cohomology, we see that zeroth homology is isomorphic to $\mathbb{C}[[\hbar]]$ as a $\mathbb{C}[[\hbar]]$ module. We define the *expectation value* $\langle f\rangle_I \in \mathbb{C}[[\hbar]]$ such that $[f]_I = \langle f\rangle_I [1]_I$.

We begin by addressing the first problem. For simplicty, take $V = \mathbb{R}$. Our complex is then $\mathbb{C}[[x, \hbar]]\xi \xrightarrow{\Delta} \mathbb{C}[[x, \hbar]]$, with differential given by $\Delta = -ax\frac{\partial}{\partial \xi} + \hbar\frac{\partial^2}{\partial x \partial \xi}$. Given an element $f(x)\xi$ in degree -1, we see that $\Delta(f\xi) = -axf(x) + \hbar f'(x)$. Therefore, we see that the element $f\xi$ is closed $f(x) = e^{\frac{ax^2}{2\hbar}}$. However, $f \in \mathbb{C}[[x, \hbar]]$, so we see that the degree -1 cohomology vanishes.

Now we want to compute the boundaries $[x^n]$ for all n. Note that since $\Delta(x^{n-1}\xi) = -ax^n + \hbar(n-1)x^{n-2}$, we have that $[x^n] = \frac{\hbar}{a}(n-1)[x^{n-2}]$ in cohomology. Applying this recursively, we see that

$$[x^n] = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \left(\frac{\hbar}{a}\right)^k (2k-1)!!, & \text{if } n = 2k \end{cases}.$$

This is the same result obtained earlier when we computed the moment $\langle X^n \rangle$ in a free field theory of a single scalar field.

In the presence of a nontrivial interaction I, Feynman diagrams give a combinatorial description of the expectation values to be computed. We first fix notation. Denote elements of $\mathbb{C}[[x_1, \cdots, x_N]]$ as a symmetric tensor in $T(V^\vee)$. For $\vec{\imath} \in \{1, \cdots, N\}^m$, $x_{\vec{i}}$ denotes the monomial $x_{i_1} \cdots x_{i_m}$. The Taylor coefficients of our interaction term are defined by specifying $I^{(m)}_{\vec{i}} = \frac{\partial^m I}{\partial x_{i_1} \cdots \partial x_{i_m}}|_{(x)=0}$. Each $I^{(m)}$ is a symmetric m-tensor and we have that $\frac{\partial I(x)}{\partial x_i} = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{\vec{j} \in \{1, \cdots, N\}^m} I^{(m+1)}_{i,\vec{j}} x_{\vec{j}}$. This term is the coefficient of $\frac{\partial}{\partial x_i}$ in ι_{dI}, which appears in the BV Laplacian.

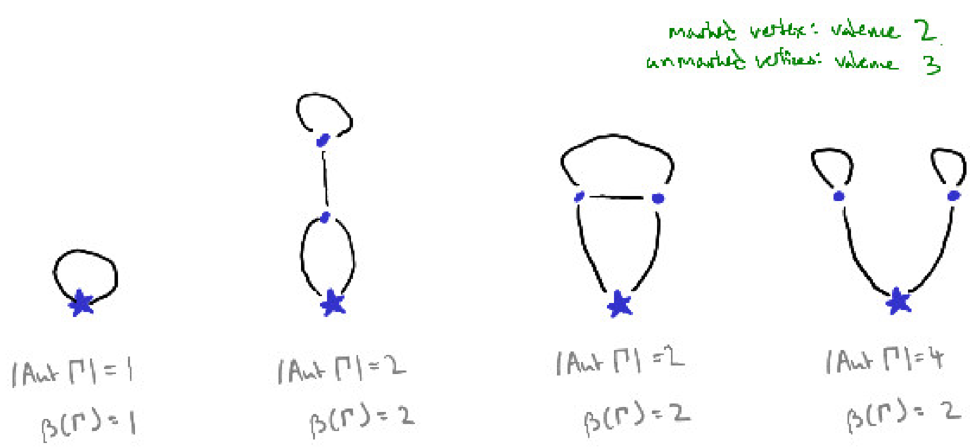
Definition 4. For us, a *Feynman diagram* is a finite connected graph (with self-loops and parallel edges allowed) such that:

- the graph has precisely one *marked vertex* with valence n, labelend by an n-tensor $f \in (\mathbb{C}^N)^{\otimes n}$, and whose incident edges are totally ordered. We denote the marked vertex by $\star$.
- a number of *internal vertices*, which are required to have valence 3 or more. We denote internal vertices by $\bullet$.
- a number of univalent *external vertices*. Denote these by $\circ$.

Definition 5. An *automorphism* of a Feynman diagram is a permutation of the half-edges that separately permutes internal and external vertices, but not the marked vertex. In particular, it does not change which half edges are part of the same edge nor the data of which half-edges are incident on a given vertex.

Definition 6. The *first Betti number* $\beta(\Gamma)$ of a Feynman diagram Γ is the number of full edges, minus the number of unmarked vertices.

Some examples of Feynman diagrams are:



Definition 7. The *evaluation* $\mathrm{ev}(\Gamma)$ of a Feynman diagram Γ on $f \in \mathbb{C}[[x_1, \cdots, x_N]]$ (whose expectation value we wish to compute) is defined as follows. Suppose we are given a labeling of the half-edges of Γ by $\{1, \cdots, N\}$. We associate a product of matrix coefficients to such a labeled Feynman diagram as follows

- the marked vertex contributes $f_{\vec{i}}$, where $\vec{i}$ denotes the vector of labesls formed by reading labels on the half-edges incident on the marked vertex in the prescribed order. Indeed recall that part of the data of a Feynman diagram was a total ordering of these edges.
- each internal vertex with valence m contributes $-I^{(m)}_{\vec{i}}$ where $\vec{i}$ is the vector of labels gotten by reading the labels on incident half-edges in any order. Note that since these tensors are symmetric, this assignment is well-defined.
- each external vertex with incident half edge labeled by i contributed the variable x_i.
- each internal edge with half-edges labeled i, j contributes $a^{ij} = a^{ji}$. Here, upper indices denote entries of the inverse matrix.
- each external edge with half-edges labeled i, j contributes δ_{ij}.

Thus, our evaluation operation sends a labeled Feynman diagram to a monomial in $\mathbb{C}[[x_1, \cdots, x_N, \hbar]]$. We then define the *evaluation* $\mathrm{ev}(\Gamma)$ of an unlabeled diagram to be the sum over all possible labelings as a labeled Feynman diagram.

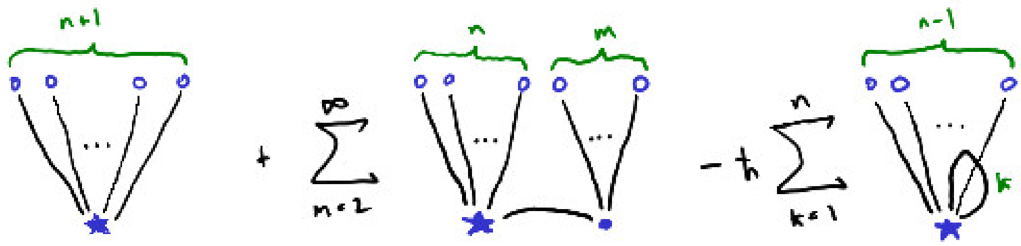
Our evaluation lets us define a map sending Feynman diagrams to power series in $\mathbb{C}[[x_1, \cdots, x_N, \hbar]]$ by $\Gamma \mapsto \frac{\mathrm{ev}(\Gamma)\hbar^{\beta(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}$. For each $(n+1)$-tensor $(f_{i,\vec{i}})_{i\in\{1,\cdots,N\},\vec{i}\in\{1,\cdots,N\}^n}$, consider the element $\sum_{i,\vec{i},j} f_{i,\vec{i}} x_{\vec{i}} (a^{-1})_{i,j} \xi_j$ in degree 1. These topologically span the degree 1 piece of the power series algebra, so we see that the boundaries in degree 0 are spanned by their images under the differential:

$$(\Delta + \{I, -\}) \left(\sum_{i,\vec{i},j} f_{i,\vec{i}} x_{\vec{i}} (a^{-1})_{i,j} \xi_j \right)$$

$$= \sum_{i,\vec{i}} f_{i,\vec{i}} x_i x_{\vec{i}} - \sum_{m=2}^{\infty} \sum_{i,\vec{i},j,\vec{j}} \frac{1}{m!} I^{(m+1)}_{j,\vec{j}} x_{\vec{j}} (a^{-1})_{i,j} f_{i,\vec{i}} - \hbar \sum_{i,\vec{j}} \sum_{k=1}^{n} f_{i,\vec{j}} (a^{-1})_{i,j_k} x_{j_1,\cdots,\hat{j_k},\cdots,j_n}$$

Thus, we see that in cohomology, we can write the class $[\sum_{i,\vec{i}} f_{i,\vec{i}} x_i x_{\vec{i}}]$ as a sum of other terms, each of which has a greater degree of x or $\hbar$. Thus, we see that the sum gotten from applying recursion relation describing the class of f converges in the usual power series topology.

Using our diagrammatic notation, we see that boundaries satisfy the following relationship:



We therefore have the following procedure for evaluating $\langle \sum_{\vec{i}} f_{\vec{i}} x_{\vec{i}} \rangle$, that is $\left\langle \frac{\mathrm{ev}(\Gamma)\hbar^{\beta(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \right\rangle$. Start with an edge incident to the marked vertex of the Feynman diagram. The recursion relation for boundaries gives us two options. Following this edge, we may return back to the marked vertex; this indicates the presence of a loop, which increases the Betti number of the graph, hence the factor of $\hbar$ in the recursion relation. Alternatively, following the edge reaches an internal vertex of valence 3 or more, and this corresponds to another Feynman diagram. For the aforementioned reason, this process terminates. Furthermore, the Feynman diagrams at the end of the process are those with no external vertices, as these are the only diagrams from which we cannot produce more external vertices or increase the Betti number.

We arrive at the following:

Proposition 1.

$$\left\langle \sum_{\vec{i}} f_{\vec{i}} x_{\vec{i}} \right\rangle = \sum_{\substack{\text{Feynman diagrams } \Gamma \\ \text{with no external vertices} \\ \text{and marked vertex labeled by } f}} \frac{\mathrm{ev}(\Gamma)\hbar^{\beta(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \in \mathbb{C}[[\hbar]].$$

Thus, we see that the algorithms for computing expectation values under the quantum BV formalism involve the same combinatorics as the algorithms used by physicists. For comfort, we examine the following example:

Example 1. Consider the example of ϕ^4 theory from before, but for simplicity, take the parameters $m = \lambda = 1$. The quantum BV complex of study is then

$$\mathbb{C}[[x,\hbar]]\xi \xrightarrow{Q} \mathbb{C}[[x,\hbar]], \qquad Q = x\frac{\partial}{\partial \xi} - \frac{x^3}{3!}\frac{\partial}{\partial \xi} - \hbar\frac{\partial^2}{\partial x \partial \xi}.$$

The boundaries in zeroth cohomology are spanned by $Q(x^n\xi) = x^{n+1} - x^{n+3}/3! - \hbar n x^{n-1}$ so we have that $[x^{n+1}] = \frac{1}{3!}[x^{n+3}] + \hbar n[x^{n-1}]$. Suppose we want to compute the expectation value $\langle x^n \rangle$. A Feynman diagram for this correlator will have all vertices 4-valent except the marked, n-valent vertex.

We claim that the Feynman sum $c_n = \sum_{\substack{\text{Feynman diagrams } \Gamma \\ \text{for} \langle x^n \rangle}} \frac{\text{ev}(\Gamma)\hbar^{\beta(\Gamma)}}{|\text{Aut}(\Gamma)|}$ satisfies the given recursion relation. Indeed, following an edge incident to the marked vertex, we may either return to the marked vertex or reach a 4-valent vertex. In the former case, deleting the loop creates a Feynman diagram for $\langle x^{n-1} \rangle$ and changes the Betti number of the Feynman diagram, so it introduces a factor of $\hbar$. In the latter case, deleting the vertex introduces extra edges so we get a Feynman diagram for $\langle x^{n+3} \rangle$ from one for $\langle x^{n+1} \rangle$. Keeping track of how these vertex deletions affect the size of automorphism groups recovers the desired factors (eg, there are n ways a loop can be deleted to get a Feynman diagram for $\langle x^{n-1} \rangle$).

3.3 BV quantization

Having seen that the BV method for computing expectation values recovers the combinatorial algorithms familiar to physicists, it remains to discuss the mechanism for passing between the classical and quantum problems. Namely, we wish to articulate a sense in which the quantum BV complex for $\mu = e^{-S/\hbar} dx_1 \wedge \cdots \wedge dx_n$ is very close to functions on the derived critical locus of S. Have that the BV Laplacian is $\Delta_\mu = -\frac{1}{\hbar}\iota_{dS} + \Delta_{Leb}$. Suppose that $\hbar \neq 0$ so that we can multiply by $\hbar$. This yields the quantum BV complex $(\text{Sym}(T[1]M), -\iota_{dS} + \hbar\Delta_{Leb})$. Now, varying $\hbar$, we see that at $\hbar = 0$, we recover exactly the P_0 algebra of functions on the derived critical locus of S. Thus, varying $\hbar$ moves us from the classical problem of describing functions on the derived critical locus at $\hbar = 0$ to the quantum problem of computing expectation values at $\hbar \neq 0$.

This is encoded in the BD algebra structure of a BV complex. Let A^q be a BD algebra. Then we can restrict to $\hbar = 0$ by considering $A_{\hbar=0} = A^q \underset{\mathbb{R}[[\hbar]]}{\otimes} \mathbb{R}[[\hbar]]/(\hbar)$. The induced differential on $A_{\hbar=0}$ is a derivation, so we get a P_0 algebra. We can also restrict to $\hbar \neq 0$ by setting $A_{\hbar\neq 0} = A^q \underset{\mathbb{R}[[\hbar]]}{\otimes} \mathbb{R}((\hbar))$, which is just a cochain complex. In particular, we don't get an algebra structure in cohomology like we did for $A_{\hbar=0}$. This motivates the following definition:

Definition 8. A *BV quantization* of a P_0 algebra A is a BD algebra A^q such that $A_{\hbar=0} = A$.

A typical approach to construct such a gadget for a P_0-algebra (A, d) is to search for BV Laplacians Δ such that $d + \hbar\Delta$ makes $A[[\hbar]]$ a BD-algebra.

4 The BV formalism in infinite dimensions

In previous sections, we began with the assumption that our space of fields was describable by a finite dimensional manifold. However, this is rarely the case. Physical fields are solutions to Euler-Lagrange equations, which constitute a system of elliptic PDE. The formal geometry of the solution space of such a system is studied via *elliptic formal moduli problems*. In the following section, we assume some background on characteristic zero derived deformation theory [7].

4.1 The classical BV formalism

Following the usual ideas of derived deformation theory, the formal neighborhood of a point of a moduli space in characteristic zero is controlled by a L_∞-algebra. Therefore, thinking of our formal moduli problems in the language of L_∞ algebras costs us no information. Indeed, an L_∞-algebra $\mathfrak{g}$ determines a formal moduli problem $B\mathfrak{g} : \mathrm{Art}_k \to \mathrm{sSets}$ given by sending a local Artin algebra R with maximal ideal $\mathfrak{m}$ to the simplicial set of Maurer-Cartan elements in the dg lie algebra $\mathfrak{g} \otimes \mathfrak{m}$. Conversely, given a pointed formal moduli problem $\mathcal{M}$, the associated L_∞-algebra $\mathfrak{g}_{\mathcal{M}}$ is the shifted tangent complex $T[-1]\mathcal{M}_P$.

As we may restrict our Euler-Lagrange equations to fields living over open subsets of spacetime, we wish to study a sheaf of L_∞-algebras on spacetime. More precisely, the following definition captures the kind of objects in consideration

Definition 9. Let M be a manifold. An *elliptic L_∞-algebra* on M consists of the following data

- a graded vector bundle $L \to M$, with space of sections $\mathscr{L}$
- a differential operator $d : \mathscr{L} \to \mathscr{L}$ of cohomological degree 1 that squares to zero, making $(\mathscr{L}, d)$ into an elliptic complex.
- polydifferential operators $\ell_n : \mathscr{L}^{\otimes n} \to \mathscr{L}$ for $n \geq 2$, which are alternating, of cohomological degree -2, and make $\mathscr{L}$ into an L_∞-algebra

We saw earlier that functions on the derived critical locus of an action functional carried a P_0 structure, so (granting nondegeneracy), the derived critical locus has a shifted symplectic structure of cohomological degree -1. Accordingly, a classical field theory in the BV formalism will be specified by an elliptic moduli problem equipped with a symplectic form of cohomological degree -1. We wish to identify the extra structure on an elliptic L_∞-algebra $\mathfrak{g}$ that endows the corresponding formal moduli problem with a symplectic form.

In accordance with the usual dictionary between formal moduli problems and L_∞-algebras, a L_∞-module over the L_∞ algebra $\mathfrak{g}_{\mathcal{M}}$ corresponds to a dg vector bundle on the formal moduli problem $\mathcal{M}$. Furthermore, sections of a dg vector bundle E on $\mathcal{M}$ corresponds to Chevalley-Eilenberg cochains $C^\bullet(\mathfrak{g}_{\mathcal{M}}, E_{\mathcal{M}})$ with coefficients in the L_∞-module $E_{\mathcal{M}}$ corresponding to E.

Thus, we see that the complex of 2-forms on the formal moduli problem $\mathcal{M}$ is identified with Chevalley-Eilenberg cochains on the associated L_∞-algebra $\mathfrak{g}_{\mathcal{M}}$ with values in $\wedge^2(\mathfrak{g}_{\mathcal{M}}^\vee[-1])$. By the usual Koszul rule of signs, we see that the given $\mathfrak{g}_{\mathcal{M}}$-module is isomorphic to $\mathrm{Sym}^2(\mathfrak{g}_{\mathcal{M}}^\vee)[-2]$.

Work of Schwarz and others [1] develops the notion of shifted symplectic structures in the setting of dg manifolds. Here, a symplectic form is required to be closed under both the internal differential on 2-forms and the deRham differential. Our use of shifted symplectic structures here follows that of [10] in the setting of formal derived stacks. One can show that the definitions given in the two settings coincide, and a Darboux lemma in the formal derived stacky setting allows us to take our form to have constant coefficients. This automatically implies that the symplectic form is closed under the deRham differential, so we simply require it to be invariant under the internal differential on $\mathrm{Sym}^2(\mathfrak{g}_{\mathcal{M}}^\vee)[-2]$. This will be equivalent to an invariance condition.

Thus, we will define a classical field theory to be an elliptic L_∞-algebra equipped with a nondegenerate invariant pairing of cohomological degree -3. We first discuss what it means to have an invariant pairing.

Definition 10. Let M be a manifold, E an elliptic L_∞-algebra on M. An *invariant pairing on E of cohomological degree k* is a symmetric bilinear map $\langle -, - \rangle_E : E \otimes E \to \mathrm{Dens}(M)[k]$ such that

- (nondegeneracy) the pairing induces a bundle isomorphism $E \to E^\vee \otimes \mathrm{Dens}(M)[k]$
- (invariance) let $\mathscr{E}_c$ denote the compactly supported sections of E. The pairing induces an inner product $\langle -, - \rangle$ on $\mathscr{E}_c$; we require this to be an invariant pairing on the L_∞ algebra $\mathscr{E}_c$. That is, the map $\mathfrak{g}^{\otimes(n+1)} \to \mathbb{R}$ given by $\alpha_1 \otimes \cdots \otimes \alpha_{n+1} \mapsto \langle \ell_n(\alpha_1, \cdots, \alpha_n), \alpha_{n+1} \rangle$ is graded anti-symmetric in the α_i.

Definition 11. A *formal pointed elliptic moduli problem with a symplectic form of cohomological degree k* on M is an elliptic L_∞-algebra on M with an invariant pairing of cohomological degree $k-2$.

Definition 12. A *(perturbative) classical field theory* in the BV formalism is a formal pointed elliptic moduli problem on M with a symplectic form of cohomological degree -1.

4.2 Quantum observables

We now discuss the structure exhibited by quantum observables in a BV theory. The author warns that the content of this section is very impressionistic. In particular, we do not discuss here what a BV quantization of a classical BV theory is. The legitimately interested reader is encouraged to stop reading and look at [3] from which the material is drawn.

Quantizations of observables are expressed in terms of quantizations of the algebras of classical observables over each $U \subset M$ open. Keeping track of how observables on disjoint opens can combine leads us to the structure of a factorization algebra on M.

Definition 13. Let M be a manifold. A *prefactorization algebra $\mathcal{F}$* on M is the assignment of a cochain complex $\mathcal{F}(U)$ to every $U \subset M$ open, together with cochain maps

$$m^V_{U_1, \cdots, U_n} : \mathcal{F}(U_1) \otimes \cdots \mathcal{F}(U_n) \to \mathcal{F}(V),$$

whenever the U_i are disjoint open subsets of V, invariant under the ordering of the U_i. We also want a natural associativity condition to be satisfied: for U_{ij} disjoint opens in V_i, which are disjoint opens in W, we want the structure map from the product of all the $\mathcal{F}(U_{ij})$ to $\mathcal{F}(W)$ to factor through the product of the $\mathcal{F}(V_i)$.

Definition 14. A *factorization algebra* $\mathcal{F}$ on M is a prefactorization algebra satisfying a local-to-global property. In particular, we want $\mathcal{F}$ to be defined completely on a large open set by its values on sets of a nice cover.

Definition 15. Suppose we have a classical field theory defined as above by an elliptic L_∞-algebra $\mathscr{L}$ with an invariant pairing. We define the prefactorization algebra of classical observables $\mathrm{Obs}^{cl}_{\mathscr{L}}$ to be such that $\mathrm{Obs}^{cl}_{\mathscr{L}}(U) = C^\bullet(\mathscr{L}(U))$.

We saw earlier that functions on $B\mathscr{L}(U)$ should have a shifted Poisson structure on its functions. In this infinite dimensional setting, we still have a homotopical imprint of this Poisson structure.

Theorem 1. There exists a P_0-factorization algebra $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{L}} \subset \mathrm{Obs}^{cl}_{\mathscr{L}}$ together with a weak equivalence of factorization algebras $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{L}} \sim \mathrm{Obs}^{cl}_{\mathscr{L}}$. That is, for each $U \subset M$ open, we have a quasi-isomorphism $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{L}}(U) \to \mathrm{Obs}^{cl}_{\mathscr{L}}(U)$.

Following the finite dimensional discussion, we expect that a quantization of the factorization algebra of observables should involve BD-factorization algebras

Definition 16. Let $\mathscr{L}$ be a classical field theory on M and $\widetilde{\mathrm{Obs}}^{cl}_{\mathscr{L}}$ the P_0-factorization algebra of observables. A *strong quantization* of this P_0-factorization algebra is a BD-factorization algebra Obs^q with a quasi-isomorphism of P_0-algebras $\mathrm{Obs}^q(U) \mod \hbar \to \widetilde{\mathrm{Obs}}^{cl}_{\mathscr{L}}$.

We conclude this note by stating the following conjecture, due to [3], which is a theorem in the case of free field theory:

Conjecture 1. Suppose we are given a classical field theory on M and a BV quantization of the theory. Then, the factorization algebra of quantum observables Obs^q is a strong quantization of the P_0 factorization algebra Obs^{cl}.

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