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Star-unitary transformation and stochasticity: emergence of white, $1/f$ noise through resonances

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Star-unitary transformation and stochasticity: emergence of white, $1/f$ noise through resonances

by

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Star-unitary transformation and stochasticity: emergence of white, $1/f$ noise through resonances

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In this thesis we consider the problem of stochasticity in Hamiltonian dynamics. It was shown by Poincaré that nonintegrable systems do not have constants of motion due to resonances. Divergences due to resonances appear when we try to solve the Hamiltonian by perturbation. In recent years, Prigogine's group showed that there may exist a new way of solving the Hamiltonian by introducing a non-unitary transformation Λ which removes the divergences systematically. In this thesis we apply this Λ transformation to the problem of stochasticity.

To this end, first we study classical Friedrichs model, which describes the interaction between a particle and field. For this model we derive the Λ transformation for general functions of particle modes, and show that the Langevin and Fokker-Planck equations can be derived through the transformed particle density function. It is also shown that the Gaussian white noise structure can be derived through the removal of divergences due to resonances. We

extend this to the quantum case, and show that the same structure can be preserved if we keep the normal order of creation and destruction operators. We also study the extended Friedrichs model. This model can be mapped from the case in which a small system is weakly interacting with a reservoir. In this model we show that low frequency $1/f$ noise is derived due to the sum of resonances effect.

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Chapter 1

Introduction

The contents of chapter 1 - chapter 4 are based on the papers [1] [2], in which the author and G. Ordonez closely collaborated. Author deeply appreciated that.

In this thesis we study the connection between Hamiltonian dynamics and irreversible, stochastic behavior, like decay, Brownian motion and noises.

In classical physics the basic laws are time reversible. If we know the Hamiltonian, then we get Hamilton's equations of motion which describe the time evolution of the system in a time reversible, deterministic way. On the other hand, we see time irreversibility and stochastic behavior everywhere. How to bridge the gap between theory and this reality is one of main theme of this thesis. We explicitly show that how resonance is related to the irreversibility and stochasticity.

A simple stochastic irreversible equation is the Langevin equation, describing Brownian motion of a particle. For example, we can write a Langevin equation for a particle with mass m under the harmonic potential $m\omega^2 x^2/2$, where x is the position of the particle. Adding a phenomenological damping

term $\gamma dx(t)/dt$ with constant γ and a white noise term $\xi(t)$ to the harmonic oscillator equation, we get

$$m\frac{d^2}{dt^2}x(t) + m\gamma\frac{d}{dt}x(t) + m\omega^2 x(t) = \xi(t). \tag{1.1}$$

The damping and noise terms represent the effect of the environment of the particle. To study the same system dynamically, we construct a Hamiltonian which serves as a model of the particle and its environment [3]. This can be done both in classical and quantum mechanics. As shown by Mori and others (see [4] and references therein), starting with the Hamiltonian (or Heisenberg) equations of motion, one can derive generalized Langevin-type equations. For example, for the particle in the harmonic potential one can model the environment as a set of harmonic oscillators (bath) linearly coupled to the particle (the Caldeira-Legget model [3]). This gives the exact equation [5, 6]

$$\begin{aligned} &m\frac{d^2}{dt^2}x(t) + \int_0^t dt'\gamma_b(t-t')\frac{dx(t')}{dt'} + m\omega^2 x(t) \\ &= \xi_b(t) - x(0)\gamma_b(t), \end{aligned} \tag{1.2}$$

where $\gamma_b(t)$ and $\xi_b(t)$ are functions of the bath degrees of freedom and their coupling to the particle. This equation is equivalent to the original Hamiltonian equations, and therefore it is both time-reversal invariant and deterministic [1].

In contrast, the Langevin equation (1.1) has broken-time symmetry and contains a random force. The question is how to relate this phenomenological equation to Hamiltonian dynamics, represented by Eq. (1.2). A main difference

[1] Eq. (1.2) is a special case of the "generalized master equation;" see Ref. [7].

between Eqs. (1.1) and (1.2) is that the first is "Markovian," for the x and p, and it has no memory terms, while the second is "non-Markovian:" it has a memory term that makes the value of $x(t)$ dependent on the velocities at all the previous times t'.

A standard argument to derive Eq. (1.1) from Eq. (1.2) is to focus on time scales of the order of the relaxation time for the weak coupling case. Then the function $\gamma_b(t)$ can be approximated by the delta function $\gamma\delta(t-0\pm)$ [6]. In this procedure time-symmetry is broken as one has to choose between the delta functions $\delta(t-0\pm)$ for $t>0$ or $t<0$, respectively. Further approximations like averging out rapidly oscillating terms [8] show that the term $\xi_b(t)$ can be replaced by the white noise term $\xi(t)$. Then for $t>0$ we obtain Eq. (1.1) from Eq. (1.2) with $\gamma>0$ [for $t<0$ we change $\gamma \Rightarrow -\gamma$]. All this amounts to a "Markovian" approximation, where the memory terms in both γ_b and the noise correlation are dropped. This corresponds to the $\lambda^2 t$ approximation [9–12].

So, is Eq. (1.1) the result of an approximation? If this is so, this implies that both irreversibility and stochasticity are results of approximations. In chapter 4 we point out that irreversible and stochastic equations such as (1.1) can be viewed in a different way: they describe the time evolution of components of dynamics. This view has been put forward by Prigogine and collaborators [14]-[24]. We will show that indeed there are components exactly obeying Langevin (or Fokker-Planck) time evolution. These components are obtained by isolating resonant contribution. Thus we can also say that

resonance gives Gaussian white noise. We will consider the Friedrichs model, and we will consider classical mechanics. Later, for the $1/f$ type noise, we consider extended Friedrichs model. In this model we also show that the sum of resonance contributions gives the $1/f$ noise.

The outline of deriving the irreversible and stochastic components is the following. The system is described by a Hamiltonian

$$H = H_0 + \lambda V \tag{1.3}$$

where H_0 describes the unperturbed system without any couplings, V describes the interactions and λ is the dimensionless coupling constant. The associated Liouville operator or Liouvillian, i.e., the Poisson bracket with the Hamiltonian, is $L_H = i\{H, \}$. This is also written as $L_H = L_0 + \lambda L_V$. The time evolution of any function of the phase space variables $f(x, p)$ is given by

$$\frac{df}{dt} = iL_H f \tag{1.4}$$

We distinguish between integrable and nonintegrable systems.

For integrable systems one can introduce a canonical "unitary" transformation U that simplifies enormously the equations of motion. The transformation U^{-1} is applied to phase-space variables. The new variables describe non-interacting particles (quasiparticles). In the new representation we have periodic motion and there are no irreversible, stochastic processes. We obtain trivial Markovian equations with zero damping and no noise [see Eq. (3.33)]. After solving the trivial equations, we can transform the solutions back to the original phase-space variables.

But integrable systems are exceptional. Most systems are nonintegrable, at least in the sense of Poincaré. This means that U is not an analytic function of the coupling constant at $\lambda = 0$, and hence it cannot be constructed by perturbation expansions. This is due to resonances that produce vanishing denominators leading to divergences.

Prigogine and collaborators have introduced a "star-unitary" transformation Λ, that has essentially the same structure as the canonical transformation U, but with regularized denominators. As we will discuss later, star-unitarity is an extension of unitarity from integrable to nonintegrable systems. The regularization in Λ eliminates the divergences due to resonances. At the same time, it brings us to a new description that is no more equivalent to free motion. We have instead a "kinetic" description in terms of quasiparticles that obey simpler Markovian equations, but still interact. After solving these equations we can (at least in principle) go back to the original variables to solve the original equations of motion. The main point is that Markovian irreversible equations such as (1.1) describe components of the motion of the particles. In this thesis we will focus on the components leading to Brownian motion.

How is it possible that Hamiltonian dynamics can yield irreversible equations? This is because the Λ-transformed phase-space functions involve generalized functions, or distributions (examples are the "Gamow modes" presented in Sec. 3.3). If the initial unperturbed functions formed a Hilbert space, the transformed functions are no more in this Hilbert space. In its transformed

domain L_H behaves as a dissipative collision operator with complex eigenvalues [19, 20]. The extension of L_H involves an extension of real frequencies to complex frequencies, which breaks time-symmetry. We remark that even though the Hamiltonian H is time reversal invariant, breaking of time symmetry can occur in the *solutions* of the equations of motion. For example, the harmonic oscillator coupled to the bath can be exponentially damped either towards the future or the past. The two types of solution are possible. This is closely connected to the existence of eigenfunctions of L_H that break time-symmetry. We also note that damping and other irreversible processes occur only for systems that can have resonances, and are nonintegrable in the sense of Poincaré.

In order to construct Λ we specify the following requirements:

(1) The Λ transformation is obtained by analytic continuation of the unitary operator U.

(2) When there are no resonances, Λ reduces to U.

(3) Λ is analytic with respect to the coupling constant λ at $\lambda = 0$.

(4) Λ preserves the measure of the phase space.

(5) Λ maps real variables to real variables.

(6) Λ leads to closed Markovian kinetic equations.

In addition, to obtain a specific form of Λ we consider the "simplest" extension of U. This will be seen more precisely later, when we study our specific Hamiltonian model [see Eq. (A.24) and comments below].

In quantum mechanics, the Λ transformation has been used to define a dressed unstable particle state that has strict exponential decay. This state has a real average energy and gives an uncertainty relation between the lifetime and energy [17, 22, 25]. The dressed state defined by Λ has an exact Markovian time evolution, without the Zeno [26] or long tail periods [27].

The Friedrichs model we will consider describes a charged particle in a harmonic potential, coupled to a field bath. We are interested in the infinite volume limit of the system $L \to \infty$. The emission of the field from the particle leads to radiation damping. Conversely, the particle is excited when it absorbs the field. Depending on the initial state of the field, we have two distinct situations: (a) The field is in a "thermodynamic" state and (b) the field is in a "non-thermodynamic" state. The first case occurs when the the average action $\langle J_k \rangle$ of each field mode k satisfies [28]

$$\langle J_k \rangle \sim O(L^0) \tag{1.5}$$

Then the total energy is proportional to the volume L. This corresponds to the "thermodynamic limit." It does not necessarily imply that the field is in thermal equilibrium; it just means that we have a finite (non vanishing) energy density in the limit $L \to \infty$. The existence of the thermodynamic limit requires an initially random distribution of the phases of the field modes [29]. The second ("non-thermodynamic") case occurs when we have $\langle J_k \rangle \sim O(L^{-1})$, i.e., we have a vanishing energy density.

Here we will consider the thermodynamic case. In addition to the

damped oscillation, the particle undergoes an erratic motion due to the excitation caused by the thermodynamic field. This erratic motion includes a Brownian motion component, which is Markovian. The initial randomness of the phases of the field modes is a necessary condition for the appearance of Brownian motion. In addition it is essential that the field *resonates* with the particle. We need Poincaré resonances. Under these conditions Λ permits us to isolate the damping and the Brownian component of the motion. In phenomenological theories Brownian motion is attributed to a Gaussian white noise source, which creates fluctuations. In our formulation the effects of fluctuations are originated in the nondistributive property of Λ with respect to multiplication of dynamical variables. Nondistribuitivity is a consequence of the removal of Poincaré divergences. Later for the extended Friedrichs model, we show that the sum of resonance effects gives $1/f$ noise effect.

The organization of this thesis is as follows. In chapter 2 we introduce Λ transformation as an extended version of unitary transformation in the non-integrable systems. In chapter 3 we introduce the Friedrichs Hamiltonian and find Λ transformation. In chapter 4 we show that this transformation leads to Langevin and Fokker-Planck equations, and classical Gaussian white noise. In chapter 5 we briefly discuss the properties of quantum noise. In chapter 6 we introduce an extended Friedrichs model for the electron waveguide cavity, and derive the $1/f$ noise as sum of resonance contributions. In conclusion we summarize our results.

Chapter 2

Non-integrable systems and star-unitary transformation

2.1 Integrable and non-Integrable systems

We study a system of particles with Hamiltonian

$$H = H_0 + \lambda V, \qquad H = H(p, q) \tag{2.1}$$

where H_0 is the unperturbed Hamiltonian describing non-interacting particles, and V is the interaction. We assume the coupling constant λ is dimensionless. We consider mainly classical systems, but our statements apply as well to quantum systems. The Hamiltonian is a function of the momenta p and positions q of the particles. Often one introduces a new set of variables $J = J(p, q)$, $\alpha = \alpha(p, q)$ (action-angle variables) such that

$$H_0 = H_0(J), \qquad V = V(J, \alpha) \tag{2.2}$$

This means that when there is no interaction between the particles ($\lambda = 0$) the energy H only depends on the action variables J.

Integrable systems are systems for which we can go to a new representation $J \Rightarrow \bar{J}$, $\alpha \Rightarrow \bar{\alpha}$ such that

$$H = \bar{H}_0(\bar{J}) \tag{2.3}$$

The Hamiltonian can be written as a new function $\bar{H}_0$ depending only on the new actions. Typically, $\bar{H}_0$ will have the same form as H_0, but with renormalized parameters (such as frequencies).

The change of representation is expressed as a "canonical transformation" U,

$$\bar{J} = U^{-1}J, \qquad \bar{\alpha} = U^{-1}\alpha \tag{2.4}$$

The operator U is "unitary," $U^{-1} = U^{\dagger}$, where we define Hermitian conjugation through the inner product

$$\langle\langle f|\rho\rangle\rangle = \int d\Gamma f^*(\Gamma)\rho(\Gamma). \tag{2.5}$$

which is the ensemble average of f.[1] Here Γ is the set of all phase space variables, $d\Gamma$ is the phase-space volume element and $*$ means complex conjugate. The Hermitian conjugate is defined by

$$\langle\langle f|U\rho\rangle\rangle = \langle\langle \rho|U^{\dagger}f\rangle\rangle^* \tag{2.6}$$

The operator U is distributive with respect to products. For any two variables A and B we have [34]

$$UAB = (UA)(UB) \tag{2.7}$$

This property together with Eq. (2.3) lead to

$$UH(J,\alpha) = U\bar{H}_0(\bar{J}) = \bar{H}_0(U\bar{J}) = \bar{H}_0(J) \tag{2.8}$$

[1] One can introduce a Hilbert-space structure in classical mechanics through the Segal-Bargmann representation [24, 30].

The transformed Hamiltonian UH is the unperturbed Hamiltonian $\bar{H}_0$ depending only on the original action variables. In other words, U eliminates interactions.

The solution of the total Hamiltonian H can be found easily through the unitary transformation U. We look for the solution of the Hamiltonian through the Liouville equation which describes the time evolution of the statistical ensemble. For systems with many degrees of freedom the Liouville equation has an advantage since it can summarize the behavior of the whole system in a single equation. The statistical ensemble ρ satisfies the Louville equation

$$i\frac{\partial}{\partial t}\rho = i\{H, \rho\} = L_H \rho \tag{2.9}$$

where $L_H = i\{H, \}$ is the Liouville operator or Liouvillian. Similar to H, L_H may be split into a free Liouvillian plus interaction: $L_H = L_0 + \lambda L_V$, where $L_0 = L_{H_0}$. Applying U on both sides of the Liouville equation we get

$$\begin{aligned} i\frac{\partial}{\partial t}U\rho &= UL_H U^{-1} U\rho \\ \Rightarrow i\frac{\partial}{\partial t}\bar{\rho} &= \bar{L_0}\bar{\rho} \end{aligned} \tag{2.10}$$

where

$$\bar{\rho} = U\rho, \quad \bar{L_0} = UL_H U^{-1}. \tag{2.11}$$

$\bar{L_0}$ has the form of the non-interacting Liouvillian. Indeed We have

$$\begin{aligned} \bar{L_0}\bar{\rho} &= iU\{H, \rho\} = i\{UH, U\rho\} \\ &= L_{\bar{H}}\bar{\rho}. \end{aligned} \tag{2.12}$$

where in the second equality we used the property of preservation of the Poisson bracket by canonical transformations [34]. The transformed Liouvillian $\bar{L_0}$ does not contain any interaction terms, and the ensemble average over this transformed density function $\bar{\rho}$ can be easily calculated when the solution for $\bar{H}$ is known.

For non-integrable systems there exist, by definition, no transformation U. This happens, for example, when there appear divergences (divisions by zero) when we try to construct U as an expansion in the coupling constant λ (called "perturbation expansion"). Vanishing denominators appear when frequencies of the systems become equal. We have "Poincaré resonances" leading to divergences [16],

$$\frac{1}{\omega_1 - \omega_k} \to \infty \quad \text{for } \omega_1 \to \omega_k \tag{2.13}$$

We define now a transformation Λ with the following properties:

(1) The Λ transformation is obtained by analytic extension of the unitary operator U which diagoanlize the Hamiltonian .

(2) When there are no resonances, Λ reduces to U.

(3) Λ is analytic with respect to the coupling constant λ at $\lambda = 0$.

(4) Λ preserves the measure of the phase space.

(5) Λ maps real variables to real variables.

(6) Λ leads to closed Markovian kinetic equations.

Our method corresponds to the elimination of Poincaré resonances on the level of distributions. This extension of U is obtained by regularization of the denominators,

$$\frac{1}{\omega_1 - \omega_k} \Rightarrow \frac{1}{\omega_1 - \omega_k \pm i\epsilon} \tag{2.14}$$

where ϵ is an infinitesimal. As discussed below, the sign of $i\epsilon$ is determined by a time ordering depending on the correlations that appear as the particles interact. The regularization breaks time symmetry.

The Λ transformation permits us to find new units obeying kinetic equations. Indeed, applying the Λ transformation to the equations of motion, we discover irreversibility and stochasticity. Irreversibility appears because of the analytic continuation of U (from real to complex frequencies) and stochasticity because of the non-distributive property

$$\Lambda AB \neq (\Lambda A)(\Lambda B). \tag{2.15}$$

Hence we have fluctuations. Irreversibility and stochasticity are closely related to Poincaré resonances [13–15].

We introduce the new distribution function $\tilde{\rho} = \Lambda\rho$ into the Liouville equation. Then we obtain

$$i\frac{\partial}{\partial t}\tilde{\rho} = \tilde{\theta}\tilde{\rho} \tag{2.16}$$

where

$$\tilde{\theta} = \Lambda L_H \Lambda^{-1} \tag{2.17}$$

This is a kinetic equation describing irreversible stochastic phenomena. This equation in general contains diffusive terms, which map trajectories to ensembles. We have an intrinsically statistical formulation in terms of probabilities.

The next section will be devoted to a brief description of the main steps involved in the construction of the Λ transformation (more details can be found in Refs. [22, 25] and references therein for quantum mechanics, and in Ref. [24] for classical mechanics).

2.2 Construction of the Λ transformation

Our formulation is based on the "dynamics of correlations" induced by the Liouville equation [7]. The Liouville operator $L_H = L_0 + \lambda L_V$ is separated into a part describing free motion $L_0 = i\{H_0, \}$ and an interaction $\lambda L_V = i\{\lambda V, \}$. We then define correlation subspaces by decomposing the density operator ρ into independent components

$$\rho = \sum_\nu P^{(\nu)} \rho \tag{2.18}$$

where $P^{(\nu)}$ are projectors to the orthogonal eigenspaces of L_0

$$L_0 P^{(\nu)} = P^{(\nu)} L_0 = w^{(\nu)} P^{(\nu)} \tag{2.19}$$

$w^{(\nu)}$ being the real eigenvalues of L_0. The projectors are orthogonal and complete:

$$P^{(\mu)} P^{(\nu)} = P^{(\mu)} \delta_{\mu\nu}, \qquad \sum_\nu P^{(\nu)} = 1 \tag{2.20}$$

The complement projectors $Q^{(\nu)}$ are defined by

$$P^{(\nu)} + Q^{(\nu)} = 1 \tag{2.21}$$

They are orthogonal to $P^{(\nu)}$, i.e., $Q^{(\nu)}P^{(\nu)} = P^{(\nu)}Q^{(\nu)} = 0$, and satisfy $[Q^{(\nu)}]^2 = Q^{(\nu)}$.

As seen in Eq. (2.19) the unperturbed Liouvillian L_0 commutes with the projectors. Therefore the unperturbed Liouville equation is decomposed into a set of independent equations,

$$i\frac{\partial}{\partial t}P^{(\nu)}\rho = L_0 P^{(\nu)}\rho = w^{(\nu)}P^{(\nu)}\rho \tag{2.22}$$

The interaction λL_V induces transitions from one subspace to another subspace. To each subspace $P^{(\nu)}$ we associate a "degree of correlation" d_ν: we first define the vacuum of correlation as the set of all distributions belonging to the $P^{(0)}$ subspace. This subspace by definition has a degree of correlation $d_0 = 0$. Usually this subspace has the eigenvalue $w^{(0)} = 0$, i.e., it contains the invariants of unperturbed motion. The degree of correlation d_ν of a subspace $P^{(\nu)}$ is then defined as the minimum number of times we need to apply the interaction L_V on the vacuum of correlations $P^{(0)}$ in order to make a transition to $P^{(\nu)}$. Dynamics is seen as a dynamics of correlations.

Our method involves the extension of U to Λ, from integrable to non-integrable systems. This is applicable to systems, which, depending on certain parameters, can be either integrable or non-integrable in the sense of Poincaré. For example, a system contained in a finite box with periodic boundary conditions will have a discrete spectrum of frequencies. We can then avoid any

resonances. The system is integrable, and we can construct U by perturbation expansions. However, when we take the limit of an infinite volume, the spectrum of frequencies becomes continuous and resonances are unavoidable. The system becomes non-integrable in Poincaré's sense, as the perturbation expansion of U gives divergent terms. We can remove the divergences by regularization of the denominators, obtaining the Λ transformation.

Let us consider first integrable systems, where we may introduce the canonical transformation U that eliminates the interactions. In the Liouvillian formulation the relation (2.8) gives

$$UL_H U^{-1} = \bar{L}_0 \tag{2.23}$$

Namely, the transformed Liouvillian is the unperturbed Liouvillian $\bar{L}_0 = i\{\bar{H}_0, \}$. Similar to L_0 we have

$$\bar{L}_0 P^{(\nu)} = P^{(\nu)} \bar{L}_0 = \bar{w}^{(\nu)} P^{(\nu)} \tag{2.24}$$

where $\bar{w}^{(\nu)}$ are renormalized eigenvalues. In the U representation there are no transitions from one degree of correlation to another.

We write U in terms of "kinetic operators" (we use bars for the integrable case)

$$\begin{aligned} \bar{\chi}^{(\nu)} &\equiv P^{(\nu)} U^{-1} P^{(\nu)} \\ \bar{C}^{(\nu)} \bar{\chi}^{(\nu)} &\equiv Q^{(\nu)} U^{-1} P^{(\nu)} \end{aligned} \tag{2.25}$$

We have as well the hermitian conjugate components

$$[\bar{\chi}^{(\nu)}]^{\dagger} \equiv P^{(\nu)} U P^{(\nu)} \tag{2.26}$$
$$[\bar{\chi}^{(\nu)}]^{\dagger} \bar{D}^{(\nu)} \equiv P^{(\nu)} U Q^{(\nu)}$$

where $\bar{D}^{(\nu)} \equiv [\bar{C}^{(\nu)}]^{\dagger}$. The operators $\bar{C}^{(\nu)}$ and $\bar{D}^{(\nu)}$ are called, respectively, "creation" and "destruction" operators [14] as they can create or destroy correlations, leading to transitions from one subspace $P^{(\nu)}$ to a different subspace $P^{(\mu)}$. The $\bar{\chi}^{(\nu)}$ operator, on the other hand, is diagonal, as it leads to transitions within each subspace, i.e., it maps $P^{(\nu)}$ to $P^{(\nu)}$. Using Eq. (2.21) we obtain

$$U^{-1} P^{(\nu)} = (P^{(\nu)} + \bar{C}^{(\nu)}) \bar{\chi}^{(\nu)} \tag{2.27}$$
$$P^{(\nu)} U = [\bar{\chi}^{(\nu)}]^{\dagger} (P^{(\nu)} + \bar{D}^{(\nu)}) \tag{2.28}$$

Now, from the commutation relation in Eq. (2.24) we derive a closed equation for the $\bar{C}^{(\nu)}$ operators. From Eq. (2.24), we have

$$L_H U^{-1} P^{(\nu)} = U^{-1} P^{(\nu)} U L_H U^{-1} P^{(\nu)}, \tag{2.29}$$

Substituting Eq. (2.27) into Eq. (2.29), we get

$$L_H (P^{(\nu)} + \bar{C}^{(\nu)}) \bar{\chi}^{(\nu)} = (P^{(\nu)} + \bar{C}^{(\nu)}) \bar{\chi}^{(\nu)} U L_H (P^{(\nu)} + \bar{C}^{(\nu)}) \bar{\chi}^{(\nu)}. \tag{2.30}$$

Let us write

$$\bar{\Phi}_C^{(\nu)} \equiv P^{(\nu)} + \bar{C}^{(\nu)}. \tag{2.31}$$

Multiplying $\bar{\Phi}_C^{(\nu)}$ on both sides of Eq. (2.30) by left, we get

$$\bar{\Phi}_C^{(\nu)} L_H \bar{\Phi}_C^{(\nu)} \bar{\chi}^{(\nu)} = \bar{\Phi}_C^{(\nu)} \bar{\chi}^{(\nu)} U L_H (P^{(\nu)} + \bar{C}^{(\nu)}) \bar{\chi}^{(\nu)} = L_H \bar{\Phi}_C^{(\nu)} \bar{\chi}^{(\nu)} \tag{2.32}$$

where we used $(\bar{\Phi}_C^{(\nu)})^2 = \bar{\Phi}_C^{(\nu)}$ and Eq. (2.30) again. The relation $(\bar{\Phi}_C^{(\nu)})^2 = \bar{\Phi}_C^{(\nu)}$ comes from the fact

$$(P^{(\nu)})^2 = P^{(\nu)}, \;\; P^{(\nu)} \bar{C}^{(\nu)} = 0, \tag{2.33}$$

$$(\bar{C}^{(\nu)})^2 = 0, \;\; \bar{C}^{(\nu)} P^{(\nu)} = 0. \tag{2.34}$$

So we have

$$\bar{\Phi}_C^{(\nu)} L_H \bar{\Phi}_C^{(\nu)} = L_H \bar{\Phi}_C^{(\nu)} \tag{2.35}$$

$$(w^{(\nu)} - L_0) \bar{\Phi}_C^{(\nu)} = \lambda L_V \bar{\Phi}_C^{(\nu)} - \bar{\Phi}_C^{(\nu)} \lambda L_V \bar{\Phi}_C^{(\nu)}. \tag{2.36}$$

From Eq. (2.36) we get a closed equation for $\bar{\Phi}_C^{(\nu)}$,

$$\bar{\Phi}_C^{(\nu)} = P^{(\nu)} + \sum_{\mu(\neq\nu)} P^{(\mu)} \frac{-1}{w^{(\mu)} - w^{(\nu)}} [\lambda L_V \bar{\Phi}_C^{(\nu)} - \bar{\Phi}_C^{(\nu)} \lambda L_V \bar{\Phi}_C^{(\nu)}] \tag{2.37}$$

or

$$\begin{aligned} \bar{C}^{(\nu)} &= \sum_{\mu(\neq\nu)} P^{(\mu)} \frac{-1}{w^{(\mu)} - w^{(\nu)}} \\ &\times \; [\lambda L_V P^{(\nu)} + \lambda L_V \bar{C}^{(\nu)} - \bar{C}^{(\nu)} \lambda L_V \bar{C}^{(\nu)}]. \end{aligned} \tag{2.38}$$

We call Eq. (2.37) the nonlinear Lippmann-Schwinger equation. The $\bar{\chi}^{(\nu)}$ operators are obtained from the relation

$$P^{(\nu)} U U^{-1} P^{(\nu)} = P^{(\nu)} \tag{2.39}$$

which leads to [22]

$$\bar{A}^{(\nu)} = \bar{\chi}^{(\nu)}[\bar{\chi}(\nu)]^{\dagger} \tag{2.40}$$

where $\bar{A}^{(\nu)} \equiv (P^{(\nu)} + \bar{D}^{(\nu)}\bar{C}^{(\nu)})^{-1}$ (the inverse is defined in each subspace: $\bar{A}^{(\nu)}[\bar{A}^{(\nu)}]^{-1} = P^{(\nu)}$). The general solution of Eq. (2.39) is

$$\bar{\chi}^{(\nu)} = [\bar{A}^{(\nu)}]^{1/2} \exp(\bar{B}^{(\nu)}) \tag{2.41}$$

where $\bar{B}^{(\nu)} = -[\bar{B}^{(\nu)}]^{\dagger}$ is an arbitrary antihermitian operator. For the integrable case we have one more condition on U: the distributivity property Eq. (2.7). This condition fixes the operator $\bar{\chi}^{(\nu)}$ (see Appendix A). Note that for integrable systems the denominator in Eq. (2.38) is always non-vanishing: there are no Poincaré resonances.

Now we go to nonintegrable systems. Due to Poincaré divergence we cannot eliminate the interactions among particles through a canonical transformation. However we may still introduce a representation for which the dynamics is closed within each correlation subspace. We introduce the transformation Λ such that the transformed Liouvillian in Eq. (2.17) commutes with the projectors $P^{(\nu)}$

$$\tilde{\theta}P^{(\nu)} = P^{(\nu)}\tilde{\theta} \tag{2.42}$$

This allows us to obtain closed Markovian equations

$$i\frac{\partial}{\partial t}P^{(\nu)}\tilde{\rho} = \tilde{\theta}P^{(\nu)}\tilde{\rho} \tag{2.43}$$

In contrast to the integrable case, the $P^{(\nu)}$ projectors are no more eigenprojectors of the transformed Liouvillian $\tilde{\theta}$. Hence we can have transitions within

each subspace. As we shall show, the Λ transformation makes a direct connection between dynamics and kinetic theory. The operator $\tilde{\theta}$ is indeed a generalized "collision operator." Collision operators are familiar in kinetic theory. They are dissipative operators with complex eigenvalues, the imaginary parts of which give, for example, damping or diffusion rates. The Liouville operator is related to $\tilde{\theta}$ through a similitude relation. This means that L_H itself has complex eigenvalues. This is possible because we are extending the domain of L_H to distributions outside the Hilbert space [18, 20, 22].

To construct Λ, the basic idea is to extend the canonical transformation U through analytic continuation. Similar to Eq. (2.27) we write Λ in terms of kinetic operators

$$\begin{aligned} \Lambda^{-1} P^{(\nu)} &= (P^{(\nu)} + C^{(\nu)})\chi^{(\nu)} \\ P^{(\nu)}\Lambda &= [\chi^{(\nu)}]^{\star}(P^{(\nu)} + D^{(\nu)}) \end{aligned} \tag{2.44}$$

In order to avoid Poincaré divergences, Λ can no more be unitary. Instead, it is "star unitary"

$$\Lambda^{-1} = \Lambda^{\star} \tag{2.45}$$

where the $\star$ applied to operators means "star-hermitian" conjugation, defined below.

From the commutation relation Eq. (2.42) we arrive again at the equation (2.38) for the creation operator. But the denominator in Eq. (2.38) may now vanish due to Poincaré resonances. We regularize it by adding $\pm i\epsilon$. We

obtain the equation

$$\Phi_C^{(\nu)} = P^{(\nu)} + \sum_{\mu(\neq\nu)} P^{(\mu)} \frac{-1}{w^{(\mu)} - w^{(\nu)} - i\epsilon_{\mu\nu}} [\lambda L_V \Phi_C^{(\nu)} - \Phi_C^{(\nu)} \lambda L_V \Phi_C^{(\nu)}] \tag{2.46}$$

or

$$\begin{aligned} C^{(\nu)} &= \sum_{\mu(\neq\nu)} P^{(\mu)} \frac{-1}{w^{(\mu)} - w^{(\nu)} - i\epsilon_{\mu\nu}} \\ &\times [\lambda L_V P^{(\nu)} + \lambda L_V C^{(\nu)} - C^{(\nu)} \lambda L_V C^{(\nu)}] \end{aligned} \tag{2.47}$$

where the sign if $i\epsilon$ is chosen according to the "$i\epsilon$-rule:" [20, 22]

$$\begin{aligned} \epsilon_{\mu\nu} &= +\epsilon \text{ if } d_\mu \geq d_\nu \\ \epsilon_{\mu\nu} &= -\epsilon \text{ if } d_\mu < d_\nu \end{aligned} \tag{2.48}$$

where $\epsilon > 0$. This rule means, essentially, that transitions from lower to higher correlations are oriented towards the future and transitions from higher to lower correlations are oriented towards the past. We could also choose the other branch with $\epsilon < 0$, where the roles of past and future are exchanged. The main point is that regularization of the denominators breaks time symmetry.

For the $D^{(\nu)}$ operators we have

$$\begin{aligned} D^{(\nu)} &= [P^{(\nu)} \lambda L_V + D^{(\nu)} \lambda L_V - D^{(\nu)} \lambda L_V D^{(\nu)}] \\ &\times \sum_{\mu(\neq\nu)} \frac{1}{w^{(\nu)} - w^{(\mu)} - i\epsilon_{\nu\mu}} P^{(\mu)} \end{aligned} \tag{2.49}$$

The $i\epsilon$-rule leads to well defined perturbation expansions for $C^{(\nu)}$, $D^{(\nu)}$ and $A^{(\nu)} = [P^{(\nu)} + D^{(\nu)} C^{(\nu)}]^{-1}$. Recall that we have the relation $\bar{D}^{(\nu)} = [\bar{C}^{(\nu)}]^\dagger$

for the integrable case. For the non-integrable case, due to $i\epsilon$ rule, these operators are no more related by hermitian conjugation. They are related by star hermitian conjugation, which is obtained by hermitian conjugation plus the change $\epsilon_{\mu\nu} \Rightarrow \epsilon_{\nu\mu}$. Then we have

$$D^{(\nu)} = [C^{(\nu)}]^\star, \quad A^{(\nu)} = [A^{(\nu)}]^\star \tag{2.50}$$

Similar to the integrable case the χ operators are given by

$$\chi^{(\nu)} = [A^{(\nu)}]^{1/2} \exp(B^{(\nu)}) \tag{2.51}$$

where $B^{(\nu)} = -[B^{(\nu)}]^\star$ is an arbitrary anti-star-hermitian operator. In contrast to the integrable case we have no distributivity condition to derive $B^{(\nu)}$. However, the conditions on Λ stated before lead to a well defined $\chi^{(\nu)}$ operator.

Chapter 3

The Friedrichs model and Λ transformation

3.1 The classical Friedrichs model

We consider a classical system consisting of a charged harmonic oscillator coupled to a classical scalar field in one-dimensional space. A quantum version of this model has been studied by Friedrichs [31], among others. We write the Hamiltonian of the system in terms of the oscillator and field modes $\bar{q}_1$ and $\bar{q}_k$,

$$\begin{aligned} H &= H_0 + \lambda V \\ &= \omega_1 \bar{q}_1^* \bar{q}_1 + \sum_k \omega_k \bar{q}_k^* \bar{q}_k + \lambda \sum_k \bar{V}_k (\bar{q}_1^* \bar{q}_k + \bar{q}_1 \bar{q}_k^*), \end{aligned} \tag{3.1}$$

with a given constant frequency $\omega_1 > 0$ for the harmonic oscillator (particle), $c = 1$ for the speed of light, $\omega_k = |k|$ for the field, and a dimensionless coupling constant λ [1]. When λ is small we can treat the interaction potential as a perturbation. We assume the system is in a one-dimensional box of size L with periodic boundary conditions. Then the spectrum of the field is discrete, i.e., $k = 2\pi j/L$ where j is an integer. The volume dependence of the interaction

[1] In Ref. [24], we used a dimensionless Hamiltonian dividing the Hamiltonian by a constant $\omega_0 J_0$. In this Hamiltonian, all variables become dimensionless. This is more convenient to deal with the Segal-Bargmann representation used in [24].

V_k is given by

$$\bar{V}_k = \sqrt{\frac{2\pi}{L}}\bar{v}_k \tag{3.2}$$

where $\bar{v}_k = O(1)$. We assume that $\bar{v}_k$ is real and even: $\bar{v}_k = \bar{v}_{-k}$. Furthermore, we assume that for small k

$$\bar{v}_k \sim \omega_k^{1/2}. \tag{3.3}$$

To deal with the continuous spectrum of the field we take the limit $L \to \infty$. In this limit we have

$$\frac{2\pi}{L}\sum_k \to \int dk, \quad \frac{L}{2\pi}\delta_{k,0} \to \delta(k). \tag{3.4}$$

The normal coordinates $\bar{q}_1$, $\bar{q}_k$ satisfy the Poisson bracket relation

$$i\{\bar{q}_\alpha,\ \bar{q}^*_\beta\} = \delta_{\alpha\beta}. \tag{3.5}$$

where

$$i\{f,g\} = \sum_{r=1,k}\Big[\frac{\partial f}{\partial \bar{q}_r}\frac{\partial g}{\partial \bar{q}^*_r} - \frac{\partial g}{\partial \bar{q}_r}\frac{\partial f}{\partial \bar{q}^*_r}\Big] \tag{3.6}$$

The normal coordinates are related to the position x_1 and the momentum p_1 of the particle as

$$\bar{q}_1 = \sqrt{\frac{m\omega_1}{2}}(x_1 + \frac{ip_1}{m\omega_1}), \tag{3.7}$$

$$x_1 = \frac{1}{\sqrt{2m\omega_1}}(\bar{q}_1 + \bar{q}^*_1),$$

$$p_1 = -i\sqrt{\frac{m\omega_1}{2}}(\bar{q}_1 - \bar{q}^*_1) \tag{3.8}$$

and to the field $\phi(x)$ and its conjugate field $\pi(x)$ as

$$\phi(x) = \sum_k \big(\frac{1}{2\omega_k L}\big)^{1/2}(\bar{q}_k e^{ikx} + \bar{q}_k^* e^{-ikx}), \tag{3.9}$$

$$\pi(x) = -i\sum_k \big(\frac{\omega_k}{2L}\big)^{1/2}(\bar{q}_k e^{ikx} - \bar{q}_k^* e^{-ikx}). \tag{3.10}$$

The field $\phi(x)$ corresponds to the transverse vector potential in electromagnetism, while $\pi(x)$ corresponds to the transverse displacement field. Our Hamiltonian can be seen as a simplified version of a classical dipole molecule interacting with a classical radiation field in the dipole approximation [32]. For simplicity we drop processes associated with the interactions proportional to $\bar{q}_1\bar{q}_k$ and $\bar{q}_1^*\bar{q}_k^*$, which correspond to "virtual processes" in quantum mechanics. This approximation corresponds to the so-called the rotating wave approximation [33].

We note that we have $\omega_k = \omega_{-k}$ degeneracy in our Hamiltonian. To avoid some complexity due to this degeneracy, we rewrite our Hamiltonian in terms of new variables as [24]

$$H = \omega_1 q_1^* q_1 + \sum_k \omega_k q_k^* q_k + \lambda \sum_k V_k(q_1^* q_k + q_1 q_k^*), \tag{3.11}$$

where

$$q_1 \equiv \bar{q}_1, \quad q_k \equiv \begin{cases} (\bar{q}_k + \bar{q}_{-k})/\sqrt{2}, & \text{for } k > 0, \\ (\bar{q}_k - \bar{q}_{-k})/\sqrt{2}, & \text{for } k \le 0, \end{cases} \tag{3.12}$$

$$V_k \equiv \begin{cases} \sqrt{2}\bar{V}_k, & \text{for } k > 0 \\ 0, & \text{for } k \le 0, \end{cases} \tag{3.13}$$

$$v_k = \sqrt{\frac{L}{2\pi}} V_k. \tag{3.14}$$

In this form the variable q_k with the negative argument k is completely decoupled from the other degrees of freedom.

We define action and angle variables J_s, α_s through the relation

$$q_s = \sqrt{J_s} e^{-i\alpha_s}, \quad s = 1,\, k \tag{3.15}$$

For our model we can have both integrable and nonintegrable cases. The first occurs when the spectrum of frequencies of the field is discrete, the second when it is continuous. We consider first the integrable case.

3.2 Integrable case: Unitary transformation

In this Section, we present the properties of the canonical transformation U that diagonalizes the Hamiltonian in the discrete spectrum case, when the size of the the box L is finite. Later we will extend U to Λ through analytic continuation.

We assume that $\omega_1 \neq \omega_k$ for all k. In this case, the system is integrable in the sense of Poincaré. We can find the new normal modes $\bar{Q}_s$, $\bar{Q}_s^*$ that diagonalize the Hamiltonian through U. The new normal modes are related to the original normal modes as

$$\bar{Q}_s = U^\dagger q_s \quad \text{for } s = 1, k. \tag{3.16}$$

in one-to-one correspondence. The operator U is "unitary," $U^{-1} = U^\dagger$, where Hermitian conjugation is defined through the inner product

$$\langle\langle f|\rho\rangle\rangle = \int d\Gamma f(\Gamma)^* \rho(\Gamma). \tag{3.17}$$

which is the ensemble average of f. Here Γ is the set of all phase space variables and $d\Gamma$ is the phase-space volume element. For an operator O the Hermitian conjugate is defined by

$$\langle\langle f|O\rho\rangle\rangle = \langle\langle \rho|O^\dagger f\rangle\rangle^*. \tag{3.18}$$

In terms of the new normal modes the Hamiltonian is diagonalized as

$$H = \sum_s \bar{\omega}_s \bar{Q}_s^* \bar{Q}_s \tag{3.19}$$

where $\bar{\omega}_\alpha$ are renormalized frequencies.

The new normal modes satisfy the Poisson bracket relation

$$i\{\bar{Q}_r,\ \bar{Q}_s^*\} = \delta_{rs}. \tag{3.20}$$

Since the interaction is bilinear in the normal modes, the new normal modes can be found explicitly through a linear superposition of the original modes [24]. For the particle we obtain, from the equation $i\{H, \bar{Q}_1\} = -\bar{\omega}_1 \bar{Q}_1$,

$$\bar{Q}_1 = \bar{N}_1^{1/2}(q_1 + \lambda \sum_k \bar{c}_k q_k) \tag{3.21}$$

where

$$\bar{c}_k \equiv \frac{V_k}{\bar{\omega}_1 - \omega_k}, \tag{3.22}$$

$$\bar{N}_1 \equiv (1 + \bar{\xi})^{-1}, \quad \bar{\xi} \equiv \lambda^2 \sum_k \bar{c}_k^2. \tag{3.23}$$

The renormalized frequency $\bar{\omega}_1$ is given by the root of the equation

$$\eta(\bar{\omega}_1) = 0, \quad \eta(z) \ \equiv \ z - \omega_1 - \sum_{k'} \frac{\lambda^2 |V_{k'}|^2}{z - \omega_{k'}} \tag{3.24}$$

that reduces to ω_1 when $\lambda = 0$. For the field modes one can also find explicit forms, but as in this paper we will focus on the particle, we will not present them here (see [24]).

The perturbation expansion of Eq. (3.21) yields

$$\bar{Q}_1 = U^\dagger q_1 = q_1 + \sum_k \frac{\lambda V_k}{\omega_1 - \omega_k} q_k + O(\lambda^2) \tag{3.25}$$

When the spectrum is discrete, the denominator never vanishes; each term in the perturbation series is finite. This implies integrability in the sense of Poincaré: U can be constructed by a perturbation series in powers of λ^n with $n \geq 0$ integer. In other words, U is analytic at $\lambda = 0$.

Since the transformation U is canonical, it is distributive with respect to multiplication

$$U^{-1} q_r q_s^* = [U^{-1} q_r][U^{-1} q_s^*] = \bar{Q}_r \bar{Q}_s^* \tag{3.26}$$

Hence we have

$$UH = U[\sum_s \bar{\omega}_s \bar{Q}_s^* \bar{Q}_s] = \sum_s \bar{\omega}_s q_s^* q_s = \bar{H}_0 \tag{3.27}$$

The transformed Hamiltonian UH has the same form of the unperturbed Hamiltonian H_0, with renormalized frequencies.

The canonical transformation can also be introduced on the level of statistical ensembles ρ. These obey the Liouville equation

$$i\frac{\partial}{\partial t}\rho = i\{H,\, \rho\} = L_H \rho, \tag{3.28}$$

where $L_H = i\{H, \}$ is the Liouville operator or Liouvillian. Similar to H, L_H may be split into a free Liouvillian plus interaction: $L_H = L_0 + \lambda L_V$, where $L_0 = L_{H_0}$. Applying U on both sides of the Liouville equation we get

$$\begin{aligned} i\frac{\partial}{\partial t}U\rho &= UL_HU^{-1}U\rho \\ \Rightarrow i\frac{\partial}{\partial t}\bar{\rho} &= \bar{L_0}\bar{\rho} \end{aligned} \tag{3.29}$$

where

$$\bar{\rho} = U\rho, \quad \bar{L_0} = UL_HU^{-1}. \tag{3.30}$$

$\bar{L_0}$ has the form of the non-interacting Liouvillian. Indeed we have

$$\begin{aligned} &\bar{L_0}\bar{\rho} = iU\{H, \rho\} = i\{UH, U\rho\} \\ &= \left[\sum_s \bar{\omega}_s(q_s^*\frac{\partial}{\partial q_s^*} - q_s\frac{\partial}{\partial q_s})\right]U\rho \end{aligned} \tag{3.31}$$

where in the second equality we used Eq. (3.27) and the property of preservation of the Poisson bracket by canonical transformations [34]. The transformed Liouvillian $\bar{L_0}$ does not contain any interaction terms, and the ensemble average over this transformed density function $\bar{\rho}$ can be easily calculated. For example for

$$i\frac{\partial}{\partial t}\int d\Gamma\, x_1 U\rho = i\frac{\partial}{\partial t}\langle x_1\rangle \tag{3.32}$$

and similarly for $\langle p_1\rangle$ we get, after substituting Eq. (3.8) and integrating by parts,

$$\frac{\partial}{\partial t}\langle x_1\rangle = \frac{1}{\bar{m}}\langle p_1\rangle, \quad \frac{\partial}{\partial t}\langle p_1\rangle = -\bar{m}\bar{\omega}_1^2\langle x_1\rangle. \tag{3.33}$$

These are the equations for the free harmonic oscillator (with renormalized frequency $\bar{\omega}_1$ and renormalized mass $\bar{m} = m\omega_1/\bar{\omega}_1$). The interaction with the field is eliminated.

Note that the normal modes are eigenfunctions of the Liouvillian $\bar{L}_0$,

$$\bar{L}_0 q_1 = -\bar{\omega}_1 q_1, \qquad \bar{L}_0 q_1^* = \bar{\omega}_1 q_1^*. \tag{3.34}$$

This leads to

$$L_H \bar{Q}_1 = -\bar{\omega}_1 \bar{Q}_1, \qquad L_H \bar{Q}_1^* = \bar{\omega}_1 \bar{Q}_1^*. \tag{3.35}$$

For products of modes we have

$$\begin{aligned} \bar{L}_0 q_1^{*m} q_1^n &= [(m-n)\bar{\omega}_1]\, q_1^{*m} q_1^n, \\ L_H \bar{Q}_1^{*m} \bar{Q}_1^n &= [(m-n)\bar{\omega}_1]\, \bar{Q}_1^{*m} \bar{Q}_1^n. \end{aligned} \tag{3.36}$$

Finally, we note that from distributive property Eq. (3.26) we have

$$U^\dagger q_1^{*m} q_1^n = (U^\dagger q_1^{*m})(U^\dagger q_1^n). \tag{3.37}$$

3.3 Nonintegrable case: Gamow modes

Now we consider the continuous spectrum case, where the particle frequency ω_1 is inside the range of the continuous spectrum ω_k. In this case, by analytic continuation of $\bar{Q}_1$ and $\bar{Q}_1^*$ we can get new modes which are eigenfunctions of the Liouvillian with complex eigenvalues. These modes are called Gamow modes. Gamow states have been previously introduced in quantum

mechanics to study unstable states [35]-[40]. In classical mechanics, Gamow modes have been introduced in Ref. [24]. In this Section we present the main properties of Gamow modes, which will be used for the construction of Λ.

When we go to the continuous limit we restrict the strength of the coupling constant λ so that

$$\int dk \frac{\lambda^2 |v_k|^2}{\omega_k} < \omega_1, \tag{3.38}$$

Then the harmonic oscillator becomes unstable. In this case we have radiation damping. If Eq. (3.38) is not satisfied, then we go outside the range of applicability of the "rotating wave approximation" (see comment after Eq. (3.10)) as the Hamiltonian becomes not bounded from below, and gives no radiation damping [41].

In the continuous spectrum case, divergences appear in the construction of U, due to resonances. For example, the denominator in Eq. (3.25) may now vanish at the Poincaré resonance $\omega_1 = \omega_k$. We have a divergence in the perturbation expansion in λ. To deal with this divergence, we regularize the denominator by adding an infinitesimal $\pm i\epsilon$. Then we get

$$Q_1 = q_1 + \lambda \sum_k \frac{\lambda V_k}{\omega_1 - \omega_k \pm i\epsilon} q_k + O(\lambda^2). \tag{3.39}$$

In the continuous limit the summation goes to an integral. We take the limit $L \to \infty$ first and $\epsilon \to \infty$ later. Then the denominator can be interpreted as a distribution under the integration over k

$$\frac{1}{\omega_1 - \omega_k \pm i\epsilon} \to \mathcal{P}\frac{1}{\omega_1 - \omega_k} \mp i\pi\delta(\omega_1 - \omega_k) \tag{3.40}$$

where $\mathcal{P}$ means principal part.

The introduction of $i\epsilon$ in the continuous limit is related to a change of the physical situation. In the discrete case the boundaries of the system cause periodicity in the motion of the particle and the field. In contrast, in the continuous case the boundaries play no role. In the continuous limit we can have damping of the particle, as the field emitted from the particle goes away and never comes back. And we can have Brownian motion, due to the interaction with the continuous set of field modes. The continuous limit may be well approximated by a discrete system during time scales much shorter than the time scale for which the field goes across the boundaries.

In the continuous limit the solutions of the Hamilton equations have broken time symmetry (although the Hamiltonian is always time reversal invariant). We can have damping of the particle either toward the future or toward the past. This corresponds to the existence of the two branches $\pm i\epsilon$ in Eq. (3.39). Breaking of time symmetry is connected to resonances [16].

As shown in [38], continuing the perturbation expansion (3.39) to all orders one obtains new renormalized modes (Gamow modes) associated with the complex frequency

$$z_1 \equiv \tilde{\omega}_1 - i\gamma \tag{3.41}$$

or its complex conjugate z_1^*. Here $\tilde{\omega}_1$ is the renormalized frequency of the particle, and $2\gamma > 0$ is the damping rate. The complex frequencies are solutions

of the equation

$$\eta^{\pm}(\omega) = \omega - \omega_1 - \int dk \frac{\lambda^2 v_k^2}{(z-\omega_k)^{\pm}_{\omega}} = 0, \tag{3.42}$$

The + (−) superscript indicates that the propagator is first evaluated on the upper (lower) half plane of z and then analytically continued to $z = \omega$.

The new modes for the $-i\epsilon$ branch in Eq. (3.39) are given by

$$\tilde{Q}_1 = N_1^{1/2}[q_1 + \lambda \sum_k c_k q_k], \tag{3.43}$$

$$c_k = \frac{V_k}{(z-\omega_k)^{+}_{z_1}}, \quad N_1 = (1 + \lambda^2 \sum_k c_k^2)^{-1}. \tag{3.44}$$

$$\tag{3.45}$$

and its complex conjugate, satisfying

$$L_H \tilde{Q}_1 = -z_1 \tilde{Q}_1, \qquad L_H \tilde{Q}_1^* = z_1^* \tilde{Q}_1^* \tag{3.46}$$

The mode $\tilde{Q}_1^*$ decays for $t > 0$ as

$$e^{iL_H t}\tilde{Q}_1^* = e^{iz_1^* t}\tilde{Q}_1^* = e^{(i\tilde{\omega}_1 - \gamma)t}\tilde{Q}_1^* \tag{3.47}$$

(and similarly $\tilde{Q}_1$).

The modes for the $+i\epsilon$ branch are given by

$$Q_1^* = N_1^{1/2}[q_1^* + \lambda \sum_k c_k q_k^*] \tag{3.48}$$

and its complex conjugate, satisfying

$$L_H Q_1^* = z_1 Q_1^*, \qquad L_H Q_1 = -z_1^* Q_1 \tag{3.49}$$

These modes decay for $t < 0$.

The modes we have introduced have quite different properties from the usual canonical variables. Their Poisson brackets vanish

$$i\{Q_1, Q_1^*\} = i\{\tilde{Q}_1, \tilde{Q}_1^*\} = 0 \tag{3.50}$$

However the modes $\tilde{Q}_1$ and Q_1^* are duals; they form a generalized canonical pair

$$i\{\tilde{Q}_1, Q_1^*\} = 1 \tag{3.51}$$

This algebra corresponds to an extension of the usual Lie algebra including dissipation [24]. An analogue of this algebra has been previously studied in quantum mechanics [35]-[40].

3.4 The Λ transformation

As seen in the previous Section, we eliminate Poincaré divergences in the renormalized particle modes by analytic continuation of frequencies to the complex plane, leading to Gamow modes. Similar to Eq. (3.16), the Gamow modes are generated by the Λ transformation (see Appendix A.2),

$$\begin{aligned} \tilde{Q}_1 &= \Lambda^\dagger q_1, \quad \tilde{Q}_1^* = \Lambda^\dagger q_1^* \\ Q_1 &= \Lambda^{-1} q_1, \quad Q_1^* = \Lambda^{-1} q_1^* \end{aligned} \tag{3.52}$$

Note that $\Lambda^\dagger \neq \Lambda^{-1}$ is not unitary. Instead, it is "star-unitary,"

$$\Lambda^{-1} = \Lambda^\star \tag{3.53}$$

where the $\star$ applied to operators means "star conjugation."

The general definition of star conjugation is given in Refs. [14, 22, 24]. Here we will restrict the action of Λ to particle modes. In this case performing star conjugation simply means taking hermitian conjugation and changing $i\epsilon \Rightarrow -i\epsilon$, so we have, e.g., $[\Lambda^\star(i\epsilon)]q_1 = [\Lambda^\dagger(-i\epsilon)]q_1$. Due to star-unitarity, the existence of the star-conjugate transformation $\Lambda^\star$ guarantees the existence of the inverse Λ^{-1}.

We are interested not only in the renormalized modes, but also the renormalized products of modes,

$$\Lambda^\dagger q_1^{*m} q_1^n, \quad \Lambda^{-1} q_1^{*m} q_1^n \tag{3.54}$$

This will allow us to calculate renormalized functions of the particle variables (expandable in monomials), which will lead us to the Langevin and Fokker-Planck equations. For the integrable case, renormalized products of modes can be easily calculated thanks to the distributive property (3.37). However, as shown below, for the nonintegrable case products of Gamow modes give new Poincaré divergences. Hence, due to the requirement (2) stated in the Introduction, the Λ transformation has to be non-distributive.

Before going to the calculation of Eq. (3.54), we discuss the main features of Λ. The requirement (1) given in the Introduction means that transformed functions obtained by Λ are expressed by analytic continuations of the functions obtained by U, where real frequencies (such as $\bar{\omega}_1$) are replaced by complex frequencies (such as z_1). The requirement (6) is one of the most im-

portant, and we will now explain it in more detail. To obtain closed Markovian equations, we first operate Λ on the Liouville equation, to obtain

$$\begin{aligned} i\frac{\partial}{\partial t}\Lambda\rho &= \Lambda L_H \Lambda^{-1}\Lambda\rho \\ \Rightarrow i\frac{\partial}{\partial t}\tilde{\rho} &= \tilde{\theta}\tilde{\rho} \end{aligned} \tag{3.55}$$

where

$$\tilde{\rho} = \Lambda\rho, \quad \tilde{\theta} = \Lambda L_H \Lambda^{-1}. \tag{3.56}$$

Closed Markovian equations involve a projection (or a part) of the ensemble $\tilde{\rho}$. In order for Λ to give this type of equations, we require that the transformed Liouvillian $\tilde{\theta}$ in Eq. (3.55) leaves subspaces corresponding to projections of $\tilde{\rho}$ invariant. We will represent these subspaces by projection operators $P^{(\nu)}$, which are complete and orthogonal in the domain of $\tilde{\theta}$,

$$P^{(\mu)}P^{(\nu)} = P^{(\mu)}\delta_{\mu\nu}, \qquad \sum_{\nu} P^{(\nu)} = 1 \tag{3.57}$$

We choose $P^{(\nu)}$ as eigenprojectors of L_0. We have $L_0 P^{(\nu)} = w^{(\nu)}P^{(\nu)}$ where $w^{(\nu)}$ are the eigenvalues. In the Friedrichs model the $P^{(\nu)}$ subspaces consist of monomials (or superposition of monomials) of field and particle modes. As an example consider the eigenvalue equations

$$\begin{aligned} L_0 q_1^* q_k &= w^{(1k)} q_1^* q_k \\ L_0 q_1^* q_k q_l^* q_l &= w^{(1k)} q_1^* q_k q_l^* q_l. \end{aligned} \tag{3.58}$$

where $w^{(1k)} \equiv \omega_1 - \omega_k$. Both monomials belong to the same subspace with eigenvalue $w^{(1k)}$. Denoting the corresponding projector by $P^{(1k)}$ we have

$$q_1^* q_k = P^{(1k)} q_1^* q_k, \quad q_1^* q_k q_l^* q_l = P^{(1k)} q_1^* q_k q_l^* q_l \tag{3.59}$$

One may introduce a Hilbert space structure for the eigenfunctions of L_0, including suitable normalization constants in the Segal-Bargmann representation [24]. The basis function $f_{\vec{m},\vec{n}}(\vec{q}^*, \vec{q})$ for the Hilbert space is given by

$$f_{\vec{m},\vec{n}}(\vec{q}^*, \vec{q}) = \prod_a \frac{(q_a^*)^{m_a} q_a^{n_a}}{\sqrt{m_a! n_a!}} e^{-|q_a|^2}, \tag{3.60}$$

where $\vec{q} \equiv q_1, q_{k1}, q_{k2}, \cdots$. With this basis we can find isomorphism between classical representation and corresponding quantum representation on the level of Liouville formalism [24]. The derivation of Λ from $C^{(\nu)}$ and $\chi^{(\nu)}$ with Segel-Bargmann representation are given in [24]. In this section we derive Λ through analytic continuation of U.

The invariance property of $\tilde{\theta}$ is

$$P^{(\nu)}\tilde{\theta} = \tilde{\theta}P^{(\nu)}. \tag{3.61}$$

Thanks to this commutation property, we obtain from Eq. (3.55) closed Markovian equations for the projections of $\tilde{\rho}$,

$$i\frac{\partial}{\partial t}P^{(\nu)}\tilde{\rho}(t) = \tilde{\theta}P^{(\nu)}\tilde{\rho}(t). \tag{3.62}$$

The commutation relation (3.61) together with the other requirements give a well-defined transformation Λ. Details on this have been presented in Refs. [22, 25] for quantum mechanics and in [23, 24] for classical mechanics, for bilinear variables. The main idea is to associate a "degree of correlation" with each subspace $P^{(\nu)}$. Dynamics induces transitions among different $P^{(\nu)}$ subspaces. We have a "dynamics of correlations" [7]. This allows us to perform

the regularization of denominators of U in a systematic way, depending on types of transitions (from lower to higher correlations or vice versa), which leads to Λ. Here we will present a short derivation of the transformed products in Eq. (3.54), based on the results of Refs. [22–25].

Note that for the integrable case, the transformed Liouvillian $\bar{L}_0$ has the same eigenfunctions as L_0 (see, e.g., Eq. (3.36)). This is connected to the fact that in the integrable case we can reduce the equation of motion to a collection of independent units. For the nonintegrable case L_0 and $\tilde{\theta}$ share eigenfunctions only in special cases, when there are no degeneracies of L_0. An example is given by the modes q_1, q_1^* (see Appendix A.2). In general, however, the subspaces $P^{(\nu)}$ include degenerate eigenfunctions of L_0 (see Eq. (3.58)). As we will see, in this case, $P^{(\nu)}$ are not eigenprojectors of $\tilde{\theta}$. This has an important physical consequence: we can have transitions inside each subspace, corresponding to kinetic processes, including damping and diffusion of the particle, which involve an exchange of energy with the field.

Let us now consider the transformed product $\Lambda^\dagger q_1^* q_1$. Later we will generalize this to obtain the expressions (3.54). If $\Lambda^\dagger$ were distributive, $\Lambda^\dagger q_1^* q_1$ could be expressed as the product $\tilde{Q}_1^* \tilde{Q}_1 = (\Lambda^\dagger q_1^*)(\Lambda^\dagger q_1)$. However, as we show now, this expression gives Poincaré divergences in the thermodynamic limit.

We have

$$\begin{aligned}
& \tilde{Q}_1^* \tilde{Q}_1 \\
= \; & |N_1|(q_1^* + \lambda \sum_k c_k^* q_k^*)(q_1 + \lambda \sum_k c_k q_k) \\
= \; & |N_1|(q_1^* q_1 + \lambda q_1^* \sum_k c_k q_k + \lambda q_1 \sum_k c_k^* q_k^* \\
+ \; & \lambda^2 \sum_{k,k'}{}' c_k^* c_{k'} q_k^* q_{k'} + \lambda^2 \sum_k |c_k|^2 q_k^* q_k). \qquad (3.63)
\end{aligned}$$

where the prime in the summation means $k \neq k'$. Going to the continuous limit and taking the ensemble average with an ensemble ρ the last term becomes

$$\sum_k |c_k|^2 \langle q_k^* q_k \rangle \rightarrow \int dk \Big| \frac{\lambda v_k}{(z - \omega_k)_{z_1}^+} \Big|^2 \langle J_k \rangle \qquad (3.64)$$

where $\langle J_k \rangle = \langle\langle q_k^* q_k | \rho \rangle\rangle$. This term has a non-vanishing finite value in the limit $L \rightarrow \infty$ if the thermodynamic limit condition in Eq. (1.5) is satisfied. It is non-negligible as compared to the average of the $q_k^* q_{k'}$ term in Eq. (3.63) if ρ belongs to a class of ensembles with δ-function singularities in the wave vector k, such that the point contribution $k = k'$ is as important as the integration over k' [7, 19, 20]

$$\sum_{k'} \langle q_k^* q_{k'} \rangle \sim \langle J_k \rangle \sim O(L^0). \qquad (3.65)$$

Both terms in Eq. (3.65) give contributions of the same order. For ensembles in this class, we have well defined intensive and extensive variables in the thermodynamic limit [7]. A typical example of this class of ensembles is the Maxwell-Boltzmann distribution.

To lowest order we have in Eq. (3.64),

$$\frac{\lambda v_k}{(z-\omega_k)^+_{z_1}} = \frac{\lambda v_k}{\omega_1 - \omega_k + i\epsilon} + O(\lambda^3) \tag{3.66}$$

which leads to

$$\begin{aligned} \Big|\frac{\lambda v_k}{(z-\omega_k)^+_{z_1}}\Big|^2 &= \frac{\lambda^2 v_k^2}{|\omega_1 - \omega_k + i\epsilon|^2} + O(\lambda^4) \\ &= \frac{\pi}{\epsilon}\lambda^2 v_k^2 \delta(\omega_1 - \omega_k) + O(\lambda^4) \to \infty. \end{aligned} \tag{3.67}$$

This diverges when $\epsilon \to 0$. Hence Eq. (3.64) is nonanalytic at $\lambda = 0$ due to the resonance at $\omega_1 = \omega_k$. We have Poincaré divergence in the perturbation series of $(\Lambda^\dagger q_1^*)(\Lambda^\dagger q_1)$.

We note that in non-thermodynamic situations, we have $\langle J_k \rangle \sim O(1/L)$. The energy density goes to zero in the infinite volume limit. In this case the appearance of the Poincaré divergence in Eq. (3.64) has no effect on the particle. This is consistent with the results we will discuss in the next Section: the non-distributivity of Λ is related to the appearance of fluctuations in Brownian motion. And Brownian motion of the particle appears only when the particle is surrounded by a field described by the thermodynamic limit.

For quantum mechanics the situation is different. We can have fluctuations even in non-thermodynamic situations [22] due to vacuum effects. For example we obtain, for a two-level atom, an energy fluctuation of the dressed excited state which is of the order of the decay rate. This gives an uncertainty relation between energy and lifetime.

Coming back to the thermodynamic limit case, we conclude that $\Lambda^\dagger q_1^* q_1$ cannot be expressed as the product Eq. (3.63) since Λ is, by definition, analytic in the coupling constant. To make this transformed product analytic, we replace $|c_k|^2$ in Eq. (3.63) by a suitable analytic function ξ_k. So we have [2]

$$\Lambda^\dagger q_1^* q_1 = \tilde{Q}_1^* \tilde{Q}_1 + \sum_k b_k q_k^* q_k \tag{3.68}$$

where

$$b_k = \lambda^2 |N_1| (-|c_k|^2 + \xi_k) \tag{3.69}$$

To determine ξ_k we note that, in the integrable case $\bar{c}_k$ is real, and the coefficient of $q_k^* q_k$ in $U^\dagger q_1^* q_1$ is $\bar{c}_k^2$ (see Eq. (3.21)). In the nonintegrable case c_k is complex. Taking into account the requirements (1)-(3) and (5) in the Introduction we conclude that a suitable extension of $\bar{c}_k^2$ to the nonintegrable case is the linear superposition

$$\xi_k = r c_k^2 + \text{c.c.}, \quad r + r^* = 1 \tag{3.70}$$

where r is a complex constant to be determined. The relation $r + r^* = 1$ guarantees that ξ_k reduces to $\bar{c}_k^2$ in the integrable case [see also the comments below Eq. (A.24)]. As shown in Appendix A.3 using the requirement (4) we obtain

$$r = \frac{\exp(-ia/2)}{2\cos(a/2)}, \quad N_1 = |N_1| \exp(-ia) \tag{3.71}$$

[2] Neglecting $O(1/L)$ terms, the second term in Eq. (3.68) may be expressed in terms of renormalized field modes as $\sum_k b_k \tilde{Q}_k^* \tilde{Q}_k$ (see Eq. (4.52)).

By including the term b_k in Eq. (3.68) we have removed the Poincaré divergence in the product of Gamow modes. As a consequence,

$$\Lambda^\dagger q_1^* q_1 \neq (\Lambda^\dagger q_1^*)(\Lambda^\dagger q_1) \tag{3.72}$$

This shows the non-distributive property of Λ.

For weak coupling the approximate value of b_k is given by [22],

$$b_k \approx \frac{2\pi}{L} \frac{\lambda^2 v_k^2 \gamma^2}{[(\omega_k - \tilde{\omega}_1)^2 + \gamma^2]^2}. \tag{3.73}$$

This has a sharp peak at $\omega_k = \tilde{\omega}_1$ with a width γ. It corresponds to the line shape of emission and absorption of the field by the renormalized particle.

To find more general transformed products $\Lambda^\dagger q_1^{*m} q_1^n$, we apply the same logic that led to Eq. (3.68). Whenever $|c_k|^2$ appears in $\tilde{Q}_1^{*m} \tilde{Q}_1^n$, we replace it with ξ_k. This leads to (see Appendix A.4).

$$\Lambda^\dagger q_1^{*m} q_1^n = \sum_{a=0}^{min(m,n)} \frac{m! n! Y^a}{(m-a)!(n-a)! a!} \tilde{Q}_1^{*m-a} \tilde{Q}_1^{n-a} \tag{3.74}$$

where

$$Y \equiv \sum_k b_k q_k^* q_k \tag{3.75}$$

Note that $b_k \sim O(1/L)$. Hence $Y \sim O(L^0)$ only if the field obeys the thermodynamic limit condition, Eq. (1.5). Otherwise Y vanishes as $1/L$ and $\Lambda^\dagger$ becomes distributive. Also, when there are no resonances, z_1 becomes real and both b_k and Y vanish. Then $\Lambda^\dagger$ reduces to $U^\dagger$ (see Eq. (3.37)). In short, both thermodynamic limit and resonances are necessary to obtain non distributivity of $\Lambda^\dagger$ in Eq. (3.74).

For $\Lambda^{-1}q_1^{*m}q_1^n$ we obtain the expression (3.74) with $\tilde{Q}_1, \tilde{Q}_1^*$ replaced by Q_1, Q_1^*, respectively.

The Λ transformation we have presented satisfies all our requirements (1)-(5) stated in the Introduction. Now we show that Λ also satisfies the requirement (6). Using Eq. (A.73) in Appendix A.7 with $q_1' = 0$, we find

$$\tilde{\theta}^\dagger q_1^{*m}q_1^n = [(mz_1^* - nz_1)q_1^*q_1 - 2i\gamma mnY]q_1^{*m-1}q_1^{n-1} \tag{3.76}$$

and similarly

$$\tilde{\theta}q_1^{*m}q_1^n = [(mz_1 - nz_1^*)q_1^*q_1 + 2i\gamma mnY]q_1^{*m-1}q_1^{n-1} \tag{3.77}$$

Both the l.h.s. and the r.h.s. of these two equations belong to the same eigenspace $P^{(mn)}$ of L_0 with eigenvalue $(m-n)\omega_1$. This illustrates the statement that $\tilde{\theta}$ leaves the subspaces $P^{(\nu)}$ invariant, satisfying the requirement (6). Also it shows that $\tilde{\theta}$ in Eq. (3.77) satisfies Eq. (2.42), which leads to Eq. (2.47). Note that $q_1^{*m}q_1^n$ are not eigenfunctions of $\tilde{\theta}$, so $P^{(mn)}$ is not an eigenprojector of $\tilde{\theta}$. The two terms inside brackets in Eqs. (3.76), (3.77) describe the decay of the particle modes (through emission of the field) and the absorption of field modes, respectively.

Chapter 4

Classical white noise

4.1 The relation between the Langevin equation and Λ transformed variables

Now we discuss the relation between the solution of the Langevin equation for a Brownian harmonic oscillator and $\Lambda^\dagger$ transformed variables for the Friedrichs model. We will focus on the $\Lambda^\dagger$ transformation, so that the transformed variables decay for $t > 0$ [1] (see Eq. (3.47)). $\Lambda^\dagger$ transformed variables A generate the kinetic equation (3.55) as we have $\langle\langle \Lambda^\dagger A|\rho\rangle\rangle = \langle\langle A|\tilde{\rho}\rangle\rangle$. Remarkably, the time evolution of the Brownian oscillator variables is the same as the evolution of $\Lambda^\dagger$ transformed variables.

Let us first write the Langevin equations for the Brownian harmonic oscillator with mass and frequency $\tilde{m}$ and $\tilde{\omega}_1$. As we will see in Eq. (4.45) these mass and frequency correspond to the renormalized mass and frequency of the particle due to the interaction with the field. For the particle position

[1] Note that variables A evolve as $\exp(iL_H t)A$, while states ρ evolve as $\exp(-iL_H t)\rho$. In Refs. [22, 24, 25] we considered transformed *states* that decay for $t > 0$. For this reason in those papers we used the Λ^{-1} transformation rather than $\Lambda^\dagger$ (see Eqs. (3.49), (3.52)).

$x_1(t)$ and momentum $p_1(t)$, we have ($t \geq 0$)

$$\frac{d}{dt}x_1(t) = -\gamma x_1(t) + \frac{p_1(t)}{\tilde{m}} + A(t), \tag{4.1}$$

$$\frac{d}{dt}p_1(t) = -\gamma p_1(t) - \tilde{m}\tilde{\omega}_1^2 x_1(t) + B(t). \tag{4.2}$$

These equations describe the damped harmonic oscillator with random momentum and force terms $A(t)$ and $B(t)$, respectively. The symmetrical random momentum and force terms are appropriate for comparison with the Friedrichs model since the Hamiltonian is symmetrical under rescaled position and momentum exchange. If the Hamiltonian were not symmetric under position and momentum exchange, e.g. if $q_1 q_k$ and $q_1^* q_k^*$ terms were included in the interaction, then the Langevin equations with asymmetric random terms would be more appropriate for the comparison [7, 42, 43].

We assume that $A(t)$ and $B(t)$ have the Gaussian white noise properties [11, 34]. Specifically,

(1) The averages of $A(t)$ and $B(t)$ over an ensemble of Brownian particles having the given position and momentum x_0 and p_0 at $t = 0$ vanish.

$$\langle A(t)\rangle_{x_0,p_0} = 0, \quad \langle B(t)\rangle_{x_0,p_0} = 0. \tag{4.3}$$

[from now on, $\langle\ \rangle$ means $\langle\ \rangle_{x_0,p_0}$].

(2) We assume that the correlation between the values of $A(t)$ and $A(t')$ is that of a white noise.

$$\langle A(t)A(t')\rangle = A_c^2\delta(t - t') \tag{4.4}$$

where A_c is a real constant to be determined. We assume the same for $B(t)$, i.e.

$$\langle B(t)B(t')\rangle = B_c^2\delta(t-t'). \tag{4.5}$$

Assuming that the noise comes from the thermal bath with temperature T, these constants A_c and B_c can be calculated explicitly (see Appendix A.6),

$$A_c^2 = \frac{2\gamma k_B T}{\tilde{m}\tilde{\omega}_1^2}, \quad B_c^2 = 2\tilde{m}\gamma k_B T. \tag{4.6}$$

(3) We assume that all higher averages of the random variable $A(t)$ can be expressed in terms of the second moments, i.e. $A(t)$ is a "Gaussian noise,"

$$\langle A(t_1)A(t_2)...A(t_{2n+1})\rangle = 0, \tag{4.7}$$

$$\langle A(t_1)A(t_2)...A(t_{2n})\rangle \tag{4.8}$$
$$= \sum_{\text{all pairs}} \langle A(t_{i_1})A(t_{i_2})\rangle....\langle A(t_{i_{2n-1}})A(t_{i_{2n}})\rangle.$$

In Eq. (4.8), the sum is over all sets of possible pairings. For example, we have

$$\begin{aligned}\langle A(t_1)A(t_2)A(t_3)A(t_4)\rangle &= \langle A(t_1)A(t_2)\rangle\langle A(t_3)A(t_4)\rangle \\ &+\langle A(t_1)A(t_3)\rangle\langle A(t_2)A(t_4)\rangle \\ &+\langle A(t_1)A(t_4)\rangle\langle A(t_2)A(t_3)\rangle.\end{aligned} \tag{4.9}$$

We assume the same property for $B(t)$, and we assume that $A(t)$ and $B(t)$ are not correlated. In other words,

$$\begin{aligned}&\langle A(t_1)...A(t_m)B(t_1')...B(t_n')\rangle \\ &= \langle A(t_1)...A(t_m)\rangle\langle B(t_1')...B(t_n')\rangle.\end{aligned} \tag{4.10}$$

Multiplying Eq. (4.1) by $\sqrt{\tilde{m}\tilde{\omega}_1/2}$ and Eq. (4.2) by $i/\sqrt{2\tilde{m}\tilde{\omega}_1}$ and adding them, we get

$$\frac{d}{dt}q_L(t) = -iz_1 q_L(t) + R(t), \tag{4.11}$$

where $z_1 \equiv \tilde{\omega}_1 - i\gamma$ and

$$q_L(t) \equiv \sqrt{\frac{\tilde{m}\tilde{\omega}_1}{2}}\left(x_1(t) + i\frac{p_1(t)}{\tilde{m}\tilde{\omega}_1}\right), \tag{4.12}$$

$$R(t) \equiv \sqrt{\frac{\tilde{m}\tilde{\omega}_1}{2}}\left(A(t) + i\frac{B(t)}{\tilde{m}\tilde{\omega}_1}\right). \tag{4.13}$$

q_L is a Langevin mode and $R(t)$ is a "complex noise." $R(t)$ has the following properties.

(1) $R^*(t)$ and $R(t')$ have the delta function correlation.

$$\langle R^*(t)R(t')\rangle = \tilde{m}\tilde{\omega}_1 A_c^2 \delta(t-t') \tag{4.14}$$

This can be proved directly from the definition of $R(t)$.

(2) $R(t)$ has the Gaussian property

$$\begin{aligned}&\langle R^*(t_1)...R^*(t_m)R(t'_1)...R(t'_n)\rangle \\ &= \delta_{mn}\sum_{\text{all pairs}}\langle R^*(t_{i_1})R(t'_{j_1})\rangle\cdots\langle R^*(t_{i_n})R(t'_{j_n})\rangle.\end{aligned} \tag{4.15}$$

The proof is shown in Appendix A.5.

The solution of Eq. (4.11) is given by

$$q_L(t) = q_{La}(t) + q_{Lr}(t), \tag{4.16}$$

where

$$q_{La}(t) \equiv q_L(0)e^{-iz_1 t}, \tag{4.17}$$

$$q_{Lr}(t) \equiv e^{-iz_1 t}\int_0^t dt' R(t')e^{iz_1 t'}. \tag{4.18}$$

The term $q_{La}(t)$ describes the damped harmonic oscillator without noise, and the term $q_{Lr}(t)$ describes the behavior due to the noise. Now we have the explicit form of $q_L(t)$. The position and momentum can be found from the relation Eq. (4.12).

Using the properties of the noise $R(t)$, we show next that the time evolution of $\langle q_L^{*m}(t)q_L^n(t)\rangle$ is the same as the time evolution of Λ transformed modes $\langle \Lambda^\dagger q_1^{*m} q_1^n\rangle$. First we calculate $\langle q_L^{*m}(t)q_L^n(t)\rangle$. We consider the case $m \geq n$. The case $m < n$ can be calculated similarly. We have

$$\begin{aligned}
&\langle q_L^{*m}(t)q_L^n(t)\rangle \\
&= \langle (q_{La}^*(t) + q_{Lr}^*(t))^m (q_{La}(t) + q_{Lr}(t))^n\rangle \\
&= \sum_{k=0}^{m}\sum_{l=0}^{n} \frac{m!}{(m-k)!k!}\frac{n!}{(n-l)!l!} \\
&\times q_{La}^{*m-k}(t)q_{La}^{n-l}(t)\langle q_{Lr}^{*k}(t)q_{Lr}^l(t)\rangle.
\end{aligned} \tag{4.19}$$

The quantity $\langle q_{Lr}^{*k}(t)q_{Lr}^l(t)\rangle$ is non-zero only when $k = l$, as we can see from Eq. (4.15). Considering the fact that the number of sets of all possible pairs

in $\langle R^*(t_1)...R^*(t_l)R(t_1')...R(t_l')\rangle$ is $l!$, we have

$$\begin{aligned}
&\langle q_{Lr}^k(t)q_{Lr}^l(t)\rangle \\
&= l!\delta_{kl}\bigg(\langle e^{iz_1^*t}\int_0^t dt_1 R^*(t_1)e^{-iz_1^*t_1} \\
&\times e^{-iz_1t}\int_0^t dt_2 R(t_2)e^{iz_1t_2}\rangle\bigg)^l \\
&= l!\delta_{kl}\bigg(\frac{\tilde{m}\tilde{\omega}_1 A_c^2(1-e^{-2\gamma t})}{2\gamma}\bigg)^l \\
&= l!\delta_{kl}\Big(\frac{k_BT}{\tilde{\omega}_1}\Big)^l(1-e^{-2\gamma t})^l
\end{aligned} \tag{4.20}$$

Substituting Eq. (4.17) and Eq. (4.20) into Eq. (4.19), we get

$$\begin{aligned}
&\langle q_L^{*m}(t)q_L^n(t)\rangle \\
&= e^{i(mz_1^*-nz_1)t}\sum_{l=0}^{n}\frac{m!n!}{(m-l)!(n-l)!l!} \\
&\times q_L^{*m-l}(0)q_L^{n-l}(0)\Big(\frac{k_BT}{\tilde{\omega}_1}\Big)^l(e^{2\gamma t}-1)^l.
\end{aligned} \tag{4.21}$$

Now we can compare the above expression with the time-evolved transformed products $e^{iL_Ht}\Lambda^\dagger q_1^{*m}q_1^n$. We have (see Eq. (3.46) and Eq. (3.74))

$$\begin{aligned}
e^{iL_Ht}\Lambda^\dagger q_1^{*m}q_1^n &= \sum_{a=0}^{n}\frac{m!n!}{(m-a)!(n-a)!a!} \\
&\times e^{i(mz_1^*-nz_1)t}e^{2\gamma at}\tilde{Q}^{*m-a}\tilde{Q}^{n-a}Y^a
\end{aligned} \tag{4.22}$$

Writing

$$e^{2\gamma at} = \sum_{l=0}^{a}\frac{a!}{l!(a-l)!}(e^{2\gamma t}-1)^l \tag{4.23}$$

and $l' = a - l$ we have

$$
\begin{aligned}
e^{iL_H t}\Lambda^\dagger q_1^{*m} q_1^n &= \sum_{l=0}^{n}\sum_{l'=0}^{n-l} \frac{m!n!}{(m-l-l')!(n-l-l')!l!l'!} \\
&\times e^{i(mz_1^* - nz_1)t}\tilde{Q}^{*m-l-l'}\tilde{Q}^{n-l-l'}Y^{l+l'}(e^{2\gamma t}-1)^l \\
&= \sum_{l=0}^{n} \frac{m!n!}{(m-l)!(n-l)!l!} e^{i(mz_1^* - nz_1)t} \\
&\times \sum_{l'=0}^{n-l} \frac{(m-l)!(n-l)!}{(m-l-l')!(n-l-l')!l'!}\tilde{Q}^{*m-l-l'}\tilde{Q}^{n-l-l'}Y^{l'} \\
&\times Y^l(e^{2\gamma t}-1)^l
\end{aligned} \tag{4.24}
$$

Using Eq. (3.74) again we obtain

$$
\begin{aligned}
& e^{iL_H t}\Lambda^\dagger q_1^{*m} q_1^n \\
&= e^{i(mz_1^* - nz_1)t} \sum_{l=0}^{n} \frac{m!n!}{(m-l)!(n-l)!l!} \\
&\times (\Lambda^\dagger q_1^{*m-l} q_1^{n-l}) Y^l (e^{2\gamma t}-1)^l
\end{aligned} \tag{4.25}
$$

Comparing Eq. (4.21) and Eq. (4.25), we see the direct correspondences

$$
\begin{aligned}
\frac{k_B T}{\tilde{\omega}_1} &\Leftrightarrow Y = \sum_k b_k q_k^* q_k \\
\langle q_L^{*m}(t) q_L^n(t)\rangle &\Leftrightarrow e^{iL_H t}\Lambda^\dagger (q_1^{*m} q_1^n).
\end{aligned} \tag{4.26}
$$

The form and time evolution of the ensemble average of Langevin equation variables are the same as those of Λ transformed variables. Furthermore, if we take the ensemble average of $\Lambda^\dagger q_1^{*m} q_1^n$, we see a closer correspondence. Let us assume that the field action J_k obeys the Boltzmann distribution. The initial distribution $\tilde{\rho}_0(\Gamma)$ has the form (with $\beta \equiv 1/(k_B T)$)

$$
\tilde{\rho}_0(\Gamma) = C\rho_{01}(J_1, \alpha_1) \exp(-\beta \sum_k \omega_k J_k) \tag{4.27}
$$

where C is a normalization constant, k_B is Boltzmann's constant and T is the temperature. Noting that

$$d\Gamma = dx_1 dp_1 \prod_k dx_k dp_k = dJ_1 \frac{d\alpha_1}{2\pi} \prod_k dJ_k \frac{d\alpha_k}{2\pi}, \tag{4.28}$$

the average of J_k over this ensemble is

$$\begin{aligned} \langle J_k \rangle &= \frac{\int d\Gamma J_k \rho_{01}(J_1, \alpha_1) \exp(-\beta \sum_k \omega_k J_k)}{\int d\Gamma \rho_{01}(J_1, \alpha_1) \exp(-\beta \sum_k \omega_k J_k)} \\ &= \frac{1}{\omega_k \beta} = \frac{k_B T}{\omega_k}. \end{aligned} \tag{4.29}$$

To calculate $\sum_k b_k \langle J_k \rangle$, we need the form of b_k. The approximate value of b_k is given in Eq. (3.73), which for the weak coupling case is approximated by the delta function $(2\pi/L)\delta(\omega_k - \tilde{\omega}_1)$ [22]. So we get

$$\sum_k b_k \langle J_k \rangle = \sum_k b_k \frac{k_B T}{\omega_k} \approx \frac{k_B T}{\tilde{\omega}_1}. \tag{4.30}$$

Note that ${\omega_k}^{-1}$ does not make any divergence for small k since b_k is proportional to $v_k^2 \sim \omega_k$ for small k. In short, we obtain a complete correspondence between Λ transformed modes and Langevin modes (see Eq. (4.26)).

4.2 Derivation of the Fokker-Planck equation

Using the above results we can now derive the Fokker-Planck equation for the transformed density function $\tilde{\rho} = \Lambda \rho$. We start with the transformed equation (see Eq. (3.56))

$$i \frac{\partial}{\partial t} \tilde{\rho} = \tilde{\theta} \tilde{\rho}. \tag{4.31}$$

We derive the Fokker-Planck equation for $q_1,\ q_1^*$. We follow the standard derivation found in textbooks (see [11, 34]), but now in terms of Λ. Consider a test function $G(q_1, q_1^*)$, which is smooth and vanishes at $|q_1| = \infty$. Multiplying this on both sides of Eq. (4.31) and integrating over the phase space, we have

$$\begin{aligned}\int d\Gamma G(q_1, q_1^*) i\frac{\partial}{\partial t}\tilde{\rho}(\Gamma, t) &= \int d\Gamma G(q_1, q_1^*)\tilde{\theta}(\Gamma)\tilde{\rho}(\Gamma, t) \\ &= \int d\Gamma d\Gamma' G(q_1, q_1^*)\tilde{\theta}(\Gamma)\delta(\Gamma - \Gamma')\tilde{\rho}(\Gamma', t). \end{aligned} \tag{4.32}$$

In Eq. (4.32), $\tilde{\theta}(\Gamma)$ means $\tilde{\theta}$ that acts on Γ variables. We expand $G(q_1, q_1^*)$ near q_1' and $q_1'^*$.

$$\begin{aligned}&\int d\Gamma G(q_1, q_1^*) i\frac{\partial}{\partial t}\tilde{\rho}(\Gamma, t) \\ &= \int d\Gamma d\Gamma' \Bigg\{ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{1}{m!n!}\left(\frac{\partial^m}{\partial(q_1'^*)^m}\frac{\partial^n}{\partial(q_1')^n} G(q_1', q_1'^*)\right) \\ &\times (q_1^* - q_1'^*)^m (q_1 - q_1')^n \Bigg\}\tilde{\theta}(\Gamma)\delta(\Gamma - \Gamma')\tilde{\rho}(\Gamma', t)\end{aligned} \tag{4.33}$$

Integrating by parts, Eq. (4.33) becomes

$$\begin{aligned}&\int d\Gamma G(q_1, q_1^*) i\frac{\partial}{\partial t}\tilde{\rho}(\Gamma, t) \\ &= \int d\Gamma' G(q_1', q_1'^*) \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^{m+n}}{m!n!}\frac{\partial^m}{\partial(q_1'^*)^m}\frac{\partial^n}{\partial(q_1')^n} \\ &\times \left[\int d\Gamma (q_1^* - q_1'^*)^m (q_1 - q_1')^n \tilde{\theta}(\Gamma)\delta(\Gamma - \Gamma')\right]\tilde{\rho}(\Gamma', t)\end{aligned} \tag{4.34}$$

We call the quantities inside the brackets in Eq. (4.34) the "moments" of order $m+n$. The moments are calculated explicitly in Appendix A.7. They are given

by

$$\int d\Gamma (q_1^* - q_1^{'*})^m (q_1 - q_1')^n \tilde{\theta}(\Gamma)\delta(\Gamma - \Gamma') = \begin{cases} z_1 q_1', & m=0,\ n=1 \\ -z_1^* q_1^{'*}, & m=1,\ n=0 \\ 2i\gamma \sum_k b_k q_k^{'*} q_k', & m=1,\ n=1 \\ 0, & \text{for all other } m \text{ and } n. \end{cases} \tag{4.35}$$

Substituting Eq. (4.35) into Eq. (4.34), we get

$$\begin{aligned} &\int d\Gamma G(q_1, q_1^*) i\frac{\partial}{\partial t}\tilde{\rho}(\Gamma, t) \\ &= \int d\Gamma' G(q_1', q_1^{'*})(-\frac{\partial}{\partial q_1'})(z_1 q_1')\,\tilde{\rho}(\Gamma', t) \\ &+ \int d\Gamma' G(q_1', q_1^{'*})(-\frac{\partial}{\partial q_1^{'*}})(-z_1^* q_1^{'*})\,\tilde{\rho}(\Gamma', t) \\ &+ \int d\Gamma' G(q_1', q_1^{'*})(\frac{\partial^2}{\partial q_1' \partial q_1^{'*}})(2i\gamma \sum_k b_k q_k^{'*} q_k')\,\tilde{\rho}(\Gamma', t) \end{aligned} \tag{4.36}$$

By changing the integration variable Γ' to Γ in the right hand side of Eq. (4.36) and eliminating i on both sides, we have

$$\begin{aligned} &\int d\Gamma G(q_1, q_1^*)\frac{\partial}{\partial t}\tilde{\rho}(\Gamma, t) \\ = &\int d\Gamma G(q_1, q_1^*)\Big\{\frac{\partial}{\partial q_1}(iz_1 q_1) + \frac{\partial}{\partial q_1^*}(-iz_1^* q_1^*) \\ + &\frac{\partial}{\partial q_1 \partial q_1^*}(2\gamma \sum_k b_k q_k^* q_k)\Big\}\tilde{\rho}(\Gamma, t). \end{aligned} \tag{4.37}$$

Now suppose that $\tilde{\rho}(\Gamma, t)$ is factorized at $t = 0$. In other words, we write $\tilde{\rho}(\Gamma, 0)$ as

$$\tilde{\rho}(\Gamma, 0) = f_1(x_1, p_1) \prod_k f_k(x_k, p_k). \tag{4.38}$$

As shown in Appendix A.8, this factorized form of $\tilde{\rho}$ enables us to write Eq. (4.36) as

$$\begin{aligned} & \int d\Gamma_1 G(q_1, q_1^*) \frac{\partial}{\partial t} \int d\Gamma_f \tilde{\rho}(\Gamma, t) \\ = & \int d\Gamma G(q_1, q_1^*) \{ \frac{\partial}{\partial q_1}(i z_1 q_1) + \frac{\partial}{\partial q_1^*}(-i z_1^* q_1^*) \\ + & \frac{\partial^2}{\partial q_1 \partial q_1^*}(2\gamma \sum_k b_k \langle q_k^* q_k \rangle) \int d\Gamma_f \tilde{\rho}(\Gamma, t). \end{aligned} \tag{4.39}$$

In Eq. (4.39), $d\Gamma_1$, $d\Gamma_f$ and $\langle q_k^* q_k \rangle$ means

$$d\Gamma_1 = dx_1 dp_1, \quad d\Gamma_f = \prod_k dx_k dp_k, \tag{4.40}$$

$$\langle q_k^* q_k \rangle = \langle J_k \rangle = \int d\Gamma q_k^* q_k \tilde{\rho}(\Gamma). \tag{4.41}$$

Since $G(q_1, q_1^*)$ is an arbitrary test function, we can write Eq. (4.39) as

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{\rho}_1(\Gamma_1, t) = & \Big\{ i z_1 \frac{\partial}{\partial q_1} q_1 - i z_1^* \frac{\partial}{\partial q_1^*} q_1^* \\ & + 2\gamma \sum_k b_k \langle J_k \rangle \frac{\partial}{\partial q_1 \partial q_1^*} \Big\} \tilde{\rho}_1(\Gamma_1, t), \end{aligned} \tag{4.42}$$

where

$$\tilde{\rho}_1(\Gamma_1, t) = \int d\Gamma_f \tilde{\rho}(\Gamma, t). \tag{4.43}$$

Eq. (4.42) is our Fokker-Planck equation for the normal modes. This equation is applicable for any initial field configuration obeying the thermodynamic limit condition. In the non-thermodynamic case, the diffusion term containing b_k vanishes, and the equation describes damping of the oscillator without Brownian motion. For the special case where the field has the Maxwell-Boltzmann

distribution, using the approximation (4.30) we get

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{\rho}_1(\Gamma_1,t) &\approx \{iz_1\frac{\partial}{\partial q_1}q_1 - iz_1^*\frac{\partial}{\partial q_1^*}q_1^* \\ &+ \frac{2\gamma k_B T}{\tilde{\omega}_1}\frac{\partial^2}{\partial q_1 \partial q_1^*}\}\ \tilde{\rho}_1(\Gamma_1,t). \end{aligned} \tag{4.44}$$

The Fokker-Planck equation for other variables can be also derived from Eq. (4.42) by changing variables. For example, the Fokker-Planck equation for the position and momentum x_1 and p_1 is given by

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{\rho}_1(\Gamma_1,t) &= \Big\{ -\frac{\partial}{\partial x_1}(\frac{p_1}{\tilde{m}} - \gamma x_1) + \frac{\partial}{\partial p_1}(\tilde{m}\tilde{\omega}_1^2 x_1 + \gamma p_1) \\ &+ \frac{D_x}{2}\frac{\partial^2}{\partial x_1^2} + \frac{D_p}{2}\frac{\partial^2}{\partial p_1^2}\Big\}\ \tilde{\rho}_1(\Gamma_1,t), \end{aligned} \tag{4.45}$$

where

$$\begin{aligned}\tilde{m} &= m\omega_1/\tilde{\omega}_1, \\ D_x &= \frac{2\gamma}{\tilde{m}\tilde{\omega}_1}\sum_k b_k\langle J_k\rangle \approx \frac{2\gamma k_B T}{\tilde{m}\tilde{\omega}_1^2}, \\ D_p &= 2\tilde{m}\gamma\tilde{\omega}_1\sum_k b_k\langle J_k\rangle \approx 2\tilde{m}\gamma k_B T. \end{aligned} \tag{4.46}$$

[the approximate values are applicable for the Maxwell-Boltzmann distribution.] The Fokker-Planck equation for the action variable J_1 is given (after integration over the angle variable α_1) by

$$\begin{aligned}&\frac{\partial}{\partial t}\tilde{\rho}(J_1,t) \\ &= \Big\{2\gamma\frac{\partial}{\partial J_1}(J_1 - \frac{k_B T}{\tilde{\omega}_1}) + D_J\frac{\partial^2}{\partial J_1^2}J_1\Big\}\tilde{\rho}(J_1,t), \end{aligned} \tag{4.47}$$

where

$$D_J = 2\gamma\sum_k b_k\langle J_k\rangle \approx \frac{2\gamma k_B T}{\tilde{\omega}_1}. \tag{4.48}$$

Eqs. (4.45) and (4.47) coincide (in the weak-coupling approximation) with the equations for Brownian motion of an oscillator in an anharmonic lattice derived in Ref. [7]. Eq. (4.47) (in its exact form) was first proposed by T. Petrosky [41].

Note that Eq. (4.45) is symmetric with respect to rescaled position x_1 and momentum p_1. The reason is that the Hamiltonian considered here is symmetric in rescaled x_1 and p_1 to begin with. The same is true for the anharmonic lattice model considered in Ref. [7]. In contrast, the Kramers (Fokker-Planck) equation [7, 47] derived from the Ornstein-Uhlenbeck phenomenological theory of Brownian motion [42] is not symmetric, because the Brownian force breaks the position-momentum symmetry (see comment below Eq. (4.2)). In spite of the difference, for the case $\gamma \ll \omega_1$, Eq. (4.45) gives the same solution as the Kramers equation. The solutions of Eq. (4.45) can be found in Ref. [7].

In phenomenological theories the character of the noise has to be assumed, more or less independently of the dynamical forces. One of the goals of the dynamical approach is to deduce the character of the noise from the Hamiltonian [7], [44]-[46]. For the model considered in this paper we have shown that one can extract a Gaussian white noise component of dynamics through the Λ transformation.

4.3 Behavior of the original variables

In previous sections we showed that the time evolution of Λ transformed modes is the same as the noise average of the solution of the Langevin equa-

tion, and the Λ transformed distribution function satisfies the Fokker-Planck equation. In this Section we investigate the behavior of the original variables and how the noise emerges from the original equations. This can be found in the literature (see, for instance, Ref. [4] for a derivation based on Langevin-Mori equations.). In our case we analyze this problem starting from the exact solution of the equations of motion. We will show that after introducing some approximations we can recover the effects of white noise, in agreement with the results of the previous sections.

We first write the correlation function $\langle q_L^*(t+\tau)q_L(t)\rangle$ obtained from the Langevin equation (see Sec. 4.1) and then we compare it with the correlation function obtained from the classical Friedrichs model term by term. As shown in Eqs. (4.16)-(4.18), the time evolution of the Langevin mode q_L is given by

$$q_L(t) = q_{La}(t) + q_{Lr}(t), \tag{4.49}$$

The correlation between the noise components is given by ($t \geq 0$, $\tau \geq 0$)

$$\begin{aligned}
&\langle q_{Lr}^*(t+\tau)q_{Lr}(t)\rangle \\
&= \langle e^{iz_1^*(t+\tau)} \int_0^{t+\tau} dt' R^*(t') e^{-iz_1^* t'} \\
&\times e^{-iz_1 t} \int_0^t dt'' R(t'') e^{iz_1 t''} \rangle \\
&= e^{-2\gamma t} e^{iz_1^*\tau} \int_0^{t+\tau} dt' \int_0^t dt'' e^{-iz_1^* t'} e^{iz_1 t''} \\
&\times \langle R^*(t') R(t'')\rangle
\end{aligned} \tag{4.50}$$

Using Eq. (4.14) we get

$$\begin{aligned}\langle q^*_{Lr}(t+\tau)q_{Lr}(t)\rangle &= e^{-2\gamma t}e^{iz_1^*\tau}\int_0^t dt'\frac{2\gamma k_BT}{\tilde{\omega}_1}e^{2\gamma t'} \\ &= \frac{k_BT}{\tilde{\omega}_1}e^{iz_1^*\tau}(1-e^{-2\gamma t}). \end{aligned} \tag{4.51}$$

In the classical Friedrichs model we can directly calculate the exact time evolution of q_1 in terms of the renormalized field modes [22, 24, 31]

$$\tilde{Q}_k = q_k + \frac{\lambda V_k}{\eta^-(\omega_k)}\left[q_1 + \sum_{k'}\frac{\lambda V_{k'}}{\omega_k - \omega_{k'} - i\epsilon}q_{k'}\right] \tag{4.52}$$

as

$$\begin{aligned} q_1(t) &= \sum_k \frac{\lambda V_k}{\eta^+(\omega_k)}\tilde{Q}_k(t) = \sum_k \frac{\lambda V_k}{\eta^+(\omega_k)}\tilde{Q}_{k0}e^{-i\omega_k t} \\ &= \sum_k \frac{\lambda V_k}{\eta^+(\omega_k)}q_{k0}e^{-i\omega_k t} + \sum_k \frac{\lambda^2 V_k^2}{|\eta^+(\omega_k)|^2}q_{10}e^{-i\omega_k t} \\ &+ \sum_k \frac{\lambda^2 V_k^2}{|\eta^+(\omega_k)|^2}\sum_{k'}\frac{\lambda V_{k'}}{\omega_k - \omega_{k'} - i\epsilon}q_{k'0}e^{-i\omega_k t}, \end{aligned} \tag{4.53}$$

where q_{r0} are the initial values of q_r and ϵ is a positive infinitesimal quantity. We choose our initial condition assuming that the particle is surrounded by a thermal field. In classical mechanics q_{10} can be determined exactly since q_{10} is a function of the initial position and momentum of the particle. For the q_{k0} we need more care. Let us first write q_{k0} in terms of action and angle variables,

$$q_{k0} = \sqrt{J_{k0}}\, e^{-i\alpha_{k0}}. \tag{4.54}$$

When we have a thermal field, J_{k0} follows the Boltzmann distribution

$$\begin{aligned}\rho(\{J_{k0}\}) &\sim \exp(-\beta\sum_k \omega_k J_{k0}), \\ \langle J_{k0}\rangle &= \frac{k_BT}{\omega_k}. \end{aligned} \tag{4.55}$$

For almost all phase points $\{J_{10}, ...J_{k0}..., \alpha_{10}, ..., \alpha_{k0}, ...\}$ out of the Boltzmann distribution ensemble, any two different angles α_{k0} and $\alpha_{k'0}$ have no correlation. In other words, the sequence of angles $\{\alpha_{k_n 0}\}$ is completely random for almost all cases. This randomness property for α_{k0} is essential. The thermodynamic limit $L \to \infty$ with $J_{k0} = O(L^0)$ only exists if α_{k0} is uniformly distributed over $[-\pi, \pi]$ and the sequence of angles $\{\alpha_{k_n 0}\}$ is completely random [29]. Since the summation over a sequence of random phases for each mode is proportional to the square root of the number of modes, which in turn is proportional to L and since $V_k \sim L^{-1/2}$, the term

$$\begin{aligned} &\sum_k \frac{\lambda V_k}{\eta^+(\omega_k)} q_{k0} e^{-i\omega_k t} \\ &= \sum_k \frac{\lambda V_k}{\eta^+(\omega_k)} \sqrt{J_{k0}}\, e^{-i(\omega_k + \alpha_{k0})t} \end{aligned} \tag{4.56}$$

in Eq. (4.53) is $O(L^0)$. It shows very irregular time evolution as the number of modes increases.

Note that if α_{k0} is a smooth function of k, then for the first term of Eq. (4.53) we have in the limit $L \to \infty$

$$\begin{aligned} &\sum_k \frac{\lambda V_k}{\eta^+(\omega_k)} q_{k0} \\ &= \sqrt{\frac{L}{2\pi}} \frac{2\pi}{L} \sum_k \frac{\lambda v_k}{\eta^+(\omega_k)} \sqrt{J_{k0}} e^{-i\alpha_{k0}} \\ &\to \sqrt{\frac{L}{2\pi}} \int dk \frac{\lambda v_k}{\eta^+(\omega_k)} \sqrt{J_{k0}}\, e^{-i\alpha_{k0}} \end{aligned} \tag{4.57}$$

and if the integral is $O(1)$, this expression diverges as $O(\sqrt{L})$.

Coming back to Eq. (4.53), we approximate for weak coupling $\lambda \ll 1$,

$$\eta^+(\omega_k) \approx \omega_k - z_1. \tag{4.58}$$

We can separate the pole contribution at $\omega_k = z_1$ and the branch cut contribution from each term. The pole contribution gives the exponential decaying part and the cut contribution gives classical Zeno effect and non-exponential behavior [24]. In our case, we will only consider the pole contributions in Eq. (4.53) and compare the result with the solution of the Langevin equation. As we will see, they show a close correspondence. Taking the pole contribution at $\omega_k = z_1$ in the last two terms of Eq. (4.53), we get

$$\begin{aligned} q_1(t) \approx\ & q_{10}e^{-iz_1t} + \sum_k \frac{\lambda V_k}{\omega_k - z_1} q_{k0} e^{-i\omega_k t} \\ & - \sum_k \frac{\lambda V_k}{\omega_k - z_1} q_{k0} e^{-iz_1 t}. \end{aligned} \tag{4.59}$$

The first term in Eq. (4.59) is exactly the same as the first term in Eq. (4.49). We define the remaining terms in Eq. (4.59) as

$$q_{1r}(t) \approx \sum_k \frac{\lambda V_k}{\omega_k - z_1} q_{k0}(e^{-i\omega_k t} - e^{-iz_1 t}). \tag{4.60}$$

Now we calculate the correlation between $q^*_{1r}(t+\tau)$ and $q_{1r}(t)$ and compare it to Eq. (4.50). We have

$$\begin{aligned} & \langle q^*_{1r}(t+\tau) q_{1r}(t) \rangle \\ & \approx \langle \sum_k \frac{\lambda V_k}{\omega_k - z_1^*} q^*_{k0}(e^{i\omega_k(t+\tau)} - e^{iz_1^*(t+\tau)}) \\ & \times (\sum_l \frac{\lambda V_l}{\omega_l - z_1} q_{l0}(e^{-i\omega_l t} - e^{-iz_1 t}) \rangle. \end{aligned} \tag{4.61}$$

The bra-ket $\langle\,\rangle$ in Eq. (4.61) means the ensemble average. For the normalized thermal field ensemble we have

$$\begin{aligned}\langle q^*_{k0} q_{k'0}\rangle &= \int d\Gamma_0 q^*_{k0} q_{k'0} (\prod_l \beta\omega_l) \exp(-\beta \sum_l \omega_l J_{l0}) \\ &= \delta_{kk'} \frac{k_B T}{\omega_k}. \end{aligned} \tag{4.62}$$

Using this result and going to the continuous limit we obtain from Eq. (4.61)

$$\begin{aligned}&\langle q^*_{1r}(t+\tau) q_{1r}(t)\rangle \\ &\approx \int_0^\infty d\omega \frac{\lambda^2 v_\omega^2}{|\omega - z_1|^2} \frac{k_B T}{\omega} \\ &\times (e^{i\omega\tau} - e^{-iz_1 t} e^{i\omega(t+\tau)} - e^{iz_1^*(t+\tau)} e^{-i\omega t} + e^{iz_1^*\tau} e^{-2\gamma t})\end{aligned} \tag{4.63}$$

For $\gamma \ll \tilde{\omega}_1$ the integrand is sharply peaked around $\omega = \tilde{\omega}_1$. Neglecting the tails of the integrand we can extend the integration range to $(-\infty, \infty)$. By adding a contour integral on the lower (or upper) infinite semicircle we obtain the pole contribution at $\omega = z_1$ (or $\omega = z_1^*$)

$$\begin{aligned}&\langle q^*_{1r}(t+\tau) q_{1r}(t)\rangle_{pole} \\ &\approx \int_{-\infty}^\infty d\omega \frac{\lambda^2 v_\omega^2}{|\omega - z_1|^2} \frac{k_B T}{\omega} \\ &\times (e^{i\omega\tau} - e^{-iz_1 t} e^{i\omega(t+\tau)} - e^{iz_1^*(t+\tau)} e^{-i\omega t} + e^{iz_1^*\tau} e^{-2\gamma t}) \\ &\approx \frac{k_B T}{\tilde{\omega}_1} (e^{iz_1^*\tau} - e^{iz_1^*\tau} e^{-2\gamma t} - e^{iz_1^*\tau} e^{-2\gamma t} + e^{iz_1^*\tau} e^{-2\gamma t}) \\ &= \frac{k_B T}{\tilde{\omega}_1} e^{iz_1^*\tau} (1 - e^{-2\gamma t}).\end{aligned} \tag{4.64}$$

where we used

$$\begin{aligned} v_{z_1} &\approx v_{\tilde{\omega}_1}, \qquad \frac{k_B T}{z_1} \approx \frac{k_B T}{\tilde{\omega}_1} \\ \gamma &\approx \pi \lambda^2 v_{\tilde{\omega}_1}^2. \end{aligned} \tag{4.65}$$

We see that Eq. (4.64) has the same form as Eq. (4.50).

So we derived the Langevin correlation functions from dynamics in terms of the original variables and generic initial conditions representing the Maxwell-Boltzmann distribution. In this sense we obtained noise from the dynamics, and showed that the pole part gives the white noise structure. As already mentioned, the derivations in this Section involved a few approximations. However, the Λ transformed variables exactly capture this stochastic behavior as we have shown in previous sections.

Chapter 5

Quantum noise

In this chapter we briefly consider the quantum formalism of noise. This is done by G. Ordonez [49], and we will use this formalism in the next chapter.

5.1 Extension from classical noise to quantum noise

We now consider the quantum Friedrichs model with the rotating wave approximation,

$$H = \omega_1 a_1^\dagger a_1 + \sum_k \omega_k a_k^\dagger a_k + \lambda \sum_k V_k (a_1^\dagger a_k + a_k^\dagger a_1), \tag{5.1}$$

$$[a_\alpha,\, a_\beta^\dagger] = \delta_{\alpha,\beta}. \tag{5.2}$$

As shown in Appendix A, the interaction L_V preserves the normal ordering of creation and annihilation operators. Furthermore, the classical algebra with possion bracket is parallel to the quantum algebra with commutator. Hence, as long as we write the creation and annihilation operators in normal order, the results in quantum and classical mechanics are the same.

From the equation we obtain

$$
\begin{aligned}
& e^{iL_H t}\Lambda^\dagger {a_1^\dagger}^m a_1^n \\
& = e^{i(mz_1^* - nz_1)t} \sum_{l=0}^{min(m,n)} \frac{m!n!}{(m-l)!(n-l)!l!}\Lambda^\dagger {a_1^\dagger}^{m-l} a_1^{n-l} \\
& \times [\sum_k b_k a_k^\dagger a_k (e^{2\gamma t} - 1)]^l. \qquad (5.3)
\end{aligned}
$$

Now we show that, as in the classical case, this solution is generated by the quantum Langevin equation

$$\frac{d}{dt}a_L(t) = -iz_1 a_L(t) + \hat{R}(t) \qquad (5.4)$$

where $a_L(t)$ is a Langevin annihilation operator and $\hat{R}(t)$ is a Gaussian white noise operator.

We assume that

$$[a_L(t),\, a_L^\dagger(t)] = 1, \qquad (5.5)$$

$$[a_L(0),\, \hat{R}(t')] = 0, \qquad (5.6)$$

$$[a_L(0),\, \hat{R}(t')^\dagger] = 0. \qquad (5.7)$$

Eq. (5.6) and Eq. (5.7) is possible if $\hat{R}(t)$ depends only on the field and not on the particle. We also assume that $\hat{R}(t)$ satisfies the properties (1) and (2), with the white noise average defined as

$$\langle \hat{R}^\dagger(t)\hat{R}(t')\rangle = Tr(\hat{R}^\dagger(t)\hat{R}(t')\rho_F) \qquad (5.8)$$

The solution of Eq. (5.4) is

$$a_L(t) = a_{La}(t) + a_{Lr}(t) \qquad (5.9)$$

where

$$a_{La}(t) \equiv e^{-iz_1 t} a_L(0), \tag{5.10}$$

$$a_{Lr}(t) \equiv e^{-iz_1 t} \int_0^t dt' \hat{R}(t') e^{iz_1 t'}. \tag{5.11}$$

Then we have

$$\begin{aligned}
&\langle a_L^\dagger(t)^m a_L(t)^n \rangle \\
&= \langle (a_{La}^\dagger(t) + a_{Lr}^\dagger(t))^m \times (a_{La}(t) + a_{Lr}(t))^n \rangle \\
&= \sum_{k=0}^{m} \sum_{l=0}^{n} \frac{m!}{(m-k)!k!} \frac{n!}{(n-l)!l!} \\
&\times a_{La}^\dagger(t)^{m-k} a_{La}(t)^{n-l} \langle a_{Lr}^\dagger(t)^k a_{Lr}(t)^l \rangle
\end{aligned} \tag{5.12}$$

where we used Eq. (5.6) and Eq. (5.7). Note that we still keep the normal order of both Langevin mode a_{La} and noise operator a_{Lr}. Using the properties of $\hat{R}(t)$, we get

$$\langle a_{Lr}^\dagger(t)^k a_{Lr}(t)^l \rangle = \delta_{kl} l! (\frac{k_B T}{\tilde{\omega}_1})^l (e^{2\gamma t} - 1)^l. \tag{5.13}$$

Substituting Eq. (5.13) into Eq. (5.12) we get

$$\begin{aligned}
&\langle a_L^\dagger(t)^m a_L(t)^n \rangle \\
&= e^{i(mz_1^* - nz_1)t} \sum_{l=0}^{min(m,n)} \frac{m!n!}{(m-l)!(n-l)!l!} \\
&\times a_L^\dagger(0)^{m-l} a_L(0)^{n-l} [\frac{k_B T}{\tilde{\omega}_1} (e^{2\gamma t} - 1)]^l
\end{aligned} \tag{5.14}$$

which corresponds to Eq. (5.3).

In short, just as in the classical case, the Λ transformed operators evolve like the quantum Langevin operators.

5.2 Langevin equations for $\tilde{x}_1$

The position and momentum operators are given by

$$x_1 = \frac{1}{\sqrt{2m_1\omega_1}}(a_1 + a_1^\dagger), \tag{5.15}$$

$$p_1 = -i\sqrt{\frac{m_1\omega_1}{2}}(a_1 - a_1^\dagger). \tag{5.16}$$

Defining

$$\tilde{x}_1 = \Lambda^\dagger x_1,\ \tilde{p}_1 = \Lambda^\dagger p_1 \tag{5.17}$$

$$e^{iL_H t}\tilde{x}_1 = \tilde{x}_1(t),\ e^{iL_H t}\tilde{p}_1 = \tilde{p}_1(t), \tag{5.18}$$

we have

$$\tilde{x}_1 = \frac{1}{\sqrt{2m_1\omega_1}}(\tilde{A}_1 + \tilde{A}_1^\dagger), \tag{5.19}$$

$$\tilde{p}_1 = -i\sqrt{\frac{m_1\omega_1}{2}}(\tilde{A}_1 - \tilde{A}_1^\dagger) \tag{5.20}$$

where $\tilde{A}_1$ and $\tilde{A}_1^\dagger$ are Gamow modes. The time evolution of $\tilde{A}_1$ and $\tilde{A}_1^\dagger$ are

$$e^{iL_H t}\tilde{A}_1 = e^{-iz_1 t}\tilde{A}_1 = \tilde{A}_1(t), \tag{5.21}$$

$$e^{iL_H t}\tilde{A}_1^\dagger = e^{-iz_1 t}\tilde{A}_1^\dagger = \tilde{A}_1^\dagger(t). \tag{5.22}$$

Using Eq. (5.21) and Eq. (5.22) we get

$$\frac{d}{dt}\tilde{x}_1(t) = -\gamma\tilde{x}_1(t) + \frac{1}{\tilde{m}_1}\tilde{p}_1(t) \tag{5.23}$$

where $\tilde{m}_1 = m_1\omega_1/\tilde{\omega}_1$.

For the Λ transformation of x_1^2 we get

$$\begin{aligned}
\tilde{x_1^2} &= \Lambda^\dagger x_1^2 \\
&= \Lambda^\dagger \frac{1}{2m_1\omega_1}(a_1^\dagger a_1 + a_1 a_1^\dagger + a_1^\dagger a_1^\dagger + a_1 a_1) \\
&= \Lambda^\dagger \frac{1}{2m_1\omega_1}(2a_1^\dagger a_1 + 1 + a_1^\dagger a_1^\dagger + a_1 a_1) \\
&= \frac{1}{2m_1\omega_1}(2\tilde{A}_1^\dagger \tilde{A}_1 + 2\sum_k b_k a_k^\dagger a_k + 1 + \tilde{A}_1^\dagger \tilde{A}_1^\dagger + \tilde{A}_1 \tilde{A}_1) \qquad (5.24)
\end{aligned}$$

where we used

$$\Lambda^\dagger a_1^\dagger a_1 = \tilde{A}_1^\dagger \tilde{A}_1 + \sum_k b_k a_k^\dagger a_k. \qquad (5.25)$$

Using $[\tilde{A}_1, \tilde{A}_1^\dagger] = 0$, we get

$$\frac{d}{dt}\tilde{x_1^2}(t) = -2\gamma \tilde{x_1^2}(t) + \frac{2}{\tilde{m}_1}(\tilde{x_1 p_1} - \frac{i}{2}) + \frac{2\gamma}{\tilde{m}_1\tilde{\omega}_1}(\sum_k b_k a_k^\dagger a_k + \frac{1}{2}). \qquad (5.26)$$

Unlike the classical case, we have $\frac{i}{2}$ and $\frac{1}{2}$ terms due to the vacuum fluctuation. Note that both thermal diffusion and vacuum fluctuation diffusion present.

Chapter 6

$1/f$ noise

6.1 The Hamiltonian

First we consider a single particle inside a system in which a small region is weakly coupled to a large region. (see fig 6.1) In fig 6.1, Ω_1 is the volume of

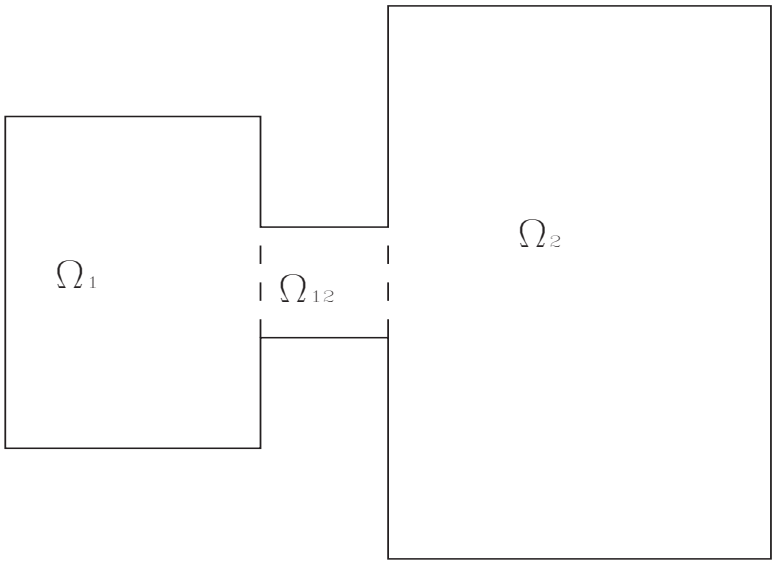

Figure 6.1: The system

a small region, Ω_2 is the volume of a large region, and Ω_{12} is the interaction region. We assume that the whole region $\Omega_1 + \Omega_2 + \Omega_{12}$ is surrounded by an infinitely hard wall. We also assume that the relation among Ω_1, Ω_2 and Ω_{12} is

$$\Omega_{12} < \Omega_1 \ll \Omega_2. \tag{6.1}$$

The Hamiltonian for a non-relativistic single particle is written as

$$H_T = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r}). \tag{6.2}$$

For the potential $V(\vec{r})$ we assume the following.

$$V(\vec{r}) = \begin{cases} 0 & \text{if } \vec{r} \in \Omega_1 \\ V_{12}(\vec{r}) > 0 & \text{if } \vec{r} \in \Omega_{12} \\ 0 & \text{if } \vec{r} \in \Omega_2 \\ \infty & \text{if } \vec{r} \notin \Omega_1 \cup \Omega_{12} \cup \Omega_2 \end{cases} \tag{6.3}$$

Now we show that this Hamiltonian can be mapped into an extended Friedrichs model. The mapping of a waveguide Hamiltonian to the extended Friedrichs model was already done by Subbiah and Petrosky [50], and the same method can be applied to our Hamiltonian. The derivation will mainly follow the method shown in [50].

To map the above Hamiltonian into the extended Friedrichs model, we consider two auxiliary Hamiltonians H_a and H_b. Let us define two regions Ω_a and Ω_b as

$$\Omega_a \equiv \Omega_1 \cup \Omega_{12}, \tag{6.4}$$

$$\Omega_b \equiv \Omega_2 \cup \Omega_{12}. \tag{6.5}$$

We define H_a is the Hamiltonian of the region Ω_a with surrounding hard wall. Similarly, H_b is defined as the Hamiltonian of the region Ω_b with surrounding hard wall.

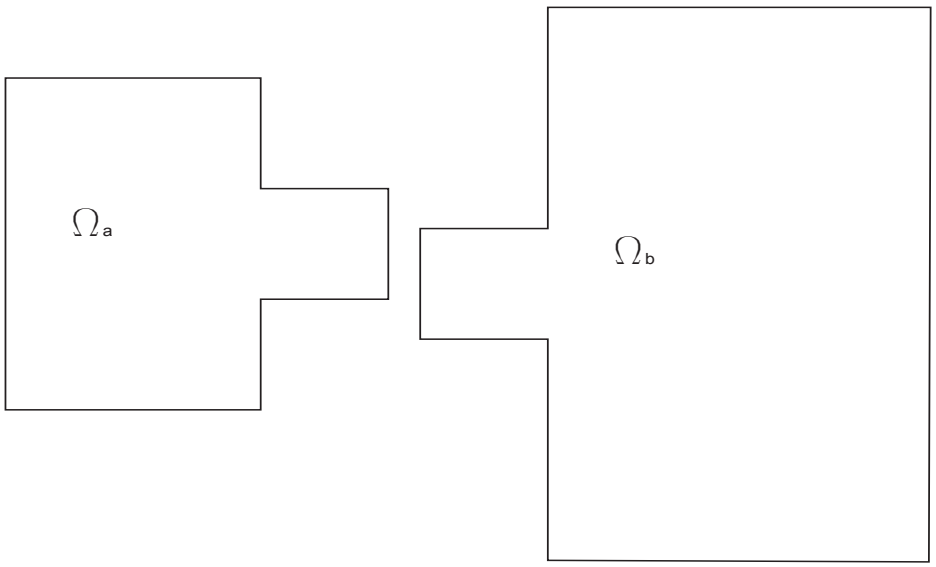

Figure 6.2: Ω_a and Ω_b

We write the eigenstates of H_a and H_b as

$$H_a|\vec{j}\rangle_a = \omega_{\vec{j}}|\vec{j}\rangle_a, \tag{6.6}$$

$$H_b|\vec{k}\rangle_b = \mu_{\vec{k}}|\vec{k}\rangle_b \tag{6.7}$$

where $\omega_{\vec{j}}$ and $\mu_{\vec{k}}$ are the eigenenergies of $|\vec{j}\rangle_a$ and $|\vec{k}\rangle_b$, respectively. Note that in the position representation $\langle\vec{r}|\vec{j}\rangle_a$ and $\langle\vec{r}|\vec{k}\rangle_b$ vanish at the boundary.

In Eq. (6.6) and Eq. (6.7) we used the vector notation $\vec{j}$ and $\vec{k}$ because there can be more than one independent quantum numbers. For example, in the case of a structureless particle in three dimension, there are three independent quantum numbers.

The set of $\{|\vec{j}\rangle_a\}$ makes a complete orthonormal basis inside the region Ω_a, and the set of $\{|\vec{k}\rangle_b\}$ makes a complete orthonormal basis inside the region Ω_b. But $|\vec{j}\rangle_a$ and $|\vec{k}\rangle_b$ are generally not orthogonal, and the set of $\{|\vec{j}\rangle_a, |\vec{k}\rangle_b\}$ forms an overcomplete basis over the total region $\Omega_a \cup \Omega_b$.

Our goal is to find a new complete orthogonal basis for the total Hamiltonian. Specifically, we construct a modified $\vec{k}$ state $|\psi_{\vec{k}}\rangle$ such that $\{|\vec{j}\rangle_a, |\psi_{\vec{k}}\rangle\}$ forms the total Hamiltonian system. In other words, we can construct the projection operator

$$P_a = \sum_{\vec{j}} |\vec{j}\rangle_a \langle\vec{j}|_a \tag{6.8}$$

and its complementary projection operator

$$Q_a = \sum_{\vec{k}} |\psi_{\vec{k}}\rangle\langle\psi_{\vec{k}}| \tag{6.9}$$

satisfying the relation

$$P_a + Q_a = 1_T \tag{6.10}$$

and

$$P_a^2 = P_a, \;\; Q_a^2 = Q_a, \;\; Q_a P_a = P_a Q_a = 0. \tag{6.11}$$

In Eq. (6.10), 1_T is the identity operator of the total system.

To construct Q_a, we introduce two supplementary projection operators P_{12} and P_2 as

$$P_{12} = \Theta_{\Omega_{12}} P_a, \;\; P_1 = \Theta_{\Omega_1} P_a. \tag{6.12}$$

In Eq. (6.13), Θ_{Vol} is an indicator function which is 1 inside Vol and 0 outside Vol. In the position representation,

$$\Theta_{Vol}(\vec{r}) = \begin{cases} 1, & \vec{r} \in Vol \\ 0, & \vec{r} \notin Vol \end{cases} \tag{6.13}$$

P_{12} and P_1 satisfy the following relations.

$$P_{12}^2 = P_{12}, \;\; P_1^2 = P_1, \;\; P_{12}P_1 = P_1 P_{12} = 0, \tag{6.14}$$

$$P_{12} + P_1 = P_a. \tag{6.15}$$

Also, for

$$P_b = \sum_{\vec{k}} |\vec{k}\rangle_b \langle \vec{k}|_b, \tag{6.16}$$

we have

$$P_{12}P_b = P_b P_{12} = P_{12}, \tag{6.17}$$

$$P_1 P_b = P_b P_1 = 0, \tag{6.18}$$

$$P_1 + P_b = 1_T. \tag{6.19}$$

Note that all the above equations hold in the L^2 space.

From Eq. (6.15) and Eq. (6.19) we get

$$Q_a = 1_T - P_a = P_1 + P_b - (P_{12} + P_1) = P_b - P_{12}. \tag{6.20}$$

From Eq. (6.17) and Eq. (6.20) we get

$$Q_a P_b = P_b Q_a = Q_a. \tag{6.21}$$

With these preparation, we construct $|\psi_{\vec{k}}\rangle$. We define $|\psi_{\vec{k}}\rangle$ as

$$|\psi_{\vec{k}}\rangle \equiv \lim_{\epsilon \to +0} |\psi_{\vec{k}}(\mu_{\vec{k}} + i\epsilon)\rangle \tag{6.22}$$

where

$$|\psi_{\vec{k}}(z)\rangle \equiv P_b(1 - R_b(z)P_{12}[P_{12}R_b(z)P_{12}]^{-1}P_{12})|\vec{k}\rangle_b. \tag{6.23}$$

$R_b(z)$ is the resolvent operator

$$R_b(z) \equiv \frac{1}{z - H_b}. \tag{6.24}$$

In Eq. (6.23) we can also choose "$-i\epsilon$". In this thesis we choose the "+" branch.

From Eq. (6.17) and Eq. (6.18) we see that

$$P_{12}|\psi_{\vec{k}}\rangle = P_1|\psi_{\vec{k}}\rangle = 0, \tag{6.25}$$

$$P_a|\psi_{\vec{k}}\rangle = (P_{12} + P_1)|\psi_{\vec{k}}\rangle = 0. \tag{6.26}$$

So $|\psi_{kv}\rangle$ is orthogonal to the P_a subspace and entirely in Q_a subspace.

$$Q_a|\psi_{\vec{k}}\rangle = |\psi_{\vec{k}}\rangle. \tag{6.27}$$

In the appendix, the following relations are shown.

$$\langle\psi_{\vec{k}'}|\psi_{\vec{k}}\rangle = \delta_{\vec{k}\vec{k}'}, \tag{6.28}$$

$$Q_a = \sum_{\vec{k}} |\psi_{\vec{k}}\rangle\langle\psi_{\vec{k}}|, \tag{6.29}$$

$$\langle\psi_{\vec{k}'}|H_b|\psi_{\vec{k}}\rangle = \mu_{\vec{k}}\delta_{\vec{k}\vec{k}} \tag{6.30}$$

Using the fact that $\langle\vec{r}|\psi_{\vec{k}}\rangle$ vanishes at the boundary of Ω_b and Eq. (6.30) we also show in the appendix that

$$\langle\vec{j}|_a H_T|\vec{j}'\rangle_a = \omega_{\vec{j}}\delta_{\vec{j}\vec{j}'}, \tag{6.31}$$

$$\langle\psi_{\vec{k}}|H_T|\psi_{\vec{k}'}\rangle = \mu_{\vec{k}}\delta_{\vec{k},\vec{k}'}. \tag{6.32}$$

From Eq. (6.31) and Eq. (6.32), the total Hamiltonian is given by

$$\begin{aligned}
H_T &= (P_a + Q_a)H_T(P_a + Q_a) \\
&= (\sum_{\vec{j}} |\vec{j}\rangle_a\langle\vec{j}|_a + \sum_{\vec{k}} |\psi_{\vec{k}}\rangle\langle\psi_{\vec{k}}|)H_T(\sum_{\vec{j}} |\vec{j}\rangle_a\langle\vec{j}|_a + \sum_{\vec{k}} |\psi_{\vec{k}}\rangle\langle\psi_{\vec{k}}|) \\
&= \sum_{\vec{j}} \omega_{\vec{j}}|\vec{j}\rangle_a\langle\vec{j}|_a + \sum_{\vec{k}} \mu_{\vec{k}}|\psi_{\vec{k}}\rangle\langle\psi_{\vec{k}}| \\
&+\lambda\sum_{\vec{j}\vec{k}}(V_{\vec{j}\vec{k}}|\vec{j}\rangle_a\langle\psi_{\vec{k}}| + V^*_{\vec{j}\vec{k}}|\psi_{\vec{k}}\rangle\langle\vec{j}|_a).
\end{aligned} \tag{6.33}$$

where

$$\lambda V_{\vec{j}\vec{k}} = \langle\psi_{\vec{k}}|H_T|\vec{j}\rangle_a. \tag{6.34}$$

6.2 Diagonalization of the Hamiltonian

In this section we diagonalize the extended Friedrichs model Eq. (6.33). The diagonalization procedure is very similar to the diagonalization procedure

of the original Friedrichs model.

First, we note that in our Hamiltonian $\Omega_1 \ll \Omega_2 < \Omega_b$. In the Limit $\Omega_2 \to \infty$, the index $\vec{k}$ becomes continuous. For the case the spectrum of $\omega_{\vec{j}}$ is inside the spectrum of $\mu_{\vec{k}}$, we expect that the particle in any state of $|\vec{j}\rangle_a$ becomes unstable and decays to the continuum. In other words, if a particle is in the Ω_1 region, it will eventually escape to the Ω_b region.

With this in mind, we look for the diagonalized Hamiltonian of the form

$$H_T = \sum_{\vec{k}} \bar{\mu}_{\vec{k}} |\phi_{\vec{k}}\rangle\langle\phi_{\vec{k}}|. \tag{6.35}$$

Note that the $\vec{j}$ spectrum disappeared inside the continuum. We write $|\phi_{\vec{k}}\rangle$ as

$$|\phi_{\vec{k}}\rangle = g_{\vec{k}\vec{k}}|\psi_{\vec{k}}\rangle + \sum_{\vec{k}'(\neq\vec{k})} g_{\vec{k}\vec{k}'}|\psi_{\vec{k}'}\rangle + \sum_{\vec{j}} h_{\vec{k}\vec{j}}|\vec{j}\rangle_a. \tag{6.36}$$

Using

$$H_T|\phi_{\vec{k}}\rangle = \sum_{\vec{k}'} \bar{\mu}_{\vec{k}'}|\phi_{\vec{k}'}\rangle\langle\phi_{\vec{k}'}|\phi_{\vec{k}}\rangle = \bar{\mu}_{\vec{k}}|\phi_{\vec{k}}\rangle \tag{6.37}$$

and substituting Eq. (6.36) into Eq. (6.37), we obtain

$$g_{\vec{k}\vec{k}}(\bar{\mu}_{\vec{k}} - \bar{\mu}_{\vec{k}}) - \lambda \sum_{\vec{j}} V^*_{\vec{j}\vec{k}} h_{\vec{k}\vec{j}} = 0, \tag{6.38}$$

$$\sum_{\vec{k}'(\neq\vec{k})} (g_{\vec{k}\vec{k}'}(\bar{\mu}_{\vec{k}} - \mu_{\vec{k}'}) - \lambda \sum_{\vec{j}} V^*_{\vec{j}\vec{k}'} h_{\vec{k}\vec{j}}) = 0, \tag{6.39}$$

$$\sum_{\vec{j}} (h_{\vec{k}\vec{j}}(\bar{\mu}_{\vec{k}} - \omega_{\vec{j}}) - \lambda g_{\vec{k}\vec{k}} V_{\vec{j}\vec{k}} - \lambda \sum_{\vec{k}'(\neq\vec{k})} V_{\vec{j}\vec{k}'} g_{\vec{k}\vec{k}'}) = 0. \tag{6.40}$$

Before solving Eq. (6.38) Eq. (6.40), we examine the properties of $V_{\vec{j}\vec{k}}$. Let us write $V_{\vec{j}\vec{k}}$ in a polar form

$$V_{\vec{j}\vec{k}} = \frac{1}{\sqrt{\Omega_b}} v_{\vec{j}\vec{k}} e^{-i\alpha_{\vec{j}\vec{k}}}. \tag{6.41}$$

In Eq. (6.41), $\frac{1}{\sqrt{\Omega_b}} v_{\vec{j}\vec{k}} > 0$ is the magnitude of $V_{\vec{j}\vec{k}}$ and the real function $\alpha_{\vec{j}\vec{k}}$ is the phase. When the interaction region is chaotic, we expect that $\alpha_{\vec{j}\vec{k}}$ will vary rapidly and irregularly with respect to $\vec{j}$ and $\vec{k}$. For $v_{\vec{j}\vec{k}}$, we assume that $v_{\vec{j}\vec{k}}$ vary more slowly. Note also $v_{\vec{j}\vec{k}} \sim O(\Omega_b^0)$.

With this information about $V_{\vec{j}\vec{k}}$, we now solve Eq. (6.38) -Eq. (6.40) for $g_{\vec{k}\vec{k}}$, $g_{\vec{k}\vec{k}'}$ and $h_{\vec{j}\vec{k}}$. Neglecting the $1/\Omega_b$ volume dependent terms, we get

$$g_{\vec{k}\vec{k}} = 1, \;\; \bar{\mu}_{\vec{k}} = \mu_{\vec{k}}, \tag{6.42}$$

$$g_{\vec{k}\vec{k}'} = \frac{\lambda}{\mu_{\vec{k}} - \mu_{\vec{k}'}} \sum_{\vec{j}} V^*_{\vec{j}\vec{k}'} h_{\vec{k}\vec{j}} \tag{6.43}$$

$$\begin{aligned} h_{\vec{k}\vec{j}} &= \frac{1}{\eta_{\vec{j}}(\mu_{\vec{k}})} (\lambda V_{\vec{j}\vec{k}} + \lambda^2 \sum_{\vec{k}'(\neq k)} \sum_{\vec{j}'(\neq \vec{j})} \frac{V_{\vec{j}\vec{k}'} V^*_{\vec{j}'\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'}} h_{\vec{k}\vec{j}'}) \\ &= \frac{1}{\eta_{\vec{j}}(\mu_{\vec{k}})} (\lambda V_{\vec{j}\vec{k}} + \lambda^2 \sum_{\vec{k}'(\neq \vec{k})} \sum_{\vec{j}'(\neq \vec{j})} \frac{V_{\vec{j}\vec{k}'} V^*_{\vec{j}'\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'}} \frac{\lambda V_{\vec{j}\vec{k}}}{\eta_{\vec{j}'}(\mu_{\vec{k}})}) + O(\lambda^4). \end{aligned} \tag{6.44}$$

where

$$\eta_{\vec{j}}(\mu_{\vec{k}}) = \mu_{\vec{k}} - \omega_{\vec{j}} - \lambda^2 \sum_{\vec{k}'(\neq \vec{k})} \frac{|V_{\vec{j}\vec{k}'}|^2}{\mu_{\vec{k}} - \mu'_{\vec{k}}}. \tag{6.45}$$

To avoid divergences, we change $\mu_{\vec{k}} \to \mu_{\vec{k}} + i\epsilon$ in the denominator. With this $i\epsilon$ regularization Eq. (6.44) becomes

$$h_{\vec{k}\vec{j}} = \frac{1}{\eta^+_{\vec{j}}(\mu_{\vec{k}})} (\lambda V_{\vec{j}\vec{k}} + \sum_{\vec{k}'(\neq \vec{k})} \sum_{\vec{j}'(\neq \vec{j})} \frac{\lambda V_{\vec{j}\vec{k}'} \lambda V^*_{\vec{j}'\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'} + i\epsilon} \frac{\lambda V_{\vec{j}'\vec{k}}}{\eta^+_{\vec{j}'}(\mu_{\vec{k}})}) + O(\lambda^4) \tag{6.46}$$

where $\eta^+_{\vec{j}}(\mu_{\vec{k}})$ is

$$\eta^+_{\vec{j}}(\mu_{\vec{k}}) \equiv \eta_{\vec{j}}(\mu_{\vec{k}} + i\epsilon). \tag{6.47}$$

For the complex argument z, $\eta^+_{\vec{j}}(z)$ means that this is defined by the analytic continuation from the upper half plane. The approximate value of $\eta^+_{\vec{j}}(\mu_{\vec{k}})$ is

$$\eta^+_{\vec{j}}(\mu_{\vec{k}}) \approx \mu_{\vec{k}} - z_{\vec{j}} \tag{6.48}$$

where

$$\eta^+_{\vec{j}}(z_{\vec{j}}) = 0, \tag{6.49}$$

$$z_{\vec{j}} = \tilde{\omega}_{\vec{j}} - i\gamma_{\vec{j}}, \;\; \tilde{\omega}_{\vec{j}} = \omega_{\vec{j}} + O(\lambda^2), \tag{6.50}$$

$$\gamma_{\vec{j}} \approx \pi \sum_{\vec{k}} \lambda^2 |V_{\vec{j}\vec{k}}|^2 \delta(\mu_{\vec{k}} - \tilde{\omega}_{\vec{j}}). \tag{6.51}$$

With these properties of η^+ in mind, the second term of Eq. (6.46) becomes

$$\begin{aligned}
&\sum_{\vec{k}'(\neq\vec{k})} \frac{\lambda V_{\vec{j}\vec{k}'} \lambda V^*_{\vec{j}'\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'} + i\epsilon} \sum_{\vec{j}'(\neq\vec{j})} \frac{\lambda V_{\vec{j}'\vec{k}}}{\eta^+_{\vec{j}'}(\mu_{\vec{k}})} \\
&= \sum_{\vec{k}'(\neq\vec{k})} P\left(\frac{\lambda V_{\vec{j}\vec{k}'} \lambda V^*_{\vec{j}'\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'}} \right) \sum_{\vec{j}'(\neq\vec{j})} \frac{\lambda V_{\vec{j}'\vec{k}}}{\eta^+_{\vec{j}'}(\mu_{\vec{k}})} \\
&+ \sum_{\vec{k}'(\neq\vec{k})} i\pi\delta(\mu_{\vec{k}} - \mu_{\vec{k}'}) \lambda V_{\vec{j}\vec{k}'} \lambda V^*_{\vec{j}'\vec{k}'} \sum_{\vec{j}'(\neq\vec{j})} \frac{\lambda V_{\vec{j}'\vec{k}}}{\eta^+_{\vec{j}'}(\mu_{\vec{k}})}.
\end{aligned} \tag{6.52}$$

From the form of $V_{\vec{j}\vec{k}}$ (Eq. (6.41)), we see that Eq. (6.52) contains the summation over rapidly changing phases. (In more than 1D, $\delta(\mu_{\vec{k}} - \mu_{\vec{k}'})$ does not mean $\vec{k} = \vec{k}'$. So the term containing delta function also has sum over rapidly

changing phases.) In the sense of distribution, these terms are very close to zero. If we neglect these terms, the coefficients $g_{\vec{k}\vec{k}}$, $g_{\vec{k}\vec{k}'}$ and $h_{\vec{k}\vec{j}}$ are given by

$$g_{\vec{k}\vec{k}} = 1, \;\; \bar{\mu}_{\vec{k}} = \mu_{\vec{k}}, \tag{6.53}$$

$$g_{\vec{k}\vec{k}'} = \frac{1}{\mu_{\vec{k}} - \mu_{\vec{k}'} + i\epsilon} \sum_{\vec{j}} \frac{\lambda V^*_{\vec{j}\vec{k}'} \lambda V_{\vec{j}\vec{k}}}{\eta^+_{\vec{j}}(\mu_{\vec{k}})}, \tag{6.54}$$

$$h_{\vec{k}\vec{j}} = \frac{\lambda V_{\vec{j}\vec{k}}}{\eta^+_{\vec{j}}(\mu_{\vec{k}})}. \tag{6.55}$$

Substituting these coefficients into Eq. (6.36), we finally get

$$|\phi_{\vec{k}}\rangle = |\psi_{\vec{k}}\rangle + \sum_{\vec{j}} \frac{\lambda V_{\vec{j}\vec{k}}}{\eta^+_{\vec{j}}(\mu_{\vec{k}})} (|\vec{j}\rangle_a + \sum_{\vec{k}'(\neq\vec{k})} \frac{\lambda V^*_{\vec{j}\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'} + i\epsilon} |\psi_{\vec{k}}\rangle). \tag{6.56}$$

The inverse relation for $|\vec{j}\rangle_a$ is given by

$$|\vec{j}\rangle_a = \sum_{\vec{k}} \frac{\lambda V^*_{\vec{j}\vec{k}}}{\eta^-_{\vec{j}}(\mu_{\vec{k}})} |\phi_{\vec{k}}\rangle. \tag{6.57}$$

6.3 The derivation of $1/f$ noise for the number fluctuation

In the previous section we derived the single particle Hamiltonian in the extended Friedrichs model form. In this section we extend this to the many non-interacting particle case. As a first approximation, we can just superpose single particle wavefunctions and treat the probability density as a number density. Actually, the conduction electrons in metals are well described by the classical mechanics, and when we neglect the electron-electron interaction, it can be reduced to a one-electron problem [51]. If we want to see the quantum

effect of non-interacting N identical particles, we have to consider the boson and fermion cases separately. In this thesis we consider the boson case and go to the classical limit. The distinction between bosons or fermions plays a role only when we calculate the number density.

In the non-interacting many particle case, we can write the Hamiltonian as a sum of single particle Hamiltonians.

$$H = \sum_{i}^{N} H^{(i)} \tag{6.58}$$

The general N particle wavefunction can be written as a linear combination of products of eigenfunctions of $H^{(i)}$,

$$\Psi(x_1, ..., x_N; t) = \sum_{m_1} \cdots \sum_{m_N} c(m_1, \cdots, m_N; t) \varphi_{m_1}(x_1) \varphi_{m_2}(x_2) \cdots \varphi_{m_N}(x_N) \tag{6.59}$$

Using the matrix element of the Hamiltonian Eq. (6.58) has the following property

$$\langle m_1, \cdots, m_N | H | m'_1, \cdots, m'_N \rangle = \sum_{i}^{N} \langle m_i | H^{(i)} | m'_i \rangle \prod_{r(\neq i)}^{N} \delta_{m_r m'_r}, \tag{6.60}$$

from the Schrödinger equation the coefficients $c(m_1, \cdots, m_N; t)$ satisfy the following relation.

$$i\hbar \partial_t \, c(m_1, \cdots, m_N; t) = \sum_{i=1}^{N} \sum_{m_j} \langle m_j | H^{(i)} | m'_j \rangle \, c(m_1, \cdots, m'_j, \cdots, m_N; t). \tag{6.61}$$

We can interpret the right-hand side as the one-particle hamiltonian causing transitions of particle j from level m'_j to level m_j.

We now translate this idea to the occupation number representation. In many identical particle case, we ask "How many particles are in level m_1, m_2, and so on?" Before the transition there were $n_{m'_j}$ particles in level m'_j and n_{m_j} particles in level m_j. After the transition, there are $n_{m'_j} - 1$ particles in level m'_j and $n_{m_j} + 1$ particles in level m_j. This process can be interpreted as the destruction of a particle in level m'_j and the creation of the particle in level m_j.

We can express the above idea by writing the creation and destruction operators $a_m^\dagger$ and a_m. In the boson case, for the eigenkets characterized by the occupation numbers we define $a_m^\dagger$ and a_m as

$$a_m^\dagger|..., n_m, ...\rangle = (n_m)^{1/2}|..., n_m - 1, ...\rangle \tag{6.62}$$

$$a_m|..., n_m, ...\rangle = (n_m + 1)^{1/2}|..., n_m + 1, ...\rangle. \tag{6.63}$$

With these definitions, the Hamiltonian of non-interacting many boson in our system can be written as

$$H = \sum_{\vec{j}} \omega_{\vec{j}} a_{\vec{j}}^\dagger a_{\vec{j}} + \sum_{\vec{k}} \mu_{\vec{k}} b_{\vec{k}}^\dagger b_{\vec{k}} + \lambda \sum_{\vec{j}\vec{k}} (V_{\vec{j}\vec{k}} a_{\vec{j}}^\dagger b_{\vec{k}} + V_{\vec{j}\vec{k}}^* b_{\vec{k}}^\dagger a_{\vec{j}}). \tag{6.64}$$

where $a_{\vec{j}}^\dagger$ and $a_{\vec{j}}$ are the creation-destruction operators for $|\vec{j}\rangle_a$, $b_{\vec{k}}^\dagger$ and $b_{\vec{k}}$ be the creation-destruction operators for $|\psi_{\vec{k}}\rangle$. This creation-destruction operators satisfy the following commutation relations.

$$[a_{\vec{j}}, a_{\vec{j}'}^\dagger] = \delta_{\vec{j}\vec{j}'}, \quad [b_{\vec{k}}, b_{\vec{k}'}^\dagger] = \delta_{\vec{k}\vec{k}'}, \tag{6.65}$$

$$[a_{\vec{j}}, a_{\vec{j}'}] = [b_{\vec{k}}, b_{\vec{k}'}] = [a_{\vec{j}}, b_{\vec{k}}^\dagger] = 0. \tag{6.66}$$

The diagonalized Hamiltonian can be written as

$$H = \sum_{\vec{k}} \mu_{\vec{k}} B^{\dagger}_{\vec{k}} B_{\vec{k}} \tag{6.67}$$

where $B^{\dagger}_{\vec{k}}$ and $B_{\vec{k}}$ is the creation-destruction operators for $|\psi_{\vec{k}}\rangle$. The commutation relation is given by

$$[B_{\vec{k}}, B^{\dagger}_{\vec{k}}] = \delta_{\vec{k}\vec{k}'}, \tag{6.68}$$

and the relation between the original operators and B operators are given by

$$B^{\dagger}_{\vec{k}} = b_{\vec{k}} + \sum_{\vec{j}} \frac{\lambda V_{\vec{j}\vec{k}}}{\eta^{+}_{\vec{j}}(\mu_{\vec{k}})} (a^{\dagger}_{\vec{j}} + \sum_{\vec{k}'(\neq\vec{k})} \frac{\lambda V^{*}_{\vec{j}\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'} + i\epsilon} b^{\dagger}_{\vec{k}'}), \tag{6.69}$$

$$a^{\dagger}_{\vec{j}} = \sum_{\vec{k}} \frac{\lambda V^{*}_{\vec{j}\vec{k}}}{\eta^{-}_{\vec{j}}(\mu_{\vec{k}})} B^{\dagger}_{\vec{k}}. \tag{6.70}$$

The time evolution of the operators can be calculated from Eq. (6.69) and Eq. (6.70). From the Heisenberg equation of motion, we have (we take $\hbar = 1$ for now.)

$$\frac{d}{dt} B^{\dagger}_{\vec{k}} = i[H, B^{\dagger}_{\vec{k}}] = i\mu_{\vec{k}} B^{\dagger}_{\vec{k}}, \tag{6.71}$$

$$B^{\dagger}_{\vec{k}}(t) = B^{\dagger\,(0)}_{\vec{k}} e^{i\mu_{\vec{k}} t}. \tag{6.72}$$

Substituting Eq. (6.72) into Eq. (6.70), we get

$$\begin{aligned} a^{\dagger}_{j}(t) = & \sum_{\vec{k}} \frac{\lambda V^{*}_{\vec{j}\vec{k}}}{\eta^{-}_{\vec{j}}(\mu_{\vec{k}})} e^{i\mu_{\vec{k}} t} \\ & \times \Big(b^{\dagger\,(0)}_{\vec{k}} + \sum_{\vec{j}'} \frac{\lambda V_{\vec{j}'\vec{k}}}{\eta^{+}_{\vec{j}}(\mu_{\vec{k}})} a^{\dagger\,(0)}_{\vec{j}'} + \sum_{\vec{j}'} \frac{\lambda V_{\vec{j}'\vec{k}}}{\eta^{+}_{\vec{j}'}(\mu_{\vec{k}})} \sum_{\vec{k}'(\neq\vec{k})} \frac{\lambda V^{*}_{\vec{j}\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'} + i\epsilon} b^{\dagger\,(0)}_{\vec{k}'} \Big). \end{aligned} \tag{6.73}$$

Using Eq. (6.48) and expanding Eq. (6.73), we have

$$
\begin{aligned}
a^{\dagger}_{\vec{j}}(t) \approx & \sum_{\vec{k}} \frac{\lambda V^{*}_{\vec{j}\vec{k}}}{\mu_{\vec{k}} - z^{*}_{\vec{j}}} b^{\dagger(0)}_{\vec{k}} e^{i\mu_{\vec{k}}t} + \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{|\mu_{\vec{k}} - z_{\vec{j}}|^2} a^{\dagger(0)}_{\vec{j}} e^{i\mu_{\vec{k}}t} \\
& + \sum_{\vec{k}} \frac{\lambda V^{*}_{\vec{j}\vec{k}}}{\mu_{\vec{k}} - z^{*}_{\vec{j}}} e^{i\mu_{\vec{k}}t} \sum_{\vec{j}'(\neq\vec{j})} \frac{\lambda V_{\vec{j}'\vec{k}}}{\mu_{\vec{k}} - z_{\vec{j}'}} a^{\dagger(0)}_{\vec{j}'} \\
& + \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{|\mu_{\vec{k}} - z^{*}_{\vec{j}}|^2} e^{i\mu_{\vec{k}}t} \sum_{\vec{k}'(\neq\vec{k})} \frac{\lambda V^{*}_{\vec{j}\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'} + i\epsilon} b^{\dagger(0)}_{\vec{k}'} \\
& + \sum_{\vec{k}} \frac{\lambda V^{*}_{\vec{j}\vec{k}}}{\mu_{\vec{k}} - z^{*}_{\vec{j}}} e^{i\mu_{\vec{k}}t} \sum_{\vec{j}'(\neq\vec{j})} \frac{\lambda V_{\vec{j}'\vec{k}}}{\eta^{+}_{\vec{j}'}(\mu_{\vec{k}})} \sum_{\vec{k}'(\neq\vec{k})} \frac{\lambda V^{*}_{\vec{j}\vec{k}'}}{\mu_{\vec{k}} - \mu_{\vec{k}'} + i\epsilon} b^{\dagger(0)}_{\vec{k}'}. \qquad (6.74)
\end{aligned}
$$

We keep only the dominant terms in Eq. (6.74) assuming λ is small. Note that in the continuous limit

$$
\begin{aligned}
& \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{|\mu_{\vec{k}} - z_{\vec{j}}|^2} e^{i\mu_{\vec{k}}t} = \int_{\mu_{min}}^{\infty} d\mu M(\mu) \frac{\lambda^2 |v_{\vec{j}}(\mu)|^2}{|\mu - z_{\vec{j}}|^2} e^{i\mu t} \\
& = \int_{-\infty}^{\infty} d\mu M(\mu) \frac{\lambda^2 |v_{\vec{j}}(\mu)|^2}{|\mu - z_{\vec{j}}|^2} e^{i\mu t} - \int_{-\infty}^{\mu_{min}} d\mu M(\mu) \frac{\lambda^2 |v_{\vec{j}}(\mu)|^2}{|\mu - z_{\vec{j}}|^2} e^{i\mu t} \\
& = e^{iz^{*}_{\vec{j}}t} + O(\lambda^2) \qquad (6.75)
\end{aligned}
$$

by taking the pole $\mu = z_{\vec{j}}$. ($M(\mu)$ is angle-integrated Jacobian.) Note that for $t \gg 1/\gamma_{\vec{j}}$, The right-hand side of Eq. (6.75) becomes $O(\lambda^2)$. We see in Eq. (6.75) that for $t \gg 1/\gamma_{\vec{j}}$ $a^{\dagger}_{\vec{j}}(t)$ is approximated by

$$
a^{\dagger}_{\vec{j}}(t) \approx \sum_{\vec{k}} \frac{\lambda V^{*}_{\vec{j}\vec{k}}}{\mu_{\vec{k}} - z^{*}_{\vec{j}}} b^{\dagger(0)}_{\vec{k}} e^{i\mu_{\vec{k}}t}. \qquad (6.76)
$$

With Eq. (6.76), we calculate the total number fluctuation inside the small subsystem Ω_a. First we write the $\vec{j}$th level number operator inside the

subsystem

$$N_{\vec{j}} = a^{\dagger}_{\vec{j}} a_{\vec{j}} = \sum_{\vec{k}} \frac{\lambda V^{*}_{\vec{j}\vec{k}}}{\mu_{\vec{k}} - z^{*}_{\vec{j}}} b^{\dagger(0)}_{\vec{k}} e^{i\mu_{\vec{k}} t} \sum_{\vec{k}'} \frac{\lambda V_{\vec{j}\vec{k}'}}{\mu_{\vec{k}'} - z_{\vec{j}}} b^{(0)}_{\vec{k}'} e^{-i\mu_{\vec{k}'} t}$$
$$= \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{|\mu_{\vec{k}} - z_{\vec{j}}|^2} b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}} + \sum_{\vec{k} \neq \vec{k}'} \frac{\lambda V^{*}_{\vec{j}\vec{k}} \lambda V_{\vec{j}\vec{k}'}}{(\mu_{\vec{k}} - z^{*}_{\vec{j}})(\mu_{\vec{k}'} - z_{\vec{j}})} b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}'} e^{i(\mu_{\vec{k}} - \mu_{\vec{k}'})t}. \tag{6.77}$$

In Eq. (6.77), we separate the time independent part and time dependent part. Let us write

$$N_{c\vec{j}} = \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{|\mu_{\vec{k}} - z_{\vec{j}}|^2} b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}}, \quad N_{f\vec{j}} = \sum_{\vec{k} \neq \vec{k}'} \frac{\lambda V^{*}_{\vec{j}\vec{k}} \lambda V_{\vec{j}\vec{k}'}}{(\mu_{\vec{k}} - z^{*}_{\vec{j}})(\mu_{\vec{k}'} - z_{\vec{j}})} b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}'} e^{i(\mu_{\vec{k}} - \mu_{\vec{k}'})t}. \tag{6.78}$$

$N_{c\vec{j}}$ is constant with time and $N_{f\vec{j}}$ is fluctuating with time. In the case that the total number density in region Ω_b is finite and the particle gas in Ω_b is in thermal equilibrium, we can write

$$\langle b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}'} \rangle = \langle b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}'} \rangle \delta_{\vec{k}\vec{k}'} \tag{6.79}$$

where $\langle\rangle$ represents the thermal equilibrium ensemble average. Furthermore, when the particles behaves like classical gas, we can write

$$\mu_{\vec{k}} \langle b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}} \rangle = C_0 k_B T. \tag{6.80}$$

In the case of classical ideal gas in 3D, $C_0 = 3/2$.

From Eq. (6.79) we see that

$$\langle N_{f\vec{j}} \rangle = 0. \tag{6.81}$$

So, $N_{f\vec{j}}$ acts as the time fluctuating quantity with ensemble average zero.

The total number operator inside Ω_a is written as

$$N_{atot} = \sum_{\vec{j}} N_{\vec{j}} = \sum_{\vec{j}} N_{c\vec{j}} + \sum_{\vec{j}} N_{f\vec{j}}. \tag{6.82}$$

The time correlation function is

$$\begin{aligned}\langle N_{atot}(t+\tau)N_{atot}(t)\rangle &= \langle(\sum_{\vec{j}} N_{c\vec{j}} + \sum_{\vec{j}} N_{f\vec{j}}(t+\tau))(\sum_{\vec{j}} N_{c\vec{j}} + \sum_{\vec{j}} N_{f\vec{j}}(t))\rangle \\ &= \langle\sum_{\vec{j}} N_{c\vec{j}} \sum_{\vec{j}'} N_{c\vec{j}'}\rangle + \langle\sum_{\vec{j}} N_{f\vec{j}}(t+\tau) \sum_{\vec{j}'} N_{f\vec{j}'}(t)\rangle. \end{aligned}\tag{6.83}$$

In Eq. (6.83) we used Eq. (6.81) and the fact that $N_{c\vec{j}}$ is time independent. The first term in Eq. (6.83) becomes

$$\begin{aligned}\langle\sum_{\vec{j}} N_{c\vec{j}} \sum_{\vec{j}'} N_{c\vec{j}'}\rangle &= \langle(\sum_{\vec{j}} N_{c\vec{j}})^2\rangle \\ &= \langle\bigg(\sum_{\vec{k}} \frac{\lambda^2|V_{\vec{j}\vec{k}}|^2}{|\mu_{\vec{k}} - z_{\vec{j}}|^2} b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}}\bigg)^2\rangle. \end{aligned}\tag{6.84}$$

By taking the residue of the pole $\mu = z_{\vec{k}}$ and using the fact

$$\begin{aligned}\gamma_{\vec{j}} &\approx \pi \sum_{\vec{k}} |V_{\vec{j}\vec{k}}|^2 \delta(\mu_{\vec{k}} - \tilde{\omega}_{\vec{j}}) \approx \frac{\pi}{(2\pi)^3} \int d\mu M(\mu) |\bar{v}_{\vec{j}}(\mu)|^2 \delta(\mu - \tilde{\omega}_{\vec{j}}) \\ &\approx \frac{\pi}{(2\pi)^3} M(\tilde{\omega}_{\vec{j}}) |\bar{v}_{\vec{j}}(\tilde{\omega}_{\vec{j}})|^2, \end{aligned}\tag{6.85}$$

and Eq. (6.80), we get

$$\langle\sum_{\vec{j}} N_{c\vec{j}} \sum_{\vec{j}'} N_{c\vec{j}'}\rangle \approx \bigg(\sum_{\vec{j}} \frac{C_0 k_B T}{\omega_{\vec{j}}}\bigg)^2. \tag{6.86}$$

Here we used $\frac{\gamma_{\vec{j}}}{\tilde{\omega}_{\vec{j}}} = O(\lambda^2)$ and $\tilde{\omega}_{\vec{j}} = \tilde{\omega}_{\vec{j}} + O(\lambda^2)$.

For the term $\langle\sum_{\vec{j}} N_{f\vec{j}}(t+\tau)\sum_{\vec{j}'} N_{f\vec{j}'}(t)\rangle$, we have

$$\begin{aligned}
&\langle\sum_{\vec{j}} N_{f\vec{j}}(t+\tau)\sum_{\vec{j}'} N_{f\vec{j}'}(t)\rangle = \sum_{\vec{j},\vec{j}'}\langle N_{f\vec{j}}(t+\tau)N_{f\vec{j}'}(t)\rangle \\
&= \sum_{\vec{j}\vec{j}'}\langle\sum_{\vec{k}\neq\vec{k}'}\frac{\lambda V^*_{\vec{j}\vec{k}}\lambda V_{\vec{j}\vec{k}'}b^{\dagger(0)}_{\vec{k}}b^{(0)}_{\vec{k}'}}{(\mu_{\vec{k}}-z^*_{\vec{j}})(\mu_{\vec{k}'}-z_{\vec{j}})}e^{i(\mu_{\vec{k}}-\mu_{\vec{k}'})(t+\tau)} \\
&\times\sum_{\vec{k}_1\neq\vec{k}'_1}\frac{\lambda V^*_{\vec{j}'\vec{k}_1}\lambda V_{\vec{j}'\vec{k}'_1}b^{\dagger(0)}_{\vec{k}_1}b^{(0)}_{\vec{k}'_1}}{(\mu_{\vec{k}_1}-z^*_{\vec{j}'})(\mu_{\vec{k}'_1}-z_{\vec{j}'})}e^{i(\mu_{\vec{k}_1}-\mu_{\vec{k}'_1})(t)}\rangle. \qquad (6.87)
\end{aligned}$$

For the condition $\vec{k}\neq\vec{k}'$ and $\vec{k}_1\neq\vec{k}'_1$, we have

$$\langle b^{\dagger(0)}_{\vec{k}}b^{(0)}_{\vec{k}'}b^{\dagger(0)}_{\vec{k}_1}b^{(0)}_{\vec{k}'_1}\rangle = \langle b^{\dagger(0)}_{\vec{k}}b^{(0)}_{\vec{k}'}b^{\dagger(0)}_{\vec{k}'}b^{(0)}_{\vec{k}}\rangle\delta_{\vec{k}\vec{k}'_1}\delta_{\vec{k}'\vec{k}_1}. \qquad (6.88)$$

With Eq. (6.88) we get

$$\begin{aligned}
&\langle N_{f\vec{j}}(t+\tau)N_{f\vec{j}'}(t)\rangle \\
= &\sum_{\vec{k}\neq\vec{k}'}\frac{\lambda V^*_{\vec{j}\vec{k}}\lambda V_{\vec{j}'\vec{k}}\lambda V_{\vec{j}\vec{k}'}\lambda V_{\vec{j}'\vec{k}}\langle b^{\dagger(0)}_{\vec{k}}b^{(0)}_{\vec{k}'}b^{\dagger(0)}_{\vec{k}'}b^{(0)}_{\vec{k}}\rangle}{(\mu_{\vec{k}}-z^*_{\vec{j}})(\mu_{\vec{k}'}-z_{\vec{j}})(\mu_{\vec{k}'}-z^*_{\vec{j}'})(\mu_{\vec{k}}-z_{\vec{j}'})}e^{i(\mu_{\vec{k}}-\mu_{\vec{k}'})\tau}. \qquad (6.89)
\end{aligned}$$

When $\vec{j}\neq\vec{j}'$, Eq. (6.89) contain the sum over rapidly changing phases, which is close to zero. So we can write

$$\langle N_{f\vec{j}}(t+\tau)N_{f\vec{j}'}(t)\rangle = \langle N_{f\vec{j}}(t+\tau)N_{f\vec{j}}(t)\rangle\delta_{\vec{j}\vec{j}'}. \qquad (6.90)$$

With Eq. (6.90), Eq. (6.87) becomes much simpler.

$$\begin{aligned}
&\langle\sum_{\vec{j}} N_{f\vec{j}}(t+\tau)\sum_{\vec{j}'} N_{f\vec{j}'}(t)\rangle = \sum_{\vec{j}}\langle N_{f\vec{j}}(t+\tau)N_{f\vec{j}}(t)\rangle \\
&= \sum_{\vec{j}}\sum_{\vec{k}\neq\vec{k}'}\frac{\lambda^2|V_{\vec{j}\vec{k}}|^2\lambda^2|V_{\vec{j}\vec{k}'}|^2\langle b^{\dagger(0)}_{\vec{k}}b^{(0)}_{\vec{k}'}b^{\dagger(0)}_{\vec{k}'}b^{(0)}_{\vec{k}}\rangle}{|\mu_{\vec{k}}-z_{\vec{j}}|^2|\mu_{\vec{k}'}-z_{\vec{j}}|^2}e^{i(\mu_{\vec{k}}-\mu_{\vec{k}'})\tau} \qquad (6.91)
\end{aligned}$$

In the classical gas limit, we can write ($\vec{k} \neq \vec{k}'$)

$$\begin{aligned}
&\langle b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}'} b^{\dagger(0)}_{\vec{k}'} b^{(0)}_{\vec{k}} \rangle = \langle b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}} \rangle \langle b^{(0)}_{\vec{k}'} b^{\dagger(0)}_{\vec{k}'} \rangle + \langle b^{\dagger(0)}_{\vec{k}} b^{(0)}_{\vec{k}} \rangle \\
&= \frac{(C_0 k_B T)^2}{\mu_{\vec{k}} \mu_{\vec{k}'}} + \frac{C_0 k_B T}{\mu_{\vec{k}}}.
\end{aligned} \tag{6.92}$$

Here we used Eq. (6.80). Neglecting $1/\Omega_b$ difference ($\vec{k} = \vec{k}'$ case) , Eq. (6.91) can be written as

$$\begin{aligned}
&\sum_{\vec{j}} \langle N_{f\vec{j}}(t+\tau) N_{f\vec{j}}(t) \rangle \\
&= \sum_{\vec{j}} \left| \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2 C_0 k_B T}{|\mu_{\vec{k}} - z_{\vec{j}}|^2 \mu_{\vec{k}}} e^{i\mu_{\vec{k}}\tau} \right|^2 \\
&+ \sum_{\vec{j}} \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2 C_0 k_B T}{|\mu_{\vec{k}} - z_{\vec{j}}|^2 \mu_{\vec{k}}} e^{i\mu_{\vec{k}}\tau} \sum_{\vec{k}'} \frac{\lambda^2 |V_{\vec{j}\vec{k}'}|^2}{|\mu_{\vec{k}'} - z_{\vec{j}}|^2} e^{-i\mu_{\vec{k}'}\tau}.
\end{aligned} \tag{6.93}$$

By changing the summation into integration in the $\Omega_b \to \infty$ limit and taking the pole and using Eq. (6.85), We can write the above expression as

$$\begin{aligned}
&\sum_{\vec{j}} \langle N_{f\vec{j}}(t+\tau) N_{f\vec{j}}(t) \rangle \\
&\approx \sum_{\vec{j}} \frac{(C_0 k_B T)^2}{\omega^2_{\vec{j}}} e^{-2\gamma_{\vec{j}}|\tau|} + \sum_{\vec{j}} \frac{C_0 k_B T}{\omega_{\vec{j}}} e^{-2\gamma_{\vec{j}}|\tau|}.
\end{aligned} \tag{6.94}$$

Substituting Eq. (6.86) and Eq. (6.94) into Eq. (6.83), we get

$$\begin{aligned}
&\langle N_{atot}(t+\tau) N_{atot}(t) \rangle \\
&= \left(\sum_{\vec{j}} \frac{C_0 k_B T}{\omega_{\vec{j}}} \right)^2 + \sum_{\vec{j}} \frac{(C_0 k_B T)^2}{\omega^2_{\vec{j}}} e^{-2\gamma_{\vec{j}}|\tau|} + \sum_{\vec{j}} \frac{C_0 k_B T}{\omega_{\vec{j}}} e^{-2\gamma_{\vec{j}}|\tau|}.
\end{aligned} \tag{6.95}$$

The power spectrum of the number fluctuation is expressed by the Fourier transformation of the correlation function Eq. (6.94). (Wiener-Khintchine theorem.) The spectral density $S(\omega)$ is

$$S(\omega) = \int_{-\infty}^{\infty} d\tau \left(\sum_{\vec{j}} \frac{(C_0 k_B T)^2}{\omega_{\vec{j}}^2} e^{-2\gamma_{\vec{j}}|\tau|} + \sum_{\vec{j}} \frac{C_0 k_B T}{\omega_{\vec{j}}} e^{-2\gamma_{\vec{j}}|\tau|} \right) e^{-i\omega\tau}$$
$$= \sum_{\vec{j}} \frac{(C_0 k_B T)^2}{\omega_{\vec{j}}^2} \frac{4\gamma_{\vec{j}}}{\omega^2 + 4\gamma_{\vec{j}}^2} + \sum_{\vec{j}} \frac{(C_0 k_B T)}{\omega_{\vec{j}}} \frac{4\gamma_{\vec{j}}}{\omega^2 + 4\gamma_{\vec{j}}^2} \tag{6.96}$$

To calculate the spectral density $S(\omega)$, we again approximate $\sum_{\vec{j}}$ to $\int d\vec{j}$. From the Euler's summation formula, when a function $f(x)$ has a continuous derivative $f'(x)$, the sum over integer can be written as

$$\sum_{n=0}^{N} f(n) = \int_0^N f(x)dx + \int_0^N f'(x)(x - [x] - \frac{1}{2})dx + \frac{1}{2}(f(0) + f(N)) \tag{6.97}$$

where $[x]$ is the greatest integer $\leq x$. The function $(x - [x] - \frac{1}{2})$ has zero average over any interval $[m, m+1)$ where m is an integer, so the summation is well-approximated to the integral when $f'(x)$ does not vary much between integer point and $f(0)$, $f(N)$ is very small compared to the integral.

In our case the quantum number $\vec{j}$ is determined by the boundary condition and the potential of the interaction region. If a lot of quantum state contribute to the decay mode with significant $\gamma_{\vec{j}}$, then the condition $f(0)$ and $f(N)$ is small compared to the integration is easily satisfied. Note that our summation is over all positive terms. To see that $f'(x)$ does not vary much between integers, we should see how our function changes with slightly different boundary conditions.

Let the characteristic lengths of Ω_a for each dimension be L_{ax}, L_{ay} and L_{az}. Roughly speaking, when we write

$$\sum_{\vec{j}} \frac{4\gamma_{\vec{j}}}{\omega_{\vec{j}}^2} \frac{1}{\omega^2 + 4\gamma_{\vec{j}}^2} \approx \frac{\Omega_a}{(2\pi)^3} \frac{(2\pi)^3}{\Omega_a} \sum_{\vec{j}} f(\frac{2\pi j_x}{L_{ax}}, \frac{2\pi j_y}{L_{ay}}, \frac{2\pi j_x}{L_{az}})$$
$$\approx \frac{\Omega_a}{(2\pi)^3} \sum_{j_x} \sum_{j_y} \sum_{j_z} f(\frac{2\pi j_x}{L_{ax}}, \frac{2\pi j_y}{L_{ay}}, \frac{2\pi j_x}{L_{az}}) \frac{2\pi}{L_{ax}} \frac{2\pi}{L_{ay}} \frac{2\pi}{L_{az}}, \tag{6.98}$$

the approximation to integration means we assume the value of $f(2\pi(j_x + r)/L_{ax})$ $(0 < r < 1)$ does not vary much compared to the integer values $f(2\pi(j_x)/L_{ax})$ and $f(2\pi(j_x + 1)/L_{ax})$. Since we can write

$$\frac{j_x + r}{L_{ax}} = \frac{j_x}{L_{ax} \frac{j_x}{j_x + r}}, \tag{6.99}$$

the effect of substituting non-integer value is similar to the effect of changing the characteristic length L_{ax} to $L_{ax} j_x/(j_x + r)$. When high quantum numbers contribute to the sum such that the effect of slight change of the wall does not affect much to the integrating function, we can approximate the sum to the integration. First we can write

$$\frac{p_x^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{2\pi j_x}{L_{ax}} \right)^2 = \frac{1}{2} k_B T, \tag{6.100}$$

$$j_x = \sqrt{\frac{2mk_BT}{\hbar^2}} \frac{L_{ax}}{2\pi}. \tag{6.101}$$

The decay constants $\gamma_{\vec{j}}$ are related to the collisions at the boundary. When the characteristic length L_{ax} changes to $L_{ax} j_x/(j_x + r)$, there is a time scale in which the number of collisions changes due to the length change. The number of collision per unit time in x direction is v_x/L_{ax}, where v_x can be estimated

from

$$\frac{1}{2}mv_x^2 = \frac{1}{2}k_BT, \;\; v_x = \sqrt{\frac{k_BT}{m}}. \tag{6.102}$$

The collision number change per unit time is

$$\Delta = \frac{v_x}{L_{ax}(\frac{j_x}{j_x+r})} - \frac{v_x}{L_{ax}} \approx r\frac{v_x}{L_{ax}j_x} = r\frac{2\pi\hbar}{\sqrt{2}mL_{ax}^2}. \tag{6.103}$$

For the electron mass and $L_{ax} \sim 1cm$, we have

$$\begin{aligned} &\Delta \sim 7 \,\text{collision number change/sec} \\ &= 1 \text{ collision number change } /0.14 \text{ sec.} \end{aligned} \tag{6.104}$$

So, very conservatively speaking, the integration approximation works well when τ is less than 0.14 sec or $\omega/2\pi$ is larger than 7Hz for the particle mass is about the electron mass and $L_{ax} \sim 1cm$. The integral approximation would fail in the case only a small number of modes contributes, or the case that characteristic length L_{ax} is small. The integration approximation would not work for the nanoscale waveguides.

With this in mind, the spectral density $S(\omega)$ becomes

$$\begin{aligned} S(\omega) &= \sum_{\vec{j}} \frac{(C_0k_BT)^2}{\omega_{\vec{j}}^2}\frac{4\gamma_{\vec{j}}}{\omega^2+4\gamma_{\vec{j}}^2} + \sum_{\vec{j}} \frac{(C_0k_BT)}{\omega_{\vec{j}}}\frac{4\gamma_{\vec{j}}}{\omega^2+4\gamma_{\vec{j}}^2} \\ &= \int d\vec{j}\frac{(C_0k_BT)^2}{\omega_{\vec{j}}^2}\frac{4\gamma_{\vec{j}}}{\omega^2+4\gamma_{\vec{j}}^2} + \int d\vec{j}\frac{C_0k_BT}{\omega_{\vec{j}}}\frac{4\gamma_{\vec{j}}}{\omega^2+4\gamma_{\vec{j}}^2} \\ &= C_1\int dj_r j_r^2\frac{\bar{\gamma}_{j_r}}{\omega_{j_r}^2}\frac{1}{\omega^2+4\bar{\gamma}_{j_r}^2} + C_2\int dj_r j_r^2\frac{\bar{\gamma}_{j_r}}{\omega_{j_r}}\frac{1}{\omega^2+4\bar{\gamma}_{j_r}^2} \\ &= C_1\int d\bar{\gamma}_{j_r}\frac{dj_r}{d\bar{\gamma}_{j_r}}j_r^2\frac{\bar{\gamma}_{j_r}}{\omega_{j_r}^2}\frac{1}{\omega^2+4\bar{\gamma}_{j_r}^2} + C_2\int d\bar{\gamma}_{j_r}\frac{dj_r}{d\bar{\gamma}_{j_r}}j_r^2\frac{\bar{\gamma}_{j_r}}{\omega_{j_r}}\frac{1}{\omega^2+4\bar{\gamma}_{j_r}^2} \end{aligned} \tag{6.105}$$

where $\bar{\gamma}_{j_r}$ is the angle averaged value of $\gamma_{\vec{j}}$. ω_{j_r} is related with j_r through the relation

$$\omega_{j_r} = p^2/2m \sim j_r^2. \tag{6.106}$$

For the decay constant $\bar{\gamma}_{j_r}$ we can first estimate it as

$$\bar{\gamma}_{j_r} \sim \exp(-D\sqrt{V - \omega_{j_r}}) \sim \exp(j_r). \tag{6.107}$$

due to the tunnelling effect. Substituting Eq. (6.106) and Eq. (6.107) into Eq. (6.105), we get

$$\begin{aligned} S(\omega) &= C_{s1} \int_{\bar{\gamma}_{min}}^{\bar{\gamma}_{max}} d\bar{\gamma} \frac{1}{\log(C_{s2}\,\bar{\gamma})(\omega^2 + 4\bar{\gamma}^2)} \\ &+ C'_{s1} \int_{\bar{\gamma}_{min}}^{\bar{\gamma}_{max}} d\bar{\gamma} \frac{1}{(\omega^2 + 4\bar{\gamma}^2)} \end{aligned} \tag{6.108}$$

where C_{s1}, C_{s2} and C'_{s1} are some constants independent of $\bar{\gamma}$ and ω, and $\bar{\gamma}_{max}$ and $\bar{\gamma}_{min}$ are the maximum and minimum decay constant, respectively. Noting that the integral

$$\int_{\gamma_{min}}^{\gamma_{max}} d\gamma \frac{\gamma^p}{\omega^2 + \gamma^2} = \frac{1}{\omega^{1-p}} \int_{\gamma_{min}/\omega}^{\gamma_{max}/\omega} dy \frac{y^p}{1 + y^2} \sim O(1/\omega^{1-p}) \tag{6.109}$$

for $\gamma_{min} \ll \omega \ll \gamma_{max}$, we have

$$S(\omega) \sim O(\frac{1}{\omega}) \tag{6.110}$$

in Eq. (6.108).

If we assume that $\bar{\gamma}_{j_r}$ is related by the power law, (see Eq. (6.85)),

$$\bar{\gamma}_{j_r} \sim \sum_{\vec{k}} \lambda^2 |V_{j_r\vec{k}}|^2 \delta(\mu_{\vec{k}} - \tilde{\omega}_{j_r}) \sim \int d\mu \mu^2 \lambda^2 |\bar{v}_{j_r}(\mu)|^2 \delta(\mu - \omega_{j_r}) \sim \omega_{j_r}^h, \ (h > 1), \tag{6.111}$$

then Eq. (6.105) becomes

$$\begin{aligned} & S(\omega) \\ & \sim C_{p1} \int_{\bar{\gamma}_{min}}^{\bar{\gamma}_{max}} d\bar{\gamma} \frac{\bar{\gamma}^{-1/2h}}{\omega^2 + \bar{\gamma}^2} + C_{p2} \int_{\bar{\gamma}_{min}}^{\bar{\gamma}_{max}} d\bar{\gamma} \frac{\bar{\gamma}^{1/2h}}{\omega^2 + \bar{\gamma}^2} \end{aligned} \tag{6.112}$$

When the first term is dominant, it gives

$$S(\omega) \sim O(\omega^{-(1+\frac{1}{2h})}), \tag{6.113}$$

and when the second term (due to the quantum effect) is dominant it gives

$$S(\omega) \sim O(\omega^{-(1-\frac{1}{2h})}). \tag{6.114}$$

This is $1/f^\beta$ spectrum, where $0.5 < \beta < 1.5$.

So, in this chapter we derived the $1/f$ noise from Hamiltonian, and showed that $1/f$ noise coming from the sum of resonances effect, as other people speculated [52].

6.4 Λ transformation for the extended Friedrichs model

In this section we construct Λ transformation for the extended Friedrichs model

$$H = \sum_{\vec{j}} \omega_{\vec{j}} a_{\vec{j}}^\dagger a_{\vec{j}} + \sum_{\vec{k}} \mu_{\vec{k}} b_{\vec{k}}^\dagger b_{\vec{k}} + \lambda \sum_{\vec{j}\vec{k}} (V_{\vec{j}\vec{k}} a_{\vec{j}}^\dagger b_{\vec{k}} + V_{\vec{j}\vec{k}}^* a_{\vec{j}} b_{\vec{k}}^\dagger). \tag{6.115}$$

To construct Λ, we first derive the unitary transformation U for the discrete case and analytically continue U to λ in the $\Omega_b \to \infty$ limit.

Suppose that Ω_b is large but finite such that $\mu_{\vec{k}}$s are discrete. Furthermore, we assume that $\omega_{\vec{j}} \neq \mu_{\vec{k}}$. In this case, we can find new creation and destruction operators $\bar{A}^\dagger_{\vec{j}}$, $\bar{A}_{\vec{j}}$, $\bar{B}^\dagger_{\vec{k}}$ and $\bar{B}_{\vec{k}}$ which diagonalize the Hamiltonian.

We can write

$$H = \sum_{\vec{j}} \bar{\omega}_{\vec{j}} \bar{A}^\dagger_{\vec{j}} \bar{A}_{\vec{j}} + \sum_{\vec{k}} \bar{\mu}_{\vec{k}} \bar{B}^\dagger_{\vec{k}} \bar{B}_{\vec{k}}. \tag{6.116}$$

The original operators and new operators are related through unitary transformation U.

$$\bar{A}^\dagger_{\vec{j}} = U^\dagger a^\dagger_{\vec{j}}, \;\; \bar{B}^\dagger_{\vec{k}} = U^\dagger b^\dagger_{\vec{k}}, \tag{6.117}$$

$$UH = \sum_{\vec{j}} \bar{\omega}_{\vec{j}} a^\dagger_{\vec{j}} a_{\vec{j}} + \sum_{\vec{k}} \bar{\mu}_{\vec{k}} b^\dagger_{\vec{k}} b_{\vec{k}}. \tag{6.118}$$

The explicit forms of $\bar{A}^\dagger_{\vec{j}}$ and $\bar{B}^\dagger_{\vec{k}}$ are found by writing

$$\bar{A}^\dagger_{\vec{j}} = p_{\vec{j}\vec{j}} a^\dagger_{\vec{j}} + \sum_{\vec{j}'(\neq\vec{j})} p_{\vec{j}\vec{j}'} a^\dagger_{\vec{j}'} + \sum_{\vec{k}} q_{\vec{j}\vec{k}} b^\dagger_{\vec{k}}, \tag{6.119}$$

$$\bar{B}^\dagger_{\vec{k}} = r_{\vec{k}\vec{k}} b^\dagger_{\vec{k}} + \sum_{\vec{k}'(\neq\vec{k})} r_{\vec{k}\vec{k}'} b^\dagger_{\vec{k}'} + \sum_{\vec{j}} s_{\vec{k}\vec{j}} a^\dagger_{\vec{j}}. \tag{6.120}$$

and substitute the above expressions into

$$[H, \bar{A}^\dagger_{\vec{j}}] = \bar{\omega}_{\vec{j}} \bar{A}^\dagger_{\vec{j}}, \;\; [H, \bar{B}^\dagger_{\vec{k}}] = \bar{\mu}_{\vec{k}} \bar{B}^\dagger_{\vec{k}}. \tag{6.121}$$

Solving these equations, for $\bar{A}^\dagger_{\vec{j}}$ we get

$$\bar{A}^\dagger_{\vec{j}} = \bar{N}^{1/2}_{\vec{j}} \left(a^\dagger_{\vec{j}} + \sum_{\vec{k}} \frac{\lambda V^*_{\vec{j}\vec{k}}}{\bar{\omega}_{\vec{j}} - \omega_{\vec{k}}} b^\dagger_{\vec{k}} \right) \tag{6.122}$$

where

$$\bar{N}_{\vec{j}} = \left(1 + \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{(\omega_{\vec{j}} - \omega_{\vec{k}})^2}\right)^{-1}, \tag{6.123}$$

$$\bar{\omega}_{\vec{j}} - \omega_{\vec{j}} - \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{\bar{\omega}_{\vec{j}} - \omega_{\vec{k}}} = 0. \tag{6.124}$$

Note that in Eq. (6.122) there are no $a^\dagger_{\vec{j}'}$ dependence. This comes from the fact

$$\sum_{\vec{k}} V_{\vec{j}\vec{k}} V^*_{\vec{j}'\vec{k}} \approx 0. \tag{6.125}$$

For $\bar{B}^\dagger_{\vec{k}}$ we have

$$\bar{B}^\dagger_{\vec{k}} = \bar{N}^{1/2}_{\vec{k}} \left(b^\dagger_{\vec{k}} + \sum_{\vec{j}} \frac{\lambda V_{\vec{j}\vec{k}}}{\eta_{\vec{j}}(\bar{\mu}_{\vec{k}})} (a^\dagger_{\vec{j}} + \sum_{l(\neq\vec{k})} \frac{\lambda V^*_{\vec{j}l}}{\bar{\mu}_{\vec{k}} - \mu_l + i\epsilon} b^\dagger_l) \right) \tag{6.126}$$

where

$$\bar{N}_{\vec{k}} = 1 + O(1/\Omega_b), \;\; \bar{\mu}_{\vec{k}} = \mu_{\vec{k}} + O(1/\Omega_2). \tag{6.127}$$

In the $\Omega_2 \to \infty$ limit ($\Omega_2 < \Omega_b \to \infty$ as well), $\mu_{\vec{k}}$ becomes continuous variables. In this case, if we keep $\bar{\omega}_{\vec{j}}$ real, then $\bar{N}_{\vec{j}} \to 0$ as we see in Eq. (6.123). The $\vec{j}$ modes disappear and we go back to the Friedrichs solution in section 6.2.

But there is another choice if we give up the real eigenvalues. If we allow $\bar{\omega}_{\vec{j}}$ to be complex, then we can construct Gamow modes as we did for the Friedrichs model (Section 3.3).

The construction is quite straightforward. By analytic continuation, we have the Gamow modes

$$A^{\dagger}_{\vec{j}} = N^{1/2}_{\vec{j}}\left(a^{\dagger}_{\vec{j}} + \sum_{\vec{k}} \frac{\lambda V^{*}_{\vec{j}\vec{k}}}{(z-\omega_{\vec{k}})^{+}_{z_{\vec{j}}}} b^{\dagger}_{\vec{k}}\right), \tag{6.128}$$

$$\tilde{A}_{\vec{j}} = N^{1/2}_{\vec{j}}\left(a_{\vec{j}} + \sum_{\vec{k}} \frac{\lambda V_{\vec{j}\vec{k}}}{[(z-\omega_{\vec{k}})^{+}_{z_{\vec{j}}}]^{*}} b_{\vec{k}}\right) \tag{6.129}$$

where

$$N_{\vec{j}} = \left(1 + \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{[(z-\omega_{\vec{k}})^{+}_{z_{\vec{j}}}]^2}\right)^{-1}. \tag{6.130}$$

In Eq. (6.130), $z_{\vec{j}}$ is the solution of the equation

$$\eta^{+}_{\vec{j}}(z) = 0. \tag{6.131}$$

The Λ transformation for the $\vec{j}$ modes can be found rather easily as done in section 3.4. We have

$$\begin{aligned} &\Lambda^{\dagger} a^{\dagger}_{\vec{j}} = \tilde{A}^{\dagger}_{\vec{j}}, \;\; \Lambda^{\dagger} a_{\vec{j}} = \tilde{A}_{\vec{j}}, \\ &\Lambda a^{\dagger}_{\vec{j}} = A^{\dagger}{}_{\vec{j}}, \;\; \Lambda a_{\vec{j}} = A_{\vec{j}} \end{aligned} \tag{6.132}$$

$$\Lambda^{\dagger} a^{\dagger m}_{\vec{j}} a^{n}_{\vec{j}} = \sum_{l=0}^{min(m,n)} \frac{m!n!}{(m-l)!(n-l)!l!} (\tilde{A}^{\dagger}_{\vec{j}})^{m-l} \tilde{A}^{n-l}_{\vec{j}} Y^{l}_{\vec{j}} \tag{6.133}$$

where $Y_{\vec{j}}$ is given by

$$Y_{\vec{j}} \equiv \sum_{\vec{k}} \bar{b}_{\vec{j}\vec{k}} b^{\dagger}_{\vec{k}} b_{\vec{k}}. \tag{6.134}$$

In Eq. (6.134), $\bar{b}_{\vec{j}\vec{k}}$ is defined as

$$\bar{b}_{\vec{j}\vec{k}} \equiv \frac{\lambda^2}{|1+\xi_{\vec{j}}|}[(r_{\vec{j}} \frac{|V_{\vec{j}\vec{k}}|^2}{(z-\mu_{\vec{k}})^{+}_{z_{\vec{j}}}} + c.c) - \frac{|V_{\vec{j}\vec{k}}|^2}{|(z-\mu_{\vec{k}})^{+}_{z_{\vec{j}}}|^2}] \tag{6.135}$$

with

$$\xi_{\vec{j}} \equiv \sum_{\vec{k}} \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{[(z-\mu_{\vec{k}})^+_{z_{\vec{j}}}]^2}, \tag{6.136}$$

$$r_{\vec{j}} = \frac{1}{2} + \frac{|1+\xi_{\vec{j}}| - 1 - (\xi_{\vec{j}} + \xi^{c.c})/2}{\xi_{\vec{j}} - \xi^{c.c}_{\vec{j}}}. \tag{6.137}$$

Approximately, $\bar{b}_{\vec{j}\vec{k}}$ can be written as

$$\begin{aligned}
\bar{b}_{\vec{j}\vec{k}} &\approx \frac{\lambda^2 |V_{\vec{j}\vec{k}}|^2}{2} \left(\frac{1}{(z-\mu_{\vec{k}})^+_{z_{\vec{j}}}} - \frac{1}{(z-\mu_{\vec{k}})^-_{z^*_{\vec{j}}}} \right)^2 \\
&= \frac{(2\pi)^3}{\Omega_b} \frac{1}{(2\pi)^3} \frac{2\lambda^2 |v_{\vec{j}\vec{k}}|^2 \gamma^2_{\vec{j}}}{[(z-\mu_{\vec{k}})^+_{z_{\vec{j}}} (z-\mu_{\vec{k}})^-_{z^*_{\vec{j}}}]^2} \\
&\approx \frac{(2\pi)^3}{\Omega_b} \frac{1}{(2\pi)^3} \frac{2\lambda^2 |v_{\vec{j}\vec{k}}|^2 \gamma^2_{\vec{j}}}{[(z_{\vec{j}}-\mu_{\vec{k}})(z^*_{\vec{j}}-\mu_{\vec{k}})]^2}.
\end{aligned} \tag{6.138}$$

The above expression has a sharp peak around $\mu_{\vec{k}} = \tilde{\omega}_{\vec{j}}$. Since $\sum_{\vec{k}} \bar{b}_{\vec{j}\vec{k}} = 1$, we can approximately write

$$\bar{b}_{\vec{j}\vec{k}} \approx \frac{(2\pi)^3}{\Omega_b} \delta(\mu_{\vec{k}} - \tilde{\omega}_{\vec{j}}). \tag{6.139}$$

Since the Gamow modes are decoupled and their multiplication does not give any non-analyticity (we assume that $z_{\vec{j}}$s are not degenerate.), Λ and $\Lambda^\dagger$ are distributive for $\vec{j}$ and $\vec{j}'$.

$$\Lambda^\dagger (a^{\dagger m}_{\vec{j}} a^n_{\vec{j}} a^{\dagger\ m'}_{\vec{j}'} a^{n'}_{\vec{j}'}) = (\Lambda^\dagger a^{\dagger m}_{\vec{j}} a^n_{\vec{j}})(\Lambda^\dagger a^{\dagger m'}_{\vec{j}} a^{n'}_{\vec{j}'}). \tag{6.140}$$

The quantity which corresponds to $\langle N_{atot}(t+\tau) N_{atot} \rangle$ can be obtained through Λ transformation. For this we first consider the unitary (discrete) case and extend to the Λ transformation.

The autocorrelation of the unitary transformed number operator N_{atot} is

$$\begin{aligned} G_U &= Tr[(e^{iL_H(t+\tau)}U^\dagger N_{atot})(U^\dagger N_{atot})\rho_{eq}] \\ &= Tr[e^{iL_Ht}(e^{iL_H\tau}U^\dagger N_{atot})(U^\dagger N_{atot})\rho_{eq}] \\ &= Tr[e^{iL_Ht}U^\dagger[((U^\dagger)^{-1}e^{iL_H\tau}U^\dagger N_{atot})N_{atot}]\rho_{eq}]. \end{aligned} \tag{6.141}$$

By analytically continuation of the above expression, we get

$$G_\Lambda = Tr[e^{iL_Ht}\Lambda^\dagger[((\Lambda^\dagger)^{-1}e^{iL_H\tau}\Lambda^\dagger N_{atot})N_{atot}]\rho_{eq}]. \tag{6.142}$$

The quantity inside the Trace becomes

$$\begin{aligned} & e^{iL_Ht}\Lambda^\dagger[((\Lambda^\dagger)^{-1}e^{iL_H\tau}\Lambda^\dagger N_{atot})N_{atot}] \\ &= e^{iL_Ht}\Lambda^\dagger[\sum_{\vec{j}}(e^{-2\gamma_{\vec{j}}\tau}a^\dagger_{\vec{j}}a_{\vec{j}}+Y_{\vec{j}})\sum_{\vec{j}'}a^\dagger_{\vec{j}'}a_{\vec{j}'}] \\ &= e^{iL_Ht}\Lambda^\dagger[\sum_{\vec{j}\vec{j}'}{}' e^{-2\gamma\tau}a^\dagger_{\vec{j}}a_{\vec{j}}a^\dagger_{\vec{j}'}a_{\vec{j}'}+\sum_{\vec{j}}e^{-2\gamma_{\vec{j}}\tau}a^\dagger_{\vec{j}}a_{\vec{j}}a^\dagger_{\vec{j}}a_{\vec{j}}+\sum_{\vec{j}\vec{j}'}Y_{\vec{j}}(1-e^{-2\gamma_{\vec{j}}\tau})a^\dagger_{\vec{j}'}a_{\vec{j}'}] \\ &= e^{iL_Ht}[\sum_{\vec{j}\vec{j}'}{}' e^{-2\gamma_{\vec{j}}\tau}(\tilde{A}^\dagger_{\vec{j}}\tilde{A}_{\vec{j}}+Y_{\vec{j}})(\tilde{A}^\dagger_{\vec{j}'}\tilde{A}_{\vec{j}'}+Y_{\vec{j}'}) \\ &+\sum_{\vec{j}}e^{-2\gamma_{\vec{j}}\tau}((\tilde{A}^\dagger_{jv})^2\tilde{A}^2_{\vec{j}}+4\tilde{A}^\dagger_{\vec{j}}\tilde{A}_{\vec{j}}Y_{\vec{j}}+2(Y_{\vec{j}})^2)+\sum_{\vec{j}}e^{-2\gamma_{\vec{j}}\tau}(\tilde{A}^\dagger_{\vec{j}}\tilde{A}_{\vec{j}}+Y_{\vec{j}}) \\ &+\sum_{\vec{j}}Y_{\vec{j}}(1-e^{-2\gamma_{\vec{j}}\tau})\sum_{\vec{j}'}(\tilde{A}^\dagger_{\vec{j}'}\tilde{A}_{\vec{j}'}+Y_{\vec{j}'})] \end{aligned} \tag{6.143}$$

where $\sum'_{\vec{j}\vec{j}'}$ means sum without $\vec{j}=\vec{j}'$ cases. In Eq. (6.143), $\tilde{A}^\dagger_{\vec{j}}$ and $\tilde{A}_{\vec{j}}$ decays exponentially with time t. For large t, these Gamow modes can be neglected.

Then Eq. (6.143) becomes

$$\begin{aligned}
&\text{rhs of Eq. (6.143)} \\
&= \sum_{\vec{j}\vec{j}'}{}' e^{-2\gamma_{\vec{j}}\tau} Y_{\vec{j}} Y_{\vec{j}'} + 2\sum_{\vec{j}} e^{-2\gamma_{\vec{j}}\tau}(Y_{\vec{j}})^2 + \sum_{\vec{j}} e^{-2\gamma_{\vec{j}}\tau} Y_{\vec{j}} + \sum_{\vec{j}\vec{j}'} Y_{\vec{j}} Y_{\vec{j}'}(1 - e^{-2\gamma_{\vec{j}}\tau}) \\
&= (\sum_{\vec{j}} Y_{\vec{j}})^2 + \sum_{\vec{j}} e^{-2\gamma_{\vec{j}}\tau}(Y_{\vec{j}})^2 + \sum_{\vec{j}} e^{-2\gamma_{\vec{j}}\tau} Y_{\vec{j}}. \qquad (6.144)
\end{aligned}$$

Here the last term comes from the quantum commutation relation. Substituting Eq. (6.144) into Eq. (6.142), we get

$$G_\Lambda = \langle (\sum_{\vec{j}} Y_{\vec{j}})^2 \rangle + \langle \sum_{\vec{j}} e^{-2\gamma_{\vec{j}}\tau}(Y_{\vec{j}})^2 \rangle + \langle \sum_{\vec{j}} e^{-2\gamma_{\vec{j}}\tau} Y_{\vec{j}} \rangle. \qquad (6.145)$$

Since ensemble average of $Y_{\vec{j}}$ is

$$\langle Y_{\vec{j}} \rangle = \langle \sum_{\vec{k}} \bar{b}_{\vec{j}\vec{k}} b^\dagger_{\vec{k}} b_{\vec{k}} \rangle = \sum_{\vec{k}} \bar{b}_{\vec{j}\vec{k}} \frac{C_0 k_B T}{\mu_{\vec{k}}} \approx \frac{C_0 k_B T}{\omega_{\vec{j}}}, \qquad (6.146)$$

we get

$$G_\Lambda \approx (\sum_{\vec{j}} \frac{C_0 k_B T}{\omega_{\vec{j}}})^2 + \sum_{\vec{j}} e^{-2\gamma_{\vec{j}}\tau} \frac{(C_0 k_B T)^2}{\omega^2_{\vec{j}}} + \sum_{\vec{j}} e^{-2\gamma_{\vec{j}}\tau} \frac{C_0 k_B T}{\omega_{\vec{j}}}. \qquad (6.147)$$

This is the same result as we obtained in the previous section (Eq. (6.95)). This again shows that Λ separates the resonant part of the Hamiltonian.

Chapter 7

Conclusion

In this thesis we studied the stochasticity in Hamiltonian dynamics.

We showed that a star-unitary transformation can be constructed for the classical Friedrichs model, and showed that this transformation leads to stochastic equations such as Langevin and Fokker-Planck equations. We also derived the Gaussian white noise through this star-unitary transformation. This derivation shows that resonances are crucial elements for the Gaussian white noise structure.

We showed that the quantum noise has similar Gaussian white noise properties when we keep the normal ordering.

Finally, we studied the extended Friedrichs model which describes the electron waveguide. We showed that in this model $1/f$ noise are present in the low frequency region due to the sum of resonant effects, and constructed the Λ transformation which extracts the resonant effects.

Appendices

Appendix A

A.1 The U operator

In this appendix we give an example of the construction of the canonical transformation operator U and show how $\bar{\chi}^{(\nu)}$ is fixed. We consider the Friedrichs model presented in Sec. 3.

We shall consider the transformed product $U^{-1}a_1^*a_1$. The monomial $a_1^*a_1$ is an eigenfunction of L_0 with eigenvalue $w^{(0)} = 0$, so it belongs to the $P^{(0)}$ subspace with $d_0 = 0$. Using Eq. (2.27) we have

$$U^{-1}a_1^*a_1 = U^{-1}P^{(0)}a_1^*a_1 = (P^{(0)} + \bar{C}^{(0)})\bar{\chi}^{(0)}a_1^*a_1 \tag{A.1}$$

The operator U^{-1} is a function of the Liouville operator. This operator preserves the number of a^* and a modes in a given monomial: it will map a monomial with m modes a^* and n modes a to a superposition of monomials with the same numbers m, n of modes a^*, a, respectively. The same is true for the operators $\bar{C}$ and $\bar{\chi}$. So we have

$$\begin{aligned} \bar{\chi}^{(0)}a_1^*a_1 &= \sum_{s=1,k} a_s^*a_s\bar{\chi}^{(0)}_{ss;11} \\ \bar{C}^{(0)}a_1^*a_1 &= \sum_{s=1,k}{\sum_{s'=1,k}}' a_s^*a_{s'}\bar{C}^{(0)}_{ss';11} \end{aligned} \tag{A.2}$$

where $\bar{\chi}^{(0)}_{ss;11}$ and $\bar{C}^{(0)}_{ss';11}$ are coefficients, and the prime in the summation over s' means $s \neq s'$ (recall that $\bar{\chi}^{(0)}$ is a diagonal operator, while $\bar{C}^{(0)}$ is off-diagonal). We have as well

$$\begin{aligned} \bar{C}^{(0)}\bar{\chi}^{(0)}a_1^* a_1 &= \sum_s \bar{C}^{(0)} a_s^* a_s \bar{\chi}^{(0)}_{ss;11} \\ &= \sum_{s=1,k} \sum_{r,r'}{}' a_r^* a_{r'} \bar{C}^{(0)}_{rr';ss} \bar{\chi}^{(0)}_{ss;11} \end{aligned} \tag{A.3}$$

Note that each change from a mode a_k to a mode a_1 or viceversa involves an interaction of order $L^{-1/2}$ in volume. Hence each index change involves an $L^{-1/2}$ factor. For example we have

$$\begin{aligned} \bar{\chi}^{(0)}_{11;11} &\sim O(L^0) \\ \bar{\chi}^{(0)}_{kk;11} &\sim O(L^{-1}) \\ \bar{C}^{(0)}_{1k;11} &\sim O(L^{-1/2}) \\ \bar{C}^{(0)}_{1k';kk} &\sim O(L^{-3/2}), \text{ etc.} \end{aligned} \tag{A.4}$$

Each summation over field modes gives an L factor. Taking all the volume factors into account we get

$$U^{-1}a_1^* a_1 = \sum_s a_r^* a_s \bar{\chi}^{(0)}_{ss;11} + \sum_{s,s'}{}' a_s^* a_{s'} \bar{C}^{(0)}_{ss';11} \bar{\chi}^{(0)}_{11;11} + O(1/L) \tag{A.5}$$

The $\bar{C}^{(0)}_{ss';kk}$ coefficients give $O(1/L)$ contributions. To determine the coefficients $\bar{C}^{(0)}_{ss';11}$ we use Eq. (2.38) [22]. This leads to

$$\bar{C}^{(0)}_{1k;11} = \bar{C}^{(0)}_{k1;11} = \bar{c}_k, \quad \bar{C}^{(0)}_{kk';11} = \bar{c}_k \bar{c}_{k'} \tag{A.6}$$

For the operator $\bar{\chi}^{(0)}_{11;11}$ we have

$$\begin{aligned} \bar{\chi}^{(0)}_{11;11} &= [\bar{A}^{(0)}]^{1/2}_{11;11}[\exp(\bar{B}^{(0)})]_{11;11} + O(1/L) \\ &= [\bar{A}^{(0)}]^{1/2}_{11;11} + O(1/L) \end{aligned} \tag{A.7}$$

where in the second line we used the antihermiticity of $\bar{B}^{(0)}$, which leads to $\bar{B}^{(0)}_{11;11} = 0$. The second line in Eq. (A.7) may be explicitly evaluated [22] using Eq. (A.6). The result is

$$\bar{\chi}^{(0)}_{11;11} = \bar{N}_1 \tag{A.8}$$

With the results obtained so far we have

$$U^{-1}a_1^* a_1 = N_1(a_1^* + \sum_k \lambda \bar{c}_k a_k^*)(a_1 + \sum_{k'} \lambda \bar{c}_{k'} a_{k'}) + \sum_k (\bar{\chi}^{(0)}_{kk;11} - N_1 \lambda^2 \bar{c}_k^2) a_k^* a_k \tag{A.9}$$

Now, from the distributive property of U we conclude that the second term in the r.h.s. has to vanish, i.e.,

$$\bar{\chi}^{(0)}_{kk;11} = \bar{N}_1 \bar{c}_k^2 \tag{A.10}$$

Indeed, if this is so we obtain the expected result $U^{-1}a_1^* a_1 = (U^{-1}a_1^*)(U^{-1}a_1)$.

With this result we have completed the calculation of all the coefficients of the expansion of $U^{-1}a_1^* a_1$ using the formulation in terms of kinetic operators. The advantage of the method followed here is that it permits a straightforward extension to the nonintegrable case.

A.2 Relation between Λ and the Gamow modes

In this Appendix we show that $\Lambda^{-1} q_1 = Q_1$. In a similar way, one can show the other equations in. First we show that q_1 is an eigenfunction of the collision operator,

$$\tilde{\theta}\, q_1 = \tilde{\theta}_{1,1}\, q_1 \tag{A.11}$$

where $\tilde{\theta}_{1,1}$ is the eigenvalue.

Due to the form of the interaction, any operator (such as $\tilde{\theta}$) that is a function of L_V will preserve the number of q and q^* in a given monomial (it will give a superposition of monomials with the same number of q and q^*). So we have

$$\tilde{\theta}\, q_1 = \tilde{\theta}_{1,1}\, q_1 + \sum_k \tilde{\theta}_{k,1}\, q_k \tag{A.12}$$

where $\tilde{\theta}_{i,j}$ are coefficients. The modes q_1 and q_k belong to different eigenspaces of L_0. From the commutation relation Eq. (3.61) we conclude that the coefficients $\tilde{\theta}_{k,1}$ must be zero, and hence q_1 is an eigenfunction of $\tilde{\theta}$.

Eq. (A.11) implies that

$$L_H\, \Lambda^{-1} q_1 = \tilde{\theta}_{1,1}\, \Lambda^{-1} q_1 \tag{A.13}$$

So $\Lambda^{-1} q_1$ is one of the Gamow modes Q_1 or $\tilde{Q}_1$. Which one is chosen depends on the sign of ϵ used in the regularization of U. We choose the sign of ϵ so

that $\exp(-iL_H t)\Lambda^{-1}q_1$ decays for $t > 0$ (see footnote [40]). Thus we obtain the desired result (see Eq. (3.49))

$$\Lambda^{-1}q_1 = Q_1, \quad \tilde{\theta}_{1,1} = -z_1. \tag{A.14}$$

A.3 Proof of Eq. (3.71)

From the requirement (4) in the Introduction, Λ preserves the measure of phase space. This means that

$$\int d\Gamma \Lambda \rho = \int d\Gamma \Lambda^\dagger \rho = 1 \tag{A.15}$$

for any normalized ensemble ρ.

Now consider the ensemble

$$\rho = C_1 q_1^* q_1 \exp(-J/J_0) \tag{A.16}$$

where C_1 is the normalization factor given by

$$C_s = \Big[\int d\Gamma q_s^* q_s \exp(-J/J_0) \Big]^{-1} \tag{A.17}$$

with

$$J = \sum_{s=1,k} q_s^* q_s \tag{A.18}$$

and J_0 a constant that makes the argument of the exponential dimensionless. The factor $\exp(-J/J_0)$ ensures the existence of a finite norm of ρ, (see the

Segal-Bargmann representation in [24, 30]). The total action J is an invariant of motion, because we have $L_0 J = 0$ and $L_V J = 0$. Since $\Lambda^\dagger$ can be expressed as a perturbation expansion, $\Lambda^\dagger = 1 + O(\lambda L_V)$, we get

$$\Lambda^\dagger J = J, \quad \Lambda^\dagger \exp(-J/J_0) = \exp(-J/J_0) \tag{A.19}$$

The operator L_V is a differential operator. Applying the chain rule of differentiation and Eq. (A.19) we conclude that

$$\Lambda^\dagger q_1^* q_1 \exp(-J/J_0) = (\Lambda^\dagger q_1^* q_1) \exp(-J/J_0) \tag{A.20}$$

Inserting the ensemble Eq. (A.16) in Eq. (A.15) and using Eq. (3.68) we get

$$\begin{aligned} C_1 \int d\Gamma |N_1| [q_1^* q_1 + \lambda^2 \sum_k (r c_k^2 + \text{c.c.}) q_k^* q_k] \\ \times \exp(-J/J_0) = 1 \end{aligned} \tag{A.21}$$

where the off-diagonal terms such as $q_1^* q_k$ appearing in the product $\tilde{Q}_1^* \tilde{Q}_1$ in Eq. (3.68) vanish due to the integration over angles in phase space. We can write Eq. (A.21) as

$$C_1 |N_1| [C_1^{-1} + \lambda^2 \sum_k (r c_k^2 + \text{c.c.}) C_k^{-1}] = 1 \tag{A.22}$$

Since $C_1 = C_k$ for any k (see Eq. (A.17)), Eq. (A.21) leads to

$$|N_1| [1 + \lambda^2 \sum_k (r c_k^2 + \text{c.c.})] = 1 \tag{A.23}$$

This equation plus the condition $r + r^* = 1$ yield the result (3.71). With this result we can write

$$\Lambda^\dagger q_1^* q_1 = Q^{(0)} \tilde{Q}_1^* \tilde{Q}_1 + P^{(0)} (r \tilde{Q}_1^* Q_1 + \text{c.c.}) \tag{A.24}$$

where $P^{(0)}$ is the projector to angle-independent monomials (such as $q_s^* q_s$) and $Q^{(0)} = 1 - P^{(0)}$. This shows that the transformed product of particle modes can be simply written as a combination of renormalized particle modes (i.e., Gamow modes).

The derivation followed here is similar to the derivation followed in Ref. [22], where we used the Λ transformation to define dressed unstable states in quantum mechanics. The only difference is that in [22] the relation $r + r^* = 1$ was derived from the requirement that the dressed unstable state has an energy fluctuation of the order of the inverse lifetime. This fluctuation is a quantum effect. Here we are dealing with classical mechanics, so we postulate $r + r^* = 1$ as a basic condition. An alternative derivation, presented in Appendix A of Ref. [25], started with the analogue of Eq. (A.24), as a postulate. All the derivations give the same result (3.71). Note that a condition different from $r + r^* = 1$ would not allow us to express $\Lambda^\dagger q_1^* q_1$ as a combination of Gamow modes only, and it would lead to energy fluctuations different from the inverse lifetime in the quantum case, which would be unphysical.

A.4 Proof of Eq. (3.74)

In this appendix we show that Eq. (3.74) removes all the non-analytic $|c_k|^2$ terms, replacing them by $\xi_k = r c_k^2 + c.c.$ First we derive recursive formulas to calculate $\Lambda^\dagger q_1^{*m} q_1^n$. We start with Eq. (3.74) for $m \geq n$

$$\Lambda^\dagger q_1^{*m} q_1^n = \sum_{l=0}^{n} \frac{m! n!}{(m-l)!(n-l)! l!} \tilde{Q}_1^{*m-l} \tilde{Q}_1^{n-l} Y^l. \tag{A.25}$$

(the $n > m$ case can be calculated by taking the complex conjugate of Eq. (A.25)). We have as well

$$\begin{aligned}&\Lambda^\dagger q_1^{*m+1} q_1^n \\ &= \sum_{l=0}^{n} \frac{(m+1)!n!}{(m+1-l)!(n-l)!l!} \tilde{Q}_1^{*m+1-l} \tilde{Q}_1^{n-l} Y^l. \end{aligned} \tag{A.26}$$

For $l > 0$ we have the identity

$$\frac{(m+1)!}{(m+1-l)!l!} = \frac{m!}{(m-l)!l!} + \frac{m!}{(m+1-l)!(l-1)!}. \tag{A.27}$$

Inserting this in Eq. (A.26) we get

$$\begin{aligned}&\Lambda^\dagger q_1^{*m+1} q_1^n = \tilde{Q}_1^{*m+1} \tilde{Q}_1^n \\ &+ \sum_{l=1}^{n} \Big[\frac{m!}{(m-l)!l!} + \frac{m!}{(m+1-l)!(l-1)!} \Big] \\ &\times \frac{n!}{(n-l)!} \tilde{Q}_1^{*m+1-l} \tilde{Q}_1^{n-l} Y^l. \end{aligned} \tag{A.28}$$

The first term plus the second term give

$$(\Lambda^\dagger q_1^{*m} q_1^n) \Lambda^\dagger q_1^* \tag{A.29}$$

(note that $\Lambda^\dagger q_1^* = \tilde{Q}_1^*$). The third term may be written as (with $l' = l - 1$)

$$\begin{aligned}&\sum_{l'=0}^{n-1} \frac{m!}{(m-l')!(l')!} \frac{n(n-1)!}{(n-l'-1)!} \tilde{Q}_1^{*m-l'} \tilde{Q}_1^{n-l'-1} Y^{l'+1} \\ &= nY \Lambda^\dagger q_1^{*m} q_1^{n-1}. \end{aligned} \tag{A.30}$$

Therefore

$$\begin{aligned}&\Lambda^\dagger q_1^{*m+1} q_1^n \\ &= (\Lambda^\dagger q_1^{*m} q_1^n) \Lambda^\dagger q_1^* + nY \Lambda^\dagger q_1^{*m} q_1^{n-1} \quad (m \geq n). \end{aligned} \tag{A.31}$$

For $m > n$ we have, from Eq. (A.25),

$$\begin{aligned}\Lambda^\dagger q_1^{*m} q_1^{n+1} &= \sum_{l=0}^{n} \frac{m!(n+1)!}{(m-l)!(n+1-l)!l!} \\ &\times \tilde{Q}_1^{*m-l} \tilde{Q}_1^{n+1-l} Y^l. \end{aligned} \tag{A.32}$$

Using Eq. (A.27) we get

$$\begin{aligned}&\Lambda^\dagger q_1^{*m} q_1^{n+1} \\ &= \tilde{Q}_1^{*m} \tilde{Q}_1^{n+1} + \frac{m!}{(m-n-1)!} \tilde{Q}_1^{*m-n-1} Y^{n+1} \\ &+ \sum_{l=1}^{n} \frac{m!}{(m-l)!} \Big[\frac{n!}{(n-l)!l!} + \frac{n!}{(n+1-l)!(l-1)!} \Big] \\ &\times \tilde{Q}_1^{*m-l} \tilde{Q}_1^{n+1-l} Y^l. \end{aligned} \tag{A.33}$$

Adding the first and the third terms we get

$$(\Lambda^\dagger q_1^{*m} q_1^n) \Lambda^\dagger q_1 \tag{A.34}$$

(note that $\Lambda^\dagger q_1 = \tilde{Q}_1$). Adding the second and fourth terms we get (with $l' = l - 1$)

$$\begin{aligned}&\sum_{l'=0}^{n} \frac{m(m-1)!}{(m-l'-1)!} \frac{n!}{(n-l')!(l')!} \tilde{Q}_1^{*m-l'-1} \tilde{Q}_1^{n-l'} Y^{l'+1} \\ &= mY \Lambda^\dagger q_1^{*m-1} q_1^n. \end{aligned} \tag{A.35}$$

Therefore

$$\Lambda^\dagger q_1^{*m} q_1^{n+1} \tag{A.36}$$
$$= (\Lambda^\dagger q_1^{*m} q_1^n)\Lambda^\dagger q_1 + mY\Lambda^\dagger q_1^{*m-1} q_1^n \quad (m > n).$$

Eqs. (A.31) and (A.36) plus their complex conjugates

permit one to construct $\Lambda^\dagger q_1^{*m} q_1^n$ recursively.

Now we prove the analyticity of $\Lambda^\dagger q_1^{*m} q_1^n$ at $\lambda = 0$ from the recursive relations. In the recursive relation, we show that if the lower order terms in m and n like $\Lambda^\dagger q^{*m} q_1^n$, $\Lambda^\dagger q_1^{*m} q_1^{n-1}$ and $\Lambda^\dagger q_1^{*m-1} q_1^{n-1}$ are analytic, then the higher order terms $\Lambda^\dagger q^{*m+1} q_1^n$ and $\Lambda^\dagger q_1^{*m} q_1^n$ are also analytic. Then from mathematical induction,

the analyticity of $\Lambda^\dagger q_1^{*m} q_1^n$ is proved for general m and n (the $m < n$ case can be shown in the same way). In Eq. (A.31),

$$\begin{aligned}
&\Lambda^\dagger q_1^{*m+1} q_1^n = (\Lambda^\dagger q^{*m} q_1^n)\Lambda^\dagger q_1^* + nY\Lambda^\dagger q_1^{*m} q_1^{n-1} \\
&= \left(\sum_{l=0}^{n} \frac{m!n!}{(m-l)!(n-l)!l!} \tilde{Q}_1^{*m-l} \tilde{Q}_1^{n-l} Y^l \right) \tilde{Q}_1^* \\
&+ nY \left(\sum_{l=0}^{n-1} \frac{m!(n-1)!}{(m-l)!(n-1-l)!l!} \tilde{Q}_1^{*m-l} \tilde{Q}_1^{n-1-l} Y^l \right)
\end{aligned} \tag{A.37}$$

Suppose that the quantities inside large parenthesis are analytic in λ. The additional non-analytic terms appear whenever additional products $\tilde{Q}_1^* \tilde{Q}_1$ ap-

pear. Since

$$\tilde{Q}_1 = N_1^{1/2}(q_1 + \lambda \sum_k c_k q_k), \tag{A.38}$$

each $\tilde{Q}_1^* \tilde{Q}_1$ produces a $|c_k|^2$ term, which is non-analytic in λ. Let us denote the non-analytic part of a function $f(\lambda)$ as $Fn(f(\lambda))$.

The non-analytic part in the first term in the right hand side of Eq. (A.37) is made by the additional $\tilde{Q}_1^*$ multiplied by $\tilde{Q}_1^{n-l}$, which generates $n-l$ terms $|c_k|^2$:

$$\begin{aligned}
&Fn\left[(\sum_{l=0}^{n} \frac{m!n!}{(m-l)!(n-l)!l!} \tilde{Q}_1^{*m-l} \tilde{Q}_1^{n-l} Y^l) \tilde{Q}_1^*\right] \\
&= \sum_{l=0}^{n} \frac{m!n!}{(m-l)!(n-l)!l!} \tilde{Q}_1^{*m-l} \tilde{Q}_1^{n-l-1} Y^l \\
&\times (n-l)\lambda^2 |N_1| \sum_k |c_k|^2 q_k^* q_k \\
&= \sum_{l=0}^{n-1} \frac{m!n!}{(m-l)!(n-l-1)!l!} \tilde{Q}_1^{*m-l} \tilde{Q}_1^{n-l-1} Y^l \\
&\times \lambda^2 |N_1| \sum_k |c_k|^2 q_k^* q_k.
\end{aligned} \tag{A.39}$$

The non-analytic part in the second term in the right hand side of Eq. (A.37) is coming from Y. Since

$$\begin{aligned}
Y &= \sum_k b_k q_k^* q_k \\
&= \sum_k \lambda^2 |N_1| (-|c_k|^2 + r c_k^2 + r^* c_k^{*2}) q_k^* q_k,
\end{aligned} \tag{A.40}$$

the non-analytic function $|c_k|^2$ appears inside Y.

$$
\begin{aligned}
&Fn\left[nY\sum_{l=0}^{n-1}\frac{m!(n-1)!}{(m-l)!(n-1-l)!l!}\tilde{Q}_1^{*m-l}\tilde{Q}_1^{n-1-l}Y^l\right] \\
&= -\lambda^2|N_1|\sum_k |c_k|^2 q_k^* q_k \\
&\times\sum_{l=0}^{n-1}\frac{m!n!}{(m-l)!(n-1-l)!a!}\tilde{Q}_1^{*m-l}\tilde{Q}_1^{*n-1-l}Y^l. \qquad \text{(A.41)}
\end{aligned}
$$

The non-analytic parts from the first term and second term in Eq. (A.36) exactly cancels out. So, the left hand side of Eq. (A.36) is analytic in λ.

Next, we show that the left hand side of Eq. (A.36) is analytic in λ. The non-analytic part of the first term in the right hand side of Eq. (A.36) is

$$
\begin{aligned}
&Fn\left[(\Lambda^\dagger q_1^{*m}q_1^{n-1})\Lambda^\dagger q_1\right] \\
&= Fn\left[(\sum_{l=0}^{n-1}\frac{m!(n-1)!}{(m-l)!(n-1-l)!l!}\tilde{Q}_1^{*m-l}\tilde{Q}_1^{n-l-1}Y^l)\tilde{Q}_1\right] \\
&= \sum_{l=0}^{n-1}\frac{m!(n-1)!}{(m-l)!(n-1-l)!l!}\tilde{Q}_1^{*m-l-1}\tilde{Q}_1^{n-l-1}Y^l \\
&\times(m-l)\lambda^2|N_1|\sum_k |c_k|^2 q_k^* q_k \\
&= \sum_{l=0}^{n-1}\frac{m!(n-1)!}{(m-l-1)!(n-1-l)!l!}\tilde{Q}_1^{*m-l-1}\tilde{Q}_1^{n-l-1}Y^l \\
&\times\lambda^2|N_1|\sum_k |c_k|^2 q_k^* q_k \qquad \text{(A.42)}
\end{aligned}
$$

The non-analytic part of the second term in the right hand side of Eq. (A.36) is

$$
\begin{aligned}
& Fn\left[mY\Lambda^{\dagger}q_1^{*m-1}q_1^{n-1}\right] \qquad (A.43)\\
&= -m\lambda^2|N_1|\sum_k |c_k|^2 q_k^* q_k \\
&\times \sum_{l=0}^{n-1} \frac{(m-1)!(n-1)!}{(m-l-1)!(n-l-1)!l!}\tilde{Q}_1^{*m-1-l}\tilde{Q}_1^{n-1-l}Y^l \\
&= -\lambda^2|N_1|\sum_k |c_k|^2 q_k^* q_k \\
&\times \sum_{l=0}^{n-1} \frac{m!(n-1)!}{(m-l-1)!(n-l-1)!l!}\tilde{Q}_1^{*m-1-l}\tilde{Q}_1^{n-1-l}Y^l.
\end{aligned}
$$

Again, the non-analytic parts of the first and second terms of Eq. (A.36) exactly cancel out. The right hand side of Eq. (A.36) is analytic in λ.

Therefore from the mathematical induction $\Lambda^{\dagger}q_1^{*m}q_1^n$ is analytic in λ.

A.5 Gaussian property of complex noise

In this section we show the Gaussian property of $R(t)$, Eq. (4.15).

$$
\begin{aligned}
& \langle R^*(t_1)...R^*(t_m)R(t'_1)...R(t'_n)\rangle \\
&= \delta_{mn} \sum_{\text{all pairs}} \langle R^*(t_{i_1})R(t'_{j_1})\rangle\langle R^*(t_{i_2})R(t'_{j_2})\rangle \\
&...\langle R^*(t_{i_n})R(t'_{j_n})\rangle. \qquad (A.44)
\end{aligned}
$$

This property can be proved directly using the Gaussian properties of $A(t)$ and $B(t)$ and using the fact that for $\langle A(t_1)....A(t_{2n})\rangle$ the number of sets of products

of possible pairs is $(2n-1)(2n-3)...3\cdot 1 = (2n)!/(2^n n!)$ and similarly for $B(t)$. Here we follow a simpler derivation using the polar coordinates representation of the complex noise. We write $R(t)$ as

$$R(t) = \tilde{A}(t) + i\tilde{B}(t) = S(t)e^{i\alpha(t)}, \tag{A.45}$$

$$\tilde{A}(t) = \sqrt{\frac{\tilde{m}\tilde{\omega}_1}{2}}A(t), \quad \tilde{B}(t) = \sqrt{\frac{1}{2\tilde{m}\tilde{\omega}_1}}B(t). \tag{A.46}$$

where $S(t)$, $\alpha(t)$ are real. First we show that if the following properties hold for $S(t)$ and $\alpha(t)$, then all the Gaussian white noise properties of $\tilde{A}(t)$ and $\tilde{B}(t)$ are reproduced.

(1) $S(t)$ is Gaussian white noise, i.e.

$$\langle S(t_1)S(t_2)\rangle = \tilde{m}\tilde{\omega}_1 A_c^2 \delta(t_1 - t_2) = \langle R^*(t_1)R(t_2)\rangle, \tag{A.47}$$

$$\begin{aligned}&\langle S(t_1)...S(t_{2n})\rangle \\ &= \sum_{\text{all pairs}} \langle S(t_{i_1})S(t_{i_2})\rangle \cdots \langle S(t_{i_{2n-1}})S(t_{i_{2n}})\rangle\end{aligned} \tag{A.48}$$

Due to the properties of α_k (see Eq. (A.50) below), the odd number products of $S(t)$ play no role when we calculate Eq. (A.44)

(2) $\alpha(t)$ is a random variable in $[-\pi, \pi]$ independent of $S(t)$ satisfying

the relations

$$\langle \cos[\alpha(t_1)] ... \cos[\alpha(t_{2n+1})] \rangle = 0, \tag{A.49}$$

$$\langle \sin[\alpha(t_1)] ... \sin[\alpha(t_{2n+1})] \rangle = 0, \tag{A.50}$$

$$\langle \cos^2[\alpha(t)] \rangle = \langle \sin^2[\alpha(t)] \rangle = \frac{1}{2}, \tag{A.51}$$

$$\langle f(\alpha(t_1)) g(\alpha(t_2)) \rangle = \langle f(\alpha(t_1)) \rangle \langle g(\alpha(t_2)) \rangle \quad \text{for } t_1 \neq t_2, \tag{A.52}$$

where f and g are arbitrary functions.

For

$$\tilde{A}(t) = S(t) \cos \alpha(t), \tag{A.53}$$

we have

$$\begin{aligned} &\langle \tilde{A}(t_1) ... \tilde{A}(t_{2n+1}) \rangle \\ &= \langle S(t_1) ... S(t_{2n+1}) \rangle \langle \cos \alpha(t_1) ... \cos \alpha(t_{2n+1}) \rangle = 0, \end{aligned} \tag{A.54}$$

which comes from the property (2). Also we have

$$
\begin{aligned}
&\langle \tilde{A}(t_1)...\tilde{A}(t_{2n})\rangle \\
&= \langle S(t_1)...S(t_{2n})\rangle \langle \cos\alpha(t_1)...\cos\alpha(t_{2n})\rangle \\
&= \sum_{\text{all pairs}} \langle S(t_{i_1})S(t_{i_2})\rangle ...\langle S(t_{i_{2n-1}})S(t_{i_{2n}})\rangle \\
&\times \langle \cos\alpha(t_1)...\cos\alpha(t_{2n})\rangle \\
&= \sum_{\text{all pairs}} \tilde{m}\tilde{\omega}_1 A_c^2 \delta(t_{i_1}-t_{i_2})...\tilde{m}\tilde{\omega}_1 A_c^2 \delta(t_{i_{2n-1}}-t_{i_{2n}}) \\
&\times \langle \cos^2\alpha(t_1)\rangle \langle \cos^2\alpha(t_3)\rangle ...\langle \cos^2\alpha(t_{2n-1})\rangle \\
&= \sum_{\text{all pairs}} \tilde{m}\tilde{\omega}_1 A_c^2 \delta(t_{i_1}-t_{i_2})...\tilde{m}\tilde{\omega}_1 A_c^2 \delta(t_{i_{2n-1}}-t_{i_{2n}}) \\
&\times \frac{1}{2}\times\frac{1}{2}...\times\frac{1}{2} \\
&= \sum_{\text{all pairs}} \langle \tilde{A}(t_{i_1})\tilde{A}(t_{i_2})\rangle ...\langle \tilde{A}(t_{i_{2n-1}})\tilde{A}(t_{i_{2n}})\rangle
\end{aligned}
\tag{A.55}
$$

So the Gaussian white noise properties of $\tilde{A}(t)$ are recovered. We can also show that the same properties can be derived for $\tilde{B}(t) = S(t)\sin(t)$. Now Eq. (A.44) follows immediately, since

$$
\begin{aligned}
&\langle R^*(t_1)...R^*(t_m)R(t'_1)...R(t'_n)\rangle \\
&= \langle S(t_1)...S(t_m)S(t'_1)...S(t'_n)\rangle \\
&\times \langle e^{-i[\alpha(t_1)+...+\alpha(t_m)-\alpha(t'_1)-...-\alpha(t'_n)]}\rangle .
\end{aligned}
\tag{A.56}
$$

The angle average is non-zero only when

$$
\alpha(t_1)+...+\alpha(t_m)-\alpha(t'_1)-...-\alpha(t'_n)=0. \tag{A.57}
$$

This is possible only when $m = n$ and

$$t_{i_1} = t'_{j_1}, ..., t_{i_m} = t'_{j_m}. \tag{A.58}$$

Hence only the pairings of $S(t_i)$ with $S(t'_j)$ give non-vanishing contributions. This leads to Eq. (A.44).

A.6 Calculation of the noise constants A_c and B_c

In this appendix we determine the noise constants A_c and B_c. We assume that the noises $A(t)$ and $B(t)$ come from the thermal bath with temperature T. In this case, we expect that the system reaches thermal equilibrium for $t \to \infty$. Furthermore, from the equipartition theorem we expect that

$$\frac{1}{2}\tilde{m}\tilde{\omega}_1^2\langle x_1^2\rangle_{eq} = \frac{\langle p_1^2\rangle_{eq}}{2\tilde{m}} = \frac{1}{2}k_B T, \tag{A.59}$$

where k_B is Boltzmann's constant. Substituting the relations

$$\begin{aligned} x_1(t) &= \sqrt{\frac{1}{2\tilde{m}\tilde{\omega}_1}}(q_L(t) + q_L^*(t)), \\ p_1(t) &= -i\sqrt{\frac{\tilde{m}\tilde{\omega}_1}{2}}(q_L(t) - q_L^*(t)) \end{aligned} \tag{A.60}$$

into Eq. (A.59), we get the conditions

$$\langle q_L^2(t)\rangle_{eq} + \langle q_L^{*2}(t)\rangle_{eq} = 0, \tag{A.61}$$

$$\tilde{\omega}_1\langle q_L^*(t) q_L(t)\rangle_{eq} = k_B T. \tag{A.62}$$

For

$$\begin{aligned}\langle q_L^2(t)\rangle &= \langle (q_{La}(t)+q_{Lr}(t))^2\rangle \\ &= \langle (q_{La}^2(t)+2q_{La}(t)q_{Lr}(t)+q_{Lr}^2(t)\rangle \\ &= q_{La}^2(t)+\langle q_{Lr}^2(t)\rangle \end{aligned} \tag{A.63}$$

and

$$\begin{aligned}&\langle q_{Lr}^2(t)\rangle \\ &= \langle e^{-2iz_1t}\int_0^t\int_0^t dt_1dt_2\frac{\tilde{m}\tilde{\omega}_1}{2}(A(t_1)+i\frac{B(t_1)}{\tilde{m}\tilde{\omega}_1}) \\ &\times(A(t_2)+i\frac{B(t_2)}{\tilde{m}\tilde{\omega}_1})e^{iz_1(t_1+t_2)}\rangle \\ = \;& e^{-2iz_1t}\int_0^t dt_1\frac{\tilde{m}\tilde{\omega}_1}{2}(A_c^2-\frac{B_c^2}{(\tilde{m}\tilde{\omega}_1)^2})e^{2iz_1t_2} \\ = \;& \frac{\tilde{m}\tilde{\omega}_1}{2}(A_c^2-\frac{B_c^2}{(\tilde{m}\tilde{\omega}_1)^2})\frac{(1-e^{-2iz_1t})}{2iz_1}, \end{aligned} \tag{A.64}$$

we have

$$\begin{aligned}\langle q_L^2(t)\rangle_{eq} &= \lim_{t\to\infty}(\langle q_{La}^2(t)\rangle+\langle q_{Lr}^2(t)\rangle) \\ &= \lim_{t\to\infty}\langle q_{Lr}^2(t)\rangle = \frac{\tilde{m}\tilde{\omega}_1}{2}(A_c^2-\frac{B_c^2}{(\tilde{m}\tilde{\omega}_1)^2})\frac{1}{2iz_1}. \end{aligned} \tag{A.65}$$

From Eq. (A.61) we get

$$A_c^2 = \frac{B_c^2}{(\tilde{m}\tilde{\omega}_1)^2}. \tag{A.66}$$

On the other hand we have

$$\begin{aligned}
&\langle q_L^*(t) q_L(t) \rangle_{eq} \\
&= \lim_{t\to\infty} (\langle q_{La}^*(t) q_{La}(t) \rangle + \langle q_{Lr}^*(t) q_{Lr}(t) \rangle) \\
&= \lim_{t\to\infty} \langle q_{Lr}^*(t) q_{Lr}(t) \rangle,
\end{aligned} \tag{A.67}$$

$$\begin{aligned}
&\langle q_{La}^*(t) q_{Lr}(t) \rangle \\
&= \langle e^{-2\gamma t} \int_0^t \int_0^t dt_1 dt_2 \frac{\tilde{m}\tilde{\omega}_1}{2} (A(t_1) + i\frac{B(t_1)}{\tilde{m}\tilde{\omega}_1}) \\
&\times (A(t_2) - i\frac{B(t_2)}{\tilde{m}\tilde{\omega}_1}) e^{iz_1 t_1 - 1 z_1^* t_2)} \rangle \\
&= e^{-2\gamma t} \int_0^t dt_1 \frac{\tilde{m}\tilde{\omega}_1}{2} (A_c^2 + \frac{B_c^2}{(\tilde{m}\tilde{\omega}_1)^2} e^{2\gamma t_1}) \\
&= \frac{\tilde{m}\tilde{\omega}_1 A_c^2 (1 - e^{-2\gamma t})}{2\gamma}.
\end{aligned} \tag{A.68}$$

Substituting this to Eq. (A.62), we get

$$A_c^2 = \frac{2\gamma k_B T}{\tilde{m}\tilde{\omega}_1^2}. \tag{A.69}$$

A.7 Calculation of the moments

In this appendix we calculate the moments in Eq. (4.35) for $m \geq n$. The case $m < n$ can be calculated in a similar way. We have

$$\int d\Gamma (q_1^* - q_1^{'*})^m (q_1 - q_1')^n \tilde{\theta}(\Gamma)\delta(\Gamma - \Gamma') \tag{A.70}$$
$$= \int d\Gamma [\tilde{\theta}^\dagger(\Gamma)(q_1^* - q_1^{'*})^n (q_1 - q_1')^m]^* \delta(\Gamma - \Gamma')$$
$$= -\int d\Gamma [(\Lambda^\dagger)^{-1} L_H \Lambda^\dagger (q_1^* - q_1^{'*})^m (q_1 - q_1')^n] \delta(\Gamma - \Gamma'),$$

where we used the relation $L_H^\dagger = L_H$ and $L_H^* = -L_H$. The quantity inside the brackets in Eq. (A.70) is

$$I = (\Lambda^\dagger)^{-1} L_H \Lambda^\dagger (q_1^* - q_1^{'*})^m (q_1 - q_1')^n$$
$$= -i\frac{d}{dt}(\Lambda^\dagger)^{-1} e^{iL_H t} \Lambda^\dagger (q_1^* - q_1^{'*})^m (q_1 - q_1')^n \mid_{t=0}$$
$$= \sum_{l=0}^{m}\sum_{j=0}^{n} (-q_1^{'*})^l (-q_1')^j \frac{m!n!}{(m-l)!(n-j)!l!j!}$$
$$\times (-i\frac{d}{dt})(\Lambda^\dagger)^{-1} e^{iL_H t} \Lambda^\dagger q_1^{*m-l} q_1^{n-j} \mid_{t=0} \tag{A.71}$$

Using Eq. (4.25), we have

$$I = \sum_{l=0}^{m}\sum_{j=0}^{n} (-q_1^{'*})^l (-q_1')^j \frac{m!n!}{(m-l)!(n-j)!l!j!} (-i\frac{d}{dt})$$
$$\times \sum_{a=0}^{min(m-l,n-l)} \frac{(m-l)!(n-l)!}{(m-l-a)!(n-j-a)!a!}$$
$$\times e^{i((m-l)z_1^* - (n-j)z_1)t} q_1^{*m-l-a} q_1^{n-j-a} Y^a$$
$$\times (e^{2\gamma t} - 1)^a \mid_{t=0} . \tag{A.72}$$

Because of the $(e^{2\gamma t}-1)^a$ term, the only non-vanishing terms in Eq. (A.72) at $t=0$ are for $a=0$ or $a=1$. So the above equation becomes

$$
\begin{aligned}
I &= \sum_{l=0}^{m}\sum_{j=0}^{n}(-q_1^{'*})^l(-q_1')^j\frac{m!n!}{l!j!(m-l)!(n-j)!} \\
&\times(-i\frac{d}{dt})(e^{iz_1^*t}q_1^*)^{m-l}(e^{-iz_1t}q_1)^{n-j}\ |_{t=0} \\
&+\sum_{l=0}^{m-1}\sum_{j=0}^{n-1}(-q_1^{'*})^l(-q_1')^j\frac{m!n!}{l!j!(m-l-1)!(n-j-1)!} \\
&\times(-i\frac{d}{dt})e^{i(z_1-z_1^*)t}(e^{iz_1^*t}q_1^*)^{m-l-1}(e^{-iz_1t}q_1)^{n-j-1}Y \\
&\times(e^{2\gamma t}-1)\ |_{t=0} \\
&=(-i\frac{d}{dt})(e^{iz_1^*t}q_1^*-q_1^{'*})^m(e^{-iz_1t}q_1-q_1')^n\ |_{t=0} \\
&+(-i\frac{d}{dt})mnY(1-e^{-2\gamma t}) \\
&\times(e^{iz_1^*t}q_1^*-q_1^{'*})^{m-1}(e^{-iz_1t}q_1-q_1')^{n-1}\ |_{t=0}\ . \qquad \text{(A.73)}
\end{aligned}
$$

Substituting Eq. (A.73) into Eq. (A.70) and integrating with $\delta(\Gamma-\Gamma')$, we get Eq. (4.35).

A.8 Factorization property

We show the factorization of Eq. (4.39) when $\tilde{\rho}(\Gamma,0)$ has the form

$$
\begin{aligned}
\tilde{\rho}(\Gamma,0) &= f_1(x_1,p_1)\prod_k f_k(x_k,p_k) \\
&= g_1(q_1^*,q_1)\prod_k g_k(q_k^*,q_k) \qquad \text{(A.74)}
\end{aligned}
$$

In Eq. (4.39), by integrating by parts, we can write

$$\begin{aligned}
&\int d\Gamma G(q_1, q_1^*) \frac{\partial^2}{\partial q_1 \partial q_1^*} \sum_k b_k J_k \tilde{\rho}(\Gamma, t) \\
&= \int d\Gamma \frac{\partial^2}{\partial q_1 \partial q_1^*} G(q_1, q_1^*) \sum_k b_k J_k \tilde{\rho}(\Gamma, t) \\
&= \int d\Gamma \left(\frac{\partial^2}{\partial q_1 \partial q_1^*} G(q_1, q_1^*) \right) \sum_k b_k J_k e^{-i\tilde{\theta}t} \tilde{\rho}(\Gamma, 0).
\end{aligned} \tag{A.75}$$

Let us expand

$$\frac{\partial^2}{\partial q_1 \partial q_1^*} G(q_1, q_1^*) = \sum_{m,n} G_{mn} q_1^{*m} q_1^n. \tag{A.76}$$

We have

$$\begin{aligned}
&\int d\Gamma q_1^{*m} q_1^n \sum_k b_k J_k e^{-i\tilde{\theta}t} \tilde{\rho}(\Gamma, 0) \\
&= \int d\Gamma [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n} \sum_k b_k J_k]^* \tilde{\rho}(\Gamma, 0).
\end{aligned} \tag{A.77}$$

Since

$$L_0 \sum_k b_k J_k = 0, \quad L_V \sum_k b_k J_k = O(1/\sqrt{L}) \tag{A.78}$$

and Λ is expressed in terms of L_0 and L_V, $\tilde{\theta} = \Lambda L_H \Lambda^{-1}$ treats $\sum_k b_k J_k$ like constant. Neglecting $O(1/\sqrt{L})$ terms, we can write

$$\int d\Gamma q_1^{*m} q_1^n \sum_k b_k J_k e^{-i\tilde{\theta}t} \tilde{\rho}(\Gamma, 0)$$
$$= \int d\Gamma \sum_k b_k J_k [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^* \tilde{\rho}(\Gamma, 0). \tag{A.79}$$

In Eq. (A.79), $[(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^*$ can be written as (see Eq. (4.25))

$$[(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^* = (\Lambda^\dagger)^{-1} \left(e^{iL_H t} \Lambda^\dagger q_1^{*m} q_1^n \right)$$
$$= \sum_{l=0}^{min(m,n)} \frac{m!n!}{(m-l)!(n-l)!l!}$$
$$\times e^{i(mz_1^* - nz_1)t} q_1^{*m-l} q_1^{n-l} Y^l (e^{2\gamma t} - 1)^l \tag{A.80}$$

Since

$$\sum_k b_k J_k (\sum_k b_k J_k)^l = \sum_k b_k J_k (\sum_{k' \neq k} b_{k'} J_{k'})^l + O(1/L), \tag{A.81}$$

we can write

$$\sum_k b_k J_k [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^*$$
$$= \sum_k b_k J_k [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^*_{f-k} + O(1/L). \tag{A.82}$$

In Eq. (A.82), $[\ ]_{f-k}$ means that we exclude the kth field mode.

With Eq. (A.82) and neglecting $O(1/L)$ terms, Eq. (A.79) becomes

$$\begin{aligned}
&\int d\Gamma \sum_k b_k J_k [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^* \tilde{\rho}(\Gamma, 0) \\
&= \sum_k \int d\Gamma b_k J_k [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^*_{f-k} \tilde{\rho}(\Gamma, 0) \\
&= \sum_k \int d\Gamma b_k J_k [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^*_{f-k} g_1(\Gamma_1) \prod_k g_k(\Gamma_k) \\
&= \sum_k \int d\Gamma_k b_k J_k g_k(\Gamma_k) \\
&\int d\Gamma_{f-k} [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^*_{f-k} g_1(\Gamma_1) \prod_{k' \neq k} g_{k'}(\Gamma_{k'})
\end{aligned} \tag{A.83}$$

where $\Gamma_s = (q_s, q_s^*)$, $s = 1, k$. For any k we have

$$\int d\Gamma_k g(\Gamma_k) = 1. \tag{A.84}$$

Then we can write

$$\begin{aligned}
&\int d\Gamma_k b_k J_k g_k(\Gamma_k) \\
&= \int d\Gamma b_k J_k g_1(\Gamma_1) \prod_k g_k(\Gamma_k) = b_k \langle J_k \rangle, \qquad \text{(A.85)} \\
&\int d\Gamma_{f-k} [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^*_{f-k} g_1(\Gamma_1) \prod_{k' \neq k} g_{k'}(\Gamma_{k'}) \\
&= \int d\Gamma [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^*_{f-k} g_1(\Gamma_1) \prod_{k'} g_{k'}(\Gamma_{k'}) \\
&= \int d\Gamma q_1^{*m} q_1^n e^{-i\tilde{\theta}t} \tilde{\rho}(\Gamma, 0), \qquad \text{(A.86)}
\end{aligned}$$

and Eq. (A.83) can be written as

$$\begin{aligned}
&\int d\Gamma \sum_k b_k J_k [(e^{-i\tilde{\theta}t})^\dagger q_1^m q_1^{*n}]^* \tilde{\rho}(\Gamma, 0) \\
&= \sum_k b_k \langle J_k \rangle \int d\Gamma q_1^{*m} q_1^n e^{-i\tilde{\theta}t} \tilde{\rho}(\Gamma, 0) \\
&= \sum_k b_k \langle J_k \rangle \int d\Gamma q_1^{*m} q_1^n \tilde{\rho}(\Gamma, t). \qquad \text{(A.87)}
\end{aligned}$$

This equation, together with Eqs. (A.75) and (A.76), leads to Eq. (4.39).

A.9 preservation of normal ordering in Λ

We show that L_V (hence Λ) preserves the normal ordering.

We have

$$\begin{aligned}
L_V (a_1^\dagger)^m a_1^n &= [\sum_k \lambda V_k (a_1^\dagger a_k + a_1 a_k^\dagger),\ (a_1^\dagger)^m a_1^n] \\
&= \sum_k \lambda V_k (-n (a_1^\dagger)^m a_1^{n-1} a_k + m (a_1^\dagger)^{m-1} a_1^n a_k^\dagger). \qquad \text{(A.88)}
\end{aligned}$$

So the normal ordering is preserved in L_V.

In classical case we have

$$\begin{aligned}
L_V q_1^{*m} q_1^n &= i\{V, q_1^{*m} q_1^n\} \\
&= \sum_k \lambda V_k (m q_1^{*m-1} q_1^n - n q_1^{*m} q_1^{n-1} q_k). \qquad \text{(A.89)}
\end{aligned}$$

This has the same form as Eq. (A.88), hence in both classical $\Lambda^\dagger q_1^{*m} q_1^n$ and quantum case $\Lambda^\dagger (a_1^\dagger)^m a_1^n$ have the same form.

Appendix B

B.1 Proof of orthogonality of $|\psi_{\vec{k}}\rangle$

In this appendix we prove the orthogonality of $|\psi_{\vec{k}}\rangle$ states. Most part of Appendix B is reproduced from Subbiah's thesis [50].

From the definition of $|\psi_{\vec{k}}\rangle$, we have

$$\begin{aligned}
&\langle\psi_{\vec{k}'}|\psi_{\vec{k}}\rangle = (\langle\vec{k}'|_b - \langle\vec{k}'|A^-(\mu_{\vec{k}'})R_b^-(\mu_{\vec{k}'})) \times (|\vec{k}\rangle_b - R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b) \\
&= \langle\vec{k}'|_b\vec{k}\rangle_b - \langle\vec{k}'|_bA^-(\mu_{\vec{k}'})R_b^-(\mu_{\vec{k}'})|\vec{k}\rangle_b - \langle\vec{k}'|R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b \\
&+\langle\vec{k}'|_bA^-(\mu_{\vec{k}'})R_b^-(\mu_{\vec{k}'})R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b
\end{aligned} \tag{B.1}$$

where $A^{\pm}(\mu_{\vec{k}})$ is defined by

$$A^{\pm}(\mu_{\vec{k}}) = P_{12}[P_{12}R^b(\mu_{\vec{k}} \pm i\epsilon)P_{12}]^{-1}P_{12}. \tag{B.2}$$

Using the fact that

$$R_b^-(\mu_{\vec{k}'})R_b^+(\mu_{\vec{k}}) = \frac{1}{\mu_{\vec{k}'} - \mu_{\vec{k}} - i\epsilon}(R_b^+(\mu_{\vec{k}}) - R_{(}^-\mu_{kv'})), \tag{B.3}$$

the 4th term of Eq. (B.1) becomes

$$\begin{aligned}
&\langle\vec{k}'|_bA^-(\mu_{\vec{k}'})R_b^-(\mu_{\vec{k}'})R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b \\
&= -\frac{1}{\mu_{\vec{k}'} - \mu_{\vec{k}} - i\epsilon}(\langle\vec{k}'|_bA^-(\mu_{\vec{k}'})R_b^-(\mu_{\vec{k}'})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b \\
&-\langle\vec{k}'|_bA^-(\mu_{\vec{k}'})R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b).
\end{aligned} \tag{B.4}$$

Using the definition of $A^{\pm}(\mu_{\vec{k}})$, Eq. (B.4) is again simplified to

$$\begin{aligned}&\langle\vec{k'}|_b A^-(\mu_{\vec{k'}})R_b^-(\mu_{\vec{k'}})R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b \\ &= -\frac{1}{\mu_{\vec{k'}}-\mu_{\vec{k}}-i\epsilon}(\langle\vec{k'}|_b A^+(\mu_{\vec{k}})|\vec{k}\rangle_b - \langle\vec{k'}|_b A^-(\mu_{\vec{k'}})|\vec{k}\rangle_b). \end{aligned} \tag{B.5}$$

Substituting Eq. (B.5) into Eq. (B.1), then from the definition of $R_b^{\pm}$ the second and third terms of Eq. (B.1) exactly cancel out with the 4th term. So we get the result

$$\langle\psi_{\vec{k'}}|\psi_{\vec{k}}\rangle = \delta_{\vec{k}\vec{k'}}. \tag{B.6}$$

B.2 Proof of $Q_a = \sum_{\vec{k}} |\psi_{\vec{k}}\rangle\langle\psi_{\vec{k}}|$

We prove the completeness of $|\psi_{\vec{k}}\rangle$ in Q_a subspace. First we show the identity

$$Q_a R_Q(z)|\vec{k}\rangle_b = \frac{1}{z-\mu_{\vec{k}}}|\psi_{\vec{k}}(z)\rangle \tag{B.7}$$

where $R_Q(z)$ is defined by

$$R_Q(z) \equiv \frac{1}{z - Q_a H_b Q_a}. \tag{B.8}$$

To prove Eq. (B.7), we consider the identity

$$I_2 \equiv (z - Q_a H_b Q_a) P_b R_b(z)(1 - A(z)R_b(z)) \tag{B.9}$$

where

$$A(z) = P_{12}[P_{12}R_b(z)P_{12}]^{-1}P_{12}. \tag{B.10}$$

Using $Q_a = P_b - P_{12}$, $R_b(z) = (z - H_b)^{-1}$ and

$$P_{12}P_bR_b(z)(1 - A(z)R_b(z)) = 0, \tag{B.11}$$

I_2 becomes

$$\begin{aligned} I_2 &= (z - H_b + P_{12}H_b)P_bR_b(z)(1 - A(z)R_b(z)) \\ &= P_b(1 - A(z)R_b(z)) + P_{12}H_bP_bR_b(z)(1 - A(z)R_b(z)) \\ &= P_b(1 - A(z)R_b(z)) - P_{12}(z - H_b)P_bR_b(z)(1 - A(z)R_b(z)) \\ &= P_b(1 - A(z)R_b(z)) - P_{12}(1 - A(z)R_b(z)) \\ &= P_b - P_{12} = Q_a \end{aligned} \tag{B.12}$$

In Eq. (B.12) we used the fact

$$P_bA(z) = P_bP_{12}A(z) = P_{12}A(z). \tag{B.13}$$

From Eq. (B.12), Eq. (B.9) is proved, and from Eq. (B.9) we get Eq. (B.7).

Now we prove the completeness in Q_a space. From Eq. (B.7) we have

$$\frac{\epsilon}{\pi}Q_aR_Q^+(E)|\vec{k}\rangle_b\langle\vec{k}|_bR_Q^-(E)Q_a = \delta(E - \mu_{\vec{k}})|\psi_{\vec{k}}\rangle\langle\psi_{\vec{k}}| \tag{B.14}$$

where we used the relation

$$\delta(x) = \frac{1}{\pi}\frac{\epsilon}{x^2 + \epsilon^2} \tag{B.15}$$

for the positive infinitesimal ϵ. Taking the summation over $\vec{k}$ on both sides of Eq. (B.14) and integrating over E, we get

$$Q_a = \int dE\frac{\epsilon}{\pi}Q_aR_Q^+(E)R_Q^-(E)Q_a = \sum_{\vec{k}}|\psi_{\vec{k}}\rangle\langle\psi_{\vec{k}}| \tag{B.16}$$

where we used Eq. (B.15) again for the first equality.

B.3 Proof of $\langle\psi_{\vec{k}'}|H_b|\psi_{\vec{k}}\rangle$

We prove the relation

$$\langle\psi_{\vec{k}'}|H_b|\psi_{\vec{k}}\rangle = \mu_{\vec{k}}\delta_{\vec{k}\vec{k}'}. \tag{B.17}$$

First, we note that

$$H_bP_b = P_bH_b = H_b. \tag{B.18}$$

From the definition of $|\psi_{\vec{k}}\rangle$, we have

$$H_b|\psi_{\vec{k}}\rangle = H_bP_b|\vec{k}\rangle_b - H_bR_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b \tag{B.19}$$

where $A^\pm(E)$ is defined in Eq. (B.2). Using Eq. (B.18) and adding and subtracting $\mu_{\vec{k}}$ term, we get

$$\begin{aligned} H_b|\psi_{\vec{k}}\rangle = {} & \mu_{\vec{k}}P_b|\vec{k}\rangle_b - \mu_{\vec{k}}R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b \\ & -(H_b - \mu_{\vec{k}})R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b. \end{aligned} \tag{B.20}$$

Since $\mu_{\vec{k}} - H_b$ is the inverse of $R_b^+(\mu_{\vec{k}})$, Eq. (B.20) is changed to

$$H_b|\psi_{\vec{k}}\rangle = \mu_{\vec{k}}P_b|\vec{k}\rangle_b - \mu_{\vec{k}}R_b^+(\mu_{\vec{k}})A^+(\mu_{\vec{k}})|\vec{k}\rangle_b + A^+(\mu_{\vec{k}})|\vec{k}\rangle_b. \tag{B.21}$$

Using the definition of $|\psi_{\vec{k}}\rangle$, the first two terms of Eq. (B.21) can be combined to give

$$H_b|\psi_{\vec{k}}\rangle = \mu_{\vec{k}}|\psi_{\vec{k}}\rangle + A^+(\mu_{\vec{k}})|\vec{k}\rangle. \tag{B.22}$$

Since $\langle\psi_{\vec{k}}|P_{12}=0$, from the definition of A we have

$$\langle\psi_{\vec{k}'}|A^{+}(\mu_{\vec{k}})|\psi_{\vec{k}}\rangle=0. \tag{B.23}$$

Therefore, we get

$$\langle\psi_{\vec{k}'}|H_b|\psi_{\vec{k}}\rangle=\mu_{\vec{k}}\delta_{\vec{k}\vec{k}'}. \tag{B.24}$$

B.4 Proof of the relations $\langle\vec{j}|_a H_T|\vec{j}'\rangle_a=\omega_{\vec{j}}\delta_{\vec{j}\vec{j}'}$, $\langle\psi_{\vec{k}}|H_T|\psi_{\vec{k}'}\rangle=\mu_{\vec{k}\vec{k}'}\delta_{\vec{k}\vec{k}'}$.

First we prove the relation

$$\langle\vec{j}|_a H_T|\vec{j}'\rangle_a=\omega_{\vec{j}}\delta_{\vec{j}\vec{j}'}. \tag{B.25}$$

In $\vec{r}$ representation, we have

$$\langle\vec{j}|_a H_T|\vec{j}'\rangle_a=\int d\vec{r}\langle\vec{j}|_a\vec{r}\rangle(-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r}))\langle\vec{r}|\vec{j}'\rangle_a. \tag{B.26}$$

We write

$$\langle\vec{j}|_a\vec{r}\rangle=\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r}) \tag{B.27}$$

$$\langle\vec{r}|\vec{j}'\rangle_a=\langle\vec{r}|\vec{j}'\rangle\Theta_{\Omega_a}(\vec{r}) \tag{B.28}$$

where $\langle\vec{j}|\vec{r}\rangle$ and $\langle\vec{r}|\vec{j}'\rangle$ are the analytically continued functions of $\langle\vec{j}|_a\vec{r}\rangle$ and $\langle\vec{r}|\vec{j}'\rangle_a$ outside their supports. Note that $\langle\vec{j}|\vec{r}\rangle$ and $\langle\vec{r}|\vec{j}'\rangle$ are still zero at the

boundary. Then Eq. (B.26) becomes

$$\begin{aligned}
&\int d\vec{r}\langle\vec{j}|_a\vec{r}\rangle(-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r}))\langle\vec{r}|\vec{j'}\rangle_a \\
&=\int d\vec{r}\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r}))\langle\vec{r}|\vec{j'}\rangle\Theta_{\Omega_a}(\vec{r}) \\
&=\int d\vec{r}\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r}))\langle\vec{r}|\vec{j'}\rangle \\
&-\frac{\hbar^2}{2m}\int d\vec{r}\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(-\frac{\hbar^2}{2m})(\nabla\langle\vec{r}|\vec{j'}\rangle)\cdot(\nabla\Theta_{\Omega_a}(\vec{r})) \\
&-\frac{\hbar^2}{2m}\int d\vec{r}\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(\langle\vec{r}|\vec{j'}\rangle)(\nabla^2\Theta_{\Omega_a}(\vec{r})). \qquad \text{(B.29)}
\end{aligned}$$

In Eq. (B.29), $\nabla\Theta_{Ome_a}(\vec{r})$ gives delta function at the boundary, and $\nabla^2\Theta_{\Omega_a}(\vec{r})$ gives the derivative of the delta function at the boundary.

The first term in Eq. (B.29) gives

$$\begin{aligned}
&\int d\vec{r}\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(-\frac{\hbar^2}{2m}\nabla^2+V(\vec{r}))\langle\vec{r}|\vec{j'}\rangle \\
&=\int d\vec{r}\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(\omega_{\vec{j'}})\langle\vec{r}|\vec{j'}\rangle=\omega_{\vec{j'}}\delta_{\vec{j}\vec{j'}}. \qquad \text{(B.30)}
\end{aligned}$$

The second term in Eq. (B.29) becomes

$$\int d\vec{r}\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(-\frac{\hbar^2}{2m})(\nabla\langle\vec{r}|\vec{j'}\rangle)\cdot(\nabla\Theta_{\Omega_a}(\vec{r}))=0 \qquad \text{(B.31)}$$

since $\langle\vec{j}|\vec{r}\rangle$ is zero at the boundary.

The third term in Eq. (B.29) becomes

$$\begin{aligned}
&\int d\vec{r}\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(\langle\vec{r}|\vec{j'}\rangle)(\nabla^2\Theta_{\Omega_a}(\vec{r})) \\
&= -\int d\vec{r}[\nabla(\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})\langle\vec{r}|\vec{j'}\rangle)]\cdot\nabla\Theta_{\Omega_a}(\vec{r}) \\
&= -\int d\vec{r}[(\nabla\langle\vec{j}|\vec{r}\rangle)\Theta_{\Omega_a}(\vec{r})\langle\vec{r}|\vec{j'}\rangle) + \langle\vec{j}|\vec{r}\rangle(\nabla\Theta_{\Omega_a}(\vec{r}))\langle\vec{r}|\vec{j'}\rangle \\
&+\langle\vec{j}|\vec{r}\rangle\Theta_{\Omega_a}(\vec{r})(\nabla\langle\vec{r}|\vec{j'}\rangle)]\cdot\nabla\Theta_{\Omega_a}(\vec{r}) \\
&= 0 \qquad\qquad (B.32)
\end{aligned}$$

since inside the square bracket is zero at the boundary. For this we used the relation

$$\int dx(x\delta(x))(x\delta(x)) = 0. \qquad (B.33)$$

Eq. (B.33) can be shown by substituting delta sequence into Eq. (B.33) or by using the relation

$$\int dx(x-a)(x-b)\delta(x-a)\delta(x-b) = 0\cdot(b-a)\delta(b-a) = 0. \qquad (B.34)$$

Using Eq. (B.30), Eq. (B.31) and Eq. (B.32) we get the desired result

$$\langle\vec{j}|_a H_T|\vec{j'}\rangle_a = \omega_{\vec{j}}\delta_{\vec{j}\vec{j'}}. \qquad (B.35)$$

The relation

$$\langle\psi_{\vec{k}}|H_T|\psi_{\vec{k'}}\rangle = \mu_{\vec{k}}\delta_{\vec{k}\vec{k'}} \qquad (B.36)$$

can be also shown using the same method.

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