Optimal Portfolio Choice: Beyond the Traditional Expected Utility Maximization Paradigm

by

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This thesis focuses on two major portfolio selection approaches: the traditional mean-variance approach and the heuristic approach based on risk budgeting. The main results from mean-variance are reviewed, as well as some novel results, followed by new contributions in the area of calculating expected functionals of the optimal wealth in a log-normal market. The available theory behind the risk budgeting approach is revisited, with the main arguments for and against the approach explained. The equally weighted portfolio, referred to as the risk parity portfolio, is compared against other heuristically derived portfolios and the more traditional mean-variance portfolio.

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Chapter 1

Introduction

This text focuses on the mean-variance and the risk budgeting approaches to portfolio selection, and is divided into three parts.

In the first chapter, the Markowitz approach to portfolio optimization is reviewed, and the basics of expected utility theory are presented, with different findings throughout the academic literature. In addition, some novel results about the probabilistic properties of the optimal wealth process are revisited. The main new contribution presented follows, where a new method of calculating expected functionals of the optimal terminal wealth in log-normal markets with one risky and one risk-free asset is derived.

The second chapter focuses on heuristic, risk-based approaches to portfolio choice. More specifically, the basics of risk budgeting are presented, with basic mathematical formulations. A brief review of the main points of the existing literature is presented. Different existing heuristics of optimizing for the risk budgeting portfolio are included.

The third and last chapter focuses on one specific form of the risk budgeting approach: the equal-risk-contribution portfolio referred to as risk parity. Existing research findings on the effectiveness and comparison of risk parity with other approaches are included, including other heuristic approaches, as well as the traditional mean-variance paradigm.

Chapter 2

Traditional Approaches in Portfolio Theory

2.1 Mean-Variance Optimization

Modern portfolio theory has seen exponential growth after the groundbreaking work by Markowitz (1952a). By taking into account the expected returns and variances and covariances between different assets, Markowitz derives the efficient frontier and optimal allocation to each asset. The expected return is used as the investor seeks to maximize the return of their portfolio. By considering the covariances between assets, the effect of the dynamics between prices of different assets, as well as volatilities of individual assets, are incorporated.

This method, known as the mean-variance approach, remains one of the widely used to this day, despite the availability of other methods that consider higher moments of the terminal wealth (such as Fama (1965)). As Elton and Gruber (1997) point out, the inclusion of higher moments, which adds additional computational and data requirements, is not shown to have a great effect in reaching the investor's goals. The intuitive interpretation of the theory is another reason why Markowitz's work has become the foundation of modern portfolio theory and is extensively used in practice.

2.2 Expected Utility Theory

Different approaches stemming from Markowitz's work have been presented by Sharpe (1964), Merton (1969), Roll (1977), and many others. Among these, one of the most influential is the work of Merton. In his sequence of papers (Merton (1969, 1972, 1975)), the original approach of expected utility maximization is introduced and analyzed in continuous time. Namely, the most basic market considered is one composed of a risk-free and risky asset, where the price dynamics of the latter are described by a geometric Brownian motion. By using a diffusion process to describe the price dynamics, this work opened the door to the extensive mathematical literature on diffusion processes and its use in portfolio optimization. Another very interesting and valuable contribution is the consideration of a general utility function. In the approach proposed by Markowitz and its extensions, the quadratic utility is standard. However, the ability to theoretically assign any utility to the proposed framework makes Merton's approach extremely powerful, both from a theoretical and a practical perspective.

Regardless of whether we consider the original mean-variance optimization proposed by Markowitz, an extension to it, or we consider another approach entirely, the use of utility as a concept is imperative if we are to decide what is the optimal portfolio allocation for a specific investor. The utility function used can be understood as the embodiment of the individual preferences of the investor, and thus the optimal allocations in various assets should heavily rely on the utility function used to hopefully reflect the investor's interests. This is why it is important to understand different utility functions and portfolio optimization using those.

In the original paper, Markowitz (1952a) outlines the mean-variance hypothesis for choosing the optimal portfolio without focusing on the investor's preferences in the form of a utility function. As Constantinides and Malliaris (1995) point out, it is only assumed that there exists a utility function with certain properties: it favors higher expected returns and lower variance of the terminal wealth, as well as some more technical properties, such as monotonicity. More specifically, a quadratic utility is assumed in this work. Later that same year, Markowitz (1952b) presents another paper where he argues, mostly with heuristic arguments, for a concave-convex shape of the utility function as applied to wealth. However, this is not connected immediately to his work on optimal portfolio choice.

It was Merton (1969) who first developed and outlined the expected utility framework while considering a general, strictly concave utility function $U(\cdot)$. In a market with one risk-free and one risky asset whose price is governed by a geometric Brownian motion, he proposes the objective to be $\mathbb{E}[U(X_T)|\mathcal{F}_t]$ (his original writing displays this with consumption and in slightly different notation). It is obvious that now the utility $U(\cdot)$, representing the investor's individual preferences, plays a central role in determining the optimal portfolio, as would be expected.

Following Merton, other researchers have also demonstrated some results in connection to the use of utilities in portfolio theory. Hanoch and

Levy (1970) study the quadratic and cubic utilities, and conclude that the mean-variance criterion and other commonly used procedures are insufficient to guarantee the dominance of a portfolio, and they can be improved. A cubic utility is found to be favorable in some situations. Furthermore, results on extensions of the Merton problem have been derived for certain forms of utility functions. Stoikov and Zariphopoulou (2005) consider incomplete markets and provide explicit solutions for the optimal consumption and investment strategies under a power utility. Kraft (2005) also considers a power utility function, and provides closed-form solutions under the Heston model with stochastic volatility of the price dynamics of the risky asset.

While various stochastic optimization techniques and approaches can be used to solve expected utility problems in continuous time, explicit solutions such as the ones mentioned above are rarely available. In the cases with a special form of the utility function as pointed out above, the problem becomes homogeneous and the value function inherits the same form and is separable in time and space (see, for example, Merton (1969)). Outside these utility classes, only general results exist that either characterize the solution as the solution to the associated Hamilton-Jacobi-Bellman equation or relate it to the solution of the dual problem.

Going beyond the class of homothetic utilities and deriving easier-to-interpret solutions has been an open topic for many years. Recently, Mostovyi et al. (2020) proposed a rather general class of utilities, the ones that have completely monotonic inverse marginals. Such utilities cover the homothetic ones and also approximate very well the general utilities. The complete monotonicity yields an elegant representation via the Laplace transformation of an underlying measure (for example, the power utility corresponds to a Dirac such measure in a finite interval $(1, a)$). In the paper mentioned above, the authors established that under rather weak assumptions in the market, the inverse marginal utilities inherit the complete monotonicity and can be, thus, represented in closed form. Herein, we revisit this class and derive closed-form expressions for expected functionals of the optimal wealth process. Detailed results appear in Lalkov and Zariphopoulou (2020) and are new.

2.3 Log-Normal Markets

The classic portfolio problem, as laid out by Merton (1969), has an investor looking at investing over a period $[0, T]$ with a fixed horizon T. There are two assets to choose from: a risk-free asset (for example, a savings account) and a risky asset (for example, a stock). The risk-free asset offers a constant interest rate $r > 0$, and its price can be described by

$$dB_t = rB_t dt, \tag{2.1}$$

where B_t is the price at time t, $t \geq 0$. On the other hand, the price dynamics of the risky asset, denoted at time t ($t \geq 0$) as S_t, is modeled as a geometric Brownian motion. Let's define a standard Wiener process W_t on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\})$ with $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$ being the corresponding filtration. Then, the stochastic price process for the risky asset mentioned above is of the form

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right), \tag{2.2}$$

where the drift μ and volatility σ are fixed constants and $S_0 > 0$. Additionally, it is useful to define the expression

$$\lambda = \frac{\mu - r}{\sigma}, \tag{2.3}$$

which is well-defined since we assume $\mu, r, \sigma \in \mathbb{R}_+$.

The goal of the investor is to maximize the expected utility of the terminal wealth, denoted by X_T. They start with an initial investment at some time $t \in [0, T)$, and with an initial wealth of $X_t = x$. At any point in time $s \in [t, T]$, the present value of funds invested in the two assets are π_s^0 and π_s for the risk-free and the risky asset, respectively. Hence the total present value of the investment at time t can be expressed as $X_s^\pi = \pi_s^0 + \pi_s$, where the pair (π_s^0, π_s) is taken from the set of admissible strategies $\mathcal{A}$ that are self-financing. Any strategy π_s is said to be admissible if $\pi_s \in \mathcal{F}_s$, $\mathbb{E}_{\mathbb{P}} \left[\int_t^T \pi_s^2 ds \right] < \infty$, and $X_s^\pi \geq 0$, $s \in [t, T]$. With an initial wealth $X_t = x$ and $s \in [t, T]$, the wealth process X_s^π can be expressed as

$$dX_s^\pi = \sigma \pi_s \left(\lambda ds + dW_s \right). \tag{2.4}$$

Representing the preferences of the investor at time T, the utility function at this time is $U : \mathbb{R}_+ \to \mathbb{R}_+$. It is assumed to satisfy the well-known conditions: strict concavity, strictly increasing, $C^4(0, \infty)$,

$$\lim_{x \to 0} U'(x) = \infty \qquad \text{and} \qquad \lim_{x \to \infty} U'(x) = 0. \tag{2.5}$$

The inverse of the marginal utility $U'(\cdot)$ is

$$I(x) = (U')^{(-1)}(x), \tag{2.6}$$

for which we assume that, for some $\varepsilon > 0$, it satisfies

$$I(x) \leq \varepsilon + x^{-\varepsilon}. \tag{2.7}$$

Maximizing the expected utility of terminal wealth, the investor's value function is then of the form

$$u(x,t) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U(X_T^\pi)|X_t^\pi = x\right]. \tag{2.8}$$

For future reference, we define the risk aversion (RA) and risk tolerance (RT) coefficients, respectively, as:

$$RA(x) = -\frac{U'(x)}{U''(x)} \qquad \text{and} \qquad RT(x) = -\frac{U''(x)}{U'(x)}, \tag{2.9}$$

where $x > 0$ and U is the utility at time T, as defined above. The risk tolerance coefficient is assumed to be strictly increasing for $x > 0$ and $RT(0) := \lim_{x \downarrow 0} RT(x) = 0$. Furthermore, for intermediate times $t \in [0,T]$, we define the local (absolute) risk aversion $\gamma(x,t)$ and the local (absolute) risk tolerance $r(x,t)$ as

$$\gamma(x,t) = -\frac{u_{xx}(x,t)}{u_x(x,t)} \qquad \text{and} \qquad r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}, \tag{2.10}$$

where u is the value function as defined in (2.8).

The solution to the problem described by (2.8) has been the focus of research for many scholars, and it has been successfully solved with explicit forms of the optimal policies $\pi_s^{*,x}$ as the result. A unique approach to this problem is provided in the work of Musiela and Zariphopoulou (2010) and Kallblad and Zariphopoulou (2014), whose main results are provided below.

The first result provides the definition of the spatial function associated with the optimal wealth process.

Proposition 2.3.1. *Let $I : \mathbb{R}_+ \to \mathbb{R}_+$ be the inverse terminal utility function as given in (2.6), with the growth condition as provided in (2.7). Furthermore, we define $H : \mathbb{R} \times [0,T] \to \mathbb{R}_+$ by*

$$u_x(H(x,t),t) = \exp\left(-x - \frac{1}{2}\lambda^2(T-t)\right), \tag{2.11}$$

where $u(x,t)$ is the value function as seen in (2.8) and λ is as defined in (2.3). Then, the following hold.

i. The spatial function $H(x,t)$ solves the heat equation

$$H_t + \frac{1}{2}\lambda^2 H_{xx} = 0 \tag{2.12}$$

with the terminal condition

$$H(x,T) = I(e^{-x}). \tag{2.13}$$

ii. The function $H(x,t)$ is strictly increasing for $t \in [0,T]$, and it is of full range: $\lim_{x\to-\infty} H(x,t) = 0$ and $\lim_{x\to\infty} H(x,t) = \infty$.

iii. The local absolute risk tolerance function $r \in C^{2,1}(\mathbb{R}_+ \times (0,T])$, as defined in (2.10), satisfies

$$r(x,t) = H_x\left(H^{(-1)}(x,t),t\right), \tag{2.14}$$

with $H(x,t)$ as the solution to (2.12) under the terminal condition (2.13).

After defining the spatial function $H(x,t)$, now we turn to results involving the spatial inverse, $H^{(-1)}(x,t)$. For proofs about any proposition, see work by Musiela and Zariphopoulou (2010) and Kallblad and Zariphopoulou (2014).

Proposition 2.3.2. *The spatial inverse $H^{(-1)} : \mathbb{R}_+ \times [0,T] \to \mathbb{R}$ satisfies the following:*

$$H_t^{(-1)}(x,t) = \frac{1}{2}\lambda^2 r_x(x,t), \tag{2.15}$$

$$H_x^{(-1)}(x,t) = \gamma(x,t), \tag{2.16}$$

where $r(x,t)$ and $\gamma(x,t)$ are the local absolute risk tolerance function and the local absolute risk aversion functions, as defined in (2.10). Hence, the spatial and temporal increments of the spatial inverse function $H^{(-1)}$, respectively, can be written as

$$H^{(-1)}(y,t) - H^{(-1)}(x,t) = \int_x^y \gamma(z,t)dz \tag{2.17}$$

and

$$H^{(-1)}(x,t) - H^{(-1)}(x,0) = \frac{1}{2}\lambda^2 \int_0^t r_x(x,s)ds. \tag{2.18}$$

Following the results introduced above, the optimal wealth and portfolio processes can be expressed in terms of the spatial function H and the spatial inverse $H^{(-1)}$. This is summarized in the following proposition.

Proposition 2.3.3. *The optimal wealth, $X_t^{*,x}$, $t \in [0,T]$, with $X_{t=0} = x$, and the optimal portfolio process, $\pi_t^{*,x}$, are given by*

$$X_t^{*,x} = H\left(H^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right) \tag{2.19}$$

and

$$\pi_t^{*,x} = \frac{\lambda}{\sigma} H_x\left(H^{(-1)}(X_t^{*,x}, t), t\right) \tag{2.20}$$

$$= \frac{\lambda}{\sigma} H_x\left(H^{(-1)}(x,0) + \lambda^2 t + \lambda W_t, t\right), \tag{2.21}$$

where H is the spatial function and a solution to (2.12) under the terminal condition (2.13).

Now, we present the results of Monin and Zariphopoulou (2014) on the probabilistic properties of the optimal wealth and associated portfolio process. For proofs, please refer to the full text by Monin and Zariphopoulou (2014).

The first result provides the cumulative distribution function and the probability density function for the optimal wealth process.

Proposition 2.3.4. *Let $r(x,t)$ and $\gamma(x,t)$ be the local absolute risk tolerance and local absolute risk aversion, respectively, as defined in (2.10). Also, let λ be as defined in (2.3). The following statements hold.*

i. Denoting the cumulative distribution function of the standard normal distribution by Φ, the cumulative distribution for the optimal wealth process $X_t^{*,x}$ at time t, for $t \in (0,T]$ and $X_0^{*,x} = x$, is of the form

$$P\{X_t^{*,x} \leq y\} = \Phi\left(\frac{1}{\lambda\sqrt{t}} \int_x^y \gamma(z,t)dz + \frac{\lambda}{2\sqrt{t}} \int_0^t r_x(x,s)ds - \lambda\sqrt{t}\right). \tag{2.22}$$

ii. Denoting the standard normal density by ϕ, the density for the optimal wealth process $X_t^{*,x}$ at time t, for $t \in (0, T]$ and $X_0^{*,x} = x$, is given by

$$f(y,t;x,0) = \frac{1}{\lambda\sqrt{t}}\gamma(y,t)\phi\left(\frac{1}{\lambda\sqrt{t}}\int_x^y \gamma(z,t)dz + \frac{\lambda}{2\sqrt{t}}\int_0^t r_x(x,s)ds - \lambda\sqrt{t}\right). \tag{2.23}$$

Corollary 2.3.1. *The optimal wealth at time T, $X_T^{*,x}$, has a cumulative distribution function of the form*

$$P\{X_T^{*,x} \leq y\} = \Phi\left(\frac{1}{\lambda\sqrt{T}}\ln\left(\frac{U'(x)}{U'(y)}\right) + \frac{\lambda}{2\sqrt{T}}\int_0^T r_x(x,s)ds - \lambda\sqrt{T}\right). \tag{2.24}$$

The next result provides the expectation of a functional of the optimal wealth, which was then used by Monin and Zariphopoulou (2014) to provide expressions for the expectation and variance of the optimal terminal wealth. For proofs, please refer to their original work.

Proposition 2.3.5. *Let H be the spatial function as defined in (2.11), (2.12) and (2.13), and let λ be as given in (2.3). Take a function of polynomial growth $g : \mathbb{R}_+ \to \mathbb{R}$ and let $G : \mathbb{R} \times (0,T] \to \mathbb{R}$ be given by*

$$G(x,t) = g\left(H(x,t)\right). \tag{2.25}$$

Then, we have the following, $\forall t \in (0,T]$:

$$\mathbb{E}\left[g(X_t^{*,x})\right] = \left(G(\cdot,t) * \xi(\cdot,t)\right)(z)\Big|_{z=H^{(-1)}(x,0)+\lambda^2 t}, \tag{2.26}$$

where $*$ represents convolution, and ξ is of the form

$$\xi(x,t) = \frac{1}{\sqrt{2\lambda^2\pi t}}e^{-x^2/2\lambda^2\pi t}. \tag{2.27}$$

Corollary 2.3.2. *The expectation and variance of the optimal wealth* $X_t^{*,x}$ *at any time* $t \in (0, T]$ *are respectively given by*

$$\mathbb{E}[X_t^{*,x}] = H\left(H^{(-1)}(x,0) + \lambda^2 t, 0\right) \tag{2.28}$$

and

$$Var(X_t^{*,x}) = v\left(H^{(-1)}(x,0) + \lambda^2 t, t\right), \tag{2.29}$$

where $v : \mathbb{R} \times (0, T] \to \mathbb{R}_+$ *satisfies*

$$v(x,t) = \left(H^2(\cdot, t) * \xi(\cdot, t)\right)(x) - H^2(x, 0), \tag{2.30}$$

and ξ *is as in (2.27).*

This section is concluded with some new results derived by the author and T. Zariphopoulou. The findings presented below are related to the representation and study of expected functionals of the optimal terminal wealth for the rich class of completely monotonic inverse marginal (CMIM) utility functions. For further reading on the results listed below, the reader is referred to Lalkov and Zariphopoulou (2020).

The proposition 2.3.5 provides a way in which one can calculate functionals of the optimal wealth at any given time t. The following result shows a different way of calculating this for the optimal terminal wealth.

Proposition 2.3.6. *Let* $H(\cdot, \cdot)$ *be as outlined in Proposition 2.3.1 and* $H^{(-1)}(\cdot, \cdot)$ *be its spatial inverse, and let* λ *be as in (2.3). Define the following two functions:*

$$F(x,t) \triangleq \mathbb{E}[f(X_T^*) | X_t^* = x], \tag{2.31}$$

$$G(x,t) \triangleq F(H(x,t),t). \tag{2.32}$$

Assume $H(x,T)$ to be absolutely monotonic on $x \in [0,\infty)$ and $f(\cdot)$ to be non-negative and non-decreasing.

Then, there exists some non-negative and bound measure $\nu(\cdot)$, such that

$$\mathbb{E}[f(X_T^*)|X_t^* = x] = \int_0^\infty e^{zH^{(-1)}(x,t)+\lambda^2\left(\frac{1}{2}z^2-z\right)(T-t)} d\nu(z). \tag{2.33}$$

Proof. Using the Feynman-Kac formula, the function $F(\cdot,\cdot)$ as defined above solves a differential equation of the form

$$\begin{cases} F_t(x,t) + \lambda^2 R(x,t) F_x(x,t) + \frac{1}{2}\lambda^2 R^2(x,t) F_{xx}(x,t) = 0 \\ F(x,T) = f(x) \end{cases}, \tag{2.34}$$

for $x \in \mathbb{R}$ and $t \in [0,T]$, with the underlying wealth process (under the same probability measure)

$$dX_t^{*,x} = \lambda^2 R(x,t)dt + \lambda R(x,t) dW_t, \tag{2.35}$$

where

$$R(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)} \tag{2.36}$$

is the local absolute risk tolerance function.

Rewriting the equation (2.34) in terms of $G(\cdot,\cdot)$ as it is defined above, recalling that

$$R(x,t) = H_x(H^{(-1)}(x,t),t), \tag{2.37}$$

and using the fact that $H(\cdot,\cdot)$ is a solution to the equation

$$\begin{cases} H_t + \frac{1}{2}\lambda^2 H_{xx} = 0 \\ H(x,T) = I(e^{-x}) \end{cases}, \tag{2.38}$$

we obtain the following:

$$\begin{cases} G_t(x,t) + \lambda^2 G_x(x,t) + \frac{1}{2}\lambda^2 G_{xx}(x,t) = 0 \\ G(x,T) = f(H(x,T)) \end{cases}. \tag{2.39}$$

Assume that the function $H(x,T)$ is absolutely monotonic on $[0,\infty)$. By Bernstein's theorem on monotone functions, there exists some non-decreasing and bounded measure $\mu(\cdot)$, such that

$$H(x,T) = \int_0^\infty e^{xz} d\mu(z). \tag{2.40}$$

Taking the equation (2.38) with (2.40) as the boundary condition, we can deduce

$$H(x,t) = \int_0^\infty e^{xz+\frac{1}{2}\lambda^2 z^2 (T-t)} d\mu(z) \tag{2.41}$$

as the unique solution.

Assume that the function $f(\cdot)$ is non-negative and non-decreasing. Then, the function $G(x,T)$ can immediately be deduced to be absolutely monotonic, since it is a composition of an absolutely monotonic function and a non-decreasing, non-negative function on $[0,\infty)$:

$$G(x,T) = f(H(x,T)) = (f \circ H)(x,T). \tag{2.42}$$

Again by Bernstein's theorem on monotone functions, there exists some non-decreasing and bounded measure $\nu(\cdot)$, such that

$$G(x,T) = \int_0^\infty e^{xz} d\nu(z). \tag{2.43}$$

Let's assume that the solution to equation (2.39) with the terminal condition (2.43) is of the form

$$G(x,t) = \int_0^\infty e^{xz} R(t) d\nu(z). \tag{2.44}$$

Substituting into the equation, re-expressing and solving gives us

$$G(x,t) = \int_0^\infty e^{xz+\lambda^2\left(\frac{1}{2}z^2+z\right)(T-t)} d\nu(z). \tag{2.45}$$

Noting that

$$F(x,t) = G\left(H^{(-1)}(x,t), t\right), \tag{2.46}$$

we deduce our final result:

$$F(x,t) = \int_0^\infty e^{zH^{(-1)}(x,t)+\lambda^2\left(\frac{1}{2}z^2+z\right)(T-t)} d\nu(z). \tag{2.47}$$

□

If the preferences of an investor are known, we can deduce the spatial inverse $H^{(-1)}(\cdot,\cdot)$, obtain the measure $\nu(\cdot)$, and calculate the moments of their optimal terminal wealth using this result. An example follows below.

Example 2.3.1. Let an investor's preferences be described by

$$H(x,T) = \sum_{i=1}^{m} A_i e^{\alpha_i x}, \tag{2.48}$$

where $x > 0$ is the investor's initial wealth, $\alpha_i \in \mathbb{R}$ and $A_i \in \mathbb{R}$ are some constants. Furthermore, we set

$$f(x) = x^n, \qquad n \in \mathbb{Z}_+. \tag{2.49}$$

Then, we would have

$$G(x,T) = f(H(x,T)) = [H(x,T)]^n. \tag{2.50}$$

Using the multinomial theorem, we can write

$$G(x,T) = \left[\sum_{i=1}^{m} A_i e^{\alpha_i x}\right]^n = \sum_{k_1+\ldots+k_m=n} \frac{n! \cdot \prod_{i=1}^{m} A_i^{k_i}}{k_1! \ldots k_m!} \exp\left(x \sum_{i=1}^{m} \alpha_i k_i\right), \tag{2.51}$$

where the sum is taken over all possible combinations of non-negative integers $k_1, ..., k_n$ that sum to n. Defining

$$C_{n,m} = \frac{n! \cdot \prod_{i=1}^{m} A_i^{k_i}}{k_1! \ldots k_m!} \qquad \text{and} \qquad d_m = \sum_{i=1}^{m} \alpha_i k_i, \tag{2.52}$$

we can abbreviate the previous expression as

$$G(x,T) = \sum_{k_1+\ldots+k_m=n} C_{n,m} e^{x d_m}. \tag{2.53}$$

It can be verified that

$$G_{n,m}(x,t) = C_{n,m} e^{x d_m + \lambda^2 \left(\frac{1}{2} d_m^2 + d_m\right)(T-t)} \tag{2.54}$$

is a solution to the equation (2.43), and by the principle of superposition, so is

$$G(x,t) = \sum_{k_1+\ldots+k_m=n} C_{n,m} e^{x d_m + \lambda^2 \left(\frac{1}{2} d_m^2 + d_m\right)(T-t)}. \tag{2.55}$$

Finally, we obtain the moments of optimal terminal wealth as

$$\mathbb{E}[(X_T^*)^n | X_t^* = x] = \sum_{k_1+\ldots+k_m=n} C_{n,m} e^{d_m H^{(-1)}(x,t) + \lambda^2 \left(\frac{1}{2} d_m^2 + d_m\right)(T-t)}. \tag{2.56}$$

Chapter 3

Risk Budgeting

3.1 Overview

The sound theoretical foundation and intuitive construction of the mean-variance approach of modern portfolio theory, as discussed in the previous chapter, explain its popularity and wide use in investment management practice. However, the theory originally presented by Markowitz (1952a) in discrete time and considerably extended by Merton (1969) into a continuous-time setting requires a pair of crucial assumptions. One is that the variance, and thus the standard deviation, of prices of assets, is a good measure for risk. The other assumption is that the prices of the assets themselves follow a certain process that we assume – in the case of Merton (1969), he assumes a log-normal market and thus a geometric Brownian motion to describe price dynamics. The question of whether these assumptions are reasonable arises naturally, and the portfolio optimization approach becomes inherently susceptible to erroneous assumptions about the markets. The latter assumption has been extensively studied, such as in the work of Lo and MacKinlay (1988). By considering weekly stock returns and performing a simple volatility-based specification test, the authors reject the random walk hypothesis for the stock prices. Since Brownian motion is simply a limit of a symmetric random walk in time and space, this reasoning could be extended to the hypothesis that asset prices follow a geometric Brownian motion.

Despite the extensive literature and numerous theoretical results based on the mean-variance approach, there certainly are limitations to its practical implementation. Among many others, Maillard et al. (2010) mention the fact that the mean-variance solutions tend to output a portfolio that is concentrated on a small group of assets out of the many more made available in a portfolio selection problem. While this is certainly a limitation of the approach, a far more grievous one is the fact that the very solution to the mean-variance portfolio implementation problem depends explicitly on the expected returns of the assets. Merton (1980) himself tried to address the issue of modeling expected asset returns by introducing different estimation techniques and possible directions of research. Not much has been done on that front in literature and the task has been proven to be an extremely difficult one. Hence the restriction of using the mean-variance approach, and the possibility of it not necessarily producing the optimal portfolio in practice.

An alternative approach to the mean-variance framework for portfolio choice is provided by what is known as risk budgeting. According to Berkelaar et al. (2006), risk budgeting is best understood in the context of risk management, where risk management itself is composed of three stages. The first one is risk measurement and is concerned with identifying and measuring risk, whereas the second one is risk attribution, where we try to identify sources of risk and its sensitivity to exposures to these sources. The third stage that is outlined is risk allocation, and determining future risk exposures is its focus. If we consider this last stage of risk management and reformulate its main focus into a question asking for the optimal risk allocations given a well-defined objective function and constraints, the analogy with the traditional asset allocation approach in portfolio optimization is immediately observable.

If we are only concerned with risk concerning asset prices and allocating it in an optimal way to compose a portfolio (note that optimal here might have a different meaning, depending on an investor's objectives and preferences), then it becomes clear that we are not factoring in the expected return of the assets. As the estimation of future returns has proven to be a notoriously difficult task, some research has been done in the direction of developing optimal portfolios from a risk budgeting perspective.

The concept of risk in the setting of investment management and portfolio construction has to be well-outlined before attempting to understand the problem and obtain optimal solutions. Lee and Lam (2001) outline two different types of risks concerning portfolio choice, and more specifically, asset returns: statistical risk and information risk. The authors point to the standard deviation as the default measure of what they call statistical risk of the asset returns in the mean-variance framework, as well as the standard measure used in industry – the tracking error. With regards to information risk, they define the hit rate as the percentage of times the returns are positive, and the information ratio as the well-known fraction of active return to the tracking error. Litterman (1996), on the other hand, provides a more detailed overview of the commonly used statistical risk measures. Volatility, or standard deviation, is advantageous because it is somewhat observable, at least for past data. However, it does not provide information about what to be prepared for in a rare event, or what these rare events might mean. It is Value at Risk (VaR) that takes an advantage there, and although not easily verifiable in data from the real world, it attempts to assign numerical values to rare market events. Research on alternative methods outside the mean-variance framework has seen the use of the measures of risk mentioned herein, as well as others.

3.2 Risk Budgeting Portfolios

A standard formulation of a risk budgeting portfolio is provided by, among others, Bruder and Roncalli (2012). Given a portfolio of n assets $x = (x_1, ..., x_n)$, the weight of the i-th asset x_i, and some coherent and convex risk measure $\mathcal{R}(x_1, ..., x_n)$, the following Euler decomposition holds:

$$\mathcal{R}(x_1, ..., x_n) = \sum_{i=1}^{n} x_i \cdot \frac{\partial \mathcal{R}(x_1, ..., x_n)}{\partial x_i} = \sum_{i=1}^{n} RC_i(x_1, ..., x_n). \tag{3.1}$$

The decomposition above means that the total risk of the portfolio, as given by the risk measure used, is the sum of the risk contributions coming from individual assets, where the risk contribution of the i-th asset is denoted by RC_i.

To completely specify our risk budget and apply the heuristic for portfolio choice, we take a certain set of risk budgets for each asset: $\{b_1, ..., b_n\}$. To construct a risk budgeting (RB) portfolio, for the i-th asset we would have a linear equation of the form

$$RC_i(x_1, ..., x_n) = x_i \cdot \frac{\partial \mathcal{R}(x_1, ..., x_n)}{\partial x_i} = b_i, \tag{3.2}$$

and we would have to solve the system of non-linear equations in order to obtain the correct weights $x = (x_1, ..., x_n)$.

In the case of the volatility risk measure

$$\mathcal{R}(x) = \sigma(x) = \sqrt{x^\top \Sigma x}, \tag{3.3}$$

for which one can verify that it satisfies the decomposition (3.1), the system of linear equations will have equations of the form

$$RC_i(x_1, ..., x_n) = x_i \cdot \frac{(\Sigma x)_i}{\sqrt{x^\top \Sigma x}}. \tag{3.4}$$

3.3 Risk Budgeting Optimization Heuristics

There have been numerous approaches to obtaining the optimal risk budgeting portfolio. One of the heuristics proposed was outlined by Blitz and Hottinga (2001), where the tracking error is used as a risk measure. After defining the tracking error as the annual standard deviation of the return differences of a portfolio and its benchmark index, the authors introduce four partial tracking errors based on four different investment decisions. The total tracking error is then taken to be the one resulting from considering all these investment decisions.

To come up with an optimal tracking error allocation, or optimal allocation of risk, the authors suggest three steps. Firstly, independent investment decisions are made. In a probabilistic sense, this means that we split up the investment opportunities such that the returns from each one are independent of the returns of the others. Next, we obtain the forecasting capabilities for each investment decision and we rank them. Quantitatively, the forecasting capabilities are defined as information ratios, which is the ratio between the expected out-performance and the tracking error. This is somewhat analogous to the Sharpe ratio if we were to use the return volatility as the risk measure. In the last step, the authors propose allocating the tracking error proportional to the information ratios for each investment decision.

The analysis and methods used in this writing are, as the authors suggest, susceptible to personal belief, which comes through the calculation and use of expected information ratios to allocate tracking errors. The method appears to be straightforward to implement but to calculate expected information ratios, we would still need to involve expected returns. Thus, it appears that this risk budgeting heuristic goes back to the use of expected return, whose absence was the reason to adopt the approach in the first place. An assumption that might not be valid in some settings is the independence of the returns for different asset groups. If we are looking into securities within an asset class, then the assumption taken into consideration becomes unrealistic.

Lee and Lam (2001) extend the simple heuristic provided by Blitz and Hottinga (2001), by considering a correlation matrix between different asset classes or assets. The impact of the correlation between assets and the uncertainty of the estimate of the information ratio is studied in more detail by Molenkamp (2004), where examples are provided where the realized values differ significantly from the estimated ones. A shortfall constraint is then introduced into the model, such that a minimum required return is imposed when solving for the optimal risk allocations.

Chapter 4

Risk Parity

4.1 Fundamentals

As was discussed in the previous chapter, in a general risk budgeting setting, we assign specific risk budgets for each asset being considered for the portfolio choice, $\{b_1, ..., b_n\}$. A special case of this heuristic are equal-risk-contribution (ERC) portfolios, sometimes referred to by the term risk parity.

To define this portfolio properly, one has the define what risk contribution by a specific asset is. Following (3.2) and the discussion in the previous chapter on risk budgeting, the risk contribution of the i-th asset to the overall risk of the portfolio would be given by

$$RC_i(x_1, ..., x_n) = x_i \cdot \frac{\partial \mathcal{R}(x_1, ..., x_n)}{\partial x_i}, \tag{4.1}$$

where $x = (x_1, ..., x_n)$ is a portfolio made up of n assets, and $\mathcal{R}(\cdot)$ denotes the risk measure being used. The Euler decomposition outlined in (3.1) holds for the overall risk of the portfolio, $\mathcal{R}(x_1, ..., x_n)$.

As the name suggests, the risk contributions from all assets are to be equal in an ERC portfolio. Following the work of Maillard et al. (2010), the

ERC portfolio can be written as

$$x^* = \left\{ x \in [0,1]^n : \sum x_i = 1, RC_i(x_1, ..., x_n) = RC_j(x_1, ..., x_n) \quad \forall i, j \right\}. \tag{4.2}$$

If we take the risk measure to be the volatility of the portfolio, the risk contribution of the i-th asset would be of the form (3.4), and we could rewrite (4.2) as

$$x^* = \left\{ x \in [0,1]^n : \sum x_i = 1, x_i(\Sigma x)_i = x_j(\Sigma x)_j \quad \forall i, j \right\}, \tag{4.3}$$

where $\Sigma \in \mathbb{R}^{n \times n}$ is the matrix of covariances between individual asset prices.

4.2 Two Popular RP Portfolios

Explicit solutions to two special cases of the general ERC portfolio are presented by Maillard et al. (2010), with volatility as the risk measure.

The first case is when there are two assets ($n = 2$). In this case, the optimal portfolio is described by

$$x^* = \left\{ x \in [0,1]^2 : x_1 + x_2 = 1, x_1(\Sigma x)_1 = x_2(\Sigma x)_2 \right\}. \tag{4.4}$$

Denoting the volatility of the assets by σ_1 and σ_2, and the correlation between them by ρ, the problem above is readily solved to yield the optimal ERC portfolio:

$$x^* = \left(\frac{\sigma_1^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}}, \frac{\sigma_2^{-1}}{\sigma_1^{-1} + \sigma_2^{-1}} \right). \tag{4.5}$$

While it is impossible to obtain the explicit form of the solution for a general case with $n > 2$ assets, another special case can be considered. If the correlation between all assets are assumed to be equal, $\rho_{ij} = \rho$, $\forall i, j$, then the optimal ERC portfolio can be computed to obtain the explicit form:

$$x^* = (x_1^*, ..., x_n^*), \qquad x_i^* = \frac{\sigma_i^{-1}}{\sum_{j=1}^{n} \sigma_j^{-1}}. \tag{4.6}$$

For these two special cases, it is interesting to note that the solution does not depend on the correlation between assets. While the assumption required for the special case might be unrealistic, the correlation does not play a factor in the case with only two assets either.

4.3 RP Portfolios versus Popular Heuristic Ones

A commonly used heuristic portfolio is the equally weighted portfolio. As the name suggests, the allocation between different assets is equally spread out, and it is common to refer to this portfolio as the "$1/n$" portfolio. Following out previous notation, the equally weighted portfolio is fully characterized by

$$x^* = \left\{ x \in [0,1]^n : \sum x_i = 1, x_i = \frac{1}{n} \quad \forall i \right\}, \tag{4.7}$$

and there is no need of any optimization or solving. The work by DeMiguel et al. (2009) demonstrates that because of the errors in estimation of volatility and expected return of assets, this seemingly naïve strategy can outperform more sophisticated approaches, such as mean-variance optimization. However, Kritzman et al. (2010) go against this view and point to the small historical sample used for estimating returns and volatility as the reason behind the former conclusions. The assumption that an investor would take such a small historical sample for parameter estimation is labeled as an unrealistic one by the authors.

Another heuristic portfolio is the minimum-variance portfolio, where the goal is to obtain asset allocations that would minimize the variance of the entire portfolio. This is similar to the mean-variance portfolio, except there is no incorporation of the expected returns of different assets. The minimum-variance portfolio with volatility as the adopted risk measure can be written as

$$x^* = \left\{ \text{argmin} \sqrt{x^\top \Sigma x} : \sum x_i = 1 \quad \forall i \right\}. \tag{4.8}$$

Scholars have studied this portfolio extensively, as it appears that because it only minimizes variance without considering expected returns, it is more robust to errors in predicting future returns.

Clarke et al. (2006) look at the US equity markets, and conclude that while minimum-variance portfolios appear to be more robust when not incorporating expected returns, they do show signs of value bias and small-size bias. These can be controlled using different methodologies, and the authors mention imposition of factor neutrality constraints through stock characteristics or factor return sensitivity.

There are other more analytic results for the minimum-variance portfolio presented by Clarke et al. (2011). An explicit solution for the optimal portfolio is obtained under a single-index model, of the form $x_i = \frac{\sigma_{MV}^2}{\sigma_i^2}\left(1 - \frac{\beta_i}{\beta_{LS}}\right)$, where σ_{MV}^2 is the ex ante variance of the minimum-variance portfolio, σ_i^2 is the (ex ante) idiosyncratic return variance for asset i, β_i is the ex ante market beta for asset i, and β_{LS} is the long-short threshold beta. In this setting, it is thus shown that the allocation weights depend on two individual asset parameters and two parameters related to the entire portfolio. In long-only minimum-variance portfolios, it is demonstrated that 80%-90% of portfolio risk is systematic.

An interesting direction to explore is the relation between the risk parity portfolio and the heuristic portfolios provided above. Let us take into account the formulation and result provided by Maillard et al. (2010). First, we consider the following formulation of the optimal portfolio for some quantity c, which can, but does not need to be a constant:

$$x^*(c) = \text{argmin}\sqrt{x^\top \Sigma x}$$
$$\text{s.t.} \begin{cases} \sum_{i=1}^{n} \ln(x_i) \geq c \\ \mathbf{1}^\top x = 1 \\ x \geq \mathbf{0} \end{cases} \tag{4.9}$$

. It is clear that if we take $c = -\infty$, then the first constraint disappears, and we end up with the minimum-variance portfolio: $x^*_{MV} = x^*(-\infty)$. On the other hand, if we take $c = -n\ln(n)$, then the solution is the equally weighted portfolio, $x^*_{1/n} = x^*(-n\ln(n))$. Because of the Jensen inequality and the constraint $\mathbf{1}^\top x = 1$, we also have $\sum_{i=1}^{n} \ln(x_i) \leq -n\ln(n)$. Noting that if $c_1 \leq c_2$ then $\sigma(x^*(c_1)) \leq \sigma(x^*(c_2))$, for the case with $c \in (-\infty, -n\ln(n)]$, we can conclude that

$$\sigma_{MV} \leq \sigma(x^*(c)) \leq \sigma_{1/n}. \tag{4.10}$$

By applying the Lagrange method and solving problem (4.9), it can be readily shown that risk contributions are the same for all assets (look at Maillard et al. (2010)). Thus, we can write the previous expression as

$$\sigma_{MV} \leq \sigma_{RP} \leq \sigma_{1/n}, \tag{4.11}$$

and conclude that the overall risk of the risk parity portfolio (when using standard deviation as the risk measure) is always bounded between that of the minimum-variance and the equally weighted portfolios.

4.4 RP Portfolios versus the Mean-Variance Portfolios

Generally, risk parity portfolios are fundamentally different from traditional Markowitz portfolios because they do not include expected returns. This is the main strength of the risk parity approach. However, as Roncalli (2014) points out, some investment managers consider this a drawback, as potential returns of risk parity are not directly observed ex ante.

An extension to risk parity is possible through introducing expected return, as outlined by Roncalli (2014). The traditional Markowitz portfolio can be written as

$$x^*(\gamma) = \operatorname{argmin} \left\{ \frac{1}{2} x^\top \Sigma x - \gamma x^\top (\mu - r\mathbf{1}) \right\}, \tag{4.12}$$

which can also be expressed as

$$x^*(c) = \operatorname{argmin} \left\{ -\pi(x) + c \cdot \sigma(x) \right\}, \tag{4.13}$$

where $\pi(x) = x^\top \mu - r$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$ are the risk premium and volatility of the portfolio.

The way in which we can bring risk parity closer to mean-variance optimization is by considering the objective above as a risk measure,

$$\mathcal{R}(x) = -\pi(x) + c \cdot \sigma(x), \tag{4.14}$$

which is referred to as the Markowitz risk measure by Roncalli (2014). Taking π to be the vector of risk premiums for the assets and asset returns to be normally distributed $R \sim \mathcal{N}(\mu, \Sigma)$, we can also rewrite the measure as follows:

$$\mathcal{R}(x) = -x^\top \pi + c \cdot \sigma(x). \tag{4.15}$$

This is different than the risk measures discussed previously, because it includes not only a risk or volatility dimension, but it also incorporates a performance dimension by considering the risk premiums. The risk contribution from each asset is of the weighted form

$$\mathcal{RC}_i = (1 - \omega)\mathcal{PC}_i + \omega \mathcal{VC}_i, \tag{4.16}$$

where the performance and volatility contributions are

$$\mathcal{PC}_i = \frac{\pi_i(x)}{\pi(x)} \qquad \text{and} \qquad \mathcal{VC}_i = \frac{\sigma_i(x)}{\sigma(x)}, \tag{4.17}$$

respectively, and the normalizing constant $\omega \in (-\infty, \infty)$ is

$$\omega = \frac{c\sigma(x)}{-\pi(x) + c\omega(x)}. \tag{4.18}$$

It is interesting to note how the weight ω depends on the scaling factor c. As we increase c towards ∞, ω tends to 1, which in turn means that the portfolio that we have defined above converges to a traditional Markowitz portfolio. When $c = 0$, we have $\omega = 0$, and the portfolio only optimizes for expected return. There is a singularity around the value $c = \pi(x)/\sigma(x)$, which is the Sharpe ratio of the portfolio.

Through the examples provided by Roncalli (2014), it appears that in this setting, it only makes sense to use the risk budgeting approach when the scaling factor is above a certain value. The two approaches remain distinct, and it is not true that risk parity is a special case of the mean-variance optimization framework.

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Vita

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