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Global L^p solutions of the Boltzmann equation with an angle-potential concentrated collision kernel and convergence to a Landau solution

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To my father.

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Global L^p solutions of the Boltzmann equation with an angle-potential concentrated collision kernel and convergence to a Landau solution

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We solve the Cauchy problem associated to the space homogeneous Boltzmann equation with an angle-potential singular concentration modeling the collision kernel, proposed in [14]. The potential under consideration ranges from Coulomb to hard spheres cases, however, the motivation of such a collision kernel is to treat the (extreme) case of Coulomb potentials, on which this particular form of collision operator is well defined. We show that the scaled angle-potential singular concentration in a grazing collisions limit makes the Boltzmann operator converge in the sense of distributions to the Landau operator acting on the Boltzmann solutions, and also that solutions of this type of Boltzmann equation converge to solutions of the Landau equation that conserve mass, momentum and energy.

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Chapter 1

Introduction

The purpose of this work is to show well posed-ness of the Cauchy problem associated to an ε-family of spatially homogeneous Boltzmann equations,

$$\begin{cases} \partial_t f(v,t) = Q_B(f,f)(v,t) \text{ on } \mathbb{R}^3 \times [0,\infty) \\ f(v,0) = f_0(v) \text{ on } \mathbb{R}^3 \times \{t=0\}, \end{cases} \tag{1.1}$$

where the collision operator $Q_B = Q_{B_\varepsilon}$ is defined for very soft and Coulomb potentials in such a way that (1.1) admits global $L^1_k(\mathbb{R}^3)$ and $L^p(\mathbb{R}^3)$ solutions, and such that $Q_{B_\varepsilon}(f,f) \to Q_L(f,f)$, where Q_L is the collision integral of the *Landau equation*,

$$\begin{cases} \partial_t f(v,t) = Q_L(f,f)(v,t) \text{ on } \mathbb{R}^3 \times [0,\infty) \\ f(v,0) = f_0(v) \text{ on } \mathbb{R}^3 \times \{t=0\}. \end{cases} \tag{1.2}$$

Both $Q_{B_\varepsilon}(f,f)$, and $Q_L(f,f)$ are in non-local velocity mixing form for binary elastic interactions.

The Boltzmann collision operator, written in strong form, is defined by

$$Q_{B_\varepsilon}(f,f)(v) := \iint_{\mathbb{R}^3\times S^2} B_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\Big(f(v')\,f(v'_*) - f(v)f(v_*)\Big)\mathrm{d}\sigma\mathrm{d}v_*, \tag{1.3}$$

expressing positive and negative terms of probabilities rates due to elastic binary interactions taking a velocity pair (v', v'_*) in a precollisional state into (v, v_*), according to the elastic collision law

$$v' = v - \frac{1}{2}(|u|\sigma - u) \qquad v'_* = v_* + \frac{1}{2}(|u|\sigma - u) \qquad u = v - v_* \tag{1.4}$$

with the transition probability rate or collision kernel given by $B_\varepsilon := B_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)$, for $\sigma \in S^2$ and the unit vector notation $\hat{z} := z/|z|$ for any $z \in \mathbb{R}^3$. B_ε is assumed to be symmetric with respect to $\mu := \hat{u} \cdot \sigma = \hat{u} \cdot \hat{u'} = \cos\theta$, with the symmetrization condition that makes μ to be defined in $[0, 1]$, and $B_\varepsilon = B_\varepsilon(|u|^\gamma, \mu)\mathbb{1}_{0\le\mu\le1}$. Our work focuses on the special case where

$$B_\varepsilon(|u|^\gamma, \mu) = \frac{4}{\pi\varepsilon}\delta_0(1 - \mu - \min\{1, \varepsilon|u|^\gamma\})\mathbb{1}_{0\le\mu\le1}. \tag{1.5}$$

The Landau collision operator, derived in [40], written in strong form, is defined by

$$Q_L(f, f)(v) := \nabla_v \cdot \int_{\mathbb{R}^3} |u|^{\gamma+2}\Pi(u)\Big(f(v_*)\nabla f(v) - \nabla f(v_*)f(v)\Big)\mathrm{d}v_*, \tag{1.6}$$

for the relative velocity $u = v - v_*$, and $\Pi(u) := I_{3\times3} - \hat{u} \otimes \hat{u} \in \mathbb{R}^{3\times3}$ projects onto the space $u^\perp$. Noticeably, the angular contribution of the collision cross section has integrated the spherical part of the Boltzman equation collision kernel into to the Landau kernel $|u|^{\gamma+2}\Pi(u)$ depending just on the relative speed u between the interacting velocities v and v_* .

The choice of B_ε is crucial for our results. Our work focuses on a very special form, where there is no splitting of the potential term depending on u

from the spherical one depending on $\hat{u} \cdot \sigma$. We refer to such a collision kernel as an *angle-potential concentrated collision kernel*, which will be described in subsequent sections. It was recently introduced in [14] as a proposed simple tool to efficiently calculate the Landau equation by Discrete Simulation Monte Carlo schemes for interacting particle systems.

The requirement for Q_{B_ε} to approximate Q_L is not an unreasonable one. In fact, strong convergence $Q_{B_\varepsilon}(f, f) \to Q_L(f, f)$ has been shown for the classical Boltzmann operator in [24],[25] while weak convergence was discussed in [42]. We will rigorously prove convergence of the classical Q_{B_ε} to Q_L in the distributional sense, and show that our particular version of Q_{B_ε} (which we call Q_{g_ε}) exhibits the same behavior.

Indeed, the Boltzmann and Landau equations are closely related both physically and mathematically; not only are they evolution equations with nonlocal collision operators, but also these operators are derived from the same collisional cross section - the *Rutherford cross section* of 1911.

The Landau equation was formally derived in 1936 (see [40]) to model plasmas and other gases with very strong (Coulomb) repulsive intermolecular forces, for which the Boltzmann equation is not adequate. This is because Coulomb forces are so strong that the singularities they create in the classical collision integral $Q_B(f, f)$ are of the critical order at which the integral diverges. Hence, the Boltzmann equation in this case is ill - posed. Landau, however, was able to use the Rutherford cross section heuristically to derive a proper, convergent, collision operator that describes particle interactions in this special case. The resulting equation, (1.2), is now named after him.

The motivation behind this project is simple: on a large scale, we would like to approximate L^p solutions of the Landau equation (if they exist). To do this we must first construct the approximations themselves, in our case as L^p solutions to a converging sequence of Boltzmann equations. This is the bulk of this work. We will also show some preliminary results about the convergence of ε - solutions of (1.1) to solutions of (1.2) and discuss the difficulties associated with this problem.

1.1 Existing results

The first existence results for the Boltzmann equation had strong assumptions on the singularities of the collision kernel, especially the one in the angular cross section (see [39], [30], [20],[21]). This scenario applies to moderately soft to hard potentials only, because for very soft (especially Coulomb) potentials the cross sections have very strong singularities.

The transition probability (or collision) kernel $B(|u|^\gamma, \hat{u}\cdot\sigma)$ is often modeled as a product of a *potential* $\Phi(|u|) = |u|^\gamma$, $-3 < \gamma < 2$, and an *angular cross section*, $b(\hat{u}\cdot\sigma)$, symmetrized with respect to changes $\sigma \leftrightarrow -\sigma$:

$$\begin{aligned} B(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma &:= |u|^\gamma b(\hat{u}\cdot\sigma)\mathrm{d}\sigma = |u|^\gamma b(\cos\theta)\sin\theta\mathrm{d}\theta\mathrm{d}\omega \\ &= |u|^\gamma \sin^{-m}(\theta/2)\sin\theta\mathrm{d}\theta\mathrm{d}\omega \qquad (1.7) \end{aligned}$$

for $\omega \in S^1$ and $m > 0$.

In those cases where the singularity in $\sin(\theta/2)$ is too strong for the spherical integration to make sense, i.e., when $m \geq 4$, typically the collision kernel is truncated at very small angles, like

$$\begin{aligned} B_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma &:= |u|^\gamma b_\varepsilon(\hat{u}\cdot\sigma)\mathrm{d}\sigma \\ &= |u|^\gamma b_\varepsilon(\cos\theta)\sin\theta\mathrm{d}\theta\mathrm{d}\omega = |u|^\gamma \sin^{-m}(\theta/2)\sin\theta\mathrm{d}\theta\mathrm{d}\omega \\ &= -|u|^\gamma \frac{\mathbb{1}_{\theta\geq\sin(\varepsilon/2)}}{2\pi H_{m,\varepsilon}(\sin(\varepsilon/2))}\sin^{-m}(\theta/2)\sin\theta\mathrm{d}\theta\mathrm{d}\omega, \quad (1.8) \end{aligned}$$

where $H_{m,\varepsilon}(x)$ is the primitive of x^{-m} (see [34]).

In [42], C. Villani introduced the idea of an *H solution* of (1.1) and (1.2), which was designed exactly for these cases of strong singularities - for very soft and Coulomb potentials. For such strong singularities in Q_B and Q_L, even the weak forms of these collision operators are ill defined. An H solution solves this issue because it requires an extra assumption, namely, that solutions have finite entropy decay. This assumption is enough for Q_B and Q_L to be well defined in their weak forms. H solutions were shown to exist for very soft and Coulomb potentials for a properly ε-truncated Boltzmann equation, and, these solutions were shown to converge to H solutions of the Landau equation.

In 2015 in [26], L. Desvillettes was first to show global existence of weak L^3_{loc} solutions of the Landau equation for all soft potentials (including Coulomb). [31], [33] show local uniqueness of solutions of (1.2) for all soft potentials.

The Boltzmann equation, however, has results about local weak solutions for very soft potentials (see [38], [35]), while global solutions are only shown to exist for moderately soft potentials ([42], [16]).

1.2 Main results

Our choice of collision kernel for (1.1) is B_ε, as introduced in (1.5). It is taken from [13] and [14] and is in the form of a Dirac mass concentrated at very small collision angles and relative speeds, given by

$$g_\varepsilon(|u|^\gamma, \mu) := \frac{4}{\pi\varepsilon}\delta_0(1 - \mu - \min\{1, \varepsilon|u|^\gamma\})\mathbb{1}_{0\leq\mu\leq 1}. \tag{1.9}$$

From now on, we will denote by g_ε this specific collision kernel, and B_ε will represent more generic ones, as referred to earlier in this section. Noticeably, in g_ε, there is no splitting of the potential term from the angular term. In fact, the singularities in u and in θ are closely linked: whenever $\hat{u}\cdot\sigma = \cos\theta$ is very close to 1, $|u|^\gamma$ is forced to be greater than or equal to $\frac{1}{\varepsilon}$.

In this thesis we solve the spatially homogeneous Cauchy problem

$$\begin{cases} \dfrac{\partial f_\varepsilon}{\partial t}(v,t) = Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v,t) \\ f_\varepsilon(v,0) = f_0\,, \end{cases} \tag{1.10}$$

in $C^1((0,\infty); L^1_2(\mathbb{R}^3)$.

We will show that with g_ε as the collision kernel, the Boltzmann operator, Q_{g_ε}, converges to the Landau operator, Q_L, at a rate of up to $\sqrt{\varepsilon}$. Moreover, if solutions to this particular Cauchy problem are in L^p for p large enough, then not only do they remain in these spaces for all time, but their norms are bounded uniformly in ε. This suggests that we may be able to extract a converging subsequence and study the limit as $\varepsilon \to 0$, which will be discussed in the last section.

Before stating our results, let us introduce some notation. For any $1 \leq p < \infty$, $m \in \mathbb{R}$, $v \in \mathbb{R}^d$, $x \in \mathbb{R}$ and $f : \mathbb{R}^d \mapsto \mathbb{R}$, we define

$$\langle v \rangle := \left(1 + |v|^2\right)^{\frac{1}{2}}$$

$$L^p_m(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) : \|f\|_{L^p_m(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(v)\langle v\rangle^m|^p \, \mathrm{d}v \right)^{\frac{1}{p}} < \infty \right\}$$

$$L\log L(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \mapsto \mathbb{R} : \|f\|_{L\log L(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(v)(\log f(v))_+ \mathrm{d}v < \infty \right\}.$$

Theorem 1.1. Let $k \geq 2$ and $0 \leq f_0 \in L^1_2(\mathbb{R}^3) \cap L\log L(\mathbb{R}^3)$ such that $\|f_0\|_{L^1} = 1$. Define the following closed, convex and bounded set $F = F(f_0) \subset L^1_2(\mathbb{R}^3)$:

$$F := \left\{ f \in L^1_2 : f \geq 0, \|f\|_{L^1_2(\mathbb{R}^3)} \leq \|f_0\|_{L^1_2(\mathbb{R}^3)} \right\}.$$

Then for any $\varepsilon > 0$, there exists a unique nonnegative solution, $0 \leq f_\varepsilon = f_\varepsilon(v,t)$, to the Boltzmann equation (1.10), that lies in $C^1((0,\infty), L^1_2(\mathbb{R}^3)) \cap C([0,\infty), F)$. This solution preserves mass, momentum, energy, and has decreasing entropy.

Theorem 1.2. Let $0 \leq f_\varepsilon = f_\varepsilon(v,t) \in L^1_2(\mathbb{R}^3)$ be a weak solution to the Boltzmann equation with nonnegative initial data $0 \leq f_0 \in L^1_2(\mathbb{R}^3) \cap L\log L(\mathbb{R}^3)$.

1. If, in addition, $f_0 \in L^p(\mathbb{R}^3)$ for $1 < p < \infty$, then f_ε will remain in $L^p(\mathbb{R}^3)$ for all time, and

$$\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)} \leq \max\left\{ C^2_{p'} e^{\frac{1}{7p'} C_p \|f_0\|_{L\log L}}, \|f_0\|_{L^p} \right\}, \qquad (1.11)$$

where $C_r := 2^{3+\frac{1}{2r}}$ for $r \geq 1$.

2. If $f_0 \in L^\infty(\mathbb{R}^3)$, then f_ε will remain in $L^\infty(\mathbb{R}^3)$ for all time, and

$$\|f_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \leq \max\left\{64e^{\frac{32}{7}\|f_0\|_{L\log L}}, \|f_0\|_{L^\infty}\right\}. \tag{1.12}$$

As a corollary of Theorems 1.1 and 1.2, we are able to study the behavior of the solutions f_ε as $\varepsilon \to 0$, showing the existence of an f that solves the Landau equation, such that $f_\varepsilon \rightharpoonup^* f$ in an appropriate L^p space, as stated in the theorem below.

Theorem 1.3. For $-3 \leq \gamma < -2$, let $p > \frac{6}{8+\gamma}$ and let $0 \leq f_0 \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \cap L\log L(\mathbb{R}^3)$. Let $\nu > 0$ be small enough so that $0 < \nu < \frac{1}{2}\min\left\{-\frac{2+\gamma}{8+\gamma}, \frac{p-1}{2}, \frac{1}{2(p-1)}\right\}$. For each $\varepsilon > 0$, let $f_\varepsilon = f_\varepsilon(v,t) \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \times C^1((0,\infty)) \cap C([0,\infty))$ denote the unique nonnegative solution to the Cauchy problem (1.10), with g_ε defined as in (1.9). Then, there exists a nonnegative f such that, up to a subsequence, $f_\varepsilon \rightharpoonup^* f$ in $L^{1+\nu}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for all $t \geq 0$. Such an f satisfies the weak form of the Landau equation and preserves mass, momentum and energy. Moreover, if $p \geq 2$ then f also has a decreasing entropy, making it a global weak L^p solution of the Cauchy problem (1.2).

Chapter 2

Description of the Boltzmann and Landau equations

The formulation of the problem, in the x-uniform framework (known as the space homogeneous problem), is posed as follows: let $f = f(v,t)$, for $(v,t) \in \mathbb{R}^3 \times (0,\infty)$, be a probability density function describing the probability of finding a particle with velocity v at time t. Let v_* denote the velocity of a particle about to collide with the first, and let v' and v'_* denote their respective velocities before or after a reversible (elastic) collision. Also in the elastic case, collisions must conserve momentum and kinetic energy:

$$v' + v'_* = v + v_* \qquad \text{and} \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2. \tag{2.1}$$

Defining the relative velocities $u := v - v_*$ and $u' := v' - v'_*$ and letting $\sigma := u'/|u'| = u'/|u| \in S^2$ denote the *scattering direction,* the post - (or pre-) collisional velocities may be written as

$$v' = v'(v, v_*, \sigma) = v + \frac{1}{2}(|u|\sigma - u), \quad v'_* = v'_*(v_*, v, \sigma) = v_* - \frac{1}{2}(|u|\sigma - u).$$

One can represent $\sigma \in S^2$ as

$$\sigma = \hat{u}\cos\theta + \omega\sin\theta, \tag{2.2}$$

where $\omega \in u^{\perp}$ has unit length, and is in turn decomposed into

$$\omega = \frac{j}{|j|}\cos\phi + \frac{k}{|k|}\sin\phi. \tag{2.3}$$

Here $j, k \in u^{\perp}$ are defined as $j := (1,0,0) - u_1$, $k = j \times \hat{u}$, where u_1 is the first component of u. The Cauchy problem associated to the spatially homogeneous Boltzmann equation is written in strong form as

$$\begin{cases} \partial_t f(v,t) = Q_B(f,f)(v,t) := \displaystyle\iint_{\mathbb{R}^3\times S^2} B(|u|^{\gamma}, \hat{u}\cdot\sigma) \\ \qquad\qquad\qquad\qquad \times\Big(f\left(v',t\right) f\left(v'_*,t\right) - f(v,t)f(v_*,t)\Big)\mathrm{d}\sigma\mathrm{d}v_* \\ f(v,0) = f_0(v), \end{cases} \tag{2.4}$$

where for any $z \in \mathbb{R}^3$ we define $\hat{z} := z/|z|$. $B(|u|^{\gamma}, \hat{u}\cdot\sigma)$, known as the *collision kernel,* is often modeled as a product of the *potential* $\Phi(|u|) = |u|^{\gamma}$, and the *angular cross section*, $b(\hat{u}\cdot\sigma) = b(\cos\theta)$:

$$\begin{aligned} B(|u|^{\gamma}, \hat{u}\cdot\sigma)\mathrm{d}\sigma &:= |u|^{\gamma} b(\hat{u}\cdot\sigma)\mathrm{d}\sigma \\ &= |u|^{\gamma} b(\cos\theta)\sin\theta \mathrm{d}\theta\mathrm{d}\omega = |u|^{\gamma}\sin^{-m}(\theta/2)\sin\theta\mathrm{d}\theta\mathrm{d}\omega. \end{aligned} \tag{2.5}$$

The parameter $\gamma \in (-3,0)$ corresponds to soft potentials (repulsive forces), and $\gamma = -3$, which is only possible in (1.2), corresponds to the Coulomb potential. The case $\gamma \in (0,1]$ corresponds to hard potential and was extensively studied in [27], [28], [29] and others. Finally $\gamma = 0$ describes Maxwell molecule interactions (see for example [11], [9], [6], [7], [42]).

The cross section $b(\hat{u}\cdot\sigma)$ does not need to be defined as in (2.5), however, $b(\cdot)$ is always an even, nonnegative function defined on $-\pi/2 \le \theta \le \pi/2$ that must satisfy

$$\int_0^{\pi/2} b(\cos\theta)\sin^2(\theta/2)\sin\theta \mathrm{d}\theta < \infty. \tag{2.6}$$

Condition (2.6) is necessary (but not always sufficient) in order for Q_B to be well defined in weak form (see [1], [16] for discussion on the cancellation lemmas).

When the singularity of $b(\cos\theta)$ is mild enough, the Boltzmann operator into its gain and loss term. There is a lot of literature about how to work with the Boltzmann equation in this form. Otherwise, our tools are very limited. This is why it is so common to truncate the cross section and perform the split of Q_B in order to study the asymptotics.

In the case of (2.5), (2.6) suggests that we must have $m < 4$. However, for Coulomb potentials (when $\gamma = -3$), the cross section has been determined to be of the *Rutherford type*, where $b(\hat{u}\cdot\sigma)\mathrm{d}\sigma = b(\cos\theta)\sin\theta\mathrm{d}\theta \sim \sin^{-3}(\theta/2)\mathrm{d}\theta$ for $\theta << \pi/2$. This corresponds to $m = 4$ in (2.5), for which Q_B is not well defined in weak form.

In fact it has been shown in [34], that when $m \in [4, 6)$ in (2.5), the Landau equation arises as a grazing collisions limit of interacting particles modeled by the Boltzmann equation with a collision cross section that concentrates all interactions on the set of near zero relative velocities with positive mass. We discuss ways to truncate $b(\hat{u}\cdot\sigma)$ to obtain the grazing collisions limit in Section 2.2.

The Landau equation, written in strong form, is

$$
\begin{cases}
\partial_t f(v) = Q_L(f,f)(v) := \nabla_v \cdot \displaystyle\int_{\mathbb{R}^3} |u|^{\gamma+2}\Pi(u)\cdot \\
\qquad\qquad\qquad\qquad \cdot\Big(f(v_*)\nabla f(v) - \nabla f(v_*)f(v)\Big)\mathrm{d}v_*, \\
f(v)\Big|_{t=0} =: f_0(v),
\end{cases}
\tag{2.7}
$$

where $\Pi(u) := I_{3\times 3} - \hat{u}\otimes\hat{u} \in \mathbb{R}^{3\times 3}$ projects onto the space $u^\perp$.

For completeness let us also define Q_B and Q_L as *symmetric bilinear operators*, given by

$$
Q_B^{symm}(f,h) := \frac{1}{2}(Q_B(f,h) + Q_B(h,f))
$$
$$
Q_L^{symm}(f,h) := \frac{1}{2}(Q_L(f,h) + Q_L(h,f)).
$$

Note that $\bar{Q}_{B,L}(f,f) = Q_{B,L}^{sym}(f,f)$. In this work, however, we will be simply working with $Q_{L,B}$.

Finally, we introduce the key physical properties that are shared by solutions of (1.1) and (1.2):

(i) *Conserved mass, momentum and energy.*

$$
\frac{d}{dt}\int_{\mathbb{R}^3} f(v,t)(1,v,|v|^2)\mathrm{d}v
$$
$$
= \int_{\mathbb{R}^3} Q_{B,L}(f,f)(v,t)(1,v,|v|^2)\mathrm{d}v = 0, \tag{2.8}
$$

where $\int f(1,v,|v|^2)$ are the mass, momentum and energy of f respectfully. This is equivalent to

$$
\int_{\mathbb{R}^3} f(v,t)(1,v,|v|^2)\mathrm{d}v = \int_{\mathbb{R}^3} f_0(v)(1,v,|v|^2)\mathrm{d}v. \tag{2.9}
$$

(ii) *Decreasing entropy.* Let $\mathcal{H}(t) := \int f(v,t)\log f(v,t)\mathrm{d}v$ denote the entropy of f. Then

$$\frac{d}{dt}\mathcal{H}(t) = \int_{\mathbb{R}^3} \frac{d}{dt} f(v,t)\log f(v,t)\mathrm{d}v + \int_{\mathbb{R}^3} \frac{d}{dt} f(v,t)\mathrm{d}v$$
$$= \int_{\mathbb{R}^3} Q_{B,L}(f,f)(v,t)\log f(v,t)\mathrm{d}v = -\mathcal{D}_{B,L}(f,f)(t) \le 0, \quad (2.10)$$

with equality holding if and only if f is Maxwellian in v (according to the Boltzmann Theorem). The Boltzmann and Landau entropy dissipation term are defined respectively as

$$\mathcal{D}_B(f,f)(t) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_{S^2} B(|u|^\gamma, \hat{u}\cdot\sigma)\,(f'f'_* - ff_*) \times (\log f'f'_* - \log ff_*)\,\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \quad (2.11)$$

$$\mathcal{D}_L(f,f)(t) := \iint_{\mathbb{R}^3\times\mathbb{R}^3} |u|^2 \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*}\right)\cdot \Pi(u)\cdot \cdot \left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*}\right)\mathrm{d}v_*\mathrm{d}v \quad (2.12)$$

Both (2.9) and (2.10) are inherently true by the structure of Q_B and Q_L (see [42]).

2.1 Weak and weak-H formulations

Let $\varphi = \varphi(v,t) \in C^1(\mathbb{R}^+, C_0^\infty(\mathbb{R}^3))$ and consider the weak form of Q_B for a collision kernel $B(|u|, \hat{u}\cdot\sigma) = |u|^\gamma b(\hat{u}\cdot\sigma)$, with b even, nonnegative and symmetric about $\theta = \pi/2$. Then B is invariant under the change of variables $(v, v_*) \leftrightarrow (v_*, v)$ and $(v, v_*) \leftrightarrow (v', v'_*)$. Furthermore, the Jacobian of these transformations has an absolute value of one, therefore we may write the weak form of Q_B as

$$\int_0^t \int_{\mathbb{R}^3} Q_B(f,f)(v,s)\varphi(v,s)\mathrm{d}v\mathrm{d}s$$
$$= \int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_{S^2} B(|u|^\gamma, \hat{u}\cdot\sigma)(f'f'_* - ff_*)\varphi(v,s)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\mathrm{d}s$$
$$= \frac{1}{4}\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_{S^2} B(|u|^\gamma, \hat{u}\cdot\sigma)\cdot$$
$$\cdot (f'f'_* - ff_*)(\varphi + \varphi_* - \varphi' - \varphi'_*)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\mathrm{d}s. \quad (2.13)$$

In [42] the author ensures that the integral on right hand side of (2.13) converges by making the extra assumption that solutions of the Boltzmann equation have *finite entropy decay,* that is,

$$0 \le -\frac{d}{dt}\mathcal{H}(t) = -\frac{d}{dt}\int f(v,t)\log f(v,t)\mathrm{d}v$$
$$= \frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} B(u,\hat{u}\cdot\sigma)\cdot(f'f'_* - ff_*)\cdot$$
$$\cdot\left(\log f'f'_* - \log ff_*\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v < \infty. \quad (2.14)$$

Because of the assumption on the entropy, the variational formulation of the Boltzmann equation (1.1) with (2.13) representing $\int Q_B\varphi$ is called the *weak-H form*, and its solutions are consequently the weak-H, or H solutions.

However, if the singularities of $B(|u|^\gamma, \hat{u}\cdot\sigma)$ are mild enough - that is, if $\gamma \ge -2$ and if (2.6) holds - then the right hand side of (2.13) is well defined even without the assumption (2.14). In fact, in this case we can even go further by splitting Q_B into its *gain and loss parts,* Q_B^+ and Q_B^- (which are still well defined):

$$
\begin{aligned}
Q_B(f,f) &= \iint_{\mathbb{R}^3\times S^2} B(|u|^\gamma, \hat{u}\cdot\sigma)(f'f'_* - ff_*)\mathrm{d}\sigma\mathrm{d}v_* \\
&= \iint_{\mathbb{R}^3\times S^2} B(|u|^\gamma, \hat{u}\cdot\sigma)f'f'_*\mathrm{d}\sigma\mathrm{d}v_* - \iint_{\mathbb{R}^3\times S^2} B(|u|^\gamma, \hat{u}\cdot\sigma)ff_*\mathrm{d}\sigma\mathrm{d}v_* \\
&\qquad =: Q_B^+(f,f) - Q_B^-(f,f).
\end{aligned}
$$

Then we can break up $\int Q_B\varphi$ into

$$
\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} Q_B(f,f)(v,s)\varphi(v,s)\mathrm{d}v\mathrm{d}s \\
&\qquad = \frac{1}{4}\int_0^t \iint \int_{S^2} B(|u|, \hat{u}\cdot\sigma)f'f'_*(\varphi + \varphi_* - \varphi' - \varphi'_*)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\mathrm{d}s \\
&\qquad - \frac{1}{4}\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \int_{S^2} B(|u|, \hat{u}\cdot\sigma)ff_*(\varphi + \varphi_* - \varphi' - \varphi'_*)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\mathrm{d}s \\
&\quad = \frac{1}{2}\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} ff_* \int_{S^2} B(|u|, \hat{u}\cdot\sigma)(\varphi' + \varphi'_* - \varphi - \varphi_*)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v\mathrm{d}s,
\end{aligned}
\tag{2.15}
$$

and the right hand side of (2.15) is still well defined. Because no additional assumption on the entropy is required in this case, this form of $\int Q_B\varphi$ is stronger, and is known simply as the *weak form.* The weak formulation of (1.1) would then have (2.15) on its left hand side, and solutions to this problem are called *weak solutions.*

The precise definitions of weak and weak-H solutions of the Boltzmann equation are defined by Villani in [42] as follows:

Definition 2.1. A function $f(v,t)$ is said to be a weak solution of the Boltzmann equation with initial data $0 \le f_0 \in L^1_2(\mathbb{R}^3)$ if the following conditions are satisfied:

(i)

$$f \geq 0, f \in \mathcal{C}(\mathbb{R}^+, \mathcal{D}'), f \in L^1([0,T], L^1_{2+\gamma}),$$
$$\forall t \geq 0, f(\cdot, t) \in L^1_2(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3),$$

(ii)

$$f(v,0) = f_0(v) \text{ for a.e. } v \in \mathbb{R}^3$$

(iii)

$$\forall t \geq 0, \int f(v,t) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v = \int f_0(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v \tag{2.16}$$

$$\int f(v,t) \log f(v,t) \mathrm{d}v \leq \int f_0(v) \log f_0(v) \mathrm{d}v \tag{2.17}$$

(iv) $\forall \varphi = \varphi(v,t) \in C^1(\mathbb{R}^+, C_0^\infty(\mathbb{R}^3)), \forall t > 0,$

$$\begin{aligned} \int f(v,t)\varphi(v,t)\mathrm{d}v - \int f_0(v)\varphi(v,0)\mathrm{d}v - \int_0^t \int f(v,s)\partial_s\varphi(v,s)\mathrm{d}v\mathrm{d}s \\ = \frac{1}{2}\int_0^t \iint f(v,s)f(v_*,s)|u|^\gamma \cdot \\ \cdot \int_{S^2} b(\hat{u}\cdot\sigma)\left(\varphi' + \varphi'_* - \varphi - \varphi_*\right) \mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v. \end{aligned} \tag{2.18}$$

Definition 2.2. A function $f(v,t)$ is an H-solution of the Boltzmann equation if it satisfies all of the conditions (i)-(iii) above, and item (iv) with the right hand side of (2.18) replaced by (2.13).

Remark 2.3. The difference between weak solutions and H solutions lies ***only*** in the interpretation of $\int Q_B\varphi$. The entropy assumption, (2.14), is only a condition under which (2.13) is well defined. If there is another way to ensure the finiteness of the right hand side of (2.13) (for example if $B(|u|^\gamma, \hat{u}\cdot\sigma)$ has a special, non-traditional structure), then (2.14) is not needed. Whether the entropy assumption is needed or not, solutions to the variational problem are called H-solutions as long as $\int Q_B\varphi$ is defined as in (2.13), and they are called weak solutions if $\int Q_B\varphi$ is defined as in (2.15).

A similar definition of weak and weak-H solutions can be derived for the Landau equation. One can check in [42] that the weak-H form of $\int Q_L\varphi$ is

$$\int Q_L(f,f)\varphi(v)\mathrm{d}v = -\int \sqrt{ff_*|u|^{\gamma+2}}(\nabla\varphi - \nabla_*\varphi_*)^T\Pi(u)\cdot \\ \cdot\left((\nabla-\nabla_*)\sqrt{ff_*|u|^{\gamma+2}}\right)\mathrm{d}v_*\mathrm{d}v, \quad (2.19)$$

which is well defined given the assumption of finite entropy decay of the solutions:

$$0 \le -\frac{d}{dt}\int f(t,v)\log f(t,v)\mathrm{d}v \\ = \int ff_*|u|^{\gamma+2}\left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*}\right)^T \Pi(u)\left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*}\right)\mathrm{d}v_*\mathrm{d}v < \infty. \quad (2.20)$$

For $\gamma \ge -2$, similar to Q_B one can split Q_L into two integrals to further expand (2.19):

$$\int Q_L(f,f)\varphi\mathrm{d}v = -\frac{1}{2}\iint ff_*|u|^\gamma(\nabla\varphi - \nabla_*\varphi_*)\cdot u\mathrm{d}v_*\mathrm{d}v \\ + 2\iint ff_*|u|^{\gamma+2}\Pi(u) : (D^2\varphi + D_*^2\varphi_*)\mathrm{d}v_*\mathrm{d}v \quad (2.21)$$

Now, we define a weak solution for the Landau equation.

Definition 2.4. A function $f(v,t)$ is said to be a weak solution of the Landau equation with initial data $0 \leq f_0 \in L^1_2(\mathbb{R}^3)$ if the following conditions are satisfied:

(i)

$$
\begin{aligned}
&f \geq 0, f \in \mathcal{C}(\mathbb{R}^+, \mathcal{D}'), f \in L^1([0,T], L^1_{2+\gamma}),\\
&\forall t \geq 0, f(\cdot,t) \in L^1_2(\mathbb{R}^3) \cap L \log L(\mathbb{R}^3),
\end{aligned}
$$

(ii)

$$
f(v,0) = f_0(v) \text{ for a.e. } v \in \mathbb{R}^3
$$

(iii)

$$
\forall t \geq 0, \int f(v,t) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v = \int f_0(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} \mathrm{d}v \tag{2.22}
$$

$$
\int f(v,t) \log f(v,t) \mathrm{d}v \leq \int f_0(v) \log f_0(v) \mathrm{d}v \tag{2.23}
$$

(iv) $\forall \varphi = \varphi(v,t) \in C^1(\mathbb{R}^+, C^\infty_0(\mathbb{R}^3)), \forall t > 0,$

$$
\begin{aligned}
&\int f(v,t)\varphi(v,t)\mathrm{d}v - \int f_0(v)\varphi(v,0)\mathrm{d}v - \int_0^t \int f(v,s)\partial_s\varphi(v,s)\mathrm{d}v\mathrm{d}s\\
&= \frac{1}{2}\int_0^t \iint f(v,s)f(v_*,s)|u|^{-1}(D^2\varphi(v,s) + D^2_*\varphi(v_*,s)) : \Pi(u)\mathrm{d}v_*\mathrm{d}v\\
&\qquad - 2\int_0^t \iint f f_* |u|^{-3}(\nabla\varphi(v,s) - \nabla_*\varphi(v_*,s)) \cdot u \mathrm{d}v_*\mathrm{d}v.
\end{aligned} \tag{2.24}
$$

By now, thanks to Desvillettes in [26], we know that H-solutions of the Landau equation for all soft potentials (even if $\gamma = -3$) are weak solutions, so

$\int Q_L(f,f)\varphi$ can be defined as in (2.21), and (2.19) is not necessary. Desvillettes shows that not only does (2.20) ensure that the right hand side of (2.13) well defined, but that the right hand side of (2.15) is finite too. However, it is important to note that the weak solutions of Desvillettes for very soft potentials require extra integrability, which is obtained by using (2.20), so the entropy assumption is still needed.

2.2 The grazing collisions limit

The most common truncation of the Boltzmann collision cross section is

$$b_\varepsilon(\cos\theta) = \frac{I_{\theta\geq\varepsilon}}{|\log\sin(\varepsilon/2)|} b(\cos\theta) \tag{2.25}$$

(see [40], [24], [42],[34]). In [34], the authors were able to extend this to an even stronger θ-singularity in $b(\cos\theta)$ with a suitable truncation, and still obtain the Landau equation: for $\delta \in [0,2)$ they define

$$b_\varepsilon^\delta(\cos\theta) := \frac{I_{\theta\geq\varepsilon}}{H_\delta(\sin(\varepsilon/2))} \sin^{-(4+\delta)}(\theta/2), \tag{2.26}$$

where H_δ is such that $H'_\delta(x) = x^{-(\delta+1)}$ (if $\delta = 0$ then we recover (2.25)). Moreover, the rate of convergence of the Boltzmann collision integral Q_{B_ε} to the corresponding Landau collision term is much higher for $\delta > 0$. In a sense, the angular cross sections b_ε approximates a singular point mass distribution as $\varepsilon \longrightarrow 0$, which is a signature of the Landau derivation. This limit is called the *grazing collisions limit.*

It is important to note that one does not need to use the exact truncation (or the exact collision kernel) above to get the grazing collisions limit. In fact, it suffices for B_ε to satisfy the following condition, pointwise in $u \in \mathbb{R}^3$:

$$
\begin{aligned}
&\forall \ k \geq 2, \ \varepsilon > 0, \ \ \exists \ \ \lambda_{k,\varepsilon} > 0 \text{ such that} \\
&\qquad \beta_k[B_\varepsilon](u) := \int_0^{\frac{\pi}{2}} B_\varepsilon(|u|^\gamma, \cos\theta)\sin^k(\theta/2)\sin\theta \mathrm{d}\theta \leq \lambda_{k,\varepsilon}|u|^\gamma, \\
&\qquad\qquad \text{and } \lambda_{k,\varepsilon} \to \begin{cases} \dfrac{2}{\pi} \text{ if } k = 2 \\ 0 \text{ if } k > 2 \end{cases} \quad \text{as } \varepsilon \to 0
\end{aligned} \tag{2.27}
$$

(similar conditions are mentioned in [14], [13], [34], [42]). Indeed, one can show that if (2.27) holds, then $Q_{B_\varepsilon}(f,f) \to Q_L(f,f)$ as $\varepsilon \to 0$ in the sense of distributions. A sketch of a similar theorem can be found in [42]:

Proposition 2.5. Consider a sequence of nonnegative collision kernels, $B_\varepsilon = B_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)$, for the Boltzmann equation with $-3 \leq \gamma < -2$, satisfying property (2.27). Let $0 \leq f_\varepsilon \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, $p > \frac{6}{8+\gamma}$ be a weak solution of (1.1) with B_ε as the collision kernel. Then for any test function $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$
\lim_{\varepsilon\to 0}\left|\int \left(Q_{B_\varepsilon}(f_\varepsilon, f_\varepsilon)(v,t) - Q_L(f_\varepsilon, f_\varepsilon)(v,t)\right)\varphi(v)\mathrm{d}v\right| = 0 \tag{2.28}
$$

for all $t \geq 0$.

Proof. First we prove the following lemma, inspired by Lemma 4 of [26]:

Lemma 2.6. Let $p > 1$ and $0 \geq \alpha > \max\{-3, -6\frac{p-1}{p}\}$. If $f \in L^1 \cap L^p(\mathbb{R}^3)$ and $h(v) := |v|^\alpha$, then

$$
\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)f(v_*)|v - v_*|^\alpha \mathrm{d}v_*\mathrm{d}v \leq \mathcal{C}(\alpha, p, \|f\|_{L^p}, \|f\|_{L^1}), \tag{2.29}
$$

where

$$\mathcal{C}(\alpha, p, \|f\|_{L^p}, \|f\|_{L^1}) = \|f\|_{L^1(\mathbb{R}^3)}^2 + \frac{|S^2|}{\alpha+3}\|f\|_{L^p}^{p'}\|f\|_{L^1}^{\frac{p-2}{p-1}} + \left(\frac{2|S^2|}{\alpha p'+6}\right)^{\frac{2}{p'}}\|f\|_{L^p}^2. \quad (2.30)$$

Proof. Let $h_1(v) := |v|^\alpha \mathbb{1}_{|v|\leq 1}$ and $h_2(v) := |v|^\alpha \mathbb{1}_{|v|>1}$, so that $h_1 + h_2 = h$. Then

$$\iint f(v)f(v_*)h(v-v_*)\mathrm{d}v_*\mathrm{d}v = \int f(v) f*h_1(v)\mathrm{d}v + \int f(v) f*h_2(v)\mathrm{d}v. \quad (2.31)$$

To estimate the first term above, first suppose that $p < 2$. In this case, since $\frac{p'}{2} > 1$ we can use Young's inequality to estimate

$$\iint f(v) f*h_1(v)\mathrm{d}v \leq \|f\|_{L^p}\|f*h_1\|_{L^{p'}} \leq \|f\|_{L^p}^2\|h_1\|_{L^{\frac{p'}{2}}} = \left(\frac{2|S^2|}{\alpha p'+6}\right)^{\frac{2}{p'}}\|f\|_{L^p}^2. \quad (2.32)$$

On the other hand, if $p \geq 2$ then $1 < p' \leq 2 \leq p$. By interpolation, $f \in L^{p'}$ and $\|f\|_{L^{p'}} \leq \|f\|_{L^p}^{\frac{1}{p-1}}\|f\|_{L^1}^{\frac{p-2}{p-1}}$. Furthermore, since $\alpha > -3$, $h_1 \in L^1$. In this case,

$$\iint f(v) f*h_1(v)\mathrm{d}v \leq \|f\|_{L^p}\|f*h_1\|_{L^{p'}} \leq \|f\|_{L^p}\|f\|_{L^{p'}}\|h_1\|_{L^1} \leq \frac{|S^2|}{\alpha+3}\|f\|_{L^p}^{p'}\|f\|_{L^1}^{\frac{p-2}{p-1}}. \quad (2.33)$$

Finally, the second term of (2.31) can be estimated easily as

$$\int f(v) f*h_2(v)\mathrm{d}v \leq \|f\|_{L^1}\|f*h_2\|_{L^\infty} \leq \|f\|_{L^1}^2\|h_2\|_{L^\infty} = \|f\|_{L^1}^2, \quad (2.34)$$

which completes the proof. □

The purpose of Lemma 2.6 is to allow us to split $\int Q_{B_\varepsilon}(f,f)\varphi \mathrm{d}v$ into its gain and loss parts so that we can use the weak forms of Q_{B_ε} and Q_L, defined in (2.18) and (2.24) respectively, for the convergence.

To prove Proposition 2.5, first separate $\int Q_{B_\varepsilon}(f_\varepsilon, f_\varepsilon)\varphi \mathrm{d}v$ and $\int Q_L(f,f)\varphi \mathrm{d}v$ into a few integrals, then use Lemma 2.6 to show that they are well defined. Recall from (2.24) that

$$\begin{aligned}\int & Q_L(f_\varepsilon, f_\varepsilon)(v)\varphi(v)\mathrm{d}v \\ &= \iint f_\varepsilon f_{\varepsilon *}|u|^\gamma \left(-2(\nabla\varphi - \nabla_*\varphi_*)\cdot u + \frac{1}{2}|u|^2(D^2\varphi + D_*^2\varphi_*) : \Pi(u)\right)\mathrm{d}v_*\mathrm{d}v \\ &= \iint f_\varepsilon f_{\varepsilon *} G_L(v,v_*)\mathrm{d}v_*\mathrm{d}v, \quad (2.35)\end{aligned}$$

where we define

$$\begin{aligned}G_L(v,v_*) &:= -2|u|^\gamma(\nabla\varphi - \nabla_*\varphi_*)\cdot u \\ &\quad + \frac{1}{2}|u|^{\gamma+2}(D^2\varphi + D_*^2\varphi_*) : \Pi(u) = G_L^1(v,v_*) + G_L^2(v,v_*) \quad (2.36)\end{aligned}$$

and

$$\begin{aligned}G_L^1(v,v_*) &= -2|u|^\gamma(\nabla\varphi - \nabla_*\varphi_*)\cdot u = -4(\partial_{v_i}\varphi - \partial_{v_{*i}}\varphi_*)u_i, \\ G_L^2(v,v_*) &= \frac{1}{2}|u|^{\gamma+2}(D^2\varphi + D_*^2\varphi_*) : \Pi(u) = |u|^2(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)\Pi(u)_{ij}.\end{aligned}$$

Similarly for the Boltzmann collision term,

$$\begin{aligned}\int Q_{B_\varepsilon}(f,f)(v)\varphi(v)\mathrm{d}v &= \frac{1}{2}\iint f f_* |u|^\gamma \int_{S^2} b_\varepsilon(\hat{u}\cdot\sigma)(\varphi' + \varphi'_* - \varphi - \varphi_*)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\ &= \iint f f_* G_{B_\varepsilon}(v,v_*)\mathrm{d}v_*\mathrm{d}v, \quad (2.37)\end{aligned}$$

where

$$G_{B_\varepsilon}(v,v_*) := \frac{1}{2}|u|^\gamma \int_0^{\pi/2} B_\varepsilon(|u|^\gamma, \cos\theta) \int_{-\pi}^{\pi} (\varphi' + \varphi'_* - \varphi - \varphi_*) \mathrm{d}\phi \sin\theta \mathrm{d}\theta$$
$$= \frac{1}{2}\int_{S^2} B_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)(\varphi' + \varphi'_* - \varphi - \varphi_*)\mathrm{d}\sigma.$$

We begin by taking the second order Taylor expansion of $\varphi' + \varphi'_* - \varphi - \varphi_*$, keeping in mind that $v'_* - v_* = -(v' - v)$. We use here the Einstein notation for summation.

$$\begin{aligned}
(\varphi' - \varphi) + (\varphi'_* - \varphi_*) = \\
&= \nabla\varphi(v)\cdot(v'-v) + \frac{1}{2}\partial_{v_i v_j}\varphi(v)(v'_i - v_i)(v'_j - v_j) \\
&+ \frac{1}{6}\partial_{v_i v_j v_k}\varphi(\xi)(v'_i - v_i)(v'_j - v_j)(v'_k - v_k) \\
&+ \nabla\varphi(v_*)\cdot(v'_* - v_*) + \frac{1}{2}\partial_{v_i v_j}\varphi(v_*)(v'_i - v_i)(v'_j - v_j) \\
&+ \frac{1}{6}\partial_{v_i v_j v_k}\varphi(\zeta)(v'_i - v_i)(v'_j - v_j)(v'_k - v_k) \\
&= (\nabla\varphi(v) - \nabla\varphi_*(v_*))\cdot(v'-v) \\
&+ \frac{1}{2}(\partial_{v_i v_j}\varphi(v) + \partial_{v_{*i} v_{*j}}\varphi(v_*))(v'_i - v_i)(v'_j - v_j) \\
&+ \frac{1}{6}(\partial_{v_i v_j v_k}\varphi(\xi) - \partial_{v_{*i} v_{*j} v_{*k}}\varphi(\zeta))(v'_i - v_i)(v'_j - v_j)(v'_k - v_k),
\end{aligned} \tag{2.38}$$

where ξ and ζ are convex combinations of v, v' and v_*, v'_* respectively. Next, substitute this expansion into G_{B_ε}:

$$\begin{aligned}
G_{B_\varepsilon}(v,v_*) &= \frac{1}{2}(\nabla\varphi(v)-\nabla\varphi(v_*))\cdot\int_0^{\pi/2} B_\varepsilon(|u|^\gamma,\cos\theta)\int_{-\pi}^{\pi}(v'-v)\mathrm{d}\phi\sin\theta\mathrm{d}\theta \\
&+ \frac{1}{4}(\partial_{v_iv_j}\varphi+\partial_{v_{*i}v_{*j}}\varphi_*)\int_0^{\pi/2} B_\varepsilon(|u|^\gamma,\cos\theta)\int_{-\pi}^{\pi}(v_i'-v_i)(v_j'-v_j)\mathrm{d}\phi\sin\theta\mathrm{d}\theta \\
&+ \frac{1}{12}\int_0^{\pi/2} B_\varepsilon(|u|^\gamma,\hat{u}\cdot\sigma)\int_{-\pi}^{\pi}(\partial_{v_iv_jv_k}\varphi(\xi)-\partial_{v_{*i}v_{*j}v_{*k}}\varphi(\zeta)) \\
&\qquad \times (v_i'-v_i)(v_j'-v_j)(v_k'-v_k)\mathrm{d}\sigma \\
&=: G^1_{B_\varepsilon}(v,v_*)+G^2_{B_\varepsilon}(v,v_*)+G^3_{B_\varepsilon}(v,v_*), \quad (2.39)
\end{aligned}$$

where

$$\begin{aligned}
G^1_{B_\varepsilon}(v,v_*) &:= \frac{1}{2}(\nabla\varphi-\nabla_*\varphi_*)\cdot \\
&\quad \cdot\int_0^{\pi/2} B_\varepsilon(|u|^\gamma,\hat{u}\cdot\sigma)\int_{-\pi}^{\pi}(v'-v)\mathrm{d}\phi\sin\theta\mathrm{d}\theta, \quad (2.40) \\
G^2_{B_\varepsilon}(v,v_*) &:= \frac{1}{4}(\partial_{v_iv_j}\varphi+\partial_{v_{*i}v_{*j}}\varphi_*)\int_0^{\pi/2} B_\varepsilon(|u|^\gamma,\cos\theta) \\
&\quad \times\int_{-\pi}^{\pi}(v_i'-v_i)(v_j-v_j)\mathrm{d}\phi\sin\theta\mathrm{d}\theta \quad (2.41)
\end{aligned}$$

and

$$\begin{aligned}
G^3_{B_\varepsilon}(v,v_*) &:= \frac{1}{12}\int_0^{\pi/2} B_\varepsilon(|u|^\gamma,\cos\theta)\int_{-\pi}^{\pi}(\partial_{v_iv_jv_k}\varphi(\xi)-\partial_{v_{*i}v_{*j}v_{*k}}\varphi(\zeta)) \\
&\qquad \times (v_i'-v_i)(v_j'-v_j)(v_k'-v_k)\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\
&\le \frac{1}{6}\|D^3\varphi\|_{L^\infty}\int_0^{\pi/2} B_\varepsilon(|u|^\gamma,\cos\theta)\int_{-\pi}^{\pi}|v'-v|^3\mathrm{d}\phi\sin\theta\mathrm{d}\theta \\
&= \frac{\pi}{3}\|D^3\varphi\|_{L^\infty}|u|^3\int_0^{\pi/2} B_\varepsilon(|u|^\gamma,\cos\theta)\sin^3(\theta/2)\sin\theta\mathrm{d}\theta \\
&= \frac{\pi}{3}\|D^3\varphi\|_{L^\infty}|u|^3\beta_3[B_\varepsilon], \quad (2.42)
\end{aligned}$$

where in the end of (2.51) we used that $|v'-v| = |u|\sin(\theta/2)$, by the geometry of particle collisions. Using the representations of v', v'_* in (1.4), one can check that

$$\int_{-\pi}^{\pi}(v'-v)\mathrm{d}\phi = \pi u(\cos\theta - 1) = -2\pi u\sin^2(\theta/2), \tag{2.43}$$

$$\begin{aligned}\int_{-\pi}^{\pi}(v'_i-v_i)(v'_j-v_j)\mathrm{d}\phi &= \frac{\pi}{2}(\cos\theta-1)^2u_iu_j + \frac{\pi}{4}\Pi(u)_{ij}|u|^2\sin^2\theta \\ &= \pi\sin^4(\theta/2)(2u_iu_j - |u|^2\Pi(u)_{ij}) + \pi|u|^2\Pi(u)_{ij}\sin^2(\theta/2)/\end{aligned} \tag{2.44}$$

Then $G^1_{B_\varepsilon}$, $G^2_{B_\varepsilon}$ reduce to

$$\begin{aligned}G^1_{B_\varepsilon}(v,v_*) &= -\pi(\nabla\varphi - \nabla_*\varphi_*)\cdot u\int_0^{\pi/2}B_\varepsilon(|u|^\gamma,\cos\theta)\sin^2(\theta/2)\sin\theta\mathrm{d}\theta \\ &= -\pi(\nabla\varphi - \nabla_*\varphi_*)\cdot u\beta_2[B_\varepsilon](u)\end{aligned} \tag{2.45}$$

and

$$\begin{aligned}G^2_{B_\varepsilon}(v,v_*) &= \frac{\pi}{4}(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)(2u_iu_j - |u|^2\Pi(u)_{ij}) \\ &\qquad\times\int_0^{\pi/2}B_\varepsilon(|u|^\gamma,\cos\theta)\sin^4(\theta/2)\sin\theta\mathrm{d}\theta \\ &\quad+\frac{\pi}{4}(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)|u|^2\Pi(u)_{ij}\int_0^{\pi/2}B_\varepsilon(|u|^\gamma,\cos\theta)\sin^2(\theta/2)\sin\theta\mathrm{d}\theta \\ &= \frac{\pi}{4}(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)(2u_iu_j - |u|^2\Pi(u)_{ij})\beta_4[B_\varepsilon](u) \\ &\qquad+\frac{\pi}{4}(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)\Pi(u)_{ij}|u|^2\beta_2[B_\varepsilon](u).\end{aligned} \tag{2.46}$$

Substituting (2.45), (2.46), into (2.37),

$$\int Q_{B_\varepsilon}(f_\varepsilon, f_\varepsilon)(v)\varphi(v)\mathrm{d}v = \iint f_\varepsilon f_{\varepsilon *}((\nabla\varphi - \nabla_*\varphi_*)\cdot G^1_{B_\varepsilon}(v,v_*)\mathrm{d}v_*\mathrm{d}v$$
$$+ \iint f_\varepsilon f_{\varepsilon *}(\partial_{v_iv_j}\varphi + \partial v_{*i}v_{*j}\varphi_*)G^2_{B_\varepsilon}(v,v_*)\mathrm{d}v_*\mathrm{d}v$$
$$+ \iint f_\varepsilon f_{\varepsilon *} G^3_{B_\varepsilon}(v,v_*)\mathrm{d}v_*\mathrm{d}v$$
$$= -\pi \iint f_\varepsilon f_{\varepsilon *}(\nabla\varphi - \nabla_*\varphi_*)\cdot u\beta_2[B_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$+ \frac{\pi}{4}\iint f_\varepsilon f_{\varepsilon *}(\partial_{v_iv_j}\varphi + \partial v_{*i}v_{*j}\varphi_*)(2u_iu_j - |u|^2\Pi(u)_{ij})\beta_4[B_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$+ \frac{\pi}{4}\iint f_\varepsilon f_{\varepsilon *}(\partial_{v_iv_j}\varphi + \partial v_{*i}v_{*j}\varphi_*)\Pi(u)_{ij}|u|^2\beta_2[B_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$+ \iint f_\varepsilon f_{\varepsilon *} G^3_{B_\varepsilon}(v,v_*)\mathrm{d}v_*\mathrm{d}v =: I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon + I_4^\varepsilon, \quad (2.47)$$

with I_j^ε, $1 \le j \le 4$, defined accordingly.

We show that each I_j^ε above is bounded. Beginning with I_1^ε,

$$I_1^\varepsilon = -\pi \iint f_\varepsilon f_{\varepsilon *}(\nabla\varphi - \nabla_*\varphi_*)\cdot u\beta_2[B_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$\le \lambda_{2,\varepsilon}\pi\|D^2\varphi\|_{L^\infty}\iint ff_*|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v. \quad (2.48)$$

For I_2^ε, note that $(2u_iu_j - |u|^2\Pi(u)_{ij}) \le 4|u|^2$. Then

$$I_2^\varepsilon = \frac{\pi}{4}\iint f_\varepsilon f_{\varepsilon *}(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)(2u_iu_j - |u|^2\Pi(u)_{ij})\beta_4[B_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$\le 2\pi\lambda_{4,\varepsilon}\|D^2\varphi\|_{L^\infty}\iint f_\varepsilon f_{\varepsilon *}|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v, \quad (2.49)$$

$$I_3^\varepsilon = \frac{\pi}{4}\iint f_\varepsilon f_{\varepsilon *}\mathrm{d}v_*\mathrm{d}v(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi)\Pi(u)_{ij}|u|^2\beta_2[B_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$\le \frac{\pi}{2}\lambda_{2,\varepsilon}\|D^2\varphi\|_{L^\infty}\iint f_\varepsilon f_{\varepsilon *}|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v \quad (2.50)$$

and finally

$$I_4^\varepsilon = \iint f_\varepsilon f_{\varepsilon *} G^3_{B_\varepsilon}(v, v_*) \mathrm{d}v_* \mathrm{d}v$$
$$\leq \frac{\pi}{3} \|D^3\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *} |u|^3 \beta_3[B_\varepsilon] \mathrm{d}v_* \mathrm{d}v$$
$$\leq \frac{\pi}{3} \lambda_{3,\varepsilon} \|D^3\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *} |u|^{\gamma+3} \mathrm{d}v_* \mathrm{d}v. \quad (2.51)$$

Thus, each I_j^ε is finite by Lemma 2.6, with $\alpha = \gamma+2$ for the first three integrals and $\alpha = \gamma+3$ for I_4^ε. The requirement $p > \frac{6}{8+\gamma}$ ensures that the choice $\alpha = \gamma+2$ satisfies the hypothesis of Lemma 2.6; that is, $\alpha > -6\frac{p-1}{p}$.

Finally we prove the convergence as $\varepsilon \to 0$. Since $\lambda_{3,\varepsilon}, \lambda_{4,\varepsilon} \to 0$ as $\varepsilon \to 0$, both I_2^ε and I_3^ε vanish. Then we estimate $I_1^\varepsilon + I_3^\varepsilon - \int Q_L(f_\varepsilon, f_\varepsilon)\varphi \mathrm{d}v$ as follows:

$$
\begin{aligned}
I_2^\varepsilon - \iint f_\varepsilon f_{\varepsilon *} G_L^1(v, v_*) \mathrm{d}v_* \mathrm{d}v + I_3^\varepsilon - \iint f_\varepsilon f_{\varepsilon *} G_L^2(v, v_*) \mathrm{d}v_* \mathrm{d}v \\
= -\pi \iint f_\varepsilon f_{\varepsilon *} (\nabla\varphi - \nabla_* \varphi_*) \cdot u \beta_2[B_\varepsilon](u) \mathrm{d}v_* \mathrm{d}v \\
+ 2 \iint f_\varepsilon f_{\varepsilon *} (\nabla\varphi - \nabla_* \varphi_*) \cdot u |u|^\gamma \mathrm{d}v_* \mathrm{d}v \\
\frac{\pi}{4} \iint f_\varepsilon f_{\varepsilon *} (\partial_{v_i v_j} \varphi + \partial_{v_{*i} v_{*j}} \varphi_*) \Pi(u)_{ij} |u|^2 \beta_2[B_\varepsilon](u) \mathrm{d}v_* \mathrm{d}v \\
- \frac{1}{2} \iint f_\varepsilon f_{\varepsilon *} (\partial_{v_i v_j} \varphi + \partial_{v_{*i} v_{*j}} \varphi_*) \Pi(u)_{ij} |u|^{\gamma+2} \mathrm{d}v_* \mathrm{d}v \\
= \iint f_\varepsilon f_{\varepsilon *} (\nabla\varphi - \nabla_* \varphi_*) \cdot u \left(-\pi \beta_2[B_\varepsilon](u) - 2|u|^\gamma\right) \mathrm{d}v_* \mathrm{d}v \\
+ \iint f_\varepsilon f_{\varepsilon *} (\partial_{v_i v_j} \varphi + \partial_{v_{*i} v_{*j}} \varphi_*) \Pi(u)_{ij} \left(\frac{\pi}{4} \beta_2[B_\varepsilon](u) - \frac{1}{2}|u|^\gamma\right) \mathrm{d}v_* \mathrm{d}v \\
\leq \|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *} |u|^2 \left(-\pi \beta_2[B_\varepsilon](u) - 2|u|^\gamma\right) \mathrm{d}v_* \mathrm{d}v \\
+ 4\|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *} \left(\frac{\pi}{4} \beta_2[B_\varepsilon](u) - \frac{1}{2}|u|^\gamma\right) \mathrm{d}v_* \mathrm{d}v \to 0 \quad (2.52)
\end{aligned}
$$

as $\varepsilon \to 0$ by the Dominated Convergence Theorem.

□

Remark 2.7. (i) If $\gamma \in [-2, -1]$ then one can choose any $p > 1$. Furthermore, if $\gamma \in (-1, s)$ for $s \geq 0$, and $f \in L^1_{s+2}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, then Proposition 2.5 holds for any $p > 1$ as well.

(ii) If B_ε has an explicit formula, then one might be able to compute the rate of convergence rates. We will include the convergence rate to the Landau collision operator in the next section, when we introduce our particular choice of B_ε.

Chapter 3

An angle-potential concentrated collision kernel

Recall that for soft potentials, the traditional collision kernel, $B(|u|^\gamma, \hat{u} \cdot \sigma) = |u|^\gamma b(\hat{u} \cdot \sigma)$, has two singularities - one when $u = 0$, the other when $\hat{u} \cdot \sigma = 1$. However, the classical truncation of the kernel (2.25) and its analogues eliminate only the singularity in the cross section and not in the potential. Thus, Q_{B_ε} is still a singular integral after the truncation.

This is why the first results for (1.1) and (1.2) with negative γ were for *moderately soft potentials*, i.e., $\gamma > -2$ (see [42], [43]). This is a problem when looking for weak solutions, partly because it does not allow us to make L^p estimates on Q_B^+ and Q_B^-. We are therefore tempted to truncate the collision kernel in a way that also controls its singularity at $u = 0$, while still sending $Q_{B_\varepsilon}(f,f)$ to $Q_L(f,f)$ in the grazing collisions limit.

3.1 Description of the kernel

We discuss here the collision kernel (1.9) from the introduction, that links the two singularities in the classical $B(|u|^\gamma, \hat{u} \cdot \sigma)$. Recall the definition

$$g_\varepsilon(|u|^\gamma, \hat{u} \cdot \sigma) = g_\varepsilon(|u|^\gamma, \mu) = \frac{4}{\pi\varepsilon}\delta_0\left(1 - \mu - \min\left\{1, \varepsilon|u|^\gamma\right\}\right)\mathbb{1}_{\mu \geq 0}, \tag{3.1}$$

where $\mu := \cos\theta = \hat{u}\cdot\sigma$. Notice that, unlike the standard $B(|u|^\gamma, \hat{u}\cdot\sigma)$, this collision kernel does not separate its two variables, $|u|^\gamma$ and $\hat{u}\cdot\sigma$.

We can check that, for fixed $u \neq 0$, g_ε satisfies (2.27) from Section 2.2. Indeed, letting $m_\varepsilon(x) := \min\{1, \varepsilon|x|^\gamma\} \in (0,1]$ and $\mu_\varepsilon(x) := 1 - m_\varepsilon(x) \in [0,1)$ for any $x > 0$, we have

$$0 \leq \beta_2[g_\varepsilon](u) = \int_0^{\frac{\pi}{2}} g_\varepsilon(|u|^\gamma, \cos\theta)\sin^2(\theta/2)\sin\theta \mathrm{d}\theta = \frac{1}{2}\int_0^1 g_\varepsilon(|u|^\gamma,\mu)(1-\mu)\mathrm{d}\mu$$
$$= \frac{2}{\pi\varepsilon}(1-\mu_\varepsilon(u)) = \frac{2}{\pi\varepsilon}m_\varepsilon(u) = \frac{2}{\pi}|u|^\gamma \mathbb{1}_{\varepsilon|u|^\gamma\leq 1} + \frac{2}{\pi\varepsilon}\mathbb{1}_{\varepsilon|u|^\gamma>1}$$
$$\longrightarrow \frac{2}{\pi}|u|^\gamma \quad (3.2)$$

because $0 < \frac{2}{\pi\varepsilon}\mathbb{1}_{\{\varepsilon|u|^\gamma>1\}} < \frac{2}{\pi\varepsilon}(\varepsilon|u|^\gamma)\mathbb{1}_{\{\varepsilon|u|^\gamma>1\}} \to 0$. Similarly, for $k > 2$ and a fixed $u \neq 0$,

$$\beta_k[g_\varepsilon](u) = \int_0^{\frac{\pi}{2}} g_\varepsilon(|u|^\gamma,\mu)\sin^k(\theta/2)\sin\theta\mathrm{d}\theta$$
$$= \int_0^1 g_\varepsilon(|u|^\gamma,\mu)\left(\frac{1}{2}(1-\mu)\right)^{\frac{k}{2}}\mathrm{d}\mu = 2^{-\frac{k}{2}}\frac{4}{\pi\varepsilon}(1-\mu_\varepsilon(|u|))^{\frac{k}{2}}$$
$$= 2^{-\frac{k}{2}}\frac{4}{\pi\varepsilon}m_\varepsilon(|u|)^{\frac{k}{2}}$$
$$= \frac{2^{2-\frac{k}{2}}}{\pi}\varepsilon^{\frac{k}{2}-1}|u|^{\gamma k/2}\mathbb{1}_{\varepsilon|u|^\gamma\leq 1} + \frac{2^{2-\frac{k}{2}}}{\pi\varepsilon}\mathbb{1}_{\varepsilon|u|^\gamma>1} \to 0 \quad (3.3)$$

as $\varepsilon \to 0$ because $k > 2$ and

$$0 \leq \frac{2^{2-\frac{k}{2}}}{\pi\varepsilon}\mathbb{1}_{\varepsilon|u|^\gamma>1} \leq \frac{2^{2-\frac{k}{2}}}{\pi}\varepsilon^{\frac{k}{2}-1}|u|^{\gamma k/2}\mathbb{1}_{\varepsilon|u|^\gamma>1} \to 0.$$

Thus (2.27) holds.

The Boltzmann operator with this new cross section is written as

$$
\begin{aligned}
Q_{g_\varepsilon}(f,f)(v) &= \int_{\mathbb{R}^3}\int_{S^2}(f(v')f(v'_*) - f(v)f(v_*)g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* \\
&= \int_{R^3}\int_{S^2} f'f'_* g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}\omega\mathrm{d}v_* - \frac{8}{\varepsilon}\|f\|_{L^1(\mathbb{R}^3)}f(v), \quad (3.4)
\end{aligned}
$$

and the corresponding Boltzmann equation is

$$
\begin{cases}
\partial_t f + \dfrac{8}{\varepsilon}\|f_0\|_{L^1} f = \displaystyle\int_{R^3}\int_{S^2} f'f'_* g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* \\
f(v,0) = f_0(v).
\end{cases} \quad (3.5)
$$

In view of Proposition 2.5, it would be reasonable to expect for there to be a grazing collisions limit, and in fact we will show that this is true.

3.2 Connection with Q_L: the grazing collisions limit

Theorem 3.1. Let $f_\varepsilon \in L^1_2(\mathbb{R}^3)\cap L^p(\mathbb{R}^3)$, $p > \max\{\frac{6}{8+\gamma}, 1\}$ be a weak solution of (1.1) with the cross section g_ε defined as in (1.9) and $\gamma \in [-3, -\frac{5}{2})$. Then for all time, $|Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon) - Q_L(f_\varepsilon, f_\varepsilon)| \longrightarrow 0$ in the distributional sense as $\varepsilon \to 0$. In particular,

(i)

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\left(Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v,t) - Q_L(f_\varepsilon, f_\varepsilon)(v,t)\right)\varphi(v)\mathrm{d}v\right| \\
&\qquad \le \varepsilon^r\Big(12\|D^2\varphi\|_{L^\infty}\mathcal{C}\left((1+r)\gamma+2, p, \|f_\varepsilon\|_{L^p}, \|f_\varepsilon\|_{L^1}\right) \\
&\qquad\quad + \frac{\sqrt{2}}{3\pi}\|D^3\varphi\|_{L^\infty}\mathcal{C}\left((1+r)\gamma+3, p, \|f_\varepsilon\|_{L^p}, \|f_\varepsilon\|_{L^1}\right) \\
&\qquad\quad + \frac{\sqrt{2}}{3\pi}\|D^3\varphi\|_{L^\infty}\mathcal{C}\left((2-r)\gamma+3, p, \|f_\varepsilon\|_{L^p}, \|f_\varepsilon\|_{L^1}\right)\Big), \quad (3.6)
\end{aligned}
$$

where $\mathcal{C}$ is defined as in Lemma 2.6, and

$$r < \min\left\{\frac{1}{2}, -\frac{5p+2\gamma}{\gamma p}, -\frac{(8+\gamma)p-6}{\gamma p}\right\}. \quad (3.7)$$

In the Coulomb case, this reduces to $r < \min\{\frac{1}{2}, \frac{5}{3} - \frac{2}{p}\}$. (Note that the minimum changes when $p = 2$.)

(ii) Moreover, if we additionally assume that $p > \frac{12}{3\gamma+16}$, then the rate of convergence is exactly $\sqrt{\varepsilon}$. That is, $r = \frac{1}{2}$.

Remark 3.2. $\delta = \frac{1}{2}$ is consistent with the rate predicted in [14], where the authors showed convergence in strong form using a priori estimates assuming that the solution f_ε is twice differentiable and bounded by a Maxwellian.

Proof. We need to compute $G_1[g_\varepsilon], G_2[g_\varepsilon]$ and $G_3[g_\varepsilon]$, as defined in (2.40), (2.41) and (2.51). Let $m_\varepsilon(u) := \min\{\varepsilon|u|^\gamma, 2\}$ and let $\mu_\varepsilon(u) := 1 - m_\varepsilon(|u|)$ denote the mass of $\delta_0(1-\mu-m_\varepsilon(u))$. Using integration by parts with the mass δ_0 and recalling (2.45), (2.46), we have:

$$G_1[g_\varepsilon](v,v_*) = -2\pi u \int_0^1 g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\sin^2(\theta/2)\mathrm{d}\mu$$
$$= -2\pi u \beta_2[g_\varepsilon](u), \quad (3.8)$$

$$\begin{aligned} G_2[g_\varepsilon](v,v_*) &= \frac{1}{2}\pi(2u_iu_j - |u|^2\Pi(u)_{ij})\int_0^{\pi/2} g_\varepsilon(\cos\theta)\sin^4(\theta/2)\sin\theta\mathrm{d}\theta \\ &\quad + \frac{1}{2}\pi|u|^2\Pi(u)_{ij}\int_0^{\pi/2} g_\varepsilon(|u|^\gamma,\cos\theta)\sin^2(\theta/2)\sin\theta\mathrm{d}\theta \\ &= \frac{\pi}{2}(2u_iu_j - |u|^2\Pi(u)_{ij})\beta_4[g_\varepsilon](u) + \frac{\pi}{2}|u|^2\Pi(u)_{ij}\beta_2[g_\varepsilon](u) \quad (3.9)\end{aligned}$$

and

$$G_3[g_\varepsilon](v,v_*) \le \frac{1}{3}\|D^3\varphi\|_{L^\infty}\int_{-\pi}^{\pi}\int_0^1 g_\varepsilon(|u|^\gamma,\mu)|v'-v|^3\mathrm{d}\mu\mathrm{d}\phi$$
$$= \frac{2\pi}{3}|u|^3\|D^3\varphi\|_{L^\infty}\int_0^1 g_\varepsilon(|u|^\gamma,\mu)\sin^3(\theta/2)\mathrm{d}\mu$$
$$= \frac{2\pi}{3}|u|^3\|D^3\varphi\|_{L^\infty}\beta_3[g_\varepsilon](u). \quad (3.10)$$

Combining these estimates, the weak form of Q_{g_ε} is

$$\int Q_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)\varphi\mathrm{d}v = \frac{1}{2}\iint f_\varepsilon f_{\varepsilon *}(\nabla\varphi-\nabla_*\varphi_*)\cdot G_1[g_\varepsilon](v,v_*)\mathrm{d}v_*\mathrm{d}v$$
$$+\frac{1}{2}\iint f_\varepsilon f_{\varepsilon *}(\partial_{v_iv_j}\varphi+\partial_{v_{*i}v_{*j}}\varphi_*)G_2[g_\varepsilon](v,v_*)\mathrm{d}v_*\mathrm{d}v$$
$$+\frac{1}{2}\iint G_3[g_\varepsilon](v,v_*)\mathrm{d}v_*\mathrm{d}v$$
$$= -\pi\iint f_\varepsilon f_{\varepsilon *}(\nabla\varphi-\nabla_*\varphi_*)\cdot u\beta_2[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$+\frac{\pi}{4}\iint f_\varepsilon f_{\varepsilon *}(\partial_{v_iv_j}\varphi+\partial_{v_{*i}v_{*j}}\varphi_*)(2u_iu_j-|u|^2\Pi(u)_{ij})\beta_4[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$+\frac{\pi}{4}\iint f_\varepsilon f_{\varepsilon *}(\partial_{v_iv_j}\varphi+\partial_{v_{*i}v_{*j}}\varphi_*)\Pi(u)_{ij}|u|^2\beta_2[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v$$
$$+\frac{1}{2}\iint f_\varepsilon f_{\varepsilon *}G_3[g_\varepsilon](v,v_*)\mathrm{d}v_*\mathrm{d}v =: I_1^\varepsilon+I_2^\varepsilon+I_3^\varepsilon+I_4^\varepsilon, \quad (3.11)$$

with I_j^ε, $1\le j\le 4$ defined accordingly.

Let us check that each I_j^ε is well defined. We begin with I_1^ε and I_3^ε because these are the terms that will converge to the weak form of Q_L. For I_1^ε, note that $(\nabla\varphi - \nabla_*\varphi_*) \cdot u \le \|D^2\varphi\|_{L^\infty}|u|^2$, so

$$\begin{aligned} I_1^\varepsilon = -\pi \iint f_\varepsilon f_{\varepsilon *}(\nabla\varphi - \nabla_*\varphi_*) \cdot u\beta_2[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v \\ \le \pi\|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *}|u|^2\beta_2[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v \\ \le 2\|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *}|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v \\ + \frac{2}{\varepsilon}\|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *}|u|^2\mathrm{d}v_*\mathrm{d}v < \infty \end{aligned} \tag{3.12}$$

by Lemma 2.6 (with $\alpha = \gamma + 2$) and the fact that $|u|^2 \le 2(\langle v\rangle^2 + \langle v_*\rangle^2)$ (since $f_\varepsilon \in L^1_2$).

To estimate I_3^ε, we note that $\Pi(u)_{ij} \le 2$. Then

$$\begin{aligned} I_3^\varepsilon = \frac{\pi}{4} \iint f_\varepsilon f_{\varepsilon *}(\partial_{v_i v_j}\varphi + \partial_{v_{*i} v_{*j}}\varphi_*)\Pi(u)_{ij}|u|^2\beta_2[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v \\ \le \pi\|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *}|u|^2\beta_2[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v \\ \le \|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *}|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v \\ + \frac{1}{\varepsilon}\|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *}|u|^2\mathrm{d}v_*\mathrm{d}v < \infty \end{aligned} \tag{3.13}$$

for the same reasons as in (3.12). In the fourth line of (3.13), we used the fact that $1 < \varepsilon|u|^\gamma$ in the domain of integration to write an additional power of $\varepsilon|u|^\gamma$ in the integrand. We will be doing this for the rest of the estimates as well.

Now that we know $I_1^\varepsilon, I_3^\varepsilon < \infty$, we can estimate

$$
\begin{aligned}
&I_1^\varepsilon + I_2^\varepsilon - \int_{\mathbb{R}^3} Q_L(f_\varepsilon, f_\varepsilon)\varphi(v)\mathrm{d}v \\
&\qquad = -\pi \iint f_\varepsilon f_{\varepsilon *}(\nabla\varphi - \nabla_*\varphi_*)\cdot u\beta_2[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v \\
&\qquad \frac{\pi}{4}\iint f_\varepsilon f_{\varepsilon *}(\partial_{v_i v_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)|u|^2\Pi(u)_{ij}\beta_2[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v \\
&\qquad - \iint f_\varepsilon f_{\varepsilon *}G_L^1(v,v_*)\mathrm{d}v_*\mathrm{d}v - \iint f_\varepsilon f_{\varepsilon *}G_L^2(vv_*)\mathrm{d}v_*\mathrm{d}v \\
&\qquad = \Big(-2\iint_{\varepsilon|u|^\gamma \le 1} f_\varepsilon f_{\varepsilon *}(\nabla\varphi - \nabla_*\varphi_*)\cdot u|u|^\gamma \mathrm{d}v_*\mathrm{d}v \\
&\qquad\qquad - \iint f_\varepsilon f_{\varepsilon *}G_L^1(v,v_*)\mathrm{d}v_*\mathrm{d}v\Big) \\
&\qquad + \Big(\frac{1}{2}\iint_{\varepsilon|u|^\gamma \le 1} f_\varepsilon f_{\varepsilon *}(\partial_{v_i v_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)\Pi(u)_{ij}|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v \\
&\qquad\qquad - \iint f_\varepsilon f_{\varepsilon *}G_L^2(v,v_*)\mathrm{d}v_*\mathrm{d}v\Big) \\
&\qquad - \frac{2}{\varepsilon}\iint_{\varepsilon|u|^\gamma > 1} f_\varepsilon f_{\varepsilon *}(\nabla\varphi - \nabla_*\varphi_*)\cdot u\mathrm{d}v_*\mathrm{d}v \\
&\qquad + \frac{1}{2\varepsilon}\iint_{\varepsilon|u|^\gamma > 1} f_\varepsilon f_{\varepsilon *}(\partial_{v_i v_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)\Pi(u)_{ij}|u|^2\mathrm{d}v_* \\
&\qquad = 2\iint_{\varepsilon|u|^\gamma > 1} f_\varepsilon f_{\varepsilon *}(\nabla\varphi - \nabla_*\varphi_*)\cdot u|u|^\gamma\mathrm{d}v_*\mathrm{d}v \\
&\qquad - \frac{1}{2}\iint_{\varepsilon|u|^\gamma > 1} f_\varepsilon f_{\varepsilon *}(\partial_{v_i v_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)\Pi(u)_{ij}|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v \\
&\qquad - \frac{2}{\varepsilon}\iint_{\varepsilon|u|^\gamma > 1} f_\varepsilon f_{\varepsilon *}(\nabla\varphi - \nabla_*\varphi_*)\cdot u\mathrm{d}v_*\mathrm{d}v \\
&\qquad + \frac{1}{2\varepsilon}\iint_{\varepsilon|u|^\gamma > 1} f_\varepsilon f_{\varepsilon *}(\partial_{v_i v_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)\Pi(u)_{ij}|u|^2\mathrm{d}v_* \\
&\qquad =: Err_1 + Err_2 + Err_3 + Err_4, \qquad (3.14)
\end{aligned}
$$

with Err_j, $1 \le j \le 4$ defined accordingly. We compute the decay rate of each Err_j by using the fact $1 < \varepsilon|u|^\gamma$ in the domain of integration and by adding as many powers of ε as possible before the integral becomes too singular. For the following, we let $0 < \delta < \min\{-\frac{5}{\gamma} - \frac{2}{p}, -\frac{(8+\gamma)p-6}{\gamma p}\}$. In the Coulomb case, this means $0 < \delta < \min\{\frac{2}{3}, \frac{5}{3} - \frac{2}{p}\}$. Hence,

$$\begin{aligned} Err_1 &\leq 2\|D^2\varphi\|_{L^\infty} \iint_{\varepsilon|u|^\gamma>1} (\varepsilon|u|^\gamma)^\delta f_\varepsilon f_{\varepsilon*}|u|^{\gamma+2} \mathrm{d}v_* \mathrm{d}v \\ &= 2\varepsilon^\delta \|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon*}|u|^{(1+\delta)\gamma+2} \mathrm{d}v_* \mathrm{d}v \\ &\leq 2\varepsilon^\delta \|D^2\varphi\|_{L^\infty} \mathcal{C}\big((1+\delta)\gamma+2, p, \|f\|_{L^p}, \|f\|_{L^1}\big) \end{aligned} \tag{3.15}$$

by Lemma 2.6 with $\alpha = (1+\delta)\gamma + 2$. Our upper bound for δ is such that the lemma may be applied with this choice of α with no other restrictions on p. Similarly,

$$\begin{aligned} Err_2 &\leq 2\varepsilon^\delta \|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon*}|u|^{(1+\delta)\gamma+2} \mathrm{d}v_* \mathrm{d}v \\ &\leq 2\varepsilon^\delta \|D^2\varphi\|_{L^\infty} \mathcal{C}\big((1+\delta)\gamma+2, p, \|f\|_{L^p}, \|f\|_{L^1}\big), \end{aligned} \tag{3.16}$$

$$\begin{aligned} Err_3 &\leq \frac{2}{\varepsilon} \|D^2\varphi\|_{L^\infty} \iint_{\varepsilon|u|^\gamma} (\varepsilon|u|^\gamma)^{1+\delta} f_\varepsilon f_{\varepsilon*}|u|^2 \mathrm{d}v_* \mathrm{d}v \\ &\leq 2\varepsilon^\delta \|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon*}|u|^{(1+\delta)\gamma+2} \mathrm{d}v_* \mathrm{d}v \\ &\leq 2\varepsilon^\delta \|D^2\varphi\|_{L^\infty} \mathcal{C}\big((1+\delta)\gamma+2, p, \|f\|_{L^p}, \|f\|_{L^1}\big) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} Err_4 &\leq 2\varepsilon^\delta \|D^2\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon*}|u|^{(1+\delta)\gamma+2} \mathrm{d}v_* \mathrm{d}v \\ &\leq 2\varepsilon^\delta \|D^2\varphi\|_{L^\infty} \mathcal{C}\big((1+\delta)\gamma+2, p, \|f\|_{L^p}, \|f\|_{L^1}\big). \end{aligned} \tag{3.18}$$

Each of these error terms decays at a rate of ε^δ.

Next, we estimate I_2^ε and I_4^ε. In I_2^ε, note that $|2u_iu_j - |u|^2\Pi(u)_{ij}| \le 4|u|^2$. Then, using the same δ,

$$
\begin{aligned}
I_2^\varepsilon = \frac{\pi}{4}\iint \beta_4[g_\varepsilon](u) f_\varepsilon f_{\varepsilon *}(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)(2u_iu_j - |u|^2\Pi(u)_{ij})\mathrm{d}v_*\mathrm{d}v \\
\le 2\pi\|D^2\varphi\|_{L^\infty}\iint f_\varepsilon f_{\varepsilon *}|u|^2\beta_4[g_\varepsilon](u)\mathrm{d}v_*\mathrm{d}v \\
= 4\|D^2\varphi\|_{L^\infty}\iint_{\varepsilon|u|^\gamma\le 1} f_\varepsilon f_{\varepsilon *}|u|^{2\gamma+2}\mathrm{d}v_*\mathrm{d}v \\
+ \frac{4}{\varepsilon}\|D^2\varphi\|_{L^\infty}\iint_{\varepsilon|u|^\gamma> 1} f_\varepsilon f_{\varepsilon *}|u|^{2}\mathrm{d}v_*\mathrm{d}v \\
\le 4\varepsilon^\delta\|D^2\varphi\|_{L^\infty}\iint_{\varepsilon|u|^\gamma\le 1} (\varepsilon|u|^\gamma)^{1-\delta} f_\varepsilon f_{\varepsilon *}|u|^{(1+\delta)\gamma+2}\mathrm{d}v_*\mathrm{d}v \\
+ \frac{4}{\varepsilon}\|D^2\varphi\|_{L^\infty}\iint_{\varepsilon|u|^\gamma> 1} (\varepsilon|u|^\gamma)^{1+\delta} f_\varepsilon f_{\varepsilon *}|u|^{2}\mathrm{d}v_*\mathrm{d}v \\
\le 4\varepsilon^\delta\|D^2\varphi\|_{L^\infty}\iint_{\varepsilon|u|^\gamma\le 1} f_\varepsilon f_{\varepsilon *}|u|^{(1+\delta)\gamma+2}\mathrm{d}v_*\mathrm{d}v \\
\le 4\varepsilon^\delta\|D^2\varphi\|_{L^\infty}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f_\varepsilon f_{\varepsilon *}|u|^{(1+\delta)\gamma+2}\mathrm{d}v_*\mathrm{d}v \\
\le 4\varepsilon^\delta\|D^2\varphi\|_{L^\infty}\mathcal{C}\big((1+\delta)\gamma+2, p, \|f\|_{L^p}, \|f\|_{L^1}\big). \quad (3.19)
\end{aligned}
$$

In the seventh line of (3.19), we used the fact that $\varepsilon|u|^\gamma \le 1$ on the domain of integration, which allowed us to get rid of $1+\delta$ powers of $\varepsilon|u|^\gamma$ from the integrand in order to soften the singularity.

The last integral is estimated as follows:

$$
\begin{aligned}
I_4^\varepsilon = \frac{1}{2} \iint f_\varepsilon f_{\varepsilon *} G_3[g_\varepsilon](v, v_*) \mathrm{d}v_* \mathrm{d}v \\
&\le \frac{1}{3} \|D^3\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *} |u|^3 \beta_3[g_\varepsilon](u) \mathrm{d}v_* \mathrm{d}v \\
&= \frac{\sqrt{2}}{3\pi} \sqrt{\varepsilon} \|D^3\varphi\|_{L^\infty} \iint_{\varepsilon|u|^\gamma \le 1} f_\varepsilon f_{\varepsilon *} |u|^{3+\frac{3}{2}\gamma} \mathrm{d}v_* \mathrm{d}v \\
&\quad + \frac{\sqrt{2}}{3\pi\varepsilon} \|D^3\varphi\|_{L^\infty} \iint_{\varepsilon|u|^\gamma > 1} f_\varepsilon f_{\varepsilon *} |u|^3 \mathrm{d}v_* \mathrm{d}v \\
&=: J_1^\varepsilon + J_2^\varepsilon. \quad (3.20)
\end{aligned}
$$

By the same technique as before, one can see that J_2^ε decays at a rate of ε^δ. Indeed,

$$
\begin{aligned}
J_2^\varepsilon \le \frac{\sqrt{2}}{3\pi} \varepsilon^\delta \|D^3\varphi\|_{L^\infty} \iint_{\varepsilon|u|^\gamma} f_\varepsilon f_{\varepsilon *} |u|^{(1+\delta)\gamma+3} \mathrm{d}v_* \mathrm{d}v \\
&\le \frac{\sqrt{2}}{3\pi} \varepsilon^\delta \|D^3\varphi\|_{L^\infty} \mathcal{C}((1+\delta)\gamma + 3, p, \|f_\varepsilon\|_{L^p}, \|f_\varepsilon\|_{L^1}). \quad (3.21)
\end{aligned}
$$

It is clear that Lemma 2.6 applies here because it applies in the previous integrals when $\alpha = (1+\delta)\gamma + 2 < (1+\delta)\gamma + 3$.

Finally, we examine J_1^ε. First, notice that since the domain of integration is $\varepsilon|u|^\gamma \le 1$ and not the opposite, we cannot fabricate more powers of ε like we did in (3.13), for example. Thus, the convergence rate is at most $\sqrt{\varepsilon}$. In this case, we must choose $\delta = \frac{1}{2}$ in all of the previous integrals in order to match the $\sqrt{\varepsilon}$ in (3.21). We also need $\delta \in \left(0, \min\{-\frac{5}{\gamma} - \frac{2}{p}, -\frac{(8+\gamma)p-6}{\gamma p}\}\right)$, which means $\delta = \frac{1}{2}$ is only possible if $\frac{1}{2} < -\frac{(8+\gamma)p-6}{\gamma p}$, or equivalently, $p > \frac{12}{3\gamma+16}$. In the Coulomb case this corresponds to $p > \frac{12}{7}$.

Alternatively, if we wish not to assume that $p > \frac{12}{3\gamma+16}$, we must add powers of $|u|$ (and hence subtract powers of ε) to J_1^ε to make its the singularity softer. For $t > 0$, in order for $\alpha = 3 + t\gamma > \max\{-3, -6\frac{p-1}{p}\}$ we need $t < -3\frac{3p-2}{\gamma p}$. Then we obtain

$$
\begin{aligned}
J_1^\varepsilon = \varepsilon^{t-1}\frac{\sqrt{2}}{3\pi}\|D^3\varphi\|_{L^\infty} \iint_{\varepsilon|u|^\gamma \leq 1} (\varepsilon|u|^\gamma)^{\frac{3}{2}-t} f_\varepsilon f_{\varepsilon *}|u|^{3+t\gamma}\mathrm{d}v_*\mathrm{d}v \\
\leq \varepsilon^{t-1}\frac{\sqrt{2}}{3\pi}\|D^3\varphi\|_{L^\infty} \iint f_\varepsilon f_{\varepsilon *}|u|^{3+t\gamma}\mathrm{d}v_*\mathrm{d}v \\
\leq \varepsilon^{t-1}\frac{\sqrt{2}}{3\pi}\|D^3\varphi\|_{L^\infty}\mathcal{C}\left(3+t\gamma, p, \|f_\varepsilon\|_{L^p}, \|f_\varepsilon\|_{L^1}\right) \quad (3.22)
\end{aligned}
$$

provided that $t < \frac{3}{2}$. Therefore we need $1 < t < \min\{\frac{3}{2}, -3\frac{3p-2}{\gamma p}\}$, in which case J_1^ε decays like ε^{t-1}. Then r, the exponent of ε that defines the rate of convergence of $Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon) \to Q_L(f_\varepsilon, f_\varepsilon)$, is majorized by

$$
\begin{aligned}
r &< \min\{t-1, \delta\} \\
&< \min\left\{\frac{1}{2}, -\frac{(9+\gamma)p-6}{\gamma p}, -\frac{5p+2\gamma}{\gamma p}, -\frac{(8+\gamma)p-6}{\gamma p}\right\} \\
&= \min\left\{\frac{1}{2}, -\frac{5p+2\gamma}{\gamma p}, -\frac{(8+\gamma)p-6}{\gamma p}\right\}. \quad (3.23)
\end{aligned}
$$

□

Chapter 4

Estimates on $Q^+_{g_\varepsilon}$

Our strategy for finding L^p solutions to (1.10) relies upon the fact that the equation can be written as $f_{\varepsilon t} + \frac{8}{\varepsilon}\|f_0\|_{L^1} f_\varepsilon = Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)$ as it is in (3.5). This way,

$$\frac{\partial}{\partial t}\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)} + \frac{8}{\varepsilon}\|f_0\|_{L^1}\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)} = \|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)\|_{L^p(\mathbb{R}^3)}. \tag{4.1}$$

A Young's type inequality for L^p estimates on $Q^+_{g_\varepsilon}$ will allow us to turn (4.1) into a simple ordinary differential inequality, where a maximum principle can be applied to obtain L^p bounds on f_ε.

First, let us introduce some notation: for $\eta, \psi, \in C_B(\mathbb{R}^3)$, let

$$\begin{aligned}
\mathcal{P}_\varepsilon(\eta,\psi)(u) &:= \int_{S^2} \eta(u^-)\psi(u^+) g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma,\\
u^- &:= \frac{1}{2}(u - |u|\sigma) = v'_* - v_* = v - v'\\
u^+ &:= \frac{1}{2}(u + |u|\sigma) = v' - v_*.
\end{aligned}$$

Then

$$\int_{\mathbb{R}^3} Q^+_{g_\varepsilon}(f,h)(v)\psi(v)\mathrm{d}v = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)h(v-u)\mathcal{P}_\varepsilon(\tau_v\mathcal{R}\psi, 1)(u)\mathrm{d}u\mathrm{d}v. \tag{4.2}$$

Finally, define the following radially symmetric functions: for any $f \in L^p$, $1 \le p \le \infty$, let

$$\begin{aligned} f_p^*(u) &:= \left(\int_{R \in SO(3)} |f(Ru)|^p \mathrm{d}R \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{|S^2|} \int_{\sigma \in S^2} |f(|u|\sigma)|^p \mathrm{d}\sigma \right)^{\frac{1}{p}} \text{ if } 1 \le p < \infty, \\ f_\infty^*(u) &:= \operatorname*{ess\,sup}_{R \in SO(3)} |f(Ru)| = \operatorname*{ess\,sup}_{\sigma \in S^2} |f(|u|\sigma)|. \end{aligned}$$

Such functions satisfy the following properties:

(i) f_p^* is radial, i.e. $f_p^*(u) = f_p^*(x)$ whenever $|u| = |x|$.

(ii) If g is radial, then $(fg)_p^*(u) = f_p^* g(u)$.

(iii) If $\mathrm{d}\nu$ is a rotationally invariant measure on $\mathbb{R}^3$, then

$$\int_{\mathbb{R}^3} |f(u)|^p \mathrm{d}\nu(u) = \int_{\mathbb{R}^3} |f_p^*(u)|^p \mathrm{d}\nu(u),$$

and in particular $\|f\|_{L^p(\mathbb{R}^3)} = \|f_p^*\|_{L^p(\mathbb{R}^3)}$.

4.1 Auxiliary lemmas

Lemma 4.1.1. Let $\eta, \psi, \phi \in C_0(\mathbb{R}^3)$ and $1/p + 1/q + 1/r = 1$, with $1 \le p, q, r \le \infty$. Then,

$$\left| \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta, \psi)(u) \phi(u) \mathrm{d}u \right| \le \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta_p^*, \psi_q^*)(u) \phi_r^*(u) \mathrm{d}u. \tag{4.3}$$

Proof. This lemma and its proof are almost identical to Lemma 3 of [6]. For some $R \in SO(3)$ we begin with the changes of variable $u \longrightarrow Ru$ and then $\sigma \longrightarrow R\sigma$ in the left hand side of (4.3):

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta,\psi)(u)\phi(u)\mathrm{d}u\right| &= \left|\int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta,\psi)(Ru)\phi(Ru)\mathrm{d}u\right| \\
&= \Big|\int_{\mathbb{R}^3}\int_{S^2} \eta\left(\frac{1}{2}(Ru - |u|R\sigma)\right)\psi\left(\frac{1}{2}(Ru + |u|R\sigma)\right)\cdot \\
&\qquad \cdot g_\varepsilon(|u|^\gamma, R\hat{u}\cdot R\sigma)\mathrm{d}\sigma\phi(Ru)\mathrm{d}u\Big| \\
&\le \int_{\mathbb{R}^3}\int_{S^2} |\eta(Ru^-)||\psi(Ru^+)|g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma|\phi(Ru)|\mathrm{d}u. \quad (4.4)
\end{aligned}$$

We can characterize the rotation $R = R_{\bar{\theta},\bar{\omega}}$, where $\bar{\theta} \in [0,\pi], \bar{\omega} \in S^1$ are defined such that $R_{\bar{\theta},\bar{\omega}}\hat{u} = \hat{u}\cos\bar{\theta} + \bar{\omega}\sin\bar{\theta}$. Since R is arbitrary and the left hand side of (4.3) does not depend on R we can take the average over all possible rotations in (4.4) to get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta,\psi)(u)\psi(u)\mathrm{d}u\right| \\
&\quad \le \int_{\mathbb{R}^3}\int_{S^2}\left(\int_{R\in SO(3)} |\eta(Ru^-)||\psi(Ru^+)|\phi(Ru)|\mathrm{d}R\right) g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}u \\
&\quad \le \int_{\mathbb{R}^3}\int_{S^2}\left(\int_{SO(3)} |\eta(Ru^-)|^p\mathrm{d}R\right)^{\frac{1}{p}}\left(\int_{SO(3)} |\psi(Ru^+)|^q\mathrm{d}R\right)^{\frac{1}{q}}\cdot \\
&\qquad \cdot\left(\int_{SO(3)} |\varphi(Ru^-)|^r\mathrm{d}R\right)^{\frac{1}{r}} g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}u \\
&\quad = \int_{\mathbb{R}^3}\int_{S^2}\Big(\eta_p^*(u^-)\psi_q^*(u^+)\phi_r^*(u)\Big)g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}u \\
&\quad = \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta_p^*,\psi_q^*)(u)\phi_r^*(u)\mathrm{d}u, \quad (4.5)
\end{aligned}$$

where in the end we used Holder's inequality with the exponents p, q, r. This concludes the proof. $\square$

Now we can take advantage of the fact that η_p^*, ψ_q^* are radial to simplify the expression $\mathcal{P}_\varepsilon(\eta_p^*, \psi_q^*)$. For any function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, let $\bar{f} : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be such that $f(x) = \bar{f}(|x|)$ for all $x \in \mathbb{R}^3$. Then,

$$\begin{aligned}\mathcal{P}_\varepsilon(\eta_p^*, \psi_q^*)(u) &= \int_{S^2} \bar{\eta_p^*}(|u^-|)\bar{\psi_q^*}(|u|^+)g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma \\ &= 2\pi\int_0^1 \bar{\eta_p^*}(a_1(|u|^\gamma, \mu))\bar{\psi_q^*}(a_2(|u|^\gamma, \mu))g_\varepsilon(|u|^\gamma, \mu)\mathrm{d}\mu, \quad (4.6)\end{aligned}$$

where

$$\begin{aligned}a_1(|u|, \mu) &:= |u|\sqrt{\frac{1}{2}(1+\mu)} = |u^+|, \\ a_2(|u|, \mu) &:= |u|\sqrt{\frac{1}{2}(1-\mu)} = |u^-|.\end{aligned}$$

This motivates the introduction of a new, simpler bilinear operator defined over bounded, continuous functions of one variable: for $\eta, \psi \in C_B(\mathbb{R}^+)$ define

$$\mathcal{B}_\varepsilon(\eta, \psi)(x) := \int_0^1 \eta(a_1(x, \mu))\psi(a_2(x, \mu))g_\varepsilon(x, \mu)\mathrm{d}\mu$$

We prove the following lemma, which is the equivalent of Lemma 4 from [6]:

Lemma 4.1.2. Let $1 \le p \le \infty$. Then for any $\eta \in L^p(\mathbb{R}^+, x^2\mathrm{d}x)$ and $\psi \in L^\infty(\mathbb{R}^+)$,

$$\|\mathcal{B}_\varepsilon(\eta, \psi)\|_{L^p(\mathbb{R}^+, x^2\mathrm{d}x)} \le \frac{2^{2+\frac{1}{2p}}}{\pi\varepsilon}\|\psi\|_{L^\infty(\mathbb{R}^+)}\|\eta\|_{L^p(\mathbb{R}^+, x^2\mathrm{d}x)}. \quad (4.7)$$

Proof. If we let $\mu_\varepsilon(x) := 1 - m_\varepsilon(x) \in [-1, 1)$ be the mass of g_ε, then by definition,

$$\begin{aligned}\mathcal{B}_\varepsilon(\eta,\psi)(x) &= \frac{4}{\pi\varepsilon}\eta(a_1(x,\mu_\varepsilon(x)))\psi(a_2(x,\mu_\varepsilon(x)))\mathbb{1}_{0\le\mu_\varepsilon(x)\le 1} \\ &= \frac{4}{\pi\varepsilon}\eta(a_1(x,\mu_\varepsilon(x)))\psi(a_2(x,\mu_\varepsilon(x)))\mathbb{1}_{0\le m_\varepsilon(x)\le 1} \\ &= \frac{4}{\pi\varepsilon}\eta(a_1(x,\mu_\varepsilon(x)))\psi(a_2(x,\mu_\varepsilon(x)))\mathbb{1}_{x\ge\varepsilon^{\frac{1}{3}}}. \quad (4.8)\end{aligned}$$

The case $p = \infty$ is trivial, so we assume that $p \neq \infty$. Then

$$\begin{aligned}\|\mathcal{B}_\varepsilon(\eta,\psi)\|^p_{L^p(\mathbb{R}^+,x^2\mathrm{d}x)} &= \left(\frac{4}{\pi\varepsilon}\right)^p \int_{\varepsilon^{\frac{1}{3}}}^\infty \eta(a_1(x,\mu_\varepsilon(x)))^p\psi(a_2(x,\mu_\varepsilon(x)))^p x^2\mathrm{d}x \\ &\le \left(\frac{4}{\pi\varepsilon}\right)^p \|\psi\|^p_{L^\infty(\mathbb{R}^+)} J_{p,\varepsilon}(\eta), \quad (4.9)\end{aligned}$$

where

$$J_{p,\varepsilon}(\eta) := \int_{\varepsilon^{\frac{1}{3}}}^\infty \eta(a_1(x,\mu_\varepsilon(x)))^p x^2\mathrm{d}x.$$

We estimate the integral $J_{p,\varepsilon}(\eta)$ by performing the change of variable $a_1(x,\mu_\varepsilon(x)) = a_1(x,\varepsilon x^{-3}) \longmapsto x$ as follows:

$$\begin{aligned}a_1(x,\mu_\varepsilon(x)) &= x\sqrt{\frac{1}{2}(1+\mu_\varepsilon(x))} = x\sqrt{1-\frac{\varepsilon}{2x^3}} = \sqrt{x^2-\frac{\varepsilon}{2x}}, \\ a_1'(x,\mu_\varepsilon(x)) &= \frac{1}{2a_1(x,\mu_\varepsilon(x))}\left(2x+\frac{\varepsilon}{2x^2}\right) = \frac{4x^3+\varepsilon}{4x^2 a_1(x,\mu_\varepsilon(x))}\cdot\frac{a_1(x,\mu_\varepsilon(x)}{a_1(x,\mu_\varepsilon(x)}, \\ x^2\mathrm{d}x &= \frac{x^2}{a_1'(x,\mu_\varepsilon(x))}\mathrm{d}a_1(x,\mu_\varepsilon(x)) = \frac{4x^4}{4x^3+\varepsilon}\frac{a_1(x,\mu_\varepsilon(x))^2}{x\sqrt{1-\frac{\varepsilon}{2x^3}}}\mathrm{d}a_1(x,\mu_\varepsilon(x)) \\ &\le \frac{a_1^2}{\sqrt{1-\frac{\varepsilon}{2x^3}}}\mathrm{d}a_1 \le \sqrt{2}a_1^2\mathrm{d}a_1, \quad (4.10)\end{aligned}$$

so

$$J_{p,\varepsilon}(\eta) = \int_{\varepsilon^{\frac{1}{3}}}^{\infty} \eta(a_1(x,\mu_\varepsilon(x)))^p x^2 \mathrm{d}x$$
$$\leq \sqrt{2}\int_0^{\infty} \eta^p(a_1)a_1^2 \mathrm{d}a_1 = \sqrt{2}\|\eta\|^p_{L^p(\mathbb{R}^+,x^2\mathrm{d}x)}, \quad (4.11)$$

as was to be shown.

□

Lemma 4.1.3. Let $1 \leq p \leq \infty$. The bilinear operator $\mathcal{P}_\varepsilon$ extends to a bounded operator from $L^p(\mathbb{R}^3) \times L^\infty(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$, and

$$\|\mathcal{P}_\varepsilon(\eta,\psi)\|_{L^p(\mathbb{R}^3)} \leq \frac{C_p}{\varepsilon}\|\psi\|_{L^\infty(\mathbb{R}^3)}\|\eta\|_{L^p(\mathbb{R}^3)}. \quad (4.12)$$

where $C_p := 2^{3+\frac{1}{2p}}$ for $1 \leq p < \infty$ and $C_\infty = 8$.

Proof. Let $\eta \in L^p(\mathbb{R}^3), \psi \in L^\infty(\mathbb{R}^3)$ and $\phi \in L^{p'}(\mathbb{R}^3)$. By Lemma 4.1.1 combined with a density argument,

$$\int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta,\psi)(u)\phi(u)\mathrm{d}u \leq \int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta^*_p,\psi^*_\infty)(u)\phi^*_{p'}(u)\mathrm{d}u.$$

Since the functions $\eta^*_p, \psi^*_\infty, \phi^*_{p'}$ are radial in u, let $\bar{\eta}_p, \bar{\psi}_\infty, \bar{\phi}_{p'} : \mathbb{R}^+ \longmapsto \mathbb{R}$ such that for any $u \in \mathbb{R}^3$, $\eta^*_p(u) = \bar{\eta}_p(|u|), \psi^*_\infty(u) = \bar{\psi}_\infty(|u|), \phi^*_{p'}(u) = \bar{\phi}_{p'}(|u|)$. Then for $p \neq \infty$ $\bar{\eta}_p \in L^p(\mathbb{R}^+, x^2\mathrm{d}x), \bar{\psi}_q \in L^q(\mathbb{R}^3, x^2\mathrm{d}x)$ and $\bar{\phi}_r \in L^r(\mathbb{R}^3, x^2\mathrm{d}x)$ and

$$\int_{\mathbb{R}^3} \mathcal{P}_\varepsilon(\eta^*_p,\psi^*_\infty)(u)\phi^*_{p'}(u)\mathrm{d}u = 2\pi\int_0^\infty \mathcal{B}_\varepsilon(\bar{\eta}_p,\bar{\psi}_\infty)(x)\bar{\phi}_{p'}(x)x^2\mathrm{d}x$$
$$\leq 2\pi\|\mathcal{B}_\varepsilon(\bar{\eta}_p,\bar{\psi}_\infty)\|_{L^p(\mathbb{R}^+,x^2\mathrm{d}x)}\|\phi^*_{p'}\|_{L^{p'}(\mathbb{R}^3)}$$
$$\leq 2\pi\frac{2^{2+\frac{1}{2p}}}{\pi\varepsilon}\|\bar{\psi}_\infty\|_{L^\infty(\mathbb{R}^+)}\|\bar{\eta}_p\|_{L^p(\mathbb{R}^+,x^2\mathrm{d}x)}\|\phi\|_{L^{p'}(\mathbb{R}^3)}$$
$$= \frac{C_p}{\varepsilon}\|\psi\|_{L^\infty(\mathbb{R}^3)}\|\eta\|_{L^p(\mathbb{R}^3}\|\phi\|_{L^{p'}(\mathbb{R}^3)}.$$

For $p = \infty$ the estimate above is almost identical (by replacing $\|\bar{\eta}_p\|_{L^p(\mathbb{R}^+, x^2 \mathrm{d}x)}$ with $\|\bar{\eta}_\infty\|_{L^\infty(\mathbb{R}^+)}$). This shows us that $\mathcal{P}_\varepsilon(\eta, \psi)$ is a bounded, real-valued linear operator acting on $L^{p'}$, and therefore belongs to L^p with norm bounded by

$$\|\mathcal{P}_\varepsilon(\eta, \psi)\|_{L^p(\mathbb{R}^3)} \leq \frac{C_p}{\varepsilon} \|\psi\|_{L^\infty(\mathbb{R}^3)} \|\eta\|_{L^p(\mathbb{R}^3)}, \tag{4.13}$$

as was to be shown. $\square$

4.2 L^1_k bounds for $Q^+_{g_\varepsilon}$

The following theorem will be needed in the proof of existence of f_ε to guarantee that $Q^+_{g_\varepsilon}$ maps $L^1_k \times L^1_k$ to L^1_k for $0 \leq k \leq 2$.

Theorem 4.2.1. If $f, h \in L^1_k(\mathbb{R}^3)$ for any $k \geq 0$, then

$$\|Q^+_{g_\varepsilon}(f, h)\|_{L^1_k(\mathbb{R}^3)} \leq \frac{2^{\frac{k}{2}+5}}{\varepsilon} \|f\|_{L^1_k(\mathbb{R}^3)} \|h\|_{L^1_k(\mathbb{R}^3)}. \tag{4.14}$$

Proof. First, note that for any $k \geq 0$, by conservation of energy,

$$\begin{aligned} \langle v \rangle^k \leq \left(\langle v' \rangle^2 + \langle v'_* \rangle^2 \right)^{\frac{k}{2}} &\leq \left(2 \langle v' \rangle^2 \right)^{\frac{k}{2}} \mathbb{1}_{|v'| \leq |v'_*|} + \left(2 \langle v'_* \rangle^2 \right)^{\frac{k}{2}} \mathbb{1}_{|v'_*| < |v'|} \\ &\leq 2^{\frac{k}{2}} \left(\langle v' \rangle^k + \langle v'_* \rangle^k \right) = 2^{\frac{k}{2}} \left(\langle v' \rangle^k + \langle v' - u' \rangle^k \right). \end{aligned} \tag{4.15}$$

This means that, for any test function $\psi \in L^\infty(\mathbb{R}^3)$, defining if $\eta_k(v) :=$

$\eta(v)\langle v\rangle^k$ for $\eta : \mathbb{R}^3 \mapsto \mathbb{R}$,

$$\int_{\mathbb{R}^3} Q^+_{g_\varepsilon}(f,h)(v)\langle v\rangle^k \psi(v)\mathrm{d}v$$
$$\leq 2^{\frac{k}{2}} \iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} f_k(v')(h(v'-u')g_\varepsilon(|u|^\gamma,\hat{u}\cdot\sigma)\psi(v)\mathrm{d}\sigma\mathrm{d}v\mathrm{d}u$$
$$= 2^{\frac{k}{2}} \int_{\mathbb{R}^3} Q^+_{g_\varepsilon}(f_k,h)\psi(v)\mathrm{d}v. \quad (4.16)$$

Finally, using (4.2), Holder's inequality and (4.12), the right hand side of (4.16) can be estimated as

$$2^{\frac{k}{2}} \int_{\mathbb{R}^3} Q^+_{g_\varepsilon}(f_k,h)\psi(v)\mathrm{d}v$$
$$\leq 2^{\frac{k}{2}} \|\mathcal{P}_\varepsilon(\psi,1)\|_{L^\infty(\mathbb{R}^3)} \iint_{\mathbb{R}^3\times\mathbb{R}^3} f_k(v)h(v-u)\mathrm{d}v\mathrm{d}u$$
$$\leq \frac{2^{\frac{k}{2}+3}}{\varepsilon}\|\psi\|_{L^\infty(\mathbb{R}^3)}\|f_k\|_{L^1(\mathbb{R}^3)}\|h\|_{L^1(\mathbb{R}^\infty)}. \quad (4.17)$$

The result follows by choosing $\psi = 1$.

□

4.3 L^p bounds on $Q^+_{g_\varepsilon}$

The following theorem is an extension of Theorem 4.2.1 to weighted L^p spaces and will be needed for L^p a priori bounds on f_ε, so that we can implore (4.1).

Theorem 4.3.1. If $1 \leq p,q,r \leq \infty$ such that $1/p + 1/q = 1 + 1/r$, then the bilinear operator $Q^+_{g_\varepsilon}$ extends to a bounded operator from $L^p(\mathbb{R}^3)\times L^q(\mathbb{R}^3)$ to $L^r(\mathbb{R}^3)$, and

1. <u>If $r < \infty$</u>, then

$$\|Q^+_{g_\varepsilon}(f,h)\|_{L^r(\mathbb{R}^3)} \leq 2\frac{C_{r'}}{\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}\|h\|_{L^q(\mathbb{R}^3)}. \tag{4.18}$$

2. <u>If $r = \infty$</u>, then

$$\|Q^+_{g_\varepsilon}(f,h)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{64}{\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}\|h\|_{L^q(\mathbb{R}^3)}. \tag{4.19}$$

Proof. <u>Case 1.</u> Suppose that $r < \infty$. The case $r = 1$ is covered in Theorem 4.2.1 (take $k = 0$), so we assume that $r \in (1, \infty)$. Let $\psi \in L^{r'}(\mathbb{R}^3)$ be a test function, and define

$$K_\psi := \int_{\mathbb{R}^3} Q^+_{g_\varepsilon}(f,h)(v)\psi(v)\mathrm{d}v. \tag{4.20}$$

Since $\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1$, we can Holder's inequality and (4.2) to write

$$\begin{aligned}
K_\psi &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} f(v)h(v-u)\mathcal{P}_\varepsilon(\tau_v\mathcal{R}\psi,1)(u)\mathrm{d}u\mathrm{d}v \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(f(v)^{\frac{p}{r}}h(v-u)^{\frac{q}{r}}\right)\left(f(v)^{\frac{p}{q'}}\mathcal{P}_\varepsilon(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{q'}}\right) \\
&\qquad\qquad \times\left(g(v-u)^{\frac{q}{p'}}\mathcal{P}_\varepsilon(\tau_v\mathcal{R}\psi,1)(u)^{\frac{r'}{p'}}\right)\mathrm{d}u\mathrm{d}v \\
&\leq K^1K^2_\psi K^3_\psi,
\end{aligned} \tag{4.21}$$

where

$$K^1 = \left(\iint f(v)^p h(v-u)^q\mathrm{d}u\mathrm{d}v\right)^{\frac{1}{r}} = \|f\|_{L^p}^{\frac{p}{r}}\|h\|_{L^q}^{\frac{q}{r}} \tag{4.22}$$

and by Lemma 4.1.3,

$$\begin{aligned}
K^2_\psi &:= \left(\iint f(v)^p\mathcal{P}_\varepsilon(\tau_v\mathcal{R}\psi,1)(u)^{r'}\mathrm{d}u\mathrm{d}v\right)^{\frac{1}{q'}} \\
&= \left(\int f(v)^p\|\mathcal{P}_\varepsilon(\tau_v\mathcal{R}\psi,1)(u)^{r'}\|_{L^{r'}}^{r'}\mathrm{d}v\right)^{\frac{1}{q'}} \leq \left(\frac{C_{r'}}{\varepsilon}\right)^{\frac{r'}{q'}}\|\psi\|_{L^{r'}}^{\frac{r'}{q'}}\|f\|_{L^p}^{\frac{p}{q'}},
\end{aligned}$$

$$K_\psi^3 := \left(\iint h(v-u)^q \mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)(u)^{r'} \mathrm{d}u\mathrm{d}v\right)^{\frac{1}{p'}}$$
$$= \left(\int h(v_*)^q \int \mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)(v-v_*)^{r'} \mathrm{d}v\mathrm{d}v_*\right)^{\frac{1}{p'}}$$
$$= \left(\int h(v_*)^q \|\tau_{v_*}\mathcal{P}_\varepsilon(\tau_v \mathcal{R}\psi, 1)\|_{L^{r'}}^{r'}\right)^{\frac{1}{p'}} \leq \left(\frac{C_{r'}}{\varepsilon}\right)^{\frac{r'}{p'}} \|\psi\|_{L^{r'}}^{\frac{r'}{p'}} \|h\|_{L^q}^{\frac{q}{p'}}.$$

Gathering the estimates for K^1, K_ψ^2 and K_ψ^3,

$$\int_{\mathbb{R}^3} Q_{g_\varepsilon}^+(f,h)(v)\psi(v)\mathrm{d}v \leq \frac{C_{r'}}{\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}\|h\|_{L^q(\mathbb{R}^3)}\|\psi\|_{L^{r'}(\mathbb{R}^3)}. \tag{4.23}$$

We have shown that for all $f \in L^p(\mathbb{R}^3)$ and $h \in L^q(\mathbb{R}^3)$, $Q_{g_\varepsilon}^+(f,h) \in (L^{r'}(\mathbb{R}^3))' = L^r(\mathbb{R}^3)$, and

$$\|Q_{g_\varepsilon}^+(f,h)\|_{L^r(\mathbb{R}^3)} = \sup_{\psi \in L^{r'}(\mathbb{R}^3)} \frac{\left|\int_{\mathbb{R}^3} Q_{g_\varepsilon}^+(f,h)(v)\psi(v)\mathrm{d}v\right|}{\|\psi\|_{L^{r'}(\mathbb{R}^3)}}$$
$$\leq \frac{C_{r'}}{\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}\|h\|_{L^q(\mathbb{R}^3)}, \tag{4.24}$$

so (4.18) holds.

Case 2. If $(r,p,q) = (\infty, p, q)$ for $p, q < \infty$, then p, q are Holder conjugates. By Holder's inequality, for a.e. $v \in \mathbb{R}^3$,

$$Q_{g_\varepsilon}^+(f,h)(v) = \int_{\mathbb{R}^3}\int_{S^2} f\big(v'(v,v_*,\sigma)\big)h\big(v'_*(v,v_*,\sigma)\big)g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*$$
$$\leq \left(\int_{\mathbb{R}^3}\int_{S^2} f\big(v'(v,v_*,\sigma)\big)^p g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\right)^{\frac{1}{p}}$$
$$\times \left(\int_{\mathbb{R}^3}\int_{S^2} h\big(v'_*(v,v_*,\sigma)\big)^q g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\right)^{\frac{1}{q}}. \tag{4.25}$$

It is important to note that *the strong form of $Q_{g_\varepsilon}^+$ is symmetric in f and h.* Indeed, due to the symmetry of the collision cross section with respect to exchanging $\sigma \leftrightarrow -\sigma$ in (4.25), we see that

$$v'(v,v_*,\sigma) = \frac{v+v_*}{2} + \frac{1}{2}|u|\sigma \longleftrightarrow \frac{v+v_*}{2} - \frac{1}{2}|u|\sigma = v'_*(v,v_*,\sigma). \tag{4.26}$$

Therefore, (4.25) can be rewritten as

$$Q^+_{g_\varepsilon}(f,h)(v) \le \left(\int_{\mathbb{R}^3}\int_{S^2} f(v'_*)^p g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\right)^{\frac{1}{p}} \times \left(\int_{\mathbb{R}^3}\int_{S^2} h(v'_*)^q g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_*\right)^{\frac{1}{q}}. \tag{4.27}$$

Now, for $0 \le F \in L^1(\mathbb{R}^3)$, let us examine

$$\begin{aligned} I(F) &:= \int_{\mathbb{R}^3}\int_{S^2} F(v'_*) g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* \\ &= \int_{S^1}\int_0^{\pi/2}\int_{\mathbb{R}^3} F(v'_*) g_\varepsilon\left(\left(\frac{|v-v'_*|}{\cos(\theta/2)}\right)^\gamma, \cos\theta\right)\sin\theta \mathrm{d}v_*\mathrm{d}\theta\mathrm{d}\omega \\ &= 2\int_{S^1}\int_0^{\pi/2}\left[\int_{\mathbb{R}^3} F(v'_*) g_\varepsilon\left(\left(\frac{|v-v'_*|}{\cos(\theta/2)}\right)^\gamma, 2\cos^2(\theta/2)-1\right)\mathrm{d}v_*\right]\cdot \\ &\qquad \cdot \sin(\theta/2)\cos(\theta/2)\mathrm{d}\theta\mathrm{d}\omega. \end{aligned}$$

For fixed θ, ω, we change the variable $v'_* \mapsto v_*$. Since $|\frac{\partial v'_*}{\partial v_*}| = \frac{1}{8}(1+\cos\theta) \ge \frac{1}{8}$ on $[0,\pi/2]$, we see that

$$\begin{aligned} I(F) &\le 16\int_{S^1}\int_0^{\pi/2} \\ &\int_{\mathbb{R}^3} F(v'_*)\, g_\varepsilon\left(\left(\frac{|v-v'_*|}{\cos(\theta/2)}\right)^\gamma, 2\cos^2(\theta/2)-1\right)\mathrm{d}v'_*\sin(\theta/2)\cos(\theta/2)\mathrm{d}\theta\mathrm{d}\omega \\ &= 32\int_{\mathbb{R}^3} F(v_*)\int_{S^1}\int_0^{\frac{\pi}{4}} g_\varepsilon\left(\left(\frac{|u|}{\cos\theta}\right)^\gamma, 2\cos^2\theta - 1\right)\sin\theta\cos\theta\mathrm{d}\theta\mathrm{d}\omega\mathrm{d}v_* \\ &= 16\pi\int_{\mathbb{R}^3} F(v_*)\int_0^1 g_\varepsilon\left(\left(\frac{|u|}{\sqrt{\frac{1}{2}(1-\nu)}}\right)^\gamma, \nu\right)\mathrm{d}\nu\mathrm{d}v_* \\ &= \frac{64}{\varepsilon}\|F\|_{L^1(\mathbb{R}^3)}, \end{aligned} \tag{4.28}$$

where $\nu := 2\cos^2\theta - 1$. The last equality above is due to the fact that $\nu(0) > 0$ and $\nu(1) < 0$, thereby ensuring that the mass of g_ε lies in $\{0 < \nu < 1\}$. Then by (4.28), (4.27) becomes

$$Q^+_{g_\varepsilon}(f,h)(v) \le I(f^p)^{\frac{1}{p}} I(h^q)^{\frac{1}{q}} \le \frac{64}{\varepsilon} \|f\|_{L^p(\mathbb{R}^3)} \|h\|_{L^q(\mathbb{R}^3)}. \tag{4.29}$$

<u>Case 3.</u> If $(r,p,q) = (\infty,\infty,1)$, (4.25) is simply replaced by

$$\begin{aligned} Q^+_{g_\varepsilon}(f,h)(v) &= \int_{\mathbb{R}^3}\int_{S^2} f(v')h(v'_*)g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* \\ &\le \|f\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3}\int_{S^2} h(v'_*)g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* \\ &= \|f\|_{L^\infty(\mathbb{R}^3)} I(h) \le \frac{64}{\varepsilon}\|f\|_{L^\infty(\mathbb{R}^3)}\|h\|_{L^1(\mathbb{R}^3)}. \end{aligned} \tag{4.30}$$

By (4.26), the same argument can be used if $(r,p,q) = (\infty,1,\infty)$:

$$\begin{aligned} Q^+_{g_\varepsilon}(f,h)(v) &= \int_{\mathbb{R}^3}\int_{S^2} f(v'_*)h(v')g_\varepsilon(|u|^|\gamma|, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* \\ &\le \|h\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3}\int_{S^2} f(v'_*)g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma\mathrm{d}v_* \\ &= \|h\|_{L^\infty(\mathbb{R}^3)} I(f) \le \frac{64}{\varepsilon}\|h\|_{L^\infty(\mathbb{R}^3)}\|f\|_{L^1(\mathbb{R}^3)}. \end{aligned} \tag{4.31}$$

This proves (4.19), which completes the proof of the theorem. □

Lemma 4.3.2. For $p \in [1,\infty]$, let $0 \le f_\varepsilon \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \cap L\log L(\mathbb{R}^3)$ be a weak solution to the Boltzmann equation (1.10) with initial data $0 \le f_0 \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, such that $\|f_0\|_{L^1} = 1$.

1. <u>If $p < \infty$,</u> then for all $K > 1$,

$$\begin{aligned} \|Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)\|_{L^p(\mathbb{R}^3)} \le K^{\frac{1}{2p'}} \frac{C_{p'}}{\varepsilon} \|f_\varepsilon\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)} \\ + \frac{C_{p'}}{\varepsilon \log K}\|f_0\|_{L\log L(\mathbb{R}^3)}\|f_\varepsilon\|_{L^p(\mathbb{R}^3)}. \end{aligned} \tag{4.32}$$

2. <u>If $p=\infty$</u>, then for all $K>1$,

$$\|Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)\|_{L^\infty(\mathbb{R}^3)} \le \frac{64}{\varepsilon}\sqrt{K}\|f_\varepsilon\|^{\frac{1}{2}}_{L^\infty} + \frac{64}{\varepsilon\log K}\|f_0\|_{L\log L(\mathbb{R}^3)}\|f_\varepsilon\|_{L^\infty(\mathbb{R}^3)}. \quad (4.33)$$

Proof. <u>Case 1:</u> First, let $p<\infty$. For $K>1$ we can write $Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)$ as

$$\begin{aligned} Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon) &= A(v)+B(v) \\ &:= Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon\mathbb{1}_{f_\varepsilon\le K})(v) + Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon\mathbb{1}_{f_\varepsilon> K})(v) \\ &\le K^{\frac{1}{2p'}}Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon^{\frac{p+1}{2p}})(v) + \frac{1}{\log K}Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon(\log f_\varepsilon)_+)(v). \quad (4.34)\end{aligned}$$

We will find L^p bounds for A and B separately.

Beginning with A, we use Theorem 4.3.1 with the coefficients $\left(p,\frac{2p}{2p+1},\frac{2p}{2p+1}\right)$ to control

$$\begin{aligned}\|A\|_{L^p(\mathbb{R}^3)} &\le K^{\frac{1}{2p'}}\frac{C_{p'}}{\varepsilon}\|f_\varepsilon\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^3)}\|f_\varepsilon^{\frac{p+1}{2p}}\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^3)} \\ &= K^{\frac{1}{2p'}}\frac{C_{p'}}{\varepsilon}\|f_\varepsilon\|_{L^{\frac{2p}{p+1}}(\mathbb{R}^3)}\|f_\varepsilon\|^{\frac{p+1}{2p}}_{L^1(\mathbb{R}^3)} \le K^{\frac{1}{2p'}}\frac{C_{p'}}{\varepsilon}\|f_\varepsilon\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)} \quad (4.35)\end{aligned}$$

($\|f_\varepsilon\|_{L^1}=1$ by mass conservation), where in the end of (4.35) we used the Riesz-Thorin Interpolation Theorem to get $\|f_\varepsilon\|_{L^{\frac{2p}{p+1}}} \le \|f_\varepsilon\|^{\frac{1}{2}}_{L^1}\|f_\varepsilon\|^{\frac{1}{2}}_{L^p} = \|f_\varepsilon\|^{\frac{1}{2}}_{L^p}$.

For B, we apply Theorem 4.3.1 with the coefficients $(p,p,1)$ to obtain

$$\begin{aligned}\|B\|_{L^p(\mathbb{R}^3)} &\le \frac{C_{p'}}{\varepsilon\log K}\|f_\varepsilon\|_{L^p(\mathbb{R}^3)}\|f_\varepsilon(\log f_\varepsilon)_+\|_{L^1(\mathbb{R}^3)} \\ &\le \frac{C_{p'}}{\varepsilon\log K}\|f_\varepsilon\|_{L^p(\mathbb{R}^3)}\|f_0\|_{L\log L(\mathbb{R}^3)}. \quad (4.36)\end{aligned}$$

Together, (4.35) and (4.36) yield (4.32).

Case 2: let $p = \infty$. The term $A(v)$ can be bounded as follows:

$$A(v) = Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon \mathbb{1}_{f_\varepsilon \le K})(v) \le \sqrt{K} Q^+_{g_\varepsilon}(f_\varepsilon, \sqrt{f}_\varepsilon)(v). \tag{4.37}$$

Again by Theorem 4.3.1, using coefficients $(\infty, 1, \infty)$ we have

$$\|A\|_{L^\infty(\mathbb{R}^3)} \le \sqrt{K}\frac{64}{\varepsilon}\|f_\varepsilon\|_{L^1(\mathbb{R}^3)}\|\sqrt{f}_\varepsilon\|_{L^\infty(\mathbb{R}^3)} = \frac{64}{\varepsilon}\sqrt{K}\|f_\varepsilon\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^3)}. \tag{4.38}$$

The estimates on $B(v)$ remain the same:

$$\begin{aligned}
&\|B\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{\log K}\|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon \log f_\varepsilon)\|_{L^\infty(\mathbb{R}^3)} \\
&\le \frac{64}{\varepsilon \log K}\|f_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|f_\varepsilon(\log f_\varepsilon)_+\|_{L^1(\mathbb{R}^3)} \le \frac{64}{\varepsilon \log K}\|f_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|f_0\|_{L\log L(\mathbb{R}^3)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
\|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)\|_{L^\infty(\mathbb{R}^3)} &\le \|A\|_{L^\infty(\mathbb{R}^3)} + \|B\|_{L^\infty(\mathbb{R}^3)} \\
&\le \frac{64}{\varepsilon}\sqrt{K}\|f_\varepsilon\|^{\frac{1}{2}}_{L^\infty(\mathbb{R}^3)} + \frac{64}{\varepsilon \log K}\|f_\varepsilon\|_{L^\infty(\mathbb{R}^3)}\|f_0\|_{L\log L(\mathbb{R}^3)},
\end{aligned} \tag{4.39}$$

which proves the lemma. □

Chapter 5

A priori bounds on f_ε

5.1 Proof of Theorem 1.2

Proof. Case 1: $p < \infty$. Let $K := e^{\frac{1}{7}C_{p'}\|f_0\|_{L\log L}} > 1$ (provided $f_0 \not\equiv 0$). By (4.32),

$$\|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(\cdot,t)\|_{L^p(\mathbb{R}^3)} \le K^{\frac{1}{2p'}} \frac{C_{p'}}{\varepsilon}\|f_\varepsilon(\cdot,t)\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)} + \frac{7}{\varepsilon}\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)}. \quad (5.1)$$

Since $p \neq \infty$, up to approximation by C_0^∞ functions we can use the weak form of (1.10) to make the following calculation:

$$\begin{aligned} p\|f_\varepsilon(\cdot,t)\|^{p-1}_{L^p(\mathbb{R}^3)} &\frac{d}{dt}\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)} \\ &= \frac{d}{dt}\|f_\varepsilon(\cdot,t)\|^p_{L^p(\mathbb{R}^3)} = \int \frac{d}{dt} f_\varepsilon(v,t) f_\varepsilon^{p-1}(v,t)\mathrm{d}v \\ &= \int_{\mathbb{R}^3} Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v,t) f_\varepsilon^{p-1}(v,t)\mathrm{d}v - \frac{8}{\varepsilon}\int f_\varepsilon(v,t)^p \mathrm{d}v \\ &\le \|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(\cdot,t)\|_{L^p(\mathbb{R}^3)}\|f_\varepsilon(\cdot,t)\|^{p-1}_{L^p(\mathbb{R}^3)} - \frac{8}{\varepsilon}\|f_\varepsilon(\cdot,t)\|^p_{L^p(\mathbb{R}^3)} \\ &\le \frac{C_{p'}}{\varepsilon} K^{\frac{1}{2p'}}\|f_\varepsilon(\cdot,t)\|^{p-\frac{1}{2}}_{L^p(\mathbb{R}^3)} - \frac{1}{\varepsilon}\|f_\varepsilon(\cdot,t)\|^p_{L^p(\mathbb{R}^3)}, \end{aligned} \quad (5.2)$$

where, in the end, again we used (4.32). Multiplying both sides of (5.2) by $\frac{1}{2p}\|f_\varepsilon(\cdot,t)\|^{\frac{1}{2}-p}_{L^p(\mathbb{R}^3)}$,

$$\frac{d}{dt}\|f_\varepsilon(\cdot,t)\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)} \le \frac{C_{p'}}{2p\varepsilon}K^{\frac{1}{2p'}} - \frac{1}{2p\varepsilon}\|f_\varepsilon(\cdot,t)\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)}. \quad (5.3)$$

Then if $u(t) := \|f_\varepsilon(\cdot,t)\|_{L^p}^{\frac{1}{2}}$, by (5.3),

$$\begin{cases} u'(t) \le \dfrac{C_{p'}}{2p\varepsilon} K^{\frac{1}{2p'}} - \dfrac{1}{2p\varepsilon} u(t) \\ u(0) = \|f_0\|_{L^p(\mathbb{R}^3)}^{\frac{1}{2}}. \end{cases} \tag{5.4}$$

(1.11) follows immediately by a maximum principle. Indeed, by (5.4),

$$\frac{d}{dt}\left(e^{\frac{t}{2p\varepsilon}} u(t)\right) \le \frac{C_{p'}}{2p\varepsilon} K^{\frac{1}{2p'}} e^{\frac{t}{2p\varepsilon}}, \tag{5.5}$$

so

$$\begin{aligned} u(t) \le u(0) e^{-\frac{t}{2p\varepsilon}} + C_{p'} K^{\frac{1}{2p'}} \left(1 - e^{-\frac{t}{2p\varepsilon}}\right) \\ = \left(u(0) - C_{p'} K^{\frac{1}{2p'}}\right) e^{-\frac{t}{2p\varepsilon}} + C_{p'} K^{\frac{1}{2p'}} \end{aligned} \tag{5.6}$$

and (1.11) follows.

Case 2. $p = \infty$. Let $K := e^{\frac{64}{7}\|f_0\|_{L\log L}} > 1$ (provided $f_0 \not\equiv 0$). By (4.33),

$$\|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \le \frac{64}{\varepsilon}\sqrt{K} + \frac{7}{\varepsilon}\|f_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}. \tag{5.7}$$

Then for a.e. $v \in \mathbb{R}^3$,

$$\begin{aligned} \frac{d}{dt} f_\varepsilon(v,t) &= Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v,t) - Q^-_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v,t) \\ &\le \|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} - \frac{8}{\varepsilon} f_\varepsilon(v,t) \\ &\le \frac{64}{\varepsilon}\sqrt{K} + \frac{7}{\varepsilon}\|f_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} - \frac{8}{\varepsilon} f_\varepsilon(v,t). \end{aligned} \tag{5.8}$$

In particular, by definition of essential supremum, this means that for a.e. $v \in \mathbb{R}^3$,

$$\frac{d}{dt} f_\varepsilon(v,t) \le \frac{64}{\varepsilon}\sqrt{K} - \frac{1}{\varepsilon} f_\varepsilon(v,t). \tag{5.9}$$

By a similar computation as in (5.5) and (5.6), the maximum principle yields

$$f_\varepsilon(v,t) \le \left(\|f_0\|_{L^\infty(\mathbb{R}^3)} - 64\sqrt{K}\right) e^{-\frac{t}{\varepsilon}} + 64\sqrt{K},$$

which implies (1.12). This concludes the proof of the theorem.

□

Chapter 6

The Cauchy problem

One advantage of Theorem 4.2.1 is that it immediately gives us Lipschitz estimates on $Q[f] := Q_{g_\varepsilon}(f,f)$ acting on bounded subsets of L^1_k : if $f, h \in L^1_k$ such that $\|f\|_{L^1_k}, \|h\|_{L^1_k} \leq r$ for some $r > 0$, then

$$\begin{aligned}\|Q_{g_\varepsilon}(f,f) - Q_{g_\varepsilon}(h,h)\|_{L^1_k(\mathbb{R}^3)} &= \|Q_{g_\varepsilon}(f-h, f+h)\|_{L^1_k(\mathbb{R}^3)} \\ &\leq \frac{2^{\frac{k}{2}+5}}{\varepsilon}\|f-h\|_{L^1_k(\mathbb{R}^3)}(\|f\|_{L^1_k(\mathbb{R}^3)} + \|h\|_{L^1_k}) \\ &\leq \frac{2^{\frac{k}{2}+6}}{\varepsilon} r\|f-h\|_{L^1_k(\mathbb{R}^3)}. \qquad (6.1)\end{aligned}$$

Thanks to (6.1), we automatically have a local existence theory in L^1_k, $(0 \leq k \leq 2)$ for the Cauchy problem (1.10) by simply using the Picard iteration

$$\begin{aligned} f^{j+1}_\varepsilon(t) &= f_0 + \int_0^t Q_{g_\varepsilon}(f^j_\varepsilon, f^j_\varepsilon)\mathrm{d}s \qquad (6.2) \\ f^0_\varepsilon(t) &= f_0. \end{aligned}$$

This, however, is of no good to us because there is no guarantee in this case that each f_j (and therefore the limiting solution $f_\varepsilon = \lim_{j\to\infty} f^j_\varepsilon$ of (1.10)) will be nonnegative. Therefore, we need to narrow our search to a positive cone in L^1_k, for which we need more tools.

The following theorem is a simplified version of Theorem A1 from [15], or equivalently, Proposition 5.1 of [8]:

Theorem 6.0.1. Let E be a Banach space, F a bounded, convex and closed subset of E, and $Q : F \longrightarrow E$ a Lipschitz operator on F that satisfies the following subtangency condition: for all $f \in F$,

$$\liminf_{\delta \to 0^+} \frac{1}{\delta} dist_E(f + \delta Q[f], F) = 0. \tag{6.3}$$

Then, the equation

$$\begin{cases} \partial_t f = Q[f] \text{ on } [0, \infty) \times E \\ f(0) = f_0 \geq 0 \in F \text{ on } \{0\} \times E \end{cases} \tag{6.4}$$

has a unique solution, f, which lies in $C^1((0, \infty), E) \cap C([0, \infty), F)$.

As expected, the proof of this theorem still involves Picard type iterations, but the subtangency (6.3) of Q ensures that each iteration remains in F. The fact that F is closed in turn guarantees that the true solution, f, of (6.4) remains in F for all time.

A direct application of Theorem 6.0.1 gives us existence of unique solutions of (1.10).

6.1 Proof of Theorem 1.1

Proof. Let $E := L^1_2(\mathbb{R}^3)$, $f_0 \in E$ and

$$F := \{f \in E : f \geq 0, \|f\|_{L^1_2} \leq \|f_0\|_{L^1_2}\}. \tag{6.5}$$

Fix $\varepsilon > 0$ and define $Q : L^1_2 \mapsto L^1_2$ by $Q[f] := Q_{g_\varepsilon}(f, f)$. By (6.1) we know that Q is Lipschitz on F. If (6.3) holds, then by Theorem 6.0.1, the Boltzmann equation (1.10) has a unique solution. To show (6.3), we use some ideas from Proposition 5.1 of [8].

Fix $h \in F$. For any $\beta > 0$, it suffices to find $\delta_0 = \delta_0(h, \beta) > 0$ and $\omega_h \in F$ such that for any $\delta \in (0, \delta_0)$,

$$\frac{1}{\delta} \|h + \delta Q_{g_\varepsilon}(h, h) - \omega_h\|_{L^1_2} < \beta. \tag{6.6}$$

For $R, \delta > 0$ we define h_R and $\omega_h = \omega_h(\delta, R)$ as follows:

$$h_R := h\mathbb{1}_{|v|\le R}, \qquad \omega_h(\delta, R) := h + \delta Q_{g_\varepsilon}(h_R, h_R).$$

We will find suitable values of R, δ and use them to define $\omega_h \in F$ for which (6.6) holds.

First, we show that $\omega_h \in F$ for large $R > 0$ and small enough δ. Indeed, let $R = R(h)$ be large enough such that $\|h_R\|_{L^1_2} \le 1$ and let $\delta \le \frac{8}{\varepsilon}$. Then,

$$\omega_h \ge h - \delta Q^-_{g_\varepsilon}(h_R, h_R) = h - \frac{8\delta}{\varepsilon}\|h_R\|_{L^1} h_R \ge f\left(1 - \frac{8\delta}{\varepsilon}\right) \ge 0.$$

whenever $\delta \le \varepsilon/8$. Furthermore, since $\int Q_{g_\varepsilon}(h, h)\langle v\rangle^2 = 0$ for any $h \in L^1$,

$$\|\omega_h\|_{L^1_2} = \int \omega_h \langle v\rangle^2 = \|f\|_{L^1_2} + \delta \int Q_{g_\varepsilon}(h_R, h_R)\langle v\rangle^2 = \|f_0\|_{L^1_2},$$

which means that $\omega_h \in F$ for $\delta \le \varepsilon/8$. By (6.1), since $h \in F$,

$$\begin{aligned} \|h + \delta Q_{g_\varepsilon}(h, h) - \omega_h\|_{L^1_2} &= \delta \|Q_{g_\varepsilon}(h, h) - Q_{g_\varepsilon}(h_R, h_R)\|_{L^1_2} \\ &\le 2\frac{\delta C_1}{\varepsilon}\|f_0\|_{L^1_2}\|h - h_R\|_{L^1_2}. \end{aligned} \tag{6.7}$$

Let R be large enough so that $\|h - h_R\|_{L^1_2} \le \delta\beta\varepsilon C_1^{-1}$. Then if $0 < \delta < \delta_0 := \min\{\varepsilon/8, \beta\}$,

$$\frac{1}{\delta}\|h + \delta Q_{g_\varepsilon}(h,h) - \omega_h\|_{L^1_2} \le \delta < \beta,$$

and (6.6) follows. By Theorem 6.0.1, this means that (1.10) has a unique solution, $f_\varepsilon \in F$.

Next, we show that f_ε conserves mass, momentum, energy, and has an entropy bounded by $\|f_0\|_{L\log L}$.

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} f_\varepsilon(v)(1,v,|v|^2)\mathrm{d}v &= \int_{\mathbb{R}^3} Q_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)(v)(1,v,|v|^2)\mathrm{d}v \\ &= \int_{\mathbb{R}^3} Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)(v)(1,v,|v|^2)\mathrm{d}v - \frac{8}{\varepsilon}\|f_\varepsilon\|_{L^1}\int_{\mathbb{R}^3} f_\varepsilon(v)(1,v,|v|^2)\mathrm{d}v \end{aligned} \tag{6.8}$$

By using the symmetry of Q_{g_ε} and the geometry of collisions (namely, $v+v_* = v'+v'_*$ and $|v|^2+|v_*|^2 = |v'|^2+|v'_*|^2$), we can write the first term in the right hand side of (6.8) as

$$\begin{aligned}&\int_{\mathbb{R}^3} Q^+_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)(v)(1,v,|v|^2)\mathrm{d}v \\ &= \iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)f_\varepsilon(v')f_\varepsilon(v'_*)(1,v,|v|^2)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\ &= \iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)f_\varepsilon(v')f_\varepsilon(v'_*)\frac{(2, v+v_*, |v|^2+|v_*|^2)}{2}\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\ &= \iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)f_\varepsilon(v')f_\varepsilon(v'_*)\frac{(2, v'+v'_*, |v'|^2+|v'_*|^2)}{2}\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\ &= \iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} g_\varepsilon(|u'|^\gamma, -\sigma\cdot\hat{u})f_\varepsilon(v')f_\varepsilon(v'_*)(1,v',|v'|^2)\mathrm{d}\sigma\mathrm{d}v'_*\mathrm{d}v' \\ &= \iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)f_\varepsilon(v)f_\varepsilon(v_*)(1,v,|v|^2)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\ &= \frac{8}{\varepsilon}\|f_\varepsilon\|_{L^1}\int_{\mathbb{R}^3} f_\varepsilon(v)(1,v,|v|^2)\mathrm{d}v. \end{aligned} \tag{6.9}$$

To prove the entropy condition, first note that for any $x, y > 0$, $(x-y)(\log y - \log x) \leq 0$. Next, for $\varepsilon > 0$, $\omega \in S^1$, define

$$
\begin{aligned}
v'_\varepsilon(\omega) &:= v + \frac{|u|}{2}\left(\sigma_\varepsilon(\omega) - \hat{u}\right) \\
{v_*}'_\varepsilon(\omega) &:= v_* - \frac{|u|}{2}\left(\sigma_\varepsilon(\omega) - \hat{u}\right),
\end{aligned}
$$

with $\sigma_\varepsilon(\omega) := \hat{u}\mu_\varepsilon + \omega\sqrt{1-\mu_\varepsilon^2}$. Then,

$$
\begin{aligned}
&\frac{d}{dt}\mathcal{H}(t) = \int Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v)\log f_\varepsilon(v)\mathrm{d}v \\
&= \frac{1}{4}\iint_{\mathbb{R}^3}\int_{S^2} g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma)\left(f'_\varepsilon {f_{\varepsilon *}}' - f_\varepsilon f_{\varepsilon *}\right)\left(\log f_\varepsilon f_{\varepsilon *} - \log f'_\varepsilon {f_{\varepsilon *}}'\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\
&\qquad = \frac{1}{\pi\varepsilon}\iint_{\mathbb{R}^3}\int_{S^1}\left(f_\varepsilon(v'_\varepsilon)f_\varepsilon({v_*}'_\varepsilon) - f_\varepsilon(v)f_\varepsilon(v_*)\right)\cdot \\
&\qquad\quad \cdot\left(\log f_\varepsilon(v)f_\varepsilon(v)f_\varepsilon(v_*) - \log f_\varepsilon(v'_\varepsilon)f_\varepsilon({v_*}'_\varepsilon)\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \leq 0, \qquad (6.10)
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^3} f_\varepsilon(v)\log f_\varepsilon(v)\mathrm{d}v - \int_{\mathbb{R}^3} f_0\log f_0\mathrm{d}v = \int_0^t \frac{d}{ds}\mathcal{H}(s)\mathrm{d}s \leq 0,
$$

giving us a bounded entropy. □

Remark 6.1.1. Theorem 1.1 can be proven in the same way if one does not start with finite energy, i.e., if $f_0 \in L^1(\mathbb{R}^3)\backslash L^1_2(\mathbb{R}^3)$, by simply choosing $E := L^1(\mathbb{R}^3)$.

Chapter 7

Convergence to a Landau solution

By Theorem 1.2, there exists a constant, $\mathcal{C} = \mathcal{C}(p, \|f_0\|_{L^p}, \|f_0\|_{L^p})$ such that for all $\varepsilon > 0$, $t \geq 0$,

$$\|f_\varepsilon\|_{L^p(\mathbb{R}^3)\cap L^1(\mathbb{R}^3)} := \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} + \|f_\varepsilon\|_{L^1(\mathbb{R}^3)} \leq \mathcal{C} + 1.$$

A uniform bound such as this might help us find a limit of f_ε, up to a subsequence, at least in L^p. To do this we need the Banach-Alaoglu Theorem. However, instead of using the space $L^1_2 \cap L^p$, we will use $L^{1+\nu} \cap L^p$ for a very small $\nu > 0$. We choose to work with $L^{1+\nu}$ instead of L^1 because in order to use this theorem we need for our function space to have an "pre-dual," that is, a Banach space whose dual is our working space.

By interpolation, for any $0 \leq \nu \leq \frac{p-1}{2}$, we know that $f_\varepsilon \in L^{1+\nu}(\mathbb{R}^3)$ as well, and

$$\|f_\varepsilon\|_{L^{1+\nu}(\mathbb{R}^3)} \leq \|f_\varepsilon\|^{\theta_\nu}_{L^p(\mathbb{R}^3)} \|f_\varepsilon\|^{1-\theta_\nu}_{L^1(\mathbb{R}^3)} \leq \mathcal{C}^{\theta_\nu}.$$

Then

$$\|f_\varepsilon\|_{L^p(\mathbb{R}^3)\cap L^{1+\nu}(\mathbb{R}^3)} = \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} + \|f_\varepsilon\|_{L^{1+\nu}(\mathbb{R}^3)} \leq \mathcal{C}^{\theta_\nu}(\mathcal{C}^{1-\theta_\nu} + 1).$$

So, by the Banach-Alaoglu Theorem, there exists $f \in L^{1+\nu}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ such that, up to a subsequence, $f_\varepsilon \rightharpoonup^* f$ in $L^{1+\nu}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$. For simplicity we will write $f_\varepsilon \rightharpoonup^* f$ to also denote the subsequence.

We claim that such an f is a weak solution of the Landau equation (1.2) if $\nu < \frac{1}{2}\min\left\{-\frac{2+\gamma}{8+\gamma}, \frac{p-1}{2}, \frac{p'-1}{2}\right\}$ and $p > \frac{6}{8+\gamma}$.

7.1 Proof of Theorem 1.3

Proof. Step 1. The first step is to show that f satisfies the variational formulation of (1.10). Let $\varphi = \varphi(v,s) \in C_B^3(\mathbb{R}^3) \times C^1([0,\infty))$ such that for all $t > 0$, $D_v^2\varphi(v,t)$, $D_v^3\varphi(v,t)$ are bounded in v. Then,

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^3} &\left(\partial_s f(v,s) - Q_L(f,f)(v,s)\right)\varphi(v,s)\mathrm{d}v\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}^3} \partial_s f(v,s) - \partial_s f_\varepsilon(v,s)\varphi(v,s)\mathrm{d}v\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^3} \left(\partial_s f_\varepsilon(v,s) - Q_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)(v,s)\right)\varphi(v,s)\mathrm{d}v\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^3} \left(Q_{g_\varepsilon}(f_\varepsilon,f_\varepsilon)(v,s) - Q_L(f_\varepsilon,f_\varepsilon)(v,s)\right)\varphi(v,s)\mathrm{d}v\mathrm{d}s \\
&+ \int_0^t \int_{\mathbb{R}^3} \left(Q_L(f_\varepsilon,f_\varepsilon)(v,s) - Q_L(f,f)(v,s)\right)\varphi(v,s)\mathrm{d}v\mathrm{d}s \\
&=: J_1^\varepsilon + 0 + J_2^\varepsilon + J_3^\varepsilon. \quad (7.1)
\end{aligned}$$

We examine each integral separately:

$$\begin{aligned}
J_1^\varepsilon = \int_{\mathbb{R}^3} &\left[\varphi(v,s)(f(v,s) - f_\varepsilon(v,s)) - \varphi(v,0)(f_0 - f_0)\right] \\
&- \int_0^t \int_{\mathbb{R}^3} (f_\varepsilon(v,s) - f(v,s))\partial_s\varphi(v,s)\mathrm{d}v\mathrm{d}s \longrightarrow 0 \quad (7.2)
\end{aligned}$$

as $\varepsilon \to 0$, since $f_\varepsilon \rightharpoonup^* f$ in $L^{1+\nu} \cap L^p$. By Theorem 3.1, $J_2^\varepsilon \longrightarrow 0$.

It remains to estimate J_3^ε. Recalling the weak form of the Landau collision operator, going back to the estimates in the proof of Theorem 3.1,

$$\begin{aligned}
J_3^\varepsilon &= \int_0^t \int_{\mathbb{R}^3} Q_L(f_\varepsilon, f_\varepsilon - f)(v,s)\varphi(v,s)\mathrm{d}v\mathrm{d}v_*\mathrm{d}s \\
&\qquad + \int_0^t \int_{\mathbb{R}^3} Q_L(f_\varepsilon - f, f_\varepsilon)(v,s)\varphi(v,s)\mathrm{d}v\mathrm{d}v_*\mathrm{d}s \\
&= -2\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Big(f_\varepsilon(f_{\varepsilon*} - f_*) + (f_\varepsilon - f)f_{\varepsilon*}\Big)(\nabla\varphi - \nabla_*\varphi_*)\cdot u\mathrm{d}v_*\mathrm{d}v\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Big(f_\varepsilon(f_{\varepsilon*} - f_*) + (f_\varepsilon - f)f_{\varepsilon*}\Big)(D^2\varphi + D_*^2\varphi_*) : \Pi(u)|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v\mathrm{d}s \\
&= -4\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} (f_\varepsilon - f)f_{\varepsilon*}(\nabla\varphi - \nabla_*\varphi_*)\cdot u\mathrm{d}v_*\mathrm{d}v\mathrm{d}s \\
&\qquad + \int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} (f_\varepsilon - f)f_{\varepsilon*}(D^2\varphi + D_*^2\varphi_*) : \Pi(u)|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v\mathrm{d}s \\
&\leq 8\|D^2\varphi\|_{L^\infty}\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} (f_\varepsilon - f)f_{\varepsilon*}|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v\mathrm{d}s, \quad (7.3)
\end{aligned}$$

where in the third and fourth lines of (7.3) we exchange the variables $v \leftrightarrow v_*$.

To estimate the right hand side of (7.3), let $\alpha := \gamma + 2$, $h(v) := |v|^\alpha$, and let K_ε denote the integral in the right hand side of (7.3). Then

$$\begin{aligned}
K_\varepsilon &:= \int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} (f_\varepsilon - f)f_{\varepsilon*}|u|^{\gamma+2}\mathrm{d}v_*\mathrm{d}v\mathrm{d}s \\
&= 2\int_0^t \iint (f - f_\varepsilon)f_{\varepsilon*}|u|^\alpha\mathrm{d}v_*\mathrm{d}v\mathrm{d}s \\
&= 2\int_0^1 \iint_{\mathbb{R}^3\times\mathbb{R}^3} (f(v) - f_\varepsilon(v))(f_{\varepsilon*} * h_1(v) + f_{\varepsilon*} * h_2(v))\mathrm{d}v\mathrm{d}s,
\end{aligned}$$

where $h_1(v) := h(v)\mathbb{1}_{|v|\leq 1}$ and $h_2(v) := h(v)\mathbb{1}_{|v|>1}$, so that $h = h_1 + h_2$.

Recall from Proposition 2.5 that $h_1 \in L^{\frac{p'}{2}}(\mathbb{R}^3)$ whenever $p > \frac{6}{8+\gamma}$. First, let $p < 2$ so that $\frac{p'}{2} > 1$. Since $f_{\varepsilon *} \in L^p(\mathbb{R}^3)$, by Young's inequality,

$$\|f_{\varepsilon *} * h_1\|_{L^{p'}(\mathbb{R}^3)} \leq \|f_{\varepsilon *}\|_{L^p(\mathbb{R}^3)} \|h_1\|_{L^{\frac{p'}{2}}(\mathbb{R}^3)} < \infty. \tag{7.4}$$

If $p \geq 2$, by interpolation, $f_{\varepsilon *} \in L^{1+\nu} \cup L^p$ means $f_{\varepsilon *} \in L^{p'}$ (because $1 + \nu < p' \leq 2 \leq p$). Since $\gamma + 2 > -3$, $h_1 \in L^1$. Then

$$\|f_{\varepsilon *} * h_1\|_{L^{p'}} \leq \|f_{\varepsilon *}\|_{L^{p'}} \|h_1\|_{L^1} \tag{7.5}$$

Similarly, since $\nu < -\frac{\gamma+2}{\gamma+8}$, we see that $h_2 \in L^{\frac{\nu+1}{2\nu}}(\mathbb{R}^3) = L^{\frac{(1+\nu)'}{2}}(\mathbb{R}^3)$. Then since $F \in L^{1+\nu}(\mathbb{R}^3)$, replacing p in (7.4) with $1 + \nu$,

$$\|f_{\varepsilon *} * h_2\|_{L^{(1+\nu)'}(\mathbb{R}^3)} \leq \|f_{\varepsilon *}\|_{L^{1+\nu}(\mathbb{R}^3)} \|h_2\|_{L^{\frac{\nu+1}{2\nu}}(\mathbb{R}^3)} < \infty. \tag{7.6}$$

Since $f_\varepsilon \rightharpoonup^* f$ in $L^{1+\nu}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, $F * h_1, F * h_2 \in L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3) \cup L^{p'}(\mathbb{R}^3) = (L^{1+\nu} \cap L^p)'$ and $\left(L^{\frac{\nu+1}{\nu}}(\mathbb{R}^3) \cup L^{p'}(\mathbb{R}^3)\right)' = L^{1+\nu}(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$, it follows that $K_\varepsilon \to 0$ as $\varepsilon \to 0$. Therefore, the limiting $f \in L^{1+\nu} \cap L^p$ does indeed solve the weak form of (1.2).

In addition, following the arguments of Appendix 1 of [**?**], the Landau solution f is differentiable in time since it is contained in $F(f_0) \subset L^1_2$. That is,

$$\int_0^t \int_{\mathbb{R}^3} \partial_s f(v,s)\varphi(v,s)\mathrm{d}v\mathrm{d}s = \int_0^t \int_{\mathbb{R}^3} Q_L(f,f)(v,s)\varphi(v,s)\mathrm{d}v\mathrm{d}s \tag{7.7}$$

for all $\varphi(v,t) \in C^3_B(\mathbb{R}^3) \times C^1((0,\infty))$. This completes Step 1.

Step 2. Next, we show that f conserves mass, momentum and energy.

1. *Mass conservation.* For $R > 0$, define

$$\varphi_R(v) = \begin{cases} 1 \text{ if } |v| \leq R \\ ce^{-\frac{1}{2}(|v|-R)^2} + (1-c) \text{ if } |v| \in (R, R+1], \\ c - ce^{-\frac{1}{2}(|v|-R-2)^2} \text{ if } |v| \in (R+1, R+2], \\ 0 \text{ if } |v| > R+2, \end{cases} \tag{7.8}$$

where $c = \frac{\sqrt{e}}{2(\sqrt{e}-1)}$. Note that $D^2\varphi_R, D^3\varphi_R$ have bounds that are independent of R.

By the Monotone Convergence Theorem, since $f, \varphi_R, 1 \geq 0$,

$$\int f(v,t)\mathrm{d}v = \int f(v,t) \lim_{R\to\infty} \varphi_R(v)\mathrm{d}v = \lim_{\mathbb{R}\to\infty} \int f(v,t)\varphi_R(v)\mathrm{d}v. \tag{7.9}$$

Then

$$\begin{aligned} \frac{d}{dt}\int f(v,t)\mathrm{d}v &= \frac{d}{dt}\int f(v,t) \lim_{R\to\infty} \varphi_R(v)\mathrm{d}v \\ &= \lim_{R\to\infty} \int \partial_t f(v,t)\varphi_R(v)\mathrm{d}v = \lim_{R\to\infty} \int Q_L(f,f)(v,t)\varphi_R(v)\mathrm{d}v \\ &= -2 \lim_{R\to\infty} \iint f f_* |u|^\gamma \left(\nabla\varphi_R(v) - \nabla_*\varphi_R(v_*)\right) \cdot u\mathrm{d}v_*\mathrm{d}v \\ &+ \frac{1}{2} \lim_{R\to\infty} \iint f f_* |u|^{\gamma+2} \left(D^2\varphi_R(v) + D_*^2\varphi_R(v_*)\right) : \Pi(u)\mathrm{d}v_*\mathrm{d}v \\ &= \int Q_L(f,f) \lim_{R\to\infty} \varphi_R(v)\mathrm{d}v = \int Q_L(f,f)\mathrm{d}v = 0. \end{aligned} \tag{7.10}$$

by the Dominated Convergence Theorem. Therefore f has constant mass, i.e.,

$$\int f(v,t)\mathrm{d}v = \int f_0(v)\mathrm{d}v = 1. \tag{7.11}$$

2. *Momentum conservation.* In a similar way to the way the test functions were constructed in (7.8), we can construct another family of test functions, $\varphi_R \in C_0^\infty(\mathbb{R}^3)$, such that $\varphi_R \nearrow v_i$ for any $1 \leq i \leq 3$ whenever $v_i > 0$ and $\varphi_R \searrow v_i$ whenever $v_i \leq 0$ (where v_i denotes the i^{th} component of the vector $v \in \mathbb{R}^3$). These functions can also be constructed such that the derivatives of φ_R are bounded by a uniform constant in R. We can use the Monotone Convergence Theorem again to say that

$$\lim_{R\to\infty} \int f(v,t)\varphi_R(v)\mathrm{d}v = \int f(v,t)v_i\mathrm{d}v. \tag{7.12}$$

Then, just as in (7.10),

$$\begin{aligned}\frac{d}{dt}\int f(v,t)v_i\mathrm{d}v &= \lim_{R\to\infty}\int \frac{d}{dt}f(v,t)\varphi_R(v)\mathrm{d}v \\ &= \lim_{R\to\infty}\int Q_L(f,f)(v,t)\varphi_R(v)\mathrm{d}v \\ &= \int Q_L(f,f)(v,t)v_i\mathrm{d}v = 0, \end{aligned} \tag{7.13}$$

also by the Dominated Convergence Theorem.

3. *Energy conservation.* Let $\{\varphi_R\}_{R>0} \subseteq C_0^\infty(\mathbb{R}^3)$ be a family of nonnegative test functions such that $\varphi_R(v) \nearrow |v|^2$ on $\mathbb{R}^3$ and the second and third derivatives of φ_R are bounded uniformly in R. By using the same techniques that we used when showing conservation of mass and momentum

we see that, by (7.7) and (7.9),

$$\begin{aligned}\frac{d}{dt}\int f(v,t)|v|^2\mathrm{d}v &= \frac{d}{dt}\left(\lim_{R\to\infty}\int f(v,t)\varphi_R(v)\mathrm{d}v\right)\\
&= \lim_{R\to\infty}\int Q_L(f,f)(v,t)\varphi_R(v)\mathrm{d}v\\
&= \frac{1}{2}\lim_{R\to\infty}\iint ff_*|u|^{\gamma+2}\Pi(u)\left(\partial_{v_iv_j}\varphi_R(v)+\partial_{v_{*i}v_{*j}}\varphi_R(v_*)\right)\mathrm{d}v_*\mathrm{d}v\\
&\quad -2\lim_{R\to\infty}\iint f(v)f(v_*)|u|^\gamma\left(\nabla\varphi_R(v)-\nabla_*\varphi_R(v_*)\right)\cdot u\mathrm{d}v_*\mathrm{d}v\\
&= \frac{1}{2}\iint ff_*|u|^{\gamma+2}\Pi(u)_{ij}\left(\partial_{v_iv_j}|v|^2+\partial_{v_{*i}v_{*j}}|v_*|^2\right)\mathrm{d}v_*\mathrm{d}v\\
&\quad -2\iint ff_*|u|^\gamma\left(\nabla|v|^2-\nabla_*|v_*|^2\right)\cdot u\mathrm{d}v_*\mathrm{d}v\\
&= \int Q_L(f,f)(v,t)|v|^2\mathrm{d}v = 0. \quad (7.14)\end{aligned}$$

This implies that

$$\int_{\mathbb{R}^3} f(v,t)|v|^2\mathrm{d}v = \int_{\mathbb{R}^3} f_0(v)|v|^2\mathrm{d}v \quad (7.15)$$

for all $t\geq 0$, so energy is conserved.

Step 3. The final step is to show that f has a decreasing entropy.

For $x\geq 1$, one can check that if $\alpha\in[1,2]$, then $\log x\leq x^\alpha$. Indeed, when $x=1$, $\log x = 1 = x^\alpha$. Taking the derivatives,

$$\begin{aligned}\frac{d}{dx}\log x &= \frac{1}{x}\\ \frac{d}{dx}x^\alpha &= \frac{\alpha}{x^{\alpha-1}}.\end{aligned}$$

Then whenever $x>1$,

$$\frac{d}{dx}\log x < \frac{d}{dx}x^\alpha,$$

meaning that x^α increases faster than $\log x$.

Now fix $p \in [2,\infty]$, let $\nu < \frac{1}{2}\min\left\{-\frac{2+\gamma}{8+\gamma}, \frac{p-1}{2}, \frac{p'-1}{2}\right\}$ as before and let $\alpha = \min\{2,\nu\}$. Let $0 \le f \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ be a limiting solution to the Landau equation obtained as a weak-$*$ limit of f_ε, and recall that in particular, $f \in L^{1+\nu}(\mathbb{R}^3)$. Then,

$$\begin{aligned}\mathcal{H}(t) &= \int_{\mathbb{R}^3} f(v,t)\log f(v,t)\mathrm{d}v \\ &\le \|f\|_{L\log L(\mathbb{R}^3)} = \int f(v)(\log f(v))_+\mathrm{d}v = \int_{f\ge 1} f(v)\log f(v)\mathrm{d}v \\ &\le \int_{f\ge 1} f(v)^{1+\alpha}\mathrm{d}v \le \int_{\mathbb{R}^3} f(v)^{1+\nu}(v)\mathrm{d}v = \|f\|_{L^{1+\nu}} < \infty.\end{aligned}$$

Thus, $f \in L\log L$. To show that the entropy of f is decreasing, simply observe the weak form:

$$\begin{aligned}\int f(v,t)\log f(v,t)\mathrm{d}v - \int f_0\log f_0\mathrm{d}v &= \int_0^t \frac{d}{ds}\int f(v,s)\log f(v,s)\mathrm{d}v\mathrm{d}s \\ &= \int_0^t\int Q_L(f,f)(v,s)\log f(v,s)\mathrm{d}v\mathrm{d}s, \quad (7.16)\end{aligned}$$

and

$$\begin{aligned}&\int Q_L(f,f)(v,s)\log f(v,s)\mathrm{d}v \\ &\quad = -\frac{1}{4}\iint_{\mathbb{R}^3\times\mathbb{R}^3} ff_*|u|^{\gamma+2}\left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*}\right)\Pi(u)\left(\frac{\nabla f}{f} - \frac{\nabla_* f_*}{f_*}\right)\mathrm{d}v_*\mathrm{d}v \\ &\quad < 0 \quad (7.17)\end{aligned}$$

by the symmetry of Q_L in (7.17) above.

We have therefore shown that the Landau equation (1.2), in the case of Coulomb potentials, has weak solutions as defined in [42], which lie in $L^p(\mathbb{R}^3)$ for all times, provided the initial data is in $L^p(\mathbb{R}^3)$ as well.

□

Bibliography

[1] R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg. Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal. 152*, 4, 2000.

[2] R. Alexandre, J. Liao, and C. Lin. Some a priori estimates for the homogeneous Landau equation with soft potentials. *Kinet. Relat. Models 8*, 4, 2015.

[3] R. Alexandre, Y. Morimoto, S. Ukai, C. Xu, and T. Yang. Smoothing effect of weak solutions for the spatially homogeneous Boltzmann equation without angular cutoff. *Kyoto J. Math. 52*, 3, 2012.

[4] R. Alexandre and C Villani. On the Boltzmann equation for long-range interactions. *Communications on Pure and Applied Mathematics*, 55, 2002.

[5] R.J. Alonso, J.A. Canizo, I.M. Gamba, and C. Mouhot. A new approach to the creation and propagation of exponential moments in the Boltzmann equation. *Comm. Partial Differential Equations 38*, 1, 2013.

[6] R.J. Alonso, E. Carneiro, and I.M. Gamba. Convolution inequalities for the Boltzmann collision operator. *Comm. Math. Phys. 298*, 2, 2010.

[7] R.J. Alonso, I.M. Gamba, and M. Taskovic. On L^p propagation for solutions of the Boltzmann equation for Maxwell type interactions. *In preparation*, 2017. in preparation.

[8] R.J. Alonso, I.M. Gamba, and M.-B. Tran. The Cauchy problem for the quantum Boltzmann equation for bosons at very low temperature. *In preparation*, 2017. in preparation.

[9] A.V. Bobylev. The method of the fourier transform in the theory of the boltzmann equation for maxwell molecules. *(Russian) Dokl. Akad. Nauk SSSR 225*, 6, 1975.

[10] A.V. Bobylev. Moment inequalities for the Boltzmann equation and applications to spatially homogeneous problems. *J. Stat. Phys. 116, no 5-6*, 88, 1997.

[11] A.V. Bobylev. The theory of the nonlinear spatially uniform Boltzmann equation for Maxwell molecules. *Mathematical physics reviews, 111233*, 7, 1998.

[12] A.V. Bobylev, I.M. Gamba, and V.A. Panferov. Moment inequalities and high-energy tails for Boltzmann equations with inelastic interactions. *J. Stat. Phys. 116*, 5-6, 2004.

[13] A.V. Bobylev, I.M. Gamba, and I. Potapenko. On some properties of the Landau kinetic equation. *J. Stat. Phys. 161*, 6, 2015.

[14] A.V. Bobylev and I. Potapenko. Monte Carlo methods and their analysis for Coulomb collisions in multicomponent plasmas. *J. Comput. Phys. 246*, 2013.

[15] A. Bressan. Notes on the Boltzmann equation. *Lecture notes for a summer course, S.I.S.S.A. Trieste*, 2005.

[16] E. Carlen, M.C. Carvalho, and X. Lu. On strong convergence to equilibrium for the Boltzmann equation with soft potentials. *J. Stat. Phys.*, 135, 2009.

[17] K. Carrapatoso. On the rate of convergence to equilibrium for the Landau equation with soft potentials. *J. Math. Pures. Appl. (9) 104*, 2, 2015.

[18] K. Carrapatoso, L. Desvillettes, and L. He. Estimates for the large time behavior of the Landau equation in the Coulomb case. *arXiv:1510.08704*.

[19] K. Carrapatoso and S. Mischler. Landau equation for very soft and Coulomb potentials near Maxwellians. *arXiv:1512.01638*.

[20] C. Cercignani. Weak solutions of the Boltzmann equation and energy conservation. *Appl. Math. Lett. 8*, 2, 1995.

[21] C. Cercignani. Existence and trend to equilibrium of weak solutions of the Boltzmann equation. Proceedings of the Second World Congress of Nonlinear Analysts, part 6. *Nonlinear Anal. 30*, 2, 1997.

[22] C. Cercignani. Global weak solutions of the Boltzmann equation. *J. Stat. Phys. 118*, 1-2, 2005.

[23] C. Cercignani. Weak solutions of the boltzmann equation without angle cutoff. *J. Stat. Phys. 123*, 4, 2006.

[24] P. Degond and B. Lucquin-Desreux. The Fokker-Planck asymptotics of the Boltzmann collision operator in the Coulomb case. *Mth. Models Mth Appl. Sci.*, 2, 1992.

[25] L. Desvillettes. On asymptotics of the boltzmann equation when the collisions become grazing. *Transport Theory Statist. Phys. 21*, 3, 1992.

[26] L. Desvillettes. Entropy dissipation estimates for the Landau equation in the Coulomb case and applications. *J. Funct. Anal. 269*, 5, 2013.

[27] L. Desvillettes and C. Villani. On the spatially homogeneous Landau equation for hard potentials. I. Existence, uniqueness and smoothness. *Comm. Partial Differential Equations 25*, 1-2, 2000.

[28] L. Desvillettes and C. Villani. On the spatially homogeneous Landau equation for hard potentials. II. H-theorem and applications. *Comm. Partial Differential Equations 25*, 1-2, 2000.

[29] L. Desvillettes and C. Villani. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math. 159*, 2, 2005.

[30] R.J. DiPerna and P.-L. Lions. Global solutions of Boltzmann's equation and the entropy inequality. *Arch. Rational Mech. Anal. 114*, 1, 1991.

[31] N. Fournier. Uniqueness of bounded solutions for the homogeneous Landau equation with a Coulomb potential. *Comm. Math. Phys. 299*, 3, 2010.

[32] N. Fournier and H. Guerin. On the uniqueness for the spatially homogeneous Boltzmann equation with a strong angular singularity. *J. Stat. Phys. 131*, 4, 2008.

[33] N. Fournier and H. Guerin. Well-posedness of the spatially homogeneous Landau equation for soft potentials. *J. Funct. Anal. 256*, 8, 2009.

[34] I.M. Gamba and J. Haack. A conservative spectral method for the Boltzmann equation with anisotropic scattering and the grazing collisions limit. *J. Comput. Phys. 270*, 2014.

[35] D. Godinho. Asymptotic of grazing collisions for the spatially homogeneous Boltzmann equation for soft and Coulomb potentials. *Stochastic Process. Appl. 123*, 11, 2013.

[36] Y. Guo. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal. 169*, 4, 2003.

[37] Y. Guo. The Landau equation in a periodic box. *Comm. Math. Phys. 231*, 3, 2003.

[38] L. He. Asymptotic analysis of the spatially homogeneous Boltzmann equation: grazing collisions limit. *J. Stat. Phys. 155*, 1, 2014.

[39] S. Kaniel and M. Shinbrot. The Boltzmann equation. I. Uniqueness and local existence. *J. Stat. Phys. 155*, 1, 2014.

[40] L. Landau. The transport equation in the case of Coulomb interactions. *JETP 7, 203,*, 1937.

[41] A. Pulvirenti and G. Toscani. The theory of the nonlinear Boltzmann equation for Maxwell molecules in Fourier representation. *Ann. Mat. Pura Appl. 171, 181204.*, 4, 1996.

[42] C. Villani. On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Rational Mech. Anal. 143*, 1998.

[43] K.-C. Wu. Global in time estimates for the spatially homogeneous Landau equation with soft potentials. *J. Funct. Anal. 266*, 5, 2014.