Cartesian Closed Categories and λ-Calculi

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Chapter 1

Introduction

A λ-calculus is a formal mathematical system involving function composition that can be used to model computation: various terms can be transformed according to certain rules, which represent function application and abstraction, where each rule application corresponds to a computational step. In addition, by defining additional relations between terms, λ-calculi can be used to define various mathematical objects, such as monoids, groups, and rings. We can assign types to each term to obtain a typed λ-calculus. It turns out that typed λ-calculi are equivalent to a special class of categories, the Cartesian closed categories, which are categories with products in which functions $f : A \times B \to C$ on products and functions can be represented as functions $f : A \to C^B$ whose codomain is an exponential object – this isomorphism corresponds to partial application of functions in λ-calculus.

Lambda calculus was developed by Alonzo Church in the 1930s; Church first attempted to use the untyped λ-calculus to describe a foundation for mathematics, but his proposed system was shown to be inconsistent. Church then applied the λ-calculus to the theory of computation, using it to prove the undecidability of the *Entscheidungsproblem*. Category theory arose as an attempt to generalize the idea of a "structure-preserving transformation" which occurs in many branches of mathematics; it has been used in algebraic topology, abstract algebra, and algebraic geometry to describe relationships between various concepts and to define various objects using their universal properties. Cartesian closed categories were developed by Lawvere in order to describe the process of substitution in a categorical setting. Lambek then proved that this view of substitution is equivalent to that captured by λ-calculus.

Chapter 2

Cartesian closed categories

Definition. In category theory, a **graph** consists of a class of objects and a class of arrows or morphisms between objects. Each arrow has a source object and a target object. In other words, a graph in category theory is a directed graph in which the vertices are objects and the edges are morphisms.

We will write $f : A \to B$ to indicate that an arrow f has source A and target B.

Definition. A **category** is a graph with three additional requirements. First, for any three objects A, B, and C and any two arrows $f : A \to B$, and $g : B \to C$, we have a third arrow $gf : A \to C$ which is the composition of g and f; we can also write $g \circ f$ instead of gf. Second, we have an arrow $\mathbf{1}_A : A \to A$ such that for any objects B and C and any arrows $f : A \to B$ and $g : C \to A$, we have $f\mathbf{1}_A = f$ and $g\mathbf{1}_A = g$. Third, composition of arrows is associative – if we have arrows $f : A \to B$, $g : B \to C$ and $h : C \to D$, then $(hg)f = h(gf)$.

Example. In the category **Set**, the objects are sets and the morphisms are functions. Similarly, we can define a category **Group** whose objects are groups and whose morphisms are group homomorphisms, and we can define a category **Ring** whose objects are rings and whose morphisms are ring homomorphisms.

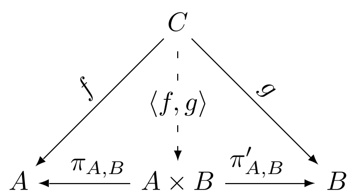
Example. For any partially-ordered set $(P, \leq)$, we can make P into a category whose objects are the elements of P. There is a single morphism between x and y if and only if $x \leq y$.

Definition. An arrow $f : A \to B$ is an **isomorphism** if there is another arrow $g : B \to A$ such that $gf = 1_A$ and $fg = 1_B$.

Definition. A category is **small** if both the class of objects and the class of arrows are sets. Given any category $\mathscr{A}$ we can define the dual category $\mathscr{A}^{\text{op}}$ by reversing all arrows. In other words, arrows $A \to B$ in $\mathscr{A}$ correspond to arrows $B \to A$ in $\mathscr{A}^{\text{op}}$. We then have $(\mathscr{A}^{\text{op}})^{\text{op}} = A$.

Definition. An object 1 is a **terminal object** if it has the property that for any object A there exists a unique arrow $\bigcirc_A : A \to 1$.

Definition. A **Cartesian category** is a category in which there is a terminal object 1, and in which we can form a product $A \times B$ of any two objects A and B. This product object has the property that there exist projection maps $\pi_{A,B} : A \times B \to A$ and $\pi'_{A,B} : A \times B \to B$, such that for any pair of arrows $f : C \to A$ and $g : C \to B$, there exists a unique third arrow $\langle f, g \rangle : C \to A \times B$ such that $\pi_{A,B}\langle f, g \rangle = f$ and $\pi'_{A,B} : \langle f, g \rangle = g$. Diagramatically, $\langle f, g \rangle$ is the unique arrow such that the following diagram commutes.



Proposition. *The universal property for products implies that for any arrow $h : C \to A \times B$, we have $\langle \pi_{A,B}h, \pi'_{A,B}h \rangle = h$, and this property is also sufficient to prove the uniqueness of $\langle f, g \rangle$.*

Proof. The map $\langle \pi_{A,B}h, \pi'_{A,B}h \rangle$ is the unique arrow with the property that $\pi_{A,B}\langle \pi_{A,B}h, \pi'_{A,B}h \rangle = \pi_{A,B}(h)$ and $\pi'_{A,B}\langle \pi_{A,B}h, \pi'_{A,B}h \rangle = \pi'_{A,B}h$. Since h also has this property, we must have $\langle \pi_{A,B}h, \pi'_{A,B}h \rangle = h$.

Conversely, if for every $h : C \to A \times B$, we have $\langle \pi_{A,B}h, \pi'_{A,B}h \rangle = h$ then if h and $\langle f, g \rangle$ both have $\pi_{A,B}\langle f, g \rangle = f$ and $\pi'_{A,B}\langle f, g \rangle = g$, then we have $\langle f, g \rangle = \langle \pi_{A,B}h, \pi'_{A,B}h \rangle = h$. □

Definition. If we have two arrows $f : A \to C$ and $g : B \to D$, we shall write $f \times g : A \times B \to C \times D$ for the arrow $\langle f\pi_{A,B}, g\pi'_{A,B} \rangle$.

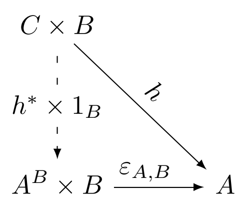
Example. The category **Set** of sets is a Cartesian category since we can define the product $A \times B$ of two sets to be the Cartesian product – i.e. the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$. The maps π and π' map each pair to the first and second coördinates respectively, and for functions $f : C \to A$ and $g : C \to B$, we define $\langle f, g \rangle : C \to A \times B$ to be the map $x \mapsto (f(x), g(x))$. Any set with one element is a terminal object. We can obtain other Cartesian categories by restricting to finite or countable sets.

Example. The category **Groups** of groups is a Cartesian category. The product of two groups G and H is the direct product, which is obtained by placing a group structure on the Cartesian product $G \times H$ – we define $(g, h) \cdot (g', h') = (gg', hh')$. The projection and pairing maps are then defined in the same way as they are for sets. The trivial group is the terminal object, since any homomorphism whose target is the trivial group maps every element of the domain to the identity. We can obtain other Cartesian categories by restricting to finite groups, countable groups, finitely-generated groups, or Abelian groups.

Similarly, we can make the category of rings into a Cartesian category by letting addition and multiplication be defined coördinatewise on the Cartesian product of any two rings. The trivial ring is the terminal object.

Example. We can make the category of directed graphs (where the objects are directed graphs and the arrows are graph homomorphisms) into a Cartesian category by defining the product of two graphs G and H to be a graph whose vertex set is $V(G) \times V(H)$ and in which there is an edge $(u, u') \to (v, v')$ if and only if there are edges $u \to v$ and $u' \to v'$. We can then project from $G \times H$ back to G and H by taking the first and second coördinates respectively, and if $f : C \to G$ and $g : C \to H$ are homomorphisms of directed graphs, then the map $\langle f, g\rangle : C \to G \times H$ that takes each vertex v of C to $(f(v), g(v))$ is a graph homomorphism, and we can get back the original maps f and g by projecting onto each coördinate. A single-vertex graph is then the terminal object. We still have a Cartesian category if we restrict to finite directed graphs.

Definition. A **Cartesian closed category** is a Cartesian category in which for any two objects A and B, we can form an exponential object A^B. This exponential object has the property that for any two objects A and B, there is an arrow $\varepsilon_{A,B} : A^B \times B \to A$, and for any three objects A, B, and C, and a unique arrow $h : C \times B \to A$, there is a unique arrow $h^* : C \to (A^B)$ such that $\varepsilon_{A,B}(h^* \times 1_B) = \varepsilon_{A,B}\langle h^*\pi_{C,B}, \pi'_{C,B}\rangle = h$. In other words, h^* is the unique arrow that makes the following diagram commute:



Proposition. *This universal property implies that for any objects A, B, and C and any arrow $k : C \to (A^B)$, we have $(\varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle)^* = k$, and this property is also sufficient to prove the uniqueness of h^*. The maps ε and $-^*$ therefore define a bijection* $\mathrm{Hom}(C \times B, A) \cong \mathrm{Hom}(C, A^B)$.

Proof. We have $\varepsilon_{A,B}((\varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle)^* \times 1_B) = \varepsilon_{A,B}(k\pi_{C,B}, \pi'_{C,B})$, and we also have $\varepsilon_{A,B}(k \times 1_B) = \varepsilon_{A,B}(k\pi_{C,B}, \pi'_{C,B})$. The universal property of the exponential object therefore implies that $k = \varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle)^*$.

Conversely, if for any arrow $k : C \to (A^B)$, we have $(\varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle)^* = k$, suppose $h : C \times B \to A$ is an arbitrary arrow. If $k : C \to (A^B)$ satisfies $\varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle = h$, we have $h^* = (\varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle)^* = k$, thus h^* is the unique arrow with the property that $\varepsilon_{A,B}(h^* \times 1_B) = h$. □

Definition. In any Cartesian closed category, we find it convenient to define the following additional maps:

1. For any objects A, B, and C, and any arrow $g : C \to A$, let $g \Leftarrow \mathbf{1}_B : (C^B) \to (A^B)$ be $(g\varepsilon_{D,B})^*$.

2. For any two objects B and C, let $\eta_{C,B} : C \to (C \times B)^B$ be $\mathbf{1}^*_{C\times B}$.

3. For any two objects A and B and any arrow $f : A \to B$, let $\ulcorner f \urcorner : 1 \to (B^A)$ be $(f\pi'_{1,A})^*$.

4. For any objects A and B and any arrow $g : 1 \to (B^A)$, let $g^{\int} : A \to B$ be $\varepsilon_{B,A}\langle g \bigcirc_A, 1_A\rangle$.

Proposition. *The maps $f \mapsto \ulcorner f \urcorner$ and $g \mapsto g^{\int}$ give an isomorphism* $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(1, B^A)$*; i.e.* $\ulcorner f \urcorner^{\int} = f$ *and* $\ulcorner g^{\int} \urcorner = g$.

Proof. Given a morphism $g : 1 \to B^A$, we have

$$\begin{aligned}
\ulcorner g^{\int} \urcorner &= (\varepsilon_{B,A}\langle g\bigcirc_A, 1_A\rangle\pi'_{1,A})^* \\
&= (\varepsilon_{B,A}\langle g \bigcirc_A \pi'_{1,A}, 1_A\pi'_{1,A}\rangle)^* \\
&= (\varepsilon_{B,A}\langle g\bigcirc_{1\times A}, 1_A\pi'_{1,A}\rangle)^* \\
&= (\varepsilon_{B,A}\langle g\pi_{1\times A}, \pi'_{1,A}\rangle)^* = g
\end{aligned}$$

Conversely, suppose we have a morphism $f : A \to B$. Then

$$\begin{aligned}
\ulcorner f \urcorner^{\int} &= \varepsilon_{B,A}\langle (f\pi'_{1,A})^*\bigcirc_A, 1_A\rangle\pi'_{1,A} \\
&= \varepsilon_{B,A}\langle (f\pi'_{1,A})^* \bigcirc_A \pi'_{1,A}, \pi'_{1,A}\rangle \\
&= \varepsilon_{B,A}\langle (f\pi'_{1,A})^*\bigcirc_{1\times A}, \pi'_{1,A}\rangle \\
&= \varepsilon_{B,A}\langle (f\pi'_{1,A})^*\pi_{1,A}, \pi'_{1,A}\rangle = f\pi'_{1,A}
\end{aligned}$$

But $\pi'_{1,A}\langle \bigcirc_A, 1_A\rangle = 1_A$, so composing both sides on the right with $\langle \bigcirc_A, 1_A\rangle$ we obtain $\ulcorner f \urcorner^{\int} = f$. □

We also have the following "distributive law" :

Proposition. *For any arrows $k : D \to A$ and $h : A \times B \to C$, we have* $h^*k = (h\langle k\pi_{D,B}, \pi'_{D,B}\rangle)^*$

Proof.

$$\begin{aligned}
h^*k &= (\varepsilon\langle h^*k\pi_{D,B}, \pi'_{D,B}\rangle)^* \\
&= (\varepsilon\langle h^*\pi_{D,B}, \pi'_{D,B}\rangle\langle k\pi_{D,B}, \pi'_{D,B}\rangle)^* \\
&= (h\langle k\pi, \pi'\rangle)^*
\end{aligned}$$

□

Example. The category of all sets has a Cartesian closed structure. We define A^B to be the set of functions from B to A, and if we have a function $f : A\times B \to C$, we define f^* to be the function $g : A \to C^B$ given by $(g(x))(y) = f(x, y)$. If we restrict to finite sets, or indeed to the class of hereditarily finite sets, we again have a Cartesian closed category.

Example. If we have a group G, the class of "G-sets", or sets for which there exists a G-action is a Cartesian closed category, since we can define a group action on $A \times B$ by $g.(a,b) = (ga, gb)$ and a group action on A^B by $(g.F)(b) = g.F(g^{-1}b)$ for any $g \in G$ and $b \in B$. If we consider the category whose objects are G-sets, and whose morphisms are functions $f : X \to Y$ such that $f(gx) = gf(x)$ for any $g \in G$ and $x \in X$, we obtain a Cartesian closed category, where $A \times B$ is the direct product of groups, and A^B is the set of functions from B to A. The maps ε and $-^*$ are defined in the same way as they are for sets. If we restrict to finite G-sets we again get a Cartesian closed category.

Example. If X is any topological space, we can define a category $\mathscr{C}$ whose objects are the open sets of X. We can make $\mathscr{C}$ into a Cartesian closed category in which there is a single arrow between U and V if and only if $U \subset V$ by defining $U \times V = U \cap V$, and $U^V = \text{Int}[U \cup (X \setminus V)]$ for any objects U and V. We then have $U^V \times V = \text{Int}[(U \cup (X \setminus V))] \cap V \subset U$, and if $U \cap V \subset A$, we have $U \subset (A \cup (X \setminus V))$, hence $U \subset \text{Int}[(A \cup (X \setminus V))]$; thus, for any objects U and V, we have an arrow $\varepsilon_{U,V}$, and for any arrow $f : U \times V \to A$, we have another arrow $f : U \to A^V$. Since there is at most one arrow between any two objects in $\mathscr{C}$, we automatically have $\varepsilon\langle h^*\pi, \pi'\rangle = h$ and $(\varepsilon\langle k\pi, \pi'\rangle)^* = k$ since both sides of each equation are arrows between the same objects. Thus $\mathscr{C}$ is Cartesian closed.

Example. The category of directed graphs can be made into a Cartesian closed category by defining the exponential graph G^H to be the set of all functions $V(H) \to V(G)$, regardless of whether they induce graph homomorphisms. If f, g are two such functions, we have $f \to g$ if and only if for all vertices u, v in H, $u \to v \implies (f(u) \to g(v))$. For example, each graph homomorphism f is adjacent to itself. For any map $f : A \times B \to C$, we define $f^* : A \to C^B$ by $f^*(a)(b) = f(a,b)$, and we define $\varepsilon_{A,B} : A^B \times B \to A$ by $\varepsilon_{A,B}(f,b) = f(b)$. If we restrict to finite graphs, we obtain a small Cartesian closed category.

Cartesian closed categories can be thought of as deductive systems in which there are certain relationships between proofs. Each object corresponds to a proposition, with the terminal object corresponding to a tautology, and each arrow $f : A \to B$ corresponds to a proof of B from the assumption A. The product $A \times B$ of two objects then corresponds to the conjunction of A and B, while the exponential object A^B corresponds to the implication $A \Leftarrow B$.

2.1 Functors

Functors are a way of describing "mappings" between categories. A functor is a graph homomorphism on the underlying graphs of the categories with some additional properties that make it preserve the structure of the categories. For a general category, we simply require that the functor preserves composition of arrows, but we will later introduce Cartesian functors and Cartesian closed functors, which additionally require the functor to preserve the Cartesian or Cartesian closed structure.

Definition. Given two categories $\mathscr{A}$ and $\mathscr{B}$, a **functor** is a map F which takes objects of $\mathscr{A}$ to objects of $\mathscr{B}$ and arrows of $\mathscr{A}$ to arrows of $\mathscr{B}$, with the property that $F(1_A) = 1_{F(A)}$ for any object $A \in \mathscr{A}$, and that $F(gf) = F(g)F(f)$ for any composable arrows f and g between objects in $\mathscr{A}$.

Example. For any category $\mathscr{A}$, **identity functor** $1_{\mathscr{A}}$ takes each object to itself and each morphism to itself.

Example. The category $\mathbf{Top}_*$ is a category whose objects are topological spaces with a distinguished base point x_0, and whose arrows are continuous functions that map base points to base points, then there is a functor from $\mathbf{Top}_*$ to the category **Group** of groups. This functor takes each object X to its fundamental group $\pi_1(X, x_0)$, and maps each arrow $f : X \to Y$ to a homomorphism that maps each homotopy class $[\gamma]$ to $[f\gamma]$.

Example. Similarly, if **Ring** is the category whose objects are rings, and whose arrows are ring homomorphisms, there is a functor $-^{\times}$ which maps each ring R to its group of units, and maps each ring homomorphism $f : R \to S$ to its restriction $f : R^{\times} \to S^{\times}$.

Definition. If $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are two functors such that $FG = 1_{\mathscr{D}}$ and $GF = 1_{\mathscr{C}}$, then F and G are **inverse functors**, and the categories $\mathscr{C}$ and $\mathscr{D}$ are considered **isomorphic**.

A functor $F : \mathscr{C} \to \mathscr{D}$ between two Cartesian categories is a Cartesian functor if it satisfies:

1. $F(1) = 1$
2. $F(A \times B) = F(A) \times F(B)$
3. $F(A^B) = F(A)^{F(B)}$
4. $F(\bigcirc_A) = \bigcirc_{F(A)}$
5. $F(\pi_{A,B}) = \pi_{F(A),F(B)}$
6. $F(\langle f, g \rangle) = \langle F(f), F(g) \rangle$
7. $F(h^*) = F(h)^*$
8. $F(\varepsilon_{A,B}) = \varepsilon_{F(A),F(B)}$

A Cartesian closed functor $F : \mathscr{C} \to \mathscr{D}$ between two Cartesian closed categories is a Cartesian functor which additionally satisfies the criteria $F(h^*) = F(h)^*$ and $F(\varepsilon_{A,B}) = \varepsilon_{F(A),F(B)}$.

2.2 Natural Transformations

Natural transformations are structure-preserving maps between functors – each natural transformation consists of a family of arrows that transform one functor to another and are compatible with the categorical structure. If each such arrow is an isomorphism, we say that the two functors are "naturally isomorphic". This notion of natural isomorphism will be used to formally define an equivalence of categories. Natural transformations will also be necessary for defining monads and comonads, which will be needed for some constructions used to demonstrate the equivalence between Cartesian closed categories and λ-calculi.

Definition. If $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{D}$ are functors, a **natural transformation** is a map $t : F \to G$ which assigns to each object A in $\mathscr{C}$ an arrow $t(A) : F(A) \to G(A)$, so that if A and B are any objects in $\mathscr{C}$ and $f : A \to B$ is an arrow in $\mathscr{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle t(A)} & & \downarrow{\scriptstyle t(B)} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

i.e. $t(B)F(f) = G(f)t(A)$.

For any functor F, we have the identity natural transformation $1_F : F \to F$ that assigns the identity arrow to each object.

Example. We have two functors π_1 : $\mathbf{Top}_* \to \mathbf{Group}$ and H_1 : $\mathbf{Top}_* \to \mathbf{Group}$ which represent taking the fundamental group and first homology respectively. Since the first homology is the abelianization of the fundamental group, we have a natural transformation α which assigns to each object X of $\mathbf{Top}_*$ an arrow $\alpha(X) : \pi_1(X, x_0) \to H_1(X)$. The arrow $\alpha(X)$ takes each element γ of $\pi_1(X, x_0)$ to the corresponding coset $[\gamma]$ in the quotient group $H_1(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$.

Example. We have two functors from the category of commutative rings with identity to **Group** : $-^\times$ which takes each ring to its group of units, and restricts each ring homomorphism to the group of units, and $\mathrm{GL}_n(-)$, which takes each ring R to the group of invertible $n \times n$ matrices with entries in R, while taking each ring homomorphism $f : R \to S$ to a map which applies f to each matrix entry. Each invertible matrix has a determinant which is a unit. Furthermore, the determinant of a matrix is given by a polynomial formula in terms of its entries, which does not depend on the ring the matrix entries come from. Thus, for any ring homomorphism $f : R \to S$, we get the same result whether we first apply f to all entries of a matrix and then calculate the determinant, or calculate the determinant and then apply f to the result. Thus the determinant is a natural transformation.

Definition. An **equivalence** between two categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair of functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$, such that there are natural transformations $\eta : 1_{\mathscr{A}} \to GF$ and $\varepsilon : FG \to 1_{\mathscr{B}}$, such that for each object A in $\mathscr{A}$, $\eta(A)$ is an isomorphism, and for each object B in $\mathscr{B}$, $\varepsilon(B)$ is an isomorphism. In other words, GF is **naturally isomorphic** to $1_{\mathscr{A}}$, and FG is naturally isomorphic to $1_{\mathscr{B}}$.

2.3 Monads, Comonads and the Kleisli category

To prove the equivalence between Cartesian closed categories and typed λ-calculi, it is necessary to consider Cartesian closed categories to which additional "indeterminate arrows" $x : 1 \to A$ are adjoined. To describe precisely what it means to adjoin an arrow to a category, we introduce the dual notions of monad and comonad, in order to introduce the Kleisli category. Considering the Kleisli category over an appropriate comonad will allow us to define precisely what it means to adjoin an indeterminate to a Cartesian closed category.

Definition. A **monad** on a category $\mathscr{A}$ consists of a triple (T, η, μ), where $T : A \to A$ is a functor and $\eta : 1_{\mathscr{A}} \to T$ and $\mu : T^2 \to T$ are natural transformations, such that for any object X, $\mu(X) \circ T(\mu(X)) = \mu(X) \circ \mu(T(X))$, and $\mu(X) \circ T(\eta(X)) = \mu(X) \circ \eta(T(X)) = 1_T(X)$. We can abbreviate these laws as $\mu \circ T\mu = \mu \circ \mu T$ and $\mu \circ T\eta = \mu \circ \eta T$.

Definition. A **comonad** on a category $\mathscr{A}$ consists of a functor $S : \mathscr{A} \to \mathscr{A}$ and two natural transformations $\varepsilon : S \to 1_{\mathscr{A}}$ and $\delta : S \to S^2$ such that for any object X, $\delta(S(X)) \circ \delta(X) = S(\delta(X)) \circ \delta(X)$ and $\varepsilon(S(X)) \circ \delta(X) = S(\varepsilon(X)) \circ \delta(X) = 1_S(X)$. We can abbreviate these laws as $\delta S \circ \delta = S\delta \circ \delta$ and $\varepsilon S \circ \delta = S\varepsilon \circ \delta$.

The **Kleisli category** can be defined over a monad or over a comonad.

Definition. The **Kleisli category of a monad** (T, η, μ) on a category $\mathscr{C}$ is another category $\mathscr{C}_T$ whose objects are the same as those in $\mathscr{C}$, but in which each arrow $A \to B$ in $\mathscr{C}_T$ corresponds to an arrow $A \to T(B)$ in $\mathscr{C}$. If we have two arrows $f : A \to T(B)$ and $g : B \to T(C)$ then their composition $g * f : A \to T(C)$ is given by $\mu(C)T(g)f$.

The identity arrow on any object A in $\mathscr{C}_T$ is then the arrow $\eta(A) : A \to T(A)$; for any arrow $f : A \to T(B)$, $f * \eta(A) = \mu(B)T(f)\eta(A) = \mu(B)\eta(T(B))f = 1_{T(B)}f = f$ and $\eta(B) * f = \mu(B)T(\eta(B))f = 1_{T(B)}f = f$. If we have arrows $f : A \to T(B)$, $g : B \to T(C)$ and $h : C \to T(D)$, then

$$\begin{aligned}(h * g) * f &= (\mu(D)T(h)g) * f \\ &= \mu(D)T(\mu(D))T(T(h))T(g)f \\ &= \mu(D)\mu(T(D))T(T(h))T(g)f \\ &= \mu(D)T(h)\mu(C)T(g)f\end{aligned}$$

and $h * (g * f) = h * (\mu(C)T(g)f) = \mu(D)T(h)\mu(C)T(g)f$. Thus, the Kleisli category over a monad is indeed a category.

Definition. Similarly, the **Kleisli category of a comonad** (S, ε, δ) over a comonad on a category $\mathscr{C}$ is a category $\mathscr{C}_S$ whose objects are the same as those of $\mathscr{C}$; an arrow $f : A \to B$ in $\mathscr{C}_S$ is an arrow $f : S(A) \to B$ in $\mathscr{C}$. If $f : A \to B$ and $g : B \to C$ are two arrows in $\mathscr{C}_S$, the composition $g * f$ is given by the arrow $gS(f)\delta(A)$ in $\mathscr{C}$.

If we have three arrows $f : A \to B$, $g : B \to C$ and $h : C \to D$ in $\mathscr{C}_S$, then we have $(h * g) * f = (h * g)S(f)\delta(A) = hS(g)\delta(B)S(f)\delta(A)$ and

$$\begin{aligned} h * (g * f) &= hS(g * f)\delta(A) = hS(gS(f)\delta(A))\delta(A) \\ &= hS(g)S(S(f))S(\delta(A))\delta(A) \\ &= hS(g)S(S(f))\delta(S(A))\delta(A) = hS(g)\delta(B)S(f)\delta(A) \end{aligned}$$

thus composition is associative. For any object B, the arrow $\varepsilon(B) : S(B) \to B$ is the identity arrow 1_B in $\mathscr{C}_S$: if $f : A \to B$ is an arrow in $\mathscr{C}_S$, then $\varepsilon(B) * f = \varepsilon(B)S(f)\delta(A) = f\varepsilon(S(A))\delta(A) = f1_{S(A)} = f$, and $f * \varepsilon_A = fS(\varepsilon(A))\delta(A) = f1_{S(A)} = f$. Thus the Kleisli category over a comonad is indeed a category.

Claim. *Given a Cartesian closed category $\mathscr{C}$ and an object A, define the functor $S_A : \mathscr{C} \to \mathscr{C}$ by $S_A(B) = A \times B$ for every object B and $S_A(f) = \langle \pi_{A,B}, f\pi'_{A,B} \rangle$ for every arrow $f : B \to C$. Define the natural transformations $\varepsilon_A : S_A \to 1_{\mathscr{C}}$ and $\delta : S \to S^2$ by $\varepsilon_A(B) = \pi'_{A,B}$ and $\delta_A(B) = \langle \pi_{A,B}, 1_{A\times B} \rangle$. Then $(S_A, \varepsilon_A, \delta_A)$ is a comonad.*

Proof. Let B be an arbitrary object of $\mathscr{A}$.

Then $\varepsilon_A(S_A(B))\delta_A(B) = \pi'_{A,A\times B}\langle \pi_{A,B}, 1_{A\times B} \rangle = 1_{A\times B} = 1_{S_A(B)}$ and

$$\begin{aligned} S_A(\varepsilon_A(B))\delta_A(B) &= \langle \pi_{A,A\times B}, \varepsilon_A(B)\pi'_{A,A\times B} \rangle \langle \pi_{A,B}, 1_{A\times B} \rangle \\ &= \langle \pi_{A,A\times B}, \pi'_{A,B}\pi'_{A,A\times B} \rangle \langle \pi_{A,B}, 1_{A\times B} \rangle \\ &= \langle \pi_{A,A\times B}\langle \pi_{A,B}, 1_{A\times B} \rangle, \pi'_{A,B}\pi'_{A,A\times B}\langle \pi_{A,B}, 1_{A\times B} \rangle \rangle \\ &= \langle \pi_{A,B}, \pi'_{A,B}1_{A\times B} \rangle \\ &= 1_{A\times B} = 1_{S(B)} \end{aligned}$$

Furthermore, $\delta(S(B))\delta(B) = S(\delta(B))\delta(B) = \langle \pi_{A,B}, \langle \pi_{A,B}, 1_{A,B} \rangle \rangle$. □

2.4 Polynomials

We will now see how the Kleisli category over the comonad described above can be given a Cartesian closed structure. In addition, we will prove a "functional completeness" result which shows how arrows in this category decompose in a nice fashion. We will also prove a universal property of this category, which describes how we can extend a functor $\mathscr{A} \to \mathscr{B}$ to the category $\mathscr{A}[x]$ obtained by adjoining an indeterminate; we will need this property to describe the equivalence between the categories of Cartesian closed categories and typed λ-calculi.

Proposition. *Given a Cartesian closed category $\mathscr{C}$ and an object A, the Kleisli category of the comonad $(S_A, \varepsilon_A, \delta_A)$ is another Cartesian closed category $\mathscr{C}_A$.*

Proof. We can put a Cartesian structure on $\mathscr{C}_A$ by defining $\bigcirc^A_C = \bigcirc_{A\times C}$, $\pi^A_{B,C} = \pi_{B,C}\pi'_{A,B\times C}$, $\pi'^A_{B,C} = \pi'_{B,C}\pi'_{A,B\times C}$, and taking products of arrows as in $\mathscr{C}$.

We then have $\pi^A * \langle f, g\rangle = \pi\pi'\langle \pi, \langle f, g\rangle\rangle = \pi\langle f, g\rangle = f$, $\pi'^A * \langle f, g\rangle = \pi'\pi'\langle \pi, \langle f, g\rangle\rangle = \pi'\langle f, g\rangle = g$ and $\langle \pi^A * h, \pi'^A * h\rangle = \langle \pi\pi'\langle \pi, h\rangle, \pi'\pi'\langle \pi, h\rangle\rangle = \langle \pi h, \pi' h\rangle = h$.

We can then extend this structure to a Cartesian closed structure. The exponential objects A^B are the same as those in $\mathscr{C}_A$, since $\mathscr{C}_A$ has the same objects as $\mathscr{C}$. Each evaluation map $\varepsilon^A_{B,C} : B^C \times C \to B$ in $\mathscr{C}_A$ is $\varepsilon_{B,C}\pi'_{A,B^C\times C}$: $A \times (B^C \times C) \to B$ in $\mathscr{C}$.

For any objects Z and B, let $\alpha : (A \times Z) \times B \to A \times (Z \times B)$ be the map $\langle \pi_{A,Z}\pi_{A\times Z,B}, \langle \pi'_{A,Z}\pi_{A\times Z,B}, \pi'_{A\times Z,B}\rangle\rangle$. Given an arrow $h : A \times (Z \times B) \to C$, we can then define h^{*_A} to be $(h\alpha_{A,B,Z})^* : A \times Z \to C^B$.

We can then show that for any $h : B \times C \to D$ and $k : B \to D^C$ in $\mathscr{A}$, $\varepsilon^A_{D,C} * \langle h^{*_A} * \pi^A_{B,C}, \pi'^A_{B,C}\rangle = h$ and $(\varepsilon^A_{D,C} * \langle k * \pi^A_{B,C}, \pi'^A_{B,C}\rangle)^{*_A} = k$:

$$
\begin{aligned}
\varepsilon^A * \langle h^{*_A} * \pi, \pi'\rangle &= \varepsilon^A * \langle (h\alpha)^* * \pi^A, \pi'^A\rangle \\
&= \varepsilon^A * \langle (h\alpha)^*\langle \pi, \pi^A\rangle, \pi'^A\rangle \\
&= \varepsilon^A * \langle (h\alpha)^*\langle \pi, \pi\pi'\rangle, \pi'\pi'\rangle \\
&= \varepsilon^A * \langle (h\alpha)^*\pi, \pi'\rangle\langle\langle \pi, \pi\pi'\rangle, \pi'\pi'\rangle \\
&= h\alpha\langle\langle \pi, \pi\pi'\rangle, \pi'\pi'\rangle \\
&= h\langle \pi, \langle \pi\pi', \pi'\pi'\rangle\rangle \\
&= h\langle \pi, \pi'\rangle = h
\end{aligned}
$$

and

$$
\begin{aligned}
(\varepsilon^A * \langle k * \pi^A, \pi'^A\rangle)^{*_A} &= (\varepsilon^A * \langle k\langle \pi, \pi\pi'\rangle, \pi'\pi'\rangle)^{*_A} \\
&= (\varepsilon^A * \langle k\langle \pi, \pi\pi'\rangle, \pi'\pi'\rangle\alpha)^* \\
&= (\varepsilon\pi'\langle \pi, \langle k\langle \pi, \pi\pi'\rangle, \pi'\pi'\rangle\alpha\rangle)^* \\
&= (\varepsilon\langle k\langle \pi, \pi\pi'\rangle, \pi'\pi'\rangle\alpha)^* \\
&= (\varepsilon\langle k\langle \pi, \pi\pi'\rangle\alpha, \pi'\rangle)^* \\
&= (\varepsilon\langle k\langle \pi\alpha, \pi\pi'\alpha\rangle, \pi'\rangle)^* = (\varepsilon\langle k\langle \pi\pi, \pi'\pi\rangle, \pi'\rangle)^* \\
&= (\varepsilon\langle k\pi, \pi'\rangle)^* = k
\end{aligned}
$$

Therefore, $\mathscr{C}_A$ is Cartesian closed.

□

Proposition. *Define a functor $H_A : \mathscr{C} \to \mathscr{C}_A$ by $H_A(B) = B$ for any object B and $H_A(f) = f\pi'_{A,B}$ for any arrow $f : B \to C$. The arrows formed by* *composition, pairing, and* $(-)^*$ *using the "indeterminate" arrow* $x := \pi_{A,1}$ *and the images under* H_A *of arrows in* $\mathscr{C}$ *are called* **polynomials**. *We can write* $\mathscr{C}[x]$ *instead of* $\mathscr{C}_A$ *and say that the arrow* x *is adjoined to* $\mathscr{C}$. *When using the notation* $\mathscr{C}[x]$, *we denote composition by ordinary juxtaposition, and write* g *for* $H_A(g)$.

Proof. To see that H_A is indeed a functor, we note that for any object B, $F(1_B) = \pi'_{A,B} = \varepsilon_A(B)$, the identity arrow in $\mathscr{C}_A$, and if we have arrows $f : B \to C$ and $g : C \to D$, then

$$\begin{aligned} H_A(g) * H_A(f) &= (g\pi'_{A,C}) * (f\pi'_{A,B}) \\ &= (g\pi'_{A,C})\langle \pi_{A,B}, f\pi'_{A,B}\rangle\langle \pi_{A,B}, 1_{A\times B}\rangle \\ &= (g\pi'_{A,C})\langle \pi_{A,B}\langle \pi_{A,B}, 1_{B\times C}\rangle, f\pi'_{A,B}\langle \pi_{A,B}, 1_{A\times B}\rangle \\ &= (g\pi'_{A,C})\langle \pi_{A,B}, f\pi'_{A,B}\rangle \\ &= gf\pi'_{A,B} = H_A(gf) \end{aligned}$$

□

The Kleisli category construction can be repeated multiple times to adjoin any finite number of indeterminates $x_1, \ldots, x_n$. If we adjoin the set X of arrows to the category $\mathscr{C}$, we call the resulting category $\mathscr{C}[X]$.

Proposition. *Given any functor* $F : \mathscr{C} \to \mathscr{D}$ *and any arrow* $b : 1 \to F(A)$ *in* $\mathscr{D}$, *there exists a unique Cartesian closed functor* $F' : \mathscr{C}_A \to \mathscr{D}$ *such that* $F'H_A = F$ *and* $F'(\pi_{A,1}) = b$.

Proof. Define $F'(B) = B$ for any object B and $F'(f) = F(f)\langle b\bigcirc_{F(B)}, 1_{F(B)}\rangle$ for any arrow $f : B \to C$ in $\mathscr{C}_A$.

We can then check that F' is a Cartesian functor:

$$\begin{aligned} F(\pi') &= F(\pi')\langle b\bigcirc, 1\rangle = \pi'\langle b\bigcirc, 1\rangle = 1 \\ F'(g * f) &= F(g\langle \pi, f\rangle)\langle b\bigcirc, 1\rangle = F(g)\langle \pi, F(f)\rangle\langle b\bigcirc, 1\rangle \\ &= F(g)\langle b\bigcirc, F(f)\langle b\bigcirc, 1\rangle\rangle = F(g)\langle b\bigcirc, 1\rangle F(f)\langle b\bigcirc, 1\rangle \\ &= F'(g)F'(f) \\ F'(\pi^A) &= F'(\pi\pi') = F(\pi\pi')\langle b\bigcirc, 1\rangle = \pi\pi'\langle b\bigcirc, 1\rangle = \pi 1 = \pi \\ F'(\pi^A) &= F'(\pi'\pi') = F(\pi'\pi')\langle b\bigcirc, 1\rangle = \pi'\pi'\langle b\bigcirc, 1\rangle = \pi' 1 = \pi' \\ F'(\langle f, g\rangle) &= F(\langle f, g\rangle)\langle b\bigcirc, 1\rangle = \langle F(f), F(g)\rangle\langle b\bigcirc, 1\rangle \\ &= \langle F(f)\langle b\bigcirc, 1\rangle, F(g)\langle b\bigcirc, 1\rangle\rangle = \langle F'(f), F'(g)\rangle \end{aligned}$$

In addition, for any objects B, C, and any arrow $h : B \times C \to D$ in $\mathscr{C}_A$, we have

$$\begin{aligned} F'(\varepsilon^A_{B,C}) &= F(\varepsilon_{B,C}\pi')\langle b\bigcirc, 1\rangle = \varepsilon_{F(B),F(C)}\pi'\langle b\bigcirc, 1\rangle \\ &= \varepsilon_{F(B),F(C)} = \varepsilon_{F'(B),F'(C)} \\ F'(h^{*_A}) &= F'((h\alpha)^*) = F((h\alpha)^*)\langle b\bigcirc, 1\rangle = (F(h)\alpha)^*\langle b\bigcirc, 1\rangle \\ &= (F(h)\alpha\langle\langle b\bigcirc, 1\rangle\pi, \pi'\rangle)^* = (F(h)\langle b\bigcirc, 1\rangle)^* = (F'(h))^* \end{aligned}$$

so F' is a Cartesian closed functor.

Furthermore, we have $F'(H_A(B)) = F'(B) = F(B)$, and for any arrow $f : B \to C$,

$$\begin{aligned} F'(H_A(f)) &= F'(f\pi'_{A,C}) \\ &= F(f\pi'_{A,C})\langle b\bigcirc_{F(B)}, 1_{F(B)}\rangle \\ &= F(f)\pi'_{A,C}\langle b\bigcirc, 1_{F(B)}\rangle = F(f)1_{F(B)} = F(f) \\ F'(\pi_{A,1}) &= F(\pi_{A,1})\langle b\bigcirc_{F(1)}, 1_{F(1)}\rangle \\ &= \pi_{A,1}\langle b\bigcirc_1, 1_1\rangle = b\bigcirc_1 = b1_1 = b \end{aligned}$$

To see that F' is unique, suppose that G is a functor with the desired properties. Then we must have $G(B) = F(B)$ on each object B, and if we have an arrow $f : B \to C$, we must have

$$\begin{aligned} G(f) &= G(f\pi'_{A,B}\langle\pi_{A,B}, 1_{A\times B}\rangle) \\ &= G((f\pi'_{A,B}) * 1) = G(f\pi'_{A,B})G(1) = G(H_A(f))G(\langle\pi_{A,B}, \pi'_{A,B}\rangle) \\ &= F(f)G(\langle\pi_{A,B}\langle\pi_{A,1}, \bigcirc_{A\times 1}\rangle, \pi'_{A,B}\rangle) \\ &= F(f)G(\langle\pi * \bigcirc, \pi'\rangle) \\ &= F(f)\langle G(\pi)G(\bigcirc), G(\pi')\rangle \\ &= G(f)\langle b\bigcirc, 1_{F(B)}\rangle \end{aligned}$$

□

Proposition. *For every polynomial $\phi(x) : 1 \to C$ in an indeterminate $x : 1 \to A$ in the Kleisli category $\mathscr{C}_A$, there is a unique arrow $g : A \to C$ in $\mathscr{A}$ such that $H_A(g) * x = \phi(x)$.*

Proof. The arrow $\phi(x)$ is an arrow $\phi : A \times 1 \to C$ in $\mathscr{C}$. So we take $g = \phi\langle 1_A, \bigcirc_A\rangle$.

We then have

$$\begin{aligned} (H_A(g) * x)\langle 1_A, \bigcirc_A\rangle &= (g\pi'\langle\pi, x\pi'\rangle)\langle\pi, 1\rangle \\ &= (gx)\pi'\langle\pi, 1\rangle = gx \end{aligned}$$

□

Proposition. *Given a polynomial $\phi(x) : 1 \to C$ in the indeterminate $x : 1 \to A$ in the Cartesian closed category $\mathscr{C}[x]$ there is a unique map $h : 1 \to C^A$ such that $\varepsilon_{C,A}\langle h, x\rangle = \phi(x)$.*

Proof. Let g be the map described previously. If we put $h = \ulcorner g \urcorner$ works, we then have $h^{\int} = g$ and $gx = h^{\int}x = \varepsilon_{C,A}\langle h, x\rangle = \phi(x)$. We can write $\lambda_{x\in A}\phi(x)$ for the map h. □

2.5 Natural numbers objects

In demonstrating the equivalence between Cartesian closed categories and typed λ-calculi, we will be considering λ-calculi in which there is a type N of natural numbers. We will now describe the corresponding notion for Cartesian closed categories.

Definition. In a Cartesian closed category, a weak **natural number object** consists of an object N and a pair of arrows $0 : 1 \to N$ and $s : N \to N$ such that. For any object A and arrows $a : 1 \to A$ and $f : A \to A$, there is an arrow $u : N \to A$ such that $u \circ 0 = a$ and $u \circ s = fu$.

If for every such structure $1 \xrightarrow{a} A \xrightarrow{f} A$ the arrow u is unique, then the category has a strong natural numbers object, or simply a natural numbers object. If $1 \xrightarrow{0} N \xrightarrow{s} N$ is a (weak) natural numbers object, then for any diagram $1 \xrightarrow{a} A \xrightarrow{f} A$, we shall define $J_A(a, f)$ to be an arrow $u : N \to A$ such that $u \circ 0 = a$ and $u \circ s = fu$.

Example. In the category **Set**, the set $\mathbf{N}$ of natural numbers together with the arrows $0 : 1 \to \mathbf{N}$ taking the singleton to 0, and the arrow $s : \mathbf{N} \to \mathbf{N}$ taking n to $n + 1$ is a natural numbers object. If we have a diagram $1 \xrightarrow{a} A \xrightarrow{f} A$, then we define $u : \mathbf{N} \to A$ so that $u(0) = a(*)$, and so that $u(n) = a^n(*)$ where a^n means applying the map a n times.

Proposition. *If $\mathscr{A}$ has a natural numbers object N, then if $\mathscr{A}[x] = \mathscr{A}_A$ is the category created by adjoining an indeterminate $x : 1 \to A$, $\mathscr{A}[x]$ has the same natural numbers object.*

Proof. Suppose we have an object B and maps $\beta(x) : 1 \to B$ and $\phi(x) : B \to B$ in $\mathscr{A}[x]$. We claim that there is a map $\chi(x) : N \to B$ in $\mathscr{A}[x]$, such that $\chi(x) * 0 = \beta(x)$ and $\chi(x) * S = \phi(x) * \chi(x)$.

More explicitly, the maps β and ϕ correspond to maps $b : 1A \times 1 \to B$ and $f : A \times B \to B$ in $\mathscr{A}$, and we want to find maps $h : A \times N \to B$ such that $h\langle \pi, 0\pi' \rangle = b$ and $h\langle \pi, S\pi' \rangle = f\langle \pi, h \rangle$.

Consider the maps $b' = (b\langle \pi', \pi \rangle)^*$ and $f' = (f\langle \pi', \varepsilon \rangle)^*$.

Since N is a natural numbers object in $\mathscr{A}$, there exist $h' : N \to B^A$ such that $h'0 = b'$ and $h'S = f'h'$.

But then if $h = \varepsilon\langle h'\pi', \pi \rangle$, we have

$$\begin{aligned}
h\langle \pi, 0\pi' \rangle &= \varepsilon\langle h'\pi', \pi \rangle\langle \pi, 0\pi' \rangle \\
&= \varepsilon\langle h'0\pi', \pi \rangle = \varepsilon\langle b'\pi', \pi \rangle = b \\
h\langle \pi, S\pi' \rangle &= \varepsilon\langle h'\pi', \pi \rangle\langle \pi, S\pi' \rangle \\
&= \varepsilon\langle h'S\pi', \pi \rangle \\
&= \varepsilon\langle f'h'\pi', \pi \rangle \\
&= \varepsilon\langle (f\langle \pi, h \rangle)^*\pi, \pi' \rangle \\
&= f\langle \pi, h \rangle
\end{aligned}$$

□

Putting $A = B \times B^B$ in the previous theorem, we see that for a Cartesian closed category to have a natural numbers object N, it is sufficient that there are maps $0 : 1 \to N$ and $S : N \to N$ such that for every object B there exists an arrow $I_B : (B \times B^B) \times N \to B$ for which $I_B\langle y, v, 0\rangle = y$, and $I_B\langle y, v, Sz\rangle = v^{\int} I_B\langle y, v, z\rangle$. For any maps $y : 1 \to B$, $v : 1 \to B^B$ and $z : 1 \to N$, we can obtain the maps J_B from the maps I_B and vice-versa by putting $J_B(y, v^{\int})z = I_B\langle y, v, z\rangle$ and using the fact that $-^{\int}$ and $\ulcorner - \urcorner$ are inverses.

2.6 The category $\mathbf{Cart_N}$

We now have all of the tools needed to define the category $\mathbf{Cart_N}$, which is one of the two equivalent categories we are studying. In the category of Cartesian closed categories with weak natural numbers objects $\mathbf{Cart_N}$, the objects are small Cartesian closed categories. The arrows are Cartesian closed functors $F : \mathscr{C} \to \mathscr{D}$ that preserve natural numbers objects – i.e. $F(N) = N$, $F(0) = 0$, $F(s) = s$, and $F(J_A(a, f)) = J_{F(A)}(F(a), F(f))$.

Chapter 3

Typed λ-calculi

A **typed λ-calculus** is a system consisting of sets of types, terms, and equations. These λ-calculi will constitute the objects of the category $\lambda - \mathbf{Calc}$. As we shall see shortly, the terms are assigned various types and can be constructed using substitution and various other term-forming operations. We will write $a \in B$ to indicate that a is of type B. As we mentioned earlier, we will include a type N of natural numbers, which will include a constant term 0, a successor operator S, and an "iterator" $I(-,-,-)$. In addition, we will include equations to ensure that that $I(a,h,n)$ yields h applied n times to a. We will also include a type 1, in which there is only one term. The types 1 and N are basic types and are guaranteed to exist; there might also be other types. Furthermore, given any two types, we can form the types $A \times B$ and B^A. Some types so constructed might turn out to be equal.

The set of terms is defined as follows:

1. For each type A, there exist countably many variables of type A, denoted $x_i^A \in A$ where $i \in \mathbf{N}$. Every variable is a term.
2. There is a constant term $*$ of type 1.
3. For any types A and B, if $a \in A$ and $b \in B$ and $c \in A \times B$, then $\langle a, b \rangle \in A \times B$, and $\pi_{A,B}(c) \in A$ and $\pi'_{A,B}(c) \in B$.
4. If $f \in B^A$ and $a \in A$ then $\varepsilon_{B,A}(f,a) \in B$. The term $\varepsilon_{B,A}(f,a)$ may be written as $f^{\int}a$ or "f of a".
5. If $x \in A$ and $\phi(x) \in B$, then $\lambda_{x \in A}\phi(x) \in B^A$.
6. The type N contains a term 0, and for any $n \in N$, we have another term $S(n) \in N$.
7. If $a \in A$ and $h \in A^A$ and $n \in N$ then $I_A(a,h,n) \in A$.

There may be other terms not constructed from these operations.

Definition. A variable is **bound** if it is in the scope of a λ, and **free** otherwise.

Definition. A term is said to be **closed** if there are no free variables.

Given a term $\phi(x)$ containing a variable x, we can substitute a term a for x by replacing all free instances of x with a, yielding a new string $\phi(a)$. However, if we allowed any term a to be substituted, we would be able to derive incorrect equations. Thus, we should only allow substitution of terms if such substitution does not bind any additional variables. We therefore, adopt the following definition, and only allow substitution of substitutible terms.

Definition. If substituting a term $a \in A$ for the variable x in $\phi(x)$ does not cause any variables that were free in a to become bound, a is said to be **substitutible** for x.

For any finite set X, we have a binary equivalence relation $=_X$ between terms; for $X = \emptyset$, $=_X$ is the same as ordinary equality. We can represent each equivalence relation $=_X$ as the set $\{(a, b) | a =_X b\}$. In order for two terms a and b to satisfy $a =_X b$, the terms a and b must be of the same type, same type, and all free variables in a and b must be elements of X. We use the other relations $=_X$ to define the ordinary equations between closed terms. In addition, if $X \subset Y$ then $a =_X b \implies a =_Y b$. Each relation $=_X$ acts like equality in that equals can be substituted for equals – given two terms that are related, we can obtain another equation between these terms by applying the term-forming operations to both sides:

1. If $a =_X a'$ and $b =_Y b'$ then $\langle a, b\rangle =_{X\cup Y} \langle a', b'\rangle$
2. If $c, c' \in A \times B$ and $c =_X c'$ then $\pi_{A,B}(c) =_X \pi_{A,B}(c')$, and $\pi'_{A,B}(c) =_X \pi'_{A,B}(c')$.
3. If $f, g \in B^A$, $a, b \in A$, and $a =_X b$ and $f =_Y g$ then $f^\int a =_X f^\int b$, and $f^\int a =_Y g^\int b$.
4. If $x \in A$ and $\phi(x) =_{X\cup\{x\}} \psi(x)$ then $\lambda_{x\in A}\phi(x) =_X \lambda_{x\in A}\psi(x)$.
5. If $a =_X a'$, $h =_Y h'$, and $k =_Z k'$ then $I_A(a, h, k) =_{X\cup Y\cup Z} I_A(a, h, k)$.

In addition, we have the following equations:

1. For any $a \in 1$, $* =_X a$ (for any X such that all free variables in a are in X).
2. For any $a \in A$ and $b \in B$, then $\pi_{A,B}(\langle a, b\rangle) =_X a$, and $\pi'_{A,B}(\langle a, b\rangle) =_X b$.
3. If $c \in A \times B$ then $\langle \pi_{A,B}(c), \pi'_{A,B}(c)\rangle =_X c$
4. If a is substitutible for x in $\phi(x)$, then $\lambda_{x\in A}\phi(x)^\int a =_X \phi(a)$.
5. If x is not in X, then $\lambda_{x\in A}(f^\int x) =_X f$.

6. If $a \in A$ and $h \in A^A$ then $I(a,h,0) =_X a$.

7. If x is not in X then $I(a,h,S(x)) =_{X \cup \{x\}} h^\int I(a,h,x)$.

8. If x' is substitutible for x in $\phi(x)$ and is not free in $\phi(x)$, then $\lambda_{x\in A}\phi(x) =_X \lambda_{x'\in A}\phi(x')$.

The pure typed λ-calculus with weak natural numbers object has exactly the types, terms, and equations listed above, without any other identifications between types.

3.1 Translations

We will now define translations, which are structure-preserving maps between typed λ-calculi. These maps preserve all type and term forming operations, and take natural numbers objects to natural numbers objects. The equivalent notion for Cartesian closed categories is that of the Cartesian closed functor.

Definition. Given two typed λ-calculi $\mathscr{L}$ and $\mathscr{L}'$, a map $\Phi : \mathscr{L} \to \mathscr{L}'$ is a **translation** if it satisfies the following properties:

1. Φ takes types to types and terms to terms. If $a \in B$ then $\Phi(a) \in \Phi(B)$.
2. If a is a closed term, then so is $\Phi(a)$.
3. For any type A, $\Phi(x_i^A) = x_i^\Phi(A)$.
4. $\Phi(1) = 1$, $\Phi(A \times B) = \Phi(A) \times \Phi(B)$, $\Phi(N) = N$, and $\Phi(B^A) = \Phi(B)^{\Phi(A)}$.
5. For terms a and b, $\langle\Phi(a), \Phi(b)\rangle = \Phi(\langle a, b\rangle)$. For $c \in A \times B$, $\Phi(\pi_{A,B}(c)) = \pi_{\Phi(A),\Phi(B)}(\Phi(c))$, and $\Phi(\pi'_{A,B}(c)) = \pi'_{\Phi(A),\Phi(B)}(\Phi(c))$.
6. If $f \in B^A$ and $a \in A$, then $\Phi(\varepsilon_{B,A}(f,a)) = \varepsilon_{\Phi(B,A)}(\Phi(f), \Phi(a))$.
7. If $x \in A$ and $\phi(x) \in B$, then $\Phi(\lambda_{x\in A}\phi(x)) = \lambda_{\Phi(x)\in\Phi(A)}\Phi(\phi(x))$.
8. $\Phi(0) = 0$. $\Phi(S(n)) = S(\Phi(n))$.
9. If $a \in A$ and $h \in A^A$ and $n \in N$ then $\Phi(I_A(a,h,n)) = I_{\Phi(A)}(\Phi(a), \Phi(h), \Phi(n))$.
10. If $a =_X a'$ then $\Phi(a) =_{\Phi(X)} \Phi(a')$.

Two translations Φ and Ψ are equal if $a =_X a'$ implies that $\Phi(a) =_{\Phi(X)} \Psi(a')$.

Definition. In the category **λ-Calc** of typed λ-calculi, the objects are typed λ-calculi and the morphisms are translations.

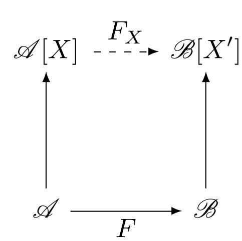
Chapter 4

A functor from Cartesian closed categories to typed lambda calculi

We can define a functor L from $\mathbf{Cart_N}$ to λ-**Calc**. This functor constitutes one half of the equivalence.

The functor L associates each Cartesian closed category $\mathscr{A}$ with a typed λ-calculus $L\mathscr{A}$. The types of $L\mathscr{A}$ correspond to the objects of $\mathscr{A}$; in particular, the type 1 corresponds to the terminal object, N corresponds to the weak natural numbers object. Furthermore, for any objects A and B, the type $A \times B$ corresponds to the product object $A \times B$, and the type B^A corresponds to the exponential object B^A. The terms of $L\mathscr{A}$ of type A are formed from variables $x_i : 1 \to A_i$ and constants, where the constants are arrows $1 \to A$ in $\mathscr{A}$ using the term-forming operations described above. Two terms are equal if they are equal in the category obtained by adjoining the set of variables that are free in either of them.

For a functor $F : \mathscr{A} \to \mathscr{B}$, $L(F)$ is defined by $L(F)(A) = F(A)$ and $L(F)(x_i) = x_i$. For any term $\phi(X)$, the functor F extends naturally to a unique functor $F_X : \mathscr{A}[X] \to \mathscr{B}[X']$ between the categories obtained by adjoining all variables free in $\phi(X)$ as arrows, so that the following diagram commutes:



We can then define $L(F)(\phi(X)) = F_X(\phi(X))$.

Proposition. *L is a functor from* $\mathbf{Cart_N}$ *to* $\lambda - \mathbf{Calc}$.

Proof. We want to show that each map $L(F)$ is a translation, that for any Cartesian closed category $\mathscr{A}$ $L(1_{\mathscr{A}})$ is the identity translation $1_{L(\mathscr{A})}$, and that $L(GF) = L(G)L(F)$ for any composable Cartesian closed functors G and F.

Since each F is a Cartesian closed functor, we have $L(F)(1) = 1$, $L(F)(A \times B) = L(F)(A) \times L(F)(B) = F(A) \times F(B)$, $L(F)(N) = N$, and $L(F)(B^A) = L(F)(B)^{L(F)(A)} = F(B)^{F(A)}$. Thus $L(F)$ preserves type-forming operations.

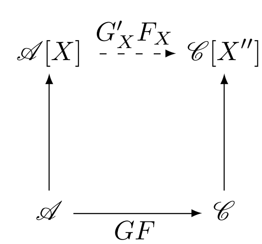
Furthermore, since F is Cartesian closed, and the extension F_X is Cartesian closed for any finite set of variables X, we have

1. $L(F)(\pi_{A,B}) = F(\pi_{A,B}) = \pi_{F(A),F(B)} = \pi_{L(F)(A),L(F)(B)}$
2. $L(F)(\pi'_{A,B}) = F(\pi'_{A,B}) = \pi'_{F(A),F(B)} = \pi'_{L(F)(A),L(F)(B)}$
3. $L(F)(\langle f, g\rangle) = F_S(\langle f, g\rangle) = \langle F_S(f), F_S(g)\rangle = \langle L(F)(f), L(F)(g)\rangle$, where S is the set of variables that are free in either f or g.
4. For any terms $f \in B^A$ and $a \in A$, $L(F)(\varepsilon_{B,A}(f, a)) = F_S(\varepsilon_{B,A}(f, a)) = \varepsilon_{F(B),F(A)}(L(F)(f), L(F)(a))$, where S is the set of variables free in either f or a.
5. For any variable $x \in A$ and term $\phi(x) \in B$, $L(F)(\lambda_{x\in A}\phi(x)) = F_X(\lambda_{x\in A}\phi(x)) = \lambda_{L(F)(x)\in F(A)}F_X(\phi(x)) = \lambda_{L(F)(x)\in L(F)(A)}L(F)(\phi(x))$
6. $L(F)(0) = F(0) = 0$, and $L(F)(S(n)) = F_X(S(n)) = S(F_X(n)) = S(L(F)(n))$ where X is the set of free variables in n.

Thus each $L(F)$ is a translation.

In addition, for any Cartesian closed category $\mathscr{A}$, $L(1_{\mathscr{A}})(A) = 1_{\mathscr{A}}(A) = A$ for each object A in $\mathscr{A}$. In addition, for each term $\phi(X)$ we have $L(1_{\mathscr{A}})(\phi(X)) = 1_{\mathscr{A}[X]}\phi(X)$. Thus $L(1_{\mathscr{A}})$ is the identity translation.

Finally, suppose we have functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{C}$. We claim that $L(GF) = L(G)L(F)$. We have $L(GF)(A) = L(G)(L(F)(A)) = G(F(A))$ for every type A in $\mathscr{A}$. Now consider a term $\phi(X)$: we have $L(GF)(\phi(X)) = (GF)_X(\phi(X))$ and $L(G)L(F)\phi(X) = L(G)(F_X(\phi(X))) = G_{X'}(F_X(\phi(X)))$, but since $G_{X'}F_X$ extends GF, making the following diagram commute



we must have $G'_X F_X = (GF)_X$. Thus $L(GF) = L(G)L(F)$. □

Chapter 5

A functor from typed lambda calculi to Cartesian closed categories

We can define a functor C from λ-**Calc** to $\mathbf{Cart_N}$ as follows:

Given a typed lambda calculus $\mathscr{L}$, the objects of $\mathscr{C}(L)$ are the types of $\mathscr{L}$.

The arrows $A \to B$ of $C(\mathscr{L})$ are equivalence classes of pairs $(x \in A, \phi(x))$ where x is a variable of type A and $\phi(x)$ is a term of type B with no free variables other than x. The arrows $(x \in A, \phi(x))$ and $(y \in A, \psi(y))$ are equivalent if and only if $\phi(x) =_{\{x\}} \psi(x)$.

To compose two arrows $(x \in A, \phi(x)) : A \to B$ and $(y \in B, \psi(y)) : B \to C$ we take $(x \in A, \psi(\phi(x)))$ where $\phi(x)$ is substituted for y in $\psi(y)$.

We have

1. $\bigcirc_A = (x \in A, *)$
2. $\pi_{A,B} = (z \in A \times B, \pi(z))$
3. $\pi'_{A,B} = (z \in A \times B, \pi'(z))$
4. $\langle (z \in C, \phi(z)), (z \in C, \psi(z)) \rangle = (z \in C, \langle \phi(z), \psi(z) \rangle)$
5. $(z \in A \times B, \chi(z))^* = (x \in A, \lambda_{y \in B} \chi(\langle x, y \rangle))$
6. $\varepsilon_{C,A} = (y \in C^A \times A, \varepsilon_{C,A}(\pi(y), \pi'(y)))$

The weak natural numbers object is then given by the object N and the arrow $0 = (x \in 1, 0)$, $S = (x \in N, S(x))$, and $I_B = (w \in (B \times B^B) \times N, I(\pi(\pi(w)), \pi'(\pi(w)), \pi'(w)))$.

For each translation Φ, we have that for each type A $C(\Phi)(A)$ is the object corresponding to $\Phi(A)$, and for an arrow $(x \in A, \phi(x))$, $C(\Phi)(x \in A, \phi(x)) = (\Phi(x) \in \Phi(A), \Phi(\phi(x)))$.

If Φ is a translation, we have $C(\Phi)(1) = 1$, $C(\Phi)(N) = N$, $C(\Phi)(A \times B) = C(\Phi)(A) \times C(\Phi)(B)$ and $C(\Phi)(A^B) = C(\Phi)(A)^{C(\Phi)(B)}$. In addition, we have

$$\begin{aligned}
C(\Phi)(\bigcirc_A) &= (\Phi(x) \in \Phi(A), \Phi(*)) \\
&= \bigcirc_{C(\Phi)(A)} \\
C(\Phi)(\pi_{A,B}) &= (\Phi(z) \in \Phi(A) \times \Phi(B), \pi(\Phi(z))) = \pi_{C(\Phi)(A),C(\Phi)(B)} \\
C(\Phi)(\pi'_{A,B}) &= (\Phi(z) \in \Phi(A) \times \Phi(B), \pi'(\Phi(z))) = \pi'_{C(\Phi)(A),C(\Phi)(B)} \\
C(\Phi)(\langle (z \in C, \phi(z)), (z \in C, \psi(z)) \rangle) &= C(\Phi)(z \in C, \langle \phi(z), \psi(z) \rangle) \\
&= (\Phi(z) \in \Phi(C), \langle \Phi(\phi(z)), \Phi(\psi(z)) \rangle) \\
&= \langle C(\Phi)(z \in C, \phi(z)), C(\Phi)(z \in C, \psi(z)) \rangle \\
C(\Phi)(z \in A \times B, \chi(z))^* &= C(\Phi)(x \in A, \lambda_{y \in B} \chi(\langle x, y \rangle)) \\
&= (\Phi(x) \in \Phi(A), \Phi(\lambda_{y \in B} \chi(\langle x, y \rangle))) \\
&= (\Phi(x) \in \Phi(A), \lambda_{\Phi(y) \in \Phi(B)} \chi(\langle \Phi(x), \Phi(y) \rangle)) \\
&= (C(\Phi)(z \in A \times B, \chi(z)))^* \\
C(\Phi)(\varepsilon_{C,A}) &= C(\Phi)(y \in C^A \times A, \varepsilon_{C,A}(\pi(y), \pi'(y))) \\
&= (\Phi(y) \in \Phi(C^A \times A), \Phi(\varepsilon_{C,A}(\pi(y), \pi'(y)))) \\
&= (\Phi(y) \in \Phi(C^A \times A), \varepsilon_{\Phi(C),\Phi(A)}(\pi(\Phi(y)), \pi'(\Phi(y)))) \\
&= \varepsilon_{C(\Phi)(C),C(\Phi)(A)}
\end{aligned}$$

Thus each $C(\Phi)$ is a Cartesian closed functor. If 1 is the identity translation for a typed lambda calculus, then $C(1)(A) = A$ and $C(1)(x \in A, \phi(x)) = (x \in A, \phi(x))$. Thus $C(1)$ is the identity functor.

We claim that if $\Phi : \mathscr{L} \to \mathscr{L}'$ and $\Psi : \mathscr{L}' \to \mathscr{L}''$ are two translations, then $C(\Psi\Phi) = C(\Psi)C(\Phi)$. For an object A, we have $C(\Psi\Phi)(A) = \Psi(\Phi(A)) = C(\Psi)C(\Phi)(A)$. In addition, for each arrow $f = (x \in A, \phi(x))$, we have $C(\Psi\Phi)(f) = (\Psi(\Phi(x)) \in \Psi(\Phi(A)), \Psi(\Phi(x))) = C(\Psi)(C(\Phi)(f))$.

Thus C is indeed a functor from **λ-Calc** to **Cart-N**.

5.1 Adjoining a parameter to a typed λ-calculus

To show that the functors L and C define an equivalence, we will need to introduce the notion of "adjoining a parameter" to a typed λ-calculus $\mathscr{L}$, and show that this operation corresponds to adjoining an indeterminate arrow to the category $C(\mathscr{L})$.

Given a typed lambda calculus $\mathscr{L}$ and a variable x of type A, we can define a language $\mathscr{L}(x)$, known as "$\mathscr{L}$ with x adjoined" – the variable x then becomes a "parameter".

The language $\mathscr{L}(x)$ has the same types as $\mathscr{L}$. The terms of $\mathscr{L}(x)$ are also the same, but terms are now considered closed if they contain no free variables other than x – i.e. the symbol x is not considered a variable. We have $a =_X b$ in $\mathscr{L}(x)$ if and only if $a =_{X \cup \{x\}} b$ in $\mathscr{L}$.

Proposition. *For any typed λ-calculus $\mathscr{L}$ and any variable x of type A, $C(\mathscr{L})[x] \equiv C(\mathscr{L}(x))$*

Proof. The category $C(\mathscr{L})[x]$ has the universal property that for any functor $F : C(\mathscr{L}) \to \mathscr{A}$ and any arrow $b : 1 \to F(A)$, there is a unique functor F' such that $F'H_x = F$ and $F'(x) = b$, where H_x is the functor which embeds $\mathscr{C}$ into $\mathscr{C}[x]$. We claim that $C(\mathscr{L}(x))$ has the same universal property, where H_x is the result of applying C to the inclusion of $\mathscr{L}$ into $\mathscr{L}(x)$.

We define $F' : C(\mathscr{L}(x)) \to \mathscr{A}$ as follows: Since the types in $\mathscr{L}$ an $\mathscr{L}(x)$ are the same, we put $F'(B) = F(B)$ for each object B in $C(\mathscr{L})$. If we have an arrow $f = (y \in B, \phi(x, y)$ from $B \to C$ in $C(\mathscr{L}(x))$, then if $\psi(y) = \lambda_{x \in A}\phi(x, y)$, we have $\phi(x, y) =_{\{x,y\}} \psi(y)^{\int} x$. Applying C to both sides, we have $f = \varepsilon_{C,A}\langle g, x\bigcirc_B\rangle$, where $g = (y \in B, \psi(y))$. But then if we are to have $F'H_x = F$ and $F'(x) = b$, we must have $F'(f) = \varepsilon_{F(C),F(A)}\langle F(g), b\bigcirc_{F(B)}\rangle$.

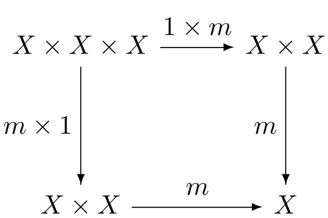
□

5.2 Examples

Suppose we have a typed lambda calculus $\mathscr{L}$ with a basic type X and basic terms $m \in X^{X \times X}$ and $u \in X$, and with the equations $\lambda_{x \in X} m^{\int}(u, x) = \lambda_{x \in X} x$, $\lambda_{x \in X} m^{\int}(x, u) = \lambda_{x \in X} x$, and

$$\lambda_{x \in X}\lambda_{y \in X}\lambda_{z \in X} m^{\int}(x, m^{\int}(y, z)) = \lambda_{x \in X}\lambda_{y \in X}\lambda_{z \in X} m^{\int}(m^{\int}(x, y), z)$$

. These equations state that m is a monoid operation on X, since we can substitute any term of type X into the λ-expressions provided no free variables become bound. Taking $C(\mathscr{L})$ gives us a Cartesian closed category in which we have arrows $u : 1 \to X$ and $m : X \times X \to X$ such that the diagram



commutes and in which $m\langle 1, u\rangle = m\langle 1, u\rangle = 1$. If we have no other equations, M is in fact the free monoid on countably many variables; otherwise, we get a monoid with countably many generators and with relations given by the other equations. We could add additional basic terms representing additonal generators if we wanted to describe a monoid which does not have a countable number of generators.

Similarly, for a typed lambda calclus $\mathscr{L}$ with a basic type X and basic terms

$m \in X^{X \times X}$, $u \in X$, and $i \in X^X$, we can take the equations

$$\lambda_{x \in X} m^{\int}(u, x) = \lambda_{x \in X} x$$
$$\lambda_{x \in X} m^{\int}(x, u) = \lambda_{x \in X} x$$
$$\lambda_{x \in X} m^{\int}(i^{\int} x, x) = \lambda_{x \in X} x$$
$$\lambda_{x \in X} \lambda_{y \in X} \lambda_{z \in X} m^{\int}(x, m^{\int}(y, z)) = \lambda_{x \in X} \lambda_{y \in X} \lambda_{z \in X} m^{\int}(m^{\int}(x, y), z)$$

Since we can substitute any terms of type x in the λ expressions provided we do not create any additional bound variables, these equations are equivalent to requiring that the terms of type M form a group, with i being a function that takes each element to its inverse. When we apply the functor C to $\mathscr{L}$, we obtain a category in which there is an object X with the same arrows as in the monoid example, but in which we also have an arrow $i : M \to M$, which satisfies $m(i \times 1_X) = 1_X$. We then obtain the free group on countably many variables, but can put in extra equations representing relations in order to obtain other groups. For example, we can add the equation $\lambda_{x \in X} \lambda_{y \in X} m^{\int}(x, y) = \lambda_{x \in X} \lambda_{y \in X} m^{\int}(y, x)$ to add the condition that the group is abelian; if we also add basic terms $n \in X^{X \times X}$, $e \in X$, and add the equations

$$\lambda_{x \in X} n^{\int}(e, x) = \lambda_{x \in X} x$$
$$\lambda_{x \in X} n^{\int}(x, e) = \lambda_{x \in X} x$$
$$\lambda_{x \in X} \lambda_{y \in X} \lambda_{z \in X} n^{\int}(x, n^{\int}(y, z)) = \lambda_{x \in X} \lambda_{y \in X} \lambda_{z \in X} n^{\int}(n^{\int}(x, y), z)$$
$$\lambda_{x \in X} \lambda_{y \in Y} n^{\int}(x, y) = \lambda_{x \in X} \lambda_{y \in Y} n^{\int}(y, x)$$
$$\lambda_{x \in X} \lambda_{y \in X} \lambda_{z \in X} n^{\int}(x, m^{\int}(y, z)) = \lambda_{x \in X} \lambda_{y \in X} \lambda_{z \in X} m^{\int}(n^{\int}(x, y), n^{\int}(x, z))$$

then we have imposed conditions which require the terms of type M to be a commutative ring with multiplicative identity e, additive identity i, multiplication operation n, and addition operation n. Without further conditions, we obtain the ring $\mathbf{Z}[\{x_n\}_{n \in \mathbf{N}}]$, but we can add additional equations to obtain some other ring which is a quotient ring of this ring.

Chapter 6

Equivalence of Cartesian closed categories and typed lambda calculi

Proposition. *The two functors L and C described previously define an equivalence of categories – i.e. there exist natural transformations $\varepsilon : CL \to \mathbf{1}_{\mathbf{Cart_N}}$ and $\eta : \mathbf{1}_{\lambda-\mathbf{Calc}} \to LC$, such that each arrow $\varepsilon(\mathscr{A}) : CL(\mathscr{A}) \to \mathscr{A}$ and $\eta(\mathscr{L}) : \mathscr{L} \to LC(\mathscr{L})$ is an isomorphism.*

Proof. Given a Cartesian closed category $\mathscr{A}$ in $\mathbf{Cart_N}$, we can define $\varepsilon : CL(\mathscr{A}) \to \mathscr{A}$ by putting $\varepsilon(\mathscr{A})(X) = X$ for each object X in $\mathscr{A}$.

In the category $CL(\mathscr{A})$, each arrow $B \to C$ is of the form $(y \in B, \phi(y))$ where $\phi(y)$ is in C.

If $f = (y \in B, \phi(y))$ is such an arrow, we define $\varepsilon(\mathscr{A})(X)$ to be the unique arrow $g : B \to C$ in $\mathscr{A}$ such that $gy = \phi(y)$.

To see that ε is indeed a natural transformation, suppose that we have two categories $\mathscr{A}$ and $\mathscr{B}$, and that we have a functor $F : \mathscr{A} \to \mathscr{B}$. We claim that $\varepsilon(\mathscr{B})CL(F) = F\varepsilon(\mathscr{A})$. For each object A of $\mathscr{A}$, we have $(\varepsilon(\mathscr{B})CL(F))(A) = (F\varepsilon(\mathscr{A}))(A) = F(A)$. For any arrow $f : B \to C$ in $CL(\mathscr{A})$, we have $f = (y \in B, \phi(y))$ where $\phi(y)$ is in C. We then have $F\varepsilon(\mathscr{A})(f) = F(g)$, where g is the unique arrow which satisfies $gy = \phi(y)$, while $\varepsilon(\mathscr{B})CL(F)(f) = \varepsilon(\mathscr{B})u$ where $u = (F(y) \in F(B), F_X(\phi(y)))$ where F_X is the unique extension of F to the category $\mathscr{A}[Y]$ obtained by adjoining all variables free in $\phi(y)$. We then have $\varepsilon(\mathscr{B})u = h$ where h is the unique arrow that satisfies $hF(y) = F_X(\phi(y))$. Applying the map F_X to both sides of $gy = \phi(y)$, we obtain $F(g)F(y) = F_X(\phi(y)$, so in fact we must have $h = F(g)$, and ε is indeed a natural transformation. In addition, each map $\varepsilon(\mathscr{A})$ is an isomorphism because for any finite set of variables X, we have a Cartesian closed functor H_X that embeds $\mathscr{A}$ embedding into the category $\mathscr{A}[X]$.

Conversely, for every object $\mathscr{L}$ in $\lambda - \mathbf{Calc}$ we can define a natural transformation $\eta : \mathbf{1}_{\lambda-\mathbf{Calc}} \to LC$ such that $\eta(\mathscr{L})(A) = A$ for each type A in $\mathscr{L}$, and $\eta(\mathscr{L})(\phi(x_1, \ldots, x_n)) = (z \in 1, h\phi(x_1, \ldots, x_n))$ in $C(\mathscr{L}(x_1, \ldots, x_n))$ for each term $\phi(x_1, \ldots, x_n)$. The map $\eta(\mathscr{L})$ is then an isomorphism because it has an inverse which takes $(z \in 1, \phi(z))$ to $\phi(*)$ for each term. To see that η is in fact a natural transformation, suppose that we have a translation $\Phi : \mathscr{L} \to \mathscr{L}'$. We claim that $\eta(\mathscr{L}')\Phi = LC(\Phi)\eta(\mathscr{L})$. For each type A, we have $(\eta(\mathscr{L}')\Phi)(A) = (LC(\Phi)\eta(\mathscr{L}))(A) = \Phi(A)$. In addition, if $\phi(x_1, \ldots, x_n)$ is a term, then we have $\eta(\mathscr{L}')\Phi(\phi(x_1, x_2, \ldots, x_n)) = (z \in 1, \Phi(\phi(x_1, \ldots, x_n)))$, and $\Phi(\eta(\mathscr{L})(\phi(x_1, x_2, \ldots, x_n))) = (\Phi(z) \in \Phi(1), \Phi(\phi(x_1, \ldots, x_n))) = (z \in \Phi(1), \Phi(\phi(x_1, \ldots, x_n)))$, since all terms of type 1 are equal. Thus η is indeed a natural transformation. $\square$

Bibliography

[1] H. P. Barendreght. *The Lambda Calculus, Its Syntax and Semantics.* North Holland, Amsterdam, 1985.

[2] J. Lambek and P. J. Scott. *Introduction to Higher-Order Categorical Logic.* Cambridge University Press, Cambridge, 1988.

[3] Saunders Maclane. *Categories for the Working Mathematician.* Springer, New York, 1971.