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Gröbner Basis Theory and its Applications for Regular and Biregular Functions

APPROVED BY

SUPERVISING COMMITTEE:

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William Beckner, Supervisor

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Irene Gamba

Gröbner Basis Theory and its Applications for Regular and Biregular Functions

by

Jenny Lee Ross, B.A.

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To Jamie. To Dagim. And to Mr. B

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Gröbner Basis Theory and its Applications for Regular and Biregular Functions

Jenny Lee Ross, M.A.
The University of Texas at Austin, 2010

Supervisor: William Beckner

This paper covers basic theory of Gröbner bases and an algebraic analysis of linear constant coefficient partial differential operators, specifically the Cauchy-Fueter operator. We will review examples and theory of regular and biregular functions in several quaternionic variables.

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Chapter 1

Introduction

When considering a system of partial differential equations with polynomial coefficients, one may look at it algebraically by viewing it as a module over a ring of differential operators. This allows the researcher to explore its properties through algebraic analysis, by understanding the algebraic structure of the associated module, with Gröbner bases playing a central role.

Gröbner bases and their applications have been gaining interest in many areas of Math and other Sciences. This paper will give an introduction to Gröbner basis theory before moving on to describe specific applications within linear constant coefficient partial differential equations. I will focus on work done by Adams, Berenstein, Loustanau, Sabini, Struppa, and Damiano [2, 1, 3, 10, 11, 13] which uses Gröbner bases to find the compatibility conditions of regular and biregular functions of several quaternionic variables.

We will first discuss the theory of Gröbner bases and linear constant coefficient partial differential operators through an algebraic analysis point of view.

1.1 Gröbner Bases

1.1.1 History of Gröbner Bases

In 1939, Wolfgang Gröbner used an ordering on monomials and found a basis for a zero-dimensional factor ring. Later he encouraged his student, Bruno Buchberger, to compute such bases for his thesis. By 1965, Buchberger had found criterion and an algorithm for computing a set of generators of ideals for commutative algebras, as well as the fundamental theorem necessary to show the termination and correctness of the algorithm [7]. He refined his work in [8, 9], and these types of sets of generators of ideals are now referred to as Gröbner Bases.

Anatoly Shirshov wrote a parallel theory of Gröbner Basis theory for Lie Algebras in 1962. It was written in Russian and never translated into English. Although his paper was written before Buchberger, it was largely overlooked. Some authors refer to Gröbner bases as Gröbner-Shirshov bases, but this is not the norm.

It wasn't until the eighties that math and computer science researches began to deeply investigate Gröbner Basis theory. Many computer algebra systems such as CoCoA [27], Macaulay2 [19], and Singular [20] have commands for computing Gröbner Bases, as well as computing syzygies, compatibility conditions, Hilbert functions, free resolutions, Extension modules, and noetherian operators. The algorithms for explicitly computing these result from the development of Gröbner Basis theory. These computer software packages save the researcher much time and hassle in computing these complicated algebraic objects, and in some cases, when dimensions are very high, they allow them to compute something that they would not otherwise have been able to compute. As George Orwell said,

> All animals are equal. But some animals are more equal than others.

A Gröbner Basis seems to be a "more equal" animal.

1.1.2 Theory of Gröbner Bases

This section will be an overview of the main concepts from the theory of Gröbner Bases. An excellent resource on this theory and its applications is Kreuzer and Robbiano's *Computational Commutative Algebra* [24]. The notation used in this report will be adopted from this surprisingly easy to read book.

1.1.2.1 Necessary Definitions and Notations

A ***monoid*** is a set together with an associative operation and an identity; for example, a group is a monoid in which each element is invertible. Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a ring of polynomials in n variables with complex coefficients. The set of all terms of R is denoted $\mathbb{T}^n$. $M = (\mathbb{C}[x_1, x_2, \ldots, x_n])^r$ is a finitely generated free R-module with canonical basis $\{e_1, \ldots, e_r\}$. A ***term*** of M is of the form te_i where $t \in \mathbb{T}^n$ and $1 \leq i \leq r$.

$$\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$$

is, therefore, the set of all terms of M. We will now define a term ordering σ on $\mathbb{T}^n$.

1.1.2.2 Orderings and Leading Terms

A relation σ on a monoid $(\Gamma, \circ)$ is called a ***monoid ordering*** if the following are satisfied for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$.

(i) (***Reflexivity***) $\gamma_1 \geq_\sigma \gamma_1$

(ii) (***Antisymmetry***) $\gamma_1 \geq_\sigma \gamma_2$ and $\gamma_2 \geq_\sigma \gamma_1$ imply $\gamma_1 = \gamma_2$

(iii) (***Transitivity***) $\gamma_1 \geq_\sigma \gamma_2$ and $\gamma_2 \geq_\sigma \gamma_3$ imply $\gamma_1 \geq_\sigma \gamma_3$

(iv) $\gamma_1 \geq_\sigma \gamma_2$ implies $\gamma_1 \circ \gamma_3 \geq_\sigma \gamma_2 \circ \gamma_3$.

(v) If we also have $\gamma \geq_\sigma 1_\Gamma$ for all $\gamma \in \Gamma$ then σ is called a ***term ordering*** on Γ.

Let $t_1, t_2 \in \mathbb{T}^n$, then $t_1 \geq_{Lex} t_2$ if and only if the first non-zero component of $log(t_1) - log(t_2)$ is positive or $t_1 = t_2$. This is called the ***Lexicographic Term Ordering*** and is denoted ***Lex***.

In other words, there is an initial ordering using Lex where the indeterminates are ordered decreasingly, by $x_1 \geq_{Lex} x_2 \geq_{Lex} \cdots \geq_{Lex} x_n$. Next you compare the exponents. For example with $n = 3$, $x_1x_2^3 \geq_{Lex} x_2^4x_3^5$, since $(1, 3, 0) - (0, 4, 5) = (1, -1, -5)$ has a positive first component. Similarly, $x_1x_3 \geq_{Lex} x_2^3$ and $x_1^2x_2^3x_3^2 \geq_{Lex} x_1^2x_2^3x_3$ since the first non-zero component is positive for each.

Let $t_1, t_2 \in \mathbb{T}^n$, then $t_1 \geq_{DegLex} t_2$ if $deg(t_1) > deg(t_2)$, or if $deg(t_1) =$

$deg(t_2)$ and $t_1 \geq_{Lex} t_2$. This is the ***Degree-Lexicographic Term Ordering*** and is denoted ***DegLex***.

With DegLex, you first compare the degree, and then if the degree of both are equal, you use Lex ordering to break the tie. Here are some examples: $x_1x_2^3x_3^3 \geq_{DegLex} x_1^3x_2x_3$ since $deg(x_1x_2^3x_3^3) = 7$ and $deg(x_1^3x_2x_3) = 5$. Also, $x_1^2x_2^2x_3^2 \geq_{DegLex} x_1x_2^3x_3^2$ since the degrees are the same, we then use Lex ordering and find the first component positive.

Let $t_1, t_2 \in \mathbb{T}^n$, then $t_1 \geq_{DegRevLex} t_2$ if $deg(t_1) > deg(t_2)$, or if $deg(t_1) = deg(t_2)$ and the last non-zero component of $log(t_1) - log(t_2)$ is negative, or if $t_1 = t_2$. This is the ***Degree-Reverse-Lexicographic Term Ordering*** and is denoted ***DegRevLex***.

With DegRevLex ordering, you first consider the degree of the terms, and if they are equal, you do the reverse of Lex ordering. As in the previous ordering, $x_1^3x_2^4x_3^5 \geq_{DegRevLex} x_1^3x_2x_3^5$ since the degree of the first is 12, and the degree of the second is 9. However, $x_1x_2^3x_3^4 \geq_{DegRevLex} x_1^4x_2^2x_3^2$ since the degrees are equal, we compare the first non-zero component, and notice it is negative.

Given a term ordering σ on $\mathbb{T}^n$, we can extend the ordering to a module ordering τ on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$; which is compatible with σ. In other words, given $t_1 \geq_\sigma t_2$, then we have $t_1e_i \geq_\tau t_2e_i$ for all e_i in the canonical basis of R^r. We can accomplish this extension either by ordering terms first and then positions, or conversely by first considering position, and then terms. The first case is called ***ToPos*** and is defined:

$$t_1e_i >_\tau t_2e_j \Longleftrightarrow t_1 >_\sigma t_2 \text{ or } (t_1 = t_2 \text{ and } i < j).$$

This first orders the terms, and then orders the position. The second method is called ***PosTo***, and it does the opposite as follows:

$$t_1e_i >_\tau t_2e_j \Longleftrightarrow i < j \text{ or } (i = j \text{ and } t_1 >_\sigma t_2).$$

Now that we have defined the possible orderings, we can define leading terms.

Let m be a non-zero element of P^r with $P = R[x_1, \ldots, x_n]$ such that $m = \sum_{i=1}^{s} c_i t_i e_{\gamma_i}$ where $c_1, \ldots, c_s \in R\backslash\{0\}$, $t_1, \ldots, t_s \in \mathbb{T}^n$, and $\gamma_1, \ldots, \gamma_s \in \{1, \ldots, r\}$. The ***leading term*** of m with respect to σ is defined as:

$$LT_\sigma(m) = t_1 e_{\gamma_1} \in \mathbb{T}^n\langle e_1, \ldots, e_r\rangle$$

Similarly, a ***leading term module*** of $M \subseteq P^r$ with respect to σ is defined as:

$$LT_\sigma(M) = \langle LT_\sigma(m) | m \in M\backslash\{0\}\rangle.$$

These definitions lead us into the following theorem in the following section.

1.1.2.3 What is a Gröbner Basis?

Theorem 1.1.1. ***(Macauley's Basis Theorem)*** *Let K be a field, let $P = K[x_1, \ldots, x_n]$ be a polynomial ring over K, let $M \subseteq P^r$ be a P-submodule,* *and let σ be a module term ordering on $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle$. We denote the set of all terms in $\mathbb{T}^n\langle e_1, \ldots, e_r\rangle \backslash LT_\sigma\{M\}$ by B. Then the residue classes of the elements of B form a basis of the K-vector space P^r/M.*

To see a proof of this theorem go to [24]. This theorem shows us that we can compute a K-basis of the quotient module P^r/M if you know the $LT_\sigma(M)$. However, the leading terms of a set of generators of M do not necessarily generate $LT_\sigma(M)$. This is one of the reasons a Gröbner Basis is so important and unique. Let G be a subset of P. The leading terms of G are $LT_\sigma(G) = \{LT_\sigma(f)|f \in G\}$.

$$LT_\sigma(G) = LT_\sigma(I) \text{ if and only if } G \text{ is a Gröbner Basis for the ideal } I$$

This works the same for modules, and is the main characterization of a Gröbner Basis. So if we have the Gröbner Basis for M, we can compute the basis of the K-vector space P^r/M. Guaranteeing that the leading terms of the Gröbner basis generate the leading terms of the module is just one aspect that makes these bases so unique.

Another interesting use relates to the division algorithm (see [24], theorem 1.6.4), which is an important tool of computational algebra used to generate the remainder of a polynomial (or vector) with respect to a set of generators of an ideal I (respectively a module M). This remainder depends on the set of generators chosen and even on the generators' order. It is possible for a polynomial (vector) that belongs to $I\ (M)$ to have a non-zero remainder with respect to a set of generators. If the generators are a Gröbner basis, it is guaranteed that the remainder is zero if the polynomial (vector) belongs to the ideal (module). This remainder calculated with respect to a Gröbner basis is called the ***normal form*** of the polynomial (vector), and it is a handy application to test membership. The central algorithm for the theory of Gröbner bases, Buchberger's Algorithm, can be found in [24] theorem 2.5.5, as well as many other sources.

Proposition 1.1.2. *(**Existence of a σ-Gröbner Basis**) Let M be a non-zero P-submodule of P^r.*

(i) Given $g_1, \ldots, g_s \in M \backslash \{0\}$ such that $LT_\sigma(M) = \langle LT_\sigma(g_1), \ldots, LT_\sigma(g_s) \rangle$, we have $M = \langle g_1, \ldots, g_s \rangle$, and the set $G = \{g_1, \ldots, g_s\}$ is a σ-Gröbner basis of M.

(ii) The module M has a σ-Gröbner basis $G = \{g_1, \ldots, g_s\} \subseteq M \backslash \{0\}$.

To see a proof of this proposition go to [24], Prop 2.4.3.

Now we know we can find a Gröbner basis for a module, and the truth is, there are many. But another crucial aspect of Gröbner basis theory, is that there exists a unique Gröbner basis that is monic, minimal, and interreduced called a ***reduced Gröbner basis***, and it only depends on the module and the ordering. To see the proof of the following theorem see Theorem 2.4.13.

Theorem 1.1.3. ***(Existence and Uniqueness of Reduced Gröbner***

Bases) *For every* P*-submodule* $M \subseteq P^r$*, there exists a unique reduced* σ*-Gröbner basis.*

1.1.2.4 Syzygies

Syzygies are very simple to define and easy to understand. They are also one of the most fundamental algebraic objects, therefore, their computation is one of the most important problems in Computational Algebra.

Let R be a ring, M an R-module, and $G = (g_1, \ldots, g_s)$ a tuple of elements of M.

(i) A ***syzygy*** of G is a tuple $(f_1, \ldots, f_s) \in R^s$ such that

$$f_1 g_1 + \cdots + f_s g_s = 0$$

(ii) The set of all syzygies of G forms an R-module that is called the ***first syzygy module*** of G, denoted $Syz(G)$ and is finitely generated.

Let $g_1, \ldots, g_s \in P^r \backslash \{0\}$, let $M = \langle g_1, \ldots, g_s \rangle \subseteq P^r$, and let the s-tuple $(g_1, \ldots, g_s) = G$. Consider the homomorphism $\lambda : P^s \longrightarrow M$ given by $\epsilon_j \to g_j$ for $j = 1, \ldots, s$, with P^s being a P-module with canonical basis $\{\epsilon_1, \ldots, \epsilon_s\}$. The kernel of this homomorphism is the syzygy module of G.

The construction of the syzygies of $f = (f_1, \ldots, f_s)$ can be done using the Gröbner basis of the module generated by the f_i as well as the division algorithm. After calculating the syzygies of f and denoting them by $g_1, \ldots, g_r$, we can continue to calculate the syzygies of $(g_1, \ldots, g_r)$, and so on. Each time, a new finitely generated module is obtained, and it is spanned by the syzygies of the generators of the previous module. Keeping in mind that the syzygies are the kernel of the map described by the generators of the module obtained at the previous step, we can create an exact complex

$$F : \cdots \to F_n \xrightarrow{\lambda_n} \cdots \xrightarrow{\lambda_2} F_1 \xrightarrow{\lambda_1} F_0 \xrightarrow{\lambda_0} M \to 0. \tag{1.1}$$

F_n is a free module R^{r_n}, with $r_1 = r$ and $r_0 = s$. The last map $\lambda_0 : F_0 \to M$ is given by $e_i \to f_i$, where e_i is the canonical basis element of $F_0 = R^s$. The syzygies that make up the images of the maps can be chosen to be minimal among the sets of generators, then the resolution is said to be ***minimal***. Hilbert proved in 1890 that this resolution is finite. This theorem, the Hilbert Syzygy Theorem, will be discussed in further detail later in this chapter.

1.2 Linear Constant Coefficient Operators

1.2.1 History

First we will look at Euler's contribution to algebraic analysis by considering his work on general solutions to linear ordinary differential equations with constant coefficients. Euler's theorem can be stated [17]:

Theorem 1.2.1. *Any smooth solution to the differential equation*

$$P(d/dx)f = 0$$

can be represented as

$$f(x) = \sum_k c_k(x) e^{z_k x}$$

where the $c_k(x)$ are the arbitrary polynomials in x of degree $m_k - 1$.

The theorem shows that to describe the kernel of the differential operator $P(d/dx)$, we need to understand the zeros of the polynomial P, which is an algebraic object.

More recent work on Algebraic Analysis began with work done by Sato, Kawai, Kashiwara, and their coworkers, see[22, 23]. Their work started in the early sixties with a deep theory of differential equations done fully with an algebraic approach.

Let's back up a little and discuss algebraic geometry and its beginnings in the late eighteen hundreds and early nineteen hundreds. Hilbert began this work with many fundamental theorems, see [11], such as The Hilbert Syzygy Theorem we have already mentioned previously. Other important theorems are the Basis theorem, listed below, and the Nullstellensatz theorem. These works cleared the way for the study of algebraic objects, such as ideals and modules in rings of polynomials, which could enhance or replace the geometric objects. One advantage of this point of view, is the invariance that is found, since all the polynomials in the ideal give the same geometrical object.

A ring (module) is called ***Noetherien*** if every ascending chain of ideals (submodules) becomes eventually stationary. A proof for the following theorem is given in [24].

Theorem 1.2.2. ***(Hilbert's Basis Theorem)*** *Every finitely generated module over a finitely generated K-algebra is Noetherian. In particular, $P =$*

$K[x_1, \dots, x_n]$ *is a Noetherian ring.*

One of the important focuses in analysis is the study of the kernel of a differential operator. However, different operators may lead to the same kernel; this is why an invariant description became of interest. Malgrange, Ehrenpreis, Palamodov, and Hörmander began the work in algebraic analysis by replacing systems of differential equations with suitable algebraic objects [**?**, 15, 25]. Let's briefly describe this approach.

1.2.2 Definitions and Notations

Let R denote the ring of complex polynomials in N variables, $R = \mathbb{C}[z_1, \dots, z_n]$. We are going to think of R as the ring of symbols of linear constant coefficient partial differential operators. We will accomplish this by replacing $z = (z_1, \dots, z_n)$ with $D = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$. The Fourier transform allows us to make this substitution of derivatives with polynomials and vice-versa in spaces of functions satisfying suitable conditions. Among these spaces is the ***Localizable Analytically Uniform spaces*** (LAU spaces). Below is a list of linear topological spaces which admit a LAU structure [11]:

(i) The space of polynomials in n variables with complex coefficients.

(ii) The space $\mathcal{O}(\Omega)$ of the holomorphic functions on an open convex $\Omega \subseteq \mathbb{C}^n$.

(ii) The space of entire functions of exponential type.

(iv) The space $\mathcal{C}^\infty(\Omega)$, where Ω is an open convex set in $\mathbb{R}^n$.

(v) The space of distributions $\mathcal{D}'(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$.

Let's consider an $r_1 \times r_2$ matrix of linear constant coefficient partial differential operators, $P(D)$ with $P = [P_{ij}]$ of elements in R. $P(D)$ defines a map

$$P(D) : S^{r_0} \longrightarrow S^{r_1},$$

with S being a sutable space described above. The kernel of this map is of great interest to analysts.

In geometry, they had already begun working in sheaves of generalized functions rather than simply in spaces, so the algebraic analysts began to do the same. It turns out that partial differential operators act on sheaves as sheaf-homomorphisms. Looking back at the previous map, given S is a sheaf, we see that $P(D)$ is a sheaf homomorphism whose kernel is also a sheaf denoted by S^P. Now let's consider

$$P(D) \cdot f = 0$$

where f is chosen from a suitable sheaf of generalized equations. Some examples of these sheaves are infinitely differentiable functions $\mathcal{E} = \mathcal{C}^{\infty}(\Omega)$, where Ω is an open convex set in $\mathbb{R}^n$, holomorphic functions $\mathcal{O}(\Omega)$, distributions $\mathcal{D}$, or hyperfunctions $\mathcal{B}$ [11]. The matrix P is called a ***symbol***. The algebraic object which has the most central role in algebraic analysis is the module

$$M := Coker(P^t) = R^{r_0}/P^t R^{r_1} = R^{r_0}/\langle P^t \rangle.$$

P^t is the transpose of the matrix P. M is a finitely generated R-module that is the quotient of the free module R^{r_0} with the submodule generated by the rows of matrix P. The module M can be described using syzygies, free resolutions, and cohomology modules. Let's consider the following theorem.

Theorem 1.2.3. ***(Hilbert Syzygy Theorem)*** *Every finitely generated R-module has a finite exact resolution of length $m \leq n$, in which all modules are finitely generated free modules as follows:*

$$0 \longrightarrow R^{r_m} \xrightarrow{P^t_{m-1}} R^{r_{m-1}} \longrightarrow \cdots \xrightarrow{P^t_1} R^{r_1} \xrightarrow{P^t} R^{r_0} \longrightarrow M \longrightarrow 0. \tag{1.2}$$

The syzygies of M are the maps in this resolution. By using the Hom functor, we can find the dual of this resolution. This amounts to taking the duals of the spaces, and using the transpose of the matrices, while reversing the arrows, giving:

$$0 \longrightarrow R^{r_0} \xrightarrow{P} R^{r_1} \xrightarrow{P_1} \ldots \longrightarrow R^{r_{m-1}} \xrightarrow{P_{m-1}} R^{r_m} \longrightarrow 0. \tag{1.3}$$

The following paragraphs explain what information we can gather from this.

1.2.2.1 Syzygies and Compatibility Conditions

Consider the matrix P_1 in complex (1.3). Looking at this algebraically, we know that P_1^t represents the syzygy module of $\langle P^t \rangle$. Analytically speaking, what does this tell us? Let's consider a smaller case. Suppose that P is a column matrix of three polynomials

$$P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

which represents the homogeneous system of equations $p_1(D)f = p_2(D)f = p_3(D)f = 0$. Let f, g_1, g_2, g_3 be differentiable functions and consider the inhomogeneous system

$$\begin{cases} p_1(D)f = g_1 \\ p_2(D)f = g_2 \\ p_3(D)f = g_3 \end{cases}$$

The g_i must satisfy a set of ***compatibility conditions***, which are a set of differential equations as follows

$$q_1(D)g_1 + q_2(D)g_2 + q_3(D)g_3 = 0,$$

in order for the system to admit a solution. These correspond to the following relations satisfied by the p_i

$$q_1p_1 + q_2p_2 + q_3p_3 = 0$$

This result goes back to the fact that the operators $p_1(D), p_2(D), p_3(D) \in \mathbb{C}[\partial x_1, \ldots, \partial x_n]$ have the same algebraic properties of their symbols, given that the commutative property

$$\partial x_i \partial x_j = \partial x_j \partial x_i$$

holds for all $i, j \in \{1, \dots, n\}$. Generally, when considering the inhomogeneous system

$$P(D)f = g, \tag{1.4}$$

the compatibility conditions are given by the syzygies of the matrix P^t, allowing us to express this condition with the new homogeneous system

$$P_1(D)g = 0. \tag{1.5}$$

The following theorem gives this result, and the proof can be found in [14].

Theorem 1.2.4. *Let S be one of the sheaves of generalized functions $\mathcal{E}, \mathcal{O}, \mathcal{B}$, or $\mathcal{D}$. The system (1.4) has a solution f on $S(U)^{r_0}$ on a convex open set U, if and only if $g \in S(U)^{r_1}$ satisfies the compatibility condition $P_1(D)g = 0$.*

Now let's consider an example given in [12] using the Cauchy-Riemann operator

$$\partial \overline{z}_j = \partial x_j + i\partial y_j, \text{ for } j = 1, \dots, n.$$

Let $z_j = x_j + iy_j$ where $i^2 = -1$ and x_j, y_j are real variables. On an open convex set of $\mathbb{C}^n$, the solutions of the system $\partial \overline{z}_j f(z_1, z_2) = 0$ are exactly the holomorphic functions on that open set. To represent this system using a matrix P, we must find the corresponding equations for the real valued functions f_1, f_2, where $f = f_1 + if_2$. This system is the ***Cauchy-Riemann system*** which characterizes the holomorphicity of a function. In the case of $n = 2$, the following system arises

$$
\begin{cases}
\partial x_1 f_1 - \partial y_1 f_2 = 0 \\
\partial y_1 f_1 + \partial x_1 f_2 = 0 \\
\partial x_2 f_1 - \partial y_2 f_2 = 0 \\
\partial y_2 f_1 + \partial x_2 f_2 = 0
\end{cases} \quad (1.6)
$$

Note the omission of the factor i since we are working with homogeneous polynomials. The symbol associated to this system is the matrix

$$
p = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \\ x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} = \begin{bmatrix} Q_1 \\ \\ Q_2 \end{bmatrix},
$$

where

$$
Q_i = \begin{pmatrix} x_i & -y_i \\ y_i & x_i \end{pmatrix}.
$$

Rather than looking at a representation of the operator acting on functions $f : \mathbb{C}^2 \longrightarrow \mathbb{C}$ we are switching to the corresponding operator acting on functions $f : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$. The rows of P are the four generators for the module $M = \langle P^t \rangle$. In [12] they used CoCoA to find its first sysygies, using the command SyzOfGens:

```
Use R::=Q[x[1..2],y[1..2]];

P:=Mat[

[x[1],-y[1]],

[y[1],x[1]],

[x[2],-y[2]],

[y[2],x[2]]];
```

```
M:=Module(P); -- in CoCoA a module associated to a matrix
                 -- is generated by its rows
S:=SyzOfGens(M,1);
Mat[
[x[2], -y[2], -x[1], y[1]],
[y[2], x[2], -y[1], -x[1]]
]
-----------------------------------------
```

The matrix obtained consists of the two blocks $[Q_2, -Q_1]$, which gives the following compatibility conditions for the system

$$\begin{cases} \partial\overline{z}_1 f = g_1 \\ \partial\overline{z}_2 f = g_2 \end{cases}$$

which can be expressed in terms of the Cauchy-Riemann operators giving the well known result $\partial\overline{z}_2 g_1 = -\partial\overline{z}_1$. All of this was obtained through a purely algebraic result.

1.2.2.2 Free Resolutions

Let's start by considering an example given in [11]. Let P be a vector of four polynomials with no common factors. With R as described previously, let $M = R/I(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4)$, where I is the ideal generated in R by the four polynomials $\mathcal{P}_1, \ldots, \mathcal{P}_4$. We are going to build Hilbert's resolution, and the resolution begins as follows:

$$R^4 \xrightarrow{P^t} R \longrightarrow M \longrightarrow 0.$$

The first map is the map from R^4 to R by multiplying by P^t. We continue to build onto this Hilbert resolution by describing the kernel of this map in R^4. Consider the vector $(-\mathcal{P}_2, \mathcal{P}_1, 0, 0)$. This vector is obviously in the kernel of this map, since multiplying by P^t gives zero. Similarly, we can easily construct $6 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ elements in the kernel for any choice of a pair $(\mathcal{P}_i, \mathcal{P}_j)$. Since the polynomials in P have no common factors, these six vectors generate the kernel. So we have extended our resolution another step:

$$R^6 \xrightarrow{P_1^t} R^4 \xrightarrow{P^t} R \longrightarrow M \longrightarrow 0.$$

Now the first map is given by the 6×4 matrix

$$P_1 = \begin{bmatrix} -\mathcal{P}_2 & \mathcal{P}_1 & 0 & 0 \\ -\mathcal{P}_3 & 0 & \mathcal{P}_1 & 0 \\ -\mathcal{P}_4 & 0 & 0 & \mathcal{P}_1 \\ 0 & -\mathcal{P}_3 & \mathcal{P}_2 & 0 \\ 0 & -\mathcal{P}_4 & 0 & \mathcal{P}_2 \\ 0 & 0 & -\mathcal{P}_4 & \mathcal{P}_3 \end{bmatrix}.$$

We now need to find the kernel of this operator that lies in R^6. By using a generic vector in R^6 and setting its product with P_1 to be zero, we find the kernel to be the 4×6 matrix

$$P_2 = \begin{bmatrix} \mathcal{P}_3 & -\mathcal{P}_2 & 0 & \mathcal{P}_1 & 0 & 0 \\ \mathcal{P}_4 & 0 & -\mathcal{P}_2 & 0 & \mathcal{P}_1 & 0 \\ 0 & \mathcal{P}_4 & -\mathcal{P}_3 & 0 & 0 & \mathcal{P}_1 \\ 0 & 0 & 0 & \mathcal{P}_4 & -\mathcal{P}_3 & \mathcal{P}_2 \end{bmatrix}.$$

This gives us the following:

$$R^4 \xrightarrow{P_2^t} R^6 \xrightarrow{P_1^t} R^4 \xrightarrow{P^t} R \longrightarrow M \longrightarrow 0.$$

Using the same method as above, we can find the kernel of the new first map to be given by the 1×4 matrix

$$P_3 = [-\mathcal{P}_4, \mathcal{P}_3, -\mathcal{P}_2, \mathcal{P}_1].$$

This operator P_3 has trivial kernel, and so we can finish our Hilbert resolution, which happens to be a well known complex, namely ***Koszul complex***, given by:

$$0 \longrightarrow R \xrightarrow{P_3^t} R^4 \xrightarrow{P_2^t} R^6 \xrightarrow{P_1^t} R^4 \xrightarrow{P^t} R \longrightarrow M \longrightarrow 0.$$

This example was given so that the reader can see how these complexes are built, noting the role of the kernels, which is directly related to the syzygies. Secifically, P_n^t represents the syzygy module of $\langle P_{n-1}^t \rangle$. So looking at the inhomogeneous system

$$P_i(D)g_i = g_{i+1}, \qquad i \in \mathbb{N},$$

we obtain a free resolution for M by iterating the calculation of syzygies, also each matrix P_i in the complex (1.3) provides the compatibility conditions for this inhomogeneous system with the g_i chosen from the same sheaf as $f = g_0$. This shows the importance of this free resolution of M. Namely, it contains all the compatibility conditions of the original system of equations. So, the free resolution is called a ***measure*** of the compatibility of the system $P(D)f = g$.

Returning to our example in the previous section involving the Cauchy-Riemann operator, [12] computed the resolution in CoCoA

```
Res(R^2/M); 0--> R^2(-2) --> R^4(-1) -->R^2
-----------------------------------
```

Giving the free resolution of the quotient by

$$0 \longrightarrow R^2(-2) \xrightarrow{P_1^t} R^4(-1) \xrightarrow{P^t} R^2 \longrightarrow M \longrightarrow 0.$$

with P and P_1 defined as previously. The exponents, called ***Betti numbers***, of the ring R gives the number of syzygies at each step. The number in parenthesis helps to indicate the degree of the syzygies, namely by taking the difference between the two parenthesis in each map. When looking at this resolution, we see that the only compatibility for this system is given by the first syzygy map. Also, notice this map is linear. It turns out that there is basically no difference between the nature of the complex for the Cauchy-Riemann system in two variables than for $n > 2$ [11].

The last theorem I would like to mention in this introductory chapter is Hartogs' phenomenon [21]. It will be referred to in other theorems in later chapters.

Theorem 1.2.5. ***(Hartogs' phenomenon for Holomorphic functions)*** *Let Ω be an open set of $\mathbb{C}^n$, $n > 1$ and let $K \subseteq \Omega$ be a compact set such that $\Omega \backslash K$ is connected. Let f be a holomorphic function on $\Omega \backslash K$ with values in $\mathbb{C}$. Then it is possible to uniquely extend f to a function $\tilde{f} : \Omega \longrightarrow \mathbb{C}$ such that $\tilde{f}$ is holomorphic on Ω.*

Hartogs' proof used purely analytical arguments. After work done by Palamodov and Ehrenpreis, an algebraic proof was done in the general setting of linear constant coefficient operators. This theorem is stated below, see [15].

Theorem 1.2.6. *Let R be a polynomial ring, let $P \in Mat(R)$ be the symbol of a linear constant coefficient partial differential operator. Denote by M the coker of the map defined by P and consider a differentiable solution f to the system $P(D)f = 0$. Suppose that f is defined on an open set $\Omega \backslash K$ where K is a compact set in Ω. Then it is possible to extend f to a function $\tilde{f}$ such that $P(D)\tilde{f} = 0$ on Ω if and only if $Ext^1_R(M,R)=0$.*

Now that we have covered some basics from Gröbner basis theory, linear constant coefficient operators, as well as, computational and analytical algebra, we would like to move onto some more recent advances in the use of Gröbner bases in systems of partial differential equations. Specifically, we will consider work done with the Cauchy-Feuter operator, looking at both regular and biregular functions. As Sir Winston Churchill said,

> This is not the end or the beginning of the end, but it is the end of the beginning.

Chapter 2

Cauchy-Feuter Operator

2.1 Introduction to Quaternions and Cauchy-Feuter Systems

It is well known that quaternions were introduced by the Irish mathematician William Rowan Hamilton on October 16th, 1843. Hamilton had worked on the theory of couplets, which at that time was considered a new algebraic representation of the Complex numbers. The couplets were pairs of real numbers, (a, b), with addition and multiplication defined as $(a, b)+(c, d) = (a+c, b+d)$ and $(a, b) \times (c, d) = (ac-bd, ad+bc)$. He idea was to extend the complex numbers to a new algebraic representation with each element having one real part and two imaginary parts. His goal being to use triplets to represent rotations in three-dimensional space, as the complex numbers could be used to describe them in two dimensions. After 10 years of trying to find a Theory of Triplets, he realized that he actually needed one real part together with three imaginary, the Quaternions.

$$\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k \mid x_t \in \mathbb{R},\ i^2 = j^2 = k^2 = ijk = -1\}.$$

2.1.1 Holomorphicity

Let's review holomorphicity in complex analysis and see how this can be extended to the case of functions defined on the space of quaternions. We have been consistently referring to the kernel of our operators, and the kernel of the Cauchy-Riemann operator on differentiable functions is the space of holomorphic functions, which are defined as follows:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

for $f : \Omega \in \mathbb{C} \to \mathbb{C}$ such that the limit exists for all $z \in \Omega$. Now, how do we extend to the Quaternionic case? Looking in [26], we can find a discussion of this. It seems natural to extend the definition as follows:

$$\lim_{h \to 0} \frac{f(q + h) - f(q)}{h},$$

such that this limit exists for every q in the domain. Looking at it this way presents a problem, since the only quaternionic functions for which this limit exists are linear functions of the form

$$f(q) = aq + b, \;\; a, b \in \mathbb{H}.$$

The Swiss mathematician Fueter found a way around this problem by generalizing the Cauchy Riemann operator, and considering the nullsolutions of a first order differential operator for differential functions on $\mathbb{H} \simeq \mathbb{R}^4$, [18].

Keeping in mind that the space we are working on is not commutative, the (left) Cauchy-Fueter operator is defined as:

$$\partial\overline{q} := \partial x_0 + i\partial x_1 + j\partial x_2 + k\partial x_3. \tag{2.1}$$

Now that we have an idea of $\mathbb{H}$-holomorphicity, we can define a regular function. A function $f : \Omega \in \mathbb{H} \longrightarrow \mathbb{H}$ is ***regular*** if and only if it satisfies

$$\frac{\partial f}{\partial \overline{q}} = 0, \ \forall \ q \in \Omega$$

This can also be said as $f = f_0 + if_1 + jf_2 + kf_3$ is regular if its real components satisfy the 4×4 system of linear constant coefficient differential equations seen below.

$$\begin{bmatrix} \partial_{x_0} & -\partial_{x_1} & -\partial_{x_2} & -\partial_{x_3} \\ \partial_{x_1} & \partial_{x_0} & -\partial_{x_3} & \partial_{x_2} \\ \partial_{x_2} & \partial_{x_3} & \partial_{x_0} & -\partial_{x_1} \\ \partial_{x_3} & -\partial_{x_2} & \partial_{x_1} & \partial_{x_0} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0. \tag{2.2}$$

This was for left regularity. Right regularity would have a similar definition and an identical theory. For the sake of simplicity, we will write (2.2) as a matrix multiplication

$$U(D)f = 0.$$

Also, when we are working in several quaternionic variables $q_i = x_{i0} + ix_{i1} + jx_{i2} + kx_{i3}$, we will use $U_i(D)f = 0$, which makes the system (2.2) become

$$\begin{bmatrix} U_1(D) \\ \vdots \\ U_m(D) \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0, \qquad \text{where } U_i(D) = \begin{bmatrix} \partial_{x_{i0}} & -\partial_{x_{i1}} & -\partial_{x_{i2}} & -\partial_{x_{i3}} \\ \partial_{x_{i1}} & \partial_{x_{i0}} & -\partial_{x_{i3}} & \partial_{x_{i2}} \\ \partial_{x_{i2}} & \partial_{x_{i3}} & \partial_{x_{i0}} & -\partial_{x_{i1}} \\ \partial_{x_{i3}} & -\partial_{x_{i2}} & \partial_{x_{i1}} & \partial_{x_{i0}} \end{bmatrix}.$$

Again, we want to consider $M = coker(P)$, with P being the $4n \times 4$ matrix made up of the U_i. Several authors have studied this, [1, 3, 4]. CocoA can be used for calculations with $n \leq 4$. Let's look at the case of $n = 1$. `Coala.FeuterMat` creates the symbol matrix for Cauchy-Feuter operators.

```
Use R::=Q[x[0..3]];
P:=Coala.FueterMat(Indets());
M:=Module(P);
Res(R^4/M);
0 --> R^4(-1) --> R^4
-----------------------------------------------
```

This gives us the resolution

$$0 \longrightarrow R^4 \xrightarrow{P^t} R^4 \longrightarrow M \longrightarrow 0.$$

We can see that there are no compatibility conditions for the Cauchy-Fueter system of one variable, since the only map is P^t.

Checking the $n = 2$ case of two quaternionic variables, [11]:

```
Use R::=Q[x[0..3]y[0..3]];
P:=Coala.FueterMat(Indets());
M:=Module(P);
Res(R^4/M);
0 --> R^4(-4) --> R^8(-3) --> R^8(-1) --> R^4
-----------------------------------------------
```

we are given

$$0 \longrightarrow R^4(-4) \xrightarrow{P_2^t} R^8(-3) \xrightarrow{P_1^t} R^8(-1) \xrightarrow{P^t} R^4 \longrightarrow M \longrightarrow 0.$$

As mentioned earlier, the exponents tell us the number of syzygies, and the numbers in parenthesis help us to know what degree the map is. We can see that we have eight syzygies from our first map, and they are of degree two, while the other syzygies are linear. We can define these maps by keeping in mind P_n^t is found from the kernel of P_{n-1}^t. Let U_1, U_2 be defined as earlier, V_1, V_2 be the symbols for the Cauchy-Fueter operators' conjugate

$$\partial q_i := \partial x_{0i} - i\partial x_{1i} - j\partial x_{2i} - k\partial x_{3i},$$

and let $D_i := U_i V_i$ be the symbol of the Laplacian operator

$$\Delta_i = \partial \overline{q}_i \partial q_i = \partial q_i \partial_i = \partial x_0^2 + \partial x_1^2 + \partial x_2^2 + \partial x_3^2.$$

Then our complex for $n = 2$ is

$$0 \longrightarrow R^4(-4) \overset{\left[\begin{array}{cc} V_1 & \mathrm{V}_2 \end{array}\right]}{\longrightarrow} R^8(-3) \overset{\left[\begin{array}{cc} -D_2 & \mathrm{U}_1 V_2 \\ U_2 V_1 & \text{-D}_1 \end{array}\right]}{\longrightarrow}$$

$$\longrightarrow R^8(-1) \overset{\left[\begin{array}{c} U_1 \\ U_2 \end{array}\right]}{\longrightarrow} R^4 \longrightarrow M \longrightarrow 0.$$

The Cauchy-Fueter operator and its conjugate commute with the Laplacian operator giving the compatibility conditions.

Doing the same for $n = 3, 4$ we get the following resolutions [11]:

for $n = 3$

$$0 \longrightarrow R^8(-6) \longrightarrow R^{36}(-5) \longrightarrow R^{60}(-4) \longrightarrow$$

$$\longrightarrow R^{40}(-3) \longrightarrow R^{12}(-1) \longrightarrow R^4 \longrightarrow M \longrightarrow 0$$

for $n = 4$

$$0 \longrightarrow R^{12}(-8) \longrightarrow R^{80}(-7) \longrightarrow R^{224}(-6) \longrightarrow R^{336}(-5) \longrightarrow$$

$$\longrightarrow R^{280}(-4) \longrightarrow R^{112}(-3) \longrightarrow R^{16}(-1) \longrightarrow R^4 \longrightarrow M \longrightarrow 0$$

Notice that the first syzygy maps are quadratic, while the rest are linear. Also, the length of the resolutions are $2n - 1$. This led to a theorem that appears in several papers from the same authors; see [5, 3, 4] for the proof.

Theorem 2.1.1. *Let M_n be the module associated to the Cauchy Fueter system in $n > 1$ variables. Then its resolution is*

$$0 \longrightarrow R^{\beta_{2n-1}}(-2n) \longrightarrow R^{\beta_{2n-2}}(-2n+1) \longrightarrow \dots$$

$$\dots \longrightarrow R^{\beta_3}(-4) \longrightarrow R^{\beta_2}(-3) \longrightarrow R^{4n}(-1) \longrightarrow R^4 \longrightarrow M_n \longrightarrow 0.$$

In particular

I) The resolution of M_n has length $2n-1$;

II) all the maps in the resolutions are linear except the first one.

Moreover:

III) the Betti number β_r at the step r is given by

$$\beta_r = 4 \begin{pmatrix} 2n-1 \\ r \end{pmatrix} \frac{n(r-1)}{r+1};$$

Also in [3, 4], the authors go on to prove the following theorem concerning the compatibility conditions.

Theorem 2.1.2. *The compatibility conditions of the system*

$$\begin{cases} \partial \overline{q}_1 & = g_1 \\ \dots & \dots \\ \partial \overline{q}_n & = g_n \end{cases}$$

with $n > 2$, are the following:

I) for each of the $2 \begin{pmatrix} n \\ 2 \end{pmatrix}$ ordered pairs of induces $r, s, 1 \leq r, s \leq n$

$$\partial \overline{q}_r \partial q_s g_s - \partial \overline{q}_s \partial q_s g_r = 0$$

II) for each of the $\begin{pmatrix} n \\ 3 \end{pmatrix}$ *triples of induces* $h, r, s, 1 \leq h, r, s \leq n$

$$\partial q_h \partial \overline{q}_r g_s + \partial q_r \partial \overline{q}_h g_s - \partial \overline{q}_s \partial q_r g_h - \partial \overline{q}_s \partial q_h g_r = 0$$

and

$$\partial q_r \partial \overline{q}_s g_h + \partial q_s \partial \overline{q}_r g_h - \partial \overline{q}_h \partial q_r g_s - \partial \overline{q}_h \partial q_s g_r = 0,$$

III) for each of the $\begin{pmatrix} n \\ 3 \end{pmatrix}$ *triples of induces* $h, r, s, 1 \leq h, r, s \leq n$

$$(D_{q_r} \partial \overline{q}_s - D_{q_s} \partial \overline{q}_r) g_h + (D_{q_s} \partial \overline{q}_h - D_{q_h} \partial \overline{q}_s) g_r + (D_{q_h} \partial \overline{q}_r - D_{q_r} \partial \overline{q}_h) g_s = 0,$$

$$(D'_{q_r} \partial \overline{q}_s - D'_{q_s} \partial \overline{q}_r) g_h + (D'_{q_r} \partial \overline{q}_s - D'_{q_s} \partial \overline{q}_r) g_h + (D'_{q_r} \partial \overline{q}_s - D'_{q_s} \partial \overline{q}_r) g_h = 0,$$

where

$$D_{q_i} = -j \partial x_{i2} + k \partial x_{i3}, \qquad D'_{q_i} = -i \partial x_{i1} + k \partial x_{i3}.$$

The syzygies of type I and II are written in terms of ∂q and $\partial \overline{q}$ and are called ***radial syzygies***. However, the ***exceptional*** syzygies of type III are written in terms of D and D' with two real variables instead of four, referred to sometimes as the "Cauchy-Riemann like" operators.

2.2 Surjectivity and Applications to Regular Functions

It has been shown that linear constant coefficients differential operators act surjectively on a large class of functional spaces. With P being a polynomial in $\mathbb{C}[z_1, \ldots, z_n]$, D being the differential operator $(-i\partial_{x_1}, \ldots, -i\partial_{x_n})$, and $\mathcal{O}(\Omega)$ is the space of holomorphic functions on a convex open set $\Omega \in \mathbb{C}^n$, it can be shown that $P(D)$ is surjective from $\mathcal{O}(\Omega)$ to itself. Similarly, the authors of [10] show, using techniques from algebraic analysis of constant coefficients differential operators, a surjectivity result on the space of regular functions.

Surjectivity of a differential operator on the space of holomorphic functions results from the triviality of its syzygies. However, triviality doesn't hold for operators on regular functions. The authors had to compute syzygies whose explicit form could lead to surjectivity, despite not being trivial.

Let $P_1(D)f = \cdots = P_k(D)f = 0$ be a system of differential equations, we want to find conditions on the square matrices $P_1, \ldots, P_k$ such that the compatibility conditions on the system, resulting from the first syzygies of the rows of

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_k \end{bmatrix},$$

can be written in terms of the P_i. Let's give some notations and definitions for the section. Let $Mat_{n,m}(R)$ denote the $n \times m$ matrices in the ring R. If $n = m$, it will be written $Mat_n(R)$, and if R is an integral domain, $Frac(R)$

will denote its field of fractions.

Let $P_1, \ldots, P_k$ be matrices in $\mathcal{R} = Mat_n(R)$. The the k-uple $(P_1, \ldots, P_k)$ is a ***left regular sequence*** if:

I) P_1 is a left regular element of $\mathcal{R}$, meaning the only $B \in \mathcal{R}$ such that $BP_1 = 0$ is $B = 0$;

II) P_i is not a zero divisor in $\mathcal{R}/(P_1, \ldots, P_{i-1})\mathcal{R}$ for all $i = 2, \ldots, k$ where $(\mathrm{P}_1, \ldots, P_{i-1})\mathcal{R}$ is the left ideal in $\mathcal{R}$ generated by $P_1, \ldots, P_{i-1}$.

Considering the case of $k = 1$, giving the square system $P_1(D)f = 0$ along with the condition that P_1 is a non-zerodivisor in $\mathcal{R}$, the following propositions with their respective proofs arises [10].

Proposition 2.2.1. *Let R be an integral domain. The following are equivalent facts for a square matrix $P \in \mathcal{R}$:*

I) $Det(P) \neq 0$;

II) P is a left regular element of $\mathcal{R}$;

III) $Syz(P) = \langle 0 \rangle \subset R^n$ where $Syz(P)$ means the first syzygies for the rows of the matrix P.

Proof.

$I) \Rightarrow II)$: Let B be a nonzero square matrix such that $BP = 0$. Any row of B is a solution to the system $(x_1, \ldots, x_n)P = (0, \ldots, 0)$. Since the $Det(P) \neq 0$, P is invertible, and therefore, the system only has trivial solutions, giving $B = 0$.

$II) \Rightarrow III)$: Let $x = (x_1, \ldots, x_n) \in Syz(P)$ be a nonzero row, then the matrix X with rows all equal to x is such that $XP = 0$. This is a contradiction since P is left regular.

$III) \Rightarrow I)$: Assume $Det(P) = 0$, then the system $xP = 0$ would have a non trivial solution $(x_1, \ldots, x_n) \in Frac(R)^n$. Let d be the product of all the nonzero denominators of the solution's elements. This gives $d(x_1, \ldots, x_n) = (s_1, \ldots, s_n)$ is a non trivial syzygy in R^n, a contradiction. □

Proposition 2.2.2. *Let R be an integral domain and set $\mathcal{R} = Mat_n(R)$. Let $P, Q \in \mathcal{R}$ be commuting square matrices. Then the following conditions are equivalent:*

I) P and Q form a left regular sequence in |mathcalR,

II) 1. Q is invertible in $Mat_n(Frac(R))$,

2. for every $A \in \mathcal{R}$ such that $AQ^{-1}P \in \mathcal{R}$, we have $AQ^{-1} \in \mathcal{R}$.

Proof.

$II \Rightarrow I$ Q being invertible in $Mat_n(Frac(R))$ means that the $Det(Q) \neq 0$, but this is the same as being a regular element according to the previous proposition (2.2.1.). Let's consider the relation $X_1P + X_2Q = 0$. Since P and Q commute, and therefore, so do Q^{-1} and P, we get $X_1Q^{-1}P = -X_2$. Since $X_1, X_2 \in \mathcal{R}$, this gives $X_1Q^{-1}P \in \mathcal{R}$, and then we have $X_1Q^{-1} \in R$. Then $X_1 = AQ$ for some A and $-AP = X_2$, which gives Q and$P$ form a left regular sequence.

$I \Rightarrow II$ If (P, Q) is left regular, then for all A such that $AP = BQ$, we have $A = CQ$. By inverting Q, II part 2 is true in the $Mat_n(Frac(R))$. □

We are able to describe the module of the first syzygies if the matrices of an overdetermined system form a regular sequence. The following theorem and proof are from [10].

Theorem 2.2.3. *Let $P_1, \ldots, P_k \in \mathcal{R}$ with $k > 1$ be matrices such that:*

1) $P_i P_j = P_j P_i$ for all $i, j = 1 \ldots k$,

2) $(P_1, \ldots, P_k)$ is a left regular sequence,

and let M be the R-submodule of the R^n generated by the rows of $P_1, \ldots, P_k$. Then the first syzygy module $Syz(M)$ is generated by the rows of the block matrix

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{k-1} \end{bmatrix}, \; B_i = \begin{bmatrix} 0 & \ldots & 0 & -P_{i+1} & P_i & 0 & \ldots & 0 \\ 0 & \ldots & 0 & -P_{i+2} & 0 & P_i & \ldots & 0 \\ & \ldots & & \ldots & & & \ldots & \\ 0 & \ldots & 0 & -P_k & 0 & 0 & \ldots & P_i \end{bmatrix}. \qquad (2.3)$$

Proof. Let P be the block matrix with the P_i's in a column, then we can see that $B \cdot P = 0$ since the P_i's commute. So the rows of matrix B are syzygies for the rows of the matrix P, i.e. they belong to the set of generators of M. Now consider the nonzero row vector

$$x = (x_{11}, \ldots, x_{1n}, \ldots, x_{k1}, \ldots, x_{kn}) \in Syz(M).$$

Then the product $x \cdot P$ is the zero vector of R^{kn}. We will show that x belongs to the R-module generated by the rows of B. Let $X \in Mat_{n,kn}(R)$ with n rows all equal to x. We see that $X \cdot P$ is the zero $n \times n$ matrix. Then we can think of X as a block matrix of k square matrices:

$$X = [X_1 \; X_2 \; \dots \; X_k]$$

and so the product $X \cdot P = 0$ means

$$X_1 P_1 + \cdots + X_k P_k = 0. \tag{2.4}$$

We now proceed by induction on k.

Case $k = 2$

Equation (2.4) for $k = 2$ is simply $X_1P_1 + X_2P_2 = 0$, which iimplies $X_2P_2 = -X_1P_1$, which is possible only if the matrix X_2 is a left multiple of P_1, since (P_1, P_2) is a regular sequence. Then there exists a square matrix $F \in \mathfrak{R}$ such that $X_2 = FP_1$. Then we get

$$0 = X_1P_1 + FP_1P_2 = X_1P_1 + FP_2P_1 = (X_1 + FP_2)P_1 \tag{2.5}$$

Since P_1 is a left regular element, this implies that $X_1 = -FP_2$, so the matrix X is of the form

$$\begin{bmatrix} X_1 & X_2 \end{bmatrix} = F \cdot \begin{bmatrix} -P_2 & P_1 \end{bmatrix} = F \cdot \tilde{B}_1 \tag{2.6}$$

where $\tilde{B}_1$ is the only block in the matrix B that we have in this case. Equation (2.6) means that a row of X is a combination of the rows of $\tilde{B}_1$ whose coefficients are a row of F.

Case $k > 2$

For this case we can rewrite (2.4) as

$$X_k P_k = -X_1 P_1 - \cdots - X_{k-1} P_{k-1}. \tag{2.7}$$

Again, since these form a regular sequence, particularly since P_k is regular in the quotient $\mathcal{R}/(P_1, \ldots, P_{k-1})\mathcal{R}$, we know that X_k has to be in the left ideal $(P_1, \ldots, P_{k-1})\mathcal{R}$:

$$X_k = F_1 P_1 + \cdots + F_{k-1} P_{k-1}. \tag{2.8}$$

plugging this into (2.7), we have

$$F_1 P_1 P_k + \cdots + F_{k-1} P_{K-1} P_K = -X_1 P_1 - \cdots - X_{k-1} P_{k-1} \tag{2.9}$$

that can be rewritten using the commutativity of the matrices as

$$(F_1 P_k + X_1) P_1 + \cdots + (F_{k-1} P_k + X_{k-1}) P_{k-1} = 0. \tag{2.10}$$

This means that the rows of $[(F_1 P_k + X_1) \ldots (F_{k-1} P_k + X_{k-1})]$ are syzygies for $[P_1 \ldots P_{k-1}]$. Since the first $k-1$ matrices commute and are still a left regular sequence, we can apply the inductive hypothesis and get their syzygies as the combination of the rows of the matrix

$$B' = \begin{bmatrix} B'_1 \\ B'_2 \\ \vdots \\ B'_{k-2} \end{bmatrix}, \; B'_i = \begin{bmatrix} 0 & \dots & 0 & -P_{i+1} & P_i & 0 & \dots & 0 \\ 0 & \dots & 0 & -P_{i+2} & 0 & P_i & \dots & 0 \\ & \dots & & \dots & & & \dots & \\ 0 & \dots & 0 & -P_{k-1} & 0 & 0 & \dots & P_i \end{bmatrix}$$

where the blocks B'_i have the same rows and columns of the blocks B_i except for the last n rows and the last n columns. Every matrix of the type $[Y_1 \dots Y_{k-1}]$ whose rows are syzygies for the $P_1, \dots, P_{k-1}$ is obtained by a suitable combination of the rows of B'. There exists a set of square matrices

$$\{F_{ji} \in \mathcal{R} \mid i = 1, \dots, k-1, j = i+1, \dots, k-1\}$$

such that we have $Y_1 = \sum_{j=2}^{k-1} F_{j1}P_j$, $Y_2 = -F_{21}P_1 + \sum_{j=3}^{k-1} F_{j2}P_j$, and generally

$$Y_l = -\sum_{i=1}^{l-1} F_{li}P_i + \sum_{j=l+1}^{k-1} F_{jl}P_j, \;\; l = 1, \dots, k-1.$$

Then using the coefficients $F_lP_k + X_l$ from (2.10) as Y_l's and defining $F_{kl} := -F_l$ for $l = 1, \dots, k-1$, (2.8) becomes $X_k = -F_{k1}P_1 - \dots - F_{kk-1}P_{k-1}$. Then we have the following:

$$X_l = -\sum_{i=1}^{l-1} F_{li}P_i + \sum_{j=l+1}^{k} F_{jl}P_j, \;\; l = 1, \dots, k.$$

It expresses the matrix X as a $\mathcal{R}$-combination of elements of B, namely

$$\left[\begin{array}{ccccc} (F_{21}\dots F_{k1}) & (F_{32}\dots F_{k2}) & \dots & (F_{k-1k-2}\ F_{kk-2}) & (F_{kk-1}) \end{array}\right] \left[\begin{array}{c} B_1 \\ B_2 \\ \dots \\ B_{k-2} \\ B_{k-1} \end{array}\right].$$

To conclude the proof, it suffices to note that x is a row of X, and so it is itself a combination of rows of B. □

It is obvious that the syzygies of the ideal generated by $P_1, \dots, P_k$ in $\mathcal{R}$ are given by the matrix B, keeping in mind that they commute and form a left regular sequence. But this theorem doesn't only say these are syzygies, it gives the module of syzygies for the module generated in R^n by the rows of $P_1, \dots, P_k$.

Theorem (2.2.3.) gives rise to the following corollaries [10].

Corollary 2.2.4. *Let P_1, P_2 be commuting matrices that for a regular sequence in $\mathcal{R} = Mat_n(R)$. Then the range of the operator $P(D) = \begin{bmatrix} P_1(D) \\ P_2(D) \end{bmatrix}$ is given by*

$$\{(g_1, g_2) \in \mathcal{C}^\infty \mid P_2(D)g_1 = P_1(D)g_2\}.$$

Corollary 2.2.5. *Let P_1, P_2 be two commuting matrices forming a regular sequence in $\mathcal{R}$. Let $\mathcal{Q} = Ker(P_2)$ and let U be an open convex (or compact convex) set, then the operator*

$$P_1(D) : \mathcal{Q}(U) \longrightarrow \mathcal{Q}(U)$$

is surjective.

Generally:

Corollary 2.2.6. *Let $P_1, \dots, P_k$ be commuting matrices forming a regular sequence in $\mathcal{R}$ and let $\mathcal{Q} = \{f | P_2(D)f = \cdots = P_k(D)f = 0\}$. Let U be an open convex (or compact convex) set, then the operator*

$$P_1(D) : \mathcal{Q}(U) \longrightarrow \mathcal{Q}(U)$$

is surjective.

Now let's go back the the quaternionic setting. Consider the pair $(\partial_{\overline{q}}, p(\partial_q))$, where $\partial_{\overline{q}}$ is the Cauchy-Fueter operator and $p(\partial_q)$ is a polynomial in ∂_q with complex coefficients. If we look at their Fourier transform $(\overline{q}, p(q))$ in the ring $R = \mathbb{C}[q, \overline{q}]$, we see it forms a regular sequence because $\overline{q}$ is not a factor of $p(q)$. We can then reformulate this into matrices.

Let the 4×4 matrix Q be the Fourier transforms of the matrix $Q(D)$ which corresponds to the operator $\partial / \partial \overline{q}$. Also, let P represent the Fourier transform of the matrix $P(D)$ corresponding to $p(\partial_q)$. Then the authors of [10] showed the following:

Proposition 2.2.7. *The pair (Q,P) form a regular sequence in the ring $\mathcal{R} = Mat_4(R)$, where $R = \mathbb{C}[x_0, \dots, x_3]$.*

This leads to the following Corollary [10]:

Corollary 2.2.8. *For any polynomial $p(q)$ with complex coefficients, the operator $p(\partial_q)$ is surjective on the space of regular functions.*

Chapter 3

Biregular Functions of Several Quaternionic Variables

3.1 Biregular Functions

We defined the left Cauchy-Fueter operator, D_l, in the last chapter. We can define the right Cauchy-Fueter operator similarly,

$$D_r = \frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k.$$

In this chapter, we will summarize work done by Struppa, Sabadini, and Damiano in [13]. They studied functions of $2n$ quaternionic variables called ***biregular functions*** that are simultaneously left regular in the first n variables, $p_1, \ldots, p_n$, and right regular in the last n variables, $q_1, \ldots, q_n$. We will first give definitions and notations, followed with specific examples, and finally discuss their findings for explicit Gröbner bases, syzygies, resolutions, and compatibility conditions for biregular functions.

Let $f : \mathbb{H}^n \times \mathbb{H}^n \longrightarrow \mathbb{H}$ be an infinitely differentiable function, then $f(p_1, \ldots, p_n, q_1, \ldots, q_n)$ is *biregular* with respect to the pairs (p_s, q_s), $s = 1, \ldots, n$ if and only if it satisfies the system

$$\begin{cases} D_{l_1}(f) & = & 0 \\ D_{r_1}(f) & = & 0 \\ & \dots & \\ D_{l_n}(f) & = & 0 \\ D_{r_n}(f) & = & 0 \end{cases} \quad (3.1)$$

where $p_s = x_{s0} + ix_{s1} + jx_{s2} + kx_{s3}$, $q_s = y_{s0} + iy_{s1} + jy_{s2} + ky_{s3}$ and

$$D_{l_s} = \frac{\partial}{\partial x_{s0}} + i\frac{\partial}{\partial x_{s1}} + j\frac{\partial}{\partial x_{s2}} + k\frac{\partial}{\partial x_{s3}}.$$

$$D_{r_s} = \frac{\partial}{\partial y_{s0}} + \frac{\partial}{\partial y_{s1}}i + \frac{\partial}{\partial y_{s2}}j + \frac{\partial}{\partial y_{s3}}k.$$

Then a quaternion equation $D_{l_s}f = 0$ or $D_{r_s}f = 0$ can be expressed with four real equations in matrix form as:

$$\begin{bmatrix} \partial_{x_{s0}} & -\partial_{x_{s1}} & -\partial_{x_{s2}} & -\partial_{x_{s3}} \\ \partial_{x_{s1}} & \partial_{x_{s0}} & -\partial_{x_{s3}} & \partial_{x_{s2}} \\ \partial_{x_{s2}} & \partial_{x_{s3}} & \partial_{x_{s0}} & -\partial_{x_{s1}} \\ \partial_{x_{s3}} & -\partial_{x_{s2}} & \partial_{x_{s1}} & \partial_{x_{s0}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0,$$

$$\begin{bmatrix} \partial_{y_{s0}} & -\partial_{y_{s1}} & -\partial_{y_{s2}} & -\partial_{y_{s3}} \\ \partial_{y_{s1}} & \partial_{y_{s0}} & \partial_{y_{s3}} & -\partial_{y_{s2}} \\ \partial_{y_{s2}} & -\partial_{y_{s3}} & \partial_{y_{s0}} & \partial_{y_{s1}} \\ \partial_{y_{s3}} & \partial_{y_{s2}} & -\partial_{y_{s1}} & \partial_{y_{s0}} \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix} = 0$$

respectively, with f being a vector with four real components. D_{l_s} and D_{r_s} will denote both the matrices and the operators associated to them. Then the system (3.1) can be written in matrix form:

$$P_n(D)f = 0$$

As before, the matrix P_n is obtained from $P_n(D)$ through the Fourier transform. When we write the entries in P_n, we will neglect the imaginary unit, and will use the same variables instead of duals. Let $\langle P_n^T \rangle$ be the module generated by the columns of the matrix P_n^t. As in the previous chapter, the primary object of interest will be the cokernel of the map the matrix P_n^t induces, which is the mudule $M_n = R^4/\langle P_n^t \rangle$, with R being the ring of polynomials in $8n$ variables, $R = \mathbb{C}[x_{s0}, \dots, x_{s3}, y_{s0}, \dots, y_{s3}]$, $s = 1, \dots, n$. Now that we have the notation fixed, let's look at some examples computed using CoCoA for $n = 1, 2$ [13].

For $n = 1$ We start by defining two 4×4 matrices that are the symbols of the operators, and then find the resolution.

```
Dl:=Mat[                              --left operator in p
  [x[0], -x[1], -x[2], -x[3]],
  [x[1], x[0], -x[3], x[2]],
  [x[2], x[3], x[0], -x[1]],
  [x[3], -x[2], x[1], x[0]]];
Dr:=Mat[                            --right operator in q
  [y[0], -y[1], -y[2], -y[3]],
  [y[1], y[0], y[3], -y[2]],
  [y[2], -y[3], y[0], y[1]],
  [y[3], y[2], -y[1], y[0]]];
```

```
ModB:=Module(B);    --resolution for biregular functions
Res(R^4/ModB);
0 --> R^4(-2) --> R^8(-1) --> R^4
------------------------------------
```

Also we can use the following command to compute the commutator.

```
  Coala.Comm(Dr,Dl);
Mat([
  [0, 0, 0, 0],
  [0, 0, 0, 0],
  [0, 0, 0, 0],
  [0, 0, 0, 0]
])
-------------------------------------
```

So let's look at what this tells us. As mentioned previously, the Betti numbers, giving the number of syzygies, are the exponents in the resolution, while the difference of the numbers in parentheses tell us the degree of the maps. Since the commutator is zero, these matrices commute. these matrices also form a regular sequence in $Mat_4(R)$, since they use two different sets of variables. Applying theorem (2.2.3.), we see that the syzygy module is generated by the rows of the matrix $P_1(D) = \begin{bmatrix} D_l \\ D_r \end{bmatrix}$ and given by $S_1(D) = (-D_r, D_l)$. This gives us the following free resolution for the module associate to biregular functions of two quaternionic variables; note in terms of degrees and Betti numbers, this complex has Koszul-like form.

$$0 \longrightarrow R^4(-2) \xrightarrow{P_1} R^8(-1) \xrightarrow{S_1} R^4 \longrightarrow M \longrightarrow 0.$$

This leads to the following proposition [13].

Proposition 3.1.1. *Let D_l and D_r be respectively the left and right Cauchy-Feuter operators acting on functions $f : \mathbb{H} \times \mathbb{H} \longrightarrow \mathbb{H}$. Consider the inhomogeneous system*

$$\begin{cases} D_l(f) &= g_l \\ D_r(f) &= g_r \end{cases}.$$

Then the only compatibility condition on the system is given by $D_r g_l = D_l g_r$ and the associated complex is Koszul-like, i.e. it has length two and its maps are constructed as in the Koszul complex. The Hartogs phenomenon holds for the solutions of the system, while the second cohomology module of the associated complex is nonzero.

Note that the Hartogs phenomenon was proved in [6].

For $n = 2$ In this case, we are considering functions in four quaternionic variables (p_1, p_2, q_1, q_2); which are left regular in p_1, p_2 and right regular in q_1, q_2. Note that every left operator commutes with every right operator, but the left operators do not commute with each other, and neither do the right operators. From the four operators $D_{l1}, D_{l2}, D_{r1}, D_{r2}$, we get four pairs (D_{li}, D_{rj}) that will commute, and two pairs (D_{l1}, D_{l2}) and (D_{r1}, D_{r2}) that will not commute. From the commuting pairs, it is expected to obtain Koszul-like syzygies, and from the others, we expect to see quadratic syzygies as was seen in our example in the previous chapter with two left Cauchy-Fueter operators. Let P_2 be the 16×4 matrix in R representing these four operators. Using CoCoA as in the previous example, we obtain [13]:

```
0 --> R^4(-8) --> R^16(-7) --> R^16(-5)(+)R^16(-6) -->
  --> R^40(-4) --> R^16(-2)(+)R^16(-3) --> R^16(-1) --> R^4
```

Now looking at the Betti numbers and degrees of the maps, we see that the first syzygies have 16 linear relations, as well as 16 quadratic relations, corresponding to 4 quaternionic syzygies each. The authors went on to compute the explicit expression of these relations using CoCoA, as well as the cohomology modules. They summarized their results in the following proposition [13].

Proposition 3.1.2. *Consider the inhomogeneous system*

$$\left\{ \begin{array}{rcl} D_{l1}(f) & = & g_{l1} \\ D_{l2}(f) & = & g_{l2} \\ D_{r1}(f) & = & g_{r1} \\ D_{r2}(f) & = & g_{r2} \end{array} \right. .$$

The compatibility conditions of the system are given by the following four linear relations:

$$D_{li}g_{rj} = D_{rj}g_{li},\ i,j \in \{1,2\}$$

plus the four quadratic relations

$$D_{li}D_{lj}g_{lj} = D_{lj}^2 g_{li},\ i,j \in \{1,2\},\ i \neq j$$

$$D_{ri}D_{rj}g_{rj} = D_{rj}^2 g_{ri},\ i,j \in \{1,2\},\ i \neq j$$

The complex associated to the module M_2 associated to the system is

$$0 \longrightarrow R^4 \xrightarrow{P_2} R^{16} \xrightarrow{S_1} R^{32} \xrightarrow{S_2} R^{40} \xrightarrow{S_3} R^{32} \xrightarrow{S_4} R^{16} \xrightarrow{S_5} R^4 \longrightarrow M_2 \longrightarrow 0$$

where the self-duality condition holds on the maps of the resolution, i.e. $S_5 = {}^tP_2,\ S_4 = {}^tS_1$, and $S_3 = {}^tS_2$. The complex is exact except at the last spot where the cohomology module is the cokernel of S_5.

3.2 Biregular Functions in $2n$ variables

The authors of [13] present preliminary lemmas and propositions that lead to a general theorem giving the length of the free resolution associated to the R-module M_n, the Betti numbers, and compatibility conditions for biregular functions of $2n$ variables.

Lemma 3.2.1. *Let $A_1, \ldots, A_n$ and B be square matrices representing $n+1$ linear constant coefficient differential operators. Let us suppose that $A_iB = BA_i$ for every $i = 1, \ldots, n$ and suppose that they form a left regular sequence in the ring of matrices. Let $S = \{(S_{j1}, \ldots, S_{jn})|\ j = 1 \ldots t\}$ be a set of generators for the module of left syzygies of the n-tuple $(A_1, \ldots, A_n)$. Then the module $Syz(A_1, \ldots, A_n, B)$ is generated by the set $S' = \{(S_{j1}, \ldots, S_{jn}, 0)|\ j = 1, \ldots t\}$ together with the set $\mathcal{K} = \{(0, \ldots, -B, \ldots, 0, A_i)|\ i = 1 \ldots n\}$.*

Proof. We can see that the elements of S' and $\mathcal{K}$ are syzygies, so we need to show they generate all the syzygies. Let $C_1, \ldots, C_n, D$ be the $n+1$ matrices such that

$$C_1A_1 + \cdots + C_nA_n + DB = 0.$$

Since $(A_1, \ldots, A_n, B)$ is a left regular sequence and $C_1A_1 + \cdots + C_nA_n = -DB$, we see that $D = T_1A_1 + \cdots + T_nA_n$ for some matrices $T_1, \ldots, T_n$. Keeping in mind that these are commuting matrices, we can substitute this equation for D and obtain

$$(C_1 + T_1B)A_1 + \cdots + (C_n + T_nB)A_n = 0.$$

Then for every $i = 1 \ldots n$, we have

$$C_i + T_iB = P_1S_{1i} + \cdots + P_tS_{ti}.$$

So we see that the $(n+1)$-tuple $(C_1, \ldots, C_n, D)$ is of the desired form. $\square$

The following lemma finds the reduced Gröbner basis for the module generated by the rows of the left and right Cauchy-Fueter operators. The term ordering used was DegRevLex. For proof, see [13].

Lemma 3.2.2. *Let $D_{l1}, \ldots, D_{ln}$ be the symbol matrices associated to n left Cauchy-Fueter operators, and let $D_{r1}, \ldots, D_{rn}$ be the symbols of n right Cauchy-Fueter operators. Let $\mathcal{B}_n$ be the module generated by the rows of such matrices. Then the reduced Gröbner basis for $\mathcal{B}_n$ is given by the rows of the matrices D_{ls} and D_{rs}, $i = 1, \ldots, n$ together with the rows of the matrices*

$$B_{ks} = D_{lk}D_{ls} - D_{ls}D_{lk} \text{ and } C_{ks} = D_{rk}D_{rs} - D_{rs}D_{rk}, \ 1 \leq r < s \leq n.$$

The authors then calculate the Hilbert series shown in the following lemma; for proof, see [13]

Lemma 3.2.3. *Let $R = \mathbb{C}[x_{i0}, \ldots, x_{i3}, y_{i0}, \ldots, y_{i3}]$ and let R^4/M_n be the R-module associated to n left Cauchy-Fueter operators and n right Cauchy-Fueter operators, with $n > 2$. Then the Hilbert series of the module R^4/M_n is given by*

$$\mathcal{H}_{R^4/M_n}(t) = 4\frac{(1+(n-1)t)^2}{(1-t)^{4n+2}}.$$

These lemmas are each used in the proof of this encompassing theorem [13].

Theorem 3.2.4. *Let $n > 2$, $R = \mathbb{C}[x_{i0}, \ldots, x_{i3}, y_{i0}, \ldots, y_{i3}|\ i = 1 \ldots n]$ and consider the system associated to n left Cauchy-Fueter operators $D_{l1}, \ldots, D_{ln}$ and n right Cauchy-Fueter operators $D_{r1}, \ldots, D_{rn}$. Let M_n be the R-module associated to the map given by all the $2n$ operators. Then the length of the minimal free resolution of M_n is $4n-2$ and the complex is exact except at the last point. The Betti numbers associated to M_n are $\gamma_0 = 4$, $\gamma_1 = 8n$ and*

$$\gamma_d = 4n^2 \sum_{i+j=d} \begin{pmatrix} 2n-1 \\ i \end{pmatrix} \begin{pmatrix} 2n-1 \\ j \end{pmatrix} \frac{ij+1-d}{ij+1+d},\ d > 1.$$

Furthermore, if we consider the inhomogeneous system

$$\begin{cases} D_{l1}(f) & = & g_{l1} \\ D_{r1}(f) & = & g_{r1} \\ & \cdots & \\ D_{ln}(f) & = & g_{ln} \\ D_{rn}(f) & = & g_{rn} \end{cases},$$

the compatibility conditions are given by the n^2 linear relations

$$D_{li} g_{rj} = D_{rj} g_{li},\ i, j \in \{1, \ldots, n\}$$

and the $4\begin{pmatrix} n \\ 2 \end{pmatrix} + 4\begin{pmatrix} n \\ 3 \end{pmatrix}$ quadratic relations given by the following

$$D_{si} D_{sj} g_{sj} = D_{sj}^2 g_{si},\ i, j \in \{1, \ldots, n\},\ i \neq j$$

$$D_{si} D_{sj} g_{sk} + D_{sj} D_{si} g_{sk} = D_{sk} D_{si} g_{sk} + D_{sk} D_{sj} g_{si},\ i, j, k \in \{1, \ldots, n\},\ i \neq j$$

and finally the $4\begin{pmatrix} n \\ 3 \end{pmatrix}$ *exceptional relations*

$$(D'_{si}D_{sj} - D'_{sj}D_{si})g_{sk} + (D'_{sj}D_{sk} - D'_{sk}D_{sj})g_{si} + (D'_{sk}D_{si} - D'_{si}D_{sk})g_{sj} = 0$$

$$(D''_{si}D_{sj} - D''_{sj}D_{si})g_{sk} + (D''_{sj}D_{sk} - D''_{sk}D_{sj})g_{si} + (D''_{sk}D_{si} - D''_{si}D_{sk})g_{sj} = 0,$$

$$1 \leq i < j < k \leq n$$

where in each line D_s *stands for either the left operator or the right operator, and the operators* D'_s *and* D''_s *are Cauchy-Riemann like operators involving only two of the four real variables corresponding to the quaternionic variable given by the index.*

3.3 Conclusion

Gröbner bases were the backbone for all of the computations mentioned throughout this paper, and of course this is only a small sampling of its applications in algebraic analysis, specifically with linear constant coefficient partial differential operators. We focused on regular and biregular functions in the quaternionic setting, however the Cauchy-Fueter system is only an example of a more general equation fundamental to physics called the Dirac Equation. One can describe the Dirac Equation defined through the Dirac operator

$$\partial_x := \sum_{i=1}^{m} e_i \frac{\partial}{\partial x_i}$$

and its variations in the framework of Clifford Algebras, and continue to use Gröbner basis theory to analyze the Dirac Operator and many others. Again, this is one application, obviously, there are many.

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Vita

Jenny Ross received a bachelor of Arts degree in Mathematics from the University of Central Arkansas in August of 2002. She taught 7th through 12th grade math for three years at two private schools. She then began at the University of Texas in Austin for a masters degree in Mathematics. Due to her high school teaching experience, she was a teaching assistant for the UTeach program while studying for her graduate degree. After completing enough coursework for a masters, she began to work at St. Edward's University in Austin, and is currently there. She directs the Math Lab as well as teaching Math for Liberal Arts, Pre-Calculus, and Algebra.

Permanent address: 500 E. Stassney Ln. Apt 232
Austin, Texas 78745

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