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Post-Contingency States Representation & Redispatch for Restoration in Power Systems Operation

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Post-Contingency States Representation & Redispatch for Restoration in Power Systems Operation

by

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DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

December 2017

Dedicated to:

My idol, *Albert Einstein*,

My teachers, *(Late) Tejendra Mohan Acharya* and *Dr. Ross Baldick*,

And my parents, *Brajadual Chakrabarti & Uma Chakrabarti*.

Acknowledgments

This doctoral dissertation and the work that has been presented here, is the culmination of last several years' (almost nine years, here at The University of Texas at Austin, but, at least twelve more years before that, for setting up the ground-work) of effort, perseverence, patience, brainstormings, planning (and replanning), and persistence of several people along with me. Restrictions on the page-limit, constraints of time, and limitations of my own memory will unfortunately, not allow me to make a mention of every single one of them. Nevertheless, I will attempt my best to devote this space, to express my humble gratitude to some of the most significant people, who stood by me and assisted, in carrying out this work. First and foremost, I would like to express my heartfelt thanks to my Ph.D. supervisor, Prof. Ross Baldick. He let me start my Ph.D. research work at a time in my life, when I was in the middle of, and suffering from a severe career crisis. He has always been an extremely supportive and encouraging mentor. He has always given me complete freedom to think about my research direction(s) and problem(s), as well as the solution approach(es). Our often long one on one discussions, which a lot of times ate into his precious time (my sincere apologies for that) form some of the most valued possessions of my life and career. Those meetings were the defining moments during which my thought process took shape; those were the times when he challenged me and let me push my boundaries. This research work is largely the result of those interactions, for which I don't have words to thank my supervisor and mentor. Apart from being and excellent guide and researcher, Prof. Baldick is one of the finest teachers and educators, with an outstanding sense of humor, and also a great human being, who is very kind, accommodating, and humble in nature. His span

of knowledge not only covers Electrical Engineering, but also several other fields, which has also enhanced my (and I am sure everyones', who comes across him) respect and admiration for him. Being Prof. Baldick's graduate student has been a great privilege and prestige for me and will forever be cherished in my life. Secondly, I would like to extend special words of thanks to some of my colleagues, co-authors in different papers, seniors, and peers with whom I have either collaborated, or had discussions, or who have guided me in this endeavor. I would like to begin by thanking my colleague and co-author, Prof. Javad Lavaei, whose work on convexification of AC-OPF problem, set the spark within me, needed to pursue my Ph.D. The material presented in appendix A is primarily extension of his work on convexification and the rest of this work, which focuses on the application of *Proximal Message Passing (PMP)* algorithm to LASCOPF problem is the extension of the work, which he co-authored [234], [233]. This leads me to make mention, next, of my collaborators, Matt Kraning, Eric Chu, and Prof. Stephen Boyd. The work on the LASCOPF problem started as a joint collaborative research project with the five of us; myself, Matt Kraning, Eric Chu, Prof. Baldick, and Prof. Boyd, to extend their previous works on ADMM and Dynamic OPF. Prof. Boyd and Prof. Baldick were kind enough to arrange my month-long stay, as a visiting researcher at Stanford University from December, 2013 to January, 2014, that let me interact with Matt, Eric, and Prof. Boyd in person and enhanced my knowledge a lot. It has been a rewarding experience to audit the Convex Optimization course, that Prof. Boyd teaches at Stanford as well. Thanks are also due to my colleague, senior, and good friend, Mahdi Kefayati for the hours of discussions, his guidence to me over issues related to Mathematical Optimization, software development, and problem solving skills. It would have been very hard for me to get a head-start on a lot of areas of study without the help from Mahdi. A few other colleagues of mine, who provided valuable insights and constructive

criticisms of this work are Mohammad Majidi, Yezhou Wang, Hector Chavez, Matthias Huber, Elisabeth Kellerer, Amin Kargarian, Anthony Papavasiliou etc. Third in line, I would like to extend my thanks to all the sources of financial support and funding. I would like to acknowledge the *National Science Foundation (NSF)* for funding this project, through grant number ECCS-1406894, without which this work could never have been completed. I would also like to acknowledge Prof. Roger Bengston, Prof. John Keto, Prof. John Markert (from the Physics department, UT Austin), Pam Dahl (from the Cockrell School of Engineering, UT Austin), my Ph.D. supervisor, Prof. Baldick, for supporting me as a Teaching Assistant (TA), Tena Nash (from the PCL library), and the Union Bank of India, for supporting me financially during the initial phase of graduate school. Along the same lines, I would also like to express my gratitude to my internship mentors. My first internship mentor, Kris Dixit, who supervised me at **Lower Colorado River Authority (LCRA)**, where I worked from June, 2009 till end of May, 2010, was extremely instrumental for building up my hands-on knowledge, the more practical and intuitive feel about the areas of Power Systems Optimization and Electricity Markets. He also trained me at getting proficient with the electricity market production cost modeling and simulation software, UPLAN Altos. My second internship was with **Trailstone Power LLC.**, from June, 2015 till August, 2015, where I was mentored by Dr. Myung Chon and Dr. Ricson Chai. I am indebted to both of them to have given me the chance and the highly empowering responsibility to apply my academic knowledge of Mathematical Optimization, particularly, Mixed Integer Programming, to build models and develop software systems, that are of direct use for the electricity trading industry. Some of the materials presented in chapters 7 and 9, and appendix B are based on the work I did there, and provide some future research directions to this work, as well. I am fortunate enough, to have gotten a chance to work in close collaboration with and under the mentorship of Dr. Feng Pan, at **Pacific Northwest National Laboratory (PNNL)** from May, 2016 till August, 2016, where I spent most of my time to build the model and software solver for an AC-OPF and the experience I gained, has been extremely useful for this dissertation work. Thanks are also due to my mentors, Mr. Rossen Tzartzev & Dr. Moses Kai, and my colleague, Mr. Mark Justiss, from the **Hal C Weaver Power Plant, The University of Texas at Austin Facilities & Services**, where I worked as a Graduate Research Assistant from June, 2017 till August, 2017. While working there, I performed studies on Transient Stability and Power System Oscillations of an islanded microgrid. That work gave me perspectives and built foundation upon which, parts of appendix A have been developed. I would also like to thank my mentors and supervisors from **Larsen & Toubro Limited (L&T)**, Mumbai, India, where I worked from July, 2006 till July, 2008. Notably, Chunduru Prabhakar, Girish Panchal, Dhanpal Govindan, Sharad Vichare, and other colleagues, who for the first time helped me get the practical industrial exposure in the power systems manufacturing sector, after graduating from college. Mr. Prabhakar was also kind enough to have written one of the recommendation letters for my graduate school applications. Thanks are also due to Michael Grant and Jacob Mattingley for help with the Matlab based CVX disciplined convex optimization modeling language and the CVXGEN custom solver, respectively, using which, most of the numerical simulation experiments of this work were done. I would also like to thank my committee members, Prof. Surya Santoso, Prof. Aristotle Arapostathis, Prof. Constantine Caramanis, Prof. Nedialko Dimitrov, and Prof. John Hasenbein for serving as examiners on my committee and offering valuable guidance and perspectives. I also want to extend my thanks to our ECE administrative staff, notably Melanie Gulick, Melody Singleton, Leticia Lira, Barry Levich, Diana Vega, and Matthew Ervin from the Physics department, for all their help, starting from the paper-works to finding TA positions, to applications for internships and filing for graduation. They are the lifelines for the graduate students. I would, next, like to express my gratitude to my parents, who form the backbone and support system for my existence. The constant help, encouragement, and support of my father (baba), Brajadual Chakrabarti and my mother (maa), Uma Chakrabarti, secifically are the fuels, without which, I couldn't have done anything at all. My parents have always been extremely supportive of my decisions to pursue academics and has stood by me in several hard times in life. Any word of thanks will be incommensurate to the significance of their presence in my life. My gratitude is also due to my elder brother (dada), Siddhartha Chakrabarti, sister-in-law (boudi), Surbhi Chakrabarti, and niece, Ahana Chakrabarti (Brishti), who has sacrificed a lot of their time, which otherwise, they would have spent with me, so that I can pursue my dream and passion of going for a training in research. Life as a graduate student can be an intensely tough experience, without the company of friends and colleagues. I want to make a special mention of my friends, Victor Steve Green, Mark Selovar, Yingue Boretz, Biswajit Ray, Soumojit Sarkar, Vaidy Krishnamurthy, Deepjyoti Deka, Akshay Vyas, Patryk Radyjowski, and Kyle Comer for hours and hours of intellectual conversations and stimulating discussions and also helping me, prepare for my conference talks, qualifying examination, and final defense examination. I also want to thank all the past and present members and office staff of Pearl Street Student Housing Cooperative and College Houses Inc., for the congenial atmosphere they created, while I was staying there, the entire time of my Ph.D. studies, from 13th February, 2011, till 15th August, 2017. Thanks are also due to my departmental colleagues and friends, Harsha Padullaparti, Tuan Ngo, Quan Nguyen, Min Lwin, Suma Basu etc., my ex-roommates, Srikanth Tondukulam, Rahul Tyagi, Naveed Arsalan, Vivek Subrahmanian, Charles Coupleux, Peter Hoai Nguyen etc., my local guardians, Robbi and Pancho Francovich, Piyush

Shivam, Chris James etc. While pursuing my graduate studies, I developed a few extra- and co-curricular activities as well. The couple that deserves special mention, are developing my burning passion for Argentine Tango and Indian Classical Music. It would probably take an entire chapter to write the names of all my wonderful friends, I came to know through these endeavors, in Austin, Houston, Dallas, Portland, Seattle, Los Angeles, Tampa, Puerto Rico, Philadelphia etc. They all have played a very significant role at keeping me well balanced and grounded, while I have been in graduate school. In conclusion, I want to sincerely apologize to all those people, whom I couldn't explicitly mention here. But, I want to say that they too have immense contribution towards the completion of this work.

Post-Contingency States Representation & Redispatch for Restoration in Power Systems Operation

Publication No. ________________

Sambuddha Chakrabarti, Ph.D.
The University of Texas at Austin, 2017

Supervisor: Ross Baldick

In this treatise, we will present a dynamic version of the *Security Constrained Optimal Power Flow (SCOPF)* problem, the *"Look Ahead Security Constrained Optimal Power Flow" (LASCOPF)* problem, with post-contingency states representation and redispatch scheme for restoration to normal operation, following an assumed outage represented in the mathematical formulation. We will also propose a distributed algorithm to solve the OPF, SCOPF, and LASCOPF problems. The objective of the problem is to minimize the cost of operation, over a number of dispatch intervals and across all contingency scenarios subject to the constraints of the network. It is, therefore, a large optimization problem, requiring an effective distributed solution method. As one of the means to address this challenge, we will be extending the *Proximal Message Passing (PMP)* algorithmic framework, which is based on another algorithm, called *Alternating Direction Method of Multipliers (ADMM)* and combine it with the *Auxiliary Problem Principle (APP)*. The resulting algorithm, which we hereafter, will call, *Auxiliary Proximal Message Passing (APMP)* is extremely scalable with respect to both network size and the number of scenarios. We implement a look-ahead

contingency planning, representing the post-contingency states of the system ahead of time, in a *Receding Horizon Control (RHC)* or, *Model Predictive Control (MPC)* type of formulation. One goal of this work is to particularly focus our attention on the trajectories of post-contingency line temperature rise, line MW flow rise, and line current rise and try to limit them through our proposed method. The reason for paying particular attention to line temperature rise and limiting the same, is the intention of the present scheme to make the most use of the existing transmission capability, without costly transmission upgrades. The means of attaining that goal is to make use of short term thermal overload rating and dynamic thermal limit, and in the event of an actual outage, modifying the dispatch in such a way, that the flows on the remaining lines can be brought back to within allowed values in a given time interval. We demonstrate the effectiveness of our distributed method with a series of numerical simulations based on some simple systems and the IEEE test systems. Finally, we conclude, with a suggestion to some possible future research directions.

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Chapter 1

Introduction

1.1 Overview and Background

1.1.1 The Premise

Maintaining economic efficiency along with continuity of reliable service, even in the presence of random outages or sudden changes is a very critical and crucial aspect of modern day power systems operation and planning. Without proper planning well ahead of time, it is difficult to ensure continuous efficient operation of the grid, especially if the abnormal conditions are large, fast, and non-deterministic. The dynamic ramp rate constraint on generators, which restricts how quickly they can adjust their generation levels is one of the most important limiting factors, affecting the ability of the grid to cope with random events, other limiting factors include generator start-up and shut-down times, network topology, and response times of relays, circuit breakers, and isolators. When large and unpredictable changes occur to the grid, the set of power flows which corresponds to the efficient operation of the new state of the grid can require large changes to generators' current generation levels. These large changes can be very costly or impossible to immediately execute. As a result, proactive schemes, which can ensure continual operation with minimal (or no) disruptions in the presence of large and unforeseen events, are required for effective grid operation.

Traditionally, the different Independent System Operators/Transmission System Operators/Independent Grid Operators (ISOs/TSOs/IGO, as they are known in different parts of the world), utilities, electricity traders etc have been solving the *Economic Dispatch (ED)* problem, the *Optimal Power Flow (OPF)* problem, and the *Security Constrained Optimal Power Flow (SCOPF)* problem to choose how to generate power from the generators most economically in order to meet the demand and also to satisfy network constraints, together with solving the *Security Constrained Unit Commitment (SCUC)* problem to most economically decide which generators to switch on or off. Although some ISOs, like the CAISO, include lookahead over multiple future dispatch intervals for their scheduling, they do not explicitly represent the post-contingency states. We will aim at explicit representation of post-contingency states in this work. We give below brief definitions of each of the above problems.

1.1.1.1 Economic Dispatch (ED) Problem

The ED problem aims at scheduling the generators' power outputs such that the demand is met at the lowest possible cost while satisfying generators' generating limits. It does not pay attention to obeying transmission limits while solving this problem. In other words, it is implicitly assumed that none of the transmission limits will be violated. The ED problem can be stated mathematically as shown in equation (1.1) and diagramatically shown in figure 1.1, where the left panel shows different sources of generating electricity, with possibly different cost curves, thus representing the varied generation portfolio while the right panel represents the power demand. In figure 1.2, we have shown a schematic diagram for the ED problem.

$$\textbf{Objective}:$$
$$\min_{\textit{Power Generation}} \text{Total Generation Cost} \tag{1.1a}$$
$$\textbf{Constraints}:$$
$$\text{Supply Demand Balance} \tag{1.1b}$$
$$\text{Generation Limit} \tag{1.1c}$$

1.1.1.2 Optimal Power Flow (OPF) Problem

The OPF problem schedules the generators' power outputs such that the load demand is met at the lowest possible cost while satisfying generators' generating limits as well as satisfying the transmission limits. The OPF problem can be stated mathematically as shown in equation (1.2) (in which the boxed constraints represent the ones that are different from ED problem) and is shown diagramatically in figure 1.3, where the trasmission towers represent the constraints pertaining to the line power flow limits. In figure 1.4, we have shown the schematic diagram for the OPF problem.

$$\textbf{Objective}:$$
$$\min_{\textit{Power Generation}} \text{Total Generation Cost} \tag{1.2a}$$
$$\textbf{Constraints}:$$
$$\text{Supply Demand Balance} \tag{1.2b}$$

Figure 1.1: Economic Dispatch: Meeting Electrical Load Demand at the Minimum Possible Cost.

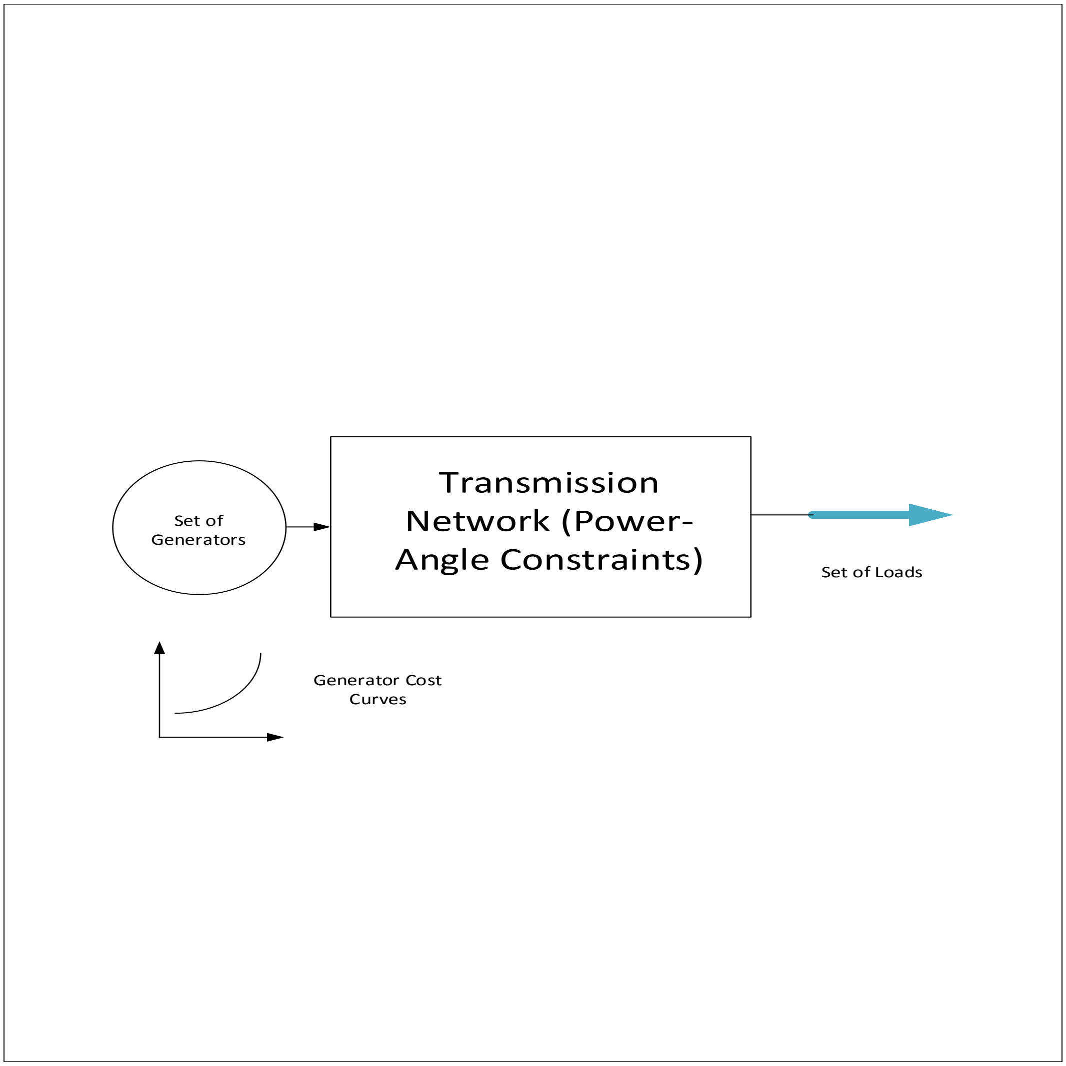


Figure 1.2: Economic Dispatch: Schematic Diagram for Economic Dispatch (ED)

Figure 1.3: OPF: Meeting Electrical Load Demand at the Minimum Possible Cost, while obeying Trasmission Constraints.

$$\text{Generation Limit} \tag{1.2c}$$

$$\boxed{\textbf{New Constraint} \rightarrow \textit{Line Power Flow Limit}} \tag{1.2d}$$

Thoughout this work, we will adopt the convention that a newly introduced constraint or issue is blocked as in (1.2d).

1.1.1.3 Security Constrained Optimal Power Flow (SCOPF) Problem

The SCOPF problem is similar to the OPF problem, but with the added set of constraints pertaining to the different scenarios corresponding to post-contingency or post-

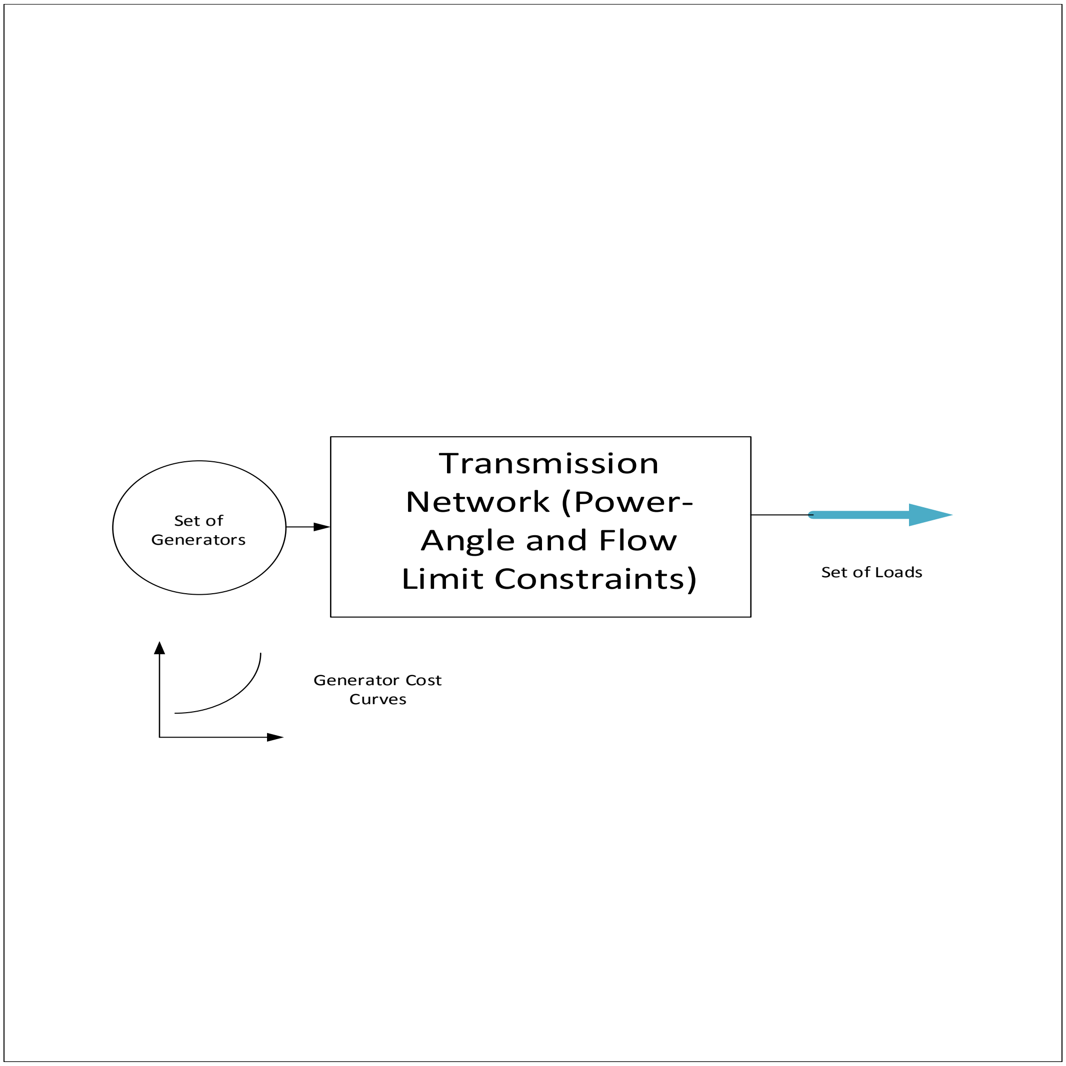


Figure 1.4: Schematic of OPF Problem

outage power flows on the transmission lines. The pre-contingency (or healthy state) line power flows correspond to the generations that abide by the post-contingency as well as pre-contingency limits. A contingency scenario, in this context, refers to a network state or topology with the outage of a certain number of network components (generators, transmission lines, loads, transformers etc.). These components are generally chosen because their outage is a credible event. In the present work, we will assume the outage of only one component for each contingency scenario, which is known as $(N-1)$ contingency scenario/analysis. This is justifiable, because the probability of more than one component going out of service in a grid simultaneously, is extremely low. For most of this work, we will mostly consider line contingencies, while also analyzing generator contingencies towards the end. The SCOPF problem can be stated mathematically as shown in equation (1.3) (in which the boxed constraints represent the ones that are different from OPF problem) and is shown diagramatically in figure 1.5, where the picture of the flashover represents the constraints to ensure security with respect to outage of any one line at a time. In figure 1.6, we have shown the schematic diagram of the SCOPF problem, where along with the base case scenario, we have also shown several contingency scenarios.

$$\textbf{Objective}:$$

$$\min_{Power\ Generation} \text{Total Generation Cost} \tag{1.3a}$$

$$\textbf{Constraints}:$$

$$\text{Supply Demand Balance} \tag{1.3b}$$

$$\text{Generation Limit} \tag{1.3c}$$

Figure 1.5: SCOPF: Meeting Electrical Load Demand at the Minimum Possible Cost, while obeying Trasmission Constraints both at the base-case as well as several contingency scenarios.

$$\text{Line Power Flow Limit (Base Case)} \tag{1.3d}$$

$$\boxed{\textbf{New Constraint} \rightarrow \textit{Line Power Flow Limit (Contingency Cases)}} \tag{1.3e}$$

As shown in the example in the next section, each of the above approaches fails to address the issue of post-contingency restoration and ensuring secure operation of the system *after* a random outage takes place.

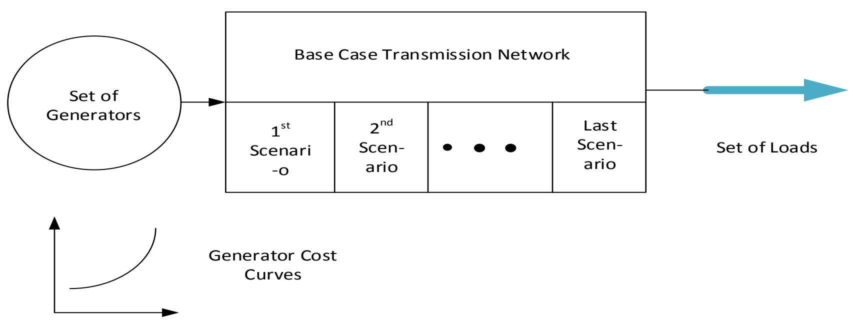


Figure 1.6: Schematic of SCOPF Problem

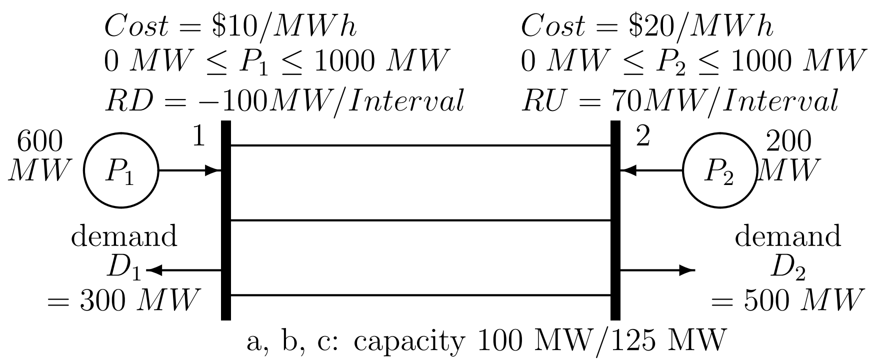


Figure 1.7: Three Bus System; Solution to the OPF.

1.1.2 Case Study

In this section, we will present a scenario that sheds light on the short-comings of the existing dispatch methods as described above. It, therefore, provides a rationale for the emphasis in this dissertation, on the need for a new dispatch scheme. The system that we consider for the purpose of illustration is a simple power system consisting of two buses, three identical transmission lines, two generators (whose power outputs are symbolized by P_1 and P_2 respectively, with generating limits ranging from 0-1000 MW each), and two loads as shown in figure 1.7 (indicated as D_1 and D_2 with values, 300 MW and 500 MW respectively). As shown in the figure, the two generators have marginal costs of \$ 10/MWh and \$ 20/MWh respectively. RD and RU refer to "Ramp-Down" and "Ramp-Up" limits respectively which are expressed in the units of MW/dispatch interval. From the figure, therefore, generator-1 can ramp down its production at a maximum rate of 100 MW from one dispatch interval to the next, while generator-2 can ramp up by a maximum of 70 MW. We assume that all three transmission lines have the same impedance and each has a long term rating and an emergency rating of 100 MW and 125 MW, respectively, to transmit power.

When OPF is solved for such a system, Generator-1 generates 600 MW and Generator-2 generates the remaining 200 MW. Hence the power flow on each line is 100 MW, which is at the nominal rated value and hence, the system operating conditions are healthy. However, if any of the lines goes out of service, the system will no longer stay healthy, since the flows on the remaining two lines would exceed both the long-term and the emergency ratings. Hence, this operating mode is not secure with respect to one line outage.

In the SCOPF case, if short-term emergency rating of 125 MW is adhered to, then the dispatches will change to 550 MW, and 250 MW, respectively, so that the total power transferred from bus-1 to bus-2 is 250 MW, which ensures security with respect to outage of any one of the lines. This is because, even if one of the lines goes out of service, the power flowing on each of the remaining lines will then be 125 MW, which is at the short term rating. So, at least for a short duration of time, (which depends on the time the short term rating is intended for) the flows are within the rating.

But, if a (permanent) outage of a line actually happens during the upcoming dispatch interval and it is required to restore to security with respect to another contingency within one further dispatch interval, then with the given ramping capability, it's **IMPOSSIBLE !!!!**.

The reason is that, although generator 1 can ramp down by 100 MW, it will be allowed to ramp down only by 70 MW, as it has to match the ramp up of generator 2, which can only ramp up by 70 MW to 375 MW, reducing imports to 125 MW, which is the secure limit with one line permanently out. The dispatch is therefore required to be as shown in figure 1.8. In this case, the total import from bus 1 to bus 2 is 195 MW. If an outage

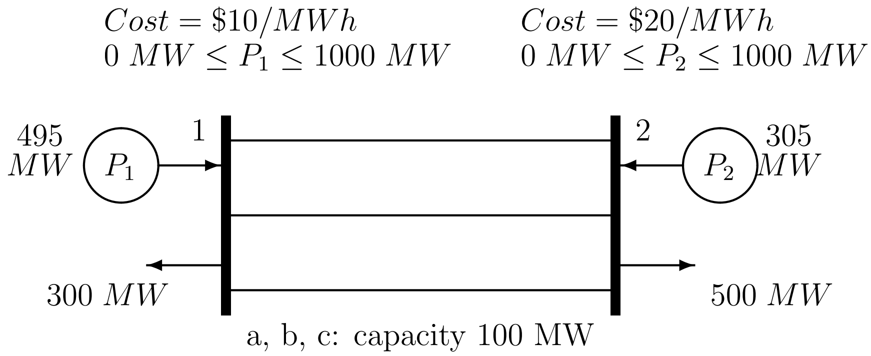


Figure 1.8: Three Bus System; Solution to the Look-Ahead SCOPF.

actually occurs, then generator 1 can very well ramp down its output by 70 MW, within one dispatch interval to bring down the power flow value to 125 MW, which is the secure value, with only two lines in operation. At the same time, generator-2 can ramp up by 70 MW, to make up for the loss of power for load at bus 2.

Thus, we can see that neither the OPF nor the SCOPF are adequate enough to solve this

The Locational Marginal Price (LMP) of electricity at the two buses/nodes is as shown in the table 1.1

Table 1.1: Locational Marginal Prices.

Problem	LMP1 (\$/MWh)	LMP2 (\$/MWh)
OPF	10	20
SCOPF	10	20

problem. Moreover, we have illustrated here restoration to security with respect to next set of outages/contingencies in just one further dispatch interval. But, in real world, it will be required to restore the system first to long term rating in possibly multiple dispatch intervals, and then to make the system secure with respect to next set of contingencies at the end of a few more intervals. In the next section, we will present some real world concerns that are actually driving the kind of issue we just presented in this case study.

1.2 Motivation

In this section, we will be describing some recent real world issues and problems, which motivates the work that we will be presenting in the next few chapters. We start with the CAISO case in the following section.

1.2.1 Contingency Modeling Enhancements by CAISO

In the 2012 stakeholder initiatives catalog, the California Independent System Operator (CAISO) and its stakeholders both highly ranked the issue of “Additional Constraints, Processes, or Products to Address Exceptional Dispatch,” which sought proactive market alternatives to the current practices of restoration followed by the ISO. The NERC reliability standard TOP-007-0R2 and the WECC (Western Electricity Coordinating Council) reliability standard TOP-007-WECC-1R1 require the ISO to restore the system to security within 30 minutes following an actual, permanent, outage. The current practice of the ISO is to restore the system through Exceptional Dispatch (ExD) and Minimum Online Capacity Constraints (MOC), following an actual outage that has resulted in the violation of Secure Operating Limits (SOL) or Interconnect Reliable Operating Limit (IROL).

Figure 1.9 and figure 1.10 show the money spent and the MWh deployed for SOL restoration by ExD and is a significant proportion. The problems with ExD and MOC are that they are non-flow based, localized, not optimized, based on operator discretion, and are not market based solutions. Therefore, in the March, 2013 issue paper [203], CAISO proposed the “preventive-corrective constraint” formulation, in which the post-contingency restoration within 30 minutes is represented within the dispatch model and the affected generators are compensated by LMP or through capacity payment reflected in the model. The CAISO has subsequently taken the stakeholder feedback and comments [204], [210], [206], [206], [208], [212] released a straw proposal [209] and three revisions of it so far [205], [207], [211] (as of 10th December, 2015). However, CAISO is not yet taking into consideration the temperature change on transmission lines, ambient temperature, and several other factors, which we will present in this work.

1.2.2 Increasing Renewable Energy Penetration

As will be shown in the subsequent chapters, in the present work we consider the dynamic adaptive line rating based on temperature rise of the transmission line relative to the ambient temperature, while considering restoration to security after the outage of a line. This means that we are going to make maximum utilization of the existent system while making it resilient to outages without costly transmission upgrades. As compared to a static transmission limit based on conservative estimates, the use of dynamic adaptive ratings allows for the dispatch of more low cost generation [380]. If we assume the generation cost to be contributed by only fuel cost, then renewable resources are the lowest cost generations. In figure 1.11, we have shown the EIA estimate of fuel mix for USA and in figure 1.12, we have shown a similar chart for only the renewable fuels. Both these charts forecast the growth through 2040 of renewables in electricity generation and especially for wind (based on the

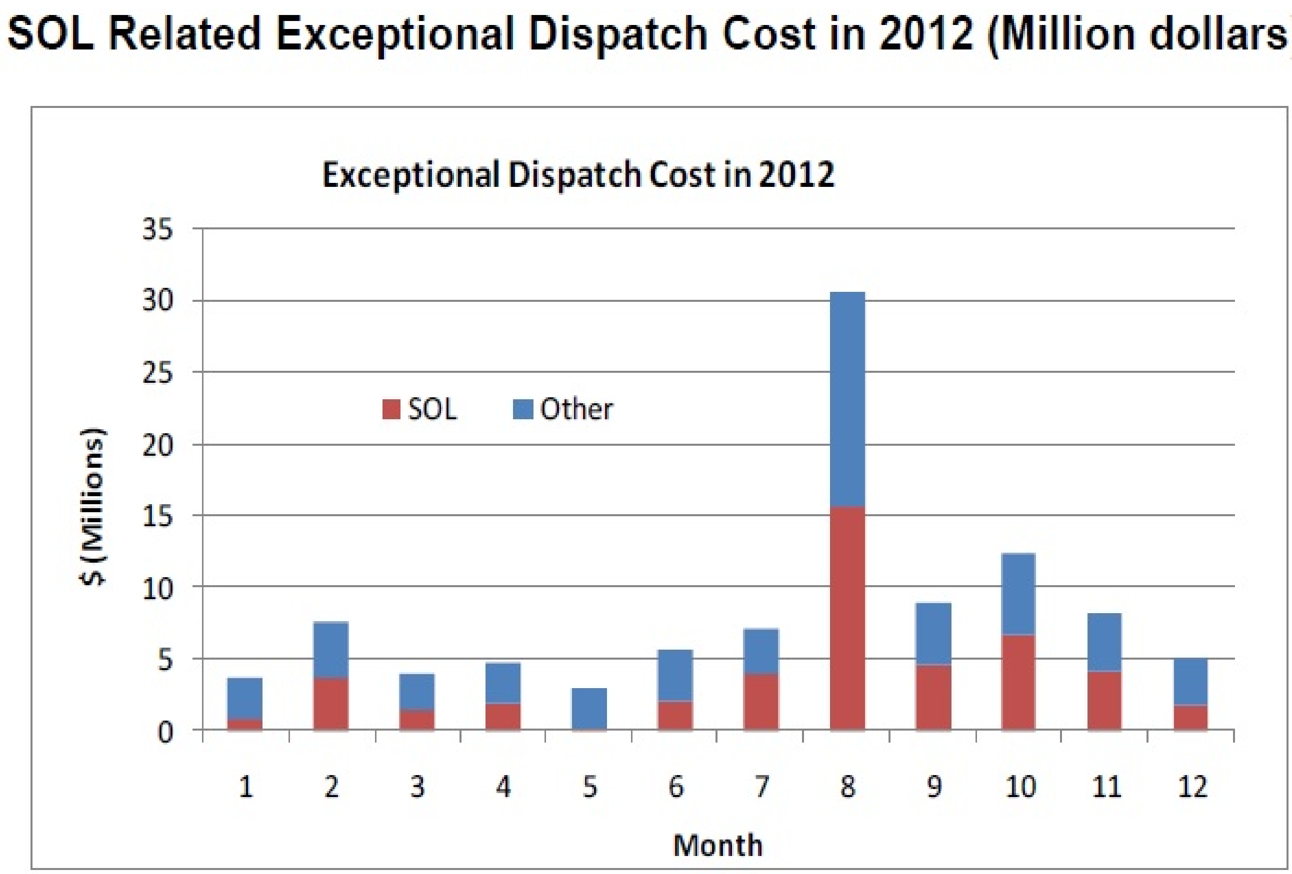


Figure 1.9: Monthly Exceptional Dispatch Dollars by CAISO in 2012 (Source: CAISO).

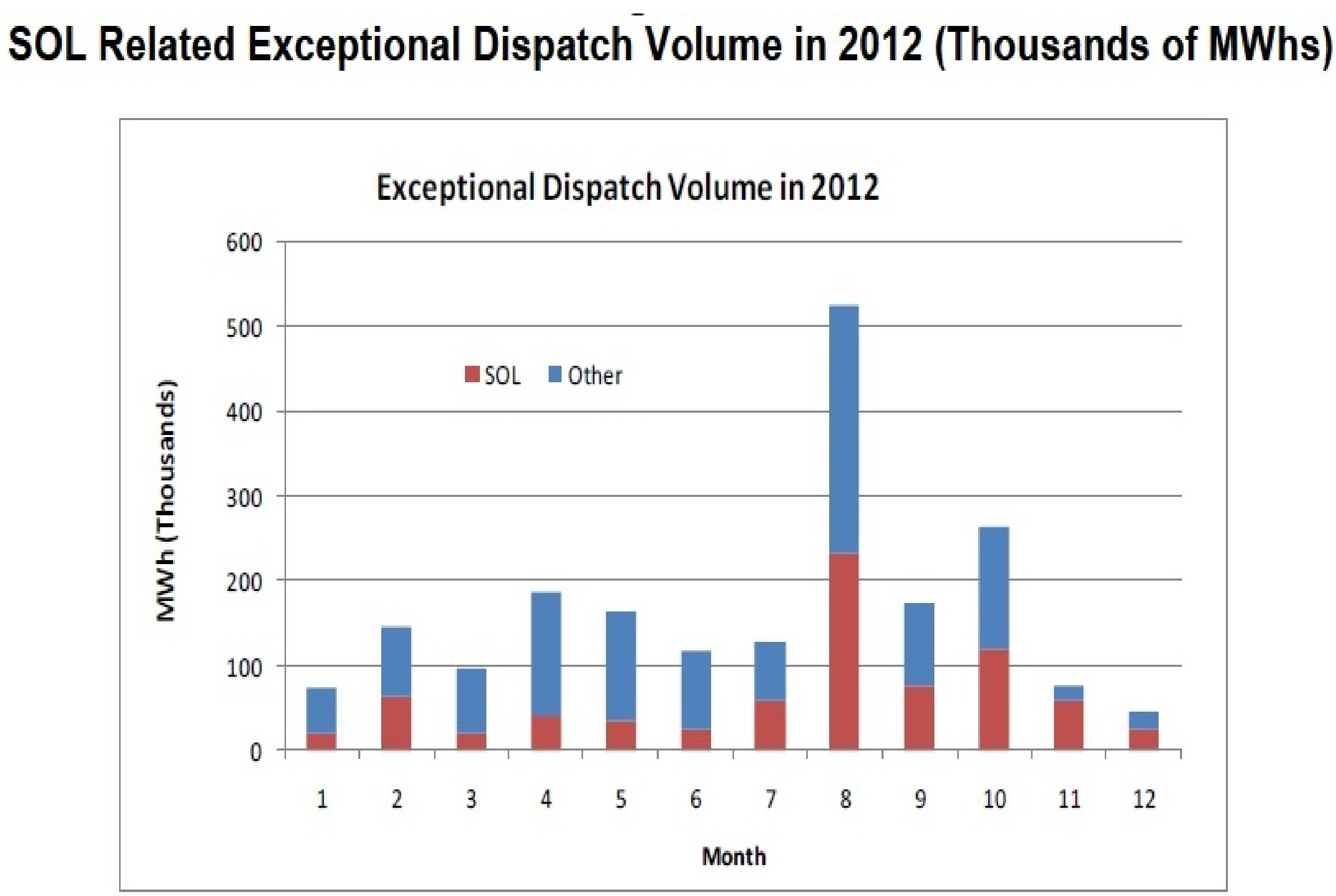


Figure 1.10: Monthly Exceptional Dispatch MWh by CAISO in 2012 (Source: CAISO).

data through 2013). In figures 1.14 and 1.13, respectively, we show the EIA estimates of the electricity generation by different fuels and by renewables only (excluding hydro and pumped storage), through 2015 for the USA. Figure 1.12 shows a steady rise in the renewable penetration. With increasing penetration of renewables, there will be less dispatchable thermal generation online. Therefore, the available ramp capacity for post-contingency actions will be more limiting. That is, the type of example illustrated in figure 1.8 will become more typical. In order to facilitate more renewables, we need newer dispatch methodologies, which the present work will focus upon.

1.2.3 Computational Challenge

The objective functions that we consider in the present work extend over a given time horizon and encode operating costs and constraints for a given device operating under a particular scenario. Hence even for a modest size network, the problem becomes quite challenging to solve. For a large network size (for example, the western interconnection, operated by WECC, the eastern interconnection, or the Texas grid operated by ERCOT

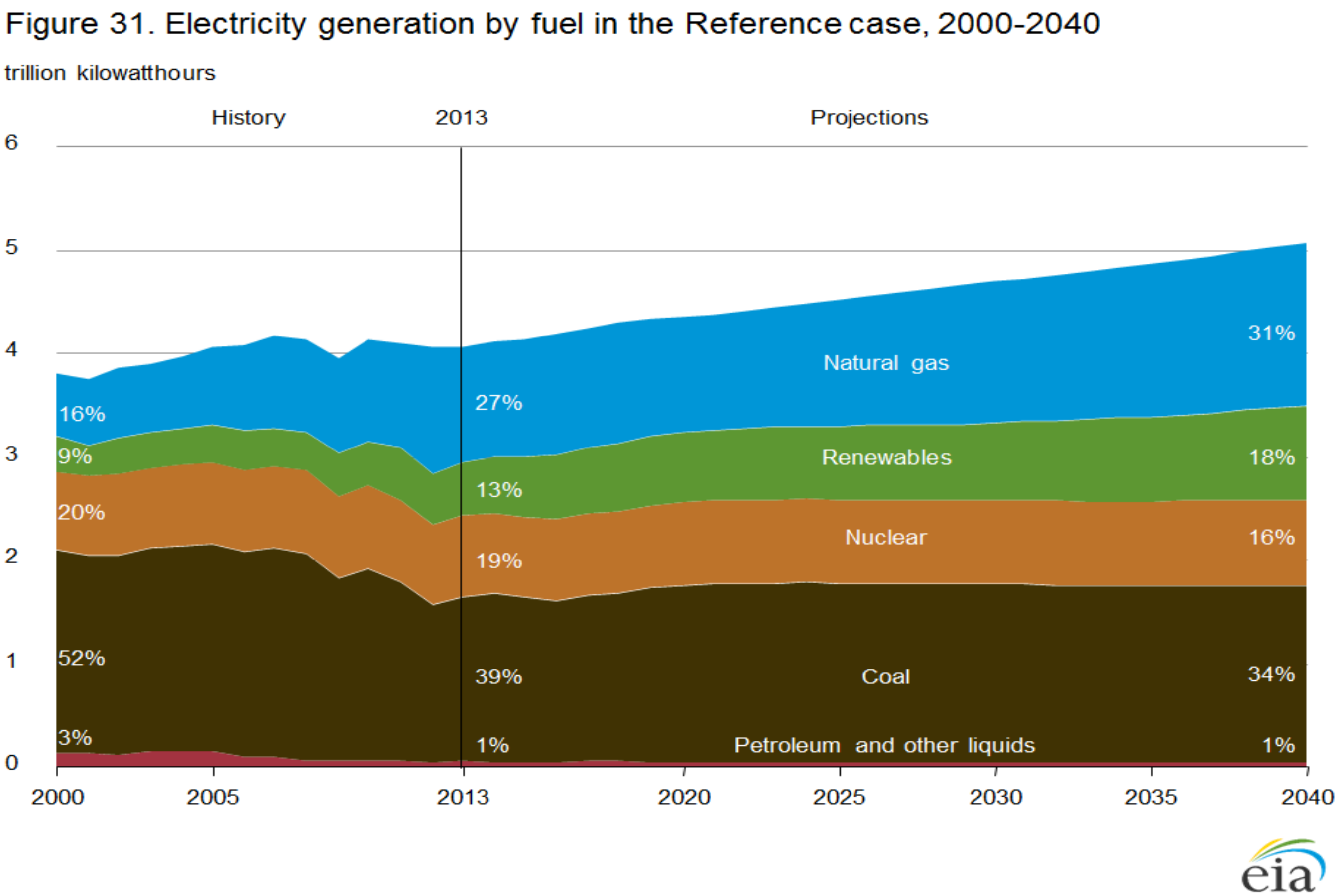


Figure 1.11: EIA Estimate of Electricity Generation till 2040 (Source:EIA).

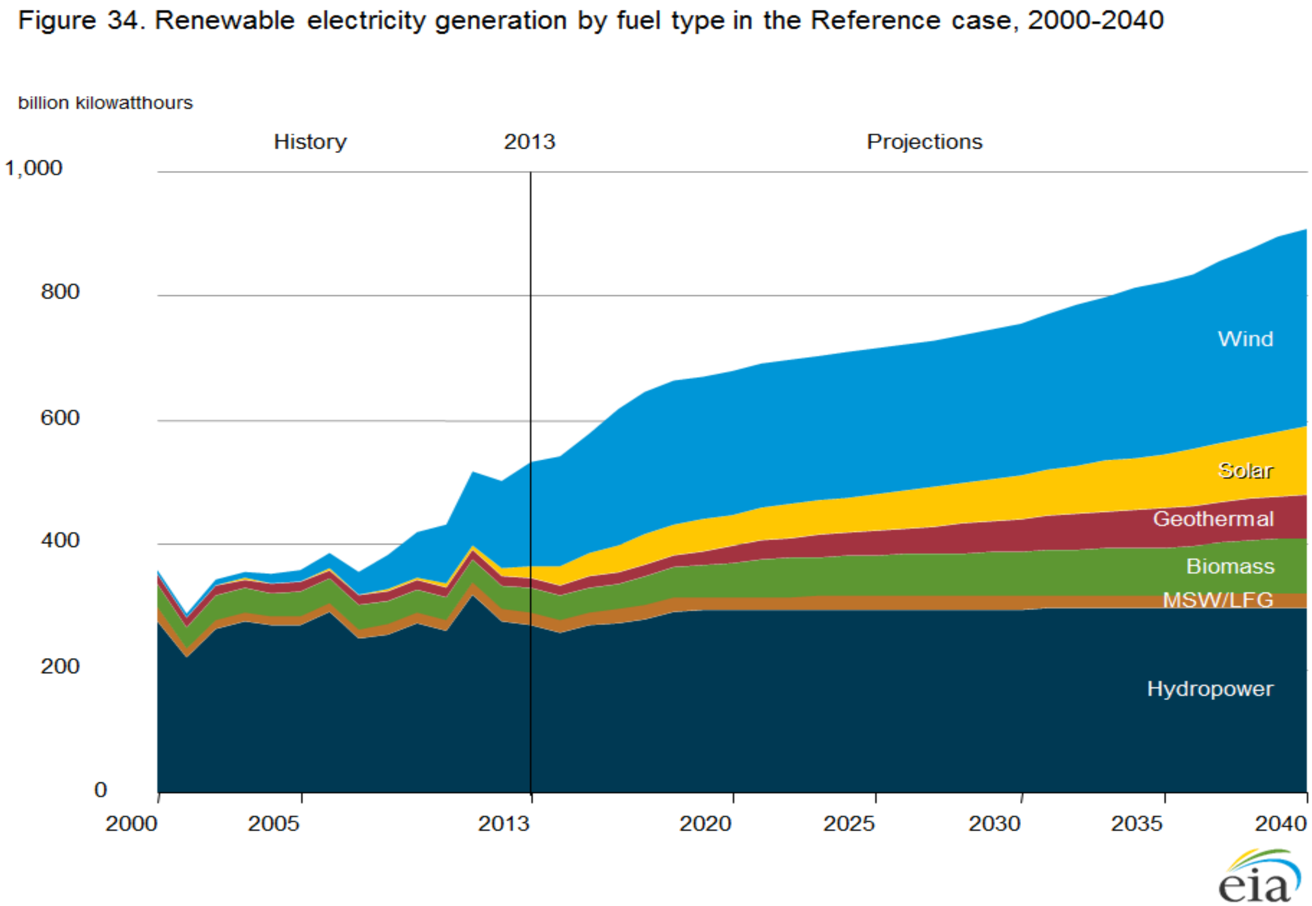


Figure 1.12: EIA Estimate of Renewable Electricity Generation till 2040 (Source:EIA).

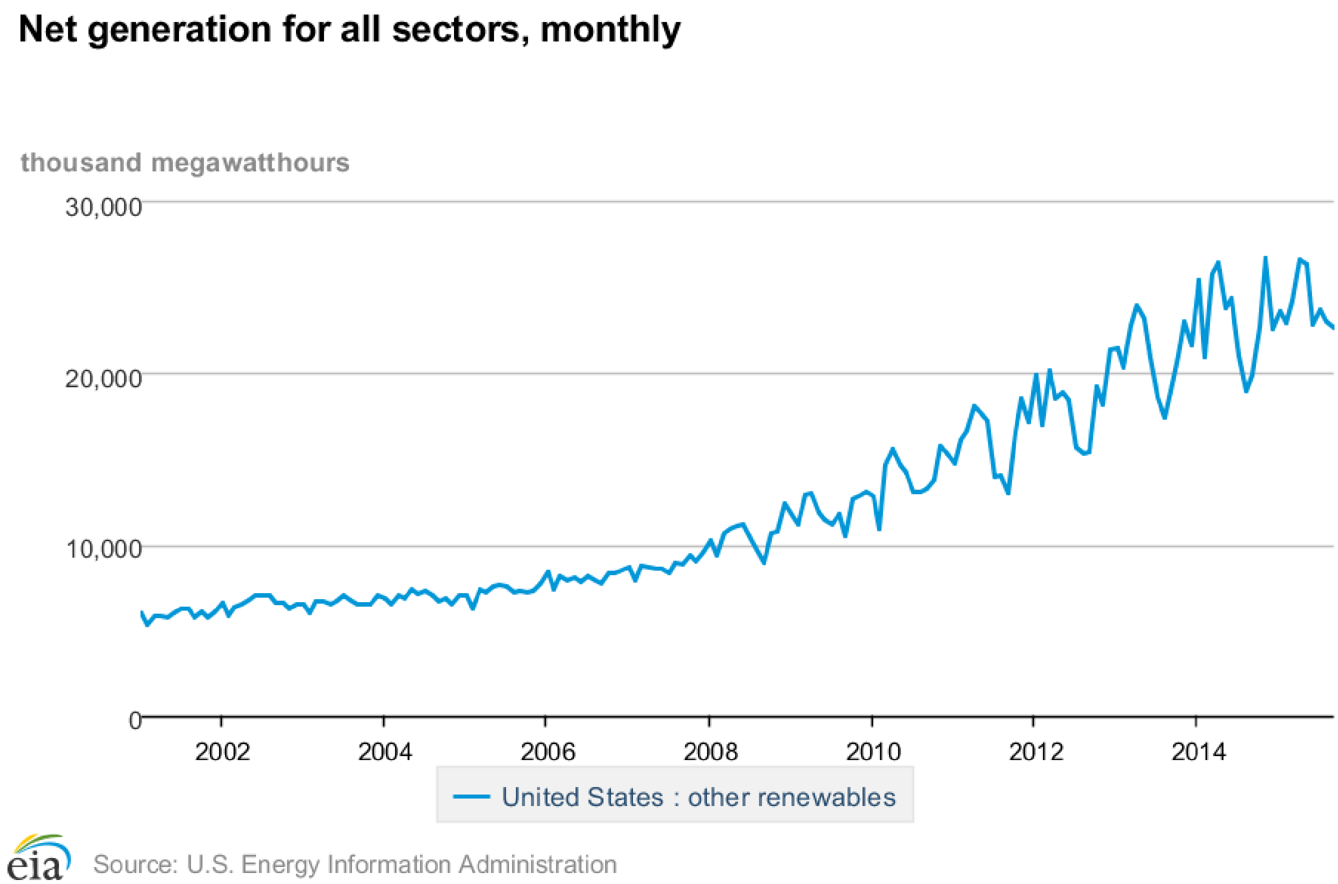


Figure 1.13: Electricity by Renewables till 2015 (Source:EIA).

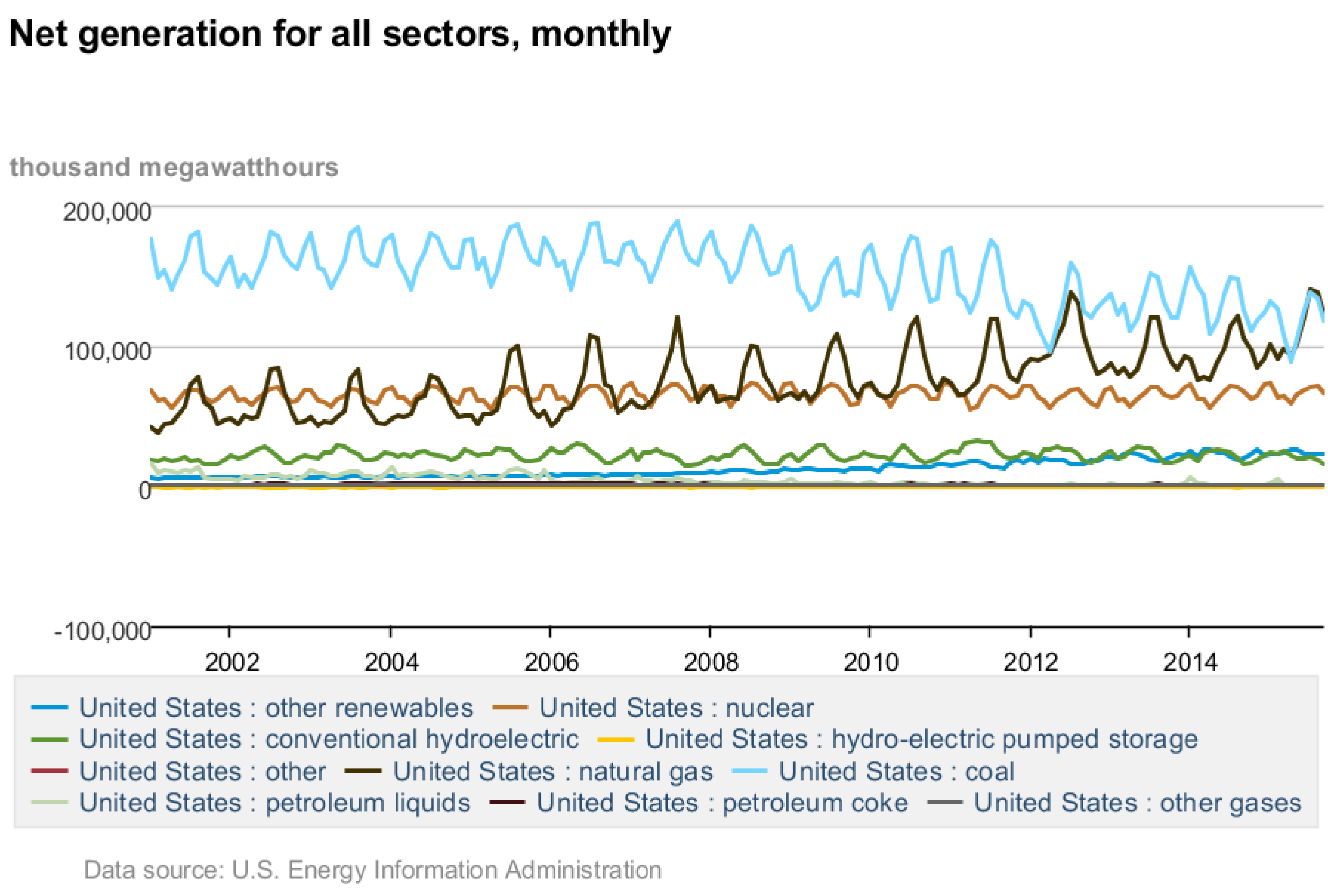


Figure 1.14: Electricity by Different Fuels till 2015 (Source:EIA).

(Electric Reliability Council of Texas) shown in figure 1.15), the whole problem becomes even larger. We therefore need an effective distributed algorithm or a combination of distributed algorithms to solve such a massive optimization problem.

The present research attempts to come up with algorithms and solution techniques in order to achieve post-fault redispatch of the system following an outage and also to ensure security of the system with respect to the next set of outages or contingencies only through computational means, instead of costly transmission upgrades or ad-hoc operator experience-based operational procedures. The goal is to minimize a composite cost function that includes the cost (and constraints) of nominal operation, as well as those associated with operation in any of the (adverse) scenarios. This results in a large optimization problem, since each variable in the network, namely, real power flow, is repeated $|\mathcal{L}|$ times, where $|\mathcal{L}|$ is the number of contingencies.

We use a suitably modified version of the proximal message passing algorithm in [234], combined with the auxiliary problem principle to solve this problem efficiently. This gives us a highly scalable and fine-grained distributed algorithm, which can ensure fast solution for big networks. There are other distributed algorithms, which have been explored and compared in the literature (See, for example, the recent work by Kargarian *et al* on the comparison of different distributed algorithms, as applied to solving the OPF problem [220]). The main reasons we use the above ones are assured convergence, less dependence on the nature of the objective function (other than it being convex, closed, and proper), and robustness.

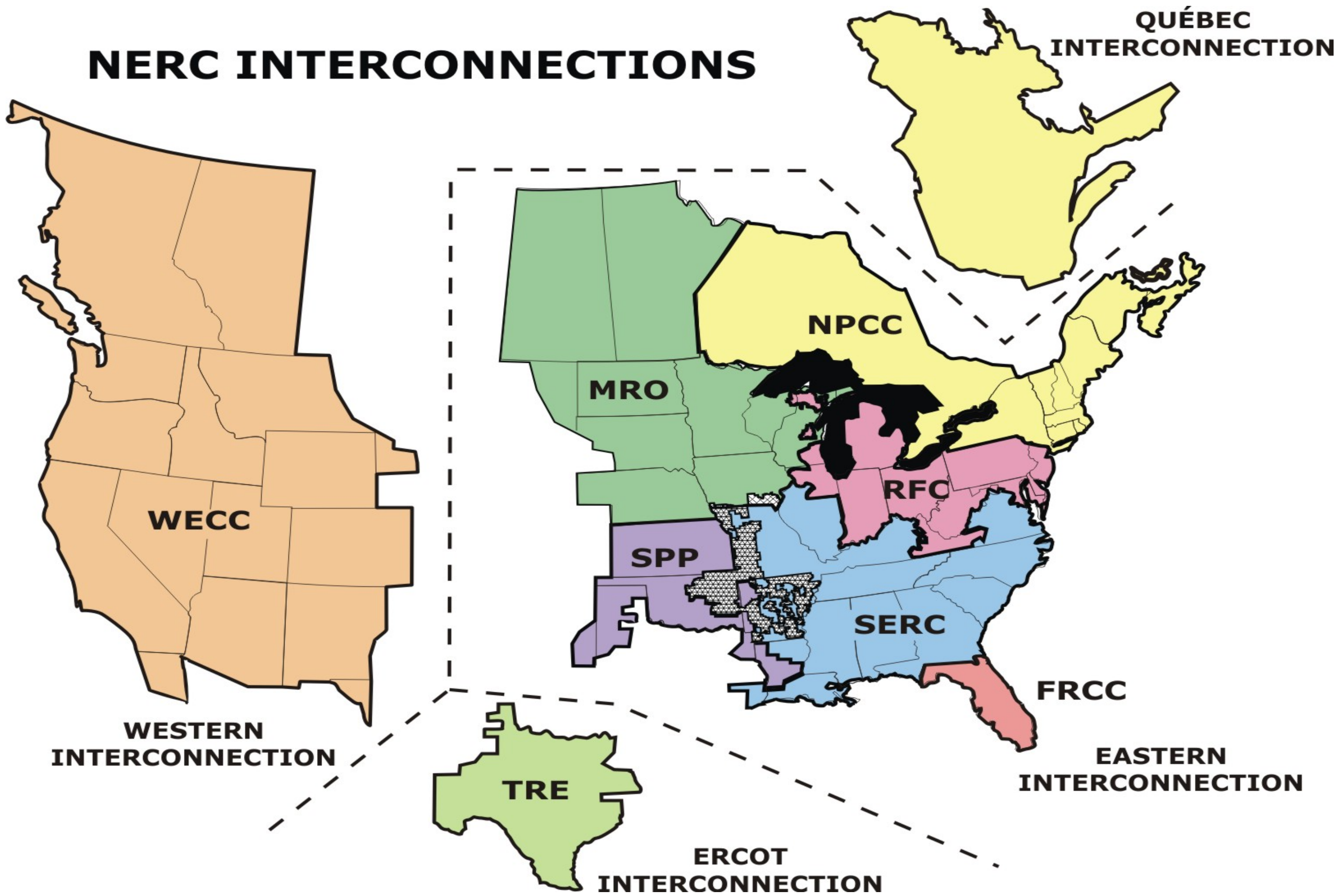


Figure 1.15: The Power Grid of the United States of America.

1.3 Problem Definition

After setting up the context and the background in sections 1.1 and 1.2, we are now ready to present a formal definition of the problem and the solution method. We will first take a look at a brief description of emergency transmission rates in the next section.

1.3.1 Emergency Transmission Rates

In this section, we'll discuss the idea of using different power ratings for transmission lines depending on the temperature rise and the capability of the system to restore itself following an actual contingency or any other event initiating a sudden change in the magnitude of line current (and hence power flow). Figure 1.16, which is based on the material presented in [269], shows three different ratings. The top most graph shows the step increase of current on a transmission line, following the outage of some other line or a major change in the system. The graph at the bottom shows the temperature of the line. As can be seen, there is a limit to the final temperature to be attained, and immediately following the step increase of current, the temperature of the line undergoes a transient phase. The details of the calculation of the temperature under steady case and transient case have been presented in the references [1], [380], while [45], [43], [8] explore the current-temperature relationship under stochastic ambient conditions. Reference [331] applies a similar concept to calculate the start-up costs of generators. In figure 1.16, initially, when I_i was flowing through the transmission line, the temperature, which was in equilibrium with the atmosphere was below the maximum that the line can withstand. When the post-fault current I_{f1} corresponds to the long term rating, the temperature after undergoing the transient approaches the max-

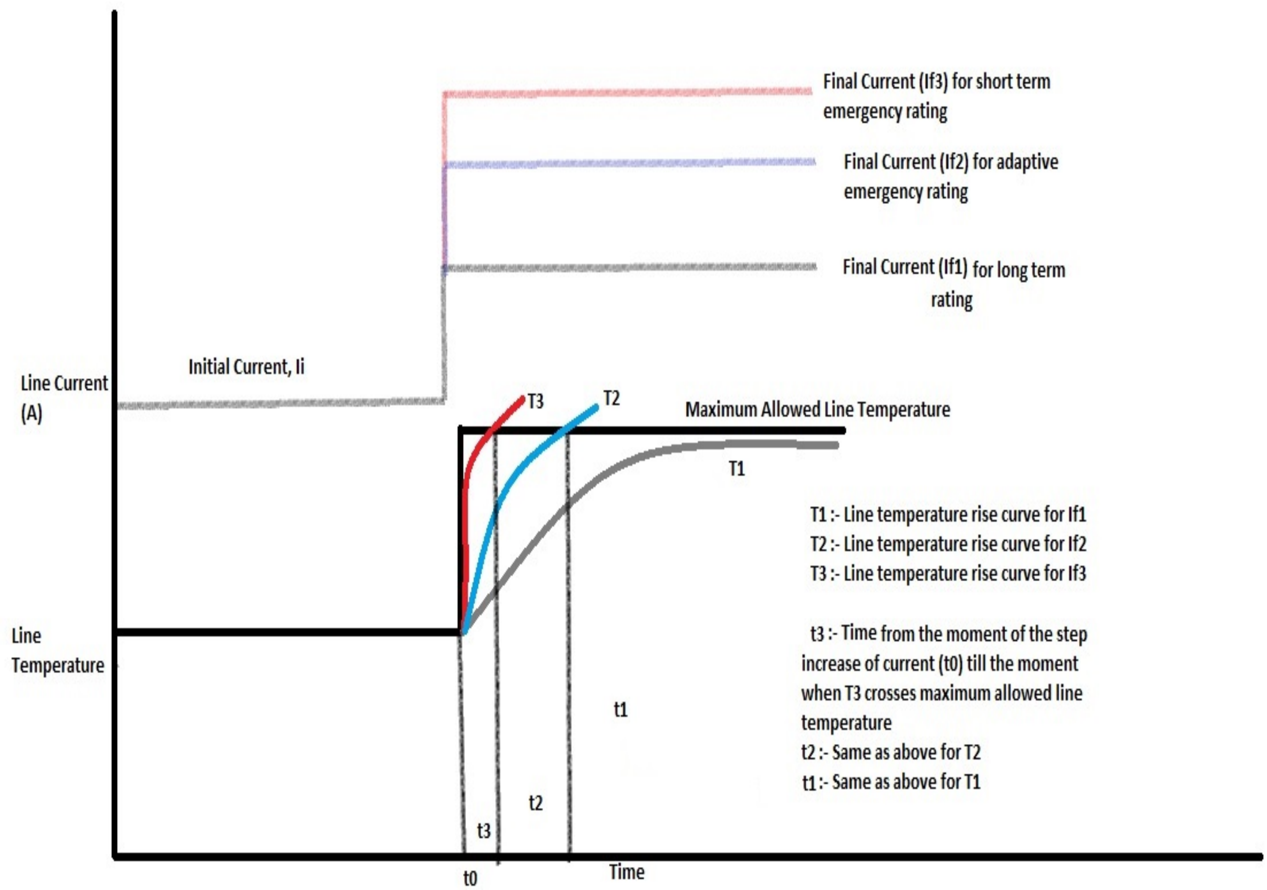


Figure 1.16: Line Temperature Rises and Different Ratings. Source: This figure is based on [ML09].

imum limit asymptotically. For practical purposes, we can think of the line reaching its maximum temperature after a long time, t_1. If, on the other hand, the power system has enough capability to redispatch itself fast enough, we can allow much higher post-fault current to flow corresponding to either the short term emergency rating (I_{f3} with the recovery scheme needed to start acting at or before t_3), or the adaptive emergency rating (I_{f2} with the recovery scheme needed to start acting at or before t_2) and our goal is to still nevertheless be able to guarantee that the system can be restored to security with respect to the next contingencies within a required time. In our new SCOPF scheme, we will make use of this idea to maximize the utilization of the existent transmission system.

1.3.2 Look-Ahead SCOPF (LASCOPF) Problem

In the present work, we consider a multiple dispatch interval security constrained optimal power flow, which we call the *Look Ahead Security Constrained Optimal Power Flow* (LASCOPF) problem, in which, at the beginning of each dispatch interval, we actually solve the SCOPF problem looking forward several dispatch intervals in the future and accounting for all the possible contingencies in those intervals. For each dispatch interval and each contingency and/or base-case scenario, we consider a power network in which devices are connected and there exists a set of *scenarios* — each corresponding to the failure and/or degradation of a set of devices — over which we must ensure efficient operation of the network. For each scenario, the scenario objective is to minimize the sum of the objective functions associated with that scenario for each device. We extend the application of the Proximal Message Passing algorithm from solving the standard static Optimal Power Flow (OPF) problem to solving the $(N-1)$ Security Constrained OPF (SCOPF). We then combine this approach with the Auxiliary Problem Principle (APP) to solve a multiple time horizon look-ahead dispatch problem, which considers the potential for multiple sequential outage of transmission lines and/or Generators. Our ultimate motive is to develop a mathematical model for the problem which considers the line temperature rise trajectory following an outage and attempts to limit the maximum value of this temperature within some safe limit decided by the continuous/short term rating of the line. In order to accurately solve this problem, we actually need a full AC-OPF representation. Nevertheless, as a starting point, we will look at a DC-OPF model of the same, which provides valuable insights to the actual and more accurate problem.

1.3.3 Stages of Development of the Look-Ahead SCOPF (LASCOPF) Problem

After having a brief discussion about modeling different devices, we derive the Mathematical Formulations for several different scenarios gradually increasing the level of complexity in our model. For almost all the cases we show the formulations in both the "Angles Eliminated" or "Power Transfer Distribution Factor (PTDF)" as well as the "Angles Included" versions, whereby we represent the different real power flows on the transmission lines in terms of the real powers injected at the buses and the voltage phase angles at the buses with respect to a system slack bus, respectively. We then reformulate all the preceding cases into a different framework, the so-called $\mathfrak{DTN}$ (Devices-Terminals-Nets) Formulation, which is particularly suitable for the ADMM (Alternating Direction Method of Multipliers) [49] based Proximal Message Passing algorithm to be applied to the problems. Thereafter, we derive the Proximal Message Passing Algorithm for the different scenarios discussed previously. We finally discuss the results of some simulation studies conducted on some simple and the IEEE test systems.

For simplicity, we consider only DC power flows in this work. The extension to AC power flow, which we will need to model line temperature rise explicitly and accurately, involves applying the AC-OPF model from [15], [242], [335], [334] and [243] to each scenario and requiring that the phase angles of a given device are equal across all scenarios in the first time period. We will present some highlights of the mathematical formulation for the AC-OPF problem in appendix A. After having presented an introduction to our work, in this chapter, we will next take a brief look at the body of literature, in the next chapter.

Chapter 2

Literature Survey and Related Previous Work

2.1 Literature Survey on Electrical Power Systems and Optimal Power Flow

As nicely stated by Grainger and Stevenson, in their classic text *Power System Analysis* [175], *"Economic operation is very important for a power system to return a profit on the capital invested. Rates fixed by regulatory bodies and the importance of conservation of fuel place pressure on power companies to achieve maximum possible efficiency..."*. These sentences adequately emphasize the importance and the associated challenges of solving the problem of economic operation of power systems.The Optimal Power Flow (OPF) problem is at the heart of every kind of Power Systems planning and operations activities. The OPF attempts to solve the power generation scheduling problem of generators, so that the supply meets the demand, while satisfying the constraints of line flow limits, generating limits of generators etc. at the minimum possible cost [194]. It has been studied for more than half a century now, started by the work of Carpentier [67] in 1962. Early works summarizing the state of art for OPF and Economic Dispatch are [180] and [81]. Some of the recent references that provide a good summary of the historical development of the problem are [61] by Cain, O'Neill, and Castillo, and [280, 280] by Momoh, El-Hawary, & Adapa. The references cited there also provides good insights into formulation and modeling particularly of the ACOPF. The OPF is an extremely hard problem to solve, owing to the typical size of the power network to which it is applied, and also to the high degree of non-convexity of the problem (which is caused primarily by the presence of cross product terms in the voltage variables & trigonometric terms [372] and also due to the non-convexity in the cost-functions of generators, mainly caused by valve switchings [376]). Over the past five and half decades, several authors have suggested different approaches to solve this extremely hard problem, some of which are due to Torres-Quintana [349, 348, 350], Wang *et al.* [368], Dommel *et al.* [106] etc.

2.2 Literature Survey on Security Constrained Optimal Power Flow

Security Constrained OPF (SCOPF) takes the concept of OPF one step further and ensures an optimal dispatch of the power system such that even in the event of an outage no line will be overloaded past its emergency ratings. The pioneering work on the Security Constrained OPF (SCOPF) was done by Stott *et al* in [340]. Chiang *et al.* applied a variation of the interior point algorithm in their work [79], which takes advantage of the structure of the problem. The ideas of applying distributed algorithms to solving SCOPF problems appears in the recent works of Phan *et al.* [301], Chakrabarti *et al.* [71], Liu *et al.* [254] etc. In the context of being able to solve look-ahead SCOPF (which we will present in this work) where the load demands change over future intervals, it is very crucial, as a simplifying assumption, that the PTDFs or, shift factors stay constant despite variation of loading (as long as we are solving the linearized or DC-SCOPF). Evidence for this assumption is presented in [19]. Equally important for solving multiple time interval look-ahead SCOPF is the concept of "Model Predictive Control" or "Receding Horizon Control," some of the good references of which are [307, 80, 184, 341, 235].

2.3 Literature Survey on Mathematical Optimization

Mathematical Optimization is a huge field of study, that can be broadly divided into "Convex Optimization" and "Non-Convex Optimization." In this work, we will mostly be concerned with Convex Optimization. Significant work on Convex Analysis, which is the theoretical field on which Convex Optimization is based is by Rockafellar [318]. The classic references for Convex Optimization are the works by Boyd & Vandenberghe [50], Baldick [17], Dattoro [98], Bertsekas [41, 42] etc.

2.4 Literature Survey on Distributed Optimization Algorithms

The Alternating Direction Method of Multipliers first originated in the 1970s with the works of Mercier-Gabay [163], Glowinski-Marocco [168], Gabay [161], Fortin and Glowinski [152] etc., followed by those during the 90's which include the works by Eckstein and Fukushima [129], Eckstein [122]. Gabay and Eckstein-Bertsekas first offered the convergence properties of the ADMM algorithm in their works [162] and [126], respectively. In that same work, Gabay also showed that there exists a more generalized method called the Douglas-Rachford method of splitting monotone operators [112], [251], of which ADMM is a special case. ADMM came into being as a result of the amalgamation of two previously proposed algorithms: Dual Decomposition (which is, in turn, based on the Dual Ascent algorithm) and the Method of Multipliers for solving augmented Lagrangian problems in a distributed manner (which is also similar in flavor to the Gauss-Siedel iterative method). ADMM combines the robustness of the augmented Lagrangian and the method of multipliers with the distributed computational capability of dual decomposition. Hestenes in [185] and Powell in [306] first proposed the augmented Lagrangian and the method of multipliers in the 1960s. Dual Decomposition also made its appearance in the 1960s in the works of Everett [136], Dantzig-Wolfe [96], Benders [30], and Dantzig [94].

The classic references for the Auxiliary Problem Principle (APP) are [84], [85]. APP was applied previously in OPF and transmission planning problems in [223], [18], [118], [24] etc.

Some previous works on Proximal Message Passing (PMP) and the prox-project algorithm include [354], [314], [282], [281], [68] etc. Combining these fields gives rise to the Distributed Computational methods for OPF problem and significant references in that field include works by Baldick and Kim [223], [224]. In [234], [254] and [71], Kraning *et al.*, Liu *et al.* and Chakrabarti *et al.*, respectively, applied the Proximal Message Passing algorithm to solving the standard Static Optimal Power Flow (OPF) Problem and solving the $(N-1)$ Security Constrained OPF (SCOPF). Our present work will build up based on the works of all these previous authors' works. In the next chapter, we will present the system of notations and conventions, to set up the tone for presenting our work.

Chapter 3

Conventions for Notations and General Model

3.1 Model

[1]In the present work, we will be using variables and parameters that are shared between different components of a network, as well as among the different scenarios and dispatch time intervals. Therefore, we will be heavily overloading our symbols and notations. Hence, it is very important to set up a consistent system of notational and modeling convention in this chapter, so that we can use it throughout the rest of the work. We will extend the network model and overload the notation from [234] to handle scenario planning. Any power system network is made up of a finite set of *terminals* $\mathcal{T}$, a finite set of *devices* $\mathcal{D}$, and a finite set of *nets* (nodes or buses) $\mathcal{N}$. The set of devices $\mathcal{D}$ and nets $\mathcal{N}$ are partitions for the set of terminals (*i.e.*, each terminal is associated with exactly one device and one net). Let us now define the notations to be used throughout, with the above idea in mind.

[1]Parts of this chapter appear in the published papers, "Security Constrained Optimal Power Flow via Proximal Message Passing," "Toward Distributed/Decentralized DC Optimal Power Flow Implementation in Future Electric Power Systems," and "A Survey of Distributed Optimization and Control Algorithms for Electric Power Systems." The author of this treatise is the first author of the first paper, contributed section V, parts of sections IX and X of the second paper, and contributed parts of section III and V of the third paper.

3.1.1 Notations and Conventions

We have categorized the entities used in the subsequent formulations into four different groups: Sets, Elements, Indices and Parameters.

3.1.1.1 Sets

$\mathcal{D}$: Set of Devices

$\mathcal{T}$: Set of Terminals

$\mathcal{N}$: Set of Nets (that is, Buses, or Nodes)

The next three sets form partitions of the set of devices:

$G \subseteq \mathcal{D}$: Set of Generators

$T \subseteq \mathcal{D}$: Set of Transmission Lines

$L \subseteq \mathcal{D}$: Set of Loads

$\mathcal{L} = \{0, 1, 2, ..., |\mathcal{L}|\}$: Set of possible $(N-1)$ Contingencies. The element, 0 indicates the base case.

$\Omega = \{0, 1, 2, ..., |\Omega|\}$: Set of Dispatch intervals or, the net Dispatch Horizon under consideration. 0 indicates the first dispatch interval under consideration. It is to be noted that the first dispatch interval under consideration, for which the calculation is done is actually the forthcoming (upcoming) one. Hence, dispatch interval -1 (which is not in this set) is the current running one. Later, when we will consider the intra-dispatch interval variation of generation (or, equivalently, variation of demand, as well), we will be more interested, for the purposes of attaining supply-demand balance (and also, other inequality constraints, at least as a simplifying first step), in the ends of each dispatch interval. In those case, it is immaterial whether we consider duration of dispatch interval -1, its end, or the duration of dispatch interval 0, as the current time (as long as we have fast enough computational power and command and control at our disposal to perform the computation and implement the results by the end of the current dispatch interval, if we consider the current interval as 0). Table 3.1 establishes the convention for numbering the look-ahead as well as present dispatch intervals. In the table, s indicates the count of dispatch interval starting from the upcoming/forthcoming interval (which we will denote by the letter, τ. As we will soon see, τ will also serve the purpose of indexing the dispatch intervals, or, more precisely, it's $(\tau + s)$). Also, in the table, Γ_{RND}, Γ_{RSD}, and Γ_{MRD} has been introduced, which will be explained in chapter 6.

$\dagger$ will be used to denote the transpose of a vector or matrix.

3.1.1.2 Elements

t: Elements of $\mathcal{T}$

g: Elements of G

D: Elements of L

T: Elements of T

N: Elements of $\mathcal{N}$

3.1.1.3 Indices

i, j: Nets,

k: Terminals,

q: Generators,

r: Transmission Lines,

d: Loads,

c: Contingencies,

τ: Dispatch Intervals,

ν: Iteration count for ADMM/PMP algorithm,

μ_{APP}: Iteration count for APP algorithm

3.1.1.4 Parameters

$R_{T_r}, X_{T_r}, Z_{T_r} = R_{T_r} + (\sqrt{-1})X_{T_r}$: Resistance, Reactance and Impedance of the r^{th} Transmission Line.

$\alpha_{g_q}, \beta_{g_q}, \gamma_{g_q}$: Quadratic, Linear, and Constant Cost Co-efficients of the q^{th} Generator.

$C_{g_q}(.), f_{dev}(.)$ will be used to denote the cost function of the ${g_q}^{th}$ generator and that of a generic device, respectively, throughout. We will introduce the other cost functions in the appropriate sections.

$\overline{P}_{g_q}, \underline{P}_{g_q}, \overline{R}_{g_q}, \underline{R}_{g_q} (= -\overline{R}_{g_q}, usually), \overline{L}_{T_r}$ denote the maximum and minimum generating limits of generators, maximum ramp-up and ramp-down limits of generators, and power carrying capacity of transmission lines respectively.

The variables are the real power P and the bus angles θ (There are no bus angles for DC tie-lines). The following is the convention we follow in order to identify the associations of any particular variable to the sets:

$$(Variable^{(ContingencyIndex)(DispatchTime\#2)}_{Net/DeviceElement_{TerminalNumber}(c)(DispatchTime\#1)})^{(IterationCount)}.$$

The above notation refers to a variable associated with a particular terminal indexed by the $TerminalNumber$ of either a net or a device for the contingency scenario indexed by $ContingencyIndex$ during $DispatchTime\#2$, as estimated by a computing unit linked to $DispatchTime\#1$ and contingency scenario, c. Sometimes we will use the net number instead of the terminal number in the above convention, when we want to indicate several devices connected to a particular net. If it is part of an iterative algorithm, then the outermost superscript indicates the iteration count. Whenever a variable is boldface, one or more of the indices will be missing and that means the boldface variable is a vector each of whose components will have all or some of the missing indices (the components themselves can be vectors or scalars). When the variable is not bold-face and still some of the indices are missing, that means it is a scalar and the missing indices are either irrelevant or their values are implied from the context (See, section 3.1.1.5 for some exaples that clarify this notational convention). Also it is to be observed that since generators and loads are single terminal devices, it is not necessary to specify the terminals for these, unless absolutely required.

3.1.1.5 Examples

In order to understand and get familiar with the above system of notational convention, we will refer to figures 3.1 and 3.2 first and then present some examples. Figure 3.1

is a modified reproduction of figure 1.7 or figure 1.8 appearing earlier. Here, the terminals have been shown as the solid dots and the buses or the nets have been shown as dashed boxes. In figure 3.2, we have shown a bipartite graph corresponding to this simple power system, where we have classified all the devices (generators: g_1, g_2; transmission lines: T_1, T_2, T_3; loads: D_1, D_2) into nodes of one category and represented them as green colored boxes and the buses or nets (N_1 and N_2) as nodes of the second category and represented them as blue colored boxes. The terminals, which connect the devices to the respective nets have been shown as the edges. We have colored all the terminals connecting to the net N_1, black and those connecting to net N_2, red. We have labeled the terminals with the letters of same colors. The labeling convention for the terminals is that we first write the name of the terminal, followed by the notation for power at the device end of the terminal and that at the net end, seperated by a comma, within a set of parantheses. We now give several examples below:

1. $P^{(c)}_{N_{i t_k}}$ indicates the real power flowing out of the k^{th} terminal, which belongs to the i^{th} net, for the contingency scenario, (c).

2. $P^{(c)}_{T_{r t_k}}$ indicates the real power flowing out of the k^{th} terminal, which belongs to the r^{th} Transmission Line (Device), for the contingency scenario, (c).

3. $P^{(c)(\tau_1)}_{g_{q t_k}(\tau)}$ indicates the real power output of the k^{th} terminal, which belongs to the q^{th} Generator (Device) during dispatch interval, τ_1, as assumed by a computing unit for

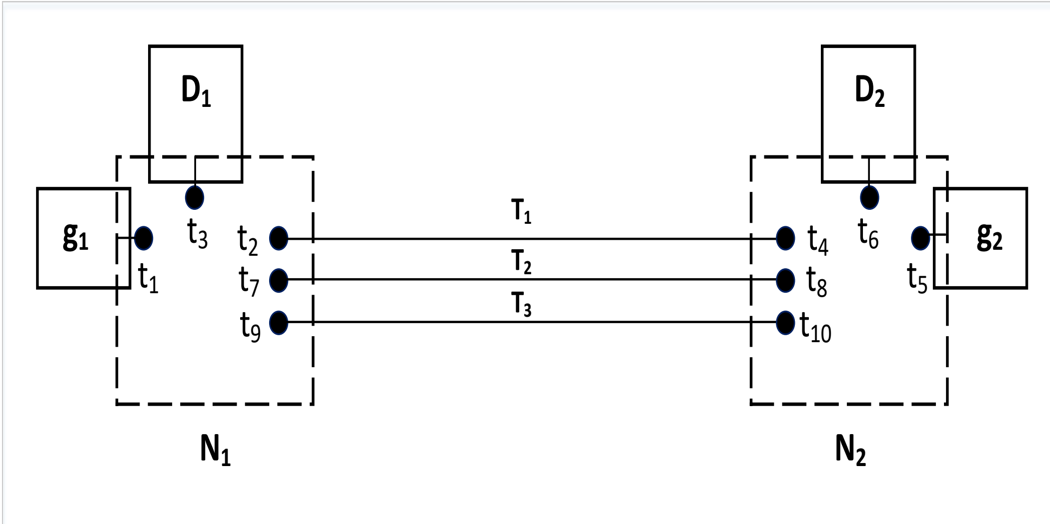


Figure 3.1: Two Bus System (modified).

dispatch interval, τ for the contingency scenario, (c).

4. $(P^{(c)(\tau_1)}_{g_{q_{t_k}}(\tau)})^{(\nu)}$ indicates the real power output of the k^{th} terminal, which belongs to the q^{th} Generator (Device) during dispatch interval, τ_1, as assumed by a computing unit for dispatch interval, τ for the contingency scenario, (c), in the ν-th iteration.

5. $\mathbf{P}^{(\tau_1)}_{(\tau)} = [P^{(\tau_1)}_{g_1(\tau)}, P^{(\tau_1)}_{g_2(\tau)}, ..., P^{(\tau_1)}_{g_{|G|}(\tau)}]^{\dagger}$ represents the vector of MW outputs of all generators in the network, in the dispatch interval, τ_1, as assumed by the computing unit associated with solving the OPF problem for the dispatch time interval τ. It is to be observed that even though some indices of each component are missing, they are all scalars. This means that indices such as contingency index, iteration count etc. are

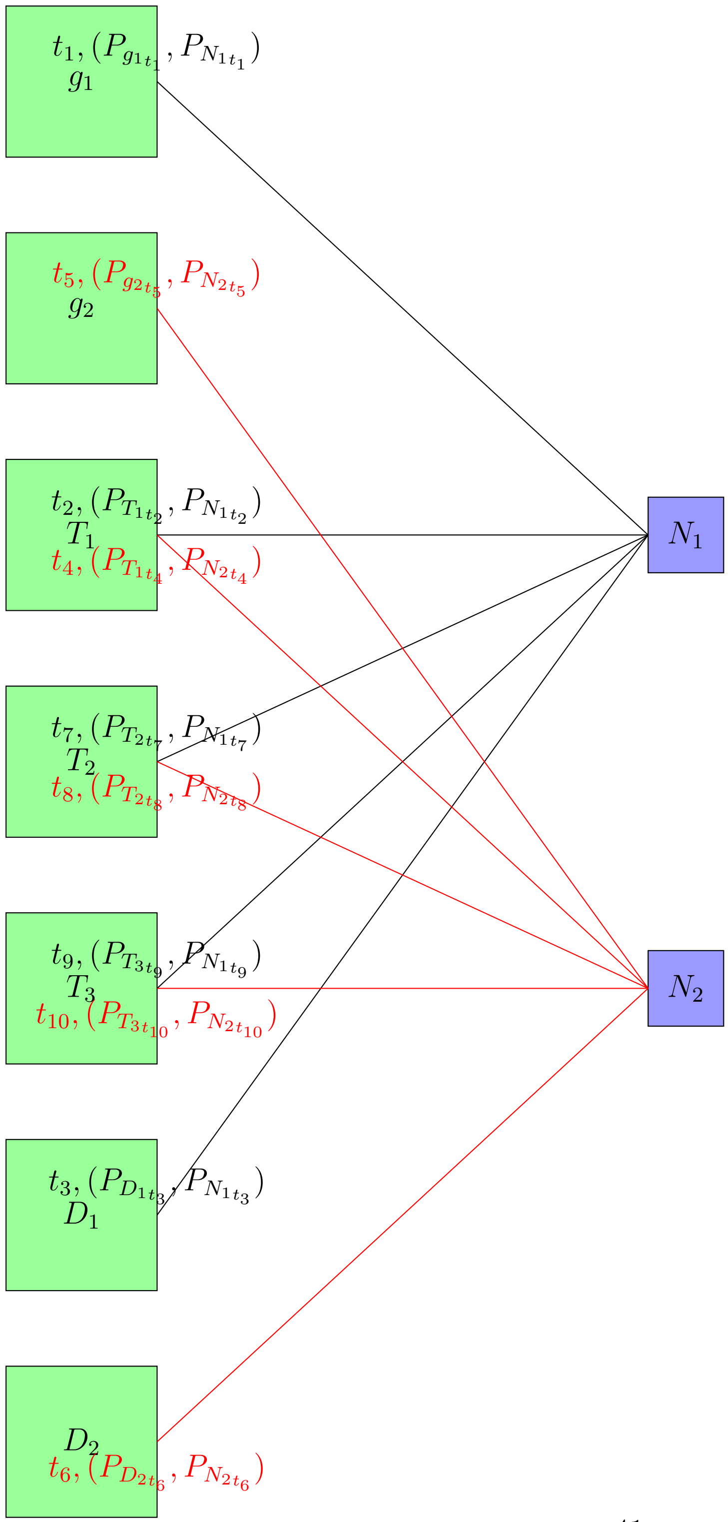


Figure 3.2: Bipartite Graph for the two bus Power System.

either not relevant or are implied in the context.

6. $\mathbf{P}_{(\tau)} = [\mathbf{P}_{(\tau)}^{(\tau_1)}, \mathbf{P}_{(\tau)}^{(\tau_2)}, ...]^\dagger$ represents the vector of MW outputs of all generators in the network, in the dispatch intervals, $\tau_1, \tau_2, ...$, and so on, as assumed by the computing unit associated with solving the OPF problem for the dispatch time interval τ. In this case, since each component of the vector is also vector, the hidden index will depend on the context and each component vector also needs to be defined.

7. Associated with each terminal $t_k \in \mathcal{T}$ is a set of *contingency power schedules* $\mathbf{P_{t_k}} = [\mathbf{P_{t_k}^{(0)}}, \mathbf{P_{t_k}^{(1)}}, \ldots, \mathbf{P_{t_k}^{(|\mathcal{L}|)}}]$, where $\mathbf{P_{t_k}^{(c)}} \in \mathbf{R}^{|\Omega|}$ for a given time horizon $|\Omega|$, if we use $\mathbf{P_{t_k}^{(c)}}$ to denote a vector that has the different schedules for different dispatch intervals as its components and $|\mathcal{L}|$ is the number of scenarios (we use scenario 0 to signify the nominal network state). We will refer to a contingency power schedule as simply a *power schedule* when it is clear from context which scenario it corresponds to. We can also reference contingency power schedules by their corresponding devices and nets, *i.e.*, $\mathbf{P_{T_r}} = [\mathbf{P_{T_r}^{(0)}}, \mathbf{P_{T_r}^{(1)}}, \ldots, \mathbf{P_{T_r}^{(|\mathcal{L}|)}}]$ refers to the set of contingency power schedules of the terminals associated with device (in this case, transmission line) $T_r \in \mathcal{D}$, where $\mathbf{P_{T_r}^{(c)}} = [\mathbf{P_{t_k}^{(c)}} \mid t_k \in T_r]$ is the set of terminal power schedules associated with device T_r in scenario c, and with a similar notation for nets.

Refering back to figure 3.2, we can now see that each of $\mathbf{P_{N_1}}$ and $\mathbf{P_{N_2}}$ is a five-dimensional vector, with entries, $[P_{N_{1t_1}}, P_{N_{1t_2}}, P_{N_{1t_3}}, P_{N_{1t_7}}, P_{N_{1t_9}}]$ and $[P_{N_{2t_4}}, P_{N_{2t_5}}, P_{N_{2t_6}}, P_{N_{2t_8}}, P_{N_{2t_{10}}}]$

respectively. There will be similar bipartite graphs in the different contingency scenarios and dispatch time intervals. We will take a deeper look at those in the subsequent chapters and appendices.

Although we are here using generators as an example device, the same remarks are valid for any of the three types of devices. Associated with each device $g_q \in \mathcal{D}$ is a family of objective functions $\{f_{g_q}(P_{g_q}^{(c)}) : \mathbf{R}^{|g_q|} \to \mathbf{R} \cup \{+\infty\} \mid c = 0, \ldots, |\mathcal{L}|\}$, where we set $f_{g_q}(P_{g_q}^{(c)}) = \infty$ to encode constraint violation on the power schedules of the device in scenario c (ie if the powers are such that they fall outside the generation limit or such that they violate the ramp rate constraints) and $|g_q|$ denotes the number of terminals of the device g_q, which for generators or loads is 1 and for transmission lines is 2. When $f_{g_q}(P_{g_q}^{(c)}) < \infty$, we say that $P_{g_q}^{(c)}$ are feasible power schedules for device g_q in scenario c and we interpret $f_{g_q}(P_{g_q}^{(c)})$ as the cost to device g_q of operating according to power schedule $P_{g_q}^{(c)}$ in scenario c.

In SCOPF, we wish to determine the set of power plans P that minimize the total cost of operating the network

$$\begin{array}{ll} \text{minimize} & f(\mathbf{P}) = \sum_{dev \in \mathcal{D}} f_{dev}(P_{dev}) \\ \text{subject to} & \hat{\mathbf{P}} = \mathbf{0}, \\ & \underline{\mathbf{P}} \leq \mathbf{P} \leq \overline{\mathbf{P}} \end{array} \tag{3.1}$$

where $\hat{\mathbf{P}} \in \mathbf{R}^{|\mathcal{N}|(|\mathcal{L}|+1)}$ is the vector of average power flow on each net in the network for all contingencies. The first constraint states that a net (or bus) is a lossless power carrier and that $(1/|N_i|) \sum_{t_k \in N_i} P_{t_k}^{(c)} = 0$ for each $N_i \in \mathcal{N}$ and $c \in \mathcal{L}$. The second constraint captures all the inequalities, such as line flow limits, ramp-rate constraints, generator maximum and minimum generating limits etc. The reason we use the same set of variables for the inequality constraints (which pertain to the devices) as that for the equality constraints (which pertain to the nodes) is because both of them are, at the fundamental level, variables associated with the terminals.

3.2 Device examples

We give some device examples in this section. For simplicity, we model all devices with convex functions. For each device, we also describe how its model behaves under a contingency. We only consider contingencies (or scenarios) in which a device fails entirely.

Fixed load. A fixed load not involved in any contingency must always have its demand met. It has the constraint that:

$$P_{\text{load}}^{(c)} = -P_{D_d}, \qquad c = 0, \ldots, |\mathcal{L}|,$$

where P_{D_d} is the fixed demand of the load and $P_{\text{load}}^{(c)}$ indicates the injection due to the load. P_{D_d} is typically a positive value, meaning consumption, so that injection $P_{\text{load}}^{(c)}$ into the network at its terminal is negative.

However, if the load demand changes during contingency c', we modify the constraint to read

$$P_{\text{load}}^{(c)} = \begin{cases} -P_{D_d} & c \neq c' \\ -P'_{D_d} & c = c', \end{cases}$$

where P'_{D_d} is the new load demand during contingency c'. This includes the possibility that the load is dropped during contingency c'. If the load is dropped, we set $-P'_{D_d} = 0$. Throughout this work, for the sign of the power injection, we use negative sign to indicate that the injection is actually a withdrawal of power from a node (either to the rest of the network, or to the load), and positive sign for indicating that the injection actually refers to pushing power into a node (from the rest of the network, or a generator, or, sometimes a load).

Generator. A generator has power generation limits,

$$\underline{P}_{g_q} \leq P_{g_q}^{(c)(\tau)} \leq \overline{P}_{g_q}, \quad c = 0, \ldots, |\mathcal{L}|$$

and ramp limits,

$$\underline{R}_{g_q} \leq P_{g_q}^{(\tau+1)} - P_{g_q}^{(\tau)} \leq \overline{R}_{g_q}, \quad \tau = 0, \ldots, |\Omega|$$

The operation limits require the powers in each of our $|\mathcal{L}|$ contingencies to obey the above inequality.

The operating cost function is usually modeled as a linear/increasing quadratic or cubic/piecewise linear function, depending on the requirement and computational power available. In this work, we will model it as an increasing quadratic function as shown below:

$$C_{g_q}(P_{g_q}) = \alpha_{g_q}(P_{g_q})^2 + \beta_{g_q}(P_{g_q}) + \gamma_{g_q}.$$

The most important feature about the *operating cost functions* are that, they are convex, increasing functions that give the cost of operating the generator at a particular power.

If a generator is dropped during contingency c', we add the constraint that $P_{g_q}^{(c')} = 0$. If fuel cost should rise sharply during contingency c', we modify $C_{g_q}^{(c')}$ to account for the new cost. If the generator performance deteriorates during contingency c', then we replace the power generation limit for $P_{g_q}^{(c')}$ with

$$\underline{P}_{g_q}^{new} \leq P_{g_q}^{(c')} \leq \overline{P}_{g_q}^{new},$$

where $\underline{P}_{g_q}^{new}$ and $\overline{P}_{g_q}^{new}$ are the new, degraded limits of the generator, g_q.

Transmission line. While using the DC approximation for a transmission line, a capacitated transmission line must satisfy the power flow equation

$$P_{T_{rt_1}} + P_{T_{rt_2}} = \ell(P_{T_{rt_1}}, P_{T_{rt_2}}),$$

and a line capacity

$$|P_{T_{rt_1}}| \leq \overline{L}_{T_r}, \qquad |P_{T_{rt_2}}| \leq \overline{L}_{T_r},$$

where $\overline{L}_{T_r}$ is the line capacity and $\ell(P_{T_{rt_1}}, P_{T_{rt_2}})$ describes the loss on the line. In the case of a lossless transmission line $\ell(P_{T_{rt_1}}, P_{T_{rt_2}}) = 0$. Although, for the DC-OPF, we assume the lines to be lossless (ie $P_{T_{rt_1}} = -P_{T_{rt_2}}$), in order to account for the temperature in our work to follow, we need to consider an estimate of the Ohmic losses, which results in the rise of line temperature and we will calculate that as a fixed fraction α' of the square of the line power flow (the fraction will vary from one line to the other).

$$\ell(P_{T_{rt_1}}, P_{T_{rt_2}}) = \alpha'(P_{T_{rt_1}})^2,$$

For AC-OPF, the calculation of losses is more straight forward. Given a transmission line with resistance R_{T_r} and the line current as I_{T_r} the loss is

$$\ell(P_{T_{rt_1}}, P_{T_{rt_2}}) = (R_{T_r})(I_{T_r})^2,$$

Note that a transmission line is a two terminal device and $P_{T_{rt_1}}$ and $P_{T_{rt_2}}$ are the power flows at the two ends of the transmission line, from the line into the rest of the network.

If a line is dropped in contingency c', then the powers no longer satisfy the power flow equations. Instead, we replace the power flow equations with

$$P^{(c')}_{T_{rt_1}} = P^{(c')}_{T_{rt_2}} = 0.$$

If the line capacity is degraded in contingency c', then we replace the line capacity constraint for $P_{T_{rt_1}}^{(c')}$ and $P_{T_{rt_2}}^{(c')}$ with

$$|P_{T_{rt_1}}^{(c')}| \leq \overline{L}_{T_r}^{(c')}, \qquad |P_{T_{rt_2}}^{(c')}| \leq \overline{L}_{T_r}^{(c')},$$

where $\overline{L}_{T_r}^{(c')}$ is the new, degraded capacity. With the notational conventions thus stated, we will now describe the conventional/traditional mathematical formulations of the problems in the next chapter.

Table 3.1: Convention for Numbering the Dispatch Intervals

Interval	Name	Definition
$s = -1$	Last Interval	Latest interval for which the dispatch targets have already been calculated.
$s = 0$	Upcoming/Forthcoming Interval	The earliest future interval for which we are calculating the dispatch targets. Note: The first contingency can occur in this interval, and worst case is assumed to be where contingency occurs at the beginning of this interval.
$s = 1$	First Interval after Upcoming	First interval after the upcoming interval for which we can respond to a contingency.
$s = \Gamma_{RND}$	Return to Normal Dispatch Interval	Interval by the end of which all flows are at or below normal or long-term ratings.
$s = \Gamma_{MRD} = \Gamma_{RND} + \Gamma_{RSD}$	Return to Secure Dispatch Interval	Interval by the end of which all flows are at or below normal ratings and the system is secure with respect to any subsequent contingency.

Note: We are not assuming contingency "occurs" in subsequent intervals. Rather, we want to be secure with respect to the next contingency.

Chapter 4

Look-Ahead SCOPF: Conventional Formulation

[1]In this chapter, we consider the conventional or traditional OPF formulation. The ultimate aim is to come up with a formulation that incorporates a look-ahead dispatch having the flavor of the Model Predictive Control or a Receding Horizon Control ([272], [370], [261], [29], [307], [80], [379]), which considers the possibility of multiple sequential outages "looking forward" several dispatch intervals and ensuring security with respect to all of them. The calculation "rolls forward" at the end of each dispatch interval, starting the calculation again and continues repeatedly in this manner. Specifically, for the look-ahead dispatch model, we consider the trajectory of temperature rise on the transmission lines following an outage and control generation in order to limit the rise.

In the next few subsections, we build up this model systematically and gradually in several steps, increasing the model complexity starting from a very simple case and culminating in our proposed model. We will be following the common system of notations that we have introduced in section 3.1.1. We will also be introducing some more notations, which

[1]Parts of this chapter appear in the published papers, "Security Constrained Optimal Power Flow via Proximal Message Passing," "Toward Distributed/Decentralized DC Optimal Power Flow Implementation in Future Electric Power Systems," and "A Survey of Distributed Optimization and Control Algorithms for Electric Power Systems." The author of this treatise is the first author of the first paper, contributed section V, parts of sections IX and X of the second paper, and contributed parts of section III and V of the third paper.

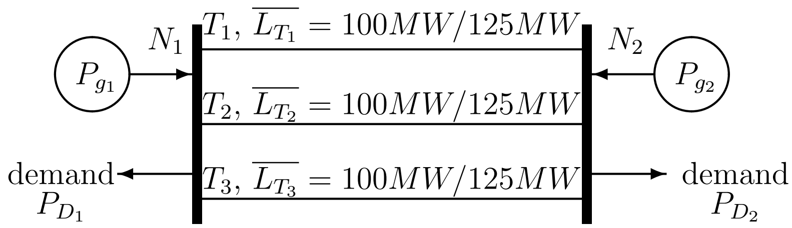


Figure 4.1: Three transmission lines, a, b, c, joining two buses, 1, and 2.

are more specific to only some particular sections.

Figure 4.1 shows the example two bus system that we have introduced in the previous chapters. We will be using this system to derive the formulations for our problems and then generalize those.

4.1 Conventional Formulation of OPF

First consider the simplest possible case of a two bus system shown in the figure. There are three transmission lines in between with equal impedances and equal power carrying capabilities (100 MW: Continuous or, Long-Term and 125 MW: Short Term). Assume that the marginal costs for generating power are \$10/MWh and \$20/MWh for Generators 1 and 2, respectively, and that the marginal costs stay the same for the entire range of generating capability. We will try to solve a very simple look-ahead dispatch calculation for such a system. We will build the model in steps, increasing the complexity and adding new constraints for making the analysis more realistic at each step. We will also be generalizing the analysis to arbitrary systems at each step. This approach will help us gain insight as to what exactly is going on physically, as well as understand how the mathematical model is fitting into the pertinent situation.

First assume that it is certain that no line outages are ever going to happen and so, we can transfer a maximum of 300 MW from Bus-1 to Bus-2. Let the power demands of the loads D_2 and D_1 to be served be 500 MW and 300 MW respectively and also let both the generators have very high or infinite generating capability. So, the Generator-1 will be generating $P_{g_1} = P_{D_1} + 300 = 600MW$, Generator-2 will be generating the remaining 500-300 i.e. 200 MW, and the LMPs (LMP: Locational Marginal Prices for Electricity, the incremental cost for generating or providing power at a particular bus, the price at which electricity is traded in the wholesale market) at buses 1 and 2 will be \$10/MWh and \$20/MWh, respectively.

4.1.1 Outline of the Formulation for OPF

From the foregoing investigation, we can generalize the above situation to more complicated multi-bus and multi-generator-load systems as follows:

$$\textbf{Objective}:$$

$$\min_{\textit{Power Generation}} \text{Total Generation Cost} \tag{4.1a}$$

$$\textbf{Constraints}:$$

$$\text{Supply-Demand Balance} \tag{4.1b}$$

$$\text{Line Power Flow Limit} \tag{4.1c}$$

$$\text{Generation Limit} \tag{4.1d}$$

4.1.2 Illustration of Simplest Two Bus System for OPF

The problem for this system, which has been just discussed previously, can be mathematically formulated as follows (the Angles Eliminated Formulation), solving which will actually give the same results we previously examined intuitively.

$$\min_{P_{g_1},P_{g_2}} C_{g_1}(P_{g_1}) + C_{g_2}(P_{g_2}) \tag{4.2a}$$

$$\text{Subject to: } P_{g_1} + P_{g_2} = P_{D_1} + P_{D_2} \tag{4.2b}$$

$$|P_{g_1} - P_{D_1}| \leq \overline{L}_{line} \tag{4.2c}$$

$$|P_{g_2} - P_{D_2}| \leq \overline{L}_{line}, \text{ Redundant Constraint} \tag{4.2d}$$

where, P_{g_1} and P_{g_2} are real powers produced by Generators-1 and 2 respectively (Decision Variables) and $\overline{L}_{line}$ is the maximum power transfer capability of the lines from Bus-1 to Bus-2, which is the sum total of all the three transmission lines between the two buses. The above formulation doesn't take into account the bus voltage angles and so it is the "angles eliminated" or "power transfer distribution factors" formulation of DC OPF. Let's now look at the other alternative formulation, taking into account the bus voltage angles.

$$\min_{P_{g_1},P_{g_2},\theta} C_{g_1}(P_{g_1}) + C_{g_2}(P_{g_2}) \tag{4.3a}$$

$$\text{Subject to: } P_{g_1} - P_{D_1} = -\frac{\theta_{N_2}}{X_{line}} \tag{4.3b}$$

$$P_{g_2} - P_{D_2} = \frac{\theta_{N_2}}{X_{line}} \tag{4.3c}$$

$$\left|\frac{\theta_{N_2}}{X_{line}}\right| \leq \overline{L}_{line} \tag{4.3d}$$

$$\left|-\frac{\theta_{N_2}}{X_{line}}\right| \leq \overline{L}_{line}, \text{ Redundant Constraint} \tag{4.3e}$$

In the above formulation, $Z_{line} = R_{line} + (\sqrt{-1})X_{line}$ and R_{line} is very small in typical transmission systems, $\mathbf{V_{N_1}} = V_{N_1}\angle 0$, $\mathbf{V_{N_2}} = V_{N_2}\angle\theta_{N_2}$, $V_{N_1} = V_{N_2} = 1\ pu$ and θ_{N_2} is very small. These are essentially the assumptions for the DC (linearized) OPF.

4.2 Conventional Formulation of $(N-1)$ SCOPF

Moving on to the next level of sophistication in our model, we will now consider incorporation of $(N-1)$ security constraints in our formulation. In the simple two bus case discussed previously, let's, for the sake of simplicity, initially assume that the short-term ratings of lines are the same as the long-term. We are again assuming equal impedances of the lines and identical maximum line flow limits of 100 MW for each line. In order to be secure with respect to a single contingency, now only 200 MW can be transferred from Bus-1 to Bus-2 and so, the remaining (500-200) i.e. 300 MW of load has to be provided by Generator-2. Therefore, being secure with respect to the single outage of the line amounts to evaluating how much power can be transferred if the line is taken out of service, without violating the limit constraints and actually allowing that very quantity of power to flow during pre-contingency.

4.2.1 Outline of the Formulation for SCOPF

Hence, the generalized SCOPF can be written down as:

$$\textbf{Objective}:$$

$$\min_{Power\ Generation} \quad \text{Total Generation Cost} \tag{4.4a}$$

Constraints :

$$\text{Supply-Demand Balance} \tag{4.4b}$$

$$\text{Generation Limit} \tag{4.4c}$$

$$\text{Line Power Flow Limit (Base Case)} \tag{4.4d}$$

$$\boxed{\textbf{New Constraint} \rightarrow \text{Line Power Flow Limit (Contingency Cases)}} \tag{4.4e}$$

The added constraint compared to (4.1) has been boxed, and we will follow the same convention in each successive model formulation.

4.2.2 $(N-1)$ SCOPF for the Two Bus System: Equal Capacities and Line Impedances

Refering back to the foregoing discussion, the SCOPF can be formulated as (Angles Eliminated Formulation, the redundant constraints have not been included in the next three formulations, the primary reason being that, we are not considering line losses yet):

$$\min_{P_{g_1},P_{g_2}} C_{g_1}(P_{g_1}) + C_{g_2}(P_{g_2}) \tag{4.5a}$$

$$\text{Subject to: } P_{g_1} + P_{g_2} = P_{D_1} + P_{D_2} \tag{4.5b}$$

$$\frac{P_{g_1} - P_{D_1}}{3} \leq 100 \tag{4.5c}$$

$$\frac{P_{g_1} - P_{D_1}}{2} \leq 100 \tag{4.5d}$$

The first inequality corresponds to the base case and the next one corresponds to the outage of a single line from the system. It should be observed that for this particular case, the constraint (4.5c) is redundant and can be omitted, thereby reducing some computational burden possibly. But in general, this is not true. Nevertheless, such kind of observation gives us important clues regarding how we can identify and get rid of redundant constraints from our problem formulation, without affecting the solution.

4.2.3 $(N-1)$ SCOPF for the Two Bus System: Unequal Capacities and Equal Line Impedances

Our next scenario to consider would be the case in which the capacities of the lines are different and they are $a \geq b \geq c$, for T_1, T_2, T_3 respectively, but the impedances are identical. Now, the SCOPF is formulated as (Angles Eliminated Formulation):

$$\min_{P_{g_1}, P_{g_2}} C_{g_1}(P_{g_1}) + C_{g_2}(P_{g_2}) \tag{4.6a}$$

$$\text{Subject to: } P_{g_1} + P_{g_2} = P_{D_1} + P_{D_2} \tag{4.6b}$$

$$\text{Base Case: } \frac{P_{g_1} - P_{D_1}}{3} \leq a \tag{4.6c}$$

$$\frac{P_{g_1} - P_{D_1}}{3} \leq b \tag{4.6d}$$

$$\frac{P_{g_1} - P_{D_1}}{3} \leq c \tag{4.6e}$$

$$\text{Outage of “}T_3\text{” : } \frac{P_{g_1} - P_{D_1}}{2} \leq a \tag{4.6f}$$

$$\frac{P_{g_1} - P_{D_1}}{2} \leq b \tag{4.6g}$$

$$\text{Outage of “}T_2\text{” : } \frac{P_{g_1} - P_{D_1}}{2} \leq a \tag{4.6h}$$

$$\frac{P_{g_1} - P_{D_1}}{2} \leq c \tag{4.6i}$$

$$\text{Outage of “}T_1\text{” : } \frac{P_{g_1} - P_{D_1}}{2} \leq b \tag{4.6j}$$

$$\frac{P_{g_1} - P_{D_1}}{2} \leq c \tag{4.6k}$$

4.2.4 $(N-1)$ SCOPF for the Two Bus System: Unequal Capacities and Line Impedances

The next step will be to consider the situation where the impedances as well as the capacities are different and let's assume that the impedances (in this case, the reactances, since we are neglecting the resistances) are X_1, X_2, X_3 of lines T_1, T_2, T_3, respectively, with capacities a, b, c, respectively). Now, the SCOPF is formulated as (Angles Eliminated Formulation):

$$\min_{P_{g_1}, P_{g_2}} C_{g_1}(P_{g_1}) + C_{g_2}(P_{g_2}) \tag{4.7a}$$

$$\text{Subject to: } P_{g_1} + P_{g_2} = P_{D_1} + P_{D_2} \tag{4.7b}$$

$$\text{Base Case: } \frac{P_{g_1} - P_{D_1}}{(X_1X_2 + X_2X_3 + X_3X_1)}(X_2X_3) \leq a \tag{4.7c}$$

$$\frac{P_{g_1} - P_{D_1}}{(X_1X_2 + X_2X_3 + X_3X_1)}(X_1X_3) \leq b \tag{4.7d}$$

$$\frac{P_{g_1} - P_{D_1}}{(X_1X_2 + X_2X_3 + X_3X_1)}(X_1X_2) \leq c \tag{4.7e}$$

$$\text{Outage of "}T_3\text{" : } \frac{P_{g_1} - P_{D_1}}{(X_1 + X_2)}(X_2) \leq a \tag{4.7f}$$

$$\frac{P_{g_1} - P_{D_1}}{(X_1 + X_2)}(X_1) \leq b \tag{4.7g}$$

$$\text{Outage of "}T_2\text{" : } \frac{P_{g_1} - P_{D_1}}{(X_1 + X_3)}(X_3) \leq a \tag{4.7h}$$

$$\frac{P_{g_1} - P_{D_1}}{(X_1 + X_3)}(X_1) \leq c \tag{4.7i}$$

$$\text{Outage of "}T_1\text{" : } \frac{P_{g_1} - P_{D_1}}{(X_3 + X_2)}(X_3) \leq b \tag{4.7j}$$

$$\frac{P_{g_1} - P_{D_1}}{(X_3 + X_2)}(X_2) \leq c \tag{4.7k}$$

4.3 Conventional Formulation of Look-Ahead SCOPF to Track Demand Variation

Now that we have taken care of the SCOPF problem in its static form, we will next focus our attention on the dynamic aspects of the problem. Specifically, we will try to develop the mathematical model for a look-ahead dispatch calculation that considers several future dispatch intervals at the onset of each current dispatch interval and takes into account the possible variations of operating parameters across different scenarios represented in different future dispatch intervals, so that at each interval the entire system is secure. At the end of each current dispatch interval, the calculation "rolls forward" and the whole look-ahead calculation is repeated.

Our primary motivation and interest is to examine the possibility of being able to represent, within the optimization framework, the post-contingency restoration of the system to security with respect to the next possible failure(s). We will again start with a very simple case of look-ahead dispatch calculation: that of variation of load across different dispatch intervals and incorporation of ramp rate constraints within the dispatch model to meet the changing load demand requirements. It is to be noted that, initially, we will not consider post-contingency restoration to security.

Let's imagine the situation in which three consecutive forthcoming dispatch intervals are considered and the load D_2 to be supplied during those intervals, takes values, respectively 290 MW, 300 MW, and 300 MW. Lets also assume that initially the system is operated such that it is secure with respect to a single line contingency and also, that each line has a maximum short-term thermal limit of 125 MW. We are, for the sake of simplicity, assuming that the line impedances and capacities are equal for the simplest two bus system that we use for the purpose of illustration. Let's also assume that each of the generators can ramp up at a maximum rate of 5 MW/dispatch interval. Thus, Generator-1 should, in the first interval, be producing 300+245=545 MW and Generator-2 should be producing 290-245=45 MW, so that, in the next interval, both can be ramped up by 5 MW each and be able to meet the increased demand of 300 MW. Observe here that, although we could have transferred 250 MW from Bus-1 to Bus-2 in the dispatch interval-1, we actually transferred only 245 MW. This is because, otherwise, Generator-2 would have been generating 40 MW and it would have been impossible to ramp it up by 10 MW by the next dispatch interval to meet 300 MW and we would have to resort to load shedding.

4.3.1 Outline of the Formulation for LASCOPF to Track Demand Variation

The generalized version of the above problem for a multi-time horizon, arbitrary network is as follows:

$$\textbf{Objective}:$$

$$\min_{Power\ Generation} \text{Total Generation Cost over } (|\Omega|+1) \text{ dispatch intervals} \tag{4.8a}$$

$$\textbf{Constraints}\ (\forall\tau):$$

$$Supply - Demand\ Balance \tag{4.8b}$$

$$\text{Generation Limit} \tag{4.8c}$$

$$\text{Line Power Flow Limit (Base Case)} \tag{4.8d}$$

$$\text{Line Power Flow Limit (Contingency Cases)} \tag{4.8e}$$

$$\boxed{\textbf{New Constraint} \rightarrow Generator\ Ramp\ Rate\ Limits\ from\ \tau\ to\ (\tau+1)} \tag{4.8f}$$

We now present pictorially the coarse grained distribution of the computation of the above problem. (For the details, the reader is directed to chapter 5). Refering to figure 4.2, the different overlapping circles here represent the different dispatch intervals. For this particular illustrative example (andalso the next one), we will consider that the value of τ is equal to 1, which means that the upcoming interval is (according to the convention introduced in table 3.1) given by $\tau = \tau + s = \tau + 0 = 1$, the next interval will be $\tau = \tau + s = \tau + 1 = 2$, and so on. The dashed circles for $\tau = 0 (= \tau + s = \tau + (-1))$ and $\tau = 4$ represent, respectively,

- The dispatch interval for the present, for which the Look-Ahead SCOPF problem was already solved and the results are known.
- The dispatch interval following the last future concerned time horizon.

We will distribute the SCOPF across each dispatch interval and then for each interval exchange messages regarding the current estimate or "belief" about the optimal values of the decision variables for the immediately preceding and succeeding intervals, eventually attempting to achieve a consensus between those. (We will be doing this through the application of the Auxiliary Problem Principle (APP) [84], [85], [223], [18], [118], [24]).

In the figure, the values shown at the two sides of the arrows are those "beliefs" among which we are trying to achieve consensus. We follow the same convention as introduced in section 3.1.1.

Hence, now there will be some consensus constraints.

For example, $\mathbf{P^{(1)}_{(1)}} = \mathbf{P^{(1)}_{(2)}}$, $\mathbf{P^{(3)}_{(2)}} = \mathbf{P^{(3)}_{(3)}}$ etc.

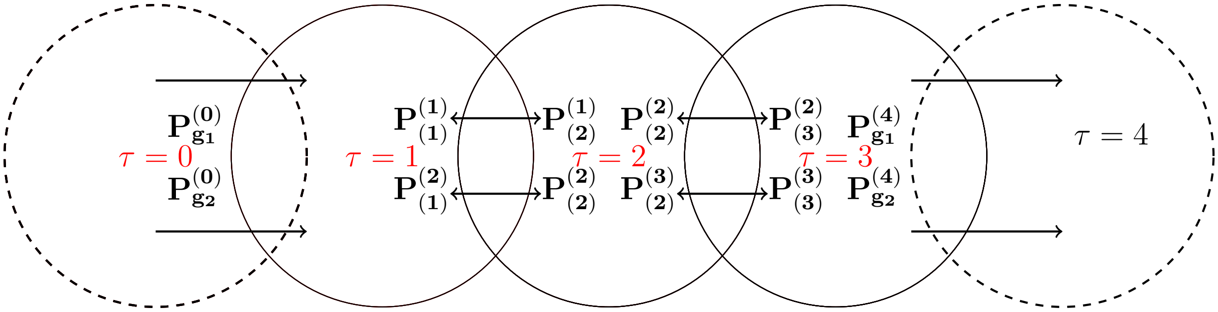


Figure 4.2: Schematic for Look-Ahead SCOPF for Demand Variation.

4.3.2 Simplest Case of Demand Variation for Two Bus System

The above problem can be mathematically expressed as follows (Angles Eliminated Formulation, henceforth, for the purpose of illustration, the two bus example problem will always be formulated in the angles eliminated format and the redundant constraints will not be included there. This is just for convenience, but all of the above are taken care of in the generalized formulation):

$$\min_{P_{g_1}{}^{(\tau)}, P_{g_2}{}^{(\tau)}} \sum_{\tau=1}^{3} C_{g_1}(P_{g_1}^{(\tau)}) + C_{g_2}(P_{g_2}^{(\tau)}) \tag{4.9a}$$

$$\textbf{Subject to} : \forall \tau \in \{1, 2, 3\}$$

$$P_{g_1}^{(\tau)} + P_{g_2}^{(\tau)} = P_{D_1}^{(\tau)} + P_{D_1}^{(\tau)} \tag{4.9b}$$

$$\frac{P_{g_1}^{(\tau)} - P_{D_1}^{(\tau)}}{3} \leq 100 \tag{4.9c}$$

$$\frac{P_{g_1}^{(\tau)} - P_{D_1}^{(\tau)}}{2} \leq 125 \tag{4.9d}$$

$$\underline{R}_{g_2} \leq P_{g_2}^{(\tau+1)} - P_{g_2}^{(\tau)} \leq \overline{R}_{g_2} \tag{4.9e}$$

$$\underline{R}_{g_1} \leq P_{g_1}^{(\tau+1)} - P_{g_1}^{(\tau)} \leq \overline{R}_{g_1} \tag{4.9f}$$

4.4 Look-Ahead SCOPF for Ensuring Security with respect to Next Outages in one Dispatch Interval

From the knowledge that we have gained so far, we will now move on to the next level where we will be thinking about the look-ahead dispatch for restoration to secure operation with respect to the next possible set of contingencies, where it is assumed that an outage can occur in upcoming dispatch interval and we would be secure with respect to that contingency (i.e. even if in the event of the contingency happening, no line is overloaded past emergency, but heating begins as early as "immediately"). Right now, for the sake of simplicity and illustration, we are assuming that it is possible reduce the flow on a line from its short term rating to a level that restores security in just one further dispatch interval, but subsequently, we will consider the situation where it is done rather gradually, over several intervals because of generation ramping limits. Restoration to security in several dispatch intervals is useful when the line ratings are based on thermal considerations (for short or medium length lines).

In order to illustrate and gradually build up the generalized mathematical model, let's consider the following scenario of the two bus system described earlier with three lines, each having continuous rating of 100 MW and short time rating of 125 MW and with load demands of $D_1 = 300MW$ and $D_2 = 500MW$, respectively. Let's consider the dispatch in several intervals. In the first dispatch interval, in order to be secure with respect to a single contingency (outage of a line), the maximum amount of power that can be transferred from bus-1 to bus-2 is 250 MW. If, now, anywhere in the first dispatch interval, one of the lines actually goes out of service, then only 125 MW can be securely transferred from bus 1 to bus 2, without exceeding short term ratings. To restore security within one further dispatch interval, Generator-1 should have enough ramp-down capability to reduce its generation by

125 MW (from 300+250 i.e. 550 MW to 300+125 i.e. 425 MW) and Generator-2 should have enough ramp-up capability to increase its generation by 125 MW (from 500-250 i.e 250 MW to 500-125 i.e. 375 MW). Now, suppose that the maximum ramp-down capability of Generator-1 and the maximum ramp-up capability of Generator-2 are respectively 100 MW and 70 MW. This means, Generator-2 can only increase its output, following the outage from interval-1 to interval-2 by 70 MW, which means Generator-1 reduces its production by 70 MW, in order to maintain power balance and in doing so, maintains a flow of 125 MW from bus-1 to bus-2 in the second dispatch interval, following outage of one transmission line. So, in the first dispatch interval, the flow from bus-1 to bus-2 was 125+70 i.e. 195 MW and so, the production from Generator-1 was 300+195 i.e 495 MW and production from Generator-2 was 500-195 i.e. 305 MW. That is, requiring restoration to security, together with ramp constraints, required flows on lines to be reduced well below the level needed for $N-1$ security alone.

Now, if the line impedances and power carrying capabilities are different, then for the first dispatch interval, the constraints will consist of a base-case and three different potential contingencies. But, for the second dispatch interval, there will be three distinct base-cases corresponding to outages of each of the lines and for each of those three, there will be two more contingency constraints corresponding to the outage of each of the remaining two lines, one at a time. Obviously, the one contingency constraint that is most stringent will be the dominating one and the other will be redundant. So, there will be a total of $3 \times 2 + 3 + 1$ i.e. 10 base case/contingency groups of constraints in the two dispatch intervals. So, if there are m possible $(N-1)$ contingency scenarios, the total number of base-case/contingency groups of constraints to be considered will be $m \times (m-1) + m + 1$ i.e. $m^2 + 1$. If contingencies of

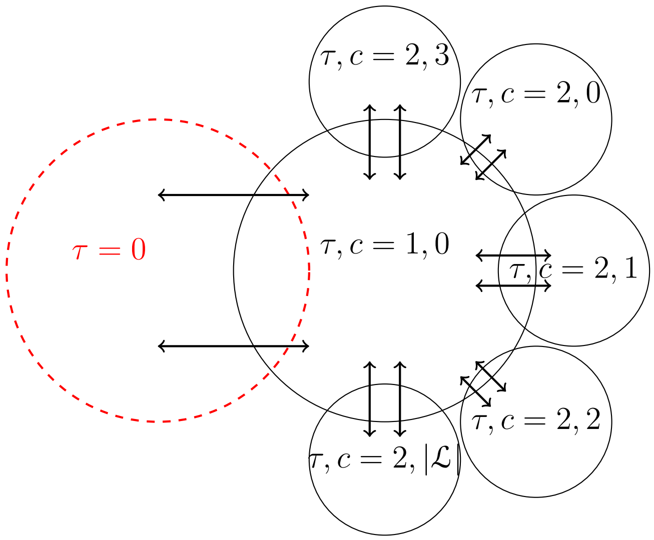


Figure 4.3: Schematic for Look-Ahead SCOPF for Post-Contingency Restoration.

all N elements are being considered, there are $N^2 + 1$ base-case/contingency groups.

4.4.1 Outline of the Formulation for LASCOPF to For Post-Contingency Restoration in One Dispatch Interval

Given below is the optimization problem description for this case:

$$\textbf{Objective}:$$

$$\min_{Power\ Generation} \text{Generation Cost over } \tau^{th} \ \& \ (\tau+1)^{th} \text{ intervals} \tag{4.10a}$$

$$\textbf{Constraints (for } \tau^{\textbf{th}} \textbf{ Dispatch Interval)}:$$

$$\text{Supply Demand Balance} \tag{4.10b}$$

$$\text{Line Power Flow Limit (Base Case)} \tag{4.10c}$$

$$\text{Line Power Flow Limit (First Contingency Cases)} \tag{4.10d}$$

$$\text{Generation Limit} \tag{4.10e}$$

$$\boxed{\textbf{Constraints (for } (\tau+1)^{\textbf{th}} \textbf{ Dispatch Interval, } \forall \textbf{FirstContingency)}:}$$

$$\boxed{\text{Supply Demand Balance}} \tag{4.10f}$$

$$\boxed{\text{Line Power Flow Limit (Base Case after first contingency)}} \tag{4.10g}$$

$$\boxed{\text{Line Power Flow Limit (Contingency Cases subsequent to first contingency)}} \tag{4.10h}$$

$$\boxed{\text{Generation Limit}} \tag{4.10i}$$

$$\text{Generator Ramp Rate Limits from } \tau \text{ to } (\tau+1) \tag{4.10j}$$

4.4.2 Look-Ahead Dispatch Model for Two Bus System

The case study that we have just presented can be mathematically formulated as:

$$\forall \tau \in \Omega$$

$$\min_{P_{g1}^{(\tau)}, P_{g2}^{(\tau)}} \sum_{s=0}^{s=1} (C_{g_1}(P_{g_1}^{(\tau+s)}) + C_{g_2}(P_{g_2}^{(\tau+s)})) \tag{4.11a}$$

$$\textbf{Subject to}: \forall \tau \in \Omega$$

$$P_{g_1}^{(\tau)} + P_{g_2}^{(\tau)} = P_{D_1} + P_{D_2} \tag{4.11b}$$

$$P_{g_1}^{(\tau+1)} + P_{g_2}^{(\tau+1)} = P_{D_1} + P_{D_2} \tag{4.11c}$$

$$\frac{P_{g_1}^{(\tau)} - P_{D_1}}{3} \leq 100 \tag{4.11d}$$

$$\frac{P_{g_1}^{(\tau)} - P_{D_1}}{2} \leq 125 \tag{4.11e}$$

$$\frac{P_{g_1}^{(\tau+1)} - P_{D_1}}{2} \leq 100 \tag{4.11f}$$

$$P_{g_1}^{(\tau+1)} - P_{D_1} \leq 125 \tag{4.11g}$$

$$\underline{R}_{g_1} \leq (P_{g_1}^{(\tau+1)} - P_{g_1}^{(\tau)}) \leq \overline{R}_{g_1} \tag{4.11h}$$

$$\underline{R}_{g_2} \leq (P_{g_2}^{(\tau+1)} - P_{g_2}^{(\tau)}) \leq \overline{R}_{g_2} \tag{4.11i}$$

If the outage occurs as well as the demand changes, then the above equations will be modified as follows:

$$\forall \tau \in \Omega$$

$$\min_{P_{g_1}^{(\tau)}, P_{g_2}^{(\tau)}} \sum_{s=0}^{s=1} (C_{g_1}(P_{g_1}^{(\tau+s)}) + C_{g_2}(P_{g_2}^{(\tau+s)})) \tag{4.12a}$$

$$\textbf{Subject to} : \forall \tau \in \Omega$$

$$P_{g_1}^{(\tau)} + P_{g_2}^{(\tau)} = P_{D_1}^{(\tau)} + P_{D_2}^{(\tau)} \tag{4.12b}$$

$$P_{g_1}^{(\tau+1)} + P_{g_2}^{(\tau+1)} = P_{D_1}^{(\tau+1)} + P_{D_2}^{(\tau+1)} \tag{4.12c}$$

$$\frac{P_{g_1}^{(\tau)} - P_{D_1}^{(\tau)}}{3} \leq 100 \tag{4.12d}$$

$$\frac{P_{g_1}^{(\tau)} - P_{D_1}^{(\tau)}}{2} \leq 125 \tag{4.12e}$$

$$\frac{P_{g_1}^{(\tau+1)} - P_{D_1}^{(\tau+1)}}{2} \leq 100 \tag{4.12f}$$

$$P_{g_1}^{(\tau+1)} - P_{D_1}^{(\tau+1)} \leq 125 \tag{4.12g}$$

$$\underline{R}_{g_1} \leq (P_{g_1}^{(\tau+1)} - P_{g_1}^{(\tau)}) \leq \overline{R}_{g_1} \tag{4.12h}$$

$$\underline{R}_{g_2} \leq (P_{g_2}^{(\tau+1)} - P_{g_2}^{(\tau)}) \leq \overline{R}_{g_2} \tag{4.12i}$$

Now, if the line impedances and power carrying capabilities are different, then for the first dispatch interval, the constraints will be the same as (4.7) consisting of a base-case and three different potential contingencies. But, for the second dispatch interval, there will be three distinct new base-cases corresponding to outages of each of the lines and for each of those three, there will be two more contingency constraints corresponding to the outage of each of the remaining two lines, one at a time. So, there will be a total of $3 \times 2 + 3 + 1$ i.e. 10 base case/contingency groups of constraints in the two dispatch intervals. So, if there are m possible $(N-1)$ contingency scenarios, the total number of base-case/contingency groups of constraints to be considered will be $m \times (m-1) + m + 1$ i.e. $m^2 + 1$. Obviously, the one contingency constraint that is most stringent for this particular system, will be the dominating one and the other will be redundant.

4.5 Mathematical Generalization of the Conventional Formulations to Arbitrary Networks

In this section, we will present the rigorous mathematical formulations for the conventional or traditional OPF, SCOPF, and LASCOPF cases for generalized power network and arbitrary number of dispatch intervals for the demand tracking. Throughout this section, we will be only presenting the DC or linearized version of the power flow model. This is primarily for the sake of simplicity and also because the DC power flow model is more intuitive than the AC model. We will eventually present the full AC OPF model in subsequent chapters of this work.:

4.5.1 Generalization of the OPF to Multi-Bus Systems

From the foregoing investigation of OPF, we can generalize the situation from two bus system to more complicated multi-bus and multi-generator-load systems as follows (Angles Included Formulation):

$$\min_{P_{g_q},\theta} \sum_{g_q \in G} C_{g_q}(P_{g_q}) \tag{4.13a}$$

$$\text{Subject to: } P_{g_{q_{N_i}}} - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}(\theta_{N_i} - \theta_{N_{\bar{i}}});\ \forall N_i \in \mathcal{N} \tag{4.13b}$$

$$|B_{T_r}(\theta_{T_{rt_1}} - \theta_{T_{rt_2}})| \leq \overline{L}_{T_r},\ \forall T_r \in T \tag{4.13c}$$

Here $J(N_i)$ is the set of all the buses that are directly connected to bus N_i. B_{T_r} is the series susceptance of the transmission line whose terminals are either indicated explicitly as subscripts of the voltage angles in the multiplier or are implicit in the node indices of the same. The alternative formulation with the angles eliminated is as follows:

$$\min_{P_{g_q}} \sum_{g_q \in G} C_{g_q}(P_{g_q}) \tag{4.14a}$$

$$\text{Subject to: } \sum_{g_q \in G} P_{g_q} = \sum_{D_d \in L} P_{D_d} \tag{4.14b}$$

$$|\Phi(\mathbf{P_g} - \mathbf{P_D})| \leq \overline{\mathbf{L}} \tag{4.14c}$$

where the bold face letters indicate vectors. The matrix $\Phi = [\mathbf{0}\ K[J_{p\theta}^{(0)}]^{-1}]$ is called the augmented Shift Factor matrix and $[J_{p\theta}{}^{(0)}]$ is the reduced power flow Jacobian Matrix (with the Slack Bus taken out). From now onward, we will use $\Phi = [\mathbf{0}\ K[J_{p\theta}^{(0)}]^{-1}]$ for the augmented shift factor matrix and $\Phi^{(c)} = [\mathbf{0}\ (K[J_{p\theta}^{(0)}]^{-1})^{(c)}]$ for the augmented shift factor matrix for a particular base-case/contingency scenario. The index $c = 0$ stands for base-case. We can also think of Φ as a $|T| \times |\mathcal{N}|$ matrix with elements $\phi(T_r, N_i)$ which are the ratios between the real power flow on line T_r, and the injection at bus N_i and withdrawal at the slack bus, where the implicit assumption is that the linearization of the line power flows holds good. Also, $(\mathbf{P_g} - \mathbf{P_D}) = [(P_{g_{N_1}} - P_{D_{N_1}}), (P_{g_{N_2}} - P_{D_{N_2}}), ..., (P_{g_{N_{|\mathcal{N}|}}} - P_{D_{N_{|\mathcal{N}|}}})]^{\dagger}$ is the vector of the real power bus injections. We will implicitly assume for the rest of this dissertation that the maximum and minimum real power generating limits are considered in the formulations, for which the constraints are : $\underline{P}_{g_q} \leq P_{g_q} \leq \overline{P}_{g_q},\ \forall g_q \in G$.

4.5.2 $(N-1)$ SCOPF for the Generalized Multi-Bus System: Unequal Capacities and Line Impedances

The generalized SCOPF for an arbitrary network can be written down as (Angles included Formulation):

$$\min_{P_{g_q},\theta} \sum_{g_q \in G} C_{g_q}(P_{g_q}) \tag{4.15a}$$

$$\text{Subject to: } P_{g_{q_{N_i}}} - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} B^{(0)}_{T_r}(\theta^{(0)}_{N_i} - \theta^{(0)}_{N_{\bar{i}}});\ \forall N_i \in \mathcal{N} \tag{4.15b}$$

$$P_{g_{q_{N_i}}} - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} B^{(c)}_{T_r}(\theta^{(c)}_{N_i} - \theta^{(c)}_{N_{\bar{i}}});\ \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L} - \{(0)\} \tag{4.15c}$$

$$|B^{(0)}_{T_r}(\theta^{(0)}_{T_{rt_1}} - \theta^{(0)}_{T_{rt_2}})| \leq \overline{L}^{(0)}_{T_r},\ \forall T_r \in T \tag{4.15d}$$

$$|B^{(c)}_{T_r}(\theta^{(c)}_{T_{rt_1}} - \theta^{(c)}_{T_{rt_2}})| \leq \overline{L}^{(c)}_{T_r},\ \forall T_r \in T, \forall (c) \in \mathcal{L} - \{(0)\} \tag{4.15e}$$

In the Angles eliminated form the formulation looks like:

$$\min_{P_{g_q}} \sum_{g_q \in G} C_{g_q}(P_{g_q}) \tag{4.16a}$$

$$\text{Subject to: } \forall (c) \in \mathcal{L}$$

$$\sum_{g_q \in G} P_{g_q} = \sum_{D_d \in L} P_{D_d} \tag{4.16b}$$

$$|\Phi^{(c)}(\mathbf{P_g} - \mathbf{P_D})| \leq \overline{\mathbf{L}}^{(c)} \tag{4.16c}$$

Here $c \in \mathcal{L} = \{0, 1, 2, ..., |\mathcal{L}|\}$ indicates the set of all the $(N-1)$ different contingencies.

4.5.3 Generalized Case of Demand Variation for Multi-Bus System

The generalized version of this problem for the angles eliminated formulation for a multi-time horizon, arbitrary network is as follows:

$$\min_{P_{g_q}^{(\tau)}} \sum_{\tau \in \Omega} \sum_{g_q \in G} C_{g_q}(P_{g_q}^{(\tau)}) \tag{4.17a}$$

$$\textbf{Subject to} : \forall \tau \in \Omega$$

$$\sum_{g_q \in G} P_{g_q}^{(\tau)} = \sum_{D_d \in L} P_{D_d}^{(\tau)} \tag{4.17b}$$

$$|\Phi^{(0)}(\mathbf{P}_{\mathbf{g}}^{(\tau)} - \mathbf{P}_{\mathbf{D}}^{(\tau)})| \leq \overline{\mathbf{L}}^{(0)} \tag{4.17c}$$

$$|\Phi^{(c)}(\mathbf{P}_{\mathbf{g}}^{(\tau)} - \mathbf{P}_{\mathbf{D}}^{(\tau)})| \leq \overline{\mathbf{L}}^{(c)}, \forall (c) \in \mathcal{L} - \{(0)\} \tag{4.17d}$$

$$\underline{R}_{g_q} \leq (P_{g_q}^{(\tau+1)} - P_{g_q}^{(\tau)}) \leq \overline{R}_{g_q}, \forall g_q \in G \tag{4.17e}$$

The equations stated below are the ones for the angles included formulation for the same problem as above:

$$\min_{P_{g_q}^{(\tau)}, \theta} \sum_{\tau \in \Omega} \sum_{g_q \in G} C_{g_q}(P_{g_q}^{(\tau)}) \tag{4.18a}$$

$$\textbf{Subject to} : \forall (c) \in \mathcal{L}, \forall \tau \in \Omega,\ \forall T_r \in T$$

$$P_{g_{q_{N_i}}}^{(\tau)} - P_{D_{d_{N_i}}}^{(\tau)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(0)}(\theta_{N_i}^{(\tau)(0)} - \theta_{N_{\bar{i}}}^{(\tau)(0)});\ \forall N_i \in \mathcal{N} \tag{4.18b}$$

$$P_{g_{q_{N_i}}}^{(\tau)} - P_{D_{d_{N_i}}}^{(\tau)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(c)}(\theta_{N_i}^{(\tau)(c)} - \theta_{N_{\bar{i}}}^{(\tau)(c)});\ \forall N_i \in \mathcal{N} \tag{4.18c}$$

$$|B_{T_r}^{(0)}(\theta_{T_{rt_1}}^{(\tau)(0)} - \theta_{T_{rt_2}}^{(\tau)(0)})| \leq \overline{L}_{T_r}^{(0)},\ \forall T_r \in T \tag{4.18d}$$

$$|B_{T_r}^{(c)}(\theta_{T_{rt_1}}^{(\tau)(c)} - \theta_{T_{rt_2}}^{(\tau)(c)})| \leq \overline{L}_{T_r}^{(c)},\ \forall T_r \in T \tag{4.18e}$$

$$\underline{R}_{g_q} \leq (P_{g_q}^{(\tau+1)} - P_{g_q}^{(\tau)}) \leq \overline{R}_{g_q}\ \forall g_q \in G \tag{4.18f}$$

In the above, $\Omega = \{0, 1, 2, 3, ...|\Omega|\}$ is the set of all the dispatch intervals.

4.5.4 Look-Ahead SCOPF for Ensuring Security with respect to Next Outages in one Dispatch Interval for Generalized Multi-Bus System

Given below are the mathematical models for the angles eliminated and angles included formulations for the general network. In this model, the superscript, $(c \rightarrow c^{'})$ denotes all the contingencies except the contingency c, and corresponds to those, when the particular outage corresponding to the scenario c has actually happened.

$$\forall \tau \in \Omega$$

$$\min_{P_{g_q}^{(\tau)}} \sum_{g_q \in G} \left(C_{g_q}(P_{g_q}^{(0)(\tau)}) + prob^{(c)} \sum_{(c) \in \mathcal{L}} C_{g_q}(P_{g_q}^{(c)(\tau+1)}) \right) \tag{4.19a}$$

$$\textbf{Subject to}: \forall \tau \in \Omega,\ \forall T_r \in T,\ \forall (c) \in \mathcal{L}$$

$$\sum_{g_q \in G} P_{g_q}^{(0)(\tau)} = \sum_{D_d \in L} P_{D_d}^{(\tau)} \tag{4.19b}$$

$$\sum_{g_q \in G} P_{g_q}^{(c)(\tau+1)} = \sum_{D_d \in L} P_{D_d}^{(\tau+1)} \tag{4.19c}$$

$$|\Phi^{(0)}(\mathbf{P_g^{(0)(\tau)}} - \mathbf{P_D^{(\tau)}})| \leq \overline{\mathbf{L}}^{(0)} \tag{4.19d}$$

$$|\Phi^{(c)}(\mathbf{P_g^{(0)(\tau)}} - \mathbf{P_D^{(\tau)}})| \leq \overline{\mathbf{L}}^{(c)} \tag{4.19e}$$

$$\forall (c) \in \mathcal{L}, \forall (c^{'}) \in [\mathcal{L} - \{c\}]$$

$$|\Phi^{(c)}(\mathbf{P_g^{(c)(\tau+1)}} - \mathbf{P_D^{(\tau+1)}})| \leq \overline{\mathbf{L}}^{(0)} \tag{4.19f}$$

$$|\Phi^{(c\rightarrow c^{'})}(\mathbf{P_g^{(c)(\tau+1)}} - \mathbf{P_D^{(\tau+1)}})| \leq \overline{\mathbf{L}}^{(c\rightarrow c^{'})} \tag{4.19g}$$

$$\underline{R}_{g_q} \leq P_{g_q}^{(c)(\tau+1)} - P_{g_q}^{(0)(\tau)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{4.19h}$$

In the angles included formulation, the problem can be framed as follows:

$$\forall \tau \in \Omega$$

$$\min_{P_{g_q}^{(\tau)},\theta} \sum_{g_q \in G} \left(C_{g_q}(P_{g_q}^{(0)(\tau)}) + prob^{(c)} \sum_{(c)\in\mathcal{L}} C_{g_q}(P_{g_q}^{(c)(\tau+1)}) \right) \tag{4.20a}$$

$$\textbf{Subject to}: \forall\tau \in \Omega,\ \forall T_r \in T,\ \forall(c) \in \mathcal{L}, \forall(c^{'}) \in [\mathcal{L} - \{c\}]$$

$$P_{g_{q_{N_i}}}^{(0)(\tau)} - P_{D_{d_{N_i}}}^{(\tau)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(0)}(\theta_{N_i}^{(0)(\tau)} - \theta_{N_{\bar{i}}}^{(0)(\tau)});\ \forall N_i \in \mathcal{N} \tag{4.20b}$$

$$P_{g_{q_{N_i}}}^{(0)(\tau)} - P_{D_{d_{N_i}}}^{(\tau)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(c)}(\theta_{N_i}^{(c)(\tau)} - \theta_{N_{\bar{i}}}^{(c)(\tau)});\ \forall N_i \in \mathcal{N} \tag{4.20c}$$

$$P_{g_{q_{N_i}}}^{(c)(\tau+1)} - P_{D_{d_{N_i}}}^{(\tau+1)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(c)}(\theta_{N_i}^{(c)(\tau+1)} - \theta_{N_{\bar{i}}}^{(c)(\tau+1)});\ \forall N_i \in \mathcal{N} \tag{4.20d}$$

$$P_{g_{q_{N_i}}}^{(c)(\tau+1)} - P_{D_{d_{N_i}}}^{(\tau+1)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(c\to c^{'})}(\theta_{N_i}^{(c\to c^{'})(\tau+1)} - \theta_{N_{\bar{i}}}^{(c\to c^{'})(\tau+1)});\ \forall N_i \in \mathcal{N} \tag{4.20e}$$

$$|B_{T_r}^{(0)}(\theta_{T_{rt_1}}^{(0)(\tau)} - \theta_{T_{rt_2}}^{(0)(\tau)})| \le \overline{L}_{T_r}^{(0)},\ \forall T_r \in T \tag{4.20f}$$

$$|B_{T_r}^{(c)}(\theta_{T_{rt_1}}^{(c)(\tau)} - \theta_{T_{rt_2}}^{(c)(\tau)})| \le \overline{L}_{T_r}^{(c)},\ \forall T_r \in T \tag{4.20g}$$

$$|B_{T_r}^{(c)}(\theta_{T_{rt_1}}^{(c)(\tau+1)} - \theta_{T_{rt_2}}^{(c)(\tau+1)})| \le \overline{L}_{T_r}^{(0)},\ \forall T_r \in T \tag{4.20h}$$

$$|B_{T_r}^{(c\to c^{'})}(\theta_{T_{rt_1}}^{(c\to c^{'})(\tau+1)} - \theta_{T_{rt_2}}^{(c\to c^{'})(\tau+1)})| \le \overline{L}_{T_r}^{(c\to c^{'})},\ \forall T_r \in T \tag{4.20i}$$

$$\underline{R}_{g_q} \le P_{g_q}^{(c)(\tau+1)} - P_{g_q}^{(0)(\tau)} \le \overline{R}_{g_q},\ \forall g_q \in G \tag{4.20j}$$

In the above formulations, $prob^{(c)}$ denotes the probability of the contingency scenario (c) to happen. Thus, we have presented the traditional formulations of the different problems in this chapter. In the next chapter, we will introduce the algorithms to solve them.

Chapter 5

Auxiliary Proximal Message Passing (APMP) Algorithm

[1]In this chapter, we will take a brief detour from the formulation of power flow problems and describe the algorithms used to solve them. We will call the algorithm we use to solve the Look-Ahead SCOPF (LASCOPF) problems, the Auxiliary Proximal Message Passing (APMP) algorithm, and it consists of two distinct components, viz:

- A coarse-grained distributed algorithmic component, which is based on the Auxiliary Problem Principle (APP).

- A fine-grained distributed algorithmic component, which, in our case is the Alternating Direction Method of Multipliers (ADMM) based Proximal Message Passing (PMP) consensus SCOPF algorithm.

[1]Parts of this chapter appear in the published papers, "Security Constrained Optimal Power Flow via Proximal Message Passing," "Toward Distributed/Decentralized DC Optimal Power Flow Implementation in Future Electric Power Systems," and "A Survey of Distributed Optimization and Control Algorithms for Electric Power Systems." The author of this treatise is the first author of the first paper, contributed section V, parts of sections IX and X of the second paper, and contributed parts of section III and V of the third paper.

The coarse-grained component of the algorithm works on the LASCOPF problem by dividing it across different dispatch intervals and splits it into several OPF, ED, or SCOPF problems corresponding to the different dispatch intervals and/or contingency scenarios, which are linked to each other through constraints like the ramp-rate constraints and thus achieve consensus by exchanging messages between them.

The fine-grained component works on each of the SCOPF, OPF, or ED problem and splits the computation across the different devices (generators, transmission lines, loads) and nodes and exchanges messages to attain consensus between the values of the decision variables. Figure 5.1 shows a schematic representation of the APMP algorithm. Based on the formulations that we have worked out for the multiple dispatch interval LASCOPFs with or without the post-contingency line temperature represented in the formulations, it can be seen that within each "coarse grain," we actually need to solve the OPF, SCOPF, or the ED. (Actually, in the line temperature represented version, as we will see, we need to solve the SCOPF for the forthcoming interval, ED for each of the contingency scenarios, from the interval succeeding the forthcoming one until one before the dispatch interval, in which the line flows are restored to within the nominal ratings, OPF from then to one interval before maximum allowed number of intervals for restoring the flows to within secure ratings, and SCOPF for the last dispatch interval. Hence the temporal sequence of coarse grains in this figure is from right to left. However, we haven't showed the last coarse grain, corresponding to the SCOPF, here). We will now describe in detail, the coarse-grained and the fine-grained components.

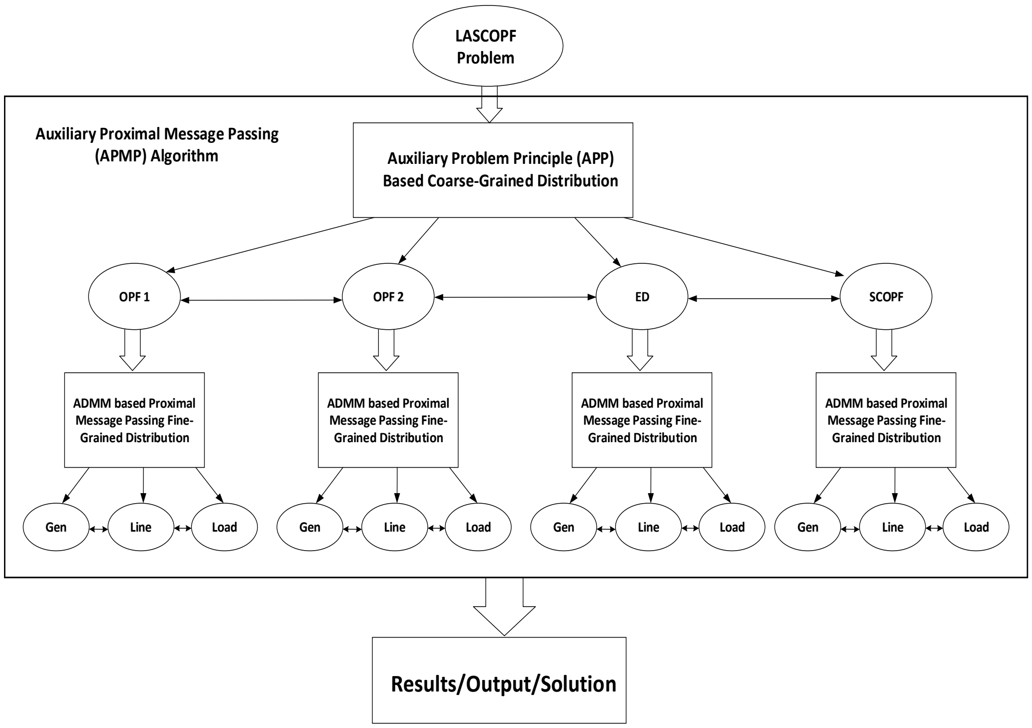


Figure 5.1: Schematic for the Auxiliary Proximal Message Passing (APMP) Algorithm.

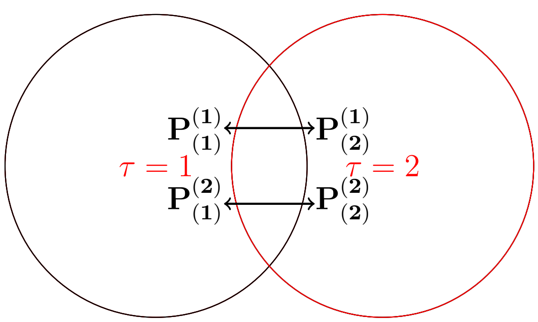


Figure 5.2: Schematic for APP Message Exchange.

5.1 APP Based Coarse-Grained Component

In the figure 5.2, the two circles represent the coarse-grained decompostion of a particular two dispatch interval SCOPF into the two individual SCOPFs, one for each of the dispatch intervals, $\tau = 1$ and $\tau = 2$. The $\mathbf{P}$ inside each of the circles represent the vector of generator outputs, which we consider to be the decision variables for this discussion. But, the contents of the circles can also represent line flow values, bus voltage angles etc. According to the convention introduced in Chapter 2, $\mathbf{P}^{(1)}_{(1)}$ and $\mathbf{P}^{(2)}_{(1)}$ stand for the vector of generator output values, as estimated by the computing unit associated with solving the SCOPF for $\tau = 1$ for itself and for the problem for $\tau = 2$, respectively. Similarly, $\mathbf{P}^{(1)}_{(2)}$ and $\mathbf{P}^{(2)}_{(2)}$ stand for the vector of generator output values estimated by the computing unit associated with solving the problem for $\tau = 2$ for the problem for $\tau = 1$ and for itself, respectively. Obviously, we will reach a consensus when the conditions, $\mathbf{P}^{(1)}_{(1)} = \mathbf{P}^{(1)}_{(2)}$ and $\mathbf{P}^{(2)}_{(1)} = \mathbf{P}^{(2)}_{(2)}$ are satisfied after a number of iterations. In order to achieve the consensus and also solve the entire problem, we will add the following terms to the objective function of each of the optimization problems:

- **Proximity from the previous iterate** is the measure of the distance of the present iterate of the decision vector from the last iterate.
- **Self-consensus term** is the term representing the deviation between the values of the decision vector belonging to a particular coarse-grain (or dispatch interval), as estimated by itself and by other coarse-grain(s) linked to it.
- **Mutual-consensus term** is the term representing the deviation between the values of the decision vector belonging to the linked coarse-grain(s), as estimated by the

coarse-grain in consideration and by other respective coarse-grain(s) linked to it.

- **Complementary slackness term** is the product of the Lagrange multipliers, corresponding to the consensus constraints and the pertinent decision variables (with the appropriate signs taken into account).

We follow the same color code as indicated above, for the respective terms. The details of the mathematical formulation and derivation pertaining to decomposing by using APMP algorithm, for the different power flow problems mentioned in the previous chapter, are presented in the subsequent sections of this chapter and the details of the APP algorithm itself can be found in references [84], [85], [223], [18], [118], [24] etc.

5.2 ADMM Based Fine-Grained Component

In order to apply the ADMM based Proximal Message Passing (PMP) algorithm, through which we achieve the fine-grained distribution, it is first necessary to reformulate the conventional formulations. In this approach, we add the indicator functions corresponding to the different constraints to the objective function and the only constraints to be satisfied are the power mismatch constraints for the nodes and the angle mismatch constraints for the devices. We first present the ones for OPF, followed by the ones for the SCOPF.

5.2.1 OPF Reformulation

The reformulated OPF is shown below

$$\min_{P,\theta} f(P) = \text{Cost of producing power}$$

$$+\text{Cost of Violating Transmission Limit}$$

$$+\text{Cost of violating Power Angle Relation} \tag{5.1a}$$

$$\text{Subject to: Power Balance Constraints}, \forall N_i \in \mathcal{N}$$

$$\text{Angle Consistency Constraints}, \forall N_i \in \mathcal{N} \tag{5.1b}$$

The above reformulation directly leads to the PMP algorithm, in which the prox-function for each device is calculated and messages are exchanged between the different devices through the nets regarding the beliefs of each of them about values of their own decision variables as well as of others. The prox function for an objective function g is defined as follows:

$$\mathbf{prox}_{g,\rho}(v) = \underset{x}{\operatorname{argmin}}(g(x) + (\rho/2)\|x - v\|_2^2).$$

In the above, ρ is a tuning parameter and v encodes the the beliefs of values of decision variables from other devices as well as nets or nodes from the previous iteration. In case of generators, g is the cost function along with the constraints of generation limit, ramp rate constraints etc, whereas for transmission lines and loads, it is the sum of indicator functions corresponding to satisfaction of Ohm's law and transmission line limit, and power consumption being equal to the load MW, respectively. Hence, each iteration consists of the following two steps, known respectively as the broadcast and gather operations:

- **Broadcast**: Calculation of prox function by each device (ie Generators, Transmission Lines, and Loads) in order to calculate the values of the next iterate of the decision variables (ie real power and bus voltage angle).

- **Gather**: Calculation of the update of Lagrange Multipliers or Dual Variables corresponding to the net power balance and angle consistency constraints at each node/net/bus.

Figure 5.3 shows the diagrammatic conventions used. The big circle represents the generators, the rectangular box stands for the transmission lines and the arrow is for the load. the small circle represents the nodes.

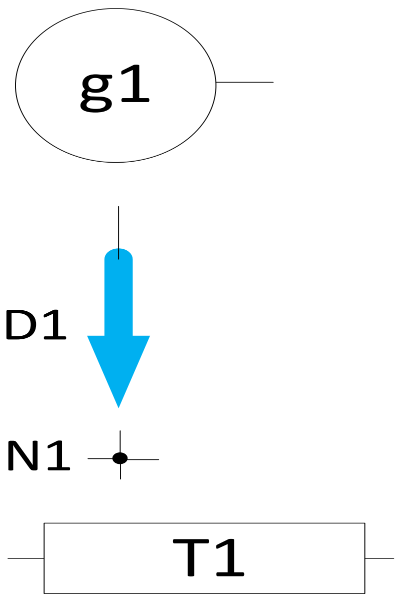
In figure 5.4, we represent the general message exchange in each iteration. The hexagonal objects stand for any device and the double headed arrows (the heads of which represent the direction or sense of exchange of messages regarding beliefs about values of decision variables by the devices and those of Lagrange multipliers by the nodes) connecting them are the terminals, through which the messages are exchanged.

Figures 5.5 and 5.6 respectively represent the broadcast and gather operations for each iteration. During the broadcast operation, the nodes send out the updated values of the Lagrange multipliers to the different devices and then the devices update the decision variables through computing the prox functions. During the gather operation, the decision variable values are sent back to the respective nodes, or are "gathered" for calculating the next update of the Lagrange multipliers.

5.2.2 SCOPF Reformulation

In this case, corresponding to each contingency scenario, we instantiate a copy of each transmission line and load. The reformulation is as follows:

$$\min_{P,\theta} f(P) = \text{Cost of producing power (Base Case)}$$



Generator: For Calculating Prox function, takes in updated scaled prices, average power mismatch, and angle inconsistency from the net, and outputs iterates of real power and voltage phase angle.

Load: For Calculating Prox function, takes in updated scaled prices, and angle inconsistency from the net, and outputs iterate of voltage phase angle.

Net: For Calculating Prox function, takes in updated power, and phase angle schedules from the devices connected to it, and outputs iterates of scaled prices.

Transmission Line: For Calculating Prox function, takes in updated scaled prices, average power mismatch, and angle inconsistency from two nets, and outputs iterates of real powers and voltage phase angles.

Figure 5.3: Conventions.

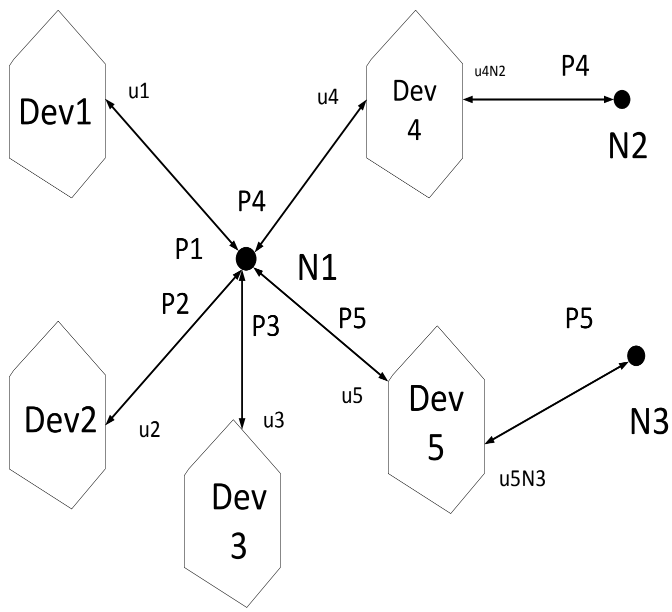


Figure 5.4: Proximal Message Passing for OPF.

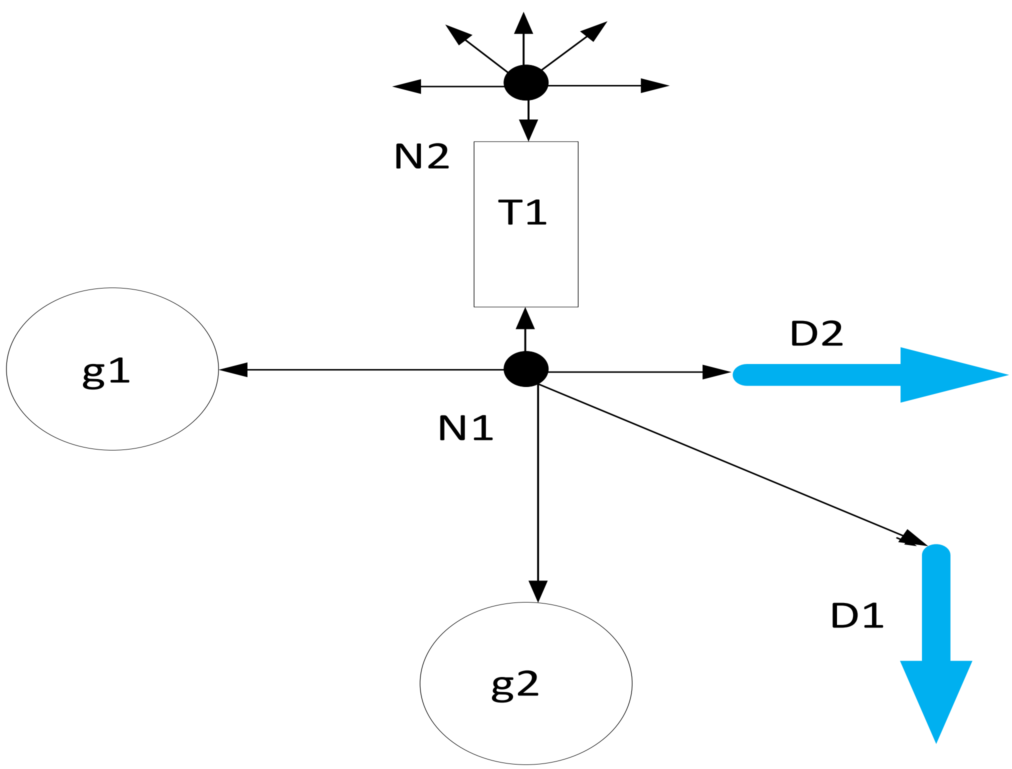


Figure 5.5: Proximal Message Passing: Broadcast.

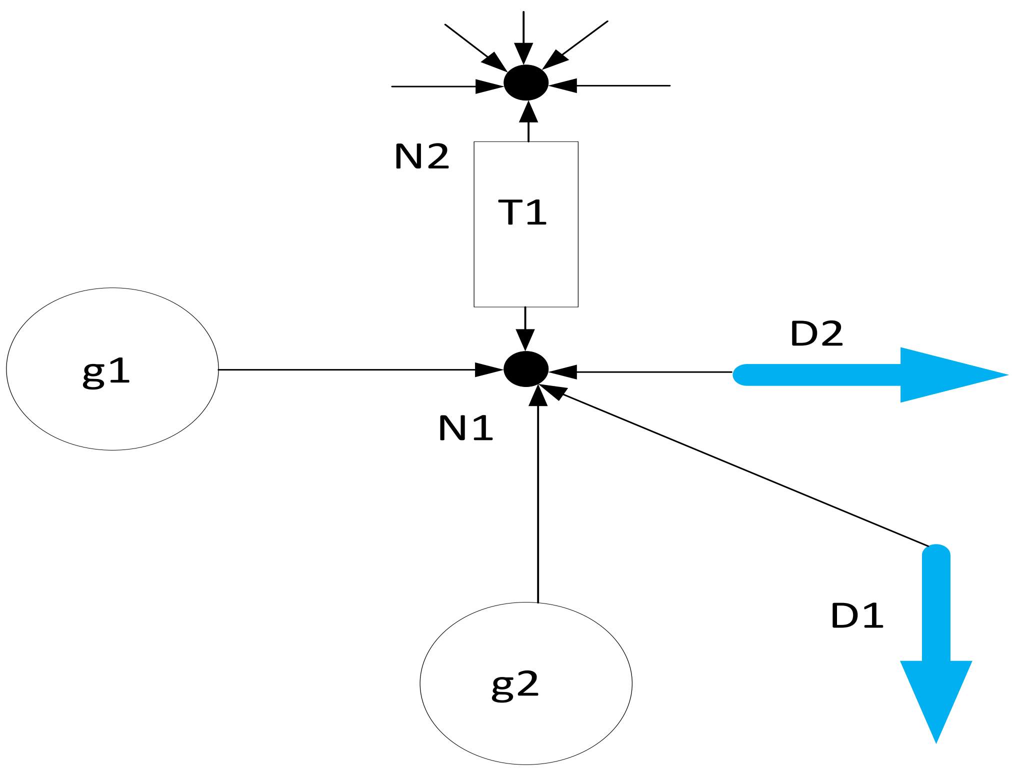


Figure 5.6: Proximal Message Passing: Gather.

$$
\begin{aligned}
&+\text{Cost of Violating Transmission Limit} \\
&\text{(Base Case \& Outage)} \\
&+\text{Cost of violating Power Angle Relation} \\
&\text{(Base Case \& Outage)} && \text{(5.2a)} \\
&\text{Subject to: Power Balance Constraint} \\
&\text{(Base Case \& Outage)}, \forall N_i \in \mathcal{N} \\
&\text{Angle Consistency Constraint} \\
&\text{(Base Case \& Outage)}, \forall N_i \in \mathcal{N} && \text{(5.2b)} \\
&\text{Injections at Base Case=Injections at each Outage Case} && \text{(5.2c)}
\end{aligned}
$$

The proximal message passing is same as before. The broadcast-gather of each iteration can be diagrammatically shown in figure 5.7. The green lines here indicate the consensus between the generation values in the diferent scenarios to be the same or "tied" to the one at the base case. It can be seen, for each scenario, we have a copy of the devices as well as the nodes. For the details of the mathematical formulations and derivations, we direct the reader to the fine grain message passing algorithm section of this chapter and the references cited throughout.

The two fine-grained models described above will be used inside each of the coarse-grains in the LASCOPF models. Figure 5.8 shows the general scheme, in which we have shown the fine-grained distribution of SCOPF within two coarse-grained distributions of the LASCOPF.

In this chapter, we will provide the detailed mathematical formulation of the Auxiliary Proximal Message Passing (APMP) algorithm, as applied to the OPF, SCOPF, and LAS-

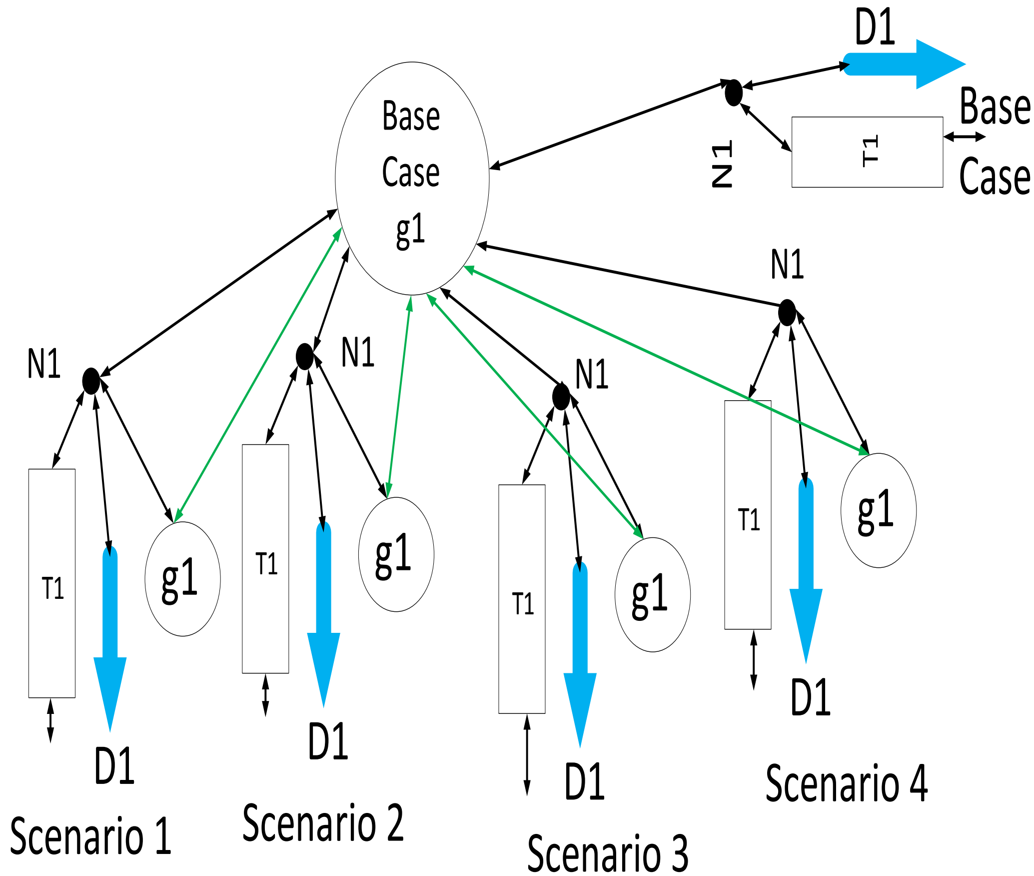


Figure 5.7: Proximal Message Passing for SCOPF.

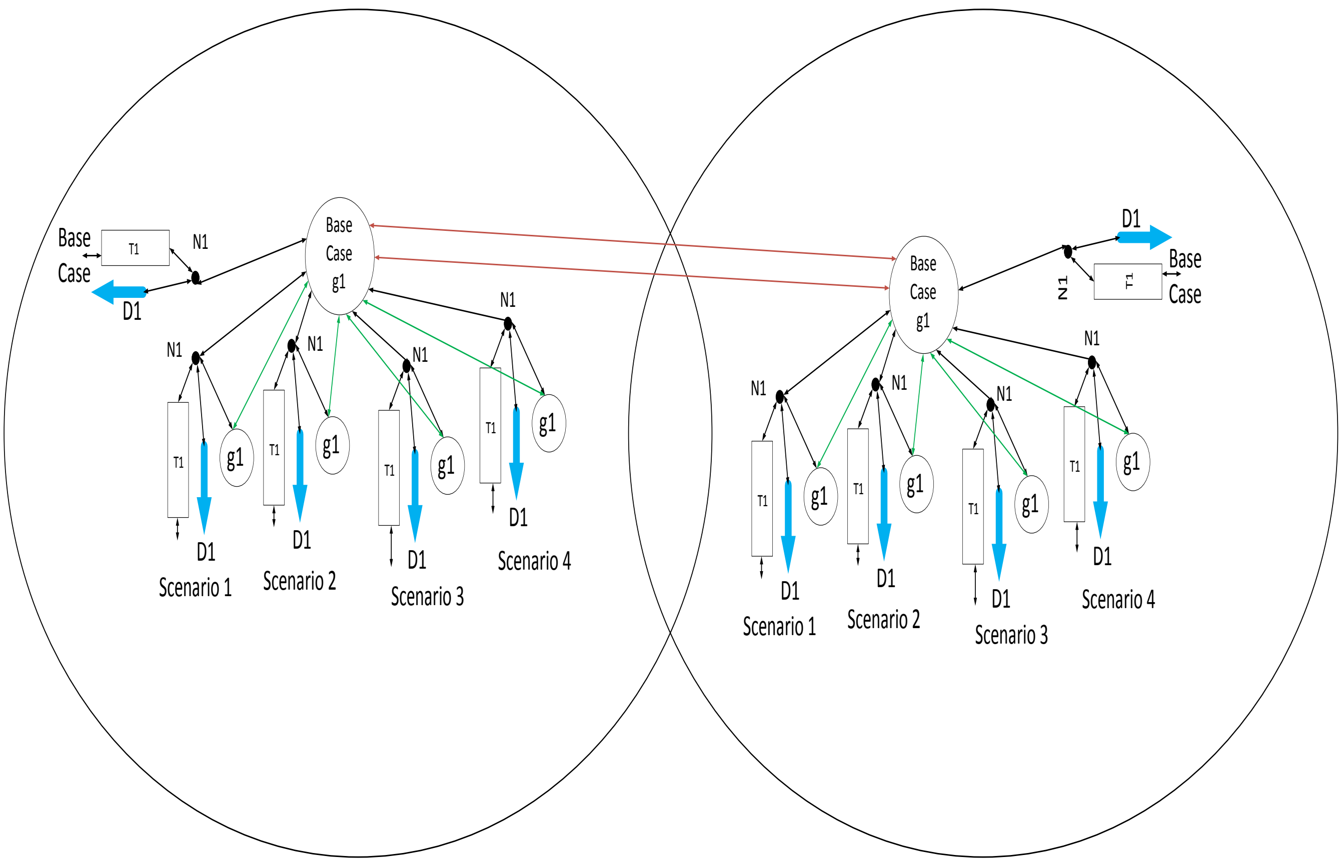


Figure 5.8: Auxiliary Proximal Message Passing for LASCOPF

COPF problems. We will start first describing the APP based coarse-grained component of the algorithm, which will be applied to the LASCOPF problems, followed by the PMP fine-grained component, which decomposes the SCOPF problems (and the OPF problems) of each dispatch interval into the device level computations. The basic idea behind the APP based coarse-grained distributed component of the algorithm is depicted in figure 5.9, where the two circles correspond to different dispatch intervals and/or scenarios, across which the computation is split, and the arrows indicate the exchange of messages between those. The messages correspond to the beliefs of each circle or coarse grain about the values of power generations and/or injections/line flows within itself and also those belonging to the neighboring circles.

5.3 Coarse-Grained Component for LASCOPF: Tracking the Demand Variation

We will now apply the coarse-grained component to the LASCOPF problems for tracking demand variation, described earlier.

5.3.1 Simplest Case of Demand Variation for Two Bus System

Let us, at this point, slightly reformulate the model stated in equation (4.9) and also several ones to follow. Refering to the figure 5.10, the different overlapping circles here represent the different dispatch intervals, and with $\tau = 0$ and $\tau = 4$ representing respectively, the previous (or, present, depending on where the time reference frame is set) dispatch interval for which the LASCOPF problem was already solved and the results are known and the dispatch interval following the concerned time horizon. We will distribute

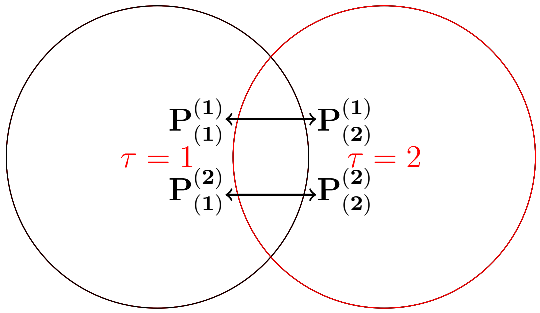


Figure 5.9: Schematic for APP Message Exchange.

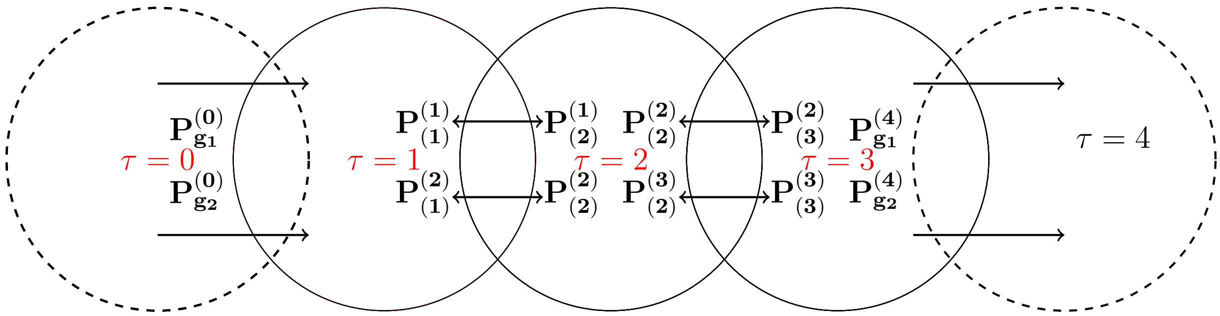


Figure 5.10: Schematic for Look-Ahead SCOPF for Demand Variation

the SCOPF across each dispatch interval and then exchange messages regarding what a particular interval thinks about the optimal values of the decision variables for the intervals immediately preceding and succeeding the current one, eventually attempting to achieve a consensus between those. In the figure, the values shown at the two sides of the arrows are those 'beliefs' among which we are trying to achieve consensus. We follow the same convention as introduced in section 3.1.1

Now there will be some consensus constraints. If we write the augmented Lagrangian taking the augmented terms for the consensus constraints into the objective, the new formulation will look like (5.3):

$$
\begin{aligned}
\min_{\mathbf{P_{(1)}},\mathbf{P_{(2)}},\mathbf{P_{(3)}}} & C_{g_1}(P^{(1)}_{g_1(1)}) + C_{g_2}(P^{(1)}_{g_2(1)}) + C_{g_1}(P^{(2)}_{g_1(2)}) + C_{g_2}(P^{(2)}_{g_2(2)}) \\
& + C_{g_1}(P^{(3)}_{g_1(3)}) + C_{g_2}(P^{(3)}_{g_2(3)}) \\
& + \frac{\gamma}{2}(||\mathbf{P^{(1)}_{(1)}} - \mathbf{P^{(1)}_{(2)}}||_2^2 + ||\mathbf{P^{(2)}_{(1)}} - \mathbf{P^{(2)}_{(2)}}||_2^2 + \\
& ||\mathbf{P^{(2)}_{(2)}} - \mathbf{P^{(2)}_{(3)}}||_2^2 + ||\mathbf{P^{(3)}_{(2)}} - \mathbf{P^{(3)}_{(3)}}||_2^2)
\end{aligned} \tag{5.3a}
$$

$$\textbf{Subject to}: \forall \tau \in \{1, 2, 3\}$$

$$\textbf{Power} - \textbf{Balance Constraints}:$$

$$P^{(\tau)}_{g_1(\tau)} + P^{(\tau)}_{g_2(\tau)} = P^{(\tau)}_{D_1} + P^{(\tau)}_{D_2} \tag{5.3b}$$

$$\textbf{Flow Limit Constraints} \ (\textbf{Base} - \textbf{Case} \ \& \ \textbf{Contingency}):$$

$$\frac{P^{(\tau)}_{g_1(\tau)} - P^{(\tau)}_{D_1}}{3} \leq 100 \tag{5.3c}$$

$$\frac{P^{(\tau)}_{g_1(\tau)} - P^{(\tau)}_{D_1}}{2} \leq 125 \tag{5.3d}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_2} \leq P^{(\tau+1)}_{g_2(\tau)} - P^{(\tau)}_{g_2(\tau)} \leq \overline{R}_{g_2} \tag{5.3e}$$

$$\underline{R}_{g_1} \leq P^{(\tau+1)}_{g_1(\tau)} - P^{(\tau)}_{g_1(\tau)} \leq \overline{R}_{g_1} \tag{5.3f}$$

$$\underline{R}_{g_2} \leq P^{(\tau)}_{g_2(\tau)} - P^{(\tau-1)}_{g_2(\tau)} \leq \overline{R}_{g_2} \tag{5.3g}$$

$$\underline{R}_{g_1} \leq P^{(\tau)}_{g_1(\tau)} - P^{(\tau-1)}_{g_1(\tau)} \leq \overline{R}_{g_1} \tag{5.3h}$$

$$\textbf{Consensus Constraints}:$$

$$P^{(\tau)}_{g_2(\tau)} = P^{(\tau)}_{g_2(\tau+1)}, P^{(\tau)}_{g_1(\tau)} = P^{(\tau)}_{g_1(\tau+1)}, \ \forall \tau \in \{1, 2\} \tag{5.3i}$$

$$P^{(\tau)}_{g_2(\tau-1)} = P^{(\tau)}_{g_2(\tau)}, P^{(\tau)}_{g_1(\tau-1)} = P^{(\tau)}_{g_1(\tau)}, \ \forall \tau \in \{2, 3\} \tag{5.3j}$$

In the above equations, when $\tau = 1$, $P_{g_2(\tau)}^{(\tau-1)}$ and $P_{g_1(\tau)}^{(\tau-1)}$ are replaced by $P_{g_2}^{(0)}$ and $P_{g_1}^{(0)}$ respectively, which are the (known) MW outputs from the last dispatch interval. When $\tau = 3$, $P_{g_2(\tau)}^{(\tau+1)}$ and $P_{g_1(\tau)}^{(\tau+1)}$ are replaced by $P_{g_2}^{(4)}$ and $P_{g_1}^{(4)}$ respectively, which are the last but one iterate values of the MW outputs from the last dispatch interval.

We will now apply the Auxiliary Problem Principle (APP)([84], [85]) to the above optimization problem to derive a set of expressions for the iterate updates. This reformulation is very similar in flavor to the ones presented previously in [223], [18], [118], [24] etc. The iterates are given as follows:

$$
\begin{aligned}
&(\mathbf{P_{(1)}}^{(\mu_{APP}+1)}, \mathbf{P_{(2)}}^{(\mu_{APP}+1)}, \mathbf{P_{(3)}}^{(\mu_{APP}+1)}) \\
&= \underset{\mathbf{P_{(1)}},\mathbf{P_{(2)}},\mathbf{P_{(3)}}}{\operatorname{argmin}} \; C_{g_1}(P_{g_1(1)}^{(1)}) + C_{g_2}(P_{g_2(1)}^{(1)}) + C_{g_1}(P_{g_1(2)}^{(2)}) + C_{g_2}(P_{g_2(2)}^{(2)}) + C_{g_1}(P_{g_1(3)}^{(3)}) + C_{g_2}(P_{g_2(3)}^{(3)}) \\
&+\frac{\beta}{2}(||\mathbf{P_{(1)}} - \mathbf{P_{(1)}}^{(\mu_{APP})}||_2^2 + ||\mathbf{P_{(2)}} - \mathbf{P_{(2)}}^{(\mu_{APP})}||_2^2 + ||\mathbf{P_{(3)}} - \mathbf{P_{(3)}}^{(\mu_{APP})}||_2^2) \\
&+\gamma[(\mathbf{P_{(1)}^{(1)}} - \mathbf{P_{(2)}^{(1)}})^{\dagger}(\mathbf{P_{(1)}^{(1)}}^{(\mu_{APP})} - \mathbf{P_{(2)}^{(1)}}^{(\mu_{APP})}) + (\mathbf{P_{(1)}^{(2)}} - \mathbf{P_{(2)}^{(2)}})^{\dagger}(\mathbf{P_{(1)}^{(2)}}^{(\mu_{APP})} - \mathbf{P_{(2)}^{(2)}}^{(\mu_{APP})}) + \\
&(\mathbf{P_{(2)}^{(2)}} - \mathbf{P_{(3)}^{(2)}})^{\dagger}(\mathbf{P_{(2)}^{(2)}}^{(\mu_{APP})} - \mathbf{P_{(3)}^{(2)}}^{(\mu_{APP})}) + (\mathbf{P_{(2)}^{(3)}} - \mathbf{P_{(3)}^{(3)}})^{\dagger}(\mathbf{P_{(2)}^{(3)}}^{(\mu_{APP})} - \mathbf{P_{(3)}^{(3)}}^{(\mu_{APP})})] \\
&+\lambda_{\mathbf{1}}^{(\mu_{APP})\dagger}(\mathbf{P_{(1)}^{(1)}} - \mathbf{P_{(2)}^{(1)}}) + \lambda_{\mathbf{2}}^{(\mu_{APP})\dagger}(\mathbf{P_{(1)}^{(2)}} - \mathbf{P_{(2)}^{(2)}}) + \lambda_{\mathbf{3}}^{(\mu_{APP})\dagger}(\mathbf{P_{(2)}^{(2)}} - \mathbf{P_{(3)}^{(2)}}) \\
&+\lambda_{\mathbf{4}}^{(\mu_{APP})\dagger}(\mathbf{P_{(2)}^{(3)}} - \mathbf{P_{(3)}^{(3)}}) \qquad (5.4a)
\end{aligned}
$$

$$\mathbf{Subject\ to} : \forall \tau \in \{1, 2, 3\}$$

$$\mathbf{Power-Balance\ Constraints} :$$

$$P_{g_1(\tau)}^{(\tau)} + P_{g_2(\tau)}^{(\tau)} = P_{D_1}^{(\tau)} + P_{D_2}^{(\tau)} \tag{5.4b}$$

$$\mathbf{Flow\ Limit\ Constraints\ (Base-Case\ \&\ Contingency)} :$$

$$\frac{P_{g_1(\tau)}^{(\tau)} - P_{D_1}^{(\tau)}}{3} \leq 100 \tag{5.4c}$$

$$\frac{P_{g_1(\tau)}^{(\tau)} - P_{D_1}^{(\tau)}}{2} \leq 125 \tag{5.4d}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_2} \leq P_{g_2(\tau)}^{(\tau+1)} - P_{g_2(\tau)}^{(\tau)} \leq \overline{R}_{g_2} \tag{5.4e}$$

$$\underline{R}_{g_1} \leq P_{g_1(\tau)}^{(\tau+1)} - P_{g_1(\tau)}^{(\tau)} \leq \overline{R}_{g_1} \tag{5.4f}$$

$$\underline{R}_{g_2} \leq P_{g_2(\tau)}^{(\tau)(\mu_{APP})} - P_{g_2(\tau)}^{(\tau)} \leq \overline{R}_{g_2},\ \tau = 3 \tag{5.4g}$$

$$\underline{R}_{g_1} \leq P_{g_1(\tau)}^{(\tau)(\mu_{APP})} - P_{g_1(\tau)}^{(\tau)} \leq \overline{R}_{g_1},\ \tau = 3 \tag{5.4h}$$

$$\underline{R}_{g_2} \leq P_{g_2(\tau)}^{(\tau)} - P_{g_2(\tau)}^{(\tau-1)} \leq \overline{R}_{g_2} \tag{5.4i}$$

$$\underline{R}_{g_1} \leq P_{g_1(\tau)}^{(\tau)} - P_{g_1(\tau)}^{(\tau-1)} \leq \overline{R}_{g_1} \tag{5.4j}$$

$$\underline{R}_{g_2} \leq P_{g_2(\tau)}^{(\tau)} - P_{g_2(\tau)}^{(0)} \leq \overline{R}_{g_2},\ \tau = 1 \tag{5.4k}$$

$$\underline{R}_{g_1} \leq P_{g_1(\tau)}^{(\tau)} - P_{g_1(\tau)}^{(0)} \leq \overline{R}_{g_1},\ \tau = 1 \tag{5.4l}$$

$$\textbf{Dual Variable Updates}:$$

$$\lambda_{\mathbf{1}}{}^{(\mu_{APP}+1)} = \lambda_{\mathbf{1}}{}^{(\mu_{APP})} + \alpha(\mathbf{P}_{\mathbf{(1)}}^{\mathbf{(1)}}{}^{(\mu_{APP}+1)} - \mathbf{P}_{\mathbf{(2)}}^{\mathbf{(1)}}{}^{(\mu_{APP}+1)}) \tag{5.4m}$$

$$\lambda_{\mathbf{2}}{}^{(\mu_{APP}+1)} = \lambda_{\mathbf{2}}{}^{(\mu_{APP})} + \alpha(\mathbf{P}_{\mathbf{(1)}}^{\mathbf{(2)}}{}^{(\mu_{APP}+1)} - \mathbf{P}_{\mathbf{(2)}}^{\mathbf{(2)}}{}^{(\mu_{APP}+1)}) \tag{5.4n}$$

$$\lambda_{\mathbf{3}}{}^{(\mu_{APP}+1)} = \lambda_{\mathbf{3}}{}^{(\mu_{APP})} + \alpha(\mathbf{P}_{\mathbf{(2)}}^{\mathbf{(2)}}{}^{(\mu_{APP}+1)} - \mathbf{P}_{\mathbf{(3)}}^{\mathbf{(2)}}{}^{(\mu_{APP}+1)}) \tag{5.4o}$$

$$\lambda_{\mathbf{4}}{}^{(\mu_{APP}+1)} = \lambda_{\mathbf{4}}{}^{(\mu_{APP})} + \alpha(\mathbf{P}_{\mathbf{(2)}}^{\mathbf{(3)}}{}^{(\mu_{APP}+1)} - \mathbf{P}_{\mathbf{(3)}}^{\mathbf{(3)}}{}^{(\mu_{APP}+1)}) \tag{5.4p}$$

The above problem can be split into three sub-problems, each corresponding to a particular dispatch time interval. In each of the subproblems, the values of the generations for the next dispatch time are guessed by the subproblem, which eventually attain consensus through the Auxiliary Problem Principle algorithm. Notice, that each of the subproblems is very similar to the classical $(N-1)$ SCOPF problems, but with two important differences. First of all, each of these have ramp rate constraints, since as contrasted to the SCOPF (which was solved for only one dispatch interval), we are solving the LASCOPF problem for multiple dispatch intervals. Hence, the generators need to respect the ramp rate limits, while changing their outputs. Secondly, it can be seen that there are some additional terms, that are added to the objective functions. These are the regularization terms for attaining consensus among different coarse grains about the values of the decision variables, which, in this case, are generator outputs. The terms follow the same color coding, which was introduced in section 5.1. The terms in blue are the ones representing the proximity from previous iterates. The terms in green are the ones for self-consensus, the ones in red are for mutual consensus, and the ones in orange are the terms corresponding to complementary slackness.

First Dispatch Interval SCOPF:

$$\mathbf{P_{(1)}}^{(\mu_{APP}+1)} = \underset{\mathbf{P_{(1)}}}{\operatorname{argmin}} C_{g_1}(P^{(1)}_{g_1(1)}) + C_{g_2}(P^{(1)}_{g_2(1)}) + \boxed{\frac{\beta}{2}||\mathbf{P_{(1)}} - \mathbf{P_{(1)}}^{(\mu_{APP})}||_2^2}$$
$$+\gamma[\boxed{\mathbf{P^{(1)}_{(1)}}^{\dagger}(\mathbf{P^{(1)}_{(1)}}^{(\mu_{APP})} - \mathbf{P^{(1)}_{(2)}}^{(\mu_{APP})})} + \boxed{\mathbf{P^{(2)}_{(1)}}^{\dagger}(\mathbf{P^{(2)}_{(1)}}^{(\mu_{APP})} - \mathbf{P^{(2)}_{(2)}}^{(\mu_{APP})})}]$$
$$+\boxed{\lambda_{\mathbf{1}}^{(\mu_{APP})\dagger}\mathbf{P^{(1)}_{(1)}} + \lambda_{\mathbf{2}}^{(\mu_{APP})\dagger}\mathbf{P^{(2)}_{(1)}}} \tag{5.5a}$$

$$\textbf{Subject to}:$$

$$\textbf{Power}-\textbf{Balance Constraints}:$$

$$P^{(1)}_{g_1(1)} + P^{(1)}_{g_2(1)} = P^{(1)}_{D_1} + P^{(1)}_{D_2} \tag{5.5b}$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base}-\textbf{Case}\ \&\ \textbf{Contingency}):$$

$$\frac{P^{(1)}_{g_1(1)} - P^{(1)}_{D_1}}{3} \leq 100 \tag{5.5c}$$

$$\frac{P^{(1)}_{g_1(1)} - P^{(1)}_{D_1}}{2} \leq 125 \tag{5.5d}$$

$$\textbf{Ramp}-\textbf{Rate Constraints}:$$

$$\underline{R}_{g_2} \leq P^{(2)}_{g_2(1)} - P^{(1)}_{g_2(1)} \leq \overline{R}_{g_2} \tag{5.5e}$$

$$\underline{R}_{g_1} \leq P^{(2)}_{g_1(1)} - P^{(1)}_{g_1(1)} \leq \overline{R}_{g_1} \tag{5.5f}$$

$$\underline{R}_{g_2} \leq P^{(1)}_{g_2(1)} - P^{(0)}_{g_2} \leq \overline{R}_{g_2} \tag{5.5g}$$

$$\underline{R}_{g_1} \leq P^{(1)}_{g_1(1)} - P^{(0)}_{g_1} \leq \overline{R}_{g_1} \tag{5.5h}$$

Second Dispatch Interval SCOPF:

$$\begin{aligned}
\mathbf{P_{(2)}}^{(\mu_{APP}+1)} = \underset{\mathbf{P_{(2)}}}{\operatorname{argmin}}\, & C_{g_1}(P^{(2)}_{g_1(2)}) + C_{g_2}(P^{(2)}_{g_2(2)}) + \boxed{\frac{\beta}{2}||\mathbf{P_{(2)}} - \mathbf{P_{(2)}}^{(\mu_{APP})}||_2^2} \\
&+\gamma[\boxed{\mathbf{P^{(1)}_{(2)}}^{\dagger}(\mathbf{P^{(1)}_{(2)}}^{(\mu_{APP})} - \mathbf{P^{(1)}_{(1)}}^{(\mu_{APP})})} + \boxed{\mathbf{P^{(2)}_{(2)}}^{\dagger}(\mathbf{P^{(2)}_{(2)}}^{(\mu_{APP})} - \mathbf{P^{(2)}_{(1)}}^{(\mu_{APP})})} \\
&+\boxed{\mathbf{P^{(2)}_{(2)}}^{\dagger}(\mathbf{P^{(2)}_{(2)}}^{(\mu_{APP})} - \mathbf{P^{(2)}_{(3)}}^{(\mu_{APP})})} + \boxed{\mathbf{P^{(3)}_{(2)}}^{\dagger}(\mathbf{P^{(3)}_{(2)}}^{(\mu_{APP})} - \mathbf{P^{(3)}_{(3)}}^{(\mu_{APP})})}] \\
&+\boxed{\lambda_{\mathbf{3}}{}^{(\mu_{APP})\dagger}\mathbf{P^{(2)}_{(2)}} + \lambda_{\mathbf{4}}{}^{(\mu_{APP})\dagger}\mathbf{P^{(3)}_{(2)}} - \lambda_{\mathbf{1}}{}^{(\mu_{APP})\dagger}\mathbf{P^{(1)}_{(2)}} - \lambda_{\mathbf{2}}{}^{(\mu_{APP})\dagger}\mathbf{P^{(2)}_{(2)}}}
\end{aligned} \tag{5.6a}$$

$$\textbf{Subject to :}$$

$$\textbf{Power – Balance Constraints :}$$

$$P^{(2)}_{g_1(2)} + P^{(2)}_{g_2(2)} = P^{(2)}_{D_1} + P^{(2)}_{D_2} \tag{5.6b}$$

$$\textbf{Flow Limit Constraints (Base – Case \& Contingency) :}$$

$$\frac{P^{(2)}_{g_1(2)} - P^{(2)}_{D_1}}{3} \leq 100 \tag{5.6c}$$

$$\frac{P^{(2)}_{g_1(2)} - P^{(2)}_{D_1}}{2} \leq 125 \tag{5.6d}$$

$$\textbf{Ramp – Rate Constraints :}$$

$$\underline{R}_{g_2} \leq P^{(3)}_{g_2(2)} - P^{(2)}_{g_2(2)} \leq \overline{R}_{g_2} \tag{5.6e}$$

$$\underline{R}_{g_1} \leq P^{(3)}_{g_1(2)} - P^{(2)}_{g_1(2)} \leq \overline{R}_{g_1} \tag{5.6f}$$

$$\underline{R}_{g_2} \leq P^{(2)}_{g_2(2)} - P^{(1)}_{g_2(2)} \leq \overline{R}_{g_2} \tag{5.6g}$$

$$\underline{R}_{g_1} \leq P^{(2)}_{g_1(2)} - P^{(1)}_{g_1(2)} \leq \overline{R}_{g_1} \tag{5.6h}$$

Third Dispatch Interval SCOPF:

$$\mathbf{P_{(3)}}^{(\mu_{APP}+1)} = \underset{\mathbf{P_{(3)}}}{\operatorname{argmin}} C_{g_1}(P^{(3)}_{g_1(3)}) + C_{g_2}(P^{(3)}_{g_2(3)}) + \boxed{\frac{\beta}{2}||\mathbf{P_{(3)}} - \mathbf{P_{(3)}}^{(\mu_{APP})}||_2^2}$$
$$+\gamma[\boxed{\mathbf{P^{(2)}_{(3)}}^{\dagger}(\mathbf{P^{(2)}_{(3)}}^{(\mu_{APP})} - \mathbf{P^{(2)}_{(2)}}^{(\mu_{APP})})} + \boxed{\mathbf{P^{(3)}_{(3)}}^{\dagger}(\mathbf{P^{(3)}_{(3)}}^{(\mu_{APP})} - \mathbf{P^{(3)}_{(2)}}^{(\mu_{APP})})}]$$
$$\boxed{-\lambda_{\mathbf{3}}{}^{(\mu_{APP})\dagger}\mathbf{P^{(2)}_{(3)}} - \lambda_{\mathbf{4}}{}^{(\mu_{APP})\dagger}\mathbf{P^{(3)}_{(3)}}} \tag{5.7a}$$

$$\textbf{Subject to}:$$

$$\textbf{Power} - \textbf{Balance Constraints}:$$

$$P^{(3)}_{g_1(3)} + P^{(3)}_{g_2(3)} = P^{(3)}_{D_1} + P^{(3)}_{D_2} \tag{5.7b}$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$\frac{P^{(3)}_{g_1(3)} - P^{(3)}_{D_1}}{3} \leq 100 \tag{5.7c}$$

$$\frac{P^{(3)}_{g_1(3)} - P^{(3)}_{D_1}}{2} \leq 125 \tag{5.7d}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_2} \leq P^{(3)(\mu_{APP})}_{g_2(3)} - P^{(3)}_{g_2(3)} \leq \overline{R}_{g_2} \tag{5.7e}$$

$$\underline{R}_{g_1} \leq P^{(3)(\mu_{APP})}_{g_1(3)} - P^{(3)}_{g_1(3)} \leq \overline{R}_{g_1} \tag{5.7f}$$

$$\underline{R}_{g_2} \leq P^{(3)}_{g_2(3)} - P^{(2)}_{g_2(3)} \leq \overline{R}_{g_2} \tag{5.7g}$$

$$\underline{R}_{g_1} \leq P^{(3)}_{g_1(3)} - P^{(2)}_{g_1(3)} \leq \overline{R}_{g_1} \tag{5.7h}$$

We have presented a coarse grained parallelization of the look-ahead SCOPF to cope with demand variation, where the entire problem is decomposed across the different dispatch time intervals. Subsequently, we will present a combination of APP and ADMM based Proximal Message Passing algorithm, which will lead to a fine grained decomposition and parallelization of the problem. In each of the sections to follow, we will present the coarse grain parallelization based on APP.

5.3.2 Generalized Case of Demand Variation for Multi-Bus Case

We will now, state the coarse-grained APP based decomposition of the problem (angles eliminated formulation; that is, the one for which the conventional formulation has been presented in equation (4.17)), following the same method from the previous section. As before, notice, that each of the subproblems is very similar to the classical $(N-1)$ SCOPF problems, but with two important differences. First of all, each of these have ramp rate constraints, since as contrasted to the SCOPF (which was solved for only one dispatch interval), we are solving the LASCOPF problem for multiple dispatch intervals. Hence, the generators need to respect the ramp rate limits, while changing their outputs. Secondly, it can be seen that there are some additional terms, that are added to the objective functions. These are the regularization terms for attaining consensus among different coarse grains about the values of the decision variables, which, in this case, are generator outputs. The terms follow the same color coding, which was introduced in section 5.1. The terms in blue are the ones representing the proximity from previous iterates. The terms in green are the ones for self-consensus, the ones in red are for mutual consensus, and the ones in orange are the terms corresponding to complementary slackness.

First Dispatch Interval SCOPF:

$$
\begin{aligned}
\mathbf{P_{(1)}}^{(\mu_{APP}+1)} = \underset{\mathbf{P_{(1)}}}{\operatorname{argmin}} \sum_{g_q \in G} C_{g_q}(P^{(1)}_{g_q(1)}) + \boxed{\frac{\beta}{2}||\mathbf{P_{(1)}} - \mathbf{P_{(1)}}^{(\mu_{APP})}||_2^2} \\
+\gamma[\boxed{\mathbf{P^{(1)}_{(1)}}^{\dagger}(\mathbf{P^{(1)}_{(1)}}^{(\mu_{APP})} - \mathbf{P^{(1)}_{(2)}}^{(\mu_{APP})})} + \boxed{\mathbf{P^{(2)}_{(1)}}^{\dagger}(\mathbf{P^{(2)}_{(1)}}^{(\mu_{APP})} - \mathbf{P^{(2)}_{(2)}}^{(\mu_{APP})})}] \\
+\boxed{\boldsymbol{\lambda_1}^{(\mu_{APP})\dagger}\mathbf{P^{(1)}_{(1)}} + \boldsymbol{\lambda_2}^{(\mu_{APP})\dagger}\mathbf{P^{(2)}_{(1)}}}
\end{aligned}
\tag{5.8a}
$$

$$\textbf{Subject to :}$$

$$\textbf{Power} - \textbf{Balance Constraints :}$$

$$\sum_{g_q \in G} P^{(1)}_{g_q(1)} = \sum_{D_d \in L} P^{(1)}_{D_d} \tag{5.8b}$$

$$\textbf{Flow Limit Constraints (Base} - \textbf{Case \& Contingency) :}$$

$$|\Phi^{(0)}(\mathbf{P}^{(1)}_{\mathbf{g(1)}} - \mathbf{P}^{(1)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{5.8c}$$

$$|\Phi^{(c)}(\mathbf{P}^{(1)}_{\mathbf{g(1)}} - \mathbf{P}^{(1)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c)}, \forall (c) \in \mathcal{L} - \{(0)\} \tag{5.8d}$$

$$\textbf{Ramp} - \textbf{Rate Constraints :}$$

$$\underline{R}_{g_q} \leq P^{(2)}_{g_q(1)} - P^{(1)}_{g_q(1)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.8e}$$

$$\underline{R}_{g_q} \leq P^{(1)}_{g_q(1)} - P^{(0)}_{g_q} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.8f}$$

For Dispatch Interval $\tau \in \{2, 3, ..., |\Omega| - 1\}$ **SCOPF:**

$$\begin{aligned}
\mathbf{P}_{(\tau)}{}^{(\mu_{APP}+1)} = \underset{\mathbf{P}_{(\tau)}}{\operatorname{argmin}} \sum_{g_q \in G} C_{g_q}(P^{(\tau)}_{g_q(\tau)}) + \boxed{\frac{\beta}{2}||\mathbf{P}_{(\tau)} - \mathbf{P}_{(\tau)}{}^{(\mu_{APP})}||^2_2} \\
+\gamma[\boxed{\mathbf{P}^{(\tau-\mathbf{1})}_{(\tau)}{}^{\dagger}(\mathbf{P}^{(\tau-\mathbf{1})}_{(\tau)}{}^{(\mu_{APP})} - \mathbf{P}^{(\tau-\mathbf{1})}_{(\tau-\mathbf{1})}{}^{(\mu_{APP})})} + \boxed{\mathbf{P}^{(\tau)}_{(\tau)}{}^{\dagger}(\mathbf{P}^{(\tau)}_{(\tau)}{}^{(\mu_{APP})} - \mathbf{P}^{(\tau)}_{(\tau-\mathbf{1})}{}^{(\mu_{APP})})} \\
+\boxed{\mathbf{P}^{(\tau)}_{(\tau)}{}^{\dagger}(\mathbf{P}^{(\tau)}_{(\tau)}{}^{(\mu_{APP})} - \mathbf{P}^{(\tau)}_{(\tau+1)}{}^{(\mu_{APP})})} + \boxed{\mathbf{P}^{(\tau+\mathbf{1})}_{(\tau)}{}^{\dagger}(\mathbf{P}^{(\tau+\mathbf{1})}_{(\tau)}{}^{(\mu_{APP})} - \mathbf{P}^{(\tau+\mathbf{1})}_{(\tau+\mathbf{1})}{}^{(\mu_{APP})})}] \\
+\boxed{\lambda_{\mathbf{2}\tau-\mathbf{1}}{}^{(\mu_{APP})\dagger}\mathbf{P}^{(\tau)}_{(\tau)} + \lambda_{\mathbf{2}\tau}{}^{(\mu_{APP})\dagger}\mathbf{P}^{(\tau+\mathbf{1})}_{(\tau)} - \lambda_{\mathbf{2}\tau-\mathbf{3}}{}^{(\mu_{APP})\dagger}\mathbf{P}^{(\tau-\mathbf{1})}_{(\tau)} - \lambda_{\mathbf{2}\tau-\mathbf{2}}{}^{(\mu_{APP})\dagger}\mathbf{P}^{(\tau)}_{(\tau)}}
\end{aligned} \tag{5.9a}$$

$$\textbf{Subject to :}$$

$$\textbf{Power} - \textbf{Balance Constraints :}$$

$$\sum_{g_q \in G} P^{(\tau)}_{g_q(\tau)} = \sum_{D_d \in L} P^{(\tau)}_{D_d} \tag{5.9b}$$

$$\textbf{Flow Limit Constraints (Base} - \textbf{Case \& Contingency) :}$$

$$|\Phi^{(0)}(\mathbf{P}^{(\tau)}_{\mathbf{g}(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{5.9c}$$

$$|\Phi^{(c)}(\mathbf{P}_{\mathbf{g}(\tau)}^{(\tau)} - \mathbf{P}_{\mathbf{D}}^{(\tau)})| \leq \overline{\mathbf{L}}^{(c)}, \forall (c) \in \mathcal{L} - \{(0)\} \tag{5.9d}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_q} \leq P_{g_q(\tau)}^{(\tau+1)} - P_{g_q(\tau)}^{(\tau)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.9e}$$

$$\underline{R}_{g_q} \leq P_{g_q(\tau)}^{(\tau)} - P_{g_q(\tau)}^{(\tau-1)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.9f}$$

For Dispatch Interval $|\Omega|$ SCOPF:

$$\mathbf{P}_{(|\boldsymbol{\Omega}|)}{}^{(\mu_{APP}+1)} = \underset{\mathbf{P}_{(|\boldsymbol{\Omega}|)}}{\operatorname{argmin}} \sum_{g_q \in G} C_{g_q}(P_{g_q(|\Omega|)}^{(|\Omega|)}) + \boxed{\frac{\beta}{2}||\mathbf{P}_{(|\boldsymbol{\Omega}|)} - \mathbf{P}_{(|\boldsymbol{\Omega}|)}{}^{(\mu_{APP})}||_2^2}$$
$$+\gamma[\boxed{\mathbf{P}_{(|\boldsymbol{\Omega}|)}^{(|\boldsymbol{\Omega}|-\mathbf{1})^\dagger}(\mathbf{P}_{(|\boldsymbol{\Omega}|)}^{(|\boldsymbol{\Omega}|-\mathbf{1})^{(\mu_{APP})}} - \mathbf{P}_{(|\boldsymbol{\Omega}|-\mathbf{1})}^{(|\boldsymbol{\Omega}|-\mathbf{1})^{(\mu_{APP})}})} + \boxed{\mathbf{P}_{(|\boldsymbol{\Omega}|)}^{(|\boldsymbol{\Omega}|)^\dagger}(\mathbf{P}_{(|\boldsymbol{\Omega}|)}^{(|\boldsymbol{\Omega}|)^{(\mu_{APP})}} - \mathbf{P}_{(|\boldsymbol{\Omega}|-\mathbf{1})}^{(|\boldsymbol{\Omega}|)\ \ (\mu_{APP})})}]$$
$$\boxed{-\lambda_{\mathbf{2}|\boldsymbol{\Omega}|-\mathbf{3}}{}^{(\mu_{APP})\dagger}\mathbf{P}_{(|\boldsymbol{\Omega}|)}^{(|\boldsymbol{\Omega}|-\mathbf{1})} - \lambda_{\mathbf{2}|\boldsymbol{\Omega}|-\mathbf{2}}{}^{(\mu_{APP})\dagger}\mathbf{P}_{(|\boldsymbol{\Omega}|)}^{(|\boldsymbol{\Omega}|)}} \tag{5.10a}$$

$$\textbf{Subject to}:$$

$$\textbf{Power} - \textbf{Balance Constraints}:$$

$$\sum_{g_q \in G} P_{g_q(|\Omega|)}^{(|\Omega|)} = \sum_{D_d \in L} P_{D_d}^{(|\Omega|)} \tag{5.10b}$$

$$\textbf{Flow Limit Constraints (Base} - \textbf{Case \& Contingency)}:$$

$$|\Phi^{(0)}(\mathbf{P}_{\mathbf{g}(|\boldsymbol{\Omega}|)}^{(|\boldsymbol{\Omega}|)} - \mathbf{P}_{\mathbf{D}}^{(|\boldsymbol{\Omega}|)})| \leq \overline{\mathbf{L}}^{(0)} \tag{5.10c}$$

$$|\Phi^{(c)}(\mathbf{P}_{\mathbf{g}(|\boldsymbol{\Omega}|)}^{(|\boldsymbol{\Omega}|)} - \mathbf{P}_{\mathbf{D}}^{(|\boldsymbol{\Omega}|)})| \leq \overline{\mathbf{L}}^{(c)}, \forall (c) \in \mathcal{L} - \{(0)\} \tag{5.10d}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_q} \leq P_{g_q(|\Omega|)}^{(|\Omega|)(\mu_{APP})} - P_{g_q(|\Omega|)}^{(|\Omega|)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.10e}$$

$$\underline{R}_{g_q} \leq P_{g_q(|\Omega|)}^{(|\Omega|)} - P_{g_q(|\Omega|)}^{(|\Omega|-1)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.10f}$$

We will now present the APP decomposition for the angles represented case (the conventional formulation for which appears in equation (4.18)). Note that, in this case also, the real power generation variables are the only ones at which we wish to attain consensus through the application of APP. The bus voltage angles can be thought of as localized to the particular dispatch interval sub-problems, whose values at any iteration are determined by the power generation or injection profiles.

First Dispatch Interval SCOPF:

$$\mathbf{P_{(1)}}^{(\mu_{APP}+1)} = \underset{\mathbf{P_{(1)}},\theta}{\operatorname{argmin}} \sum_{g_q \in G} C_{g_q}(P^{(1)}_{g_q(1)}) + \boxed{\frac{\beta}{2}||\mathbf{P_{(1)}} - \mathbf{P_{(1)}}^{(\mu_{APP})}||_2^2}$$
$$+\gamma[\boxed{\mathbf{P^{(1)}_{(1)}}^{\dagger}(\mathbf{P^{(1)}_{(1)}}^{(\mu_{APP})} - \mathbf{P^{(1)}_{(2)}}^{(\mu_{APP})})} + \boxed{\mathbf{P^{(2)}_{(1)}}^{\dagger}(\mathbf{P^{(2)}_{(1)}}^{(\mu_{APP})} - \mathbf{P^{(2)}_{(2)}}^{(\mu_{APP})})}]$$
$$+\boxed{\lambda_{\mathbf{1}}^{(\mu_{APP})\dagger}\mathbf{P^{(1)}_{(1)}} + \lambda_{\mathbf{2}}^{(\mu_{APP})\dagger}\mathbf{P^{(2)}_{(1)}}} \tag{5.11a}$$

$$\textbf{Subject to}: \forall (c) \in \mathcal{L},\ \forall T_r \in T$$

$$\textbf{Power} - \textbf{Balance Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$P^{(1)}_{g_{q_{N_i}}(1)} - P^{(1)}_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} B^{(0)}_{T_r}(\theta^{(0)(1)}_{N_i} - \theta^{(0)(1)}_{N_{\bar{i}}});\ \forall N_i \in \mathcal{N} \tag{5.11b}$$

$$P^{(1)}_{g_{q_{N_i}}(1)} - P^{(1)}_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} B^{(c)}_{T_r}(\theta^{(c)(1)}_{N_i} - \theta^{(c)(1)}_{N_{\bar{i}}});\ \forall N_i \in \mathcal{N} \tag{5.11c}$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$|B^{(0)}_{T_r}(\theta^{(0)(1)}_{T_{r t_1}} - \theta^{(0)(1)}_{T_{r t_2}})| \leq \overline{L}^{(0)}_{T_r},\ \forall T_r \in T \tag{5.11d}$$

$$|B^{(c)}_{T_r}(\theta^{(c)(1)}_{T_{r t_1}} - \theta^{(c)(1)}_{T_{r t_2}})| \leq \overline{L}^{(c)}_{T_r},\ \forall T_r \in T \tag{5.11e}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_q} \leq P^{(2)}_{g_q(1)} - P^{(1)}_{g_q(1)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.11f}$$

$$\underline{R}_{g_q} \leq P^{(1)}_{g_q(1)} - P^{(0)}_{g_q} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.11g}$$

For Dispatch Interval $\tau \in \{2, 3, ..., |\Omega|-1\}$ SCOPF:

$$\mathbf{P}_{(\tau)}{}^{(\mu_{APP}+1)} = \underset{\mathbf{P}_{(\tau)},\theta}{\text{argmin}} \sum_{g_q \in G} C_{g_q}(P^{(\tau)}_{g_q(\tau)}) + \boxed{\frac{\beta}{2}||\mathbf{P}_{(\tau)} - \mathbf{P}_{(\tau)}{}^{(\mu_{APP})}||_2^2}$$
$$+\gamma[\boxed{\mathbf{P}^{(\tau-1)^{\dagger}}_{(\tau)}(\mathbf{P}^{(\tau-1)(\mu_{APP})}_{(\tau)} - \mathbf{P}^{(\tau-1)(\mu_{APP})}_{(\tau-1)})} + \boxed{\mathbf{P}^{(\tau)^{\dagger}}_{(\tau)}(\mathbf{P}^{(\tau)(\mu_{APP})}_{(\tau)} - \mathbf{P}^{(\tau)\quad(\mu_{APP})}_{(\tau-1)})}$$
$$+\boxed{\mathbf{P}^{(\tau)^{\dagger}}_{(\tau)}(\mathbf{P}^{(\tau)(\mu_{APP})}_{(\tau)} - \mathbf{P}^{(\tau)\quad(\mu_{APP})}_{(\tau+1)})} + \boxed{\mathbf{P}^{(\tau+1)^{\dagger}}_{(\tau)}(\mathbf{P}^{(\tau+1)(\mu_{APP})}_{(\tau)} - \mathbf{P}^{(\tau+1)(\mu_{APP})}_{(\tau+1)})}]$$
$$+\boxed{\lambda_{2\tau-1}{}^{(\mu_{APP})\dagger}\mathbf{P}^{(\tau)}_{(\tau)} + \lambda_{2\tau}{}^{(\mu_{APP})\dagger}\mathbf{P}^{(\tau+1)}_{(\tau)} - \lambda_{2\tau-3}{}^{(\mu_{APP})\dagger}\mathbf{P}^{(\tau-1)}_{(\tau)} - \lambda_{2\tau-2}{}^{(\mu_{APP})\dagger}\mathbf{P}^{(\tau)}_{(\tau)}} \tag{5.12a}$$

$$\textbf{Subject to}: \forall(c) \in \mathcal{L},\ \forall T_r \in T$$

$$\textbf{Power}-\textbf{Balance Constraints}\ (\textbf{Base}-\textbf{Case}\ \&\ \textbf{Contingency}):$$

$$P^{(\tau)}_{g_{q_{N_i}}(\tau)} - P^{(\tau)}_{D_{dN_i}} = \sum_{N_{\bar{i}} \in J(N_i)} B^{(0)}_{T_r}(\theta^{(0)(\tau)}_{N_i} - \theta^{(0)(\tau)}_{N_{\bar{i}}});\ \forall N_i \in \mathcal{N} \tag{5.12b}$$

$$P^{(\tau)}_{g_{q_{N_i}}(\tau)} - P^{(\tau)}_{D_{dN_i}} = \sum_{N_{\bar{i}} \in J(N_i)} B^{(c)}_{T_r}(\theta^{(c)(\tau)}_{N_i} - \theta^{(c)(\tau)}_{N_{\bar{i}}});\ \forall N_i \in \mathcal{N} \tag{5.12c}$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base}-\textbf{Case}\ \&\ \textbf{Contingency}):$$

$$|B^{(0)}_{T_r}(\theta^{(0)(\tau)}_{T_{rt_1}} - \theta^{(0)(\tau)}_{T_{rt_2}})| \leq \overline{L}^{(0)}_{T_r},\ \forall T_r \in T \tag{5.12d}$$

$$|B^{(c)}_{T_r}(\theta^{(c)(\tau)}_{T_{rt_1}} - \theta^{(c)(\tau)}_{T_{rt_2}})| \leq \overline{L}^{(c)}_{T_r},\ \forall T_r \in T \tag{5.12e}$$

$$\textbf{Ramp}-\textbf{Rate Constraints}:$$

$$\underline{R}_{g_q} \leq P^{(\tau+1)}_{g_q(\tau)} - P^{(\tau)}_{g_q(\tau)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.12f}$$

$$\underline{R}_{g_q} \leq P^{(\tau)}_{g_q(\tau)} - P^{(\tau-1)}_{g_q(\tau)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.12g}$$

For Dispatch Interval $|\Omega|$ SCOPF:

$$\mathbf{P}_{(|\Omega|)}{}^{(\mu_{APP}+1)} = \underset{\mathbf{P}_{(|\Omega|)},\theta}{\text{argmin}} \sum_{g_q \in G} C_{g_q}(P^{(|\Omega|)}_{g_q(|\Omega|)}) + \boxed{\frac{\beta}{2}||\mathbf{P}_{(|\Omega|)} - \mathbf{P}_{(|\Omega|)}{}^{(\mu_{APP})}||_2^2}$$
$$+\gamma[\boxed{\mathbf{P}^{(|\Omega|-1)^{\dagger}}_{(|\Omega|)}(\mathbf{P}^{(|\Omega|-1)(\mu_{APP})}_{(|\Omega|)} - \mathbf{P}^{(|\Omega|-1)(\mu_{APP})}_{(|\Omega|-1)})} + \boxed{\mathbf{P}^{(|\Omega|)^{\dagger}}_{(|\Omega|)}(\mathbf{P}^{(|\Omega|)(\mu_{APP})}_{(|\Omega|)} - \mathbf{P}^{(|\Omega|)\quad(\mu_{APP})}_{(|\Omega|-1)})}]$$

$$\boxed{-\lambda_{\mathbf{2|\Omega|-3}}{}^{(\mu_{APP})\dagger}\mathbf{P}_{(|\mathbf{\Omega}|)}^{(|\mathbf{\Omega}|-\mathbf{1})} - \lambda_{\mathbf{2|\Omega|-2}}{}^{(\mu_{APP})\dagger}\mathbf{P}_{(|\mathbf{\Omega}|)}^{(|\mathbf{\Omega}|)}} \tag{5.13a}$$

$$\textbf{Subject to}: \forall (c) \in \mathcal{L},\ \forall T_r \in T$$

$$\textbf{Power} - \textbf{Balance Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$P_{g_{q_{N_i}}(|\Omega|)}^{(|\Omega|)} - P_{D_{dN_i}}^{(|\Omega|)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(0)}(\theta_{N_i}^{(0)(|\Omega|)} - \theta_{N_{\bar{i}}}^{(0)(|\Omega|)});\ \forall N_i \in \mathcal{N} \tag{5.13b}$$

$$P_{g_{q_{N_i}}(|\Omega|)}^{(|\Omega|)} - P_{D_{dN_i}}^{(|\Omega|)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(c)}(\theta_{N_i}^{(c)(|\Omega|)} - \theta_{N_{\bar{i}}}^{(c)(|\Omega|)});\ \forall N_i \in \mathcal{N} \tag{5.13c}$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$|B_{T_r}^{(0)}(\theta_{T_{rt_1}}^{(0)(|\Omega|)} - \theta_{T_{rt_2}}^{(0)(|\Omega|)})| \leq \overline{L}_{T_r}^{(0)},\ \forall T_r \in T \tag{5.13d}$$

$$|B_{T_r}^{(c)}(\theta_{T_{rt_1}}^{(c)(|\Omega|)} - \theta_{T_{rt_2}}^{(c)(|\Omega|)})| \leq \overline{L}_{T_r}^{(c)},\ \forall T_r \in T \tag{5.13e}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_q} \leq P_{g_q(|\Omega|)}^{(|\Omega|)(\mu_{APP})} - P_{g_q(|\Omega|)}^{(|\Omega|)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.13f}$$

$$\underline{R}_{g_q} \leq P_{g_q(|\Omega|)}^{(|\Omega|)} - P_{g_q(|\Omega|)}^{(|\Omega|-1)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{5.13g}$$

Thus from the above equations we can observe that after the APP decomposition, for each dispatch time interval, the problem that is solved is just the SCOPF, with some regularization terms added to the objective function in order to achieve consensus.

5.4 Coarse-Grained Component for LASCOPF: Post-Contingency Restoration in One Dispatch Interval

Described below is the application of the APP based coarse-grained component to the LASCOPF problems for post-contingency restoration in one dispatch interval, the conventional formulation for which has been presented in equations (4.11), (4.12), (4.19), (4.20)

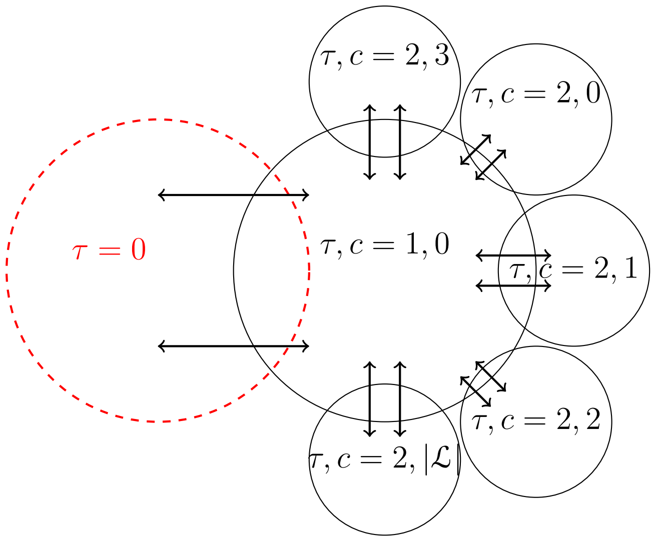


Figure 5.11: Schematic for Look-Ahead SCOPF for Post-Contingency Restoration

5.4.1 Look-Ahead Dispatch Model for Generalized Multi-Bus Case

Now, we will, as before, apply the APP algorithm to effect a coarse-grained distribution of the above problem. We will present the entire derivation for the angles eliminated version. The angles included version, can simply be written down by just modifying the power balance and flow limit constraints as before. For the sake of brevity, we are not presenting those here. The next set of equations represent the augmented Lagrangian for the optimization model of (4.19):

$$\forall \tau \in \Omega$$

$$\min \sum_{g_q \in G} \left(C_{g_q}(P^{(0)(\tau)}_{g_q(\tau)}) + \sum_{(c) \in \mathcal{L}} prob^{(c)} C_{g_q}(P^{(c)(\tau+1)}_{g_q(c)(\tau+1)}) \right)$$
$$+\frac{\gamma}{2}(||\mathbf{P}^{(\mathbf{0})(\tau)}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)}_{(\mathbf{0})(\tau+\mathbf{1})}||_2^2 + ||\mathbf{P}^{(\mathbf{0})(\tau+\mathbf{1})}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau+\mathbf{1})}_{(\mathbf{0})(\tau+\mathbf{1})}||_2^2 +$$
$$||\mathbf{P}^{(\mathbf{0})(\tau)}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)}_{(\mathbf{1})(\tau+\mathbf{1})}||_2^2 + ||\mathbf{P}^{(\mathbf{1})(\tau+\mathbf{1})}_{(\tau)} - \mathbf{P}^{(\mathbf{1})(\tau+\mathbf{1})}_{(\mathbf{1})(\tau+\mathbf{1})}||_2^2 +$$

$$||\mathbf{P}^{(\mathbf{0})(\tau)}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)}_{(\mathbf{2})(\tau+\mathbf{1})}||_2^2 + ||\mathbf{P}^{(\mathbf{2})(\tau+\mathbf{1})}_{(\tau)} - \mathbf{P}^{(\mathbf{2})(\tau+\mathbf{1})}_{(\mathbf{2})(\tau+\mathbf{1})}||_2^2 +$$

$$+ ... + ||\mathbf{P}^{(\mathbf{0})(\tau)}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)}_{(|\mathcal{L}|)(\tau+\mathbf{1})}||_2^2 + ||\mathbf{P}^{(|\mathcal{L}|)(\tau+\mathbf{1})}_{(\tau)} - \mathbf{P}^{(|\mathcal{L}|)(\tau+\mathbf{1})}_{(|\mathcal{L}|)(\tau+\mathbf{1})}||_2^2) \tag{5.14a}$$

$$\textbf{Subject to}: \forall \tau \in \Omega,\ \forall T_r \in T,\ \forall (c) \in \mathcal{L}$$

$$\textbf{Power} - \textbf{Balance Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$\sum_{g_q \in G} P^{(0)(\tau)}_{g_q(\tau)} = \sum_{D_d \in L} P^{(\tau)}_{D_d} \tag{5.14b}$$

$$\sum_{g_q \in G} P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} = \sum_{D_d \in L} P^{(\tau+1)}_{D_d} \tag{5.14c}$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$|\Phi^{(0)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{\mathbf{g}(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{5.14d}$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{\mathbf{g}(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c)} \tag{5.14e}$$

$$\forall (c) \in \mathcal{L}, \forall (c^{'}) \in [\mathcal{L} - \{c\}]$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\tau+\mathbf{1})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{5.14f}$$

$$|\Phi^{(c \to c^{'})}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\tau+\mathbf{1})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c \to c^{'})} \tag{5.14g}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)}_{g_q(\tau)} - P^{(0)(\tau)}_{g_q(\tau)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.14h}$$

$$\underline{R}_{g_q} \leq P^{(0)(\tau)}_{g_q(\tau)} - P^{(0)(\tau-1)}_{g_q} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.14i}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)(\mu_{APP})}_{g_q(c)(\tau+1)} - P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.14j}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} - P^{(0)(\tau)}_{g_q(c)(\tau+1)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.14k}$$

$$\textbf{Consensus Constraints}:$$

$$P^{(0)(\tau)}_{g_q(\tau)} = P^{(0)(\tau)}_{g_q(c)(\tau+1)}, \forall (c) \in \mathcal{L},\ \forall g_q \in G \tag{5.14l}$$

$$P^{(c)(\tau+1)}_{g_q(\tau)} = P^{(c)(\tau+1)}_{g_q(c)(\tau+1)}, \forall (c) \in \mathcal{L},\ \forall g_q \in G \tag{5.14m}$$

As before, the iterates are as follows:

$$
\begin{aligned}
&(\mathbf{P}_{(\tau)}{}^{(\mu_{APP}+1)}, \mathbf{P}_{(\mathbf{0})(\tau+\mathbf{1})}{}^{(\mu_{APP}+1)}, ..., \mathbf{P}_{(|\mathcal{L}|)(\tau+\mathbf{1})}{}^{(\mu_{APP}+1)}) \\
&= \underset{\mathbf{P}_{(\tau)},\mathbf{P}_{(\mathbf{0})(\tau+\mathbf{1})},...,\mathbf{P}_{(|\mathcal{L}|)(\tau+\mathbf{1})}}{\text{argmin}} \sum_{g_q \in G} \left(C_{g_q}(P^{(0)(\tau)}_{g_q(\tau)}) + \sum_{(c)\in[\mathcal{L}]} prob^{(c)} C_{g_q}(P^{(c)(\tau+1)}_{g_q(c)(\tau+1)}) \right) \\
&\quad + \frac{\beta}{2}(||\mathbf{P}_{(\tau)} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\tau)}||_2^2 + ||\mathbf{P}_{(\mathbf{0})(\tau+\mathbf{1})} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\mathbf{0})(\tau+\mathbf{1})}||_2^2 + \\
&\quad ||\mathbf{P}_{(\mathbf{1})(\tau+\mathbf{1})} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\mathbf{1})(\tau+\mathbf{1})}||_2^2 + ||\mathbf{P}_{(\mathbf{2})(\tau+\mathbf{1})} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\mathbf{2})(\tau+\mathbf{1})}||_2^2 \\
&\quad + ... + ||\mathbf{P}_{(|\mathcal{L}|)(\tau+\mathbf{1})} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(|\mathcal{L}|)(\tau+\mathbf{1})}||_2^2) \\
&\quad + \gamma[\sum_{c=0}^{|\mathcal{L}|}((\mathbf{P}^{(\mathbf{0})(\tau)}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)}_{(\mathbf{c})(\tau+\mathbf{1})})^{\dagger}(\mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}})}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+1)}) \\
&\quad + (\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{(\tau)} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{1})})^{\dagger}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\tau)} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{1})}))] \\
&+ \sum_{c=0}^{|\mathcal{L}|}(\lambda^{(\mu_{APP})\dagger}_{(2c+1)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)}_{(\mathbf{c})(\tau+\mathbf{1})}) + \lambda^{(\mu_{APP})\dagger}_{(2c+2)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{(\tau)} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{1})})) \qquad (5.15a)
\end{aligned}
$$

$$\textbf{Subject to}: \forall\tau \in \Omega,\ \forall T_r \in T,\ \forall(c) \in \mathcal{L}$$

$$\textbf{Power} - \textbf{Balance Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$\sum_{g_q \in G} P^{(0)(\tau)}_{g_q(\tau)} = \sum_{D_d \in L} P^{(\tau)}_{D_d} \qquad (5.15b)$$

$$\sum_{g_q \in G} P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} = \sum_{D_d \in L} P^{(\tau+1)}_{D_d} \qquad (5.15c)$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base} - \textbf{Case}\ \&\ \textbf{Contingency}):$$

$$|\Phi^{(0)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{\mathbf{g}(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \qquad (5.15d)$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{\mathbf{g}(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c)} \qquad (5.15e)$$

$$\forall(c) \in \mathcal{L}, \forall(c^{'}) \in [\mathcal{L} - \{c\}]$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\tau+\mathbf{1})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \qquad (5.15f)$$

$$|\Phi^{(c\to c^{'})}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\tau+\mathbf{1})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c\to c^{'})} \tag{5.15g}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)}_{g_q(\tau)} - P^{(0)(\tau)}_{g_q(\tau)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.15h}$$

$$\underline{R}_{g_q} \leq P^{(0)(\tau)}_{g_q(\tau)} - P^{(0)(\tau-1)}_{g_q} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.15i}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)(\mu_{APP})}_{g_q(c)(\tau+1)} - P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.15j}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} - P^{(0)(\tau)}_{g_q(c)(\tau+1)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.15k}$$

$$\textbf{Dual Variable Updates}: \forall c \in \mathcal{L}$$

$$\lambda_{(\mathbf{2c+1})}{}^{(\mu_{APP}+1)} = \lambda_{(\mathbf{2c+1})}{}^{(\mu_{APP})} + \alpha(\mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}}+\mathbf{1})}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}}+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{1})}) \tag{5.15l}$$

$$\lambda_{(\mathbf{2c+2})}{}^{(\mu_{APP}+1)} = \lambda_{(\mathbf{2c+2})}{}^{(\mu_{APP})} + \alpha(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}}+\mathbf{1})}_{(\tau)} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}}+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{1})}) \tag{5.15m}$$

The coarse grained decomposition of the iterates can be classified into two categories as below:

Iterates for Dispatch Interval τ

$$\begin{aligned}\mathbf{P}_{(\tau)}{}^{(\mu_{APP}+1)} = \underset{\mathbf{P}_{(\tau)}}{\operatorname{argmin}} \sum_{g_q \in G} C_{g_q}(P^{(0)(\tau)}_{g_q(\tau)}) + \boxed{\frac{\beta}{2}||\mathbf{P}_{(\tau)} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\tau)}||_2^2} \\ +\gamma[\sum_{c=0}^{|\mathcal{L}|}(\boxed{\mathbf{P}^{(\mathbf{0})(\tau)\dagger}_{(\tau)}(\mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}})}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{1})})} \\ +\boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})\dagger}_{(\tau)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\tau)} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{1})})})] \\ +\boxed{\sum_{c=0}^{|\mathcal{L}|}(\lambda^{(\mu_{APP})\dagger}_{(2c+1)}\mathbf{P}^{(\mathbf{0})(\tau)}_{(\tau)} + \lambda^{(\mu_{APP})\dagger}_{(2c+2)}\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{(\tau)})}\end{aligned} \tag{5.16a}$$

$$\textbf{Subject to}: \forall \tau \in \Omega,\ \forall T_r \in T,\ \forall (c) \in \mathcal{L}$$

$$\textbf{Power} - \textbf{Balance Constraints}\ (\textbf{Base} - \textbf{Case}):$$

$$\sum_{g_q \in G} P^{(0)(\tau)}_{g_q(\tau)} = \sum_{D_d \in L} P^{(\tau)}_{D_d} \tag{5.16b}$$

$$\textbf{Flow Limit Constraints (Base - Case \& Contingency)} :$$

$$|\Phi^{(0)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{\mathbf{g}(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{5.16c}$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{\mathbf{g}(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c)} \tag{5.16d}$$

$$\textbf{Ramp - Rate Constraints} :$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)}_{g_q(\tau)} - P^{(0)(\tau)}_{g_q(\tau)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.16e}$$

$$\underline{R}_{g_q} \leq P^{(0)(\tau)}_{g_q(\tau)} - P^{(0)(\tau-1)}_{g_q} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.16f}$$

Iterates for Dispatch Interval $\tau + 1$

$$\forall c \in \mathcal{L}$$

$$\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{1})}{}^{(\mu_{APP}+1)} = \underset{\mathbf{P}_{(\mathbf{c})(\tau+1)}}{\text{argmin}} \sum_{g_q \in G} C_{g_q}(P^{(c)(\tau+1)}_{g_q(c)(\tau+1)})$$

$$+ \boxed{\frac{\beta}{2}||\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{1})}||^2_2} + \gamma[\boxed{\mathbf{P}^{(\mathbf{0})(\tau)\dagger}_{(\mathbf{c})(\tau+\mathbf{1})}(\mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}})}_{(\tau)})}$$

$$+ \boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})\dagger}_{(\mathbf{c})(\tau+\mathbf{1})}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\tau)})}]$$

$$\boxed{-\lambda^{(\mu_{APP})\dagger}_{(2c+1)}\mathbf{P}^{(\mathbf{0})(\tau)}_{(\mathbf{c})(\tau+\mathbf{1})} - \lambda^{(\mu_{APP})\dagger}_{(2c+2)}\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{1})}} \tag{5.17a}$$

$$\textbf{Subject to} : \forall \tau \in \Omega,\ \forall T_r \in T,\ \forall (c) \in \mathcal{L}$$

$$\textbf{Power - Balance Constraints (Contingency Cases)} :$$

$$\sum_{g_q \in G} P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} = \sum_{D_d \in L} P^{(\tau+1)}_{D_d} \tag{5.17b}$$

$$\textbf{Flow Limit Constraints (Base - Case \& Contingency)} :$$

$$\forall (c) \in \mathcal{L}, \forall (c') \in [\mathcal{L} - \{c\}]$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\tau+\mathbf{1})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{5.17c}$$

$$|\Phi^{(c\to c^{'})}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{\mathbf{g(c)}(\tau+\mathbf{1})} - \mathbf{P}^{(\tau+\mathbf{1})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c\to c^{'})} \tag{5.17d}$$

$$\mathbf{Ramp - Rate\, Constraints}:$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)(\mu_{APP})}_{g_q(c)(\tau+1)} - P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.17e}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} - P^{(0)(\tau)}_{g_q(c)(\tau+1)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{5.17f}$$

5.5 Fine-Grained Message Passing Algorithm

In the previous sections, we explored the coarse-grained distributed message passing algorithm to split LASCOPFs (applicable to LAOPFs as well) across multiple dispatch intervals. In this section, we describe the fine-grained distributed message passing algorithm used to solve each of the SCOPFs belonging to each of the dispatch intervals under consideration. We begin by assuming that all device objective functions are convex, closed, and proper (CCP) functions. We then derive our distributed, message passing algorithm using operator splitting and the alternating directions method of multipliers (ADMM) [49]. This algorithm has guaranteed convergence for CCP functions, is fully decentralized, is self-healing and robust. The particular form that we will derive here is the ADMM based Proximal Message Passing (PMP) algorithm, used to solve the class of problems called "Consensus and Sharing."

5.5.1 Consensus Form SCOPF

Before applying ADMM to solve the SCOPF, we first replicate the power plans $\mathbf{P} \in \mathbf{R}^{|\mathcal{T}|\times(|\mathcal{L}|+1)}$ by introducing a copy, $\mathbf{z} \in \mathbf{R}^{|\mathcal{T}|\times(|\mathcal{L}|+1)}$, of the plans. We then solve the *consensus* *form SCOPF*:

$$\begin{array}{ll} \text{minimize} & f(\mathbf{P}^{(0)}) \\ \text{subject to} & \hat{\mathbf{z}}^{(c)} = \mathbf{0},\ \forall c \in \mathcal{L} \\ & \mathbf{P} = \mathbf{z} \rightarrow \text{Consensus constraint}, \end{array} \tag{5.18}$$

where $\hat{\mathbf{z}}^{(c)} \in \mathbf{R}^{|\mathcal{N}|}$ is the vector of arithmetic means of $\mathbf{z}^{(c)}$ associated with the nets and a particular contingency scenario. Because of the consensus constraint, when we solve the consensus form SCOPF, the optimal solution will agree with the solution of the original SCOPF. We introduce the indicator function $g : \mathbf{R}^{|\mathcal{T}|\times(|\mathcal{L}|+1)} \rightarrow \mathbf{R}$, defined as, $g(\mathbf{z}) = \mathcal{I}_{\{\mathbf{z}|\hat{\mathbf{z}}=\mathbf{0}\}}(\mathbf{z})$, which is 0 whenever $\hat{\mathbf{z}} = \mathbf{0}$ and $+\infty$ otherwise (if the power balance constraint is violated). Because $\hat{\mathbf{z}}$ is the vector of the average power at each net, the set $\{\mathbf{z} \mid \hat{\mathbf{z}} = \mathbf{0}\}$ can be written as $\bigcap_{N_i \in \mathcal{N}} \{\mathbf{z} \mid \hat{\mathbf{z}}_{N_i} = 0\}$, where $\hat{\mathbf{z}}_{N_i}$ is the average power at net N_i; then,

$$g(\mathbf{z}) = \sum_{N_i \in \mathcal{N}} g_{N_i}(\mathbf{z}) = \sum_{N_i \in \mathcal{N}} \mathcal{I}_{\{\mathbf{z}|\hat{\mathbf{z}}_{N_i}=0\}}(\mathbf{z}).$$

Since the summands in the last expression only involve each net N_i separately, $g(\mathbf{z})$ separates across nets completely

$$g(\mathbf{z}) = \sum_{N_i \in \mathcal{N}} \mathcal{I}_{\{\mathbf{z}_{N_i}|\hat{\mathbf{z}}_{N_i}=0\}}(\mathbf{z}_{N_i}).$$

5.5.2 ADMM Based Proximal Message Passing (PMP) Algorithm

We apply ADMM to solve the SCOPF by first forming the (scaled) augmented Lagrangian,

$$\mathcal{L}(\mathbf{P}, \mathbf{z}, \mathbf{u}) = f(\mathbf{P}) + g(\mathbf{z}) + (\rho/2)\|\mathbf{P} - \mathbf{z} + \mathbf{u}\|_2^2,$$

where $\mathbf{u} = (1/\rho)\mathbf{y}$ is the vector of the scaled dual variable $\mathbf{y}$ associated with the consensus constraint $\mathbf{P} = \mathbf{z}$. We obtained the augmented Lagrangian by completing the squares.

ADMM is then

$$\begin{aligned}
\mathbf{P}^{(\nu+1)} &:= \operatorname*{argmin}_{\mathbf{P}} \left(f(\mathbf{P}) + (\rho/2)\|\mathbf{P} - \mathbf{z}^{(\nu)} + \mathbf{u}^{(\nu)}\|_2^2 \right) \\
\mathbf{z}^{(\nu+1)} &:= \operatorname*{argmin}_{\mathbf{z}} \left(g(\mathbf{z}) + (\rho/2)\|\mathbf{P}^{(\nu+1)} - \mathbf{z} + \mathbf{u}^{(\nu)}\|_2^2 \right) \\
\mathbf{u}^{(\nu+1)} &:= \mathbf{u}^{(\nu)} + (\mathbf{P}^{(\nu+1)} - \mathbf{z}^{(\nu+1)}).
\end{aligned}$$

Note that the superscript is an iteration counter—not the contingency label.

Because of our problem structure, we can further simplify ADMM. The $\mathbf{P}$-updates separate across devices and

$$P_d^{(\nu+1)} := \operatorname*{argmin}_{P_d} \left(f_d(P_d) + (\rho/2)\|P_d - z_d^{(\nu)} + u_d^{(\nu)}\|_2^2 \right)$$

for all $d \in \mathcal{D}$. Furthermore, the $\mathbf{z}$-updates separate across nets and $\mathbf{z}_{N_i}$-update is just a Euclidean projection on to the set $\hat{\mathbf{z}}_{N_i} = 0$ and can be solved analytically, so

$$\mathbf{z}_{N_i}^{(\nu+1)} = \mathbf{P}_{N_i}^{(\nu+1)} + \mathbf{u}_{N_i}^{(\nu)} - \hat{\mathbf{P}}_{N_i}^{(\nu+1)} - \hat{\mathbf{u}}_{N_i}^{(\nu)}.$$

Substituting this expression for $\mathbf{z}_{N_i}$ in to the u-update—which also splits across nets—we obtain the **prox-project message passing algorithm/proximal message passing algorithm**:

1. *Proximal plan updates.*

$$P_d^{(\nu+1)} := \mathbf{prox}_{f_d,\rho}(P_d^{(\nu)} - \hat{P}_d^{(\nu)} - u_d^{(\nu)}), \quad \forall d \in \mathcal{D}.$$

2. *Scaled price updates.*

$$\mathbf{u}_{N_i}^{(\nu+1)} := \mathbf{u}_{N_i}^{(\nu)} + \hat{\mathbf{P}}_{N_i}^{(\nu+1)}, \quad \forall N_i \in \mathcal{N},$$

where the proximal function for a function g is given by

$$\mathbf{prox}_{g,\rho}(\mathbf{v}) = \underset{\mathbf{x}}{\operatorname{argmin}}(g(\mathbf{x}) + (\rho/2)\|\mathbf{x} - \mathbf{v}\|_2^2).$$

The net variables are vectorized because each net variable is actually a vector, whose components are the scalars of the corresponding variable associated with each terminal that is connected to the concerned net. The ADMM based Proximal Message Passing (PMP) algorithm alternates between evaluating prox functions (in parallel) on each device and performing price updates on each net. This algorithm has the following three properties:

Convergence. Since our PMP algorithm is a (simplified) version of ADMM, the convergence results for ADMM also apply to prox-project message passing. In particular, with mild conditions on device objective functions f_d—namely, that they are closed, convex, and proper—and provided a feasible solution exists, the following properties of our algorithm hold.

1. *Residual convergence.* $\hat{\mathbf{P}}^{(\nu)} \to \mathbf{0}$ as $\nu \to \infty$,
2. *Objective convergence.* $\sum_{d\in\mathcal{D}} f_d(P_d^{(\nu)}) + \sum_{N_i\in\mathcal{N}} g_{N_i}(\mathbf{P}_{N_i}^{(\nu)}) \to f^\star$ as $\nu \to \infty$,
3. *Dual variable convergence.* $\rho\mathbf{u}^{(\nu)} = \mathbf{y}^{(\nu)} \to \mathbf{y}^\star$ as $\nu \to \infty$,

where $f^\star$ is the optimal value for the (convex) SCOPF, and $\mathbf{y}^\star$ are the optimal dual variables (prices). A proof of these conditions can be found in [49].

Convergence of our algorithm guarantees that, if message passing is run long enough, power balance will be satisfied by $\mathbf{P}^{(\nu)}$. Furthermore, this $\mathbf{P}^{(\nu)}$ will minimize the total cost of operating the network.

Distributed. Note that while the scaled price updates were separated across nets, it could have also been separated across devices. As long as each device has the ability to access the average power imbalance for the nets it shares with its neighbors, this algorithm can be completely decentralized. The scaled price updates can happen locally, with each device retaining a copy of the (scaled) prices u. This means that net computation (scaled price update) can be virtualized and done on each device instead of on each net, as long as devices that share a net are able to compute their average power imbalance. Then, the algorithm consists of each device planning for each contingency and a broadcast of plans to its neighbors.

Self-healing. Because the semantics of a net is to enforce power balance, when a device connects or disconnects from a net (*i.e.*, when the network topology changes), the algorithm can continue to function without modification.

When the topology of the network changes, the current iterates $\mathbf{P}^{(\nu)}$ and $\mathbf{u}^{(\nu)}$ of our algorithm can be used as a *warm start*. Assuming topology changes are not too frequent, the message passing algorithm will simply converge to the new fixed point.

5.5.3 Stopping criterion

We can define primal and dual residuals for the PMP algorithm:

$$\mathbf{r}^{(\nu)} = \left(\hat{\mathbf{P}}^{(\nu)}, \tilde{\mathbf{\Theta}}^{(\nu)}\right), \mathbf{s}^{(\nu)} = \rho\left(((\mathbf{P}^{(\nu)} - \hat{\mathbf{P}}^{(\nu)}) - (\mathbf{P}^{(\nu-1)} - \hat{\mathbf{P}}^{(\nu-1)})), (\hat{\mathbf{\Theta}}^{(\nu)} - -\hat{\mathbf{\Theta}}^{(\nu-1)})\right).$$

The interpretation of $\mathbf{P}^{(\nu)}$ is as a power plan.

A simple terminating criterion for prox-project message passing is when

$$\|\mathbf{r}^{(\nu)}\|_2 \leq \epsilon^{\text{pri}}, \qquad \|\mathbf{s}^{(\nu)}\|_2 \leq \epsilon^{\text{dual}},$$

where ϵ^{pri} and ϵ^{dual} are, respectively, primal and dual tolerances. We can normalize both of these quantities to the network size by the relation

$$\epsilon^{\text{pri}} = \epsilon^{\text{dual}} = \epsilon^{\text{abs}} \sqrt{|\mathcal{T}|(|\mathcal{L}|+1)},$$

for some absolute tolerance $\epsilon^{\text{abs}} > 0$.

5.5.4 Choice of ρ

The value of the algorithm parameter ρ can greatly affect the convergence rate of the message passing algorithm. There are no known methods for choosing the optimal value of ρ *a priori*, except in certain special cases [166].

Empirically, however, it has been observed that the choice of ρ affects the convergence rate of the primal and dual residuals. If ρ is too large, the dual residuals converge slowly. If ρ is too small, the primal residuals converge slowly. The optimal ρ balances the convergence rate of both residuals. From this observation, several heuristics can be devised to modify ρ at each iteration such that the primal and dual residuals remain approximately the same size. In our OPF and SCOPF simulations, we change the ρ each iteration, from the start until 3000 iterations using a discretized proportional and derivative control algorithm to maintain approximate equality of primal and dual residuals and then we keep it fixed for the subsequent iterations.

For more details on ρ selection, consult [49, 234].

5.5.5 Implementation of proximal functions

Each device (generators, transmission lines, loads) computes its proximal function. In general, evaluating the proximal function requires solving an optimization problem. In some cases, as we will soon see, evaluating the proximal functions can be simply calculating some projection results, analytically. The complexity of solving this optimization problem depends on the structure of the local problem. In the case of SCOPF, the variables are the local power plans P_{dev} and the device voltage angles θ_{dev}. At most, the variables are coupled through the base case $P_{dev}^{(0)}$, as for the generators. If the power plans do not couple through the base case, then the local problem is completely separable across the contingency scenarios as well as the different dispatch intervals. That's precisely what the coarse-grain and fine-grain combination of the APMP algorithm attempts to achieve.

Because of this simple structure in the local SCOPF problems on each device, we can quickly and efficiently evaluate the proximal functions for each device.

5.6 $\mathcal{DTN}$ Reformulations

In this section we carry out the reformulations of the conventional OPF, SCOPF, and LASCOPF models that we presented earlier, in order for us to be able to solve the problems by the Proximal Message Passing method.

In the material that follows, we will be encountering objective functions that become quite lengthy and complicated in appearance as we include more constraints into our problem. The reason for this is that we include the constraints within the objective function by defining equivalent indicator functions. In order to simplify the presentation and also to clarify the meaning of each term, we will group the terms of the objective into four different categories, the fourth one of which will be used only for the look-ahead dispatch problems. We will define them for each case. These are:

- **Cost of Generation** ($C(\mathbf{P})$): This term consists of the actual total cost of generating real power by the different generators as well as the indicator functions corresponding to the lower and upper generating limits of the different generators. For this term, the real power generated is always considered at the base case.

- **Line Flow Limit Constraint** ($F(\mathbf{P})$): This term consists of the sum of the indicator functions corresponding to the constraints meant for enforcement of the real power flow on the lines being less the maximum allowed, both at the base-case as well as during the different contingencies.

- **Power-Angle Relation** ($\chi(\mathbf{P}, \theta)$):This term consists of the sum of the indicator functions corresponding to the relation of the power flow at each end of the lines and the voltage phase angles at the two ends, both at the base-case and the contingencies.

- **Ramp Constraint** ($\Delta(\mathbf{P})$):This term corresponds to the change of power output of generator from one time period to another and the maximum rate at which it can go up or down.

5.6.1 $\mathcal{DTN}$ Formulation of the OPF for the Simplest Two Bus Case

We are considering here the simplest two bus case and are going to reformulate the OPF equations, stated in (4.2) and (4.3), similar to the paradigm introduced in [234]. Let us introduce the following sign convention that we will follow throughout the rest of this work: Power coming out of a terminal is positive and going into a terminal is considered negative. Referring to the figure 3.2, let P_{t_k} refer to the real power coming out of the terminal t_k each of which is associated with exactly one device and one net. Since the loads consume real power, $P_{D_{1t_3}} = -D_1, P_{D_{2t_6}} = -D_2$. Since we have previously stated that each terminal is shared between exactly one net and one device, hence, the power and voltage angle schedules pertaining to a particular terminal can either be thought of associated with the net or the device, to which the particular terminal is connected. Therefore, $P_{D_{1t_3}} = P_{N_{1t_3}}, P_{D_{2t_6}} = P_{N_{2t_6}}, \theta_{g_{1t_1}} = \theta_{N_{1t_1}}, \theta_{g_{2t_4}} = \theta_{N_{2t_4}}$ etc. The average net real power mismatches are given by the following equations:

$$\hat{P}_{N_1} = \frac{P_{N_{1t_1}} + P_{N_{1t_2}} + P_{N_{1t_3}} + P_{N_{1t_7}} + P_{N_{1t_9}}}{5} = \hat{P}_{N_{1t_1}} = ... = \hat{P}_{N_{1t_9}} \tag{5.19a}$$

$$\hat{P}_{N_2} = \frac{P_{N_{2t_4}} + P_{N_{2t_5}} + P_{N_{2t_6}} + P_{N_{2t_8}} + P_{N_{2t_{10}}}}{5} = \hat{P}_{N_{2t_4}} = ... = \hat{P}_{N_{2t_{10}}} \tag{5.19b}$$

The voltage phase angle consistency constraints are the following:

$$\hat{\theta}_{N_1} = \frac{\theta_{N_{1t_1}} + \theta_{N_{1t_2}} + \theta_{N_{1t_3}} + \theta_{N_{1t_7}} + \theta_{N_{1t_9}}}{5} \tag{5.20a}$$

$$\hat{\theta}_{N_2} = \frac{\theta_{N_{2t_4}} + \theta_{N_{2t_5}} + \theta_{N_{2t_6}} + \theta_{N_{2t_8}} + \theta_{N_{2t_{10}}}}{5} \tag{5.20b}$$

$$\tilde{\theta}_{N_{it_k}} = \theta_{N_{it_k}} - \hat{\theta}_{N_i} = 0 \tag{5.20c}$$

In order for the line power flow limit constraint to hold, we require that $|P_{T_{r t_k}}| \leq \overline{L}_{T_r}$ for each terminal of each of the transmission lines, for which we can write indicator functions, $I_{\leq}(\overline{L}_{T_r} - |P_{T_{r t_k}}|)$ and define $I_{\leq}(x) = 0$ if $x \geq 0$ and $= \infty$ otherwise.

For any particular net, we consider the real powers flowing 'out' of a terminal 'into' the net as positive and real powers 'leaving' the net and 'entering' the terminal as negative. Two other constraints, which establish the relationship between power injection and phase angle (under DC OPF Assumptions) for each transmission line, are (shown below only for T_1):

$$-P_{T_{1 t_2}} = \frac{\theta_{T_{1 t_2}} - \theta_{T_{1 t_4}}}{X_{T_1}} \tag{5.21a}$$

$$-P_{T_{1 t_4}} = \frac{\theta_{T_{1 t_4}} - \theta_{T_{1 t_2}}}{X_{T_1}} \tag{5.21b}$$

and the corresponding indicator functions for these are $I_{=}(P_{T_{1 t_2}} + \frac{\theta_{T_{1 t_2}} - \theta_{T_{1 t_4}}}{X_{T_1}})$ and $I_{=}(P_{T_{1 t_4}} + \frac{\theta_{T_{1 t_4}} - \theta_{T_{1 t_2}}}{X_{T_1}})$, which, unlike the previously defined indicator functions are zero only when the respective arguments are zero and ∞ otherwise. Here are the different components of the objective function:

- **Cost of Generation:** $C(\mathbf{P}) = C_{g_1}(P_{g_{1 t_1}}) + C_{g_2}(P_{g_{2 t_5}})$

- **Line Flow Limit Constraint:** $F(\mathbf{P}) = \sum_{r=1}^{3} \sum_{t_k \in T_r} I_{\leq}(\overline{L}_{T_r} - |P_{T_{r t_k}}|)$

- **Power-Angle Relation:** $\chi(\mathbf{P},\theta) = \sum_{r=1}^{3} \sum_{t_k,t_{k'} \in T_r} I_{=}(P_{T_{rt_k}} + \frac{\theta_{T_{rt_k}} - \theta_{T_{rt_{k'}}}}{X_{T_r}})$

Hence, taking all these into account, the reformulated OPF is:

$$\min_{P_{t_k},\theta_{t_k}} f(\mathbf{P},\theta) = C(\mathbf{P}) + F(\mathbf{P}) + \chi(\mathbf{P},\theta) \tag{5.22a}$$

$$\text{Subject to: } \hat{P}_{N_{it_k}} = 0, \tilde{\theta}_{N_{it_k}} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T} \tag{5.22b}$$

5.6.2 $\mathcal{DTN}$ Formulation of OPF for the Generalized Multi-Bus Systems

Given below are the steps for the reformulation of the problem stated in equations (4.13) and (4.14). Let a particular net, $N_i \in \mathcal{N}$ has $|\mathcal{N}|$ number of terminals. The average power mismatch for each net is as follows:

$$\hat{P}_{N_i} = \frac{1}{|\mathcal{N}|} \sum_{t_k \in N_i \cap \mathcal{T}} P_{N_{it_k}} = \hat{P}_{N_{it_k}}$$
$$\forall N_i \in \mathcal{N}, \forall t_k \in N_i \cap \mathcal{T} \tag{5.23a}$$

Similarly, the voltage phase angle inconsistency equations are:

$$\hat{\theta}_{N_i} = \frac{1}{|\mathcal{N}|} \sum_{t_k \in N_i \cap \mathcal{T}} \theta_{N_{it_k}} \tag{5.24a}$$

$$\tilde{\theta}_{N_{it_k}} = \theta_{N_{it_k}} - \hat{\theta}_{N_i}$$
$$\forall N_i \in \mathcal{N}, \forall t_k \in N_i \cap \mathcal{T} \tag{5.24b}$$

The indicator functions corresponding to the line flow limit constraints are as follows: $\sum_{T_r \in T} \sum_{t_k \in T_r \cap \mathfrak{T}} I_{\leq}(\overline{L}_{T_r} - |P_{T_{r t_k}}|)$. The indicator functions corresponding to the defining relationship between the power injections on the lines and the phase angles are as follows: $\sum_{T_r \in T} \sum_{t_k, t_{k'} \in T_r \cap \mathfrak{T}} I_{=}(P_{T_{r t_k}} + \frac{\theta_{T_{r t_k}} - \theta_{T_{r t_{k'}}}}{X_{T_r}})$.

The indicator functions corresponding to the Generator maximum and minimum real power generating limits are $\sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|} (I_{\leq}(\overline{P}_{g_q} - P_{g_{q t_k}}) + I_{\leq}(P_{g_{q t_k}} - \underline{P}_{g_q}))$ and the Generator Cost functions are of the form $C_{g_q}(P_{g_{q t_k}}) = \alpha_{g_q}(P_{g_{q t_k}})^2 + \beta_{g_q} P_{g_{q t_k}} + \gamma_{g_q}$. As before, the different terms of the objective function in this case are:

- **Cost of Generation:** $C(\mathbf{P}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|} (C_{g_q}(P_{g_{q t_k}}) + I_{\leq}(\overline{P}_{g_q} - P_{g_{q t_k}}) + I_{\leq}(P_{g_{q t_k}} - \underline{P}_{g_q}))$

- **Line Flow Limit Constraint:** $F(\mathbf{P}) = \sum_{T_r \in T} \sum_{t_k \in T_r \cap \mathfrak{T}} I_{\leq}(\overline{L}_{T_r} - |P_{T_{r t_k}}|)$

- **Power-Angle Relation:** $\chi(\mathbf{P}, \theta) = \sum_{T_r \in T} \sum_{t_k, t_{k'} \in T_r \cap \mathfrak{T}} I_{=}(P_{T_{r t_k}} + \frac{\theta_{T_{r t_k}} - \theta_{T_{r t_{k'}}}}{X_{T_r}})$

With all the above components, the reformulated OPF for this particular case can written as:

$$\min_{P_{t_k}, \theta_{t_k}} f(\mathbf{P}, \theta) = C(\mathbf{P}) + F(\mathbf{P}) + \chi(\mathbf{P}, \theta) \tag{5.25a}$$

$$\text{Subject to: } \hat{P}_{N_{i t_k}} = 0, \tilde{\theta}_{N_{i t_k}} = 0, \forall N_i \in \mathfrak{N}, \forall t_k \in \mathfrak{T} \tag{5.25b}$$

5.6.3 $\mathcal{DTN}$ Formulation of $(N-1)$ SCOPF for the Two Bus Case: Equal Capacities and Line Impedances

Here we will reformulate the SCOPF problem stated in equation (4.5) previously. The base-case constraints are the same as before, i.e. they are defined by the constraints in (5.22). But now, in order to be secure with respect to outage of a single transmission line, one at a time, there will be some extra constraints.The contingency constraints for the net average power mismatch and phase consistency are similar in form to the ones for base case, but now we will use a superscript to denote which particular contingency case or the base case we are referring to. The components of the objective function are as follows:

- **Cost of Generation (At Base Case):**
  $C(\mathbf{P^{(0)}}) = C_1(P^{(0)}_{g_{1_{t_1}}}) + C_2(P^{(0)}_{g_{2_{t_5}}}) + I_{\leq}(\overline{P}_{g_1} - P^{(0)}_{g_{1_{t_1}}}) + I_{\leq}(P^{(0)}_{g_{1_{t_1}}} - \underline{P}_{g_1})$
  $+ I_{\leq}(\overline{P}_{g_2} - P^{(0)}_{g_{2_{t_5}}}) + I_{\leq}(P^{(0)}_{g_{2_{t_5}}} - \underline{P}_{g_2})$

- **Line Flow Limit Constraint ($(N-1)$ Secure):**
  $F(\mathbf{P^{(c)}}) = \sum_{c=0}^{3}\sum_{r=1}^{3}\sum_{t_k \in T_r}(I_{\leq}(\overline{L}^{(c)}_{T_r} - |P^{(c)}_{T_{r_{t_k}}}|))$

- **Power-Angle Relation ($(N-1)$ Secure):**
  $\chi(\mathbf{P^{(c)}}, \theta^{(\mathbf{c})}) = \sum_{c=0}^{3}\sum_{r=1}^{3}\sum_{t_k, t_{k'} \in T_r}(I_{=}(P^{(c)}_{T_{r_{t_k}}} + \frac{\theta^{(c)}_{T_{r_{t_k}}} - \theta^{(c)}_{T_{r_{t_{k'}}}}}{X^{(c)}_{T_r}}))$

Hence, the $\mathcal{DTN}$ reformulated equations for this particular case can be written down as follows:

$$\min_{P_{t_k}^{(c)},\theta_{t_k}^{(c)}} f(\mathbf{P},\theta) = C(\mathbf{P^{(0)}}) + F(\mathbf{P^{(c)}}) + \chi(\mathbf{P^{(c)}},\theta^{(c)}) \tag{5.26a}$$

$$\text{Subject to: } \hat{P}_{N_{it_k}}^{(c)} = 0, \tilde{\theta}_{N_{it_k}}^{(c)} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall (c) \in \mathcal{L} \tag{5.26b}$$

It is to be observed that the Generators' output power stays the same pre- and post-contingency (neglecting change in losses, as in this DCOPF Formulation). We will now skip the explicit treatment of the unequal line capacities and equal line impedances and proceed directly to the unequal line capacities and unequal line impedances cases. We do this partly because it is in some sense more easy to deal with the latter case and partly because the latter case is more general and the former one is just a special case of it.

5.6.4 $\mathcal{DTN}$ Formulation of $(N-1)$ SCOPF for the Two Bus Case: Unequal Capacities and Unequal Line Impedances

Here we will present the reformulation of the SCOPF problem stated in equation (4.7). In this case, since we are considering each of the transmission lines having potentially different impedances and line capacities, the power flows at each end of each of the lines will be separate decision variables both at the base case as well as at each of the contingencies. The constraint linking the decision variables at the base case to those at the contingencies for the power flows is the fact that the net injection at each net, both before and after contingency, within a particular dispatch interval remains same. The components of the objective function are as follows:

- **Cost of Generation (At Base Case):**

$$C(\mathbf{P^{(0)}}) = C_1(P^{(0)}_{g_{1t_1}}) + C_2(P^{(0)}_{g_{2t_5}}) + I_{\leq}(\overline{P}_{g_1} - P^{(0)}_{g_{1t_1}}) + I_{\leq}(P^{(0)}_{g_{1t_1}} - \underline{P}_{g_1})$$
$$+ I_{\leq}(\overline{P}_{g_2} - P^{(0)}_{g_{2t_5}}) + I_{\leq}(P^{(0)}_{g_{2t_5}} - \underline{P}_{g_2})$$

- **Line Flow Limit Constraint** $((N-1)$ **Secure):**

$$F(\mathbf{P^{(c)}}) = \textstyle\sum_{c=0}^{3}(I_{\leq}(A^{(c)} - |P^{(c)}_{T_{1t_2}}|)$$
$$+ I_{\leq}(A^{(c)} - |P^{(c)}_{T_{1t_4}}|) + I_{\leq}(B^{(c)} - |P^{(c)}_{T_{2t_7}}|) + I_{\leq}(B^{(c)} - |P^{(c)}_{T_{2t_8}}|)$$
$$+ I_{\leq}(C^{(c)} - |P^{(c)}_{T_{3t_9}}|) + I_{\leq}(C^{(c)} - |P^{(c)}_{T_{3t_{10}}}|))$$

- **Power-Angle Relation** $((N-1)$ **Secure):**

$$\chi(\mathbf{P^{(c)}}, \theta^{(\mathbf{c})}) = \textstyle\sum_{c=0}^{3}(I_{=}(P^{(c)}_{T_{1t_2}} + \frac{\theta^{(c)}_{T_{1t_2}} - \theta^{(c)}_{T_{1t_4}}}{X^{(c)}_1})$$
$$+ I_{=}(P^{(c)}_{T_{2t_7}} + \frac{\theta^{(c)}_{T_{2t_7}} - \theta^{(c)}_{T_{2t_8}}}{X^{(c)}_2}) + I_{=}(P^{(c)}_{T_{3t_9}} + \frac{\theta^{(c)}_{T_{3t_9}} - \theta^{(c)}_{T_{3t_{10}}}}{X^{(c)}_3}) + I_{=}(P^{(c)}_{T_{1t_4}} + \frac{\theta^{(c)}_{T_{1t_4}} - \theta^{(c)}_{T_{1t_2}}}{X^{(c)}_1})$$
$$+ I_{=}(P^{(c)}_{T_{2t_8}} + \frac{\theta^{(c)}_{T_{2t_8}} - \theta^{(c)}_{T_{2t_7}}}{X^{(c)}_2}) + I_{=}(P^{(c)}_{T_{3t_{10}}} + \frac{\theta^{(c)}_{T_{3t_{10}}} - \theta^{(c)}_{T_{3t_9}}}{X^{(c)}_3}))$$

Hence, the appropriate model now, is a slightly modified version of the one before.

$$\min_{P^{(c)}_{t_k}, \theta^{(c)}_{t_k}} f(\mathbf{P}, \theta) = C(\mathbf{P^{(0)}}) + F(\mathbf{P^{(c)}}) + \chi(\mathbf{P^{(c)}}, \theta^{(\mathbf{c})}) \tag{5.27a}$$
$$\text{Subject to: } \hat{P}^{(c)}_{N_{i t_k}} = 0, \tilde{\theta}^{(c)}_{N_{i t_k}} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall (c) \in \mathcal{L} \tag{5.27b}$$

The three transmission lines have capacities A, B, C and impedances X_1, X_2, X_3, respectively. Evidently, the indexed capacity of the particular transmission line being outaged in a particular contingency scenario is 0. For instance, in scenario (1), $A^{(1)}$ is zero and for the other scenarios it has the rated value. Obviously, in the above expression, there are going to be many redundant constraints. But, for the purposes of generalizations for the materials to follow, we have not eliminated any of the constraints.

5.6.5 $\mathcal{DTN}$ Formulation of $(N-1)$ SCOPF for the Generalized Multi-Bus Case: Unequal Capacities and Unequal Line Impedances

The SCOPF problem stated in equations (4.15) and (4.16) will be reformulated. The average net real power imbalance for the base case as well as the contingencies are as follows:

$$\begin{aligned} \hat{P}_{N_i}^{(c)} &= \frac{1}{|N_i|} \sum_{t_k \in N_i \cap \mathcal{T}} P_{N_{i t_k}}^{(c)} = \hat{P}_{N_{i t_k}}^{(c)} \\ &\forall N_i \in \mathcal{N}, \forall t_k \in N_i \cap \mathcal{T}, \forall (c) \in \mathcal{L} \end{aligned} \tag{5.28a}$$

and the phase consistency constraints for the base case as well as contingencies are as follows:

$$\hat{\theta}_{N_i}^{(c)} = \frac{1}{|N_i|} \sum_{t_k \in N_i \cap \mathcal{T}} \theta_{N_{i t_k}}^{(c)} \tag{5.29a}$$

$$\begin{aligned} \tilde{\theta}_{N_{i t_k}}^{(c)} &= \theta_{N_{i t_k}}^{(c)} - \hat{\theta}_{N_i}^{(c)} \\ &\forall N_i \in \mathcal{N}, \forall t_k \in N_i \cap \mathcal{T}, \forall (c) \in \mathcal{L} \end{aligned} \tag{5.29b}$$

The components of the objective function are as follows:

- **Cost of Generation (At Base Case):**
  $C(\mathbf{P^{(0)}}) = \sum_{t_k \in g_q \cap \mathcal{T}, q=1}^{|G|} (C_{g_q}(P_{g_{q t_k}}^{(0)}) +$

$I_{\leq}(\overline{P}_{g_q} - P^{(0)}_{g_{q_{t_k}}}) + I_{\leq}(P^{(0)}_{g_{q_{t_k}}} - \underline{P}_{g_q}))$

- **Line Flow Limit Constraint ($(N-1)$ Secure):**
  $F(\mathbf{P^{(c)}}) = \sum_{(c)\in\mathcal{L}} \sum_{T_r\in T} \sum_{t_k\in T_r\cap\mathfrak{T}} I_{\leq}(\overline{L}^{(c)}_{T_r} - |P^{(c)}_{T_{r t_k}}|)$

- **Power-Angle Relation ($(N-1)$ Secure):**
  $\chi(\mathbf{P^{(c)}}, \theta^{(\mathbf{c})}) = \sum_{(c)\in\mathcal{L}} \sum_{T_r\in T} \sum_{t_k, t_{k'}\in T_r\cap\mathfrak{T}} I_{=}(P^{(c)}_{T_{r t_k}} + \frac{\theta^{(c)}_{T_{r t_k}} - \theta^{(c)}_{T_{r t_{k'}}}}{X^{(c)}_{T_r}})$

So, the reformulated OPF Problem for this case is as follows:

$$\min_{P^{(c)}_{t_k}, \theta^{(c)}_{t_k}} f(\mathbf{P}, \theta) = C(\mathbf{P^{(0)}}) + F(\mathbf{P^{(c)}}) + \chi(\mathbf{P^{(c)}}, \theta^{(\mathbf{c})}) \tag{5.30a}$$

$$\text{Subject to: } \overline{P}^{(c)}_{N_{i t_k}} = 0, \tilde{\theta}^{(c)}_{N_{i t_k}} = 0, \forall N_i \in \mathfrak{N}, \forall t_k \in \mathfrak{T}, \forall (c) \in \mathcal{L} \tag{5.30b}$$

For the sake of brevity, now we will present the $\mathfrak{DTN}$ reformulations (which will eventually lead to the fine-grained decomposition) of only the generalized cases.

5.6.6 $\mathfrak{DTN}$ Formulation Applied to the Look-Ahead Dispatch: Generalized Case of Demand Variation for the Multi Bus-Case

Referring to sections 5.3.1 and 5.3.2, we will present below , the fine-grained distribution for each of $\tau = 1, \tau \in \{2, 3, .., |\Omega| - 1\}$, and $\tau = |\Omega|$ (It is to be noted that only the cost of generation and the ramp constraint components will have slightly different representations for $\tau = 1, \tau \in \{2, 3, .., |\Omega| - 1\}$, and $\tau = |\Omega|$ respectively. The other two components of the flow limit and power-angle relationship will be the same in form throughout and hence we will write only one form corresponding to all the cases for the latter two). The components of the objective for this case are:

- **Cost of Generation (At Base Case, for $\tau = 1$):**
  $C(\mathbf{P^{(0)(1)}}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|}(C_{g_q}(P^{(0)(1)}_{g_{q_{t_k}}(1)}) + I_{\leq}(\overline{P}_{g_q} - P^{(0)(1)}_{g_{q_{t_k}}(1)})$
  $+ I_{\leq}(P^{(0)(1)}_{g_{q_{t_k}}(1)} - \underline{P}_{g_q}) + \frac{\beta}{2}[(P^{(0)(1)}_{g_{q_{t_k}}(1)} - P^{(0)(1)(\mu_{APP})}_{g_{q_{t_k}}(1)})^2 + (P^{(0)(2)}_{g_{q_{t_k}}(1)} - P^{(0)(2)(\mu_{APP})}_{g_{q_{t_k}}(1)})^2]$
  $+ \gamma[P^{(0)(1)}_{g_{q_{t_k}}(1)}(P^{(0)(1)(\mu_{APP})}_{g_{q_{t_k}}(1)} - P^{(0)(1)(\mu_{APP})}_{g_{q_{t_k}}(2)}) + P^{(0)(2)}_{g_{q_{t_k}}(1)}(P^{(0)(2)(\mu_{APP})}_{g_{q_{t_k}}(1)} - P^{(0)(2)(\mu_{APP})}_{g_{q_{t_k}}(2)})]$
  $+ \lambda^{(\mu_{APP})}_{g_q(1)} P^{(0)(1)}_{g_{q_{t_k}}(1)} + \lambda^{(\mu_{APP})}_{g_q(2)} P^{(0)(2)}_{g_{q_{t_k}}(1)})$

- **Cost of Generation (At Base Case, for $\tau \in \{2, 3, .., |\Omega| - 1\}$):**
  $C(\mathbf{P^{(0)(\tau)}}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|}(C_{g_q}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}) + I_{\leq}(\overline{P}_{g_q} - P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)})$
  $+ I_{\leq}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - \underline{P}_{g_q}) + \frac{\beta}{2}[(P^{(0)(\tau-1)}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau-1)(\mu_{APP})}_{g_{q_{t_k}}(\tau)})^2 + (P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau)})^2 +$
  $(P^{(0)(\tau+1)}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(\tau)})^2]$
  $+ \gamma[P^{(0)(\tau-1)}_{g_{q_{t_k}}(\tau)}(P^{(0)(\tau-1)(\mu_{APP})}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau-1)(\mu_{APP})}_{g_{q_{t_k}}(\tau-1)}) + P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}(P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau-1)}) +$
  $P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}(P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau+1)}) + P^{(0)(\tau+1)}_{g_{q_{t_k}}(\tau)}(P^{(0)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(\tau+1)})]$
  $+ \lambda^{(\mu_{APP})}_{g_q(2\tau-1)} P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} + \lambda^{(\mu_{APP})}_{g_q(2\tau)} P^{(0)(\tau+1)}_{g_{q_{t_k}}(\tau)} - \lambda^{(\mu_{APP})}_{g_q(2\tau-3)} P^{(0)(\tau-1)}_{g_{q_{t_k}}(\tau)} - \lambda^{(\mu_{APP})}_{g_q(2\tau-2)} P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)})$

- **Cost of Generation (At Base Case, for $\tau = |\Omega|$):**
  $C(\mathbf{P^{(0)(|\Omega|)}}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|}(C_{g_q}(P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)}) + I_{\leq}(\overline{P}_{g_q} - P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)})$
  $+ I_{\leq}(P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)} - \underline{P}_{g_q}) + \frac{\beta}{2}[(P^{(0)(|\Omega|-1)}_{g_{q_{t_k}}(|\Omega|)} - P^{(0)(|\Omega|-1)(\mu_{APP})}_{g_{q_{t_k}}(|\Omega|)})^2 + (P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)} - P^{(0)(|\Omega|)(\mu_{APP})}_{g_{q_{t_k}}(|\Omega|)})^2]$
  $+\gamma[P^{(0)(|\Omega|-1)}_{g_{q_{t_k}}(|\Omega|)}(P^{(0)(|\Omega|-1)(\mu_{APP})}_{g_{q_{t_k}}(|\Omega|)} - P^{(0)(|\Omega|-1)(\mu_{APP})}_{g_{q_{t_k}}(|\Omega|-1)}) + P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)}(P^{(0)(|\Omega|)(\mu_{APP})}_{g_{q_{t_k}}(|\Omega|)} - P^{(0)(|\Omega|)(\mu_{APP})}_{g_{q_{t_k}}(|\Omega|-1)})]$

$- \lambda^{(\mu_{APP})}_{g_q(2|\Omega|-3)} P^{(0)(|\Omega|-1)}_{g_{q_{t_k}}(|\Omega|)} - \lambda^{(\mu_{APP})}_{g_q(2|\Omega|-2)} P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)})$

- **Line Flow Limit Constraint ($(N-1)$ Secure):**
  $F(\mathbf{P}^{(\mathbf{c})(\tau)}) = (\sum_{(c)\in\mathcal{L}} \sum_{T_r\in T} \sum_{t_k\in T_r\cap\mathfrak{T}} I_{\leq}(\overline{L}^{(c)}_{T_r} - |P^{(c)(\tau)}_{T_{rt_k}(\tau)}|))$

- **Power-Angle Relation ($(N-1)$ Secure):**
  $\chi(\mathbf{P}^{(\mathbf{c})(\tau)}, \theta^{(\mathbf{c})(\tau)}) = (\sum_{(c)\in\mathcal{L}} \sum_{T_r\in T} \sum_{t_k,t_{k'}\in T_r\cap\mathfrak{T}} I_{=}(P^{(c)(\tau)}_{T_{rt_k}(\tau)} + \frac{\theta^{(c)(\tau)}_{T_{rt_k}(\tau)} - \theta^{(c)(\tau)}_{T_{rt_{k'}}(\tau)}}{X^{(c)}_{T_r}}))$

- **Ramp Constraint for $\tau = 1$:**
  $\Delta(\mathbf{P}^{(\mathbf{0})(\mathbf{1})}) = \sum_{t_k\in g_q\cap\mathfrak{T},q=1}^{|G|}(I_{\leq}(\overline{R}_{g_q} - P^{(0)(2)}_{g_{q_{t_k}}(1)} + P^{(0)(1)}_{g_{q_{t_k}}(1)}) + I_{\leq}(P^{(0)(2)}_{g_{q_{t_k}}(1)} - P^{(0)(1)}_{g_{q_{t_k}}(1)} - \underline{R}_{g_q})$
  $+ I_{\leq}(\overline{R}_{g_q} - P^{(0)(1)}_{g_{q_{t_k}}(1)} + P^{(0)(0)}_{g_{q_{t_k}}}) + I_{\leq}(P^{(0)(1)}_{g_{q_{t_k}}(1)} - P^{(0)(0)}_{g_{q_{t_k}}} - \underline{R}_{g_q}))$

- **Ramp Constraint for $\tau \in \{2, 3, .., |\Omega| - 1\}$:**
  $\Delta(\mathbf{P}^{(\mathbf{0})(\tau)}) = \sum_{t_k\in g_q\cap\mathfrak{T},q=1}^{|G|}(I_{\leq}(\overline{R}_{g_q} - P^{(0)(\tau+1)}_{g_{q_{t_k}}(\tau)} + P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}) + I_{\leq}(P^{(0)(\tau+1)}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - \underline{R}_{g_q})$
  $+ I_{\leq}(\overline{R}_{g_q} - P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} + P^{(0)(\tau-1)}_{g_{q_{t_k}}(\tau)}) + I_{\leq}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau-1)}_{g_{q_{t_k}}(\tau)} - \underline{R}_{g_q}))$

- **Ramp Constraint for $\tau = |\Omega|$:**
  $\Delta(\mathbf{P}^{(\mathbf{0})(|\mathbf{\Omega}|)}) = \sum_{t_k\in g_q\cap\mathfrak{T},q=1}^{|G|}(I_{\leq}(\overline{R}_{g_q} - P^{(0)(|\Omega|)(\mu_{APP})}_{g_{q_{t_k}}(|\Omega|)} + P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)}) + I_{\leq}(P^{(0)(|\Omega|)(\mu_{APP})}_{g_{q_{t_k}}(|\Omega|)} -$
  $P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)} - \underline{R}_{g_q})$
  $+ I_{\leq}(\overline{R}_{g_q} - P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)} + P^{(0)(|\Omega|-1)}_{g_{q_{t_k}}(|\Omega|)}) + I_{\leq}(P^{(0)(|\Omega|)}_{g_{q_{t_k}}(|\Omega|)} - P^{(0)(|\Omega|-1)}_{g_{q_{t_k}}(|\Omega|)} - \underline{R}_{g_q}))$

The reformulated equations are as follows:

$$\forall \tau \in \Omega$$

$$\min_{\mathbf{P}_{\mathbf{t_k}}^{(\mathbf{c})(\tau)}, \theta_{\mathbf{t_k}}^{(\mathbf{c})(\tau)}} f(\mathbf{P}, \theta) = C(\mathbf{P}^{(\mathbf{0})(\tau)}) + F(\mathbf{P}^{(\mathbf{c})(\tau)}) + \chi(\mathbf{P}^{(\mathbf{c})(\tau)}, \theta^{(\mathbf{c})(\tau)}) + \Delta(\mathbf{P}^{(\mathbf{0})(\tau)}) \tag{5.31a}$$

$$\text{Subject to: } \hat{P}^{(c)(\tau)}_{N_{it_k}(\tau)} = 0, \tilde{\theta}^{(c)(\tau)}_{N_{it_k}(\tau)} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.31b}$$

As before, $\overline{L}^{(c)}$ now onward, unlike the previous cases, shouldn't just be interpreted as just being zero when the particular line is outaged, but should also be taken to be possibly the short time rating of the lines that are still in service.

5.6.7 $\mathcal{DTN}$ Formulation Applied to the Look-Ahead Dispatch Model for Ensuring Security with respect to Next Outage: Generalized Multi Bus-Case

Now we will be dealing with ensuring security with respect to the next possible outages, after one outage has taken place at the beginning of the first dispatch interval and since, we assume that security is achieved within one dispatch interval by virtue of the ramp-rate limit, we will deal with just two dispatch intervals in each 'roll' of the calculation, one for the present time and one for the 'look-ahead' or the second interval. We have already presented the coarse-grained distribution. Therefore, in this case, we will split each of the first three different components of the objective function further into the one for present time and one for second interval. Listed below are the components:

- **Cost of Generation (At Base Case, Present Time):**
  $C(\mathbf{P}^{(\mathbf{0})(\tau)}) = \sum_{t_k \in g_q \cap \mathcal{T}, q=1}^{|G|} (C_{g_q}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}) + I_{\leq}(\overline{P}_{g_q} - P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}) + I_{\leq}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - \underline{P}_{g_q}) + \frac{\beta}{2}[(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} -$

$P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau)})^2 + \sum_{c=0}^{|\mathcal{L}|}(P^{(c)(\tau+1)}_{g_{q_{t_k}}(\tau)} - P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(\tau)})^2]$
$+\gamma[\sum_{c=0}^{|\mathcal{L}|}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}(P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)}) + P^{(c)(\tau+1)}_{g_{q_{t_k}}(\tau)}(P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(\tau)} - P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)}))]$
$+ \sum_{c=0}^{|\mathcal{L}|}(\lambda^{(\mu_{APP})}_{g_q(2c+1)}P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} + \lambda^{(\mu_{APP})}_{g_q(2c+2)}P^{(c)(\tau+1)}_{g_{q_{t_k}}(\tau)}))$

- **Cost of Generation (Contingency Cases, Second Interval):**
  $C(\mathbf{P^{(c)(\tau+1)}}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|}(C_{g_q}(P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)}) + I_{\leq}(\overline{P}_{g_q} - P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)}) + I_{\leq}(P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)} - \underline{P}_{g_q}) + \frac{\beta}{2}[(P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)} - P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)})^2 + (P^{(0)(\tau)}_{g_{q_{t_k}}(c)(\tau+1)} - P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)})^2]$
  $+\gamma[P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)}(P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)} - P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(0)(\tau)}) + P^{(0)(\tau)}_{g_{q_{t_k}}(c)(\tau+1)}(P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)} - P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(0)(\tau)})]$
  $- \lambda^{(\mu_{APP})}_{g_q(2c+1)}P^{(0)(\tau)}_{g_{q_{t_k}}(c)(\tau+1)} - \lambda^{(\mu_{APP})}_{g_q(2c+2)}P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)})$

- **Line Flow Limit Constraint (($N-1$) Secure, Present Time):**
  $F(\mathbf{P^{(c)(\tau)}}) = \sum_{(c)\in\mathcal{L}}\sum_{T_r \in T}\sum_{t_k \in T_r \cap \mathfrak{T}} I_{\leq}(\overline{L}^{(c)}_{T_r} - |P^{(c)(\tau)}_{T_{r t_k}(\tau)}|)$

- **Line Flow Limit Constraint (($N-1$) Secure, Second Interval):**
  $F(\mathbf{P^{(c\to c')(\tau+1)}}) = \sum_{T_r \in T}\sum_{t_k \in T_r \cap \mathfrak{T}} I_{\leq}(\overline{L}^{(c)}_{T_r} - |P^{(c)(\tau+1)}_{T_{r t_k}(c)(\tau+1)}|) +$
  $\sum_{(c')\in[\mathcal{L}-\{c\}]}\sum_{T_r \in T}\sum_{t_k \in T_r \cap \mathfrak{T}} I_{\leq}(\overline{L}^{(c')}_{T_r} - |P^{(c')(\tau+1)}_{T_{r t_k}(c)(\tau+1)}|), \forall (c) \in \mathcal{L}$

- **Power-Angle Relation (($N-1$) Secure, Present Time):**
  $\chi(\mathbf{P^{(c)(\tau)}}, \theta^{(\mathbf{c})(\tau)}) = \sum_{(c)\in\mathcal{L}}\sum_{T_r \in T}\sum_{t_k, t_{k'} \in T_r \cap \mathfrak{T}} I_{=}(P^{(c)(\tau)}_{T_{r t_k}(\tau)} + \frac{\theta^{(c)(\tau)}_{T_{r t_k}(\tau)} - \theta^{(c)(\tau)}_{T_{r t_{k'}}(\tau)}}{X^{(c)}_{T_r}})$

- **Power-Angle Relation (($N-1$) Secure, Second Interval):**

$\chi(\mathbf{P}^{(\mathbf{c}\to\mathbf{c}^{'})(\tau+\mathbf{1})}, \theta^{(\mathbf{c}\to\mathbf{c}^{'})(\tau+\mathbf{1})}) = \sum_{T_r \in T} \sum_{t_k, t_{k^{'}} \in T_r \cap \mathfrak{T}} I_{=}(P^{(c)(\tau+1)}_{T_r t_k (c)(\tau+1)} + \frac{\theta^{(c)(\tau+1)}_{T_r t_k (c)(\tau+1)} - \theta^{(c)(\tau+1)}_{T_r t_{k^{'}} (c)(\tau+1)}}{X^{(c)}_{T_r}}) +$
$\sum_{(c^{'}) \in [\mathcal{L} - \{c\}]} \sum_{T_r \in T} \sum_{t_k, t_{k^{'}} \in T_r \cap \mathfrak{T}} I_{=}(P^{(c^{'})(\tau+1)}_{T_r t_k (c)(\tau+1)} + \frac{\theta^{(c^{'})(\tau+1)}_{T_r t_k (c)(\tau+1)} - \theta^{(c^{'})(\tau+1)}_{T_r t_{k^{'}} (c)(\tau+1)}}{X^{(c^{'})}_{T_r}}), \forall (c) \in \mathcal{L}$

- **Ramp Constraint for τ:**
  $\Delta(\mathbf{P}^{(\mathbf{0})(\tau)}) = \sum_{(c) \in \mathcal{L}} (\sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|} (I_{\leq}(\overline{R}_{g_q} - P^{(c)(\tau+1)}_{g_{q_{t_k}}(\tau)} + P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}) + I_{\leq}(P^{(c)(\tau+1)}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} -$
  $\underline{R}_{g_q})))$
  $+ I_{\leq}(\overline{R}_{g_q} - P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} + P^{(0)}_{g_{q_{t_k}}}) + I_{\leq}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - P^{(0)}_{g_{q_{t_k}}} - \underline{R}_{g_q})$

- **Ramp Constraint for $(\tau + 1)$:**
  $\Delta(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|} (I_{\leq}(\overline{R}_{g_q} - P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)} + P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)}) + I_{\leq}(P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)} -$
  $P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)} - \underline{R}_{g_q}))$
  $+ I_{\leq}(\overline{R}_{g_q} - P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)} + P^{(0)(\tau)}_{g_{q_{t_k}}(c)(\tau+1)}) + I_{\leq}(P^{(c)(\tau+1)}_{g_{q_{t_k}}(c)(\tau+1)} - P^{(0)(\tau)}_{g_{q_{t_k}}(c)(\tau+1)} - \underline{R}_{g_q}), \forall (c) \in \mathcal{L}$

The reformulated OPF Problems for this case are as follow:

$$\forall \tau$$

$$\min_{\mathbf{P}^{(\mathbf{c})(\tau)}_{\mathbf{t_k}}, \theta^{(\mathbf{c})(\tau)}_{\mathbf{t_k}}} f(\mathbf{P}, \theta) = C(\mathbf{P}^{(\mathbf{0})(\tau)}) + F(\mathbf{P}^{(\mathbf{c})(\tau)}) + \chi(\mathbf{P}^{(\mathbf{c})(\tau)}, \theta^{(\mathbf{c})(\tau)}) + \Delta(\mathbf{P}^{(\mathbf{0})(\tau)}) \quad (5.32a)$$

$$\text{Subject to: } \hat{P}^{(c)(\tau)}_{N_{i t_k}(\tau)} = 0, \tilde{\theta}^{(c)(\tau)}_{N_{i t_k}(\tau)} = 0, \forall N_i \in \mathfrak{N}, \forall t_k \in \mathfrak{T}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \quad (5.32b)$$

$$\forall(\tau+1)$$

$$\min_{\mathbf{P}_{\mathbf{t_k}}^{(\mathbf{c})(\tau+\mathbf{1})},\theta_{\mathbf{t_k}}^{(\mathbf{c})(\tau+\mathbf{1})}} f(\mathbf{P},\theta) = C(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}) + F(\mathbf{P}^{(\mathbf{c}\to\mathbf{c}^{'})(\tau+\mathbf{1})}) + \chi(\mathbf{P}^{(\mathbf{c}\to\mathbf{c}^{'})(\tau+\mathbf{1})}, \theta^{(\mathbf{c}\to\mathbf{c}^{'})(\tau+\mathbf{1})}) + \Delta(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}) \tag{5.33a}$$

$$\text{Subject to: } \hat{P}^{(c\to c^{'})(\tau+1)}_{N_{i t_k}(c)(\tau+1)} = 0, \tilde{\theta}^{(c\to c^{'})(\tau+1)}_{N_{i t_k}(c)(\tau+1)} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall (c) \in \mathcal{L}, \forall (c^{'}) \in [\mathcal{L} - \{c\}] \cap \{c\}, \forall \tau \in \Omega \tag{5.33b}$$

5.7 ADMM Based Proximal Message Passing Algorithm for the Fine-Grained Distributed Computation

In this section, we will present the ADMM Based Proximal Message Passing iterations for all the generalized multi-bus versions of the different models presented in the previous sections. For the details of the derivation, we refer the reader to [234] and [49], which contain a thorough explanation of the algorithm itself and also the specific applications of it to the Power Systems problems.

5.7.1 Proximal Message Passing for Generalized OPF

A slightly reformulated version of the $\mathcal{DTN}$ equations from the last section, which allows us to apply the Proximal Message Passing Algorithm is presented here:

$$\min_{P_{t_k},\theta_{t_k}} C(\mathbf{P}) + F(\mathbf{P}) + \chi(\mathbf{P},\theta) + \sum_{N_i\in\mathcal{N}} (\overline{I}(z_{N_i t_k}) + \tilde{I}(\xi_{N_i t_k})) \tag{5.34a}$$

$$\text{Subject to: } P_{t_k} = z_{t_k}, \theta_{t_k} = \xi_{t_k}, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T} \tag{5.34b}$$

where $\overline{I}(z_{N_i t_k})$ and $\tilde{I}(\xi_{N_i t_k})$ are indicator functions of the sets $\{z_{t_k} | \hat{z}_{N_i t_k} = 0\}$ and $\{\xi_{t_k} | \tilde{\xi}_{N_i t_k} = 0\}$ respectively.

5.7.1.1 Iterates for Generators

They consist of the update equations for the real power output and voltage-phase angles of the generator terminals and are as follows:

$$\begin{gathered}
(P_{g_q t_k}^{(\nu+1)}, \theta_{g_q t_k}^{(\nu+1)}) = \underset{P_{g_q t_k}, \theta_{g_q t_k}}{\operatorname{argmin}} [C_{g_q t_k}(P_{g_q t_k}, \theta_{g_q t_k}) + \\
I_{\leq}(\overline{P}_{g_q} - P_{g_{q_{t_k}}}) + I_{\leq}(P_{g_{q_{t_k}}} - \underline{P}_{g_q}) + \\
\frac{\rho}{2}(||P_{g_{q_{t_k}}} - z_{g_q t_k}^{(\nu)} + u_{g_q t_k}^{(\nu)}||_2^2 + ||\theta_{g_{q_{t_k}}} - \xi_{g_q t_k}^{(\nu)} + v_{g_q t_k}^{(\nu)}||_2^2)], \\
\forall g_q \in G, t_k \in \mathfrak{T} \cap G
\end{gathered} \tag{5.35a}$$

Here, ν, u_{t_k} and v_{t_k} are the ADMM iteration count, scaled dual variable for power balance and scaled dual variable for phase consistency constraints respectively. The parameter ρ is known as the 'penalty parameter' of the Augmented Lagrangian term.

5.7.1.2 Iterates for Transmission Lines

They consist of the update equations for the real power and voltage-phase angles of the Transmission Line terminals, which are two terminal devices (and so, unlike the previous case, we will have to update four variables here) and are as follows:

$$\begin{gathered}
(P_{T_r t_k}^{(\nu+1)}, \theta_{T_r t_k}^{(\nu+1)}, P_{T_r t_{k'}}^{(\nu+1)}, \theta_{T_r t_{k'}}^{(\nu+1)}) \\
= \underset{P_{T_r t_k}, \theta_{T_r t_k}}{\operatorname{argmin}} [\sum_{k, k' \in \mathfrak{T} \cap T_r} (I_{\leq}(\overline{L}_{T_r} - |P_{T_{r t_k}}|) + \\
I_{=}(P_{T_{r t_k}} + \frac{\theta_{T_{r t_k}} - \theta_{T_{r t_{k'}}}}{X_{T_r}}) +
\end{gathered}$$

$$\frac{\rho}{2}(||P_{T_{r t_k}} - z^{(\nu)}_{T_r t_k} + u^{(\nu)}_{T_r t_k}||^2_2 + ||\theta_{T_{r t_k}} - \xi^{(\nu)}_{T_r t_k} + v^{(\nu)}_{T_r t_k}||^2_2))],$$
$$\forall T_r \in T, t_k \in \mathfrak{T} \cap T \tag{5.36a}$$

5.7.1.3 Iterates for Loads

They consist of the update equations for the real power and voltage-phase angles of the loads (which have constant real power consumption) and are as follows:

$$P^{(\nu+1)}_{D_d t_k} = P^{(\nu)}_{D_d t_k} = -P_{D_{d t_k}}$$
$$\theta^{(\nu+1)}_{D_d t_k} = \underset{\theta_{D_d t_k}}{\operatorname{argmin}}[\frac{\rho}{2}(||\theta_{D_{d t_k}} - \xi^{(\nu)}_{D_d t_k} + v^{(\nu)}_{D_d t_k}||^2_2)],$$
$$\forall D_d \in L, t_k \in \mathfrak{T} \cap L \tag{5.37a}$$

5.7.1.4 Iterates for Nets

We are repeating just the analytical forms already derived in [234]:

$$\forall N_i \in \mathcal{N}, \forall t_k \in \mathfrak{T} \cap N_i$$
$$z^{(\nu+1)}_{N_i t_k} = u^{(\nu)}_{N_i t_k} + P^{(\nu+1)}_{N_i t_k} - \hat{u}^{(\nu)}_{N_i t_k} - \hat{P}^{(\nu+1)}_{N_i t_k} \tag{5.38a}$$
$$\xi^{(\nu+1)}_{N_i t_k} = \hat{v}^{(\nu)}_{N_i t_k} + \hat{\theta}^{(\nu+1)}_{N_i t_k} \tag{5.38b}$$
$$u^{(\nu+1)}_{N_i t_k} = u^{(\nu)}_{N_i t_k} + (P^{(\nu+1)}_{N_i t_k} - z^{(\nu+1)}_{N_i t_k}) \tag{5.38c}$$
$$v^{(\nu+1)}_{N_i t_k} = v^{(\nu)}_{N_i t_k} + (\theta^{(\nu+1)}_{N_i t_k} - \xi^{(\nu+1)}_{N_i t_k}) \tag{5.38d}$$

In the above, all the devices update their variables in parallel. Then all the nets update the first two variables in parallel and then update the next two in parallel. It is to be observed here that each $P_{N_i t_k}$ actually comes from the updates from the devices in the previous set of updates, because each of them is actually the real power output/consumption of the respective device having the same terminal in the particular net. Using the above equations and the definition of the prox-functions, the proximal message passing algorithm for this case can be written as follows:

$$
\begin{aligned}
&(P^{(\nu+1)}_{g_q t_k}, \theta^{(\nu+1)}_{g_q t_k}) \\
&= \mathbf{prox}_{C(\mathbf{P}),\rho}(P^{(\nu)}_{g_q t_k} - \hat{P}^{(\nu)}_{g_q t_k} - u^{(\nu)}_{g_q t_k}, \hat{v}^{(\nu-1)}_{g_q t_k} + \hat{\theta}^{(\nu)}_{g_q t_k} - v^{(\nu)}_{g_q t_k}), \forall g_q \in G && \text{(5.39a)} \\
&(P^{(\nu+1)}_{T_r t_k}, \theta^{(\nu+1)}_{T_r t_k}, P^{(\nu+1)}_{T_r t_{k'}}, \theta^{(\nu+1)}_{T_r t_{k'}}) \\
&= \mathbf{prox}_{F(\mathbf{P})+\chi(\mathbf{P},\theta),\rho}(P^{(\nu)}_{T_r t_k} - \hat{P}^{(\nu)}_{T_r t_k} - u^{(\nu)}_{T_r t_k}, \hat{v}^{(\nu-1)}_{T_r t_k} + \hat{\theta}^{(\nu)}_{T_r t_k} - v^{(\nu)}_{T_r t_k}), \forall T_r \in T && \text{(5.39b)} \\
&(P^{(\nu+1)}_{D_d t_k}, \theta^{(\nu+1)}_{D_d t_k}) \\
&= \mathbf{prox}_{-P_{D_d},\rho}(\hat{v}^{(\nu-1)}_{D_d t_k} + \hat{\theta}^{(\nu)}_{D_d t_k} - v^{(\nu)}_{D_d t_k}), \forall D_d \in L && \text{(5.39c)} \\
&u^{(\nu+1)}_{N_i t_k} = u^{(\nu)}_{N_i t_k} + \hat{P}^{(\nu+1)}_{N_i t_k}, \forall N_i \in \mathcal{N} && \text{(5.39d)} \\
&v^{(\nu+1)}_{N_i t_k} = \tilde{v}^{(\nu)}_{N_i t_k} + \tilde{\theta}^{(\nu+1)}_{N_i t_k}, \forall N_i \in \mathcal{N} && \text{(5.39e)}
\end{aligned}
$$

Among the above prox functions, in (5.39b) and (5.39c), the prox functions calculated are indicator functions. So we can analytically calculate the decision variable values as the projection formulas as presented below:

$$
\begin{aligned}
&(P^{(\nu+1)}_{T_r t_k}, \theta^{(\nu+1)}_{T_r t_k}, P^{(\nu+1)}_{T_r t_{k'}}, \theta^{(\nu+1)}_{T_r t_{k'}}) \\
&= \mathbf{prox}_{F(\mathbf{P})+\chi(\mathbf{P},\theta),\rho}(P^{(\nu)}_{T_r t_k} - \hat{P}^{(\nu)}_{T_r t_k} - u^{(\nu)}_{T_r t_k}, \hat{v}^{(\nu-1)}_{T_r t_k} + \hat{\theta}^{(\nu)}_{T_r t_k} - v^{(\nu)}_{T_r t_k}), \forall T_r \in T \\
&\implies P^{(\nu+1)}_{T_r t_k} = \frac{\mathcal{V}}{X_{T_r}} && \text{(5.40a)} \\
&\implies P^{(\nu+1)}_{T_r t_{k'}} = -\frac{\mathcal{V}}{X_{T_r}} && \text{(5.40b)} \\
&\implies \theta^{(\nu+1)}_{T_r t_k} = \mathcal{X}_1 && \text{(5.40c)}
\end{aligned}
$$

$$\implies \theta_{T_r t_{k'}}^{(\nu+1)} = \mathcal{X}_1 + \mathcal{V} \tag{5.40d}$$

$$\text{where } \mathcal{V} = \frac{(X_{T_r})^2(\mathcal{E} - \mathcal{B}) + 2(\mathcal{A} - \mathcal{C})}{(4 + X_{T_r}^2)} \tag{5.40e}$$

$$\mathcal{X}_1 = \frac{(2 + (X_{T_r})^2)\mathcal{B} + 2\mathcal{E} + \mathcal{C} - \mathcal{A}}{(4 + (X_{T_r})^2)} \tag{5.40f}$$

$$\text{where } \mathcal{A} = X_{T_r}(P_{T_r t_k}^{(\nu)} - \hat{P}_{T_r t_k}^{(\nu)} - u_{T_r t_k}^{(\nu)}) \tag{5.40g}$$

$$\mathcal{B} = \hat{v}_{T_r t_k}^{(\nu-1)} + \hat{\theta}_{T_r t_k}^{(\nu)} - v_{T_r t_k}^{(\nu)} \tag{5.40h}$$

$$\mathcal{C} = X_{T_r}(P_{T_r t_{k'}}^{(\nu)} - \hat{P}_{T_r t_{k'}}^{(\nu)} - u_{T_r t_{k'}}^{(\nu)}) \tag{5.40i}$$

$$\mathcal{E} = \hat{v}_{T_r t_{k'}}^{(\nu-1)} + \hat{\theta}_{T_r t_{k'}}^{(\nu)} - v_{T_r t_{k'}}^{(\nu)} \tag{5.40j}$$

If terminal t_k is connected to the slack bus, then

$$\mathcal{V} = \frac{(X_{T_r})^2\mathcal{E} + \mathcal{A} - \mathcal{C}}{(2 + (X_{T_r})^2)} \tag{5.40k}$$

$$\mathcal{X}_1 = 0 \tag{5.40l}$$

If terminal $t_{k'}$ is connected to the slack bus, then

$$\mathcal{V} = \frac{\mathcal{A} - \mathcal{C} - (X_{T_r})^2\mathcal{B}}{(2 + (X_{T_r})^2)} \tag{5.40m}$$

$$\mathcal{X}_1 = -\mathcal{V} \tag{5.40n}$$

If $-X_{T_r}\overline{L}_{T_r} \leq \mathcal{V} \leq X_{T_r}\overline{L}_{T_r}$, then the $\mathcal{V}$ minimizer is chosen among these three : $\min\{\frac{(X_{T_r})^2(\mathcal{E}-\mathcal{B})+2(\mathcal{A}-\mathcal{C})}{(4+X_{T_r}^2)}, -X_{T_r}\overline{L}_{T_r}, X_{T_r}\overline{L}_{T_r}\}$, $\min\{\frac{(X_{T_r})^2\mathcal{E}+\mathcal{A}-\mathcal{C}}{(2+(X_{T_r})^2)}, -X_{T_r}\overline{L}_{T_r}, X_{T_r}\overline{L}_{T_r}\}$, $\min\{\frac{\mathcal{A}-\mathcal{C}-(X_{T_r})^2\mathcal{B}}{(2+(X_{T_r})^2)}, -X_{T_r}\overline{L}_{T_r}, X_{T_r}\overline{L}_{T_r}\}$. Otherwise, it is chosen from among $-X_{T_r}\overline{L}_{T_r}$ and $X_{T_r}\overline{L}_{T_r}$. The $\mathcal{X}_1$ stays the same as defined above in all the cases ie as stated in equations (5.40f), (5.40l), or, (5.40n).

$$(P_{D_d t_k}^{(\nu+1)}, \theta_{D_d t_k}^{(\nu+1)})$$
$$= \mathbf{prox}_{-P_{D_d},\rho}(\hat{v}_{D_d t_k}^{(\nu-1)} + \hat{\theta}_{D_d t_k}^{(\nu)} - v_{D_d t_k}^{(\nu)}), \forall D_d \in L$$

$$\implies P_{D_d t_k}^{(\nu+1)} = -P_{D_d} \tag{5.41a}$$

$$\implies \theta_{D_d t_k}^{(\nu+1)} = \hat{v}_{D_d t_k}^{(\nu-1)} + \hat{\theta}_{D_d t_k}^{(\nu)} - v_{D_d t_k}^{(\nu)} \tag{5.41b}$$

Hence, from the above, the calculation of prox functions by the devices happen all in parallel, corresponding to equations (5.39a) to (5.39c). After this, the devices pass the most recent values of power and voltage angle schedules to the nets. The respective nets "Gather" or receive those values and calculate the updates for the dual variables according to equations (5.39d) and (5.39e) and pass those back to the connected devices via process called "Broadcast". We have shown this in figures 5.5, 5.6, and 5.4, the broadcast, gather, and the schematic for the ADMM-PMP algorithm respectively for some generalized devices.

5.7.2 Proximal Message Passing for $(N-1)$ Contingency Constrained Generalized Multi-Bus Case: Unequal Capacities and Unequal Line Impedances

The slightly reformulated $\mathcal{DTN}$ equations from previous section are:

$$\begin{aligned} \min_{P_{t_k}^{(c)}, \theta_{t_k}^{(c)}} \; & C(\mathbf{P^{(0)}}) + F(\mathbf{P^{(c)}}) + \chi(\mathbf{P^{(c)}}, \theta^{(\mathbf{c})}) \\ & + \sum_{(c)\in\mathcal{L}} \sum_{N_i\in\mathcal{N}} (\overline{I}(z_{N_i t_k}^{(c)}) + \tilde{I}(\xi_{N_i t_k}^{(c)})) \end{aligned} \tag{5.42a}$$

$$\begin{aligned} \text{Subject to: } & P_{t_k}^{(c)} = z_{t_k}^{(c)}, \theta_{t_k}^{(c)} = \xi_{t_k}^{(c)}, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \\ & \forall (c) \in \mathcal{L} \end{aligned} \tag{5.42b}$$

It is to be observed here that the power balance and the phase consistency constraints need to be satisfied for each and every contingency scenario. The different update equations of the Proximal Message Passing Algorithm in this case are as follows:

5.7.2.1 Iterates for Generators

They consist of the update equations for the real power output and voltage-phase angles of the generator terminals for both the base case and the different $(N-1)$ contingency scenarios and are as follows:

$$
\begin{aligned}
(P_{g_q t_k}^{(0)(\nu+1)}, \theta_{g_q t_k}^{(c)(\nu+1)}) = \underset{P_{g_q t_k}^{(0)}, \theta_{g_q t_k}^{(c)}}{\operatorname{argmin}} & [C_{g_q t_k}(P_{g_q t_k}^{(0)}, \theta_{g_q t_k}^{(c)})+ \\
I_{\leq}(\overline{P}_{g_q} - P_{g_{q_{t_k}}}^{(0)}) &+ I_{\leq}(P_{g_{q_{t_k}}}^{(0)} - \underline{P}_{g_q})+ \\
\sum_{(c)\in\mathcal{L}} (\frac{\rho}{2})(||P_{g_{q_{t_k}}}^{(0)} - z_{g_q t_k}^{(c)(\nu)} + u_{g_q t_k}^{(c)(\nu)}||_2^2 &+ ||\theta_{g_{q_{t_k}}}^{(c)} - \xi_{g_q t_k}^{(c)(\nu)} + v_{g_q t_k}^{(c)(\nu)}||_2^2)], \\
\forall g_q \in G, t_k &\in \mathfrak{T} \cap G
\end{aligned} \tag{5.43a}
$$

5.7.2.2 Iterates for Transmission Lines

$$
\begin{aligned}
&(P_{T_r t_k}^{(c)(\nu+1)}, \theta_{T_r t_k}^{(c)(\nu+1)}, P_{T_r t_{k'}}^{(c)(\nu+1)}, \theta_{T_r t_{k'}}^{(c)(\nu+1)}) \\
&= \underset{P_{T_r t_k}^{(c)}, \theta_{T_r t_k}^{(c)}}{\operatorname{argmin}} [\sum_{k,k' \in \mathfrak{T}\cap T_r} (I_{\leq}(\overline{L}_{T_r}^{(c)} - |P_{T_{r t_k}}^{(c)}|)+ \\
&I_{=}(P_{T_{r t_k}}^{(c)} + \frac{\theta_{T_{r t_k}}^{(c)} - \theta_{T_{r t_{k'}}}^{(c)}}{X_{T_r}^{(c)}})+ \\
&\frac{\rho}{2}(||P_{T_{r t_k}}^{(c)} - z_{T_r t_k}^{(c)(\nu)} + u_{T_r t_k}^{(c)(\nu)}||_2^2 + ||\theta_{T_{r t_k}}^{(c)} - \xi_{T_r t_k}^{(c)(\nu)} + v_{T_r t_k}^{(c)(\nu)}||_2^2))] \\
&\forall T_r \in T, t_k \in \mathfrak{T} \cap T, (c) \in \mathcal{L}
\end{aligned} \tag{5.44a}
$$

5.7.2.3 Iterates for Loads

They consist of the update equations for the real power and voltage-phase angles of the loads (which have constant real power consumption) and are as follows:

$$P_{D_d t_k}^{(c)(\nu+1)} = P_{D_d t_k}^{(c)(\nu)} = -P_{D_{d t_k}}$$
$$\theta_{D_d t_k}^{(c)(\nu+1)} = \underset{\theta_{D_d t_k}^{(c)}}{\text{argmin}}[\frac{\rho}{2}(||\theta_{D_{d t_k}}^{(c)} - \xi_{D_d t_k}^{(c)(\nu)} + v_{D_d t_k}^{(c)(\nu)}||_2^2)],$$
$$\forall D_d \in L, t_k \in \mathcal{T} \cap L, (c) \in \mathcal{L} \tag{5.45a}$$

5.7.2.4 Iterates for Nets

We are writing here just the analytical forms already derived in [234].

$$\forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T} \cap N_i, \forall (c) \in \mathcal{L}$$
$$z_{N_i t_k}^{(c)(\nu+1)} = u_{N_i t_k}^{(c)(\nu)} + P_{N_i t_k}^{(c)(\nu+1)} - \hat{u}_{N_i t_k}^{(c)(\nu)} - \hat{P}_{N_i t_k}^{(c)(\nu+1)} \tag{5.46a}$$
$$\xi_{N_i t_k}^{(c)(\nu+1)} = \hat{v}_{N_i t_k}^{(c)(\nu)} + \hat{\theta}_{N_i t_k}^{(c)(\nu+1)} \tag{5.46b}$$
$$u_{N_i t_k}^{(c)(\nu+1)} = u_{N_i t_k}^{(c)(\nu)} + (P_{N_i t_k}^{(c)(\nu+1)} - z_{N_i t_k}^{(c)(\nu+1)}) \tag{5.46c}$$
$$v_{N_i t_k}^{(c)(\nu+1)} = v_{N_i t_k}^{(c)(\nu)} + (\theta_{N_i t_k}^{(c)(\nu+1)} - \xi_{N_i t_k}^{(c)(\nu+1)}) \tag{5.46d}$$

In the above, as before, not only do all the devices update their variables in parallel, but also, except the generators, all devices have associated with them the base-case and the contingency scenarios, each of which in turn update their respective variables in parallel as well. Then all the nets and the base-case/contingency scenarios associated with them update the first two set of variables in parallel and then update the next two in parallel. For this case, the prox messages and the Proximal Message Passing Algorithm is as follows:

$$(P_{g_q t_k}^{(0)(\nu+1)}, \theta_{g_q t_k}^{(c)(\nu+1)})$$

$$= \mathbf{prox}_{C(\mathbf{P^{(0)}}),\rho}(P^{(0)(\nu)}_{g_qt_k} - \hat{P}^{(c)(\nu)}_{g_qt_k} - u^{(c)(\nu)}_{g_qt_k}, \hat{v}^{(c)(\nu-1)}_{g_qt_k} + \hat{\theta}^{(c)(\nu)}_{g_qt_k} - v^{(c)(\nu)}_{g_qt_k}), \forall g_q \in G, \forall (c) \in \mathcal{L} \tag{5.47a}$$

$$(P^{(c)(\nu+1)}_{T_rt_k}, \theta^{(c)(\nu+1)}_{T_rt_k}, P^{(c)(\nu+1)}_{T_rt_{k'}}, \theta^{(c)(\nu+1)}_{T_rt_{k'}})$$

$$= \mathbf{prox}_{F(\mathbf{P^{(c)}})+\chi(\mathbf{P^{(c)}},\theta^{(c)}),\rho}(P^{(c)(\nu)}_{T_rt_k} - \hat{P}^{(c)(\nu)}_{T_rt_k} - u^{(c)(\nu)}_{T_rt_k}, \hat{v}^{(c)(\nu-1)}_{T_rt_k} + \hat{\theta}^{(c)(\nu)}_{T_rt_k} - v^{(c)(\nu)}_{T_rt_k}),$$

$$\forall T_r \in T, \forall (c) \in \mathcal{L} \tag{5.47b}$$

$$(P^{(c)(\nu+1)}_{D_dt_k}, \theta^{(c)(\nu+1)}_{D_dt_k})$$

$$= \mathbf{prox}_{-P_{D_d},\rho}(\hat{v}^{(c)(\nu-1)}_{D_dt_k} + \hat{\theta}^{(c)(\nu)}_{D_dt_k} - v^{(c)(\nu)}_{D_dt_k}), \forall D_d \in L, \forall (c) \in \mathcal{L} \tag{5.47c}$$

$$u^{(c)(\nu+1)}_{N_it_k} = u^{(c)(\nu)}_{N_it_k} + \hat{P}^{(c)(\nu+1)}_{N_it_k}, \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L} \tag{5.47d}$$

$$v^{(c)(\nu+1)}_{N_it_k} = \tilde{v}^{(c)(\nu)}_{N_it_k} + \tilde{\theta}^{(c)(\nu+1)}_{N_it_k}, \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L} \tag{5.47e}$$

As before, for (5.47b) and (5.47c) we will have exactly identical expressions as in (5.40) and (5.41) for carrying out the calculations analytically. The ADMM-PMP algorithm has been pictorially depicted in figure 5.7. As can be seen from the schematic diagram, we have shown the problem corresponding to one base case and four different contingency scenarios. Corresponding to each scenario, we have copies of transmission lines and loads, as shown in the figure. The different generators in the different scenarios are linked to the one for the base case with the green lines, which represent the fact that the power generations in the scenarios are the same as for the base-case. The other lines and arrows represent the messages as before.

For the sake of brevity and to avoid redundancy, we will skip the next two sections on demand variation and post-contingency restoration in a single interval, with only the mention that almost all of the proximal message passing iterations will look exactly the same as before. The only exception is that, now, the generator cost functions will be augmented by the APP message terms, which we already described earlier. The diagrammatic representation of the APMP algorithm with both the fine and coarse grained components taken into consideration looks as shown in figure 5.8.

However, to gain some perspective, we will now present, in the next two sections, the purely ADMM-PMP based distributed algorithm to solve the LASCOPF problems to track demand variation and post-contingency restoration within a single dispatch interval. This will justify why we are proposing to use an AMPMP algorithm, instead of a pure ADMM-PMP algorithm for these (and more complicated problems).

5.7.3 Proximal Message Passing for the Look-Ahead Dispatch:Generalized Case of Demand Variation for the Multi Bus-Case

The reformulated $\mathcal{DTN}$ equations for each of the coarse grains, which represents the SCOPF for each look-ahead dispatch time interval, in this case are as follows:

$$\forall \tau \in \Omega$$

$$\begin{aligned}\min_{\mathbf{P}_{\mathbf{t_k}}^{(\mathbf{c})(\tau)}, \theta_{\mathbf{t_k}}^{(\mathbf{c})(\tau)}} & C(\mathbf{P}^{(\mathbf{0})(\tau)}) + F(\mathbf{P}^{(\mathbf{c})(\tau)}) + \chi(\mathbf{P}^{(\mathbf{c})(\tau)}, \theta^{(\mathbf{c})(\tau)}) + \Delta(\mathbf{P}^{(\mathbf{0})(\tau)}) \\ & + \sum_{(c)\in\mathcal{L}} \sum_{N_i \in \mathcal{N}} (\bar{I}(z_{N_i t_k}^{(c)(\tau)}) + \tilde{I}(\xi_{N_i t_k}^{(c)(\tau)}))\end{aligned} \tag{5.48a}$$

$$\begin{aligned}\text{Subject to: } P_{t_k}^{(c)(\tau)} = z_{t_k}^{(c)(\tau)}, \theta_{t_k}^{(c)(\tau)} = \xi_{t_k}^{(c)(\tau)}, \forall \tau \in \Omega, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \\ \forall (c) \in \mathcal{L}\end{aligned} \tag{5.48b}$$

The different update equations of the Proximal Message Passing Algorithm in this case are as follows:

5.7.3.1 Iterates for Generators

They consist of the update equations for the real power output and voltage-phase angles of the generator terminals for both the base case and the different $(N-1)$ contingency scenarios and are as follows:

$$
\begin{aligned}
(P^{(0)(\tau)(\nu+1)}_{g_q t_k}, \theta^{(c)(\tau)(\nu+1)}_{g_q t_k}) = \underset{P^{(0)(\tau)}_{g_q t_k}, \theta^{(c)(\tau)}_{g_q t_k}}{\operatorname{argmin}} [C_{g_q t_k}(P^{(0)(\tau)}_{g_q t_k}, \theta^{(c)(\tau)}_{g_q t_k}) + \\
I_{\leq}(\overline{P}_{g_q} - P^{(0)(\tau)}_{g_{q_{t_k}}}) + I_{\leq}(P^{(0)(\tau)}_{g_{q_{t_k}}} - \underline{P}_{g_q}) + \Delta(P^{(0)(\tau)}) \\
\sum_{(c)\in\mathcal{L}} (\frac{\rho}{2})(||P^{(0)(\tau)}_{g_{q_{t_k}}} - z^{(c)(\tau)(\nu)}_{g_q t_k} + u^{(c)(\tau)(\nu)}_{g_q t_k}||_2^2 + ||\theta^{(c)(\tau)}_{g_{q_{t_k}}} - \xi^{(c)(\tau)(\nu)}_{g_q t_k} + v^{(c)(\tau)(\nu)}_{g_q t_k}||_2^2)], \\
\forall g_q \in G, \tau \in \Omega, t_k \in \mathfrak{T} \cap G
\end{aligned}
\tag{5.49a}
$$

5.7.3.2 Iterates for Transmission Lines

$$
\begin{aligned}
&(P^{(c)(\tau)(\nu+1)}_{T_r t_k}, \theta^{(c)(\tau)(\nu+1)}_{T_r t_k}, P^{(c)(\tau)(\nu+1)}_{T_r t_{k'}}, \theta^{(c)(\tau)(\nu+1)}_{T_r t_{k'}}) \\
&= \underset{P^{(c)(\tau)}_{T_r t_k}, \theta^{(c)(\tau)}_{T_r t_k}}{\operatorname{argmin}} [\sum_{k,k' \in \mathfrak{T}\cap T_r} (I_{\leq}(\overline{L}^{(c)(\tau)}_{T_r} - |P^{(c)(\tau)}_{T_{r t_k}}|) + \\
&I_{=}(P^{(c)(\tau)}_{T_{r t_k}} + \frac{\theta^{(c)(\tau)}_{T_{r t_k}} - \theta^{(c)(\tau)}_{T_{r t_{k'}}}}{X^{(c)}_{T_r}}) + \\
&\frac{\rho}{2}(||P^{(c)(\tau)}_{T_{r t_k}} - z^{(c)(\tau)(\nu)}_{T_r t_k} + u^{(c)(\tau)(\nu)}_{T_r t_k}||_2^2 + ||\theta^{(c)(\tau)}_{T_{r t_k}} - \xi^{(c)(\tau)(\nu)}_{T_r t_k} + v^{(c)(\tau)(\nu)}_{T_r t_k}||_2^2))] \\
&\forall T_r \in T, t_k \in \mathfrak{T} \cap T, (c) \in \mathcal{L}, \tau \in \Omega
\end{aligned}
\tag{5.50a}
$$

5.7.3.3 Iterates for Loads

They consist of the update equations for the real power and voltage-phase angles of the loads (which have constant real power consumption) and are as follows:

$$
\begin{aligned}
P_{D_d t_k}^{(c)(\tau)(\nu+1)} = P_{D_d t_k}^{(c)(\tau)(\nu)} = -D_{d_{t_k}}^{(\tau)} \\
\theta_{D_d t_k}^{(c)(\tau)(\nu+1)} = \underset{\theta_{D_d t_k}^{(c)(\tau)}}{\operatorname{argmin}}[\frac{\rho}{2}(||\theta_{D_{d_{t_k}}}^{(c)(\tau)} - \xi_{D_d t_k}^{(c)(\tau)(\nu)} + v_{D_d t_k}^{(c)(\tau)(\nu)}||_2^2)], \\
\forall D_d \in L, t_k \in \mathcal{T} \cap L, (c) \in \mathcal{L}, \tau \in \Omega
\end{aligned}
\tag{5.51a}
$$

5.7.3.4 Iterates for Nets

We are writing here just the analytical forms already derived in [234].

$$\forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T} \cap N_i, \forall (c) \in \mathcal{L}, \tau \in \Omega$$

$$z_{N_i t_k}^{(c)(\tau)(\nu+1)} = u_{N_i t_k}^{(c)(\tau)(\nu)} + P_{N_i t_k}^{(c)(\tau)(\nu+1)} - \hat{u}_{N_i t_k}^{(c)(\tau)(\nu)} - \hat{P}_{N_i t_k}^{(c)(\tau)(\nu+1)} \tag{5.52a}$$

$$\xi_{N_i t_k}^{(c)(\tau)(\nu+1)} = \hat{v}_{N_i t_k}^{(c)(\tau)(\nu)} + \hat{\theta}_{N_i t_k}^{(c)(\tau)(\nu+1)} \tag{5.52b}$$

$$u_{N_i t_k}^{(c)(\tau)(\nu+1)} = u_{N_i t_k}^{(c)(\tau)(\nu)} + (P_{N_i t_k}^{(c)(\tau)(\nu+1)} - z_{N_i t_k}^{(c)(\tau)(\nu+1)}) \tag{5.52c}$$

$$v_{N_i t_k}^{(c)(\tau)(\nu+1)} = v_{N_i t_k}^{(c)(\tau)(\nu)} + (\theta_{N_i t_k}^{(c)(\tau)(\nu+1)} - \xi_{N_i t_k}^{(c)(\tau)(\nu+1)}) \tag{5.52d}$$

In the above, in this case, not only do all the devices update their variables in parallel, but also, except the generators, all devices have associated with them the base-case and the contingency scenarios in each dispatch interval, each of which in turn update their respective variables in parallel as well. For the generators, the outputs of the present dispatch interval and the next one are related through the ramps rate constraint (Δ Function is the indicator function corresponding to the fact that the change in power output of generators over successive dispatch intervals is not allowed to surpass the limits imposed by the maximum and minimum ramping capabilities, as explained in sections 5.6.6 and 5.6.7). Then all the nets and the base-case/contingency scenarios associated with them update the first two set of variables in parallel and then update the next two in parallel. For this case, the prox messages and the Proximal Message Passing Algorithm is as follows:

$$(P^{(0)(\tau)(\nu+1)}_{g_q t_k}, \theta^{(c)(\tau)(\nu+1)}_{g_q t_k})$$
$$= \mathbf{prox}_{C(P^{(0)(\tau)})+\Delta(P^{(0)(\tau)}),\rho}(P^{(0)(\tau)(\nu)}_{g_q t_k} - \hat{P}^{(c)(\tau)(\nu)}_{g_q t_k} - u^{(c)(\tau)(\nu)}_{g_q t_k}, \hat{v}^{(c)(\tau)(\nu-1)}_{g_q t_k} + \hat{\theta}^{(c)(\tau)(\nu)}_{g_q t_k} - v^{(c)(\tau)(\nu)}_{g_q t_k}),$$
$$\forall g_q \in G, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.53a}$$

$$(P^{(c)(\tau)(\nu+1)}_{T_r t_k}, \theta^{(c)(\tau)(\nu+1)}_{T_r t_k}, P^{(c)(\tau)(\nu+1)}_{T_r t_{k'}}, \theta^{(c)(\tau)(\nu+1)}_{T_r t_{k'}})$$
$$= \mathbf{prox}_{F(P^{(c)(\tau)})+\chi(P^{(c)(\tau)},\theta^{(c)(\tau)}),\rho}(P^{(c)(\tau)(\nu)}_{T_r t_k} - \hat{P}^{(c)(\tau)(\nu)}_{T_r t_k} - u^{(c)(\tau)(\nu)}_{T_r t_k}, \hat{v}^{(c)(\tau)(\nu-1)}_{T_r t_k} + \hat{\theta}^{(c)(\tau)(\nu)}_{T_r t_k} - v^{(c)(\tau)(\nu)}_{T_r t_k}),$$
$$\forall T_r \in T, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.53b}$$

$$(P^{(c)(\tau)(\nu+1)}_{D_d t_k}, \theta^{(c)(\tau)(\nu+1)}_{D_d t_k})$$
$$= \mathbf{prox}_{-D,\rho}(\hat{v}^{(c)(\tau)(\nu-1)}_{D_d t_k} + \hat{\theta}^{(c)(\tau)(\nu)}_{D_d t_k} - v^{(c)(\tau)(\nu)}_{D_d t_k}), \forall D_d \in L, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.53c}$$

$$u^{(c)(\tau)(\nu+1)}_{N_i t_k} = u^{(c)(\tau)(\nu)}_{N_i t_k} + \hat{P}^{(c)(\tau)(\nu+1)}_{N_i t_k}, \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.53d}$$

$$v^{(c)(\tau)(\nu+1)}_{N_i t_k} = \tilde{v}^{(c)(\tau)(\nu)}_{N_i t_k} + \tilde{\theta}^{(c)(\tau)(\nu+1)}_{N_i t_k}, \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.53e}$$

5.7.4 Proximal Message Passing for the Look-Ahead Dispatch Model for Ensuring Security with respect to Next Outage:Generalized Multi Bus-Case

The modified $\mathcal{DTN}$ formulation for this case is as follows:

$$\forall \tau \in \Omega$$

$$\min_{P^{(c)(\tau)}_{t_k}, \theta^{(c)(\tau)}_{t_k}} C(P^{(0)(\tau)}) + C(P^{(c)(\tau+1)}) + F(P^{(c)(\tau)}) + F(P^{(c\to c')(\tau+1)}) +$$

$$\chi(P^{(c)(\tau)}, \theta^{(c)(\tau)}) + \chi(P^{(c \to c')(\tau+1)}, \theta^{(c \to c')(\tau+1)}) + \Delta(P^{(c)(\tau)})$$

$$+ \sum_{N_i \in \mathcal{N}} \sum_{(c) \in \mathcal{L}} (\overline{I}(z_{N_i t_k}^{(c)(\tau)}) + \tilde{I}(\xi_{N_i t_k}^{(c)(\tau)}) + \sum_{(c') \in [\mathcal{L} - \{0,c\}] \cap \{c\}} (\overline{I}(z_{N_i t_k}^{(c \to c')(\tau+1)}) + \tilde{I}(\xi_{N_i t_k}^{(c \to c')(\tau+1)})) \quad (5.54a)$$

$$\text{Subject to: } P_{t_k}^{(c)(\tau)} = z_{t_k}^{(c)(\tau)}, \theta_{t_k}^{(c)(\tau)} = \xi_{t_k}^{(c)(\tau)}, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall (c) \in \mathcal{L} \quad (5.54b)$$

$$P_{t_k}^{(c \to c')(\tau+1)} = z_{t_k}^{(c \to c')(\tau+1)}, \theta_{t_k}^{(c \to c')(\tau+1)} = \xi_{t_k}^{(c \to c')(\tau+1)}, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall (c) \in \mathcal{L},$$

$$\forall (c') \in [\mathcal{L} - \{0, c\}] \cap \{c\}, \forall \tau \in \Omega \quad (5.54c)$$

The different update equations of the Proximal Message Passing Algorithm in this case are as follows:

5.7.4.1 Iterates for Generators

In the present case, they consist of the update equations for the real power output and voltage-phase angles of the generator terminals for both the base case and the different $(N-1)$ contingency scenarios for the first dispatch interval and the same for each of those contingencies as the base-case as well as the rest of the scenarios for the second interval. These are as follows:

$$(\mathbf{P}_{g_q t_k}^{(c)(\tau)(\nu+1)}, \mathbf{\Theta}_{g_q t_k}^{(c)(\tau)(\nu+1)}) = \underset{\mathbf{P}_{g_q t_k}^{(c)(\tau)}, \mathbf{\Theta}_{g_q t_k}^{(c)(\tau)}}{\operatorname{argmin}} [C_{g_q t_k}(\mathbf{P}_{g_q t_k}^{(c)(\tau)}, \mathbf{\Theta}_{g_q t_k}^{(c)(\tau)}) +$$

$$I_{\leq}(\overline{\mathbf{P}}_{g_q} - \mathbf{P}_{g_{q_{t_k}}}^{(c)(\tau)}) + I_{\leq}(\mathbf{P}_{g_{q_{t_k}}}^{(c)(\tau)} - \underline{\mathbf{P}}_{g_q}) + \Delta(\mathbf{W}\mathbf{P}_{g_{q_{t_k}}}^{(c)(\tau)})$$

$$+(\frac{\rho}{2})(||\mathbf{P}_{g_{q_{t_k}}}^{(c)(\tau)}[\mathbf{1}]^T - (\mathcal{Z}_{g_q t_k}^{(c)(\tau)(\nu)} - \mathcal{U}_{g_q t_k}^{(c)(\tau)(\nu)})||_{Frob}^2 + ||\mathbf{\Theta}_{g_{q_{t_k}}}^{(c)(\tau)} - (\mathbf{\Xi}_{g_q t_k}^{(c)(\tau)(\nu)} - \mathcal{V}_{g_q t_k}^{(c)(\tau)(\nu)})||_{Frob}^2)],$$

$$\forall g_q \in G, \tau \in \Omega, t_k \in \mathcal{T} \cap G \quad (5.55a)$$

where

$$\mathbf{P}_{g_q t_k}^{(c)(\tau)} = [P_{g_q t_k}^{(0)(\tau)} \; P_{g_q t_k}^{(1)(\tau+1)} \; ... \; P_{g_q t_k}^{(|\mathcal{L}|)(\tau+1)}]^\dagger \quad (5.56)$$

$$\boldsymbol{\Theta}^{(c)(\tau)}_{g_qt_k} = \begin{pmatrix} \theta^{(0)(\tau)}_{g_qt_k} & \theta^{(1)(\tau)}_{g_qt_k} & \theta^{(2)(\tau)}_{g_qt_k} & ... & \theta^{(|\mathcal{L}|)(\tau)}_{g_qt_k} \\ 0 & \theta^{(1)(\tau+1)}_{g_qt_k} & \theta^{(1\to2)(\tau+1)}_{g_qt_k} & ... & \theta^{(1\to|\mathcal{L}|)(\tau)}_{g_qt_k} \\ ... \\ 0 & \theta^{(|\mathcal{L}|\to1)(\tau+1)}_{g_qt_k} & \theta^{(|\mathcal{L}|\to2)(\tau+1)}_{g_qt_k} & ... & \theta^{(|\mathcal{L}|)(\tau+1)}_{g_qt_k} \end{pmatrix} \tag{5.57}$$

$$\mathbf{W} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & ... & 0 \\ 1 & 0 & -1 & 0 & 0 & ... & 0 \\ ... \\ 1 & 0 & 0 & 0 & 0 & ... & -1 \end{pmatrix} \tag{5.58}$$

$$\overline{\mathbf{P}}_{g_q} = \overline{P}_{g_q}[1\ 1\ ...\ 1]^\dagger = \overline{P}_{g_q}[\mathbf{1}] \tag{5.59}$$

$$\underline{\mathbf{P}}_{g_q} = \underline{P}_{g_q}[1\ 1\ ...\ 1]^\dagger = \underline{P}_{g_q}[\mathbf{1}] \tag{5.60}$$

$$\tag{5.61}$$

$I_{\leq} : \mathbb{R}^{(|\mathcal{L}|+1)} \to 0, \infty; I_{\leq}(\mathbf{n}) = 0, if \mathbf{n} \geq \mathbf{0}, = \infty, \text{otherwise}$

In the above, $\geq$ for vectors indicates component-wise inequality

$C_{g_qt_k}(\mathbf{P}^{(c)(\tau)}_{g_qt_k}, \boldsymbol{\Theta}^{(c)(\tau)}_{g_qt_k}) = (\frac{1}{2})P^{(c)(\tau)\dagger}_{g_qt_k}\text{diag}(\alpha_{g_qt_k})P^{(c)(\tau)}_{g_qt_k} + (\beta_{g_qt_k})[\mathbf{1}]^\dagger P^{(c)(\tau)}_{g_qt_k} + (|\mathcal{L}|+1)\gamma_{g_qt_k}$

$$\mathcal{Z}^{(c)(\tau)}_{g_qt_k} = \begin{pmatrix} z^{(0)(\tau)}_{g_qt_k} & z^{(1)(\tau)}_{g_qt_k} & z^{(2)(\tau)}_{g_qt_k} & ... & z^{(|\mathcal{L}|)(\tau)}_{g_qt_k} \\ P^{(1)(\tau+1)}_{g_qt_k} & z^{(1)(\tau+1)}_{g_qt_k} & z^{(1\to2)(\tau+1)}_{g_qt_k} & ... & z^{(1\to|\mathcal{L}|)(\tau)}_{g_qt_k} \\ ... \\ P^{(|\mathcal{L}|)(\tau+1)}_{g_qt_k} & z^{(|\mathcal{L}|\to1)(\tau+1)}_{g_qt_k} - & z^{(|\mathcal{L}|\to2)(\tau+1)}_{g_qt_k} & ... & z^{(|\mathcal{L}|)(\tau+1)}_{g_qt_k} \end{pmatrix} \tag{5.62}$$

$$\mathcal{U}^{(c)(\tau)}_{g_qt_k} = \begin{pmatrix} u^{(0)(\tau)}_{g_qt_k} & u^{(1)(\tau)}_{g_qt_k} & u^{(2)(\tau)}_{g_qt_k} & ... & u^{(|\mathcal{L}|)(\tau)}_{g_qt_k} \\ 0 & u^{(1)(\tau+1)}_{g_qt_k} & u^{(1\to2)(\tau+1)}_{g_qt_k} & ... & u^{(1\to|\mathcal{L}|)(\tau)}_{g_qt_k} \\ ... \\ 0 & u^{(|\mathcal{L}|\to1)(\tau+1)}_{g_qt_k} & u^{(|\mathcal{L}|\to2)(\tau+1)}_{g_qt_k} & ... & u^{(|\mathcal{L}|)(\tau+1)}_{g_qt_k} \end{pmatrix} \tag{5.63}$$

$$\boldsymbol{\Xi}^{(c)(\tau)}_{g_qt_k} = \begin{pmatrix} \xi^{(0)(\tau)}_{g_qt_k} & \xi^{(1)(\tau)}_{g_qt_k} & \xi^{(2)(\tau)}_{g_qt_k} & ... & \xi^{(|\mathcal{L}|)(\tau)}_{g_qt_k} \\ 0 & \xi^{(1)(\tau+1)}_{g_qt_k} & \xi^{(1\to2)(\tau+1)}_{g_qt_k} & ... & \xi^{(1\to|\mathcal{L}|)(\tau)}_{g_qt_k} \\ ... \\ 0 & \xi^{(|\mathcal{L}|\to1)(\tau+1)}_{g_qt_k} & \xi^{(|\mathcal{L}|\to2)(\tau+1)}_{g_qt_k} & ... & \xi^{(|\mathcal{L}|)(\tau+1)}_{g_qt_k} \end{pmatrix} \tag{5.64}$$

$$\mathcal{V}^{(c)(\tau)}_{g_q t_k} = \begin{pmatrix} v^{(0)(\tau)}_{g_q t_k} & v^{(1)(\tau)}_{g_q t_k} & v^{(2)(\tau)}_{g_q t_k} & ... & v^{(|\mathcal{L}|)(\tau)}_{g_q t_k} \\ 0 & v^{(1)(\tau+1)}_{g_q t_k} & v^{(1\rightarrow 2)(\tau+1)}_{g_q t_k} & ... & v^{(1\rightarrow|\mathcal{L}|)(\tau)}_{g_q t_k} \\ ... & & & & \\ 0 & v^{(|\mathcal{L}|\rightarrow 1)(\tau+1)}_{g_q t_k} & v^{(|\mathcal{L}|\rightarrow 2)(\tau+1)}_{g_q t_k} & ... & v^{(|\mathcal{L}|)(\tau+1)}_{g_q t_k} \end{pmatrix} \tag{5.65}$$

$$\tag{5.66}$$

$||(.)||^2_{Frob}$ indicates sqaure of the Frobenius Norm of a sqaure matrix i.e. the sum of sqaure of its elements.

$\Delta(\mathbf{WP}^{(c)(\tau)}_{g_{q_{t_k}}}) = I_{\leq}(\mathbf{R}^{max}_{g_{q_{t_k}}} + \mathbf{WP}^{(c)(\tau)}_{g_{q_{t_k}}}) + I_{\leq}(-\mathbf{R}^{min}_{g_{q_{t_k}}} - \mathbf{WP}^{(c)(\tau)}_{g_{q_{t_k}}})$

5.7.4.2 Iterates for Transmission Lines

There will be two sets of equations now. One for the present time and the second for the next dispatch interval. The “present time” update equations are just the same as before and are as follows:

$$\begin{gathered} (P^{(c)(\tau)(\nu+1)}_{T_r t_k}, \theta^{(c)(\tau)(\nu+1)}_{T_r t_k}, P^{(c)(\tau)(\nu+1)}_{T_r t_{k'}}, \theta^{(c)(\tau)(\nu+1)}_{T_r t_{k'}}) \\ = \underset{P^{(c)(\tau)}_{T_r t_k}, \theta^{(c)(\tau)}_{T_r t_k}}{\operatorname{argmin}} [\sum_{k,k' \in \mathfrak{T} \cap T_r} (I_{\leq}(\overline{L}^{(c)(\tau)}_{T_r} - |P^{(c)(\tau)}_{T_{r t_k}}|) + \\ I_{=}(P^{(c)(\tau)}_{T_{r t_k}} + \frac{\theta^{(c)(\tau)}_{T_{r t_k}} - \theta^{(c)(\tau)}_{T_{r t_{k'}}}}{X^{(c)}_{T_r}}) + \\ \frac{\rho}{2}(||P^{(c)(\tau)}_{T_{r t_k}} - z^{(c)(\tau)(\nu)}_{T_r t_k} + u^{(c)(\tau)(\nu)}_{T_r t_k}||^2_2 + ||\theta^{(c)(\tau)}_{T_{r t_k}} - \xi^{(c)(\tau)(\nu)}_{T_r t_k} + v^{(c)(\tau)(\nu)}_{T_r t_k}||^2_2))] \\ \forall T_r \in T, t_k \in \mathfrak{T} \cap T, (c) \in \mathcal{L}, \tau \in \Omega \end{gathered} \tag{5.67a}$$

The update equations for the next dispatch interval takes into account each of the potential contingencies to have happened and security with respect to the next set of line outages and as follows:

$$(P_{T_rt_k}^{(c\to c^{'})(\tau+1)(\nu+1)},\theta_{T_rt_k}^{(c\to c^{'})(\tau+1)(\nu+1)},P_{T_rt_{k'}}^{(c\to c^{'})(\tau+1)(\nu+1)},\theta_{T_rt_{k'}}^{(c\to c^{'})(\tau+1)(\nu+1)})$$

$$=\underset{P_{T_rt_k}^{(c\to c^{'})(\tau+1)},\theta_{T_rt_k}^{(c\to c^{'})(\tau+1)}}{\operatorname{argmin}}[\sum_{k,k^{'}\in\mathfrak{T}\cap T_r}(I_{\leq}(\overline{L}_{T_r}^{(c\to c^{'})(\tau+1)}-|P_{T_rt_k}^{(c\to c^{'})(\tau+1)}|)+$$

$$I_{=}(P_{T_rt_k}^{(c\to c^{'})(\tau+1)}+\frac{\theta_{T_rt_k}^{(c\to c^{'})(\tau+1)}-\theta_{T_rt_{k'}}^{(c\to c^{'})(\tau+1)}}{X_{T_r}^{(c\to c^{'})}})+\frac{\rho}{2}(||P_{T_rt_k}^{(c\to c^{'})(\tau+1)}-z_{T_rt_k}^{(c\to c^{'})(\tau+1)(\nu)}+u_{T_rt_k}^{(c\to c^{'})(\tau+1)(\nu)}||_2^2$$

$$+||\theta_{T_rt_k}^{(c\to c^{'})(\tau+1)}-\xi_{T_rt_k}^{(c\to c^{'})(\tau+1)(\nu)}+v_{T_rt_k}^{(c\to c^{'})(\tau+1)(\nu)}||_2^2))]$$

$$\forall T_r\in T,t_k\in\mathfrak{T}\cap T,(c)\in\mathcal{L},(c\to c^{'})\in[\mathcal{L}-\{0,c\}]\cap c,\tau\in\Omega \qquad (5.68a)$$

5.7.4.3 Iterates for Loads

Similar to the previous case, the update equations for loads, which consist of the ones for the real power and voltage-phase angles of the loads (which have constant real power consumption) also has a present dispatch time and next dispatch time components. The first one is as follows:

$$P_{D_dt_k}^{(c)(\tau)(\nu+1)}=P_{D_dt_k}^{(c)(\tau)(\nu)}=-D_{d_{t_k}}^{(\tau)}$$

$$\theta_{D_dt_k}^{(c)(\tau)(\nu+1)}=\underset{\theta_{D_dt_k}^{(c)(\tau)}}{\operatorname{argmin}}[\frac{\rho}{2}(||\theta_{D_{dt_k}}^{(c)(\tau)}-\xi_{D_dt_k}^{(c)(\tau)(\nu)}+v_{D_dt_k}^{(c)(\tau)(\nu)}||_2^2)],$$

$$\forall D_d\in L,t_k\in\mathfrak{T}\cap L,(c)\in\mathcal{L},\tau\in\Omega \qquad (5.69a)$$

The equations for the next dispatch interval are as follows:

$$P_{D_dt_k}^{(c\to c^{'})(\tau+1)(\nu+1)}=P_{D_dt_k}^{(c\to c^{'})(\tau+1)(\nu)}=-D_{d_{t_k}}^{(\tau+1)}$$

$$\theta_{D_dt_k}^{(c\to c^{'})(\tau+1)(\nu+1)}=\underset{\theta_{D_dt_k}^{(c\to c^{'})(\tau+1)}}{\operatorname{argmin}}[\frac{\rho}{2}(||\theta_{D_{dt_k}}^{(c\to c^{'})(\tau+1)}-\xi_{D_dt_k}^{(c\to c^{'})(\tau+1)(\nu)}+v_{D_dt_k}^{(c\to c^{'})(\tau+1)(\nu)}||_2^2)],$$

$$\forall D_d\in L,t_k\in\mathfrak{T}\cap L,(c)\in\mathcal{L},(c\to c^{'})\in[\mathcal{L}-\{0,c\}]\cap c,\tau\in\Omega \qquad (5.70a)$$

5.7.4.4 Iterates for Nets

We are writing here just the analytical forms already derived in [234].

$\forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T} \cap N_i, \forall (c) \in \mathcal{L}, \tau \in \Omega$

$$z_{N_it_k}^{(c)(\tau)(\nu+1)} = u_{N_it_k}^{(c)(\tau)(\nu)} + P_{N_it_k}^{(c)(\tau)(\nu+1)} - \hat{u}_{N_it_k}^{(c)(\tau)(\nu)} - \hat{P}_{N_it_k}^{(c)(\tau)(\nu+1)} \tag{5.71}$$

$$z_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} = u_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu)} + P_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} - \hat{u}_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu)} - \hat{P}_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} \tag{5.72}$$

$$\xi_{N_it_k}^{(c)(\tau)(\nu+1)} = \hat{v}_{N_it_k}^{(c)(\tau)(\nu)} + \hat{\theta}_{N_it_k}^{(c)(\tau)(\nu+1)} \tag{5.73}$$

$$\xi_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} = \hat{v}_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu)} + \hat{\theta}_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} \tag{5.74}$$

$$u_{N_it_k}^{(c)(\tau)(\nu+1)} = u_{N_it_k}^{(c)(\tau)(\nu)} + (P_{N_it_k}^{(c)(\tau)(\nu+1)} - z_{N_it_k}^{(c)(\tau)(\nu+1)}) \tag{5.75}$$

$$u_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} = u_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu)} + (P_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} - z_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)}) \tag{5.76}$$

$$v_{N_it_k}^{(c)(\tau)(\nu+1)} = v_{N_it_k}^{(c)(\tau)(\nu)} + (\theta_{N_it_k}^{(c)(\tau)(\nu+1)} - \xi_{N_it_k}^{(c)(\tau)(\nu+1)}) \tag{5.77}$$

$$v_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} = v_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu)} + (\theta_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)} - \xi_{N_it_k}^{(c\to c^{'})(\tau+1)(\nu+1)}) \tag{5.78}$$

In the above, in this case, not only do all the devices update their variables in parallel, but also, except the generators, all devices have associated with them the base-case and the contingency scenarios in each dispatch interval, each of which in turn update their respective variables in parallel as well. For the generators, the present dispatch interval and the next one are related through the ramps rate constraint (Δ Function). Then all the nets and the base-case/contingency scenarios associated with them update the first two set of variables in parallel and then update the next two in parallel. For this case, the prox messages and the Proximal Message Passing Algorithm is as follows:

$$(P_{g_qt_k}^{(0)(\tau)(\nu+1)}, \theta_{g_qt_k}^{(c)(\tau)(\nu+1)})$$

$$= \mathbf{prox}_{C(P^{(0)(\tau)})+\Delta(P^{(0)(\tau)}),\rho}(P_{g_qt_k}^{(0)(\tau)(\nu)} - \hat{P}_{g_qt_k}^{(c)(\tau)(\nu)} - u_{g_qt_k}^{(c)(\tau)(\nu)}, \hat{v}_{g_qt_k}^{(c)(\tau)(\nu-1)} + \hat{\theta}_{g_qt_k}^{(c)(\tau)(\nu)} - v_{g_qt_k}^{(c)(\tau)(\nu)}),$$

$$\forall g_q \in G, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79a}$$

$$(P_{T_rt_k}^{(c)(\tau)(\nu+1)}, \theta_{T_rt_k}^{(c)(\tau)(\nu+1)}, P_{T_rt_{k'}}^{(c)(\tau)(\nu+1)}, \theta_{T_rt_{k'}}^{(c)(\tau)(\nu+1)})$$

$$= \mathbf{prox}_{F(P^{(c)(\tau)})+\chi(P^{(c)(\tau)},\theta^{(c)(\tau)}),\rho}(P_{T_rt_k}^{(c)(\tau)(\nu)} - \hat{P}_{T_rt_k}^{(c)(\tau)(\nu)} - u_{T_rt_k}^{(c)(\tau)(\nu)}, \hat{v}_{T_rt_k}^{(c)(\tau)(\nu-1)} + \hat{\theta}_{T_rt_k}^{(c)(\tau)(\nu)} - v_{T_rt_k}^{(c)(\tau)(\nu)}),$$

$$\forall T_r \in T, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79b}$$

$$(P_{T_rt_k}^{(c\to c')(\tau+1)(\nu+1)}, \theta_{T_rt_k}^{(c\to c')(\tau+1)(\nu+1)}, P_{T_rt_{k'}}^{(c\to c')(\tau+1)(\nu+1)}, \theta_{T_rt_{k'}}^{(c\to c')(\tau+1)(\nu+1)})$$

$$= \mathbf{prox}_{F+\chi,\rho}(P_{T_rt_k}^{(c\to c')(\tau+1)(\nu)} - \hat{P}_{T_rt_k}^{(c\to c')(\tau+1)(\nu)} - u_{T_rt_k}^{(c\to c')(\tau+1)(\nu)},$$

$$\hat{v}_{T_rt_k}^{(c\to c')(\tau+1)(\nu-1)} + \hat{\theta}_{T_rt_k}^{(c\to c')(\tau+1)(\nu)} - v_{T_rt_k}^{(c\to c')(\tau+1)(\nu)}),$$

$$\forall T_r \in T, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79c}$$

$$(P_{D_dt_k}^{(c)(\tau)(\nu+1)}, \theta_{D_dt_k}^{(c)(\tau)(\nu+1)})$$

$$= \mathbf{prox}_{-D,\rho}(\hat{v}_{D_dt_k}^{(c)(\tau)(\nu-1)} + \hat{\theta}_{D_dt_k}^{(c)(\tau)(\nu)} - v_{D_dt_k}^{(c)(\tau)(\nu)}), \forall D_d \in L, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79d}$$

$$(P_{D_dt_k}^{(c\to c')(\tau+1)(\nu+1)}, \theta_{D_dt_k}^{(c\to c')(\tau+1)(\nu+1)})$$

$$= \mathbf{prox}_{-D,\rho}(\hat{v}_{D_dt_k}^{(c\to c')(\tau+1)(\nu-1)} + \hat{\theta}_{D_dt_k}^{(c\to c')(\tau+1)(\nu)} - v_{D_dt_k}^{(c\to c')(\tau+1)(\nu)}), \forall D_d \in L, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79e}$$

$$u_{N_it_k}^{(c)(\tau)(\nu+1)} = u_{N_it_k}^{(c)(\tau)(\nu)} + \hat{P}_{N_it_k}^{(c)(\tau)(\nu+1)}, \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79f}$$

$$u_{N_it_k}^{(c\to c')(\tau+1)(\nu+1)} = u_{N_it_k}^{(c\to c')(\tau+1)(\nu)} + \hat{P}_{N_it_k}^{(c\to c')(\tau+1)(\nu+1)}, \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79g}$$

$$v_{N_it_k}^{(c)(\tau)(\nu+1)} = \tilde{v}_{N_it_k}^{(c)(\tau)(\nu)} + \tilde{\theta}_{N_it_k}^{(c)(\tau)(\nu+1)}, \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79h}$$

$$v_{N_it_k}^{(c\to c')(\tau+1)(\nu+1)} = \tilde{v}_{N_it_k}^{(c\to c')(\tau+1)(\nu)} + \tilde{\theta}_{N_it_k}^{(c\to c')(\tau+1)(\nu+1)}, \forall N_i \in \mathcal{N}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{5.79i}$$

We can, therefore observe from the two foregoing sections that not only does the implementation of the algorithm become more complicated, but also, the computational burden on the processors dedicated to the generators becomes too high in the case that the ADMM-PMP is the only algorithm used to solve the LASCOPF problems. Hence we resort to the

APMP algorithm, which relieves that burden by introducing two layers of distribution to the problem. It also makes the solution more modular, which is extremely helpful for designing the simulation software. APMP creates more re-usable and modular components for the coding, thereby making it easier to understand, debug, and maintain the code. In the next chapter, we will introduce the problem, its formulation and the APMP algorithm for the multi-dispatch time interval post-contingency restoration for limiting line temperature rise.

Chapter 6

Look-Ahead SCOPF Limiting Line Temperature

6.1 Line Temperature Limiting LASCOPF

We will now proceed to a formulation in which we will consider the look-ahead dispatch over several future dispatch intervals. in which, initially, the line power flows are limited by dynamic line ratings and lines will be limited by allowable temperature rise. Actually, even if lines are limited by temperature, we still want to be secure, except temporarily for look-ahead from the time of an actual contingency until the system is restored back security. This ensures that we make the best use of the capabilities offered by the transmission grid at the best possible economy of operation. If an actual outage happens, following that outage, the post-contingency system restoration is achieved in several steps by gradually bringing down the power flows on, and temperature of, the lines, making sure that the temperatures of the remaining lines in operation never exceed the maximum allowed temperatures of the respective lines. But before doing that, we first need to understand how the temperature of a line, due to Ohmic losses changes, following an outage. Once we have that knowledge, we can then develop a dispatch method to limit that temperature rise.

6.1.1 Dynamics of Temperature Rise on Transmission Lines

6.1.1.1 Governing Equations

The temperature dynamics of a transmission line is governed by the following first order linear non-homogeneous differential equation, which is obtained from the heat balance equation (HBE) by taking into account the sources for the production and the dissipation of heat, as described in detail in [339], [1], [13], [380], [5], [20] etc. We define $\psi(t)$ to be the temperature of the conductor at time t and define ψ_{amb} to be the ambient temperature. The details of derivation of the differential equation describing ψ and its solution, presented in equations (6.1), (6.3), and (6.4), which is also the same as Newton's law of cooling, can be found in references [48], [332], [8] etc.:

$$\frac{d\psi(t)}{dt} = \alpha^{'}(P_{T_r})^2 - \beta^{'}(\psi(t) - \psi_{amb}) \tag{6.1}$$

In the above differential equation, $\alpha^{'}(P_{T_r})^2$ expresses the Ohmic loss per unit length due to the line power flow, which is P_{T_r}. We assume that P_{T_r} is constant for the moment, and so is the consequent heat generation per unit length in the conductor. The second term on the right hand side represents the heat loss per unit length from the conductor due to the combined effects of convection (mostly) and radiation (linearized). It is to be observed that we have neglected heating due to insolation, but this can be easily considered if the information pertaining to latitude, longitude, time of the year, and day etc are given. If, in the above equation, the Ohmic loss is represented in terms of the current flow on the transmission line, which is useful in the context of AC-OPF, we have:

$$\frac{d\psi(t)}{dt} = \Big(\frac{\rho_{res}}{\rho_{mass}(CS)^2 S_p}\Big)(I_{T_r})^2 - \Big(\frac{\bar{\beta}}{\rho_{mass} l(CS) S_p}\Big)(\psi(t) - \psi_{amb}) \tag{6.2}$$

where

- ρ_{res} is the resistivity of the conductor material of the T_r^{-th} transmission line,
- ρ_{mass} is the volumetric mass density of the conductor material of the T_r^{-th} transmission line,
- CS is the cross-sectional area of the T_r^{-th} transmission line,
- l is the length of the T_r^{-th} transmission line,
- $\bar{\beta}$ is the coefficient for cooling (positive),
- S_p is the specific heat of the conductor material of the T_r^{-th} transmission line, and
- I_{T_r} is the current flowing through the T_r^{-th} transmission line.

Solving the above Differential Equation we get:

$$\boxed{\psi(t) = e^{-\beta' t}(\psi_{init} - \psi_{eq}) + \psi_{eq}} \tag{6.3a}$$

$$\text{where } \boxed{\psi_{eq} = \psi_{amb} + \left(\frac{\alpha'}{\beta'}\right)(P_{T_r})^2} \tag{6.3b}$$

$$\text{or } \boxed{\psi_{eq} = \psi_{amb} + \left(\frac{\rho_{res} l}{(CS)\bar{\beta}}\right)(I_{T_r})^2} \tag{6.3c}$$

Here

- $\psi(t)$ is the instantaneous line temperature (as a function of time),

- ψ_{amb} is the ambient temperature,
- ψ_{eq} is the equilibrium line temperature for the given power flow and ambient conditions,
- ψ_{init} is the initial temperature of the line, and
- $\alpha^{'}$ is the fraction of line power that is the Ohmic loss (positive).

It is to be observed that,

- $\left(\frac{\rho_{res}}{\rho_{mass}(CS)^2 S_p}\right)$ in (6.2) and $\alpha^{'}$ in (6.1) play identical roles.
- $\left(\frac{\bar{\beta}}{\rho_{mass} l (CS) S_p}\right)$ in (6.2) and $\beta^{'}$ in (6.1) play identical roles.
- $\left(\frac{\rho_{res} l}{(CS)\bar{\beta}}\right)$ in (6.3c) and $\left(\frac{\alpha^{'}}{\beta^{'}}\right)$ (6.3b) play identical roles.

In the next equation, we state the expression for the time required to attain a particular temperature, ψ_p from an inital temperature:

$$\boxed{t_p = \frac{1}{\beta^{'}} ln \left| \frac{(\psi_{init} - \psi_{eq})}{(\psi_p - \psi_{eq})} \right|} \tag{6.4}$$

With the fundamental governing equations of temperature evolution stated above, let us now define some terms, which we will be using throughout the remainder of this chapter.

6.1.1.2 Definitions and Terminology

(a) Power System Security States: In terms of security considerations, the power system state refers to the condition of the systems in terms of the amount of power flowing through the devices (primarily transmission lines) and the allowed limits. It can further be divided into two states, viz :

- Emergency State: State in which one or more of the long-term or short-term transmission line power flow limits are violated.

- Normal State: State in which all the lines are operated at or below the normal, long-term operating power flow limits, but if an outage happens the system may or may not go to an emergency state. This can be further classified into the following two states:

  - Secure State: State in which all the lines are operated at or below the normal, long-term operating power flow limits, and following an outage, some of the flows may be above long-term limits, but all are below short-term or emergency limits.

  - Insecure State: State in which all the lines are operated at or below the normal long-term operating power flow limits, but following an outage, some of the flows would be above short-term or emergency limits.

(b) Maximum Restoration Duration (MRD): The MRD (either in second, minutes, or in terms of dispatch intervals) is the maximum time allowed to bring back the system to security, subsequent to occurence of an outage. We will use the symbol Γ_{MRD} to denote the MRD. The entire MRD for a particular line can be split into two sub-intervals as follows :

- Restoration to Normal Duration (RND): This refers to the time allowed to restore the system flows from emergency state to the normal (but, presumably insecure) state. We will use the symbol Γ_{RND} to denote RND. During the different dispatch intervals in RND, the line power flow is greater than the nominal or long-term rated value for those lines for which the immediate post contingency flow is more than the rated. Hence, for such lines, the ψ_{eq} for each dispatch interval within RND is greater than ψ^{max}, which is the maximum temperature allowed for the particular line. The control action used during the RND is known as the "corrective control".

- Restoration to Security Duration (RSD): This refers to the time allowed to restore the system flows from normal insecure state to the normal secure state. Hence $\Gamma_{MRD} = \Gamma_{RND} + \Gamma_{RSD}$. During the different dispatch intervals in RSD, the line power flow is less than the nominal or long-term rated value, but if an outage happens, some of the line flows will exceed the short-term ratings. Hence the ψ_{eq} for each dispatch interval within RSD is less than ψ^{max}, which is the maximum temperature allowed for a particular line. The control action used during the RND is known as the "preventive control".

(c) Increasing and Decreasing Sequence of Functions: Let $\{A_1, A_2, ..., A_m\}$ be the set of intervals of some independent variables, with total order imposed on them (which means, for any $\mathbf{x}_i \in A_i$ and $\mathbf{x}_j \in A_j$, whenever $i > j$, then $\mathbf{x}_i > \mathbf{x}_j$ and if $i = j$, then the two variables are either equal, greater or less). A sequence of functions $\{\psi_1, \psi_2, ..., \psi_m\}$ defined on the respective intervals, is said to form an increasing sequence, if

- $\psi_i(\mathbf{x}_i) > \psi_j(\mathbf{x}_j)$ whenever the variables belong to different intervals and $i > j$, and
- $\psi_i(\mathbf{x}_i) > \psi_i(\mathbf{x}_j)$ whenever $\mathbf{x}_i > \mathbf{x}_j$, for both variables belonging to the same interval.

The sequence is said to be decreasing, if

- $\psi_i(\mathbf{x}_i) < \psi_j(\mathbf{x}_j)$ whenever the variables belong to different intervals and $i > j$ and,
- $\psi_i(\mathbf{x}_i) < \psi_i(\mathbf{x}_j)$, whenever $\mathbf{x}_i > \mathbf{x}_j$, for both variables belonging to the same interval.

6.1.1.3 Theorems and Claims

We now state below, some lemmas, theorems, and corollaries, which we will subsequently use in our LASCOPF model.

Lemma 6.1. *If $\psi_{init} < \psi_{eq}$ in (6.3a), then $\psi(t)$ is an increasing function of t and it is decreasing, otherwise.*

Proof. Differentiating (6.3a), we get

$$\frac{d\psi(t)}{dt} = (-\beta^{'})(\psi_{init} - \psi_{eq})e^{-\beta^{'}t} \tag{6.5}$$

in which, $(-\beta^{'})$ is negative and $(\psi_{init} - \psi_{eq})$ is negative or positive, respectively, if $\psi_{init} < \psi_{eq}$ or $\psi_{init} > \psi_{eq}$ making the derivative positive or negative and hence the temperature function increasing or decreasing, respectively.

□

Theorem 6.2. *Let it be assumed that an outage happens at ϵ time units into the upcoming dispatch interval. The following are the constraints to be satisfied to ensure that the temperature of a given transmission line doesn't rise above the maximum allowed temperature* *and that the corresponding flows can be safely brought back to the rated value over several dispatch intervals within RND. (Note that these constraints depend only on the initial pre-contingency line temperature, ambient temperature, and the power flow on the line during different dispatch intervals throughout RND and are given by the relations below.)*

$$
\begin{aligned}
&E_{\Gamma}^{(\omega)}[\psi_{init}^{(\tau)}] + (1 - E_{\Gamma}^{(\omega)})[\psi_{amb}] \\
&+ \left(\frac{\alpha^{'}}{\beta^{'}}\right) \left[(P_{T_r}^{(\tau)})^2 E_0^{(\omega)} + (P_{T_r}^{(\tau+\epsilon)})^2 E_{\epsilon}^{(\omega)} + \sum_{s=1}^{\Gamma_{RND}-(\omega+1)} (P_{T_r}^{(\tau+s)})^2 E_s^{(\omega)} \right] < \psi_{T_r}^{max}
\end{aligned}
\tag{6.6a}
$$

$\forall \omega \in \{0, 1, 2, ...(\Gamma_{RND} - 1)\}$

where $E_{\Gamma}^{(\omega)}$, $E_s^{(\omega)}$, $E_{\epsilon}^{(\omega)}$, $E_0^{(\omega)}$ *are constants depending on the duration of the RND, nature of the conductor material, duration of a single dispatch interval, time of occurence of the outage etc. If the power flow on the transmission line for the forthcoming dispatch interval has been flowing for a sufficiently long time (long enough as compared to the thermal time constant of the line), then the constraints take the form:*

$$
\boxed{\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right) \sum_{s=0}^{\Gamma_{RND}-(\omega+1)} (P_{T_r}^{(\tau+s)})^2 E_s^{(\omega)} < \psi_{T_r}^{max}}
\tag{6.7}
$$

$\forall \omega \in \{0, 1, 2, ...(\Gamma_{RND} - 1)\}$

Proof. Let's focus our attention on the T_r^{-th} transmission line. Let's consider $\psi_{T_r}^{max}$ as the maximum allowed line temperature for this particular transmission line under the currently assumed ambient conditions. Throughout this proof, we will, just for the sake of convenience, consider the upcoming interval, $\tau + s = \tau + 0$.

Let's assume that a maximum of Γ_{RND} dispatch intervals are allowed for bringing the line flows to normal levels, following a contingency. Let's also assume that each dispatch interval is of duration δt time units.

The most stringent situation happens when the Γ_{RND}^{-th} dispatch interval is the one in which the flow on T_r^{-th} line is brought to a level corresponding to which the equilibrium temperature is $\psi_{T_r}^{max}$. ie

$$\psi_{eq}(P_{T_r}^{(\Gamma_{RND})}) = \psi_{T_r}^{max} = \psi_{amb} + \Big(\frac{\alpha^{'}}{\beta^{'}}\Big)(P_{T_r}^{(\Gamma_{RND})})^2 \tag{6.8}$$

Or

$$\psi_{eq}(I_{T_r}^{(\Gamma_{RND})}) = \psi_{T_r}^{max} = \psi_{amb} + \Big(\frac{\rho_{res} S_p^2 l}{(CS)\beta^{'}}\Big)(I_{T_r}^{(\Gamma_{RND})})^2 \tag{6.9}$$

For this to be true, it must be the case that at the beginning of this interval, the initial temperature of the line, (which is the final temperature at the end of the $(\Gamma_{RND}-1)^{-th}$ interval) needs to be less than $\psi_{T_r}^{max}$. That is (in the following, $\psi_{eq}(.)$ stands for the equilibrium temperature as a function of the line power flow or current flow):

$$\psi_{init}^{(\Gamma_{RND})} = e^{-\beta^{'}(\delta t)}(\psi_{init}^{(\Gamma_{RND}-1)} - \psi_{eq}(P_{T_r}^{(\Gamma_{RND}-1)})) + \psi_{eq}(P_{T_r}^{(\Gamma_{RND}-1)}) < \psi_{T_r}^{max}$$

$$\implies e^{-\beta^{'}(\delta t)}(\psi_{init}^{(\Gamma_{RND}-1)}) + (1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-1)})^2\right) < \psi_{T_r}^{max}$$

$$\implies e^{-\beta^{'}(\delta t)}\left[e^{-\beta^{'}(\delta t)}(\psi_{init}^{(\Gamma_{RND}-2)}) + (1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-2)})^2\right)\right] +$$

$$(1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-1)})^2\right) < \psi_{T_r}^{max}$$

$$\implies e^{-2\beta^{'}(\delta t)}(\psi_{init}^{(\Gamma_{RND}-2)}) + e^{-\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-2)})^2\right) +$$

$$(1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-1)})^2\right) < \psi_{T_r}^{max}$$

Continuing like this upto the first interval

$$\implies e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}(\psi_{init}^{(\Gamma_{RND}-(\Gamma_{RND}-1))}) +$$

$$e^{-(\Gamma_{RND}-2)\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-(\Gamma_{RND}-1))})^2\right) +$$

$$e^{-(\Gamma_{RND}-3)\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-(\Gamma_{RND}-2))})^2\right) + ... +$$

$$e^{-\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-2)})^2\right) +$$

$$(1-e^{\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\Gamma_{RND}-1)})^2\right) < \psi_{T_r}^{max}$$

$$\implies e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}(\psi_{init}^{(1)}) + \sum_{s=1}^{(\Gamma_{RND}-1)} e^{-(\Gamma_{RND}-(s+1))\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(s)})^2\right)$$
$$< \psi_{T_r}^{max}$$

$$\implies e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}(\psi_{init}^{(1)}) + \psi_{amb}(1-e^{-\beta^{'}(\delta t)})e^{-\Gamma_{RND}\beta^{'}(\delta t)} \sum_{s=1}^{(\Gamma_{RND}-1)} e^{(s+1)\beta^{'}(\delta t)} +$$

$$\left(\frac{\alpha^{'}}{\beta^{'}}\right)(1-e^{-\beta^{'}(\delta t)})e^{-\Gamma_{RND}\beta^{'}(\delta t)} \sum_{s=1}^{(\Gamma_{RND}-1)} e^{(s+1)\beta^{'}(\delta t)}(P_{T_r}^{(s)})^2 < \psi_{T_r}^{max}$$

$$\implies e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}(\psi_{init}^{(1)}) + \psi_{amb}(1-e^{-\beta^{'}(\delta t)})e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)} \sum_{s=1}^{(\Gamma_{RND}-1)} e^{s\beta^{'}(\delta t)} +$$

$$\left(\frac{\alpha^{'}}{\beta^{'}}\right) \sum_{s=1}^{(\Gamma_{RND}-1)} E_s^{(0)}(P_{T_r}^{(s)})^2 < \psi_{T_r}^{max}$$

$$\implies e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}(\psi_{init}^{(1)}) + \psi_{amb}(1-e^{-\beta^{'}(\delta t)})e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}.\frac{e^{\beta^{'}(\delta t)}(1-e^{(\Gamma_{RND}-1)\beta^{'}(\delta t)})}{(1-e^{\beta^{'}(\delta t)})} +$$

$$\left(\frac{\alpha^{'}}{\beta^{'}}\right) \sum_{s=1}^{(\Gamma_{RND}-1)} E_s^{(0)}(P_{T_r}^{(s)})^2 < \psi_{T_r}^{max}$$

$$\implies e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}(\psi_{init}^{(1)}) + \psi_{amb}(1-e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}) +$$

$$\left(\frac{\alpha^{'}}{\beta^{'}}\right) \sum_{s=1}^{(\Gamma_{RND}-1)} E_s^{(0)}(P_{T_r}^{(s)})^2 < \psi_{T_r}^{max} \tag{6.10a}$$

If we assume that the contingency that gives rise to the step change in power flow on the line T_r and the consequent rise in temperature takes place within the forthcoming dispatch interval, after ϵ time units from the start of the τ^{-th} dispatch interval, then $\psi_{init}^{(1)} = e^{-\beta^{'}(\delta t-\epsilon)}(\psi_{init}^{(\epsilon)} - \psi_{eq}(P_{T_r}^{(\epsilon)})) + \psi_{eq}(P_{T_r}^{(\epsilon)}) = e^{-\beta^{'}(\delta t-\epsilon)}\psi_{init}^{(\epsilon)} + (1-e^{-\beta^{'}(\delta t-\epsilon)})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\epsilon)})^2\right)$, where $\psi_{init}^{(\epsilon)}$ is the temperature of the line at the instant of the outage, which is in turn given by the relation, $\psi_{init}^{(\epsilon)} = e^{-\beta^{'}\epsilon}\psi_{init}^{(0)} + (1-e^{-\beta^{'}\epsilon})\left(\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(0)})^2\right)$. $\psi_{init}^{(0)}$ is the line temperature at the beginning of the forthcoming dispatch interval and $P_{T_r}^{(0)}$, $P_{T_r}^{(\epsilon)}$ are the line flows immediately before and after the outage, respectively. Substituting these values into (6.10a) we get (In the current work, we will assume that $\epsilon = 0$ is the worst case, and perform all our simulation studies following this assumption. We will consider non-zero ϵ in our future work.)

$$\implies e^{-(\Gamma_{RND}(\delta t)-\epsilon)\beta^{'}}\psi_{init}^{(\epsilon)} + (1-e^{-(\Gamma_{RND}(\delta t)-\epsilon)\beta^{'}})\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(\epsilon)})^2 E_{\epsilon}^{(0)} +$$

$$\left(\frac{\alpha^{'}}{\beta^{'}}\right)\sum_{s=1}^{\Gamma_{RND}-1}(P_{T_r}^{(s)})^2E_s^{(0)} < \psi_{T_r}^{max}$$

$$\implies e^{-\Gamma_{RND}\beta^{'}(\delta t)}\psi_{init}^{(0)} + (1-e^{-\Gamma_{RND}\beta^{'}(\delta t)})\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)\Bigg[(P_{T_r}^{(0)})^2E_0^{(0)} + (P_{T_r}^{(\epsilon)})^2E_\epsilon^{(0)}$$

$$+ \sum_{s=1}^{\Gamma_{RND}-1}(P_{T_r}^{(s)})^2E_s^{(0)}\Bigg] < \psi_{T_r}^{max}$$

$$\implies E_\Gamma^{(0)}[\psi_{init}^{(0)}] + (1-E_\Gamma^{(0)})[\psi_{amb}] + \left(\frac{\alpha^{'}}{\beta^{'}}\right)\Bigg[(P_{T_r}^{(0)})^2E_0^{(0)} + (P_{T_r}^{(\epsilon)})^2E_\epsilon^{(0)} + \sum_{s=1}^{\Gamma_{RND}-1}(P_{T_r}^{(s)})^2E_s^{(0)}\Bigg]$$

$$< \psi_{T_r}^{max}$$

$$\text{where } E_\Gamma^{(0)} = e^{-\Gamma_{RND}\beta^{'}(\delta t)}, E_s^{(0)} = e^{-(\Gamma_{RND}-(s+1))\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t)})$$

$$E_\epsilon^{(0)} = e^{-(\Gamma_{RND}-1)\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t-\epsilon)}), E_0^{(0)} = e^{-(\Gamma_{RND}(\delta t)-\epsilon)\beta^{'}}(1-e^{-\beta^{'}\epsilon})$$

Generalizing for any τ,

$$E_\Gamma^{(0)}[\psi_{init}^{(\tau)}] + (1-E_\Gamma^{(0)})[\psi_{amb}] + \left(\frac{\alpha^{'}}{\beta^{'}}\right)\Bigg[(P_{T_r}^{(\tau)})^2E_0^{(0)} + (P_{T_r}^{(\tau+\epsilon)})^2E_\epsilon^{(0)} + \sum_{s=1}^{\Gamma_{RND}-1}(P_{T_r}^{(\tau+s)})^2E_s^{(0)}\Bigg]$$

$$< \psi_{T_r}^{max}$$

Generalizing for any ω,

$$\psi_{init}^{(\Gamma_{RND}-\omega)} =$$

$$E_\Gamma^{(\omega)}[\psi_{init}^{(\tau)}] + (1-E_\Gamma^{(\omega)})[\psi_{amb}] + \left(\frac{\alpha^{'}}{\beta^{'}}\right)\Bigg[(P_{T_r}^{(\tau)})^2E_0^{(\omega)} + (P_{T_r}^{(\tau+\epsilon)})^2E_\epsilon^{(\omega)} + \sum_{s=1}^{\Gamma_{RND}-(\omega+1)}(P_{T_r}^{(\tau+s)})^2E_s^{(\omega)}\Bigg]$$

$$< \psi_{T_r}^{max} \tag{6.11a}$$

$$\text{where } E_\Gamma^{(\omega)} = e^{-(\Gamma_{RND}-\omega)\beta^{'}(\delta t)}, E_s^{(\omega)} = e^{-(\Gamma_{RND}-(s+\omega+1))\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t)})$$

$$E_\epsilon^{(\omega)} = e^{-(\Gamma_{RND}-(\omega+1))\beta^{'}(\delta t)}(1-e^{-\beta^{'}(\delta t-\epsilon)}), E_0^{(\omega)} = e^{-((\Gamma_{RND}-\omega)(\delta t)-\epsilon)\beta^{'}}(1-e^{-\beta^{'}\epsilon})$$

$$\omega \in \{0,1,2,...,(\Gamma_{RND}-1)\} \tag{6.11b}$$

If we assume that $\psi_{init}^{(\tau+1)}$, which is the line temperature for transmission line T_r at the beginning of the $(\tau+1)^{-th}$ dispatch interval (which is the same as the temperature at the end of the present planning horizon/dispatch interval, τ), is the steady state temperature of the line for the present planning horizon flow (which implies $P_{T_r}^{(\tau)}$ has been flowing for a sufficiently long time) and the contingency happens at the beginning of the $(\tau+1)^{-th}$ dispatch interval, then, first for the special case, when $\tau = 0, \omega = 0$, substituting $\psi_{init}^{(1)} = \psi_{amb} + (\frac{\alpha'}{\beta'})(P_{T_r}^{(0)})^2$ in equation (6.10a), we get

$$\psi_{amb} + \left(\frac{\alpha'}{\beta'}\right) \sum_{s=0}^{\Gamma_{RND}-1} (P_{T_r}^{(s)})^2 E_s^{(0)} < \psi_{T_r}^{max} \tag{6.12}$$

where $E_0^{(0)} = E_\Gamma^{(0)} = e^{-(\Gamma_{RND}-1)\beta'(\delta t)}$

Again, generalizing for any τ and ω, we get

$$\boxed{\psi_{amb} + \left(\frac{\alpha'}{\beta'}\right) \sum_{s=0}^{\Gamma_{RND}-(\omega+1)} (P_{T_r}^{(\tau+s)})^2 E_s^{(\omega)} < \psi_{T_r}^{max}} \tag{6.13}$$

where $E_0^{(\omega)} = E_\Gamma^{(\omega)} = e^{-(\Gamma_{RND}-(\omega+1))\beta'(\delta t)}$

The rest of the symbols have the same range of values and mathematical form as in (6.11).

(*Hence Proved*) □

Corollary 6.3. *In terms of the generator outputs and load demands, the post-outage line temperature limiting and restoration conditions, when the outage corresponding to contingency c is assumed to happen ϵ time units into the forthcoming dispatch interval (τ), can be stated as:*

$$E_\Gamma^{(\omega)}[\psi_{init}^{(\tau)}] + (1 - E_\Gamma^{(\omega)})[\psi_{amb}] +$$
$$\left(\frac{\alpha'}{\beta'}\right) \left[\sum_{s=0}^{\Gamma_{RND}-(\omega+1)} ((\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})^\dagger \mathbf{\Phi}_{sT_r}^{(c,\omega)} \ (\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})) \right] < \psi_{T_r}^{max} \tag{6.14a}$$

where $\mathbf{\Phi}_{sT_r}^{(c,\omega)} = \mathbf{\Phi}_{T_r}^{(c)} E_s^{(\omega)}, \forall s \in \{1, 2, ..., \Gamma_{RND} - 1\}$,

$\forall \omega \in \{0, 1, 2, ..., (\Gamma_{RND} - 1)\}, \mathbf{\Phi}_{0T_r}^{(c,\omega)} = \mathbf{\Phi}_{T_r}^{(0)} E_0^{(\omega)} + \mathbf{\Phi}_{T_r}^{(c)} E_\epsilon^{(\omega)}$,

$(\mathbf{P_g}^{(c)(\tau)} - \mathbf{P_D}^{(\tau)}) = (\mathbf{P_g}^{(\tau)} - \mathbf{P_D}^{(\tau)})$

$$\mathbf{\Phi}_{T_r}^{(c)} = \begin{pmatrix} (\Phi^{(c)}(T_r,1))^2 & \Phi^{(c)}(T_r,1)\Phi^{(c)}(T_r,2) & ... & \Phi^{(c)}(T_r,1)\Phi^{(c)}(T_r,|\mathcal{N}|) \\ \Phi^{(c)}(T_r,2)\Phi^{(c)}(T_r,1) & (\Phi^{(c)}(T_r,2))^2 & ... & \Phi^{(c)}(T_r,2)\Phi^{(c)}(T_r,|\mathcal{N}|) \\ ... & & & \\ \Phi^{(c)}(T_r,|\mathcal{N}|)\Phi^{(c)}(T_r,1) & \Phi^{(c)}(T_r,|\mathcal{N}|)\Phi^{(c)}(T_r,2) & ... & (\Phi^{(c)}(T_r,|\mathcal{N}|))^2 \end{pmatrix}$$

$$= [\Phi^{(c)}(T_r,1), ..., \Phi^{(c)}(T_r,|\mathcal{N}|)]^\dagger [\Phi^{(c)}(T_r,1), ..., \Phi^{(c)}(T_r,|\mathcal{N}|)]$$

where $\Phi^{(c)}(T_r, k)$ *refers to the element in the* $(T_r, k)^{-th}$ *position of the shift factor matrix of the network i.e. the change in power flow on the line* T_r *for unit MW injection at bus* k *and withdrawal at the system slack bus, corresponding to contingency scenario* c.

When the outage is assumed to happen at the end of the τ^{-th} *dispatch interval and the power through the transmission line before that is assumed to have been flowing for sufficiently long, then the condition is :*

$$\boxed{\psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right) \left[\sum_{s=0}^{\Gamma_{RND}-(\omega+1)} (\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})^\dagger \mathbf{\Phi}_{T_r}^{(c,\omega)} (\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)}) E_s^{(\omega)} \right] < \psi_{T_r}^{max}} \tag{6.15}$$

$\forall \omega \in \{0, 1, 2, ..., (\Gamma_{RND} - 1)\}$ *It is implicitly assumed that the above result is valid only for a preventive control scheme i.e. the case of DC-SCOPF with an* $(N-1)$ *contingency corresponding to line outage only.*

Proof. Let $\mathbf{e}_{T_r} \in \mathbb{R}^{|T|}$ be a vector, which has all entries 0, except the T_r^{-th} entry, which is 1. The power flow on the line T_r in the contingency scenario, c after ϵ time units into the τ^{-th} dispatch interval will be

$$P_{T_r}^{(c)(\tau+\epsilon)} = \mathbf{e}_{T_r}^{\dagger}(\Phi^{(c)}(\mathbf{P_g}^{(\tau)} - \mathbf{P_D}^{(\tau)})) \tag{6.16}$$

where $\Phi^{(c)} \in \mathbb{R}^{|T|\times\mathcal{N}}$ is the shift-factor matrix. In (6.16), because of preventive control for DC-SCOPF, for consideration of line contingency, $P_g^{(\tau+\epsilon)} = P_g^{(\tau)}$ i.e. the pre- and post-contingency generator power outputs are the same. Since $P_{T_r}^{(c)(\tau+1)}$ is the power flow on line T_r in the contingency scenario c during the dispatch interval $\tau+1$, hence

$$P_{T_r}^{(c)(\tau+1)} = \mathbf{e}_{T_r}^{\dagger}(\Phi^{(c)}(\mathbf{P_g}^{(c)(\tau+1)} - \mathbf{P_D}^{(\tau+1)})) \tag{6.17}$$

and the flow at any dispatch interval, $(\tau+s)$ is

$$P_{T_r}^{(c)(\tau+s)} = \mathbf{e}_{T_r}^{\dagger}(\Phi^{(c)}(\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)}))$$

$$\implies (P_{T_r}^{(c)(\tau+s)})^2 = (\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})^{\dagger}\Phi^{(c)\dagger}\mathbf{e}_{T_r}\mathbf{e}_{T_r}^{\dagger}\Phi^{(c)}(\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})$$

$$\implies (P_{T_r}^{(c)(\tau+s)})^2 = (\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})^{\dagger}\Phi^{(c)\dagger}M_{T_r}\Phi^{(c)}(\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})$$

$$\text{where } M_{T_r} = \mathbf{e}_{T_r}\mathbf{e}_{T_r}^{\dagger}, \in \mathbb{R}^{|T|\times|T|}$$

with all entries 0 except the $(T_r, T_r)^{-th}$ element, which is equal to 1

$$\implies (P_{T_r}^{(c)(\tau+s)})^2 = (\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})^{\dagger}\mathbf{\Phi}_{T_r}^{(c)}(\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})$$

where

$$\mathbf{\Phi}_{T_r}^{(c)} = \begin{pmatrix} (\Phi^{(c)}(T_r,1))^2 & \Phi^{(c)}(T_r,1)\Phi^{(c)}(T_r,2) & ... & \Phi^{(c)}(T_r,1)\Phi^{(c)}(T_r,|\mathcal{N}|) \\ \Phi^{(c)}(T_r,2)\Phi^{(c)}(T_r,1) & (\Phi^{(c)}(T_r,2))^2 & ... & \Phi^{(c)}(T_r,2)\Phi^{(c)}(T_r,|\mathcal{N}|) \\ ... & & & \\ \Phi^{(c)}(T_r,|\mathcal{N}|)\Phi^{(c)}(T_r,1) & \Phi^{(c)}(T_r,|\mathcal{N}|)\Phi^{(c)}(T_r,2) & ... & (\Phi^{(c)}(T_r,|\mathcal{N}|))^2 \end{pmatrix}$$

Hence, (6.11) $\implies$

$$E_{\Gamma}^{(\omega)}[\psi_{init}^{(\tau)}] + (1 - E_{\Gamma}^{(\omega)})[\psi_{amb}] + \left(\frac{\alpha'}{\beta'}\right)\Bigg[(\mathbf{P_g}^{(\tau)} - \mathbf{P_D}^{(\tau)})^{\dagger}(\mathbf{\Phi}_{T_r}^{(0)}E_0^{(\omega)} + \mathbf{\Phi}_{T_r}^{(c)}E_{\epsilon}^{(\omega)})(\mathbf{P_g}^{(\tau)} - \mathbf{P_D}^{(\tau)})$$

$$+\sum_{s=1}^{\Gamma_{RND}-(\omega+1)}\left(({\mathbf{P_g}}^{(c)(\tau+s)}-{\mathbf{P_D}}^{(\tau+s)})^{\dagger}\mathbf{\Phi}_{T_r}^{(c)}({\mathbf{P_g}}^{(c)(\tau+s)}-{\mathbf{P_D}}^{(\tau+s)})E_s^{(\omega)}\right)\Bigg]<\psi_{T_r}^{max}$$

$$\implies$$

$$E_{\Gamma}^{(\omega)}[\psi_{init}^{(\tau)}]+(1-E_{\Gamma}^{(\omega)})[\psi_{amb}]+$$

$$\left(\frac{\alpha^{'}}{\beta'}\right)\left[\sum_{s=0}^{\Gamma_{RND}-(\omega+1)}(({\mathbf{P_g}}^{(c)(\tau+s)}-{\mathbf{P_D}}^{(\tau+s)})^{\dagger}\mathbf{\Phi}_{sT_r}^{(c,\omega)}({\mathbf{P_g}}^{(c)(\tau+s)}-{\mathbf{P_D}}^{(\tau+s)}))\right]<\psi_{T_r}^{max} \tag{6.18a}$$

where $\mathbf{\Phi}_{sT_r}^{(c,\omega)}=\mathbf{\Phi}_{T_r}^{(c)}E_s^{(\omega)},\forall s\in\{1,2,...,\Gamma_{RND}-1\},\mathbf{\Phi}_{0T_r}^{(c,\omega)}=\mathbf{\Phi}_{T_r}^{(0)}E_0^{(\omega)}+\mathbf{\Phi}_{T_r}^{(c)}E_{\epsilon}^{(\omega)},$

$$({\mathbf{P_g}}^{(c)(\tau)}-{\mathbf{P_D}}^{(\tau)})=({\mathbf{P_g}}^{(\tau)}-{\mathbf{P_D}}^{(\tau)})$$

$$\text{and (6.12)}\implies$$

$$\boxed{\psi_{amb}+\left(\frac{\alpha^{'}}{\beta'}\right)\sum_{s=0}^{\Gamma_{RND}-(\omega+1)}({\mathbf{P_g}}^{(c)(\tau+s)}-{\mathbf{P_D}}^{(\tau+s)})^{\dagger}\mathbf{\Phi}_{T_r}^{(c,\omega)}({\mathbf{P_g}}^{(c)(\tau+s)}-{\mathbf{P_D}}^{(\tau+s)})E_s^{(\omega)}<\psi_{T_r}^{max}} \tag{6.18b}$$

$$\forall\omega\in\{0,1,2,...,(\Gamma_{RND}-1)\}$$

(*Hence Proved*) □

Corollary 6.4. *The conditions for restoration of a transmission line to secure state, following an outage, are the conditions stated in Theorem-1 and Corollary-1 along with the conditions that*

$$|\Phi^{(c)}(\mathbf{P_g^{(c)(\tau+\Gamma_{RND}+\omega)}}-\mathbf{P_D^{(\tau+\Gamma_{RND}+\omega)}})|\leq\overline{\mathbf{L}}^{(0)},\forall\omega\in\{0,1,2,...,(\Gamma_{MRD}-\Gamma_{RND}-1)\} \tag{6.19a}$$

$$|\Phi^{(c\to c^{'})}(\mathbf{P_g^{(c)(\tau+\Gamma_{MRD})}}-\mathbf{P_D^{(\tau+\Gamma_{MRD})}})|\leq\overline{\mathbf{L}}^{(c\to c^{'})} \tag{6.19b}$$

where $\overline{\mathbf{L}}^{(0)}$ *and* $\overline{\mathbf{L}}^{(c\to c^{'})}$ *are respectively, the vectors of long-term thermal line flow limits (where the elements of the vector are the remaining lines in the system after the outage corresponding to scenario c is supposed to have occured), and short-term thermal line flow* *limits (where the elements of the vector are the remaining lines in the system after the outage corresponding to scenario $c^{'}$ is supposed to have occured, following the outage of the line corresponding to scenario, c). The set of generated power (or equivalently, line power flows) satisfying these conditions forms a convex set.*

Proof. In this proof, we will consider two cases as follows:

Case I:

In this case, we consider the situations where the post-contingency flow on the transmission line is higher than the pre-contingency flow and is higher than the nominal value.

The conditions (6.18) imply that over the duration of the RND, the line flows never exceed the maximum allowed temperature. Hence, imposing constraint (6.19a) makes certain that by the end of the RND, the line flow is brought to within the nominal rating.

Therefore, no thermal upper limit constraints are needed to be enforced. Imposing constraint (6.19b) will make sure that by the end of the RSD (and hence by the end of the entire MRD), the flows are brought to within values, that make the system secure with respect to the next set of contingencies.

Case II:

In this case, we consider the situations where the post-contingency flow on the transmission line is lower than the pre-contingency flow and is less than the values, that make the system secure with respect to the next set of contingencies.

In this case, there will be a drop in the line temperature, followed by a control scheme that will try to bring the flows to within secure values by the end of the MRD. Hence, for this case, constraints (6.18) and (6.19a) will be redundant, unless at some intermediate time, the flows rise above the secure value. Constraint (6.19b) will make sure that by the end of the RSD (and hence by the end of the entire MRD), the flows are brought to within the values, that make the system secure with respect to the next set of contingencies, as before.

For establishing the convexity of the set, observe that $\mathbf{\Phi}_{T_r}^{(c)}$ is a positive semi-definite (PSD) matrix, and each of the E_s's are positive real numbers (since both $\alpha^{'}$ and $\beta^{'}$ are positive reals and δt being the duration of dispatch interval, is also positive). Hence, the LHS of the inequalities (6.18) (and also the equivalent versions from Theorem 1) are convex quadratic functions of the decision variables and so the $\leq$ type constraints give rise to sublevel sets of such functions, which are convex sets. Hence, the feasible set is convex. $(Hence\ Proved)$ $\square$

Theorem 6.5. *In order for the series of line temperatures to form an increasing sequence of curves during RND, the necessary condition to be satified is* $\psi_{T_r}^{max} > \psi_{init}^{(1)}$.

Proof. We can say that, whenever the series of line temperature curves forms an increasing sequence, then,

$$\psi_{init}^{(\Gamma_{max}-\omega)} \leq \psi_{eq}(P_{T_r}^{(\Gamma_{max}-\omega)}),\ \forall \omega \in \{0, 1, 2, ..., (\Gamma_{max} - 1)\}$$

$$\implies \text{For } \omega = \Gamma_{max} - 1, \Gamma_{max} - 2, \Gamma_{max} - 3... \text{ we have, respectively}$$

$$\psi_{init}^{(1)} = E_{\Gamma}^{(\Gamma_{max}-1)}[\psi_{init}^{(0)}] + (1 - E_{\Gamma}^{(\Gamma_{max}-1)})[\psi_{amb}] + \left(\frac{\alpha^{'}}{\beta^{'}}\right)\left[(P_{T_r}^{(0)})^2 E_0^{(\Gamma_{max}-1)} + (P_{T_r}^{(\epsilon)})^2 E_{\epsilon}^{(\Gamma_{max}-1)}\right]$$

$$< \psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(1)})^2$$

$$\implies \psi_{amb} + \left(\frac{\alpha^{'}}{\beta^{'}}\right)(P_{T_r}^{(1)})^2 >$$

$$e^{-\beta'(\delta t)}[\psi_{init}^{(0)}] + (1 - e^{-\beta'(\delta t)})[\psi_{amb}] + \left(\frac{\alpha'}{\beta'}\right)\left[(P_{T_r}^{(0)})^2(1 - e^{-\beta'\epsilon})e^{-(\delta t - \epsilon)\beta'} + (P_{T_r}^{(0)})^2(1 - e^{-\beta'(\delta t - \epsilon)})\right] \tag{6.20a}$$

$$\psi_{init}^{(2)} =$$

$$E_{\Gamma}^{(\Gamma_{max}-2)}[\psi_{init}^{(0)}] + (1 - E_{\Gamma}^{(\Gamma_{max}-2)})[\psi_{amb}] + \left(\frac{\alpha'}{\beta'}\right)\left[(P_{T_r}^{(0)})^2 E_0^{(\Gamma_{max}-2)} + (P_{T_r}^{(\epsilon)})^2 E_{\epsilon}^{(\Gamma_{max}-2)}\right.$$

$$\left. +(P_{T_r}^{(1)})^2 E_1^{(\Gamma_{max}-2)}\right] < \psi_{amb} + \left(\frac{\alpha'}{\beta'}\right)(P_{T_r}^{(2)})^2$$

Substituting from the inequality (6.20a) the upper bound for $P_{T_r}^{(1)}$

$$\implies \psi_{amb} + \left(\frac{\alpha'}{\beta'}\right)(P_{T_r}^{(2)})^2 >$$

$$e^{-\beta'(\delta t)}[\psi_{init}^{(0)}] + (1 - e^{-\beta'(\delta t)})[\psi_{amb}] + \left(\frac{\alpha'}{\beta'}\right)\left[(P_{T_r}^{(0)})^2(1 - e^{-\beta'\epsilon})e^{-(\delta t - \epsilon)\beta'} + (P_{T_r}^{(0)})^2(1 - e^{-\beta'(\delta t - \epsilon)})\right] \tag{6.20b}$$

Similarly

$$\psi_{amb} + \left(\frac{\alpha'}{\beta'}\right)(P_{T_r}^{(3)})^2 >$$

$$e^{-\beta'(\delta t)}[\psi_{init}^{(0)}] + (1 - e^{-\beta'(\delta t)})[\psi_{amb}] + \left(\frac{\alpha'}{\beta'}\right)\left[(P_{T_r}^{(0)})^2(1 - e^{-\beta'\epsilon})e^{-(\delta t - \epsilon)\beta'} + (P_{T_r}^{(0)})^2(1 - e^{-\beta'(\delta t - \epsilon)})\right] \tag{6.20c}$$

And generalizing

$$\psi_{amb} + \left(\frac{\alpha'}{\beta'}\right)(P_{T_r}^{(\Gamma_{max})})^2 >$$

$$e^{-\beta'(\delta t)}[\psi_{init}^{(0)}] + (1 - e^{-\beta'(\delta t)})[\psi_{amb}] + \left(\frac{\alpha'}{\beta'}\right)\left[(P_{T_r}^{(0)})^2(1 - e^{-\beta'\epsilon})e^{-(\delta t - \epsilon)\beta'} + (P_{T_r}^{(0)})^2(1 - e^{-\beta'(\delta t - \epsilon)})\right]$$

$$\boxed{\psi_{T_r}^{max} > \psi_{init}^{(1)}}, \text{ Since}$$

$$e^{-\beta'(\delta t)}[\psi_{init}^{(0)}] + (1 - e^{-\beta'(\delta t)})[\psi_{amb}] + \left(\frac{\alpha'}{\beta'}\right)\left[(P_{T_r}^{(0)})^2(1 - e^{-\beta'\epsilon})e^{-(\delta t - \epsilon)\beta'} + (P_{T_r}^{(0)})^2(1 - e^{-\beta'(\delta t - \epsilon)})\right]$$

$$= \psi_{init}^{(1)} \tag{6.20d}$$

$\square$

Lemma 6.6. *If the intra-dispatch interval ramping of generators is considered, the line flow on a transmission line, T_r is an affine function given as:*

$$P_{T_r} = \left[e^{\dagger}_{T_r} \mathbf{\Phi} (\mathbf{P_g}^{fin} - \mathbf{P_g}^{init}) \left(\frac{1}{\delta t} \right) \right] t + \left[e^{\dagger}_{T_r} \mathbf{\Phi} (\mathbf{P_g}^{init} - \mathbf{P_D}) \right] \tag{6.21}$$

where

- $e^{\dagger}_{T_r} \in \mathbb{R}^{|T|}$ *is a vector of all zeroes except the T_r^{-th} element, which is 1.*
- $\mathbf{\Phi}$ *is the injection shift factor matrix.*
- $\mathbf{P_g}^{fin}$ *is the vector of final values of generator ouputs at the end of the dispatch interval.*
- $\mathbf{P_g}^{init}$ *is the vector of initial values of generator outputs at the beginning of the dispatch interval.*
- $\mathbf{P_D}$ *is the vector of load magnitudes.*
- δt *is the duration of one dispatch interval.*
- t *is the intermediate time within an interval.*

The above equation can be written in a short-hand way as

$$P_{T_r} = P^{FL}_{T_r} t + P^{init}_{T_r} \tag{6.22}$$

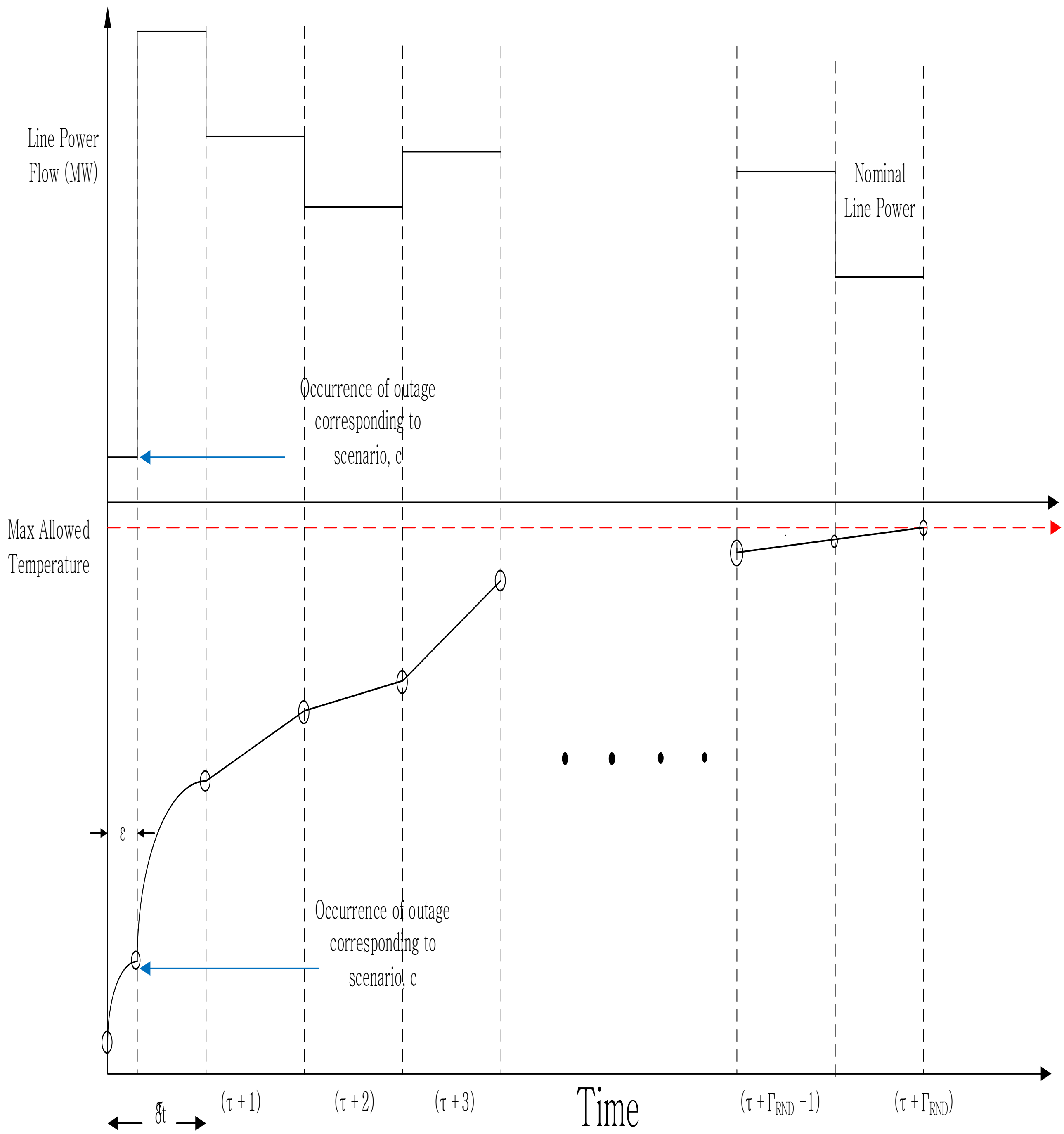


Figure 6.1: Restoration to Nominal Rating Following an Outage

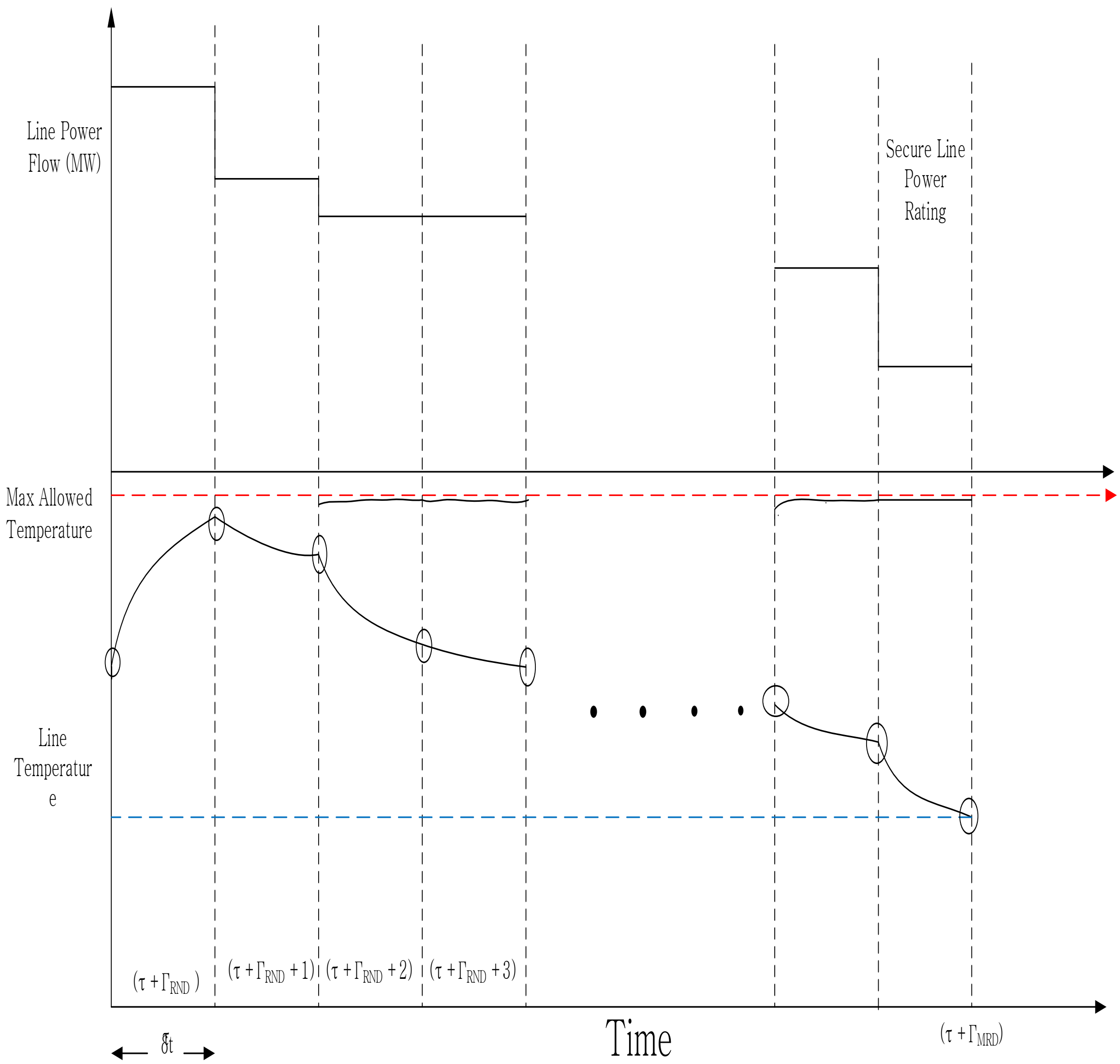


Figure 6.2: Restoration to Secure Rating Following an Outage

where $P_{T_r}^{FL} = \left[e_{T_r}^{\dagger}\mathbf{\Phi}(\mathbf{P_g}^{fin} - \mathbf{P_g}^{init})\left(\frac{1}{\delta t}\right)\right]$ *is the change of the line power flow over the course of the dispatch interval and* $P_{T_r}^{init} = \left[e_{T_r}^{\dagger}\mathbf{\Phi}(\mathbf{P_g}^{init} - \mathbf{P_D})\right]$ *is the initial power flow on the line at the beginning of the dispatch interval.*

Proof. Since the power generators' output within a dispatch interval (when intra-interval ramping is taken care of) can be expressed as

$$\mathbf{P_g} = (\mathbf{P_g}^{fin} - \mathbf{P_g}^{init})\left(\frac{1}{\delta t}\right)t + \mathbf{P_g}^{init} \tag{6.23}$$

and since the power flow on transmission line T_r can be written as

$$P_{T_r} = e_{T_r}^{\dagger}\mathbf{\Phi}(\mathbf{P_g} - \mathbf{P_D}) \tag{6.24}$$

Hence combining (6.23) and (6.24) and re-organizing the terms gives us (6.21) and (6.22). □

Lemma 6.7. *If the intra dispatch interval ramping of generators is taken into consideration, then the solution to (6.1) for the evolution of transmission line temperature, is given by the following equation:*

$$\boxed{\psi(t) = e^{-\beta' t}(\psi_{init} - \psi_{pseudo}) + \psi_{adj}} \tag{6.25}$$

where $\psi_{pseudo} = \psi_{amb} + \left(\frac{\alpha'}{\beta'^3}\right)[(P_{T_r}^{FL})^2 + (\beta' P_{T_r}^{init} - P_{T_r}^{FL})^2]$ *and* $\psi_{adj} = \psi_{amb} + \left(\frac{\alpha'}{\beta'^3}\right)[(P_{T_r}^{FL})^2 + (\beta' P_{T_r}^{init} - (1-\beta' t)P_{T_r}^{FL})^2]$*, which we will respectively call the "pseudo equilibrium temperature" and the "adjusted pseudo equilibrium temperature". It should be noted that, when* $P_{T_r}^{FL} = 0$ *ie, when the flows are assumed to stay constant throughout a dispatch interval, both these two quantities reduce to* ψ_{eq} *as defined earlier in (6.3b).*

Proof. Substituting the expression for P_{T_r} from (6.22) into (6.1), we get

$$\frac{d\psi}{dt} = \alpha^{'}(P_{T_r}^{FL}t + P_{T_r}^{init})^2 - \beta^{'}(\psi - \psi_{amb})$$

$$\implies \frac{d\psi}{dt} + \beta^{'}\psi = \alpha^{'}(P_{T_r}^{FL})^2 t^2 + (2\alpha^{'}P_{T_r}^{FL}P_{T_r}^{init})t + [\alpha^{'}(P_{T_r}^{init})^2 + \beta^{'}\psi_{amb}] \tag{6.26a}$$

$$\therefore \text{Integrating Factor} = e^{\int_0^t \beta^{'}dx} = e^{\beta^{'}t}$$

Multiplying both sides of equation 6.26a by the I.F. from above we get

$$e^{\beta^{'}t}\frac{d\psi}{dt} + \beta^{'}\psi e^{\beta^{'}t} = \alpha^{'}(P_{T_r}^{FL})^2 t^2 e^{\beta^{'}t} + (2\alpha^{'}P_{T_r}^{FL}P_{T_r}^{init})te^{\beta^{'}t} + [\alpha^{'}(P_{T_r}^{init})^2 + \beta^{'}\psi_{amb}]e^{\beta^{'}t}$$

$$\begin{aligned}
\implies \int d(e^{\beta^{'}t}\psi) &= \alpha^{'}(P_{T_r}^{FL})^2 \int t^2 e^{\beta^{'}t}dt + (2\alpha^{'}P_{T_r}^{FL}P_{T_r}^{init}) \int te^{\beta^{'}t}dt \\
&+[\alpha^{'}(P_{T_r}^{init})^2 + \beta^{'}\psi_{amb}] \int e^{\beta^{'}t}dt + C \\
\implies e^{\beta^{'}t}\psi &= \alpha^{'}(P_{T_r}^{FL})^2 \left(\frac{t^2 e^{\beta^{'}t}}{\beta'} - \frac{2te^{\beta^{'}t}}{\beta'^2} + \frac{2e^{\beta^{'}t}}{\beta'^3}\right) \\
&+(2\alpha^{'}P_{T_r}^{FL}P_{T_r}^{init}) \left(\frac{te^{\beta^{'}t}}{\beta'} - \frac{e^{\beta^{'}t}}{\beta'^2}\right) \\
&+[\alpha^{'}(P_{T_r}^{init})^2 + \beta^{'}\psi_{amb}] \left(\frac{e^{\beta^{'}t}}{\beta'}\right) + C \\
\implies \psi &= \alpha^{'}(P_{T_r}^{FL})^2 \left(\frac{t^2}{\beta'} - \frac{2t}{\beta'^2} + \frac{2}{\beta'^3}\right) \\
&+(2\alpha^{'}P_{T_r}^{FL}P_{T_r}^{init}) \left(\frac{t}{\beta'} - \frac{1}{\beta'^2}\right) \\
&+[\alpha^{'}(P_{T_r}^{init})^2 + \beta^{'}\psi_{amb}] \left(\frac{1}{\beta'}\right) + e^{-\beta^{'}t}C
\end{aligned} \tag{6.26b}$$

When $t = 0$, $\psi = \psi_{init}$,

$$\begin{aligned}
\therefore C = \psi_{init} - \alpha^{'}(P_{T_r}^{FL})^2 \left(\frac{2}{\beta'^3}\right) &+ (2\alpha^{'}P_{T_r}^{FL}P_{T_r}^{init}) \left(\frac{1}{\beta'^2}\right) \\
&-[\alpha^{'}(P_{T_r}^{init})^2 + \beta^{'}\psi_{amb}] \left(\frac{1}{\beta'}\right)
\end{aligned} \tag{6.26c}$$

Hence 6.26c and 6.26b implies

$$\psi(t) = \alpha^{'}(P_{T_r}^{FL})^2\left(\frac{t^2}{\beta'} - \frac{2t}{\beta'^2} + \frac{2}{\beta'^3}(1-e^{-\beta^{'}t})\right)$$
$$+(2\alpha^{'}P_{T_r}^{FL}P_{T_r}^{init})\left(\frac{t}{\beta'} - \frac{1}{\beta'^2}(1-e^{-\beta^{'}t})\right)$$
$$+[\alpha^{'}(P_{T_r}^{init})^2 + \beta^{'}\psi_{amb}]\left(\frac{1}{\beta'}\right)(1-e^{-\beta^{'}t}) + e^{-\beta^{'}t}\psi_{init}$$
$$\implies \psi(t) = [\alpha^{'}(P_{T_r}^{FL})^2]\left(\frac{t^2}{\beta'}\right) + \left[\frac{2\alpha^{'}P_{T_r}^{FL}(\beta^{'}P_{T_r}^{init} - P_{T_r}^{FL})}{\beta'^2}\right]t$$
$$+\left[\left(\frac{\alpha^{'}}{\beta'^3}\right)\left((P_{T_r}^{FL})^2 + (P_{T_r}^{FL} - \beta^{'}P_{T_r}^{init})^2\right) + \psi_{amb}\right](1-e^{-\beta^{'}t}) + e^{-\beta^{'}t}\psi_{init}$$
$$\implies \psi(t) = \left(\frac{\alpha^{'}}{\beta'}\right)\left[(P_{T_r}^{FL})^2t^2 + \frac{2P_{T_r}^{FL}t(\beta^{'}P_{T_r}^{init} - P_{T_r}^{FL})}{\beta'} + \frac{(\beta^{'}P_{T_r}^{init} - P_{T_r}^{FL})^2}{\beta'^2}\right]$$
$$+\left(\frac{\alpha^{'}}{\beta'^3}\right)(P_{T_r}^{FL})^2 + \psi_{amb} + e^{-\beta^{'}t}\left[\psi_{init} - \psi_{amb} - \left(\frac{\alpha^{'}}{\beta'^3}\right)\left((P_{T_r}^{FL})^2 + (\beta^{'}P_{T_r}^{init} - P_{T_r}^{FL})^2\right)\right]$$
$$\implies \psi(t) = e^{-\beta^{'}t}\left[\psi_{init} - \psi_{amb} - \left(\frac{\alpha^{'}}{\beta'^3}\right)\left((P_{T_r}^{FL})^2 + (\beta^{'}P_{T_r}^{init} - P_{T_r}^{FL})^2\right)\right]$$
$$+\psi_{amb} + \left(\frac{\alpha^{'}}{\beta'^3}\right)\left[\left(\beta^{'}P_{T_r}^{init} + (\beta^{'}t-1)P_{T_r}^{FL}\right)^2 + (P_{T_r}^{FL})^2\right]$$

From which the result follows. (6.26d)

□

Theorem 6.8. *When the intra-dispatch interval ramping of generators, and consequent intra-dispatch interval temporal variation of line flows are taken into account, the line temperature attains a maximum (or a minimum) value:*

(i) At a time, given by the solution of the equation,

$$e^{-\beta^{'}t}\left[\psi_{init} - \psi_{amb} - \left(\frac{\alpha^{'}}{\beta'^3}\right)\left((P_{T_r}^{FL})^2 + (\beta^{'}P_{T_r}^{init} - P_{T_r}^{FL})^2\right)\right] - \left(\frac{2\alpha^{'}}{\beta'^2}\right)P_{Tr}^{FL}(P_{Tr}^{FL}t + P_{Tr}^{init}) = 0 \tag{6.27}$$

(ii) The maximum (or minimum) value of line temperature at that time, is given by the equation,

$$\psi(t)_{M/m} = \psi_{amb} + (\frac{\alpha^{'}}{\beta^{'}})(P_{Tr}^{FL}t + P_{Tr}^{init})^2 \tag{6.28}$$

(iii) The above extremum of the line temperature is a maximum (or minimum) if the line flow is decreasing (or increasing) at the instant of time given by the solution of (6.27)

Proof. The rate of change of line temperature, with respect to time, as given by (6.1) for the case when intra-dispatch interval ramping of generators is considered is:

$$\frac{d\psi}{dt} = \alpha^{'}(P_{T_r}^{FL}t + P_{T_r}^{init})^2 - \beta^{'}(\psi - \psi_{amb}) \tag{6.29}$$

equating the right hand side of the above equation to zero, for the maximum (or minimum) line temperature, we get (6.28). Taking the second order derivative with respect to time, t in (6.29), and evaluating it at the extrema, we get,

$$\frac{d^2\psi}{dt^2}|_{\frac{d\psi}{dt}=0} = 2\alpha^{'}P_{T_r}^{FL}(P_{T_r}^{FL}t + P_{T_r}^{init}) - \beta^{'}\left(\frac{d\psi}{dt}\right) = 2\alpha^{'}P_{T_r}^{FL}(P_{T_r}^{FL}t + P_{T_r}^{init}) \tag{6.30}$$

From the above expression, since, $\alpha^{'}$ is positive, the second derivative evaluated at the extrema is negative (or positive) when $P_{T_r}^{FL}$, which is the rate of change of line power flow and $(P_{T_r}^{FL}t + P_{T_r}^{init})$, which is the line power flow, are of opposite signs (or same sign). This implies, the extremum of the line temperature is a maximum (or minimum) when the power

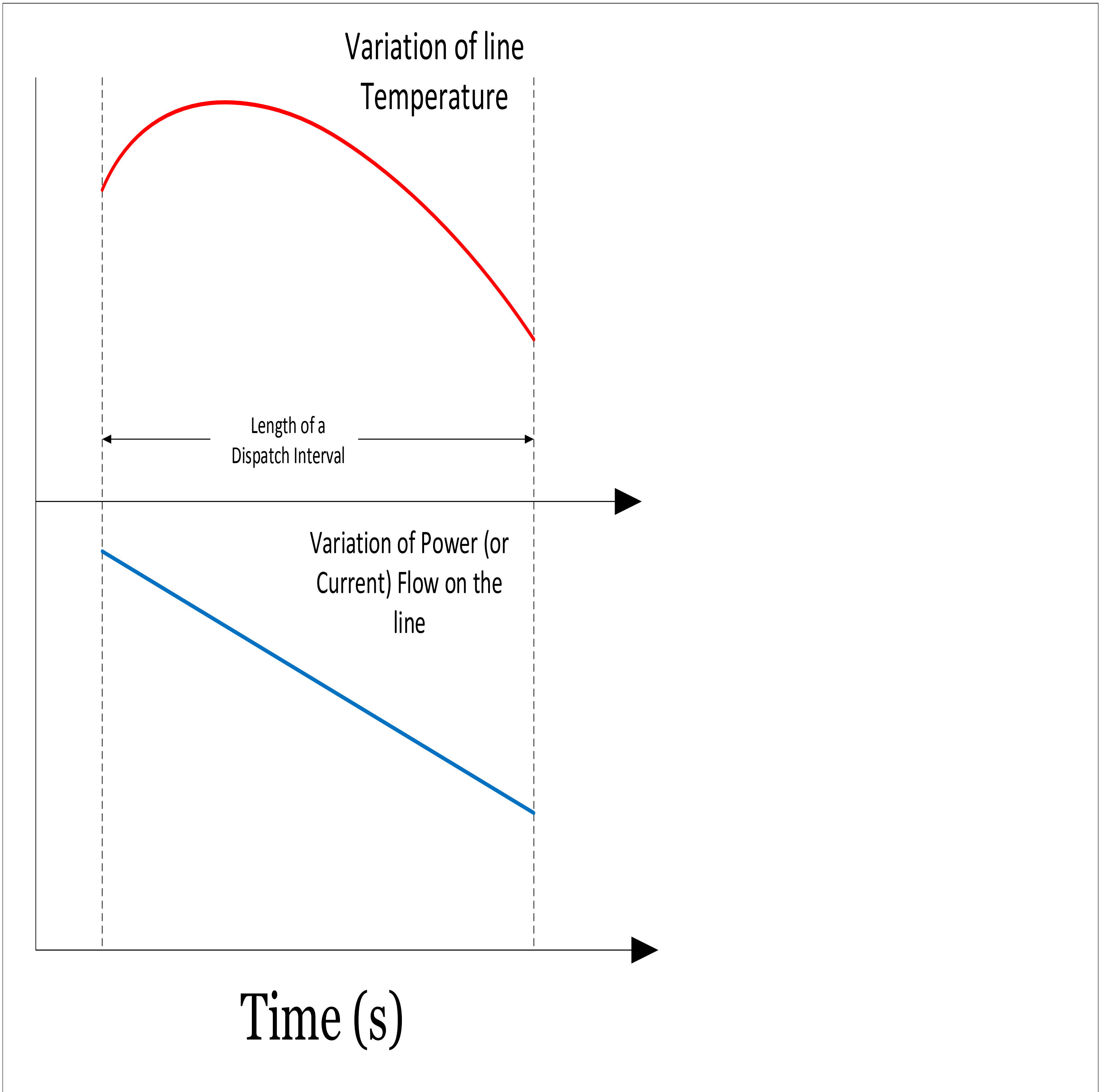


Figure 6.3: Variation of line temperature within a dispatch interval, when the line power fow (or line current) is Decreasing

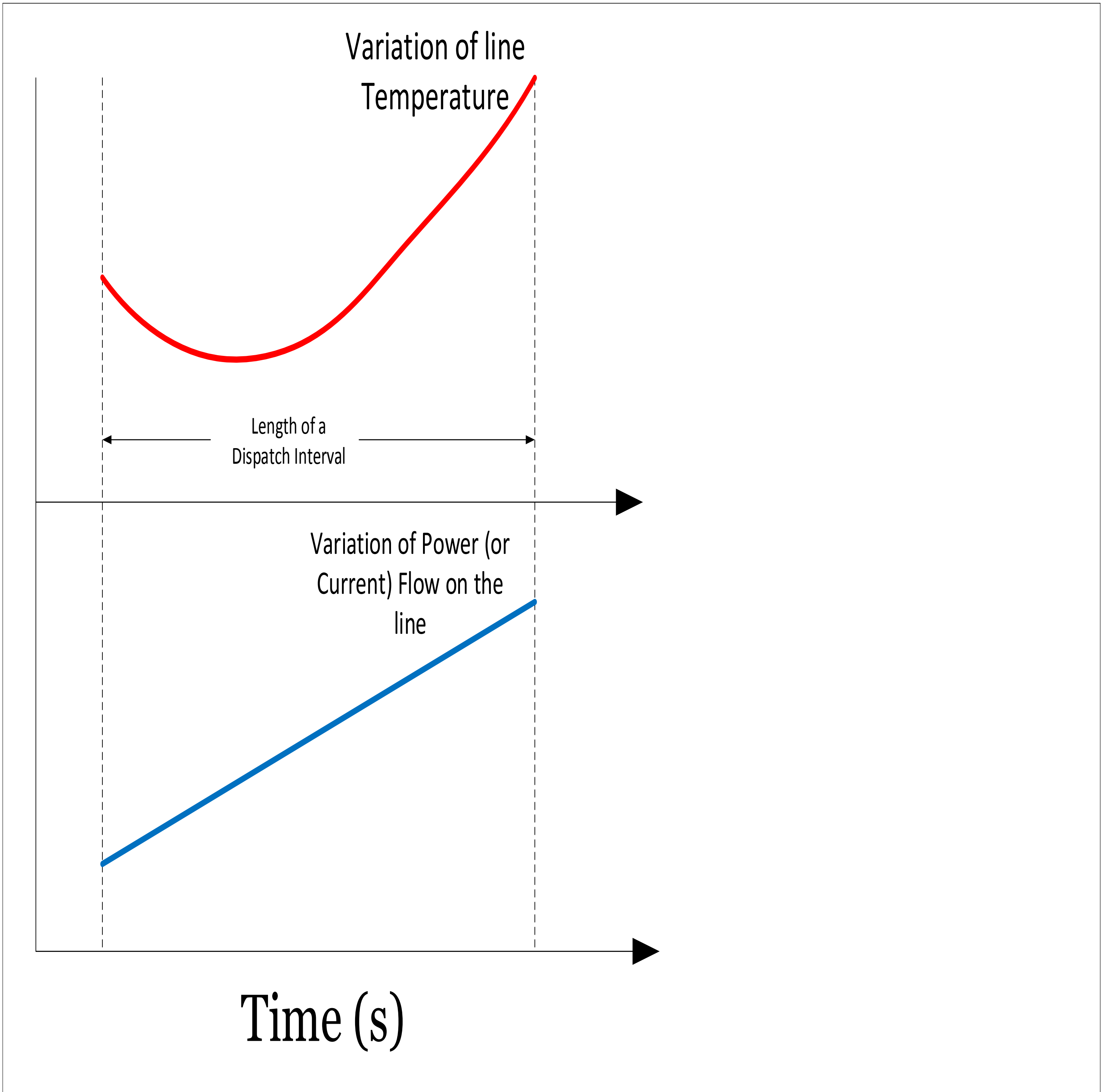


Figure 6.4: Variation of line temperature within a dispatch interval, when the line power fow (or line current) is Increasing

flow on the particular line in any direction is decreasing (or increasing).

Substituting the value of ψ from (6.28) and equating it to that from (6.25), it can be shown that, the time of attainment of the extremum of temperature is the root for t of (6.27). $\square$

Corollary 6.9. *Depending on the signs of the derivatives of the line temperature with respect to time, evaluated at the beginning and at the end of a particular dispatch interval, the constraint of limiting the line temperature will be enforced on the following as stated below:*

- *If* $\frac{d\psi}{dt}|_{t=0}$ *is positive and* $\frac{d\psi}{dt}|_{t=\delta t}$ *is positive, then* $\psi(\delta t)$ *is the maximum temperature and has to be constrained below* ψ_{max}.

- *If* $\frac{d\psi}{dt}|_{t=0}$ *is positive and* $\frac{d\psi}{dt}|_{t=\delta t}$ *is negative, then* $\psi(t)_{Max}$ *is the maximum temperature and has to be constrained below* ψ_{max}.

- *If* $\frac{d\psi}{dt}|_{t=0}$ *is negative and* $\frac{d\psi}{dt}|_{t=\delta t}$ *is positive, then either of* $\psi(\delta t)$ *or* $\psi(0)$ *is the maximum temperature and both has to be constrained below* ψ_{max}.

- *If* $\frac{d\psi}{dt}|_{t=0}$ *is negative and* $\frac{d\psi}{dt}|_{t=\delta t}$ *is negative, then* $\psi(0)$ *is the maximum temperature and has to be constrained below* ψ_{max}.

In the first case above, the entire temperature evolution function is an increasing function and hence, the temperature at the end of the dispatch interval is the maximum.

For the second case, the temperature function is increasing at the beginning and decreasing towards the end of the interval, and hence it reaches a maximum somewhere within the interval. For the third case, the temperature function transitions from decreasing to increasing, and so, depending on the initial temperature, either of the temperature at the start or the end of the dispatch interval may be the maximum one. For the fourth and last case, since the temperature is a decreasing function, the one at the start is of the maximum magnitude.

Hence, we can see that consideration of the intra-dispatch time interval ramping of the generators makes the situation more complicated and unlike the constant power flow within dispatch interval case, just constraining the end of the interval temperatures may not be adequate. We will now present the LASCOPF model for the simpler case of constant power flow within dispatch interval case. If we adhere to the simplifying assumption that, in the case of intra-dispatch interval ramping of generators, we assume that the maximum or minimum temperatures occur only at one of the ends of the interval, then we can argue that the same mathematical model as that for the constant power flow within dispatch interval case can be applied here. The only difference will be that, now, notionally, the different power values will be interpreted as the ones occuring at the ends of the dispatch intervals, rather than during the dispatch intervals.

6.1.2 The LASCOPF Model for Multi Bus Systems

Now we will consider a situation, where, on the assumption that a line outage occurs at immediately after commencement of the forthcoming dispatch time interval, we will try to work out the dispatch in such a way that the power flows on the lines are first brought down to the long-term rating and then the system is made secure with respect to the next potential failure within a total of Γ_{MRD} dispatch times, but the temperature of the lines are brought down within the maximum allowable temperature in the first Γ_{RND} dispatch intervals. The following is the formulation for this case.

$$\textbf{Objective}:$$

$$\min_{Power\ Generation} \quad Generation\ Cost\ over\ \tau^{th}\ to\ (\tau+\Gamma_{MRD})^{th}\ intervals \tag{6.31a}$$

$$\textbf{Constraints (for } \tau^{\textbf{th}} \textbf{ Dispatch Interval)}:$$

$$Supply\ Demand\ Balance \tag{6.31b}$$

$$Line\ Power\ Flow\ Limit\ (Base\ Case) \tag{6.31c}$$

$$Line\ Power\ Flow\ Limit\ (Contingency\ Cases) \tag{6.31d}$$

$$Generation\ Limit \tag{6.31e}$$

$$\boxed{\textbf{Constraints (for } (\tau+\mathbf{1}) \textbf{ to } (\tau+\mathbf{\Gamma_{RND}}) \textbf{ Dispatch Intervals}, \forall \textbf{Contingency)}:}$$

$$\boxed{Supply\ Demand\ Balance} \tag{6.31f}$$

$$\boxed{Line\ Temperature\ Maintained\ Below\ Maximum\ Allowed} \tag{6.31g}$$

$$\boxed{\textbf{Constraints (for } (\tau+\mathbf{\Gamma_{RND}}) \textbf{ to } (\tau+\mathbf{\Gamma_{MRD}}-\mathbf{1}) \textbf{ Dispatch Interval}, \forall \textbf{Contingency)}:}$$

$$\boxed{Supply\ Demand\ Balance} \tag{6.31h}$$

$$\boxed{Line\ Power\ Flow\ Below\ Nominal\ Rating} \tag{6.31i}$$

$$\boxed{\textbf{Constraints (for } (\tau+\mathbf{\Gamma_{MRD}}) \textbf{ Dispatch Interval}, \forall \textbf{Contingency)}:}$$

$$\boxed{Supply\ Demand\ Balance} \tag{6.31j}$$

$$\boxed{\textit{Line Power Flow Below Rating Ensuring Security}} \tag{6.31k}$$

$$\textit{Generation Limit} \tag{6.31l}$$

$$\textit{Generator Ramp Rate Limits from } \tau \textit{ to } (\tau+1) \tag{6.31m}$$

$$\boxed{\textit{Generator Ramp Rate Limits from } (\tau+s) \textit{ to } (\tau+s+1)} \tag{6.31n}$$

So, we can see that there is a combinatorial explosion in the number of base-case/contingency groups of constraints in the above problem. In the next three sections, we will present the proofs, the exact mathematical formulation, and an algorithmic approach to effectively distribute and carry out the computation for this problem, respectively.

6.2 The Mathematical Model of LASCOPF for Multi Bus Systems for Limiting Line Temperature Rise

Presented below, are the centralized or traditional optimization formulation, followed by the APP based coarse grained decomposition thereof, and parts of the ADMM-PMP based fine grained decomposition of some categories of the coarse-grained decomposition.

6.2.1 The Centralized Optimization Formulation

Now we will consider a situation, where, on the assumption that a line outage occurs at the beginning of the upcoming dispatch time interval, we will try to work out the dispatch in such a way that the power flows on the lines are brought down below the continuous rating as well as the system is made secure with respect to the next potential failure within a total of Γ_{MRD} dispatch times, but the temperature of the lines are brought down within the maximum allowable temperature in the first Γ_{RND} dispatch intervals. In the formulations to follow, we will use the more generalized equation (6.18a). Equation (6.18b), which is just a special case, can actually be derived from the generalized one under the special circumstance. The following formulation takes care of this issue (Angles Eliminated for the generalized multi-bus system):

$$\forall \tau \in \Omega$$

Objective Function:

$$\min_{\mathbf{P_g}} \sum_{g_q \in G} \left(C_{g_q}(P_{g_q}^{(0)(\tau)}) + \sum_{s=1}^{\Gamma_{MRD}} \sum_{(c) \in \mathcal{L}} prob^{(c)} C_{g_q}(P_{g_q}^{(c)(\tau+s)}) \right) \tag{6.32a}$$

Subject to: Supply-Demand Balance (Upcoming Interval):

$$\forall \tau \in \Omega,\ \forall T_r \in T,\ \forall (c) \in \mathcal{L},\ \forall s \in \{0, 1, 2, ..., \Gamma_{MRD}\}$$

$$\sum_{g_q \in G} P_{g_q}^{(0)(\tau)} = \sum_{D_d \in L} P_{D_d}^{(\tau)} \tag{6.32b}$$

Supply-Demand Balance (Look-Ahead Intervals):

$$\sum_{g_q \in G} P_{g_q}^{(c)(\tau+s)} = \sum_{D_d \in L} P_{D_d}^{(\tau+s)} \tag{6.32c}$$

Line Flow Limits

(Base-Case, Upcoming and Look-Ahead Intervals for No-outage case):

$$|\Phi^{(0)}(\mathbf{P_g^{(0)(\tau+s)}} - \mathbf{P_D^{(\tau+s)}})| \leq \overline{\mathbf{L}}^{(0)} \tag{6.32d}$$

Line Flow Limits

(Contingency scenarios, Upcoming and Look-Ahead Intervals for No-outage case):

$$|\Phi^{(c)}(\mathbf{P_g^{(0)(\tau+s)}} - \mathbf{P_D^{(\tau+s)}})| \leq \overline{\mathbf{L}}^{(c)} \tag{6.32e}$$

Line Flow Limits (for RSD, corresponding to long-term rating):

$$\forall (c) \in \mathcal{L} - \{0\}, \forall (c^{'}) \in [\mathcal{L} - \{0, c\}], \forall s \in \{0, 1, 2, ..., \Gamma_{MRD} - \Gamma_{RND}\}$$

$$|\Phi^{(c)}(\mathbf{P_g^{(c)(\tau+\Gamma_{RND}+s)}} - \mathbf{P_D^{(\tau+\Gamma_{RND}+s)}})| \leq \overline{\mathbf{L}}^{(0)} \tag{6.32f}$$

Line Flow Limits

(for RSD, corresponding to attaining security with respect to next set of outages):

$$|\Phi^{(c\to c')}(\mathbf{P_g^{(c)(\tau+\Gamma_{MRD})}} - \mathbf{P_D^{(\tau+\Gamma_{MRD})}})| \leq \overline{\mathbf{L}}^{(c\to c')} \tag{6.32g}$$

Line Temperature Limits (for RND):

$$\forall\tau \in \Omega,\ \forall T_r \in T,\ \forall(c) \in \mathcal{L} - \{0\},\ \forall\omega \in \{0,1,2,...(\Gamma_{max}-1)\}$$

$$E_\Gamma^{(\omega)}[\psi_{init}^{(\tau)}] + (1 - E_\Gamma^{(\omega)})[\psi_{amb}] +$$

$$\left(\frac{\alpha'}{\beta'}\right)\left[\sum_{s=0}^{\Gamma_{max}-(\omega+1)}\Big((\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})^\dagger \mathbf{\Phi}_{sT_r}^{(c,\omega)}(\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})\Big)\right] < \psi_{T_r}^{max} \tag{6.32h}$$

Generator Ramping Limits:

$$\underline{R}_{g_q} \leq P_{g_q}^{(c)(\tau+1)} - P_{g_q}^{(0)(\tau)} \leq \overline{R}_{g_q},\ \forall g_q \in G \tag{6.32i}$$

$$\underline{R}_{g_q} \leq P_{g_q}^{(c)(\tau+s+1)} - P_{g_q}^{(c)(\tau+s)} \leq \overline{R}_{g_q},\ \forall g_q \in G,\ (c) \in \mathcal{L},\ s \in \{1,2,...,\Gamma_{MRD}\} \tag{6.32j}$$

In the angles included formulation, the problem can be framed as follows:

$$\forall\tau \in \Omega$$

Objective Function:

$$\min_{\mathbf{P_g}} \sum_{g_q \in G}\left(C_{g_q}(P_{g_q}^{(0)(\tau)}) + \sum_{s=1}^{\Gamma_{MRD}}\sum_{(c)\in\mathcal{L}-\{0\}} prob^{(c)} C_{g_q}(P_{g_q}^{(c)(\tau+s)})\right) \tag{6.33a}$$

Subject to:

Supply-Demand Balance (Base-Case & Contingency Scenarios,

Upcoming & Look-Ahead Intervals for No-outage Case):

$$\forall \tau \in \Omega,\ \forall T_r \in T,\ \forall (c) \in \mathcal{L},\ \forall s \in \{0,1,2,...,\Gamma_{MRD}\}$$

$$P_{g_{q_{N_i}}}^{(0)(\tau)} - P_{D_{d_{N_i}}}^{(\tau)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(0)}(\theta_{N_i}^{(0)(\tau)} - \theta_{N_{\bar{i}}}^{(0)(\tau)});\ \forall N_i \in \mathcal{N} \tag{6.33b}$$

$$P_{g_{q_{N_i}}}^{(0)(\tau)} - P_{D_{d_{N_i}}}^{(\tau)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(c)}(\theta_{N_i}^{(c)(\tau)} - \theta_{N_{\bar{i}}}^{(c)(\tau)});\ \forall N_i \in \mathcal{N} \tag{6.33c}$$

Supply-Demand Balance (Post Contingency Cases, Look-Ahead Intervals):

$$P_{g_{q_{N_i}}}^{(c)(\tau+s)} - P_{D_{d_{N_i}}}^{(\tau+s)} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(c)}(\theta_{N_i}^{(c)(\tau+s)} - \theta_{N_{\bar{i}}}^{(c)(\tau+s)});\ \forall N_i \in \mathcal{N} \tag{6.33d}$$

Supply-Demand Balance (Post Contingency Cases, Restoration to security):

$$P_{g_{q_{N_i}}}^{(c)(\tau+\Gamma_{MRD})} - P_{D_{d_{N_i}}}^{(\tau+\Gamma_{MRD})} = \sum_{N_{\bar{i}} \in J(N_i)} B_{T_r}^{(c\to c^{'})}(\theta_{N_i}^{(c\to c^{'})(\tau+\Gamma_{MRD})} - \theta_{N_{\bar{i}}}^{(c\to c^{'})(\tau+\Gamma_{MRD})});\ \forall N_i \in \mathcal{N} \tag{6.33e}$$

Flow Limit Constraints (Base-Case & Contingency Scenarios,

Upcoming & Look-Ahead Intervals for No-outage Case):

$$|B_{T_r}^{(0)}(\theta_{T_{rt_1}}^{(0)(\tau+s)} - \theta_{T_{rt_2}}^{(0)(\tau+s)})| \le \overline{L}_{T_r}^{(0)},\ \forall T_r \in T \tag{6.33f}$$

$$|B_{T_r}^{(c)}(\theta_{T_{rt_1}}^{(c)(\tau+s)} - \theta_{T_{rt_2}}^{(c)(\tau+s)})| \le \overline{L}_{T_r}^{(c)},\ \forall T_r \in T \tag{6.33g}$$

Line Flow Limits (for RSD, corresponding to long-term rating):

$$\forall (c) \in \mathcal{L} - \{0\}, \forall (c^{'}) \in [\mathcal{L} - \{0, c\}], \forall s \in \{0,1,2,...,\Gamma_{MRD} - \Gamma_{RND}\}$$

$$|B_{T_r}^{(c)}(\theta_{T_{rt_1}}^{(c)(\tau+\Gamma_{RND}+s)} - \theta_{T_{rt_2}}^{(c)(\tau+\Gamma_{RND}+s)})| \le \overline{L}_{T_r}^{(0)},\ \forall T_r \in T \tag{6.33h}$$

Line Flow Limits

(for RSD, corresponding to attaining security with respect to next set of outages):

$$|B_{T_r}^{(c\to c^{'})}(\theta_{T_{rt_1}}^{(c\to c^{'})(\tau+\Gamma_{MRD})} - \theta_{T_{rt_2}}^{(c\to c^{'})(\tau+\Gamma_{MRD})})| \le \overline{L}_{T_r}^{(c\to c^{'})},\ \forall T_r \in T \tag{6.33i}$$

Line Temperature Limits (for RND):

$$\forall \tau \in \Omega,\ \forall T_r \in T,\ \forall (c) \in \mathcal{L} - \{0\},\ \forall \omega \in \{0,1,2,...(\Gamma_{max} - 1)\}$$

$$E_{\Gamma}^{(\omega)}[\psi_{init}^{(\tau)}] + (1 - E_{\Gamma}^{(\omega)})[\psi_{amb}] +$$

$$\left(\frac{\alpha'}{\beta'}\right)\left[\sum_{s=0}^{\Gamma_{max}-(\omega+1)} \Big((\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)})^{\dagger} \mathbf{\Phi}_{sT_r}^{(c,\omega)} (\mathbf{P_g}^{(c)(\tau+s)} - \mathbf{P_D}^{(\tau+s)}) \Big)\right] < \psi_{T_r}^{max} \quad (6.33j)$$

Generator Ramping Limits:

$$\underline{R}_{g_q} \leq P_{g_q}^{(c)(\tau+1)} - P_{g_q}^{(0)(\tau)} \leq \overline{R}_{g_q},\ \forall g_q \in G \quad (6.33k)$$

$$\underline{R}_{g_q} \leq P_{g_q}^{(c)(\tau+s+1)} - P_{g_q}^{(c)(\tau+s)} \leq \overline{R}_{g_q},\ \forall g_q \in G,\ (c) \in \mathcal{L},\ s \in 1, 2, ..., \Gamma_{MRD} \quad (6.33l)$$

6.2.2 APP Based Coarse Grained Decomposition

With the model stated above, we will now proceed with reformulating the problem, first in order for us to apply the coarse grained decomposition. Presented below is the augmented Lagrangian for the angles eliminated model for applying the APP part of the APMP algorithm.

$$\forall \tau \in \Omega$$

Objective Function :

$$\min_{\mathbf{P_g}} \sum_{g_q \in G} \left(C_{g_q}(P_{g_q(\tau)}^{(0)(\tau)}) + \sum_{s=1}^{\Gamma_{MRD}} \sum_{(c)\in\mathcal{L}} prob^{(c)} C_{g_q}(P_{g_q(c)(\tau+s)}^{(c)(\tau+s)}) \right)$$

$$+\frac{\gamma}{2}\Bigg[\sum_{(c)\in\mathcal{L}} \Big\{ ||\mathbf{P}_{(\mathbf{0})(\tau)}^{(\mathbf{0})(\tau)} - \mathbf{P}_{(\mathbf{c})(\tau+\mathbf{1})}^{(\mathbf{0})(\tau)}||_2^2 + ||\mathbf{P}_{(\mathbf{0})(\tau)}^{(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}_{(\mathbf{c})(\tau+\mathbf{1})}^{(\mathbf{c})(\tau+\mathbf{1})}||_2^2$$

$$+||\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{1})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{2})}||_2^2 + \sum_{s=2}^{\Gamma_{MRD}-1} \Big(||\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{(\mathbf{c})(\tau+\mathbf{s}-\mathbf{1})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{(\mathbf{c})(\tau+\mathbf{s})}||_2^2 +$$

$$||\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{(\mathbf{c})(\tau+\mathbf{s}+\mathbf{1})}||_2^2\Big) + ||\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}}-\mathbf{1})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}||_2^2\Big\} +$$

$$\sum_{(c)\in\mathcal{L}-\{0\}} \Big\{ \sum_{s=1}^{\Gamma_{RND}-1} ||\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{\mathbf{T}(\mathbf{0})(\tau)} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{\mathbf{T}(\mathbf{c})(\tau+\mathbf{s})}||_2^2 \Big\}\Big] \tag{6.34a}$$

$$\textbf{Subject to}: \forall(c) \in \mathcal{L}$$

$$\textbf{Power} - \textbf{Balance Constraints} \ (\textbf{Base} - \textbf{Case} \ \& \ \textbf{Contingency}):$$

$$\sum_{g_q \in G} P^{(0)(\tau)}_{g_q(0)(\tau)} = \sum_{D_d \in L} P^{(\tau)}_{D_d} \tag{6.34b}$$

$$\sum_{g_q \in G} P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} = \sum_{D_d \in L} P^{(\tau+s)}_{D_d}, \forall s \in \{1, 2, ..., \Gamma_{MRD}\} \tag{6.34c}$$

$$\textbf{Flow Limit Constraints} \ (\textbf{Base} - \textbf{Case} \ \& \ \textbf{Contingency}):$$

$$|\Phi^{(0)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{\mathbf{g}(\mathbf{0})(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{6.34d}$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{0})(\tau)}_{\mathbf{g}(\mathbf{0})(\tau)} - \mathbf{P}^{(\tau)}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c)} \tag{6.34e}$$

$$\textbf{Flow} - \textbf{Limit Constraints} \ (\textbf{For Look} - \textbf{Ahead Intervals for RND}):$$

$$\forall T_r \in T, \ \forall \omega \in \{0, 1, 2, ...(\Gamma_{RND} - 1)\}, \forall c \in \mathcal{L} - \{0\}$$

$$E^{(\omega)}_{\Gamma}[\psi^{(\tau)}_{init}] + (1 - E^{(\omega)}_{\Gamma})[\psi_{amb}] +$$

$$\left(\frac{\alpha^{'}}{\beta^{'}}\right)\left[(P^{(0)(\tau)}_{T_r(0)(\tau)})^2 E^{(\omega)}_0 + (P^{(c)(\tau+\epsilon)}_{T_r(0)(\tau)})^2 E^{(\omega)}_{\epsilon} + \sum_{s=1}^{\Gamma_{RND}-(\omega+1)} (P^{(c)(\tau+s)}_{T_r(0)(\tau)})^2 E^{(\omega)}_s\right]$$

$$< \psi^{max}_{T_r} \tag{6.34f}$$

$$\textbf{Flow} - \textbf{Limit Constraints} \ (\textbf{For Look} - \textbf{Ahead Intervals for MRD}):$$

$$\forall(c^{'}) \in [\mathcal{L} - \{0, c\}], \forall s \in \{1, 2, ..., (\Gamma_{MRD} - \Gamma_{RND})\}$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{RND}}+\mathbf{s})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{\Gamma_{RND}}+\mathbf{s})} - \mathbf{P}^{(\tau+\mathbf{\Gamma_{RND}}+\mathbf{s})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{6.34g}$$

$$|\Phi^{(c\rightarrow c')}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} - \mathbf{P}^{(\tau+\mathbf{\Gamma_{MRD}})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c\rightarrow c')} \tag{6.34h}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\forall g_q \in G,\ \forall s \in \{2,3,...,(\Gamma_{MRD}-1)\}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)}_{g_q(0)(\tau)} - P^{(0)(\tau)}_{g_q(0)(\tau)} \leq \overline{R}_{g_q} \tag{6.34i}$$

$$\underline{R}_{g_q} \leq P^{(0)(\tau)}_{g_q(0)(\tau)} - P^{(0)(\tau-1)}_{g_q} \leq \overline{R}_{g_q} \tag{6.34j}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} - P^{(0)(\tau)}_{g_q(c)(\tau+1)} \leq \overline{R}_{g_q} \tag{6.34k}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+2)}_{g_q(c)(\tau+1)} - P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} \leq \overline{R}_{g_q} \tag{6.34l}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} - P^{(c)(\tau+s-1)}_{g_q(c)(\tau+s)} \leq \overline{R}_{g_q} \tag{6.34m}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+s+1)}_{g_q(c)(\tau+s)} - P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} \leq \overline{R}_{g_q} \tag{6.34n}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+\Gamma_{MRD})}_{g_q(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD}-1)}_{g_q(c)(\tau+\Gamma_{MRD})} \leq \overline{R}_{g_q} \tag{6.34o}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP}-1)}_{g_q(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD})}_{g_q(c)(\tau+\Gamma_{MRD})} \leq \overline{R}_{g_q} \tag{6.34p}$$

$$\textbf{Consensus Constraints for MRD}:$$

$$\forall g_q \in G,\ \forall s \in \{2,3,...,(\Gamma_{MRD}-1)\}$$

$$P^{(0)(\tau)}_{g_q(0)(\tau)} = P^{(0)(\tau)}_{g_q(c)(\tau+1)} \tag{6.34q}$$

$$P^{(c)(\tau+1)}_{g_q(0)(\tau)} = P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} \tag{6.34r}$$

$$P^{(c)(\tau+1)}_{g_q(c)(\tau+1)} = P^{(c)(\tau+1)}_{g_q(c)(\tau+2)} \tag{6.34s}$$

$$P^{(c)(\tau+s)}_{g_q(c)(\tau+s-1)} = P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} \tag{6.34t}$$

$$P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} = P^{(c)(\tau+s)}_{g_q(c)(\tau+s+1)} \tag{6.34u}$$

$$P^{(c)(\tau+\Gamma_{MRD})}_{g_q(c)(\tau+\Gamma_{MRD}-1)} = P^{(c)(\tau+\Gamma_{MRD})}_{g_q(c)(\tau+\Gamma_{MRD})} \tag{6.34v}$$

$$\textbf{Consensus Constraints for RND}:$$

$$\forall T_r \in T,\ \forall s \in \{1,2,3,...,\Gamma_{RND}-1\}, c \in \mathcal{L}-\{0\}$$

$$P^{(c)(\tau+s)}_{T_r(0)(\tau)} = P^{(c)(\tau+s)}_{T_r(c)(\tau+s)} \tag{6.34w}$$

$$\tag{6.34x}$$

Stated below are the update equations for the iterates of the decision variables for the optimization problem (6.34) applying the APP algorithm.

$$\forall \tau \in \Omega$$

$$\textbf{Objective Function}:$$

$$\begin{aligned}
\big(\mathbf{P}^{(\mu_{APP}+1)}_{(\tau)}, \mathbf{P}^{(\mu_{APP}+1)}_{(c)(\tau+s)}\big) = \min_{\mathbf{P_g}} \sum_{g_q \in G} \left(C_{g_q}(P^{(0)(\tau)}_{g_q(\tau)}) + \sum_{s=1}^{\Gamma_{MRD}} \sum_{(c)\in\mathcal{L}} prob^{(c)} C_{g_q}(P^{(c)(\tau+s)}_{g_q(c)(\tau+s)}) \right) \\
+\frac{\beta}{2}\left[||\mathbf{P_{(0)(\tau)}} - \mathbf{P^{(\mu_{APP})}_{(0)(\tau)}}||_2^2 + \sum_{(c)\in\mathcal{L}} \sum_{s=1}^{\Gamma_{MRD}} \{||\mathbf{P_{(c)(\tau+s)}} - \mathbf{P^{(\mu_{APP})}_{(c)(\tau+s)}}||_2^2\} \right] \\
+\gamma\Bigg[\sum_{(c)\in\mathcal{L}} \Bigg\{ (\mathbf{P^{(0)(\tau)}_{(0)(\tau)}} - \mathbf{P^{(0)(\tau)}_{(c)(\tau+1)}})^\dagger(\mathbf{P^{(0)(\tau)(\mu_{APP})}_{(0)(\tau)}} - \mathbf{P^{(0)(\tau)(\mu_{APP})}_{(c)(\tau+1)}}) + \\
(\mathbf{P^{(c)(\tau+1)}_{(0)(\tau)}} - \mathbf{P^{(c)(\tau+1)}_{(c)(\tau+1)}})^\dagger(\mathbf{P^{(c)(\tau+1)(\mu_{APP})}_{(0)(\tau)}} - \mathbf{P^{(c)(\tau+1)(\mu_{APP})}_{(c)(\tau+1)}}) \\
+(\mathbf{P^{(c)(\tau+1)}_{(c)(\tau+1)}} - \mathbf{P^{(c)(\tau+1)}_{(c)(\tau+2)}})^\dagger(\mathbf{P^{(c)(\tau+1)(\mu_{APP})}_{(c)(\tau+1)}} - \mathbf{P^{(c)(\tau+1)(\mu_{APP})}_{(c)(\tau+2)}}) + \\
\sum_{s=2}^{\Gamma_{MRD}-1} \Big((\mathbf{P^{(c)(\tau+s)}_{(c)(\tau+s-1)}} - \mathbf{P^{(c)(\tau+s)}_{(c)(\tau+s)}})^\dagger(\mathbf{P^{(c)(\tau+s)(\mu_{APP})}_{(c)(\tau+s-1)}} - \mathbf{P^{(c)(\tau+s)(\mu_{APP})}_{(c)(\tau+s)}}) + \\
(\mathbf{P^{(c)(\tau+s)}_{(c)(\tau+s)}} - \mathbf{P^{(c)(\tau+s)}_{(c)(\tau+s+1)}})^\dagger(\mathbf{P^{(c)(\tau+s)(\mu_{APP})}_{(c)(\tau+s)}} - \mathbf{P^{(c)(\tau+s)(\mu_{APP})}_{(c)(\tau+s+1)}}) \Big) \\
+(\mathbf{P^{(c)(\tau+\Gamma_{MRD})}_{(c)(\tau+\Gamma_{MRD}-1)}} - \mathbf{P^{(c)(\tau+\Gamma_{MRD})}_{(c)(\tau+\Gamma_{MRD})}})^\dagger(\mathbf{P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP})}_{(c)(\tau+\Gamma_{MRD}-1)}} - \mathbf{P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP})}_{(c)(\tau+\Gamma_{MRD})}}) \Bigg\} + \\
\sum_{(c)\in\mathcal{L}-\{0\}} \left\{ \sum_{s=1}^{\Gamma_{RND}-1} (\mathbf{P^{(c)(\tau+s)}_{T(0)(\tau)}} - \mathbf{P^{(c)(\tau+s)}_{T(c)(\tau+s)}})^\dagger(\mathbf{P^{(c)(\tau+s)(\mu_{APP})}_{T(0)(\tau)}} - \mathbf{P^{(c)(\tau+s)(\mu_{APP})}_{T(c)(\tau+s)}}) \right\} \Bigg]
\end{aligned}$$

$$+\sum_{c=0}^{|\mathcal{L}|}\Bigg(\mathbf{\Lambda}_{2c\Gamma_{MRD}+1}^{ADJ(\mu_{APP})\dagger}(\mathbf{P}_{(0)(\tau)}^{(0)(\tau)}-\mathbf{P}_{(c)(\tau+1)}^{(0)(\tau)})+\mathbf{\Lambda}_{2c\Gamma_{MRD}+2}^{ADJ(\mu_{APP})\dagger}(\mathbf{P}_{(0)(\tau)}^{(c)(\tau+1)}-\mathbf{P}_{(c)(\tau+1)}^{(c)(\tau+1)})$$

$$+\mathbf{\Lambda}_{2c\Gamma_{MRD}+3}^{ADJ(\mu_{APP})\dagger}(\mathbf{P}_{(c)(\tau+1)}^{(c)(\tau+1)}-\mathbf{P}_{(c)(\tau+2)}^{(c)(\tau+1)})+\sum_{s=2}^{\Gamma_{MRD}-1}\Bigg[\mathbf{\Lambda}_{2c\Gamma_{MRD}+2s}^{ADJ(\mu_{APP})\dagger}(\mathbf{P}_{(c)(\tau+s-1)}^{(c)(\tau+s)}-\mathbf{P}_{(c)(\tau+s)}^{(c)(\tau+s)})$$

$$+\mathbf{\Lambda}_{2c\Gamma_{MRD}+(2s+1)}^{ADJ(\mu_{APP})\dagger}(\mathbf{P}_{(c)(\tau+s)}^{(c)(\tau+s)}-\mathbf{P}_{(c)(\tau+s+1)}^{(c)(\tau+s)})\Bigg]+\mathbf{\Lambda}_{2(c+1)\Gamma_{MRD}}^{ADJ(\mu_{APP})\dagger}(\mathbf{P}_{(c)(\tau+\Gamma_{MRD}-1)}^{(c)(\tau+\Gamma_{MRD})}-\mathbf{P}_{(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})})\Bigg)$$

$$+\sum_{c=1}^{|\mathcal{L}|}\Bigg(\sum_{s=1}^{\Gamma_{RND}-1}\Bigg[\mathbf{\Lambda}_{(c-1)(\Gamma_{RND}-1)+s}^{LA(\mu_{APP})\dagger}(\mathbf{P}_{T(0)(\tau)}^{(c)(\tau+s)}-\mathbf{P}_{T(c)(\tau+s)}^{(c)(\tau+s)})\Bigg]\Bigg) \tag{6.35a}$$

$$\textbf{Subject to}:\forall(c)\in\mathcal{L}$$

$$\textbf{Power}-\textbf{Balance Constraints}\ (\textbf{Base}-\textbf{Case}\ \&\ \textbf{Contingency}):$$

$$\sum_{g_q\in G}P_{g_q(0)(\tau)}^{(0)(\tau)}=\sum_{D_d\in L}P_{D_d}^{(\tau)} \tag{6.35b}$$

$$\sum_{g_q\in G}P_{g_q(c)(\tau+s)}^{(c)(\tau+s)}=\sum_{D_d\in L}P_{D_d}^{(\tau+s)},\forall s\in\{1,2,...,\Gamma_{MRD}\} \tag{6.35c}$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base}-\textbf{Case}\ \&\ \textbf{Contingency}):$$

$$|\Phi^{(0)}(\mathbf{P}_{\mathbf{g(0)}(\tau)}^{\mathbf{(0)}(\tau)}-\mathbf{P}_{\mathbf{D}}^{(\tau)})|\leq\overline{\mathbf{L}}^{(0)} \tag{6.35d}$$

$$|\Phi^{(c)}(\mathbf{P}_{\mathbf{g(0)}(\tau)}^{\mathbf{(0)}(\tau)}-\mathbf{P}_{\mathbf{D}}^{(\tau)})|\leq\overline{\mathbf{L}}^{(c)} \tag{6.35e}$$

$$\textbf{Flow}-\textbf{Limit Constraints}\ (\textbf{For Look}-\textbf{Ahead Intervals for RND}):$$

$$\forall T_r\in T,\ \forall\omega\in\{0,1,2,...(\Gamma_{RND}-1)\}$$

$$E_{\Gamma}^{(\omega)}[\psi_{init}^{(\tau)}]+(1-E_{\Gamma}^{(\omega)})[\psi_{amb}]+$$

$$\left(\frac{\alpha^{'}}{\beta^{'}}\right)\left[(P_{T_r(0)(\tau)}^{(0)(\tau)})^2E_0^{(\omega)}+(P_{T_r(0)(\tau)}^{(c)(\tau+\epsilon)})^2E_{\epsilon}^{(\omega)}+\sum_{s=1}^{\Gamma_{RND}-(\omega+1)}(P_{T_r(0)(\tau)}^{(c)(\tau+s)})^2E_s^{(\omega)}\right]$$

$$<\psi_{T_r}^{max} \tag{6.35f}$$

$$\textbf{Flow}-\textbf{Limit Constraints}\ (\textbf{For Look}-\textbf{Ahead Intervals for MRD}):$$

$$\forall(c^{'})\in[\mathcal{L}-\{0,c\}],\forall s\in\{1,2,...,(\Gamma_{MRD}-\Gamma_{RND})\}$$

$$|\Phi^{(c)}(\mathbf{P}_{\mathbf{g(c)}(\tau+\boldsymbol{\Gamma}_{\mathbf{RND}}+\mathbf{s})}^{\mathbf{(c)}(\tau+\boldsymbol{\Gamma}_{\mathbf{RND}}+\mathbf{s})} - \mathbf{P}_{\mathbf{D}}^{(\tau+\boldsymbol{\Gamma}_{\mathbf{RND}}+\mathbf{s})})| \leq \overline{\mathbf{L}}^{(0)} \tag{6.35g}$$

$$|\Phi^{(c\to c')}(\mathbf{P}_{\mathbf{g(c)}(\tau+\boldsymbol{\Gamma}_{\mathbf{MRD}})}^{\mathbf{(c)}(\tau+\boldsymbol{\Gamma}_{\mathbf{MRD}})} - \mathbf{P}_{\mathbf{D}}^{(\tau+\boldsymbol{\Gamma}_{\mathbf{MRD}})})| \leq \overline{\mathbf{L}}^{(c\to c')} \tag{6.35h}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\forall g_q \in G,\ \forall s \in \{2, 3, ..., (\Gamma_{MRD} - 1)\}$$

$$\underline{R}_{g_q} \leq P_{g_q(0)(\tau)}^{(c)(\tau+1)} - P_{g_q(0)(\tau)}^{(0)(\tau)} \leq \overline{R}_{g_q} \tag{6.35i}$$

$$\underline{R}_{g_q} \leq P_{g_q(0)(\tau)}^{(0)(\tau)} - P_{g_q}^{(0)(\tau-1)} \leq \overline{R}_{g_q} \tag{6.35j}$$

$$\underline{R}_{g_q} \leq P_{g_q(c)(\tau+1)}^{(c)(\tau+1)} - P_{g_q(c)(\tau+1)}^{(0)(\tau)} \leq \overline{R}_{g_q} \tag{6.35k}$$

$$\underline{R}_{g_q} \leq P_{g_q(c)(\tau+1)}^{(c)(\tau+2)} - P_{g_q(c)(\tau+1)}^{(c)(\tau+1)} \leq \overline{R}_{g_q} \tag{6.35l}$$

$$\underline{R}_{g_q} \leq P_{g_q(c)(\tau+s)}^{(c)(\tau+s)} - P_{g_q(c)(\tau+s)}^{(c)(\tau+s-1)} \leq \overline{R}_{g_q} \tag{6.35m}$$

$$\underline{R}_{g_q} \leq P_{g_q(c)(\tau+s)}^{(c)(\tau+s+1)} - P_{g_q(c)(\tau+s)}^{(c)(\tau+s)} \leq \overline{R}_{g_q} \tag{6.35n}$$

$$\underline{R}_{g_q} \leq P_{g_q(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})} - P_{g_q(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD}-1)} \leq \overline{R}_{g_q} \tag{6.35o}$$

$$\underline{R}_{g_q} \leq P_{g_q(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})(\mu_{APP}-1)} - P_{g_q(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})} \leq \overline{R}_{g_q} \tag{6.35p}$$

$$\textbf{Dual Variable Updates for MRD}:$$

$$\forall c \in \mathcal{L},\ \forall s \in \{2, 3, ..., \Gamma_{MRD} - 1\}$$

$$\boldsymbol{\Lambda}_{2c\Gamma_{MRD}+1}^{ADJ(\mu_{APP}+1)} = \boldsymbol{\Lambda}_{2c\Gamma_{MRD}+1}^{ADJ(\mu_{APP})} + \alpha(\mathbf{P}_{(0)(\tau)}^{(0)(\tau)(\mu_{APP}+1)} - \mathbf{P}_{(c)(\tau+1)}^{(0)(\tau)(\mu_{APP}+1)}) \tag{6.35q}$$

$$\boldsymbol{\Lambda}_{2c\Gamma_{MRD}+2}^{ADJ(\mu_{APP}+1)} = \boldsymbol{\Lambda}_{2c\Gamma_{MRD}+2}^{ADJ(\mu_{APP})} + \alpha(\mathbf{P}_{(0)(\tau)}^{(c)(\tau+1)(\mu_{APP}+1)} - \mathbf{P}_{(c)(\tau+1)}^{(c)(\tau+1)(\mu_{APP}+1)}) \tag{6.35r}$$

$$\boldsymbol{\Lambda}_{2c\Gamma_{MRD}+3}^{ADJ(\mu_{APP}+1)} = \boldsymbol{\Lambda}_{2c\Gamma_{MRD}+3}^{ADJ(\mu_{APP})} + \alpha(\mathbf{P}_{(c)(\tau+1)}^{(c)(\tau+1)(\mu_{APP}+1)} - \mathbf{P}_{(c)(\tau+2)}^{(c)(\tau+1)(\mu_{APP}+1)}) \tag{6.35s}$$

$$\boldsymbol{\Lambda}_{2c\Gamma_{MRD}+2s}^{ADJ(\mu_{APP}+1)} = \boldsymbol{\Lambda}_{2c\Gamma_{MRD}+2s}^{ADJ(\mu_{APP})} + \alpha(\mathbf{P}_{(c)(\tau+s-1)}^{(c)(\tau+s)(\mu_{APP}+1)} - \mathbf{P}_{(c)(\tau+s)}^{(c)(\tau+s)(\mu_{APP}+1)}) \tag{6.35t}$$

$$\boldsymbol{\Lambda}_{2c\Gamma_{MRD}+(2s+1)}^{ADJ(\mu_{APP}+1)} = \boldsymbol{\Lambda}_{2c\Gamma_{MRD}+(2s+1)}^{ADJ(\mu_{APP})} + \alpha(\mathbf{P}_{(c)(\tau+s)}^{(c)(\tau+s)(\mu_{APP}+1)} - \mathbf{P}_{(c)(\tau+s+1)}^{(c)(\tau+s)(\mu_{APP}+1)}) \tag{6.35u}$$

$$\boldsymbol{\Lambda}_{2(c+1)\Gamma_{MRD}}^{ADJ(\mu_{APP}+1)} = \boldsymbol{\Lambda}_{2(c+1)\Gamma_{MRD}}^{ADJ(\mu_{APP})} + \alpha(\mathbf{P}_{(c)(\tau+\Gamma_{MRD}-1)}^{(c)(\tau+\Gamma_{MRD})(\mu_{APP}+1)} - \mathbf{P}_{(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})(\mu_{APP}+1)}) \tag{6.35v}$$

$$\textbf{Consensus Constraints for RND}:$$

$$\forall T_r \in T,\ \forall s \in \{1,2,3,...,\Gamma_{RND}-1\}, \forall c \in \mathcal{L} - \{0\}$$

$$\mathbf{\Lambda}^{LA(\mu_{APP}+1)}_{(c-1)(\Gamma_{RND}-1)+s} = \mathbf{\Lambda}^{LA(\mu_{APP})}_{(c-1)(\Gamma_{RND}-1)+s} + \alpha(\mathbf{P}^{(c)(\tau+s)(\mu_{APP}+1)}_{T(0)(\tau)} - \mathbf{P}^{(c)(\tau+s)(\mu_{APP}+1)}_{T(c)(\tau+s)}) \tag{6.35w}$$

$$\tag{6.35x}$$

The APP based coarse grained decomposition of the iterates, in this case, can be classified into the following four different categories, as below. Each of the four categories consist of the coarse grains, that have the identical optimization formulations.:

Iterates for Dispatch Interval τ:

The optimization problem corresponding to this coarse grain or computational unit is the one for the immediately forthcoming dispatch interval and is the hardest problem in the entire series. The reason for that is, this problem not only aims to solve an $(N-1)$ SCOPF for the dispatch interval τ, but also attempts to evaluate its beliefs about the values of the line flows of (and consequently, temperature rises at the ends of) the dispatch intervals from $(\tau+1)$ to $(\tau+\Gamma_{RND}-1)$ in order to attain consensus on those values and limit the temperature rises on the lines below the allowed value for the duration of the RND. Given below, is the optimization problem formulation for this coarse grain.

$$\mathbf{P}_{(\tau)}{}^{(\mu_{APP}+1)} = \underset{\mathbf{P}_{(\tau)}}{\text{argmin}} \sum_{g_q \in G} C_{g_q}(P^{(0)(\tau)}_{g_q(\tau)}) + \boxed{\frac{\beta}{2}||\mathbf{P}_{(\tau)} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\tau)}||^2_2}$$
$$+\gamma[\sum_{c=0}^{|\mathcal{L}|}\{\boxed{\mathbf{P}^{(\mathbf{0})(\tau)\dagger}_{(\tau)}(\mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}})}_{(\tau)} - \mathbf{P}^{(\mathbf{0})(\tau)(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{1})})}$$
$$+\boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})\dagger}_{(\tau)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\tau)} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{1})})}\}$$
$$+\sum_{(c)\in\mathcal{L}-\{0\}}\{\sum_{s=1}^{\Gamma_{RND}-1}\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})^{\dagger}}_{\mathbf{T}(\mathbf{0})(\tau)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})}_{\mathbf{T}(\mathbf{0})(\tau)} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})}_{\mathbf{T}(\mathbf{c})(\tau+\mathbf{s})})\}]$$

$$+\sum_{c=0}^{|\mathcal{L}|}(\mathbf{\Lambda}_{(2c\Gamma_{MRD}+1)}^{ADJ(\mu_{APP})\dagger}\mathbf{P}_{(\tau)}^{(\mathbf{0})(\tau)}+\mathbf{\Lambda}_{(2c\Gamma_{MRD}+2)}^{ADJ(\mu_{APP})\dagger}\mathbf{P}_{(\tau)}^{(\mathbf{c})(\tau+\mathbf{1})}+\sum_{c=1}^{|\mathcal{L}|}\sum_{s=1}^{\Gamma_{RND}-1}\mathbf{\Lambda}_{(c-1)(\Gamma_{RND}-1)+s}^{LA(\mu_{APP})\dagger}\mathbf{P}_{T(0)(\tau)}^{(c)(\tau+s)}) \tag{6.36a}$$

$$\textbf{Subject to}: \forall\tau\in\Omega,\ \forall T_r\in T,\ \forall(c)\in\mathcal{L}$$

$$\textbf{Power}-\textbf{Balance Constraints}\ (\textbf{Base}-\textbf{Case}):$$

$$\sum_{g_q\in G}P_{g_q(\tau)}^{(0)(\tau)}=\sum_{D_d\in L}P_{D_d}^{(\tau)} \tag{6.36b}$$

$$\textbf{Flow Limit Constraints}\ (\textbf{Base}-\textbf{Case}\ \&\ \textbf{Contingency}):$$

$$|\Phi^{(0)}(\mathbf{P}_{\mathbf{g}(\tau)}^{(\mathbf{0})(\tau)}-\mathbf{P}_{\mathbf{D}}^{(\tau)})|\leq\overline{\mathbf{L}}^{(0)} \tag{6.36c}$$

$$|\Phi^{(c)}(\mathbf{P}_{\mathbf{g}(\tau)}^{(\mathbf{0})(\tau)}-\mathbf{P}_{\mathbf{D}}^{(\tau)})|\leq\overline{\mathbf{L}}^{(c)} \tag{6.36d}$$

$$\textbf{Flow}-\textbf{Limit Constraints}\ (\textbf{For Look}-\textbf{Ahead Intervals for RND}):$$

$$\forall T_r\in T,\ \forall\omega\in\{0,1,2,...(\Gamma_{RND}-1)\},\forall c\in\mathcal{L}-\{0\}$$

$$E_{\Gamma}^{(\omega)}[\psi_{init}^{(\tau)}]+(1-E_{\Gamma}^{(\omega)})[\psi_{amb}]+$$

$$\left(\frac{\alpha'}{\beta'}\right)\left[(P_{T_r(0)(\tau)}^{(0)(\tau)})^2E_0^{(\omega)}+(P_{T_r(0)(\tau)}^{(c)(\tau+\epsilon)})^2E_{\epsilon}^{(\omega)}+\sum_{s=1}^{\Gamma_{RND}-(\omega+1)}(P_{T_r(0)(\tau)}^{(c)(\tau+s)})^2E_s^{(\omega)}\right]$$

$$<\psi_{T_r}^{max} \tag{6.36e}$$

$$\textbf{Ramp}-\textbf{Rate Constraints}:$$

$$\underline{R}_{g_q}\leq P_{g_q(\tau)}^{(c)(\tau+1)}-P_{g_q(\tau)}^{(0)(\tau)}\leq\overline{R}_{g_q},\ \forall g_q\in G \tag{6.36f}$$

$$\underline{R}_{g_q}\leq P_{g_q(\tau)}^{(0)(\tau)}-P_{g_q}^{(0)(\tau-1)}\leq\overline{R}_{g_q},\ \forall g_q\in G \tag{6.36g}$$

(6.36h)

Iterates for Dispatch Intervals $(\tau+1)$ to $(\tau+\Gamma_{RND}-1)$

The optimization problems to be solved for these intervals corresponding to the different contingency scenarios, except the no-outage case, are just Economic Dispatch (ED) problems, with the additional ramp rate limit constraints and extra regularization terms in the objective function, corresponding to the consensus to be achieved among the beliefs of the values of decision variables to the adjacent coarse grains. The reason we are solving a series of EDs for these coarse grains is that, for these intervals, we allow the line flows to exceed the nominal rating and instead limit the temperature rise (which is enforced by the computation for the interval τ). Given below, is the optimization problem formulation for these coarse-grains.

$$\forall c \in \mathcal{L} - \{0\}, \forall s \in \{1, 2, ..., (\Gamma_{RND} - 1)\}$$

$$\begin{aligned}
\mathbf{P}^{(\mu_{\mathbf{APP}}+\mathbf{1})}_{\mathbf{(c)}(\tau+\mathbf{s})} = \underset{\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}}{\operatorname{argmin}} \sum_{g_q \in G} prob^{(c)(\tau+s)} C_{g_q}(P^{(c)(\tau+s)}_{g_q(c)}) + \boxed{\frac{\beta}{2} ||\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s})}||^2_2} \\
+\gamma[\boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s}-\mathbf{1})\dagger}_{(\mathbf{c})(\tau+\mathbf{s})} (\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s}-\mathbf{1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s}-\mathbf{1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s}-\mathbf{1})})} \\
+\boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})\dagger}_{(\mathbf{c})(\tau+\mathbf{s})} (\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s}-\mathbf{1})})} \\
+\boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})\dagger}_{(\mathbf{c})(\tau+\mathbf{s})} (\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s}+\mathbf{1})})} \\
+\boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s}+\mathbf{1})\dagger}_{(\mathbf{c})(\tau+\mathbf{s})} (\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s}+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s}+\mathbf{1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{s}+\mathbf{1})})} \\
+\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})\dagger}_{\mathbf{T}(\mathbf{c})(\tau+\mathbf{s})} (\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})}_{\mathbf{T}(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})}_{\mathbf{T}(\mathbf{0})(\tau)})] \\
+\mathbf{\Lambda}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}_{\mathbf{2c\Gamma_{MRD}}+(\mathbf{2s}+\mathbf{1})} \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{(\mathbf{c})(\tau+\mathbf{s})} + \mathbf{\Lambda}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}_{\mathbf{2c\Gamma_{MRD}}+(\mathbf{2s}+\mathbf{2})} \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s}+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{s})} \\
-\mathbf{\Lambda}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}_{\mathbf{2c\Gamma_{MRD}}+(\mathbf{2s}-\mathbf{1})} \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s}-\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{\Lambda}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}_{\mathbf{2c\Gamma_{MRD}}+\mathbf{2s}} \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{(\mathbf{c})(\tau+\mathbf{s})} \\
-\mathbf{\Lambda}^{\mathbf{LA}(\mu_{\mathbf{APP}})\dagger}_{(\mathbf{c}-\mathbf{1})(\mathbf{\Gamma_{RND}}-\mathbf{1})+\mathbf{s}} \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{\mathbf{T}(\mathbf{c})(\tau+\mathbf{s})}
\end{aligned} \tag{6.37a}$$

$$\textbf{Subject to :}$$

$$\textbf{Power} - \textbf{Balance Constraints :}$$

$$\sum_{g_q \in G} P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} = \sum_{D_d \in L} P^{(\tau+s)}_{D_d} \tag{6.37b}$$

$$\textbf{Ramp} - \textbf{Rate Constraints :}$$

$$\underline{R}_{g_q} \leq P_{g_q(c)(\tau+s)}^{(c)(\tau+s+1)} - P_{g_q(c)(\tau+s)}^{(c)(\tau+s)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{6.37c}$$

$$\underline{R}_{g_q} \leq P_{g_q(c)(\tau+s)}^{(c)(\tau+s)} - P_{g_q(c)(\tau+s)}^{(c)(\tau+s-1)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{6.37d}$$

Iterates for Dispatch Intervals $(\tau + \Gamma_{RND})$ **to** $(\tau + \Gamma_{MRD} - 1)$

The optimization problems to be solved for these intervals corresponding to the different contingency scenarios are Optimal Power Flow (OPF) problems, with the additional ramp rate limit constraints and extra regularization terms in the objective function, corresponding to the consensus to be achieved among the beliefs of the values of decision variables to the adjacent coarse grains. The reason we are solving a series of OPFs for these coarse grains is that, for these intervals, we allow the line flows to exceed the rating that willmake the system secure, and instead limit the flow by the nominal value, while eventually attempting to achieve secure dispatch by the end of the entire MRD. Given below is the optimization problem formulation for these coarse-grains.

$$\forall c \in \mathcal{L} - \{0\}, \forall s \in \{\Gamma_{RND}, \Gamma_{RND}+1, \Gamma_{RND}+2, ..., (\Gamma_{MRD}-1)\}$$

$$\begin{aligned}
\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mu_{\mathbf{APP}}+\mathbf{1})} = \underset{\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}}{\text{argmin}} \sum_{g_q \in G} prob^{(c)} C_{g_q}(P_{g_q(c)(\tau+s)}^{(c)(\tau+s)}) + \boxed{\frac{\beta}{2} ||\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mu_{\mathbf{APP}})}||_2^2} \\
+\gamma[\boxed{\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s-1})\dagger}(\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s-1})(\mu_{\mathbf{APP}})} - \mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s-1})}^{(\mathbf{c})(\tau+\mathbf{s-1})(\mu_{\mathbf{APP}})})} \\
+\boxed{\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s})\dagger}(\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})} - \mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s-1})}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})})} \\
+\boxed{\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s})\dagger}(\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})} - \mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s+1})}^{(\mathbf{c})(\tau+\mathbf{s})(\mu_{\mathbf{APP}})})} \\
+\boxed{\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s+1})\dagger}(\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s+1})(\mu_{\mathbf{APP}})} - \mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s+1})}^{(\mathbf{c})(\tau+\mathbf{s+1})(\mu_{\mathbf{APP}})})}] \\
+\mathbf{\Lambda}_{\mathbf{2c\Gamma_{MRD}+(2s+1)}}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s})} + \mathbf{\Lambda}_{\mathbf{2c\Gamma_{MRD}+(2s+2)}}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s+1})} \\
-\mathbf{\Lambda}_{\mathbf{2c\Gamma_{MRD}+(2s-1)}}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s-1})} - \mathbf{\Lambda}_{\mathbf{2c\Gamma_{MRD}+2s}}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{s})}^{(\mathbf{c})(\tau+\mathbf{s})}
\end{aligned} \tag{6.38a}$$

$$\textbf{Subject to}:$$

$$\textbf{Power} - \textbf{Balance Constraints}:$$

$$\sum_{g_q \in G} P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} = \sum_{D_d \in L} P^{(\tau+s)}_{D_d} \tag{6.38b}$$

$$\textbf{Flow} - \textbf{Limit Constraints (For Look} - \textbf{Ahead Intervals for MRD)}:$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{s})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{s})} - \mathbf{P}^{(\tau+\mathbf{s})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{6.38c}$$

$$\textbf{Ramp} - \textbf{Rate Constraints}:$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+s+1)}_{g_q(c)(\tau+s)} - P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{6.38d}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+s)}_{g_q(c)(\tau+s)} - P^{(c)(\tau+s-1)}_{g_q(c)(\tau+s)} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{6.38e}$$

Iterates for Dispatch Interval $(\tau + \Gamma_{MRD})$

The optimization problems to be solved for these intervals corresponding to the different contingency scenarios are $(N-1)$ Security Constrained Optimal Power Flow (SCOPF) problems, with the additional ramp rate limit constraints and extra regularization terms in the objective function, corresponding to the consensus to be achieved among the beliefs of the values of decision variables to the adjacent coarse grains. The security is attained with respect to the remaining set of contingencies, in each outage scenario. The beliefs of the decision variable values corresponding to the interval $(\tau + \Gamma_{MRD} + 1)$ are taken as those for the $(\mu_{APP} - 1)^{-th}$ iterate values of the $(\tau + \Gamma_{MRD})$ interval. Given below, is the optimization problem formulation for these coarse-grains.

$$\forall c \in \mathcal{L}$$

$$\mathbf{P}^{(\mu_{\mathbf{APP}}+\mathbf{1})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} = \underset{\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}}{\operatorname{argmin}} \sum_{g_q \in G} prob^{(c)} C_{g_q}(P^{(c)(\tau+\Gamma_{MRD})}_{g_q(c)(\tau+\Gamma_{MRD})})$$

$$+\boxed{\frac{\beta}{2}||\mathbf{P}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} - \mathbf{P}^{(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}||_2^2}$$

$$+\gamma[\boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}-1})\dagger}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}-1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}-1})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}-1})})}$$

$$+\boxed{\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})\dagger}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} - \mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})(\mu_{\mathbf{APP}})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}-1})})}]$$

$$-\mathbf{\Lambda}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}_{\mathbf{2\Gamma_{MRD}(c+1)-1}}\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}-1})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} - \mathbf{\Lambda}^{\mathbf{ADJ}(\mu_{\mathbf{APP}})\dagger}_{\mathbf{2\Gamma_{MRD}(c+1)}}\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}_{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} \tag{6.39a}$$

$$\textbf{Subject to :}$$

$$\textbf{Power} - \textbf{Balance Constraints :}$$

$$\sum_{g_q \in G} P^{(c)(\tau+\Gamma_{MRD})}_{g_q(c)(\tau+\Gamma_{MRD})} = \sum_{D_d \in L} P^{(\tau+\Gamma_{MRD})}_{D_d} \tag{6.39b}$$

$$\textbf{Flow} - \textbf{Limit Constraints (For Look} - \textbf{Ahead Intervals for MRD) :}$$

$$|\Phi^{(c)}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} - \mathbf{P}^{(\tau+\mathbf{\Gamma_{MRD}})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(0)} \tag{6.39c}$$

$$|\Phi^{(c\to c')}(\mathbf{P}^{(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})}_{\mathbf{g}(\mathbf{c})(\tau+\mathbf{\Gamma_{MRD}})} - \mathbf{P}^{(\tau+\mathbf{\Gamma_{MRD}})}_{\mathbf{D}})| \leq \overline{\mathbf{L}}^{(c\to c')} \tag{6.39d}$$

$$\textbf{Ramp} - \textbf{Rate Constraints :}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP})}_{g_q(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD})}_{g_q(c)(\tau+\Gamma_{MRD})} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{6.39e}$$

$$\underline{R}_{g_q} \leq P^{(c)(\tau+\Gamma_{MRD})}_{g_q(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD}-1)}_{g_q(c)(\tau+\Gamma_{MRD})} \leq \overline{R}_{g_q}, \forall g_q \in G \tag{6.39f}$$

The above coarse grained distribution and the exchange of beliefs as passing of messages can be pictorially represented as follows:

6.2.3 ADMM-PMP Based Fine Grained Distribution

We will now present the fine grained distribution, which is based on ADMM-Proximal Message Passing (PMP) algorithm. Following our style of presentation from the previous

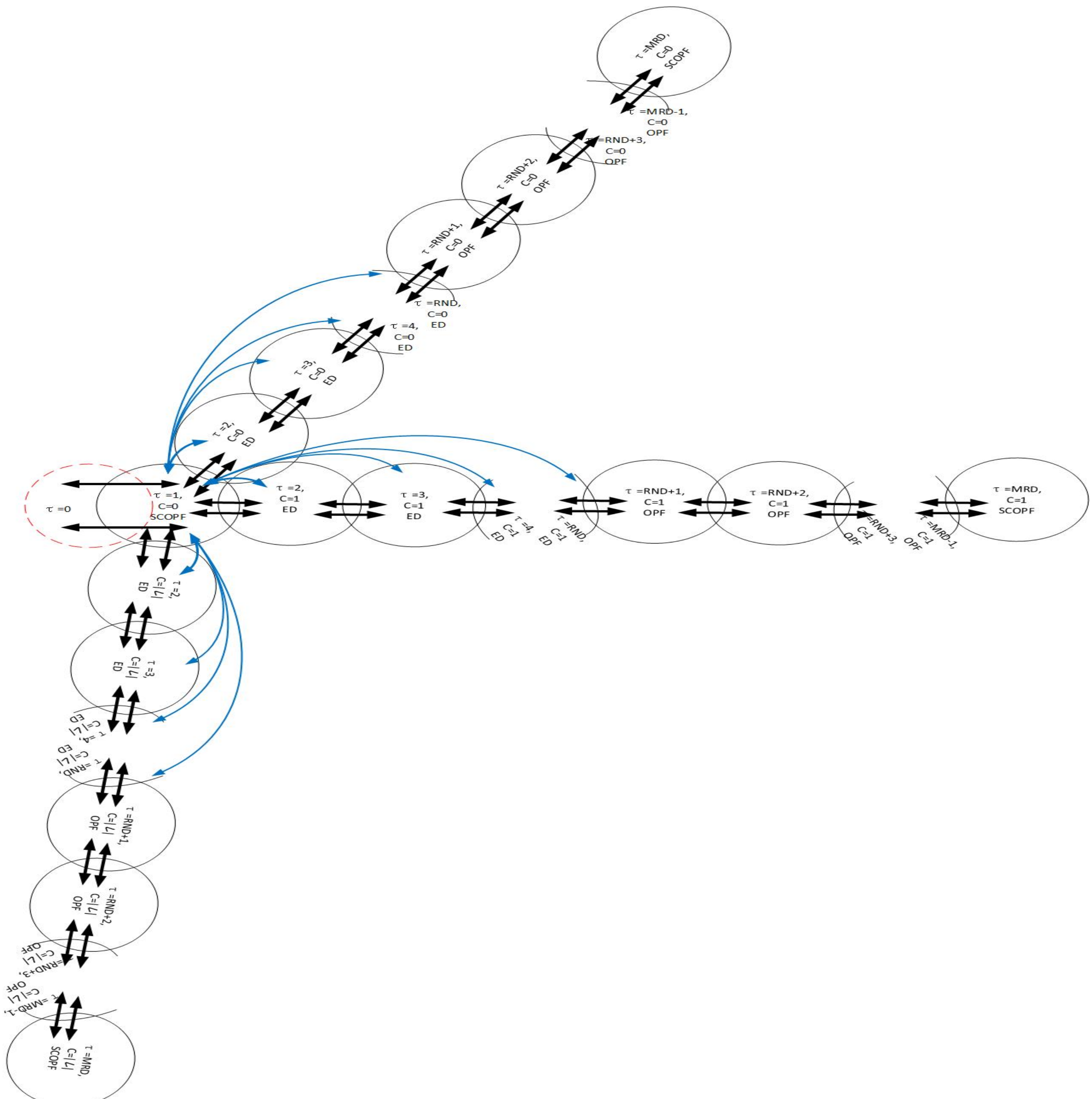


Figure 6.5: Auxiliary Proximal Message Passing for LASCOPF for post-contingency restoration

two chapters, we will first present the $\mathcal{DTN}$ reformulation, followed by the ADMM-PMP algorithmic formulation. Keeping in view the difference in nature of the problems to be solved belonging to the four different categories of the coarse grained decomposition, we will present below the formulations for each of the categories.

6.2.4 $\mathcal{DTN}$ Formulation Applied to the Temperature Limiting LASCOPF for Ensuring Security with respect to Next Outage: Generalized Multi Bus-Case

Now we will be dealing with ensuring security with respect to the next possible outages, after one outage is assumed to have taken place at the beginning of the upcoming dispatch interval and since, we assume that security is achieved in Γ_{MRD} dispatch intervals, starting from the dispatch interval following the upcoming one by virtue of generator ramping and redispatch, we will deal with $(\Gamma_{MRD} + 1)$ dispatch intervals in each 'roll' of the calculation, one for the upcoming time and the others for the Γ_{MRD} 'look-ahead' intervals. We have already presented the coarse-grained distribution. Therefore, in this case, we will split each of the components of the objective function further for upcoming time and Γ_{MRD} dispatch intervals belonging to three categories, viz:

- $(\tau + 1)$ to $(\tau + \Gamma_{RND} - 1)$
- $(\tau + \Gamma_{RND})$ to $(\tau + \Gamma_{MRD} - 1)$
- $(\tau + \Gamma_{MRD})$

.

Listed below are the components:

- **Cost of Generation (At Base Case, for τ):**
$C(\mathbf{P^{(0)(\tau)}}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|} (C_{g_q}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}) + I_{\leq}(\overline{P}_{g_q} - P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}) + I_{\leq}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - \underline{P}_{g_q}) + \frac{\beta}{2}[(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau)})^2 + \sum_{c=0}^{|\mathcal{L}|}(P^{(c)(\tau+1)}_{g_{q_{t_k}}(\tau)} - P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(\tau)})^2]$
$+\gamma[\sum_{c=0}^{|\mathcal{L}|}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)}(P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(\tau)} - P^{(0)(\tau)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)}) + P^{(c)(\tau+1)}_{g_{q_{t_k}}(\tau)}(P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(\tau)} - P^{(c)(\tau+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+1)}))]$
$+ \sum_{c=0}^{|\mathcal{L}|}(\lambda^{(\mu_{APP})}_{g_q(2c\Gamma_{MRD}+1)} P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} + \lambda^{(\mu_{APP})}_{g_q(2c\Gamma_{MRD}+2)} P^{(c)(\tau+1)}_{g_{q_{t_k}}(\tau)}))$

- **Cost of Generation (Contingency Cases, for $(\tau+1)$ to $(\tau+\Gamma_{RND}-1)$ and $(\tau+\Gamma_{RND})$ to $(\tau+\Gamma_{MRD}-1)$):**
$C(\mathbf{P^{(c)(\tau+s)}}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|} (C_{g_q}(P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)}) + I_{\leq}(\overline{P}_{g_q} - P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)}) + I_{\leq}(P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)} - \underline{P}_{g_q}) + \frac{\beta}{2}[(P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s)})^2 + (P^{(c)(\tau+s-1)}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s-1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s)})^2$
$+ (P^{(c)(\tau+s+1)}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s)})^2]$
$+\gamma[P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)}(P^{(c)(\tau+s)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s-1)}) + P^{(c)(\tau+s-1)}_{g_{q_{t_k}}(c)(\tau+s)}(P^{(c)(\tau+s-1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s-1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s-1)})$
$+P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)}(P^{(c)(\tau+s)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s+1)}) + P^{(c)(\tau+s+1)}_{g_{q_{t_k}}(c)(\tau+s)}(P^{(c)(\tau+s+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s+1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+s+1)})]$
$-\lambda^{(\mu_{APP})}_{g_q(2c\Gamma_{MRD}+2s-1)} P^{(c)(\tau+s-1)}_{g_{q_{t_k}}(c)(\tau+s)} - \lambda^{(\mu_{APP})}_{g_q(2c\Gamma_{MRD}+2s)} P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)} + \lambda^{(\mu_{APP})}_{g_q(2c\Gamma_{MRD}+2s+1)} P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)} + \lambda^{(\mu_{APP})}_{g_q(2c\Gamma_{MRD}+2s+2)} P^{(c)(\tau+s+1)}_{g_{q_{t_k}}(c)(\tau+s)})$

- **Cost of Generation (Contingency Cases, for $(\tau+\Gamma_{MRD})$):**
$C(\mathbf{P^{(c)(\tau+\Gamma_{MRD})}}) = \sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|} (C_{g_q}(P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})}) + I_{\leq}(\overline{P}_{g_q} - P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})}) + I_{\leq}(P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - \underline{P}_{g_q}) + \frac{\beta}{2}[(P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})})^2 + (P^{(c)(\tau+\Gamma_{MRD}-1)}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} -$

$P^{(c)(\tau+\Gamma_{MRD}-1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})})^2]$
$+ \gamma[P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})}(P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD}-1)}) +$
$P^{(c)(\tau+\Gamma_{MRD}-1)}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})}(P^{(c)(\tau+\Gamma_{MRD}-1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD}-1)(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD}-1)})]$
$- \lambda^{(\mu_{APP})}_{g_q(2\Gamma_{MRD}(c+1)-1)}P^{(c)(\tau+\Gamma_{MRD}-1)}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - \lambda^{(\mu_{APP})}_{g_q(2\Gamma_{MRD}(c+1))}P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})})$

- **Objective Function of Lines in τ for Temperature Constraints from $(\tau+1)$ to $(\tau+\Gamma_{RND}-1)$:**
  $C_T(\mathbf{P^{(c)(\tau)}_T}) = \sum_{(c)\in\mathcal{L}-\{0\}}\sum_{T_r\in T}\sum_{t_k\in T_r\cap\mathfrak{T}}\sum_{s=1}^{(\Gamma_{RND}-1)}\{\frac{\beta}{2}[(P^{(c)(\tau+s)}_{T_{rt_k}(\tau)} - P^{(c)(\tau+s)(\mu_{APP})}_{T_{rt_k}(\tau)})^2]+$
  $\gamma[P^{(c)(\tau+s)}_{T_{rt_k}(\tau)}(P^{(c)(\tau+s)(\mu_{APP})}_{T_{rt_k}(\tau)} - P^{(c)(\tau+s)(\mu_{APP})}_{T_{rt_k}(c)(\tau+s)})] + \lambda^{(\mu_{APP})}_{T_{rt_k}(c-1)(\Gamma_{MRD}-1)+s}P^{(c)(\tau+s)}_{T_{rt_k}(\tau)}\}$

- **Temperature Constraints of Lines in τ from $(\tau+1)$ to $(\tau+\Gamma_{RND}-1)$:**
  $\Psi_T(\mathbf{P^{(c)(\tau)}_T}) = \sum_{(c)\in\mathcal{L}-\{0\}}\sum_{T_r\in T}\sum_{t_k\in T_r\cap\mathfrak{T}}\sum_{\omega=0}^{(\Gamma_{RND}-1)}\{I_{\leq}(\psi^{max}_{T_r} - (E^{(\omega)}_{\Gamma}[\psi^{(\tau)}_{init}] + (1-E^{(\omega)}_{\Gamma})[\psi_{amb}]+$

  $(\frac{\alpha'}{\beta'})[(P^{(0)(\tau)}_{T_{rt_k}(0)(\tau)})^2E^{(\omega)}_0 + (P^{(c)(\tau+\epsilon)}_{T_{rt_k}(0)(\tau)})^2E^{(\omega)}_{\epsilon} + \sum_{s=1}^{\Gamma_{RND}-(\omega+1)}(P^{(c)(\tau+s)}_{T_{rt_k}(0)(\tau)})^2E^{(\omega)}_s]))\}$

- **Objective Function of Lines belonging to $(\tau+1)$ to $(\tau+\Gamma_{RND}-1)$ for Temperature Constraints:**
  $C_T(\mathbf{P^{(c)(\tau+s)}_T}) = \sum_{T_r\in T}\sum_{t_k\in T_r\cap\mathfrak{T}}\{\frac{\beta}{2}[(P^{(c)(\tau+s)}_{T_{rt_k}(c)(\tau+s)} - P^{(c)(\tau+s)(\mu_{APP})}_{T_{rt_k}(c)(\tau+s)})^2]+$
  $\gamma[P^{(c)(\tau+s)}_{T_{rt_k}(c)(\tau+s)}(P^{(c)(\tau+s)(\mu_{APP})}_{T_{rt_k}(c)(\tau+s)} - P^{(c)(\tau+s)(\mu_{APP})}_{T_{rt_k}(\tau)})] - \lambda^{(\mu_{APP})}_{T_{rt_k}(c-1)(\Gamma_{MRD}-1)+s}P^{(c)(\tau+s)}_{T_{rt_k}(c)(\tau+s)}\}$

- **Line Flow Limit Constraint ($(N-1)$ Secure, for τ):**
  $F(\mathbf{P^{(c)(\tau)}}) = \sum_{(c)\in\mathcal{L}}\sum_{T_r\in T}\sum_{t_k\in T_r\cap\mathfrak{T}}I_{\leq}(\overline{L}^{(c)}_{T_r} - |P^{(c)(\tau)}_{T_{rt_k}(\tau)}|)$

- **Line Flow Limit Constraint (for $(\tau+\Gamma_{RND})$ to $(\tau+\Gamma_{MRD}-1)$):**

$F(\mathbf{P^{(c)(\tau+s)}}) = \sum_{T_r \in T} \sum_{t_k \in T_r \cap \mathfrak{T}} I_{\leq}(\overline{L}_{T_r}^{(c)} - |P_{T_{rt_k}(c)(\tau+s)}^{(c)(\tau+s)}|)$

- **Line Flow Limit Constraint ($(N-1)$ Secure, for $(\tau+\Gamma_{MRD})$):**
$F(\mathbf{P^{(c \to c')(\tau+\Gamma_{MRD})}}) = \sum_{T_r \in T} \sum_{t_k \in T_r \cap \mathfrak{T}} I_{\leq}(\overline{L}_{T_r}^{(c)} - |P_{T_{rt_k}(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})}|) +$
$\sum_{(c') \in [\mathcal{L}-\{c\}]} \sum_{T_r \in T} \sum_{t_k \in T_r \cap \mathfrak{T}} I_{\leq}(\overline{L}_{T_r}^{(c')} - |P_{T_{rt_k}(c)(\tau+\Gamma_{MRD})}^{(c')(\tau+\Gamma_{MRD})}|), \forall (c) \in \mathcal{L}$

- **Power-Angle Relation ($(N-1)$ Secure, for τ):**
$\chi(\mathbf{P^{(c)(\tau)}}, \theta^{(\mathbf{c})(\tau)}) = \sum_{(c) \in \mathcal{L}} \sum_{T_r \in T} \sum_{t_k, t_{k'} \in T_r \cap \mathfrak{T}} I_{=}(P_{T_{rt_k}(\tau)}^{(c)(\tau)} + \frac{\theta_{T_{rt_k}(\tau)}^{(c)(\tau)} - \theta_{T_{rt_{k'}}(\tau)}^{(c)(\tau)}}{X_{T_r}^{(c)}})$

- **Power-Angle Relation (for $(\tau+1)$ to $(\tau+\Gamma_{RND}-1)$, $(\tau+\Gamma_{RND})$ to $(\tau+\Gamma_{MRD}-1)$):**
$\chi(\mathbf{P^{(c)(\tau+s)}}, \theta^{(\mathbf{c})(\tau+s)}) = \sum_{T_r \in T} \sum_{t_k, t_{k'} \in T_r \cap \mathfrak{T}} I_{=}(P_{T_{rt_k}(\tau+s)}^{(c)(\tau+s)} + \frac{\theta_{T_{rt_k}(\tau+s)}^{(c)(\tau+s)} - \theta_{T_{rt_{k'}}(\tau+s)}^{(c)(\tau+s)}}{X_{T_r}^{(c)}})$

- **Power-Angle Relation ($(N-1)$ Secure, for $\tau+\Gamma_{MRD}$):**
$\chi(\mathbf{P^{(c \to c')(\tau+\Gamma_{MRD})}}, \theta^{(\mathbf{c \to c'})(\tau+\mathbf{\Gamma_{MRD}})}) = \sum_{T_r \in T} \sum_{t_k, t_{k'} \in T_r \cap \mathfrak{T}} I_{=}(P_{T_{rt_k}(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})} +$
$\frac{\theta_{T_{rt_k}(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})} - \theta_{T_{rt_{k'}}(c)(\tau+\Gamma_{MRD})}^{(c)(\tau+\Gamma_{MRD})}}{X_{T_r}^{(c)}}) + \sum_{(c') \in [\mathcal{L}-\{c\}]} \sum_{T_r \in T} \sum_{t_k, t_{k'} \in T_r \cap \mathfrak{T}} I_{=}(P_{T_{rt_k}(c)(\tau+\Gamma_{MRD})}^{(c')(\tau+\Gamma_{MRD})} +$
$\frac{\theta_{T_{rt_k}(c)(\tau+\Gamma_{MRD})}^{(c')(\tau+\Gamma_{MRD})} - \theta_{T_{rt_{k'}}(c)(\tau+\Gamma_{MRD})}^{(c')(\tau+\Gamma_{MRD})}}{X_{T_r}^{(c')}}), \forall (c) \in \mathcal{L}$

- **Ramp Constraint for τ:**
$\Delta(\mathbf{P^{(0)(\tau)}}) = \sum_{(c) \in \mathcal{L}} (\sum_{t_k \in g_q \cap \mathfrak{T}, q=1}^{|G|} (I_{\leq}(\overline{R}_{g_q} - P_{g_{q_{t_k}}(\tau)}^{(c)(\tau+1)} + P_{g_{q_{t_k}}(\tau)}^{(0)(\tau)}) + I_{\leq}(P_{g_{q_{t_k}}(\tau)}^{(c)(\tau+1)} - P_{g_{q_{t_k}}(\tau)}^{(0)(\tau)} -$
$\underline{R}_{g_q}))$

$+ I_{\leq}(\overline{R}_{g_q} - P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} + P^{(0)}_{g_{q_{t_k}}}) + I_{\leq}(P^{(0)(\tau)}_{g_{q_{t_k}}(\tau)} - P^{(0)}_{g_{q_{t_k}}} - \underline{R}_{g_q}))$

- **Ramp Constraint for** $(\tau + 1)$ **to** $(\tau + \Gamma_{MRD} - 1)$**:**
$\Delta(\mathbf{P^{(c)(\tau+s)}}) = \sum_{t_k \in g_q \cap \mathcal{T}, q=1}^{|G|}(I_{\leq}(\overline{R}_{g_q} - P^{(c)(\tau+s+1)}_{g_{q_{t_k}}(c)(\tau+s)} + P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)}) + I_{\leq}(P^{(c)(\tau+s+1)}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)} - \underline{R}_{g_q})$
$+ I_{\leq}(\overline{R}_{g_q} - P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)} + P^{(c)(\tau+s-1)}_{g_{q_{t_k}}(c)(\tau+s)}) + I_{\leq}(P^{(c)(\tau+s)}_{g_{q_{t_k}}(c)(\tau+s)} - P^{(c)(\tau+s-1)}_{g_{q_{t_k}}(c)(\tau+s)} - \underline{R}_{g_q}))$

- **Ramp Constraint for** $(\tau + \Gamma_{MRD})$**:**
$\Delta(\mathbf{P^{(c)(\tau+\Gamma_{MRD})}}) = \sum_{t_k \in g_q \cap \mathcal{T}, q=1}^{|G|}(I_{\leq}(\overline{R}_{g_q} - P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} + P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})}$
$+ I_{\leq}(P^{(c)(\tau+\Gamma_{MRD})(\mu_{APP})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - \underline{R}_{g_q}))$
$+ I_{\leq}(\overline{R}_{g_q} - P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} + P^{(c)(\tau+\Gamma_{MRD}-1)}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})}) + I_{\leq}(P^{(c)(\tau+\Gamma_{MRD})}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - P^{(c)(\tau+\Gamma_{MRD}-1)}_{g_{q_{t_k}}(c)(\tau+\Gamma_{MRD})} - \underline{R}_{g_q}), \forall (c) \in \mathcal{L})$

The reformulated OPF Problems for this case are as follow:

$$\forall \tau$$

$$\begin{aligned} \min_{\mathbf{P^{(c)(\tau)}_{t_k}}, \theta^{(\mathbf{c})(\tau)}_{\mathbf{t_k}}} f(\mathbf{P}, \theta) = C(\mathbf{P^{(0)(\tau)}}) + C_T(\mathbf{P^{(c)(\tau)}_T}) + \\ \Psi_T(\mathbf{P^{(c)(\tau)}_T}) + F(\mathbf{P^{(c)(\tau)}}) + \\ \chi(\mathbf{P^{(c)(\tau)}}, \theta^{(\mathbf{c})(\tau)}) + \Delta(\mathbf{P^{(0)(\tau)}}) \end{aligned} \tag{6.40a}$$

$$\text{Subject to: } \hat{P}^{(c)(\tau)}_{N_{i_{t_k}}(\tau)} = 0, \tilde{\theta}^{(c)(\tau)}_{N_{i_{t_k}}(\tau)} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall (c) \in \mathcal{L}, \forall \tau \in \Omega \tag{6.40b}$$

$$\forall(\tau+1) \text{ to } (\tau+\Gamma_{RND}-1)$$

$$\min_{\mathbf{P_{t_k}^{(c)(\tau+s)}},\theta_{\mathbf{t_k}}^{\mathbf{(c)}(\tau+s)}} f(\mathbf{P},\theta) = C(\mathbf{P^{(c)(\tau+s)}}) + C_T(\mathbf{P_T^{(c)(\tau+s)}}) + \chi(\mathbf{P^{(c)(\tau+s)}},\theta^{\mathbf{(c)(\tau+s)}}) + \Delta(\mathbf{P^{(c)(\tau+1)}}) \tag{6.41a}$$

$$\text{Subject to: } \hat{P}^{(c)(\tau+s)}_{N_{it_k}(c)(\tau+s)} = 0, \tilde{\theta}^{(c)(\tau+s)}_{N_{it_k}(c)(\tau+s)} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall(c) \in \mathcal{L}, \forall\tau \in \Omega \tag{6.41b}$$

$$\forall(\tau+\Gamma_{RND}) \text{ to } (\tau+\Gamma_{MRD}-1)$$

$$\min_{\mathbf{P_{t_k}^{(c)(\tau+s)}},\theta_{\mathbf{t_k}}^{\mathbf{(c)}(\tau+s)}} f(\mathbf{P},\theta) = C(\mathbf{P^{(c)(\tau+s)}}) + F(\mathbf{P^{(c)(\tau+s)}}) + \chi(\mathbf{P^{(c)(\tau+s)}},\theta^{\mathbf{(c)(\tau+s)}}) + \Delta(\mathbf{P^{(c)(\tau+s)}}) \tag{6.42a}$$

$$\text{Subject to: } \hat{P}^{(c)(\tau+s)}_{N_{it_k}(c)(\tau+s)} = 0, \tilde{\theta}^{(c)(\tau+s)}_{N_{it_k}(c)(\tau+s)} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall(c) \in \mathcal{L}, \forall\tau \in \Omega \tag{6.42b}$$

$$\forall(\tau+\Gamma_{MRD})$$

$$\min_{\mathbf{P_{t_k}^{(c)(\tau+\Gamma_{MRD})}},\theta_{\mathbf{t_k}}^{\mathbf{(c)}(\tau+\mathbf{\Gamma_{MRD}})}} f(\mathbf{P},\theta) = C(\mathbf{P^{(c)(\tau+\Gamma_{MRD})}}) + F(\mathbf{P^{(c\to c^{'})(\tau+\Gamma_{MRD})}}) + \chi(\mathbf{P^{(c\to c^{'})(\tau+\Gamma_{MRD})}},\theta^{\mathbf{(c\to c^{'})(\tau+\Gamma_{MRD})}}) + \Delta(\mathbf{P^{(c)(\tau+\Gamma_{MRD})}}) \tag{6.43a}$$

$$\text{Subject to: } \hat{P}^{(c\to c^{'})(\tau+\Gamma_{MRD})}_{N_{it_k}(c)(\tau+\Gamma_{MRD})} = 0, \tilde{\theta}^{(c\to c^{'})(\tau+\Gamma_{MRD})}_{N_{it_k}(c)(\tau+\Gamma_{MRD})} = 0, \forall N_i \in \mathcal{N}, \forall t_k \in \mathcal{T}, \forall(c) \in \mathcal{L}, \forall(c^{'}) \in [\mathcal{L}-\{c\}] \cap \{c\}, \forall\tau \in \Omega \tag{6.43b}$$

The rest of the derivation for the proximal messages follow the exact same style, as presented in the last chapter. The only difference is that, now, there will be actual objective functions associated with the transmission lines from τ to $(\tau+\Gamma_{RND}-1)$ for enforcing the temperature constraints, rather than them being simply indicator functions. We will now move on to the next chapter, where we will present the approach to an object oriented programming for implementing the above-mentioned algorithms.

Chapter 7

An Object Oriented Software Design Approach to LASCOPF

[1]In this chapter, we will describe the object oriented design approach to developing the software for implementing the mathematical optimization models of the different problems, we have built in the foregoing chapters. Even though, the software architecture, we will describe in this chapter, can be used for coding in any object oriented language, since we have developed all the codes in C++11, we will implicitly assume that the architecture complies with the construct of that language. However, with minor or no modifications, the same ones can be used for Java, Python, C#, Ruby etc.

7.1 Software Model for OPF and SCOPF Solvers

We will first mention the software models for OPF and SCOPF solvers. Before we begin, we will introduce the conventions to be used throughout the chapter.

[1]Parts of this chapter appear in the published papers, "Security Constrained Optimal Power Flow via Proximal Message Passing," "Toward Distributed/Decentralized DC Optimal Power Flow Implementation in Future Electric Power Systems," and "A Survey of Distributed Optimization and Control Algorithms for Electric Power Systems." The author of this treatise is the first author of the first paper, contributed section V, parts of sections IX and X of the second paper, and contributed parts of section III and V of the third paper.

7.1.1 Conventions

The diagrammatic conventions are shown in figure 7.1. The STL vector refers to an object of the vector container class, which is part of the Standard Template Library (STL) in C++.

7.1.2 First Step: Creation of the Network Object

In this step, as it can be seen from figure 7.2, the main method instantiates the object, "networkObject" of the class, "Network" by invoking the constructor function, and passing, as an argument, the number of nodes of the network. The constructor, in turn calls another method, "setNetworkVariables". It is within this method, that the creation of the objects corresponding to Generators, Transmission Lines, Loads, and Nodes, as shown in figures 7.3, 7.4, 7.5, 7.6, and 7.7, takes place. The other method, that the main invokes is the "runSimulation" method, which actually performs the optimization calculation.

7.1.3 Instantiation of the Generator Objects and Gensolver Objects

The generator is the only object, which needs an actual solver object to solve its optimization problem, because for all other devices, as we have described previously, their optimization problems can be solved analytically as a closed form solution. The solver object "genSolver" objects corresponding to each generator object, "genObject" are created within the "setNetworkVariables" method of the network object, and the solver objects are passed

Figure 7.1: Conventions

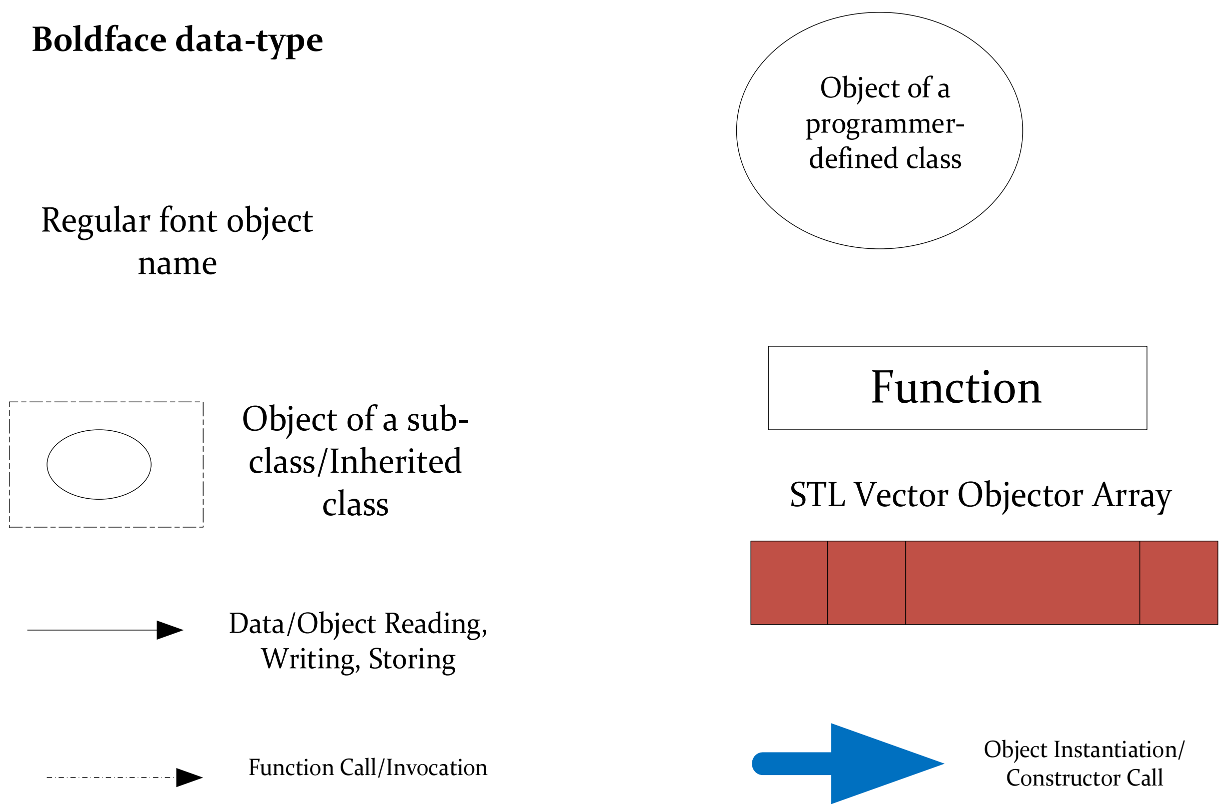
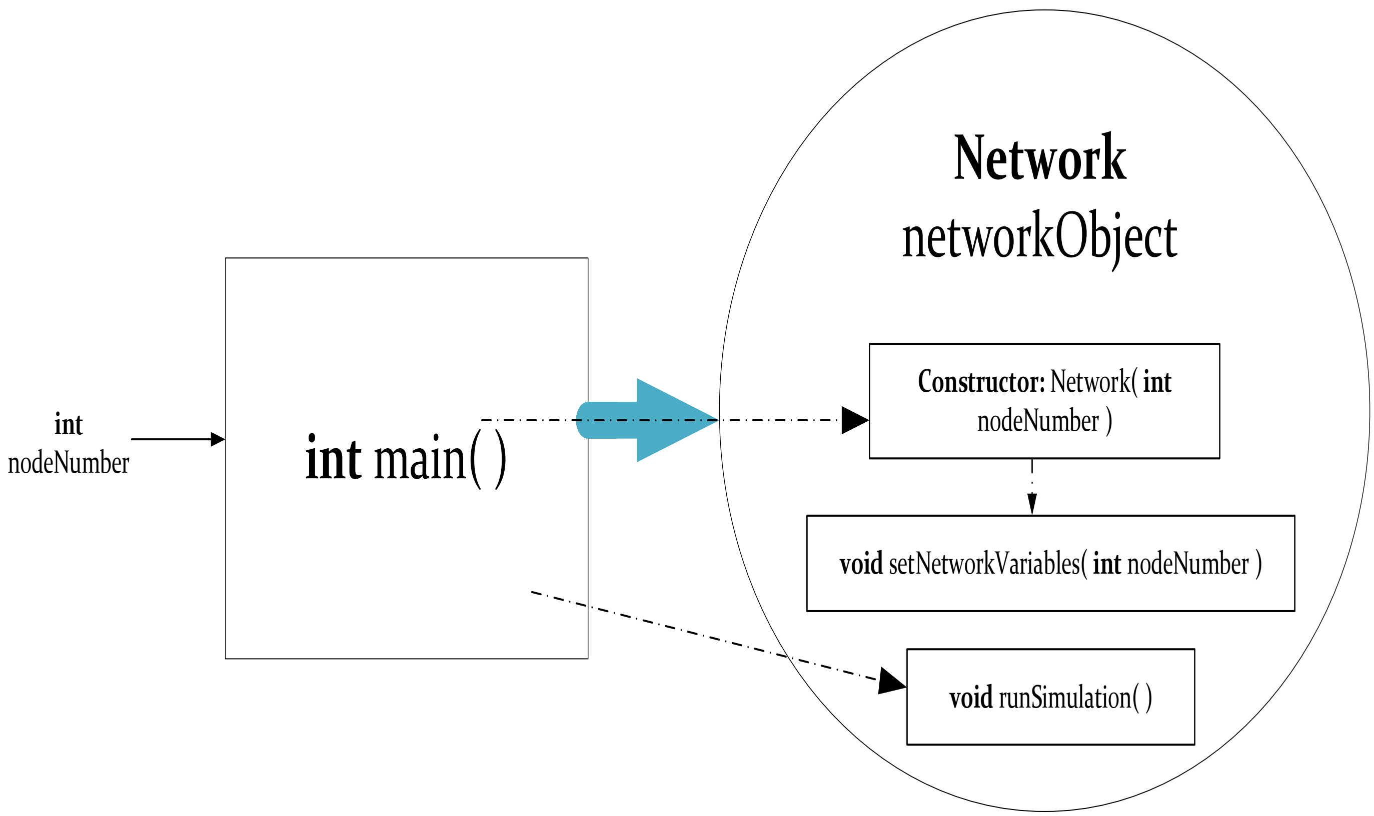


Figure 7.2: First Step: Creation of the Network Object



as arguments of the respective generator object constructors. The whole process is shown in figure 7.3.

7.1.4 Initialization of Network: Handles for Generators (OPF)

As it can be seen from figure 7.4, the objects corresponding to different devices (Generators, Transmission lines, and Loads), as well as those corresponding to nodes are created an stored in respective STL vector container class objects. In order to build up the connectivity of the network, it is necessary to mention to which nodes the generators are connected. This is done by including a data member of the generator objects, which is the pointer of the node object to which a particular generator is connected. Alternatively, we can also pass a reference to the node object as the data member instead of the pointer. Either of these two approaches is called, creating a handle.

7.1.5 Initialization of Network: Handles for Transmission Lines (OPF)

Similarly as before, for each transmission line object we create two handles, each corresponding to the node to which each end of a particular transmission line is connected. This is shown in figure 7.5.

7.1.6 Initialization of Network: Handles for Loads (OPF)

In figure 7.6, we have shown the creation of handles for loads.

Figure 7.3: Instantiation of the Generator Objects and Gensolver Objects

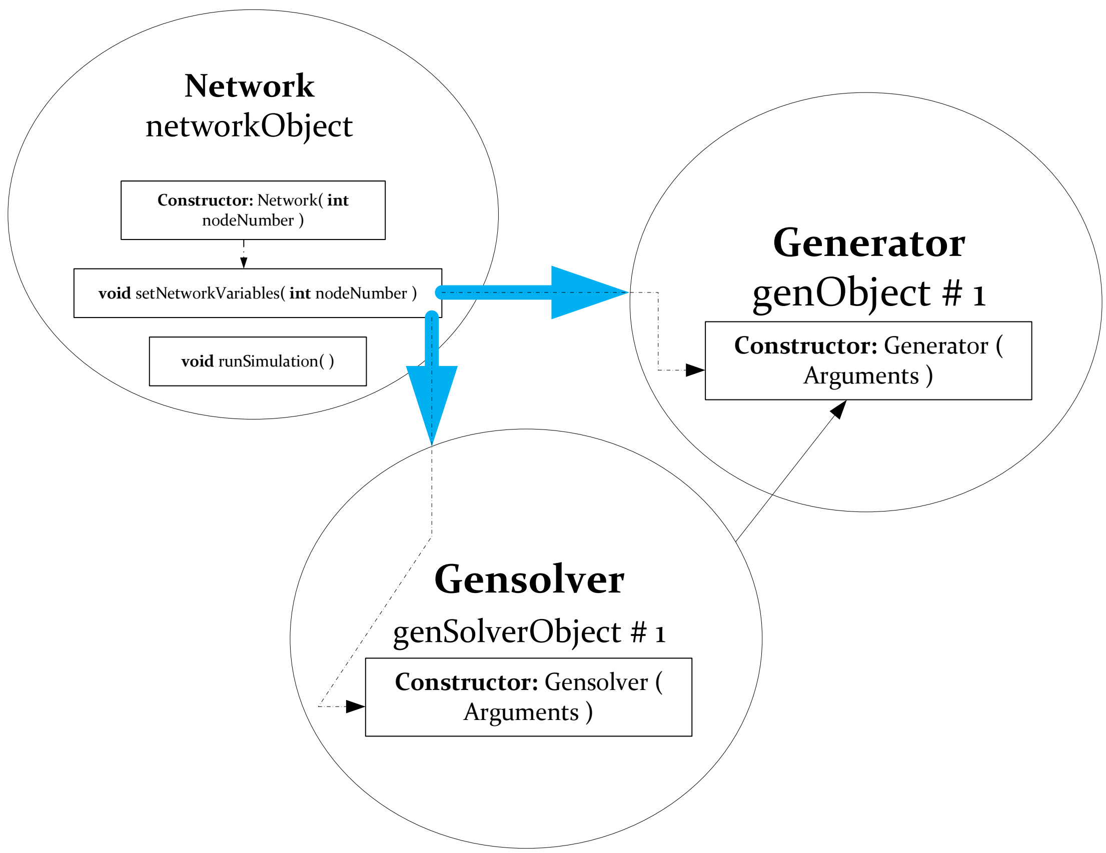


Figure 7.4: Initialization of Network: Handles for Generators (OPF)

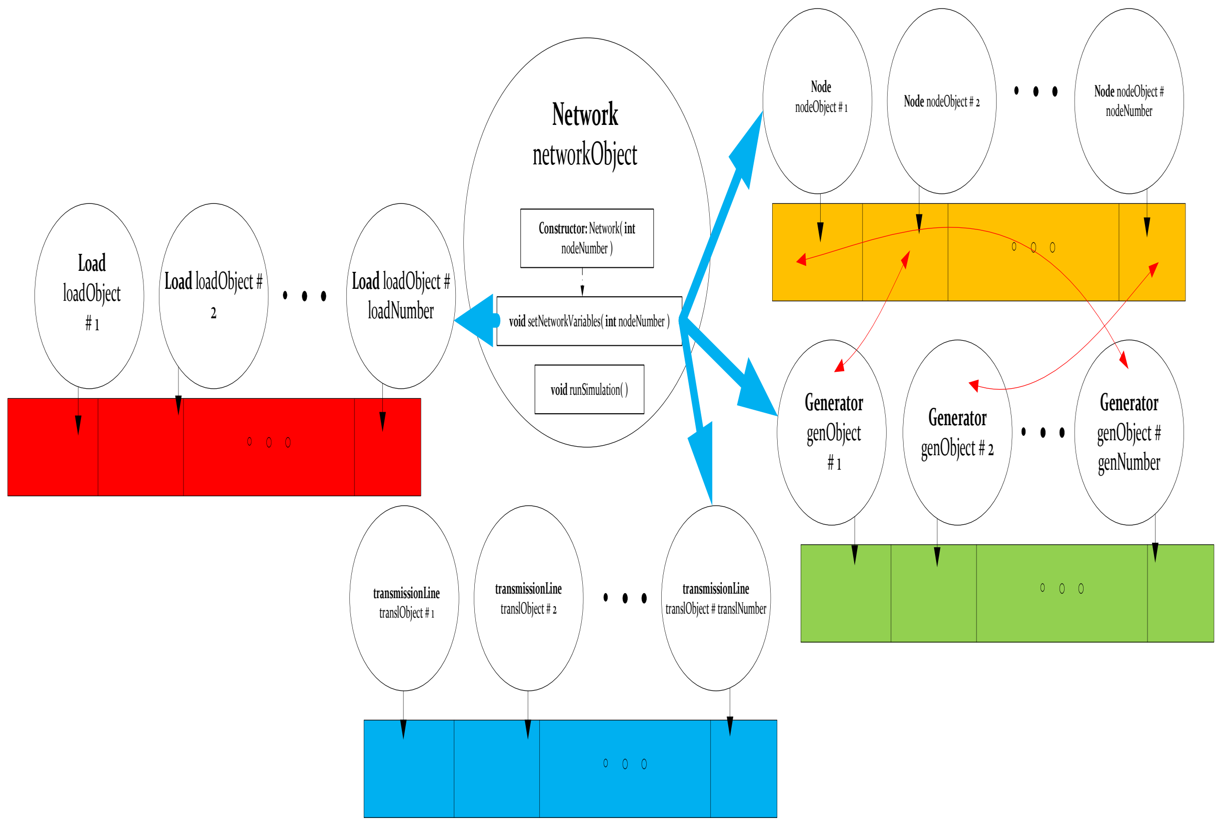


Figure 7.5: Initialization of Network: Handles for Transmission Lines (OPF)

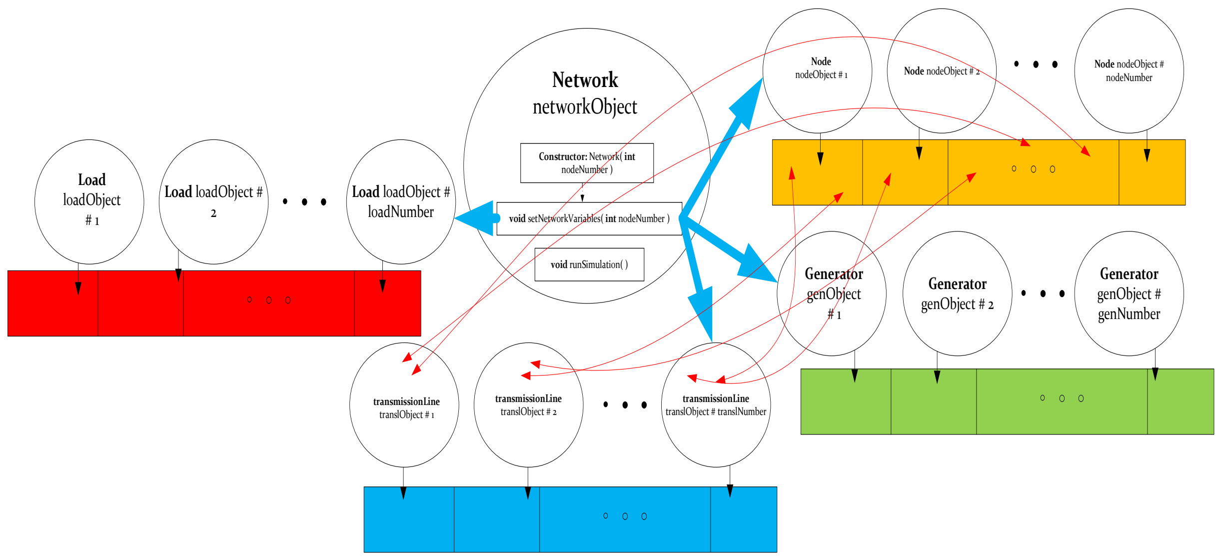
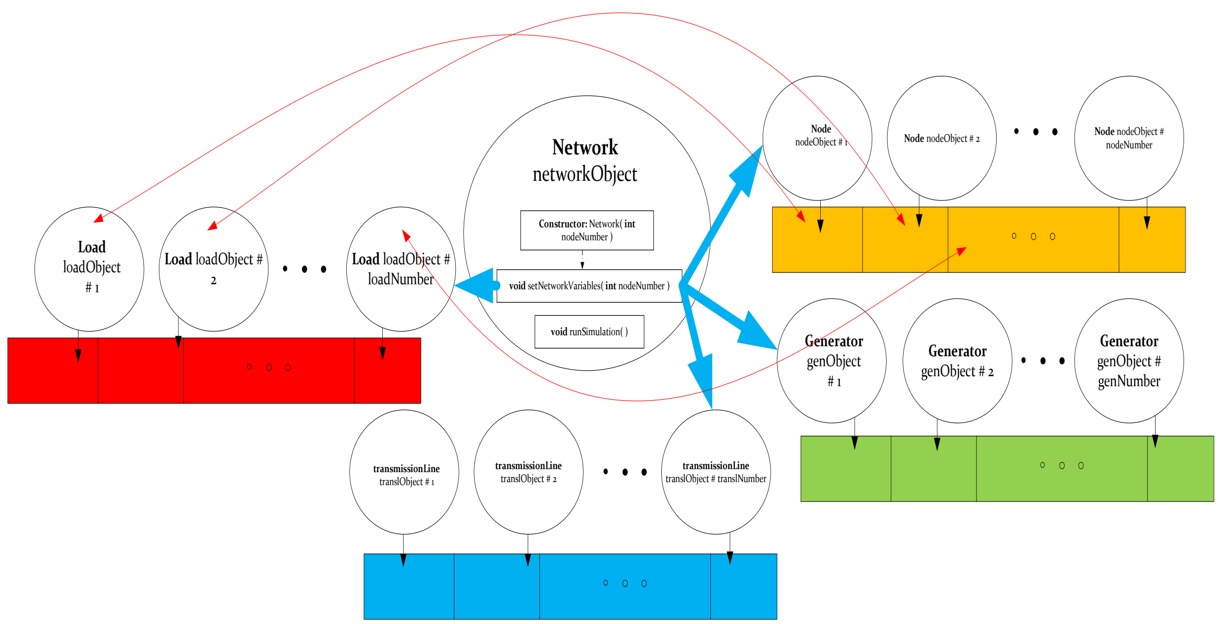


Figure 7.6: Initialization of Network: Handles for Loads (OPF)



7.1.7 Initialization of Network for OPF

Figure 7.7 shows the formation of the entire network. This completes the network initialization phase, through which we create a software image of the actual physical network.

7.1.8 Initialization of Network for SCOPF $(N-1)$ Line Contingencies

The only difference here, as shown in figure 7.8 from the OPF software is that, in this case, for each load and each transmission line, we create several copies of them, corresponding to each contingency scenario. The copies are objects of the subclass of loads and transmission lines. It is to be obsrved that, for the transmission lines, corresponding to each contingency scenario, one of the lines is outaged, which we simulate by letting the corresponding copy of the particular transmission line under consideration, have very high reactance and very low line limit. We have colored those particular copies in violet.

7.2 Software Model for LASCOPF Solvers

For the LASCOPF solvers, where we are using the APMP algorithm, we need to create multiple network objects, corresponding to each dispatch intervaland/or contingency scenario, as shown in figure 7.9. The rest of the process is exactly same as before.

Figure 7.7: Initialization of Network for OPF

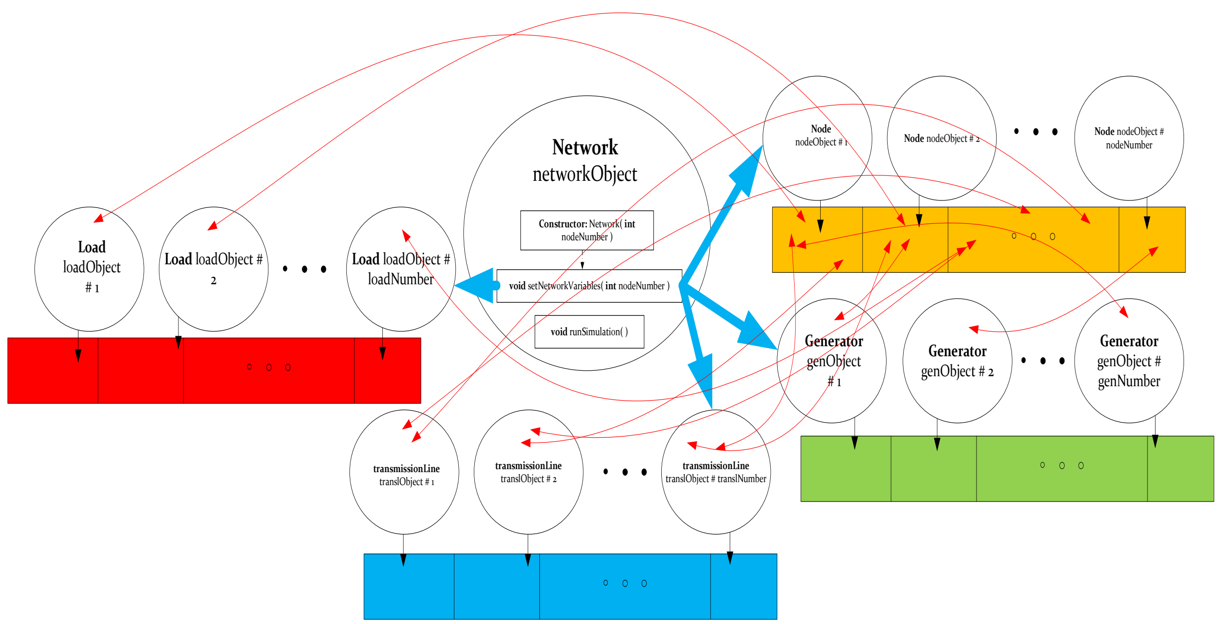


Figure 7.8: Initialization of Network for SCOPF $(N-1)$ Line Contingencies

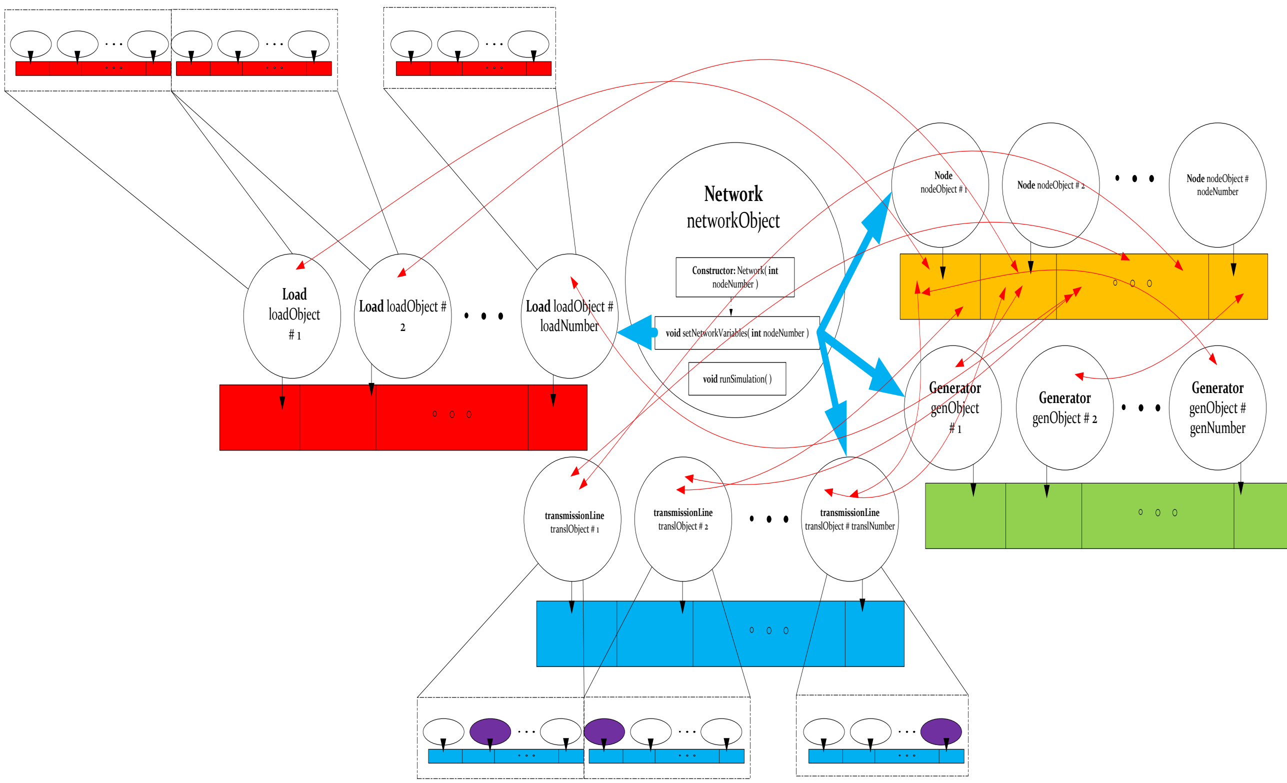
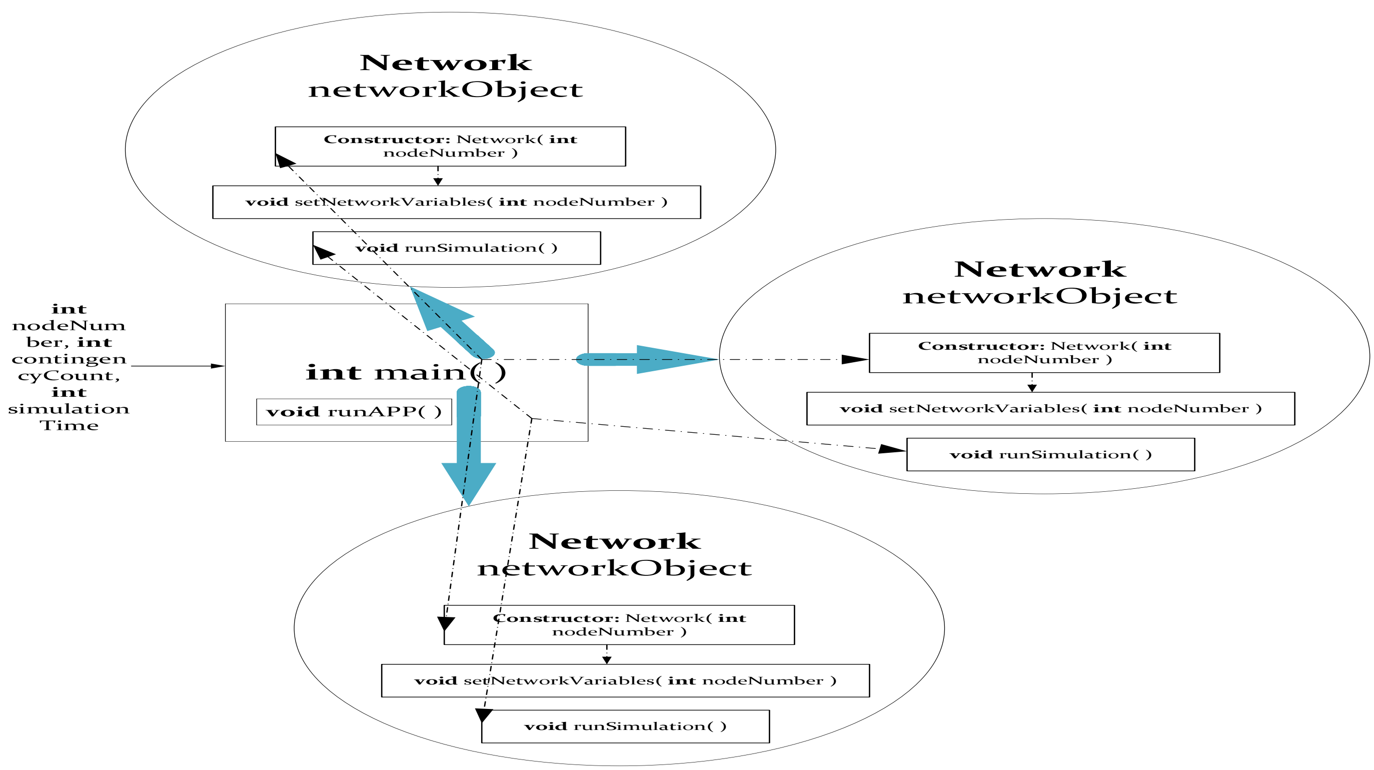


Figure 7.9: Software Model for LASCOPF Solvers

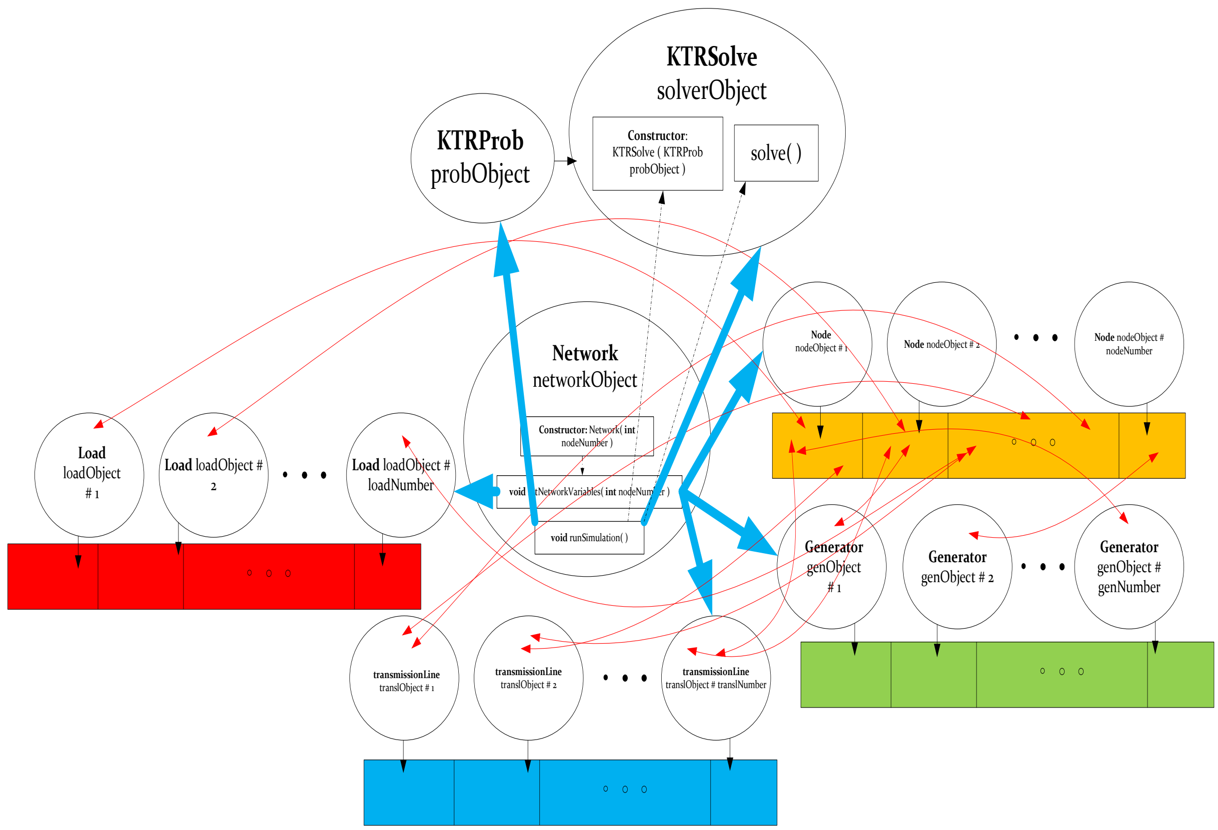


7.2.1 Software Model for Centralized Solvers

Lastly, for the sake of comparison, we have shown in figure 7.10, how a typical software, written for a centralized solver would look like. Here, we can see that we need to create a problem object and a solver object (shown in the figure as “KTRProb” and “KTRSolve” respectively). These problem object and solver objects are external to the device objects of the network, unlike the previous case.

Thus, with the software model and coding approach and methodology described in this chapter, we will now look at some of the results of the simulation studies, in the next chapter.

Figure 7.10: Software Model for Centralized Solvers



Chapter 8

Numerical Examples

8.1 Numerical examples

[1]We have subdivided this entire section into two subsections, the first one for describing the results of the OPF, and the next one for describing the results of the SCOPF. For the first section, we will describe the results for 5 bus system, and also the IEEE test systems with 14, 30, 57, and 118 buses. For the next section, we currently will describe the results for the simple systems with 3, and 5 buses. The 5 bus and 3 bus systems are shown in figures 8.2 and 8.1 respectively (The functions adjacent to the generators describe the cost functions, whereas the numbers in the parantheses are the generating limits). The details about the IEEE systems can be found at the University of Washington Power Systems Archive (https://www.ee.washington.edu/research/pstca/):

We have run all these simulations on a Dell Inspiron 17R laptop computer powered by a 4x Intel(R) Core(TM) i7-4500U CPU running at 1.80 GHz, with a RAM of capacity 8054 MB and the OS is Ubuntu 14.04.4 LTS. We have coded all the simulation programs

[1]Parts of this chapter appear in the published papers, "Security Constrained Optimal Power Flow via Proximal Message Passing," "Toward Distributed/Decentralized DC Optimal Power Flow Implementation in Future Electric Power Systems," and "A Survey of Distributed Optimization and Control Algorithms for Electric Power Systems." The author of this treatise is the first author of the first paper, contributed section V, parts of sections IX and X of the second paper, and contributed parts of section III and V of the third paper.

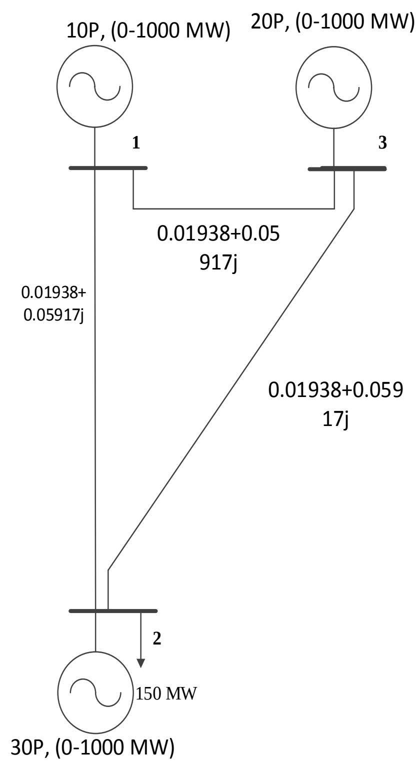


Figure 8.1: The 3 Bus Power System.

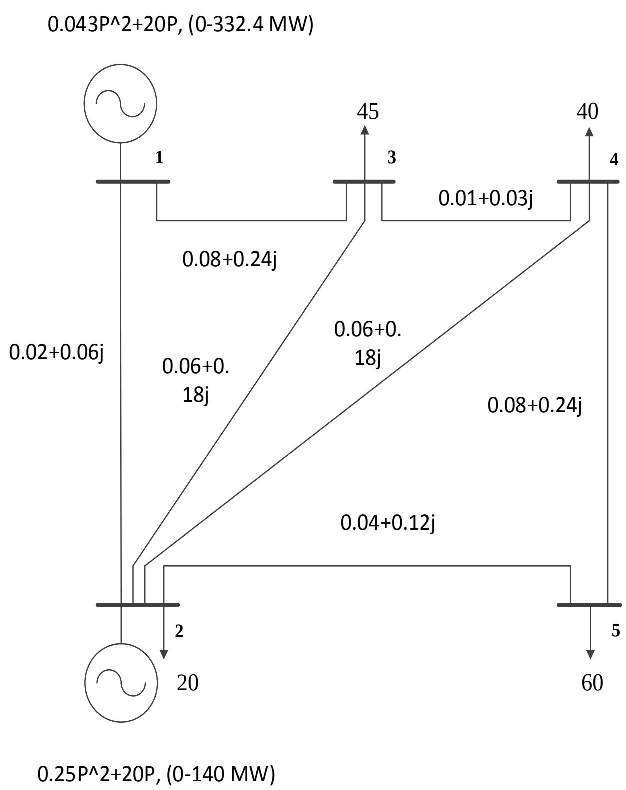


Figure 8.2: The 5 Bus Power System.

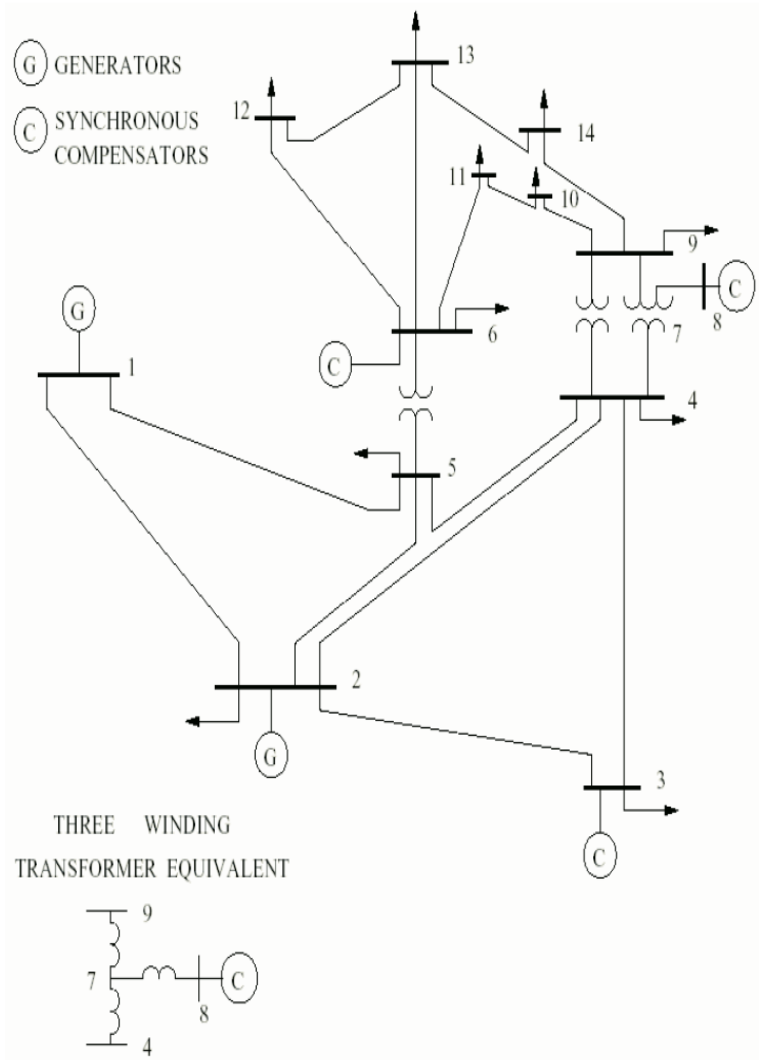


Figure 8.3: The IEEE 14 Bus Power System.

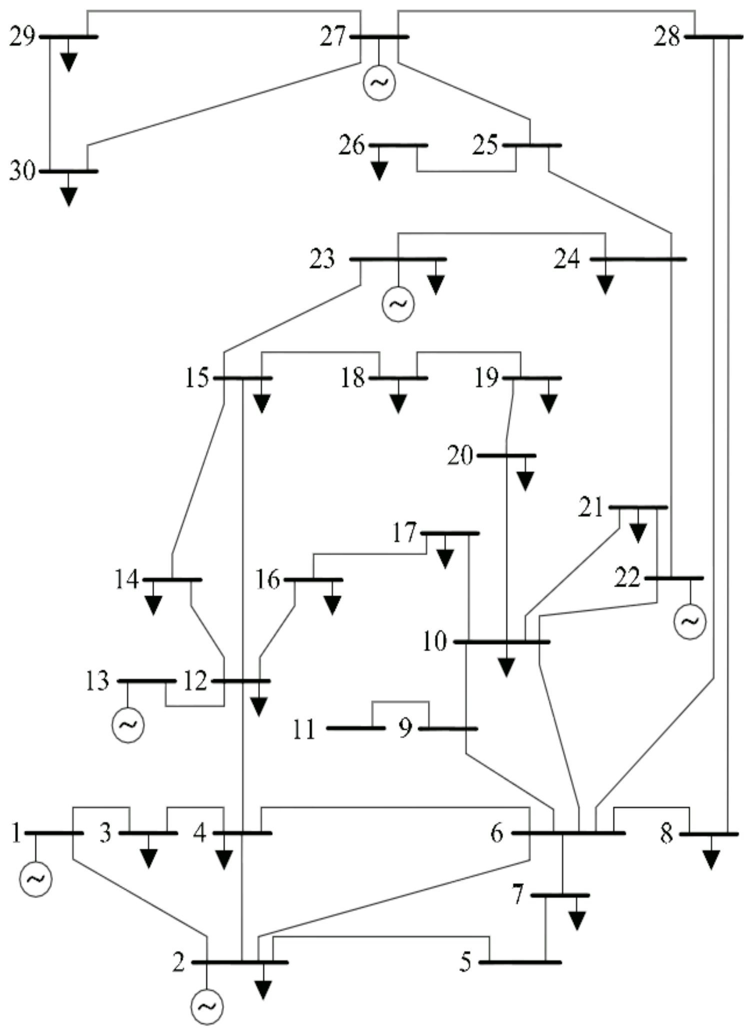


Figure 8.4: The IEEE 30 Bus Power System.

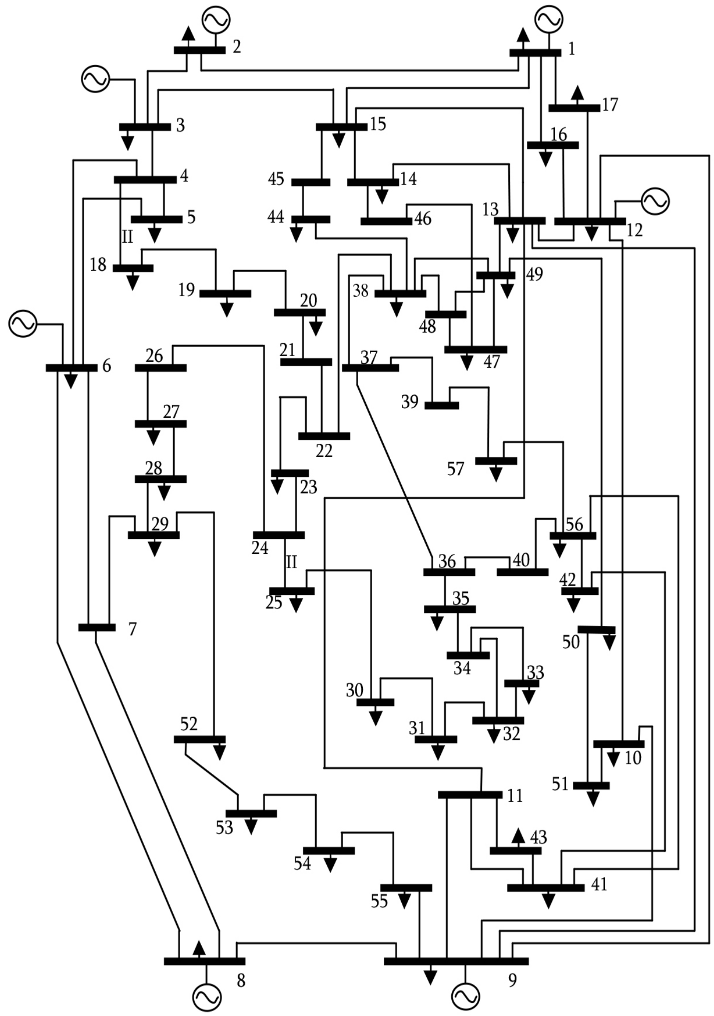


Figure 8.5: The IEEE 57 Bus Power System.

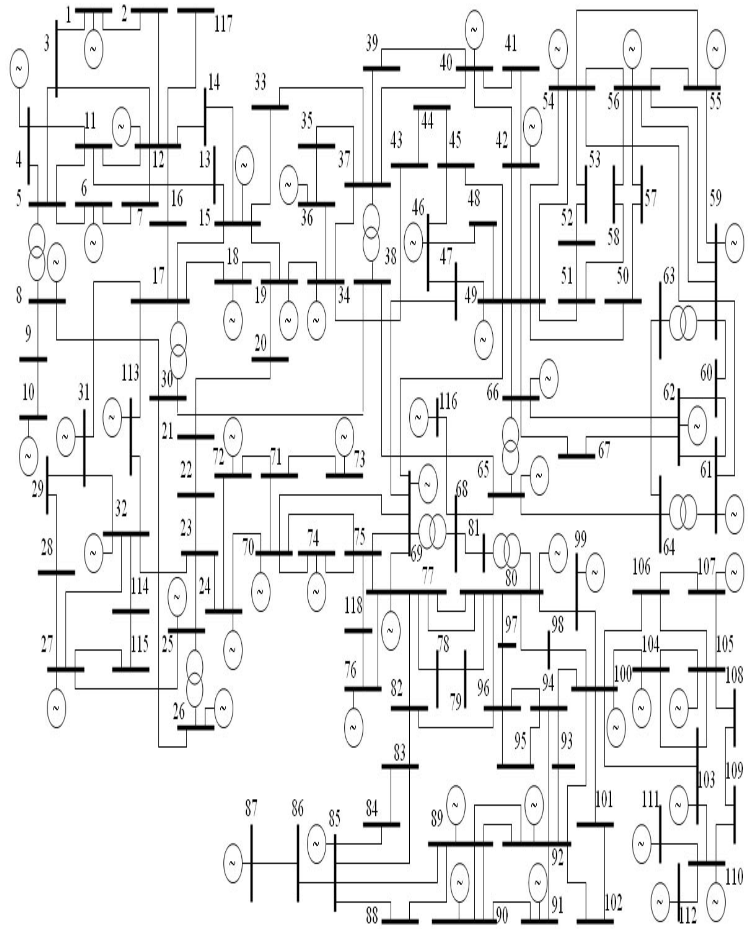


Рис.1. IEEE тестовая схема, состоящая из 118 узлов

Figure 8.6: The IEEE 118 Bus Power System.

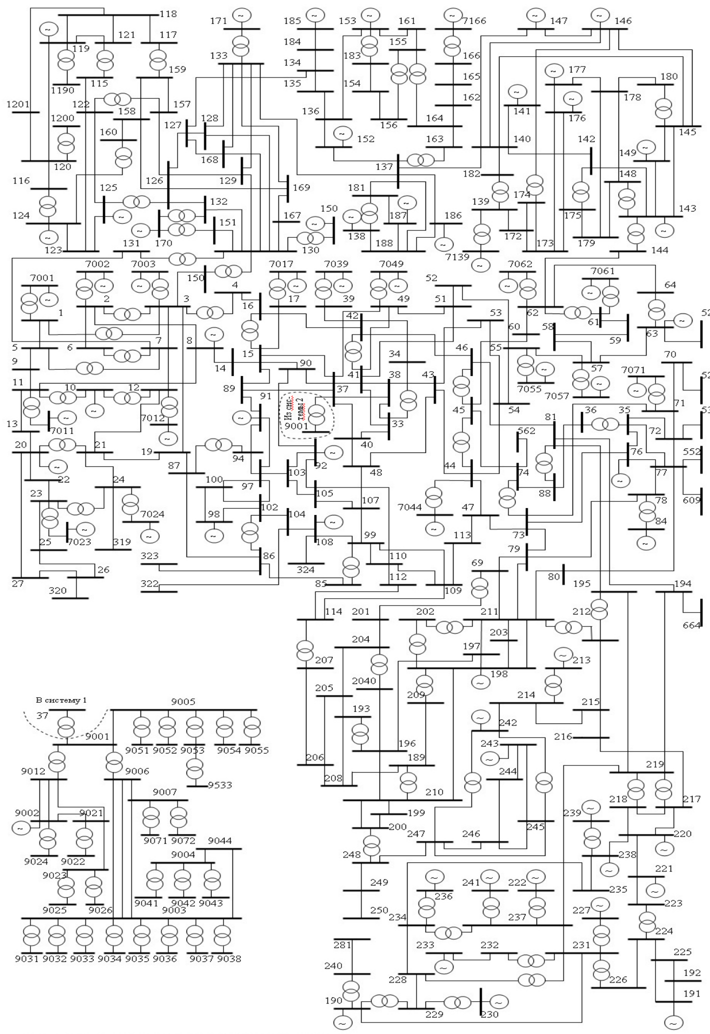


Рис.1. IEEE тестовая схема, состоящая из 300 узлов

Figure 8.7: The IEEE 300 Bus Power System.

in C++11, with the generator optimization solvers being implemented with the CVXGEN custom solvers [271]. The compiler used is GCC version 4.8.4.

For both the OPF and the SCOPF simulations, we have used the primal and dual residual tolerances, each to be 0.006, and we have used a discrete version of proportional+derivative controller to adjust the value of ρ for the first 3000 iterations, such that at each iteration the relationship $\rho \times \epsilon_{primal} = \epsilon_{dual}$ is maintained, where ϵ_{primal} and ϵ_{dual} are respectively the primal and dual residuals. After the first 3000 iterations, if the algorithm hasn't still converged, then ρ is held fixed at the last value. The initial value of ρ at the beginning of the iterations is taken as 1.

8.1.1 OPF

Here we will describe the simulation results of the OPF. For each system, we will describe a moderately unconstrained case (with higher transmission line limits) and a moderately congested case (with lower line limit). OPF Results for the different systems are as shown in the table 8.1. In this table, and the convergence characteristics plots, that follow, we have simulated the systems for two different values of transmission limits. It is assumed that all the lines have the same limits. The higher values of the limits indicate a relatively less constrained problem. The lower values are the ones that are closest to infeasibility.

In these simulations, we have maintained all the transmission lines of the system to be of exactly the same limit, for any particular case. We start with the 5 bus system: Presented below are the convergence characteristics for the primal residual, dual residual, and the objective value:

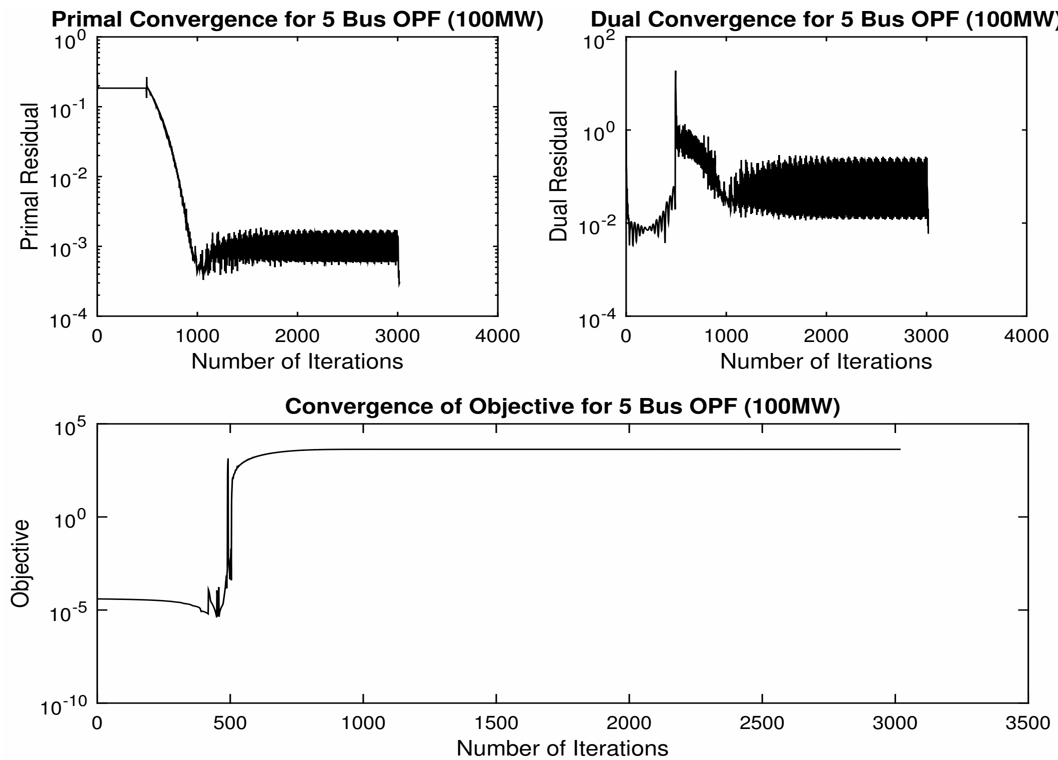


Figure 8.8: Convergence Characteristics of 5 Bus System OPF with 100 MW line capacities.

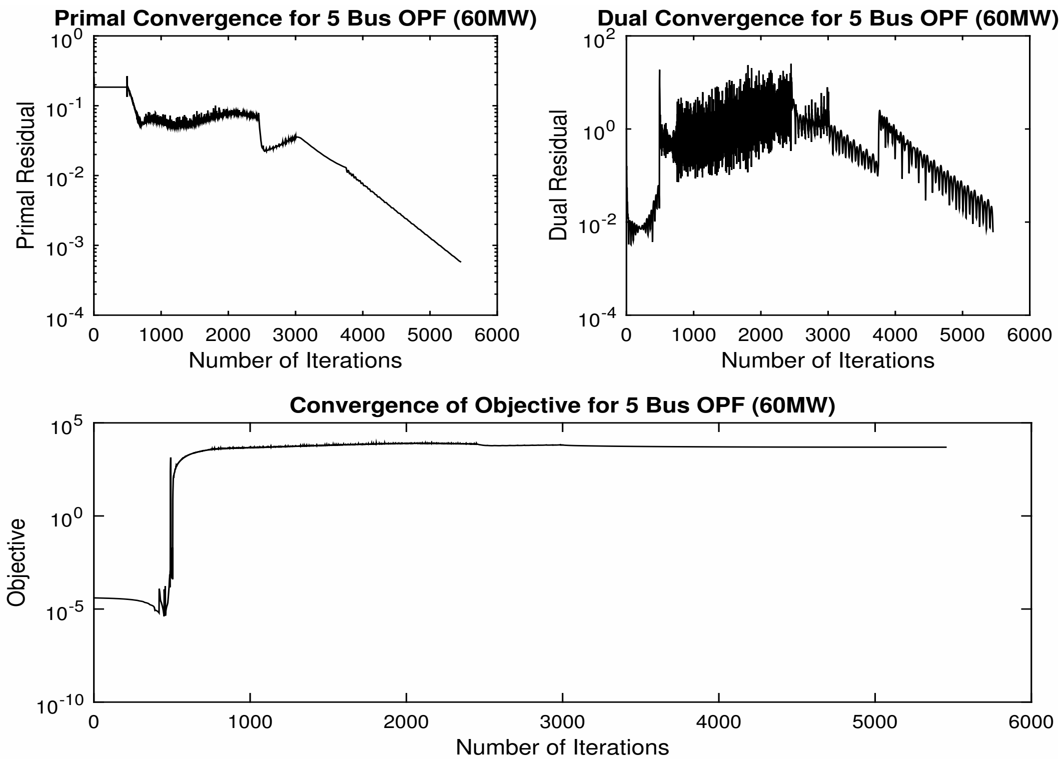


Figure 8.9: Convergence Characteristics of 5 Bus System OPF with 60 MW line capacities.

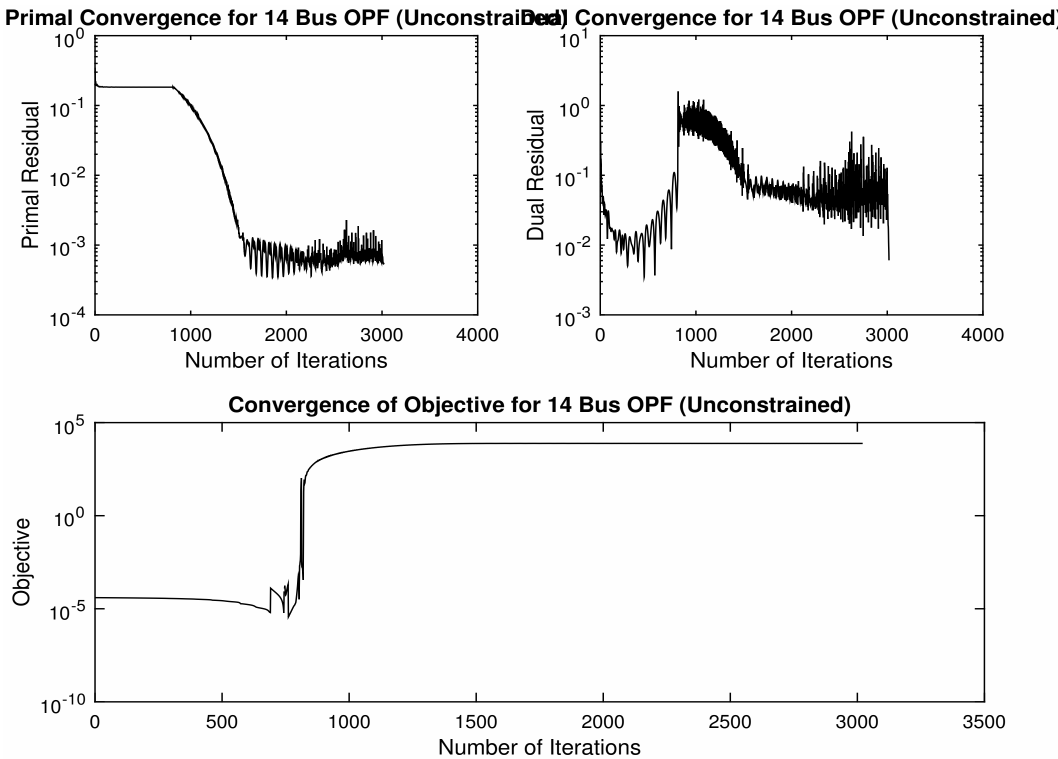


Figure 8.10: Convergence Characteristics of 14 Bus System OPF with very high (1000000 MW) line capacities.

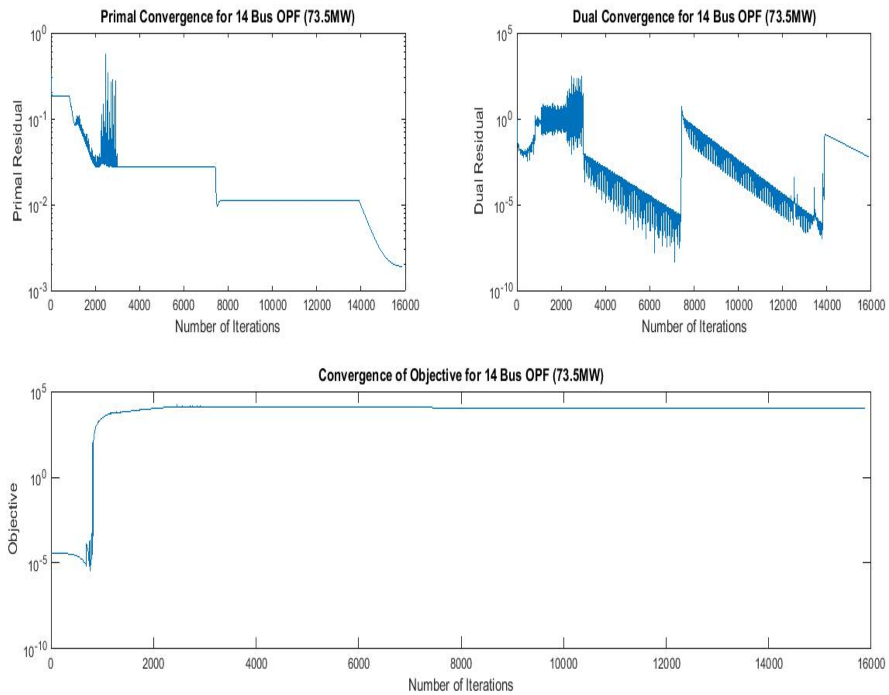


Figure 8.11: Convergence Characteristics of 14 Bus System OPF with 73.5 MW line capacities.

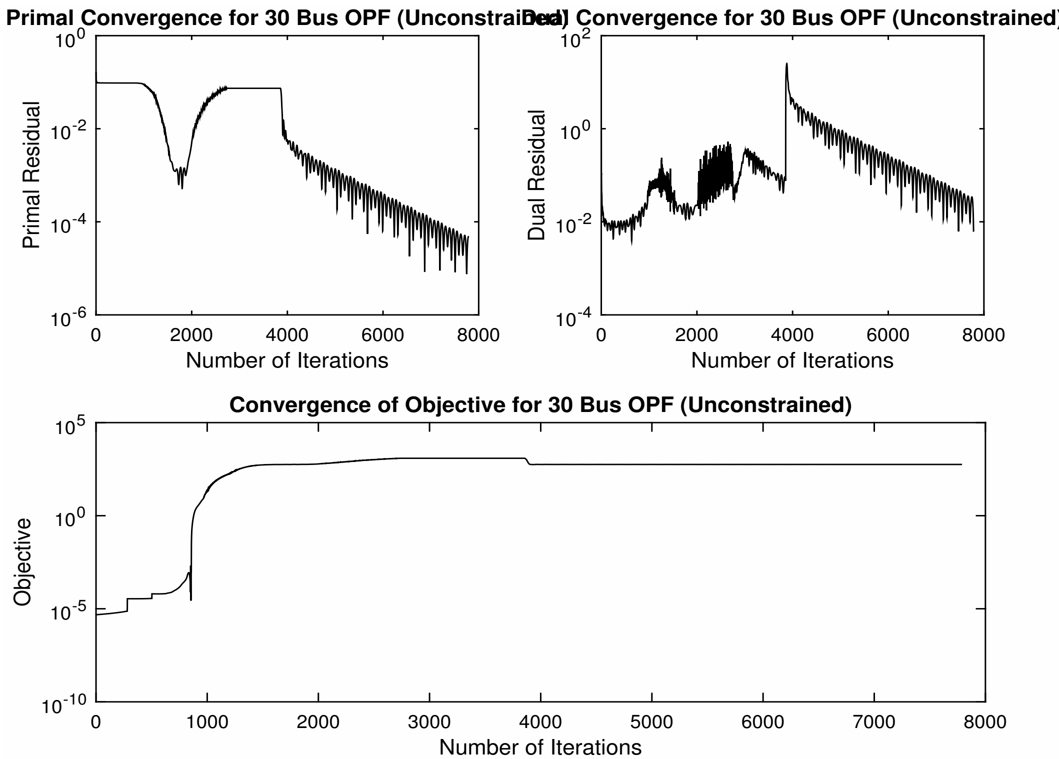


Figure 8.12: Convergence Characteristics of 30 Bus System OPF with very high (1000000 MW) line capacities.

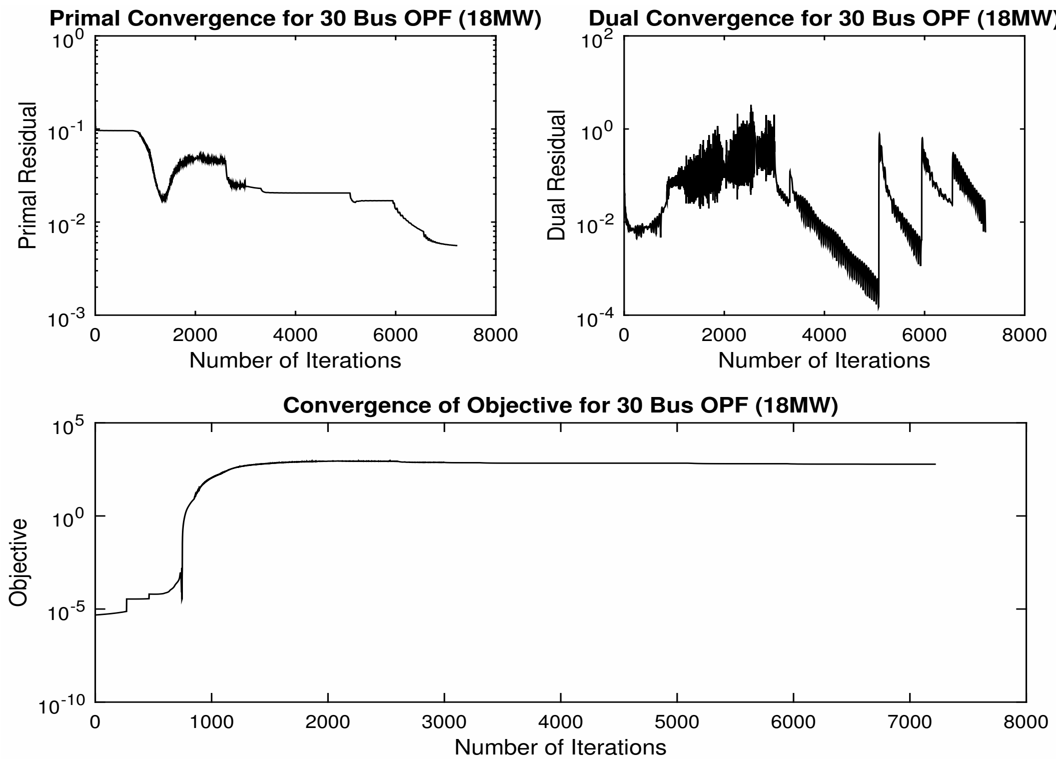


Figure 8.13: Convergence Characteristics of 30 Bus System OPF with 18 MW line capacities.

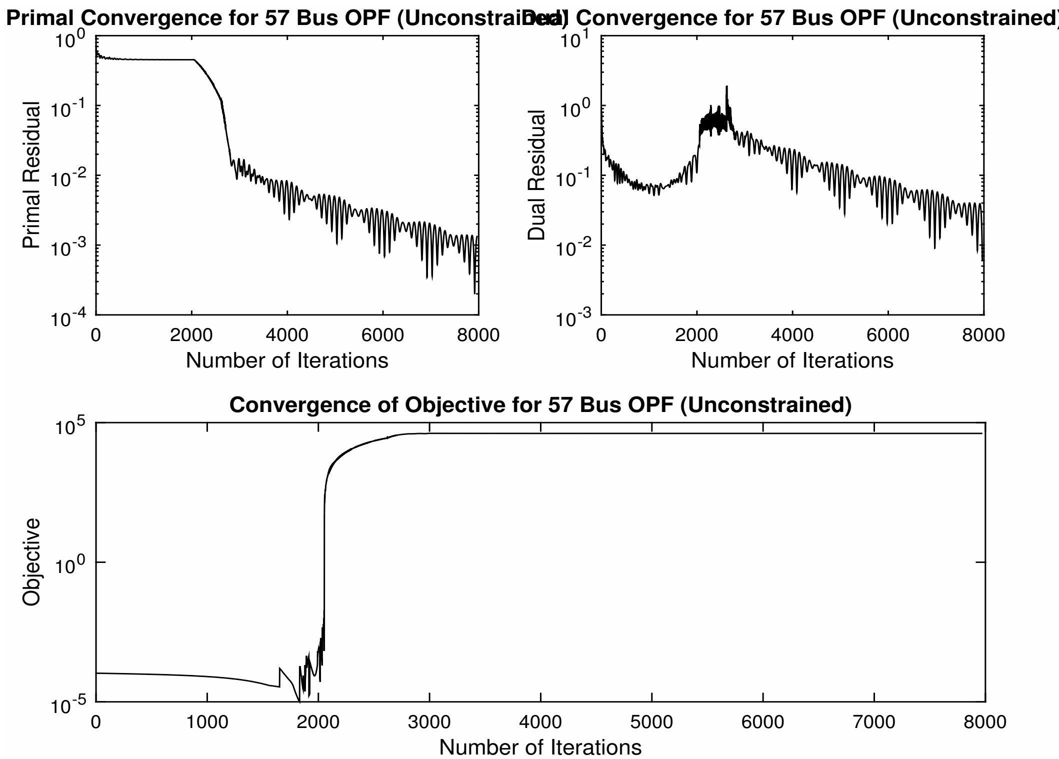


Figure 8.14: Convergence Characteristics of 57 Bus System OPF with very high (1000000 MW) line capacities.

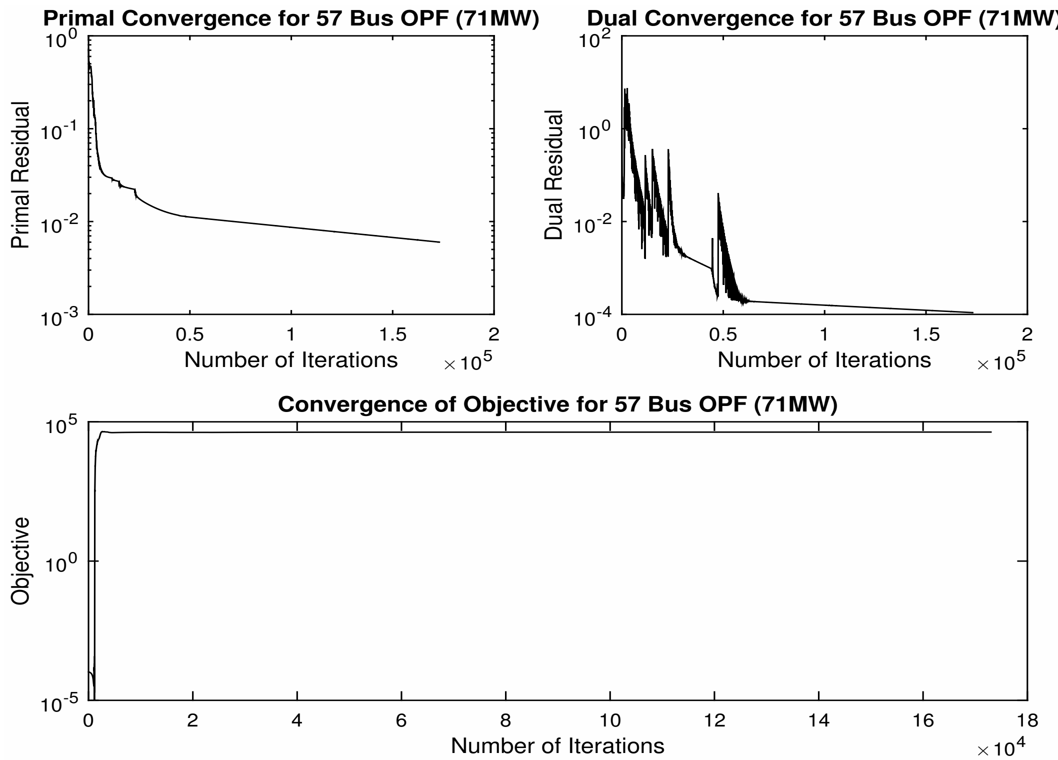


Figure 8.15: Convergence Characteristics of 57 Bus System OPF with 71 MW line capacities.

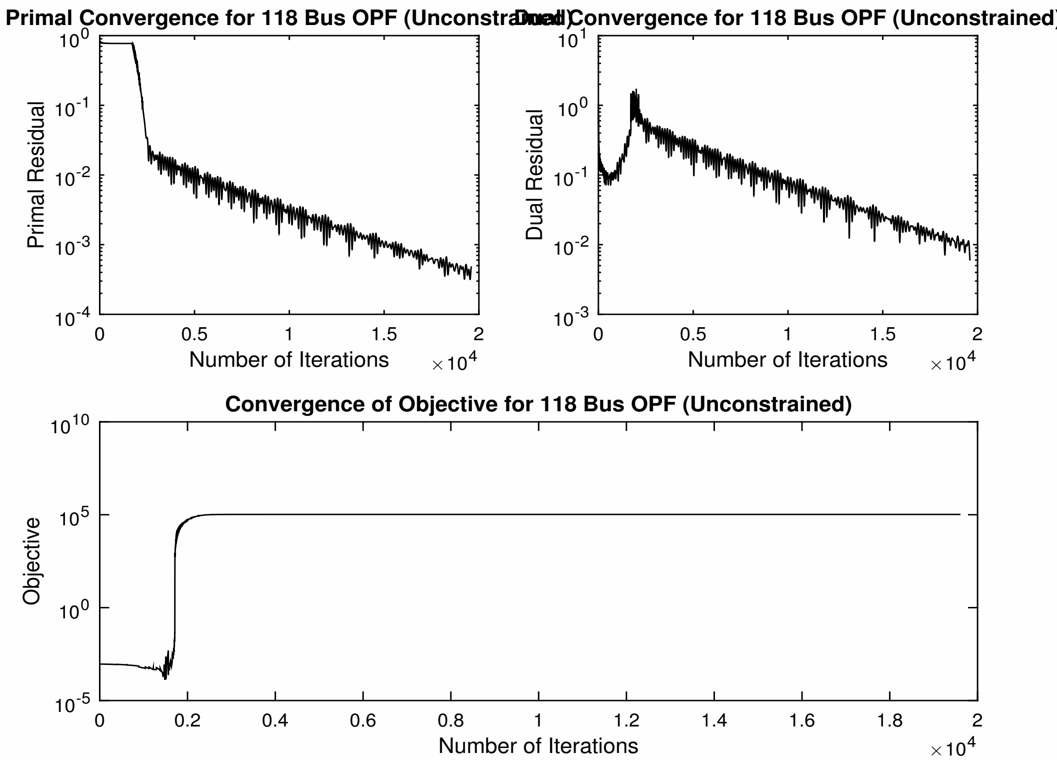


Figure 8.16: Convergence Characteristics of 118 Bus System OPF with very high (1000000 MW) line capacities.

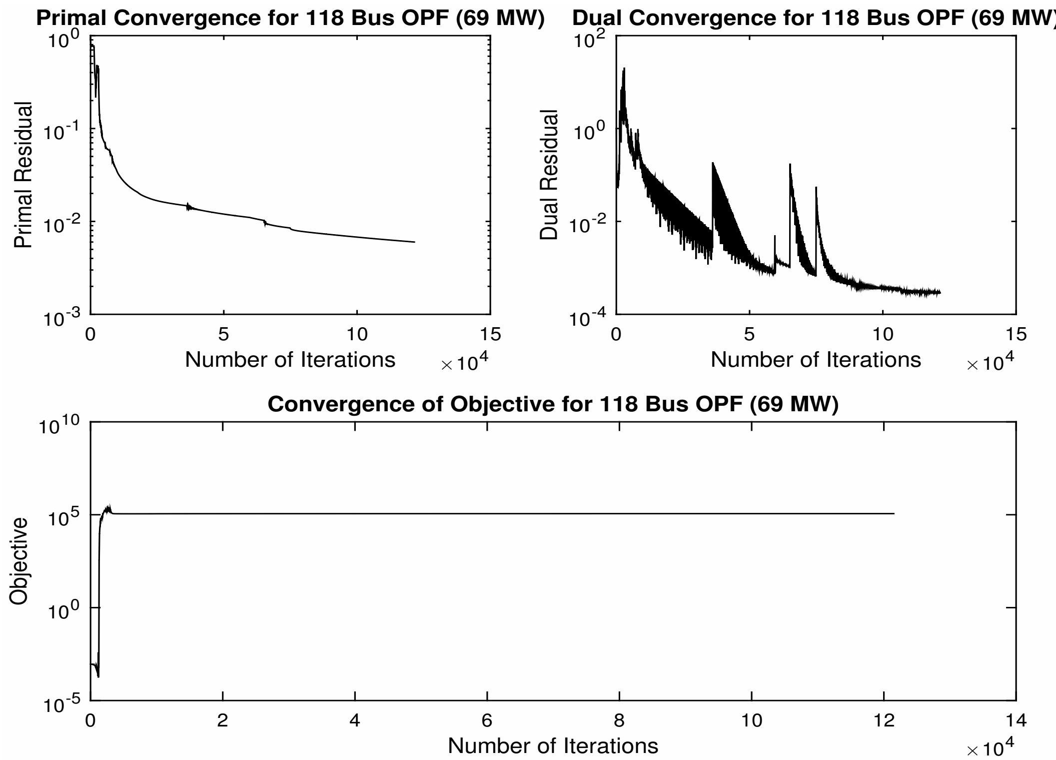


Figure 8.17: Convergence Characteristics of 118 Bus System OPF with 69 MW line capacities.

OPF Results for 5 Bus System is as shown in the table 8.1

Table 8.1: OPF Results for 5 Bus System.

Line Limits (MW)	Solution Time (s)	No. of Iterations
100	0.047913	3017
60	0.075353	5453

OPF Results for 14 Bus System is as shown in the table 8.2

Table 8.2: OPF Results for 14 Bus System.

Line Limits (MW)	Solution Time (s)	No. of Iterations
1000000	0.058781	3018
73.5	0.260373	15888

In the preceding tables and figures, we have shown the results of the OPF. Now we will describe the SCOPF results.

8.1.2 SCOPF

In this section, for each of the simulation studies, we have assumed the line ratings to be 100 MW only. We have marked the all the lines for contingency analysis for the 3 bus system, whereas, only considering lines $3-4$ and $4-5$ for the 5-bus system gave us feasible solutions.

8.1.3 APMP for LASCOPF Problems

In this section, we present the results of the APMP algorithm for the LASCOPF problem instance for tracking demand variation for the 5 bus system. All the other LASCOPF problems can be solved in the exact same manner.

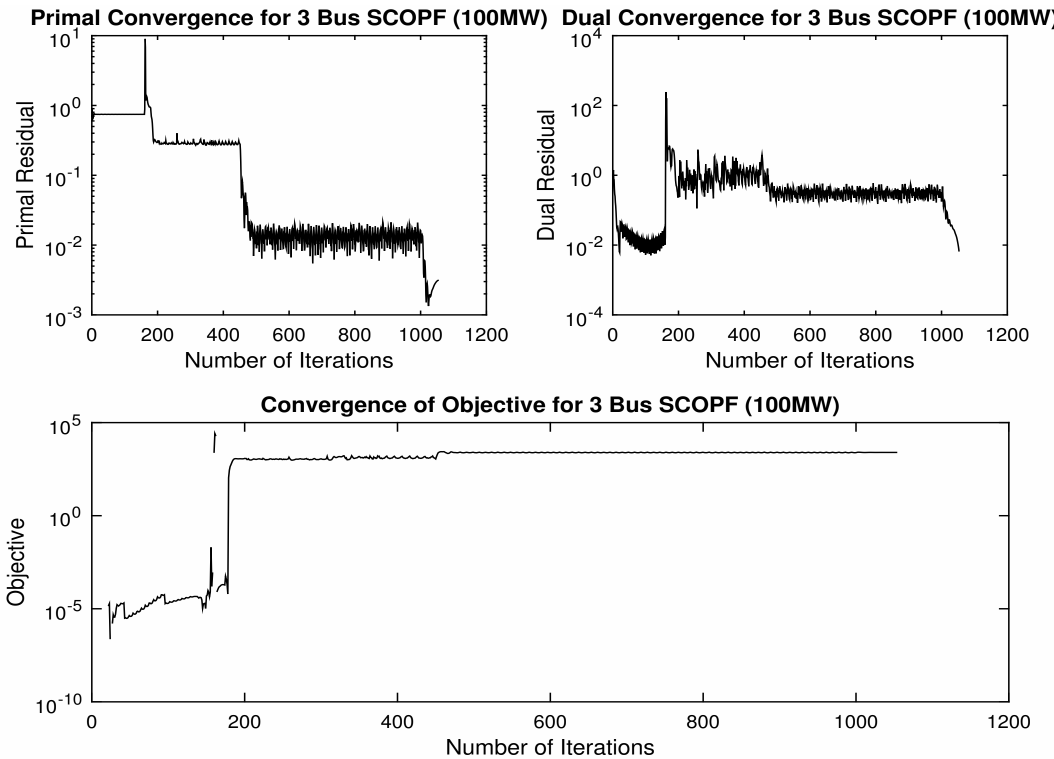


Figure 8.18: Convergence Characteristics of 3 Bus System SCOPF with 100 MW line capacities.

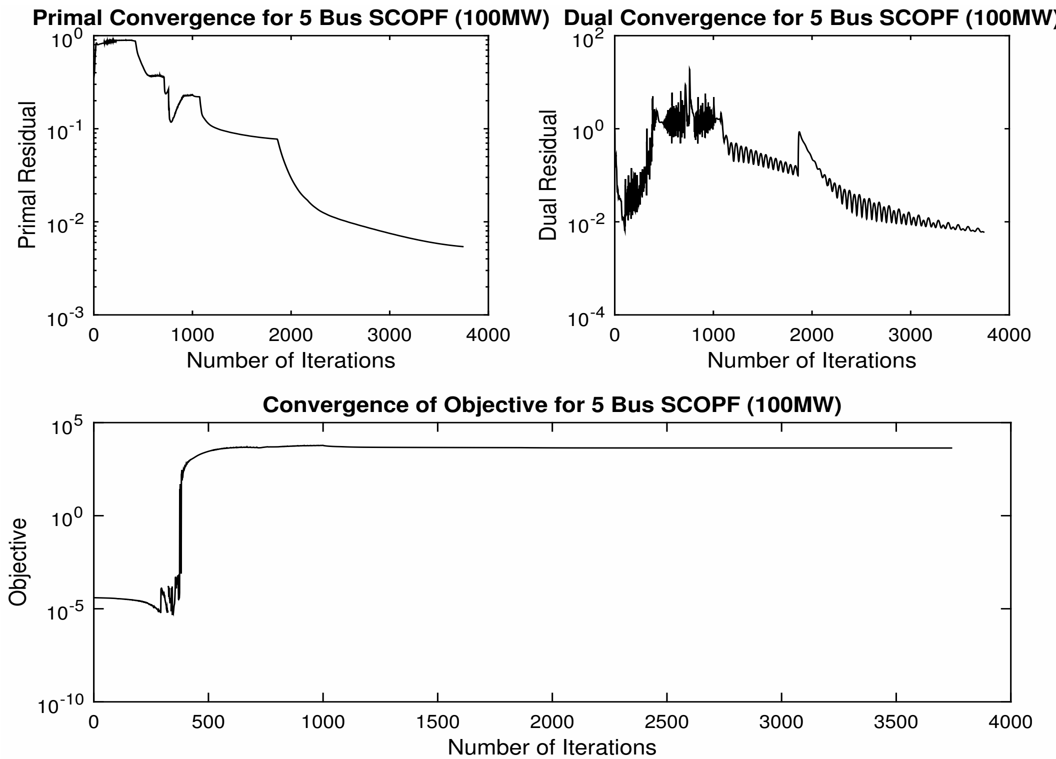


Figure 8.19: Convergence Characteristics of 5 Bus System SCOPF with 100 MW line capacities.

OPF Results for 30 Bus System is as shown in the table 8.3

Table 8.3: OPF Results for 30 Bus System.

Line Limits (MW)	Solution Time (s)	No. of Iterations
1000000	0.472536	7783
18	0.332098	7216

OPF Results for 57 Bus System is as shown in the table 8.4

Table 8.4: OPF Results for 57 Bus System.

Line Limits (MW)	Solution Time (s)	No. of Iterations
1000000	0.598247	7966
71	14.6919	173012

The figure 8.20 shows the outer APP convergence for this example problem.

8.1.3.1 APP Convergence Characteristics

All the rsults presented above have been calculated using a single thread, although, the code is written in such a way, that, if desired, it can easily be converted to a multithreaded one. Implementing fully multithreaded code is one of the proposed future works.

OPF Results for 118 Bus System is as shown in the table 8.5

Table 8.5: OPF Results for 118 Bus System.

Line Limits (MW)	Solution Time (s)	No. of Iterations
1000000	7.96253	19595
69	42.2865	121487

SCOPF Results for 3 and 5 Bus System is as shown in the table 8.6

Table 8.6: SCOPF Results for 3 and 5 Bus System.

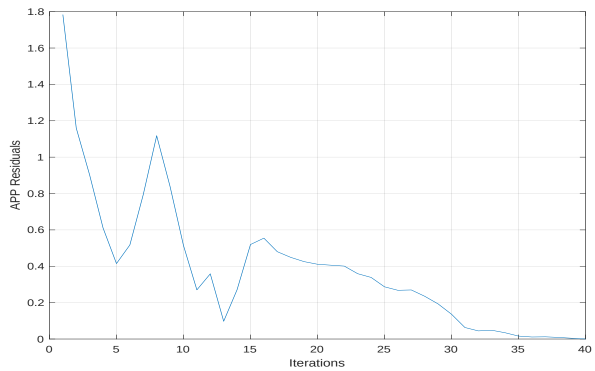
Line Limits (MW)	Solution Time (s)	No. of Iterations
100 (5 bus)	3.12735	3741
100 (3 bus)	0.893854	1053

Table 8.7: Variation of Load for 5 Bus System.

Connection Node	MW in Interval-1	MW in Interval-2	MW in Interval-3
2	20	30	20
3	45	40	43
4	40	40	45
5	60	65	65

Table 8.8: Generator Outputs 5 Bus System.

Generator	MW in Interval-1	MW in Interval-2	MW in Interval-3
Gen. 1 (Conn. Node: 1)	140.771	141.355	143.131
Gen. 2 (Conn. Node: 2)	24.229	33.6459	29.8704



Residuals.pdf

Figure 8.20: APP Convergence Characteristics of 5 Bus System LASCOPF

Chapter 9

Extensions, Future Works, & Conclusion

9.1 Proposed Future Research Works

In this chapter, we will give brief outline of the proposed work, we plan on doing as part of this ongoing research endeavor. Here are the problems we will be considering extending our present work to. In the following sections, we will describe each of them.

- AC LASCOPF formulation.
- Development of Multithreaded code.
- Modeling of generator contingency and Ancillary Services/reserve pricing.
- Unit Commitment and Combined Cycle Power Plant Scheduling.
- Dynamic Line Switching.
- Advanced Mathematics for Grid Operation Enhancing Demand Response and Renewables.
- Financial Transmission Rights (FTR)/Congestion Revenue Rights (CRR) Revenue Adequacy in the events of line switching.

- Comparison Between Existing Scheme and the Proposed New Scheme for Post-Contingency Restoration

9.2 AC LASCOPF formulation

We will be applying the recent advances on the ACOPF work presented in [15], [243], [242], [335], [334] etc. to our problem. It will be a more natural approach, since we will be able to calculate the exact Ohmic losses and temperature rise on transmission lines by solving the ACOPF. The only difference is that, we will be replacing the linearized model of (6.32) or (6.33) with ACOPF and for each problem, as described in the references, we will be solving the Semidefinite Prodrgramming (SDP) relaxation and caiculating the primal solution of the SCOPF from the dual solution. The details of this future work has been described in appendix A. We refer the reader to the appendix.

9.3 Development of Multithreaded code

So far, we have only developed the serial code. In order to increase the speed and leverage the concurrent computation capability of the computing machine, we can make use of the C++11 standard thread library and implementing the actual multithreaded code, which will adaptively decide the number of threads to be generated based on the number of generators present (for which the most intense calculations of actually iteratively solving the optimization problems is required) and also the number of processor cores present.

9.4 Modeling of generator contingency and Ancillary Services/reserve pricing

In this work, we have just modeled the line contingency. We will also be modeling the possible outage of generators, or, the generator contingency in the subsequent work. This will require us to keep certain fraction of power generation capability of the different generators in reserve, to supply the demand, when an outage of a generator actually happens. Hence, there will be pricing scheme related to such reserves or "Ancillary Services (AS)" as well.

9.5 Unit Commitment and Combined Cycle Power Plant Scheduling

In this future work, we will include the aspects of switching of generators, by extending the LASCOPF to have integer decision variables. The details of this future work has been described in appendix B. We refer the reader to the appendix. This work also paves the way for the next future work, we will mention.

9.6 Dynamic Line Switching

In this work, we have only indicated how to use the ramping capability of generators in order to relieve the line overloads after an outage. But, for this exploration, we will try to achieve the same by switching on or off different lines. The modeling for this problem will build upon the same ideas as those presented in section 9.5.

9.7 Advanced Mathematics for Grid Operation Enhancing Demand Response and Renewables

In this future work, we will explore the decision-making problem, as it pertains to deciding upon the optimal mix on the part of the consumer and the producer of electricity (hereafter, also to be referred to as the "prosumer"), regarding different conventional and non-conventional sources of electricity, as well as loads acting as sources. The preliminary ideas of this future work has been described in appendix C. We refer the reader to the appendix.

9.8 Financial Transmission Rights (FTR)/Congestion Revenue Rights (CRR) Revenue Adequacy in the events of line switching

It is known that in the event of a change in network topology, (caused primarily by line switching), the ISO can run out of money to fund the FTRs/CRRs, a condition, known as, "Revenue Inadequacy". In this future work, we will attempt to develop newer schemes of both the design and auctioning of FTRs/CRRs, such that the revenue inadequacy is circumvented, even in the event of random or planned outages. Some of the preliminary ideas of this future work has been described in appendix A. We refer the reader to the appendix.

9.9 Comparison Between Existing Scheme and the Proposed New Scheme for Post-Contingency Restoration

In this future work, we will be alluding to the example system and results thereof, mentioned in the references [209], [205], [207], [211] for the following three cases:

- Out-of-merit dispatch with LMCP reflecting opportunity cost
- Marginal congestion cost saving
- Marginal capacity value to null the incentive of uninstructed deviations to support dispatch, and
- LASCOPF for multiple dispatch interval with representation of line temperatures.

9.10 Conclusions

In this section, we will draw some conclusions regarding how we have addressed the issue of representing post-contingency states of a power system within the dispatch optimization problem and solved them. We began by mentioning where the current state of art and industrial practice are at, for power system scheduling and dispatch optimization in chapter 1. We also mentioned, in that chapter, the real world motivations, that are necessitating the need for newer and better optimization schemes,and the challenges that lie in their implementation.

Following that, in chapter 2, we have provided a comprehensive survey of the body of existing knowledge in the fields of Power Systems Optimization, Mathematical Optimization, and Distributed Optimization Algorithms, specifically in the context of their applications to power systems. We have built up this present work based on those previous works. The bibliographies presented at the end of each of the references cited in that chapter, throughout this dissertation, and in the bibliography of this work, point to several other references, which forms a rich body of literature.

Thereafter, in chapter 3, we set the conventions for symbols and notations, which we have followed in the rest of this work. As much as possible, we have tried to stick to this convention. In a few instances, where we have introduced some new notion and corresponding symbols for that, we have restated the conventions specifically for notions and symbols.

In chapter 4, we start by presenting the OPF and SCOPF formulations in terms of the intuitive pictures from a case study of a simple two bus power system, and eventually present the mathematical formulations. We thereafter, present the intuitive models, followed by the mathematical ones for the LASCOPF problems for tracking demand variation and ensuring post-contingency restoration in one dispatch interval (as a simplifying first step, before making the model more realistic). In the second half f the chapter, we have generalized the optimization formulations to power networks of arbitrary size.

We have introduced the *Auxiliary Proximal Message Passing (APMP)* algorithm in chapter 5, for solving the problems described in the previous chapter. We have reformulated all the problems presented earlier, for us to be able to apply the *APMP* algorithm to solve them.We have also presented justifications for using this algorithm to solve the problems.

In chapter 6, we have presented the dynamicsof temperature evolution of transmission lines, following the outage of a transmission line, and applied the *APMP* algorithm to develop a mathematical model, which solves the LASCOPF problem, while limiting the temperature on the transmission line. The formulation also attempts to restore the system to normal operation, followed by restoring it to secure operation, following an outage, in the most ecomnomic manner, possible.

So far, we have just discussed DC-OPF. Chapter A Extends the model developed in chapter 6, to solve the AC-OPF problem. Solving the AC-OPF is hard and hence we made use of the recent advances in Semi Definite Programming (SDP) and rank relaxation, as applied to AC-OPF, to build the LASCOPF model for post-contingency restoration for the non-linear and non-convex AC power flow problem.

Chapter 7 discusses the object oriented programming approach that has been used for developing the simulation software for implementing all the tools that we have developed in this work and chapter 8 gives the numerical results, while applying the methodologies to different power systems.

Finally, in chapter 9, we make mention of some future research directions, specifically, inclusion of *Mixed Interger Linear Programming (MILP) Optimization* and *Markov Decision Process (MDP)* based decision making, in which the LASCOPF model can be embedded.

In the following subsection, the major contributions of this work has been summarized, followed by an explanation of the roadblocks faced while carrying ouut this work.

9.10.1 Broad Overview and Contribution

- Extension of the Alternating Direction Method of Multipliers (ADMM) Based Proximal Message Passing Algorithm from solving simple OPF Problems to solving SCOPF Problems which are secure to $(N-1)$ Contingencies.

- Specifically considered outages of Transmission Lines.

- Numerical results pertaining to simple systems.

- **Contributions:** Representation of post-contingency states, post-contingency remedial and corrective actions, and line temperature evolution within dispatch optimization

- Combining Proximal Message Passing with Auxiliary Problem Principle to give rise to the APMP algorithm for solving multiple dispatch interval SCOPF.

9.10.2 Roadblocks

- **Tuning of the ADMM step-length ρ to attain convergence:** This is the most critical roadblack we have faced in the current work. We think that this factor is responsible for the following three roadblocks, as well. In general, there currently exists no provable technique of ρ tuning, which is universally applicable to any problem instance to be solved by ADMM. In our work, we have made use of a discrete time PID controller, with varying the gains, in order to vary ρ such that, we maintain the primal and dual residuals approximately equal to each other, while they approach to zero, as the iterations converge. However, we have in practice observed that it's best if we let this controlled tuning happen till 3000 iterations, and then fix the ρ. Unfortunately, for solving SCOPF for arbitrary networks and line contingencies, this approach also fails. One alternative idea, we would like to explore in future, is to use different values of rho for each contingency scenario and/or each node and tune them seprarately to explore if the iterations converge.

- **Current limited ability of the algorithm for solving SCOPF for arbitrary size and contingency scenario:** This has been already mentioned in the previous point, and can be attributed to the ad-hoc tuning of ρ.

- **High number of iterations to attain convergence:** Currently, even though for some situations in OPF and SCOPF, we get convergence via ADMM, but, it takes a lot of iterations to converge. This is also a very well known setback of ADMM. However, if the sepcific application at hand doesn't demand very high accuracy of the results, we can trade off accuracy for fewer iterations. Nevertheless, means of exploring faster convergence is an issue we haven't been able to address to, in this work and would like to consider as a future work.

- **Impact of above factors to adversely influence scalability, and exploration**

of alternative algorithms: Lastly, all of the above factors can be seen as impediments to achieve precisely what a distributed algorithm is intended for, i.e. scalability. Therefore, in our future work, we would like to compare the performance through alternative algorithms, like Consensus + Innovation (C+I), Analytical Target Cascading (ATC), Optimality Condition Decomposition (OCD) etc.

Thus, the present work comes to an end. But, I sincerely hope and wish that the material presented here will, rather, be the starting point of a plethora of inquiries, further doubts & questions, and enhancing factor for the thirst for more discovery and knowledge creation. Thus, this end is, on the contrary, the beginning of a vast field of exploration by several generations of future researchers. This, as a researcher, is my deepest desire. With that, I rest my case.

Appendices

Appendix A

Future Work: LASCOPF Modeling for AC-OPF

A.1 Introduction

In this appendix, we will first introduce the AC-OPF problem in its pristine, or, classical form (in both the angles included (which is also the polar coordinate formulation) as well as the angles eliminated (which is also the Cartesian or rectangular coordinate formulation) formulations). The AC-OPF is well known in both the Electrical Power Engineering and Operations Research/Mathematical Optimization communities as an extremely hard problem to solve because of its notorious degree of non-convexity. The non-convexity arises in polar coordinates due to the cross-terms in the voltage magnitude variables and due to the presence of trigonometric terms in the expressions for real and reactive powers, in angles included formulation. For the last fifty five years (starting from 1962 by Carpentier *et al.* [67]), several authors have tried to solve this hard non-convex problem using different approaches. Recently, Bai *et al.* [15] first reformulated the AC-OPF as a Semi-Definite Programming (SDP) optimization problem using rectangular coordinates, thereby convexifying it. Later on, Lavaei *et al.* [243, 242, 335, 334] substantially advanced the approach, building upon the work of Bai. In this chapter, our work will build upon that of Bai and Lavaei. For the sake of completeness, we will present salient parts of the convexification approach of Lavaei in the form of a series of reformulations of the classical AC-OPF formulation in terms of the rectangular or Cartesian co-ordinates, which eventually leads us to the convexification. As we will soon see, representation of the classical AC-OPF problem as an SDP OPF problem means that the non-convexity is manifested as a rank constraint and eliminating that constraint or "relaxing" the problem leads to the convexification. The resulting convexified problem is called the "SDP Rank Relaxed OPF". There are also other approaches to convexifying the AC-OPF, such as the "Second Order Cone Programming (SOCP) Convexification," [22, 213, 214] "Convex Quadratic Envelope Method," [83] etc. The material presented in this appendix will form the basis, upon which in our future work, we will use the SDP rank relaxation to build the AC-SCOPF and the AC-LASCOPF models as well. We will finally apply the APMP algorithm to distribute the computation of the above-mentioned models. In the next section, we will describe the notational conventions that are most relevant for the material we are going to present in this chapter. Some of these notations have already been introduced earlier. We will also introduce some new notations.

A.2 Classical AC OPF Formulation: Notations

First of all, let's introduce the notations to be used throughout. The system of notations follows the same pattern, as that presented in the earlier chapters.

A.2.0.1 Sets

$\mathcal{D}$: Set of Devices

$\mathcal{T}$: Set of Terminals

$\mathcal{N}$: Set of Nets (or Buses, or Nodes), $J(N_i)$: Set of buses directly connected to bus, N_i

The next three sets form partitions of the set of devices:

$G \subseteq \mathcal{D}$: Set of Generators, $G(N_i)$: Set of Generators connected to bus, N_i

$T \subseteq \mathcal{D}$: Set of Transmission Lines, $T(N_i N_j)$: Set of Transmission lines connected between buses, N_i and N_j

$L \subseteq \mathcal{D}$: Set of Loads

$\mathcal{L} = \{0, 1, 2, ..., |\mathcal{L}|\}$: Set of possible $(N-1)$ Contingencies. The element, 0 indicates the base case.

$\Omega = \{0, 1, 2, ..., |\Omega|\}$: Set of Dispatch intervals or, the net Dispatch Horizon under consideration. 0 indicates the upcoming dispatch interval under consideration. It is to be noted that the first dispatch interval under consideration, for which the calculation is done is the upcoming/forthcoming one. Hence, dispatch interval -1 (which is not in this set) is the current running one.

$\dagger$ will be used to denote the transpose of a vector or matrix.

A.2.0.2 Elements

t: Elements of $\mathcal{T}$

g: Elements of G

D: Elements of L

T: Elements of T

N: Elements of $\mathcal{N}$

A.2.0.3 Indices

i, j: Nets

k: Terminals

q: Generators

r: Transmission Lines

d: Loads

c: Contingencies

τ: Dispatch Intervals

ν: Iteration count for ADMM/PMP algorithm

μ_{APP}: Iteration count for APP algorithm

A.2.0.4 Parameters

$R_{T_r}, X_{T_r}, y_{T_r}, Z_{T_r} = R_{T_r} + (\sqrt{-1})X_{T_r}$: Resistance, Reactance, Admittance, and Impedance of the r^{th} Transmission Line.

$\alpha_{g_q}, \beta_{g_q}, \gamma_{g_q}$: Quadratic, Linear, and Constant Cost Co-efficients of the q^{th} Generator.

$C_{g_q}(.), f_{dev}(.)$ will be used to denote the cost function of the generator ${g_q}^{th}$ and that of a generic device, respectively, throughout. We will introduce the other cost functions in the appropriate sections.

Re. and Im. will be used to extract the real and imaginary parts of a complex number.

Hence, $\sum_{g_q \in G} \left(C_{g_q}(P_{g_q})\right) = \sum_{g_q \in G} \left(\alpha_{g_q}(P_{g_q})^2 + \beta_{g_q}(P_{g_q}) + \gamma_{g_q}\right)$: Total cost of Generation

$\overline{Q}_{g_q} : \forall g_q \in G$: Maximum Reactive Power Generating Capability

$\overline{P}_{g_q} : \forall g_q \in G$: Maximum Real Power Generating Capability

$\overline{V}_{N_i} : \forall N_i \in \mathcal{N}$: Maximum Bus Voltage Magnitude allowed

$\underline{Q}_{g_q}$: $\forall g_q \in G$: Minimum Reactive Power Generating Capability

$\underline{P}_{g_q}$: $\forall g_q \in G$: Minimum Real Power Generating Capability

$\underline{V}_{N_i}$: $\forall N_i \in \mathcal{N}$: Minimum Bus Voltage Magnitude allowed

$\overline{R}_{g_q}$: $\forall g_q \in G$: Maximum ramp-up rate of the generator g_q in $MW/$interval

$\underline{R}_{g_q}(= -\overline{R}_{g_q}, \text{usually})$: $\forall g_q \in G$: Maximum ramp-down rate of the generator g_q in $MW/$interval

$\overline{S}_{T_r}^{(0)}$: $\forall T_r \in T$: Maximum allowable line MVA flows corresponding to long-term rating

$\overline{P}_{T_r}^{(0)}$: $\forall T_r \in T$: Maximum allowable line MW flows corresponding to long-term rating

$\overline{I}_{T_r}^{(0)}$: $\forall T_r \in T$: Maximum allowable line current flows corresponding to long-term rating

$\overline{S}_{T_r}^{(c)}$: $\forall T_r \in T$: Maximum allowable line MVA flows corresponding to short-term rating

$\overline{P}_{T_r}^{(c)}$: $\forall T_r \in T$: Maximum allowable line MW flows corresponding to short-term rating

$\overline{I}_{T_r}^{(c)}$: $\forall T_r \in T$: Maximum allowable line current flows corresponding to short-term rating

$\overline{\Delta V}_{T_r}$: $\forall T_r \in T$: Maximum allowable magnitude of difference between bus voltages between two ends of a Transmission line

$P_{D_{d_{N_i}}}$: $\forall D_d \in L$ and $\forall N_i \in \mathcal{N}$: MW Demand at the buses

$Q_{D_{d_{N_i}}}$: $\forall D_d \in L$ and $\forall N_i \in \mathcal{N}$: MVAr Demand at the buses

The following (same as before) is the convention we follow in order to identify the associations of any particular variable to the sets:

$$\left(Variable^{(ContingencyIndex)(DispatchTime\#2)}_{Net/DeviceElement_{TerminalNumber}(c)(DispatchTime\#1)}\right)^{(IterationCount)}.$$

A.2.0.5 Variables

V_{N_i} : $\forall N_i \in \mathcal{N}$: Complex bus voltages (Cartesian Form)

P_{g_q} : $\forall g_q \in G$: Real Power Generation

Q_{g_q} : $\forall g_q \in G$: Reactive Power Generation

$S_{T_{r_{N_iN_j}}}$: $\forall T_r \in T$ and $\forall N_i, N_j \in \mathcal{N}$: Set of line MVA flows

$P_{T_{r_{N_iN_j}}}$: $\forall T_r \in T$ and $\forall N_i, N_j \in \mathcal{N}$: Set of line MW flows

$I_{T_{r_{N_iN_j}}}$: $\forall T_r \in T$ and $\forall N_i, N_j \in \mathcal{N}$: Set of line current flows

θ_{N_i} : $\forall N_i \in \mathcal{N}$: Bus voltage-phase angle (Polar Form)

$\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_{|\mathcal{N}|}$: Standard basis vectors in $\mathbb{R}^{|\mathcal{N}|}$

$\mathcal{E}_l = [\mathbf{e}_l{}^\dagger\ \mathbf{e}_l{}^\dagger]^\dagger$ and H occuring in the exponent indicates complex conjugate and Hermitian transpose of matrices and vectors.

A.3 SDP-OPF Semidefinite Programming Optimal Power Flow

In this section, we will first present the classical polar corrdinate, or, angle represented version of the full AC-OPF problem, followed by the rectangular/Cartesian coordinate, or, angles eliminated version. We will subsequently focus on the angles second version, and reformulate it in the Semi Definite Programming (SDP) format.

A.3.1 Classical AC OPF Formulation: The Primal Problem

The classical AC-OPF problem is stated below. The equality constraints are the ones pertaining to satisfying the power-balance Kirchhoff's laws for real and reactive power injection. The objective is to minimize the cost of generation. Let $y_{N_iN_{\bar{i}}}$ be the series admittance between buses N_i and $N_{\bar{i}}$. Let $y'_{N_i N_{\bar{i}}}$ be the total shunt admittance between buses N_i and $N_{\bar{i}}$ associated with the π model of all the transmission lines connecting bus N_i to bus $N_{\bar{i}}$ (which means, on each shunt leg, the shunt admittance is $\frac{y'_{N_i N_{\bar{i}}}}{2}$), and let $y_{D_{d_{N_i}}}$ be the admittance to ground of the constant impedance load (if any) connected to bus N_i. We state below, first, the angles included formulation.

$$\textbf{Objective Function}: \min_{V,P_g,Q_g,\theta} \sum_{g_q \in G} \Big(C_{g_q}(P_{g_q}) = \alpha_{g_q}(P_{g_q})^2 + \beta_{g_q} P_{g_q} + \gamma_{g_q} \Big) \tag{A.1a}$$

$$\textbf{Subject to}: \underline{P}_{g_q} \leqslant P_{g_q} \leqslant \overline{P}_{g_q},\ \forall g_q \in G \tag{A.1b}$$

$$\underline{Q}_{g_q} \leqslant Q_{g_q} \leqslant \overline{Q}_{g_q},\ \forall g_q \in G \tag{A.1c}$$

$$\begin{aligned} &P_{T_{r_{N_i N_{\bar{i}}}}} \\ &= |V_{N_i}||V_{N_{\bar{i}}}| \Bigg(\mathrm{Re}(y_{N_i N_{\bar{i}}}) \cos(\theta_{N_i} - \theta_{N_{\bar{i}}}) + \mathrm{Im}(y_{N_i N_{\bar{i}}}) \sin(\theta_{N_i} - \theta_{N_{\bar{i}}}) \Bigg),\ \forall N_i \in \mathcal{N},\ \forall N_{\bar{i}} \in J(N_i) \end{aligned} \tag{A.1d}$$

$$\begin{aligned} &Q_{T_{r_{N_i N_{\bar{i}}}}} \\ &= |V_{N_i}||V_{N_{\bar{i}}}| \Bigg(\mathrm{Re}(y_{N_i N_{\bar{i}}}) \sin(\theta_{N_i} - \theta_{N_{\bar{i}}}) + \mathrm{Im}(y_{N_i N_{\bar{i}}}) \cos(\theta_{N_i} - \theta_{N_{\bar{i}}}) \Bigg),\ \forall N_i \in \mathcal{N},\ \forall N_{\bar{i}} \in J(N_i) \end{aligned} \tag{A.1e}$$

$$\sum_{g_q \in G(N_i)} P_{g_{q_{N_i}}} - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} P_{T_{r_{N_i N_{\bar{i}}}}}\ \forall N_i \in \mathcal{N} \tag{A.1f}$$

$$\sum_{g_q \in G(N_i)} Q_{g_{q_{N_i}}} - Q_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} Q_{T_{r_{N_i N_{\bar{i}}}}}\ \forall N_i \in \mathcal{N} \tag{A.1g}$$

$$\underline{V}_{N_i} \leqslant |V_{N_i}| \leqslant \overline{V}_{N_i},\ \forall N_i \in \mathcal{N} \tag{A.1h}$$

$$|S_{T_r}| \leqslant \overline{S}^{(0)}_{T_r},\ \forall T_r \in T \tag{A.1i}$$

$$|P_{T_r}| \leqslant \overline{P}^{(0)}_{T_r},\ \forall T_r \in T \tag{A.1j}$$

$$(|I_{T_r}|)^2 \leqslant (\overline{I}_{T_r}^{(0)})^2,\ \forall T_r \in T \tag{A.1k}$$

$$|V_{N_i} - V_{N_{\bar{i}}}| \leqslant \overline{\Delta V}_{T_{r_{N_i N_{\bar{i}}}}},\ \forall T_r \in T \tag{A.1l}$$

The angles eliminated formulation can be written down as follows:

$$\textbf{Objective Function}: \min_{V,P_g,Q_g} \sum_{g_q \in G} \Big(C_{g_q}(P_{g_q}) = \alpha_{g_q}(P_{g_q})^2 + \beta_{g_q} P_{g_q} + \gamma_{g_q} \Big) \tag{A.2a}$$

$$\textbf{Subject to}: \underline{P}_{g_q} \leqslant P_{g_q} \leqslant \overline{P}_{g_q},\ \forall g_q \in G \tag{A.2b}$$

$$\underline{Q}_{g_q} \leqslant Q_{g_q} \leqslant \overline{Q}_{g_q},\ \forall g_q \in G \tag{A.2c}$$

$$\sum_{g_q \in G(N_i)} P_{g_{q_{N_i}}} - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \Bigg[\sum_{T_r \in T(N_i N_{\bar{i}})} \mathrm{Re}\Bigg\{ V_{N_i}(V_{N_i} - V_{N_{\bar{i}}})^H y^H_{T_{r_{N_i N_{\bar{i}}}}} + |V_{N_i}|^2 \Big(\frac{y'^H_{N_i N_{\bar{i}}}}{2} + y^H_{D_{d_{N_i}}} \Big) \Bigg\} \Bigg],\ \forall N_i \in \mathcal{N} \tag{A.2d}$$

$$\sum_{g_q \in G(N_i)} Q_{g_{q_{N_i}}} - Q_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \Bigg[\sum_{T_r \in T(N_i N_{\bar{i}})} \mathrm{Im}\Bigg\{ V_{N_i}(V_{N_i} - V_{N_{\bar{i}}})^H y^H_{T_{r_{N_i N_{\bar{i}}}}} + |V_{N_i}|^2 \Big(\frac{y'^H_{N_i N_{\bar{i}}}}{2} + y^H_{D_{d_{N_i}}} \Big) \Bigg\} \Bigg],\ \forall N_i \in \mathcal{N} \tag{A.2e}$$

$$\underline{V}_{N_i} \leqslant |V_{N_i}| \leqslant \overline{V}_{N_i},\ \forall N_i \in \mathcal{N} \tag{A.2f}$$

$$|S_{T_r}| \leqslant \overline{S}_{T_r}^{(0)},\ \forall T_r \in T \tag{A.2g}$$

$$|P_{T_r}| \leqslant \overline{P}_{T_r}^{(0)},\ \forall T_r \in T \tag{A.2h}$$

$$(|I_{T_r}|)^2 \leqslant (\overline{I}_{T_r}^{(0)})^2,\ \forall T_r \in T \tag{A.2i}$$

$$|V_{N_i} - V_{N_{\bar{i}}}| \leqslant \overline{\Delta V}_{T_{r_{N_i N_{\bar{i}}}}},\ \forall T_r \in T \tag{A.2j}$$

A.3.2 Network Matrices

If $y_{N_iN_{\bar{i}}}$ is the series admittance between buses N_i and $N_{\bar{i}}$. If $y'_{N_iN_{\bar{i}}}$ be the total shunt admittance between buses N_i and $N_{\bar{i}}$ associated with the π model of all the transmission lines connecting bus N_i to bus $N_{\bar{i}}$ (which means, on each shunt leg, the shunt admittance is $\frac{y'_{N_iN_{\bar{i}}}}{2}$), and if $y_{D_{d_{N_i}}}$ be the admittance to ground of the constant impedance load (if any) connected to bus N_i. Hence the Bus Admittance matrix, $\mathcal{Y} \in \mathbb{C}^{|\mathcal{N}|\times|\mathcal{N}|}$ is given by:

$$\mathcal{Y}(N_i, N_{\bar{i}}) = -y_{N_iN_{\bar{i}}},\ \textit{when}\ N_i \neq N_{\bar{i}} \tag{A.3a}$$

$$\mathcal{Y}(N_i, N_i) = \sum_{N_{\bar{i}} \in J(N_i)} \left(\frac{y'_{N_iN_{\bar{i}}}}{2} + y_{D_{d_{N_i}}} + y_{N_iN_{\bar{i}}} \right), \tag{A.3b}$$

where $J(N_i)$ denotes the buses that are directly connected to the bus N_i

$$\mathbf{I} = [I_1\ I_2\ ...\ I_{|\mathcal{N}|}]^\dagger = \mathcal{Y}\mathbf{V} = \text{Current Injection Vector} \tag{A.3c}$$

Let us now define the other Network matrices to be used in the subsequent formulations of the problem:

$$\mathbf{Y}_{N_i} = \mathbf{e}_{N_i}\mathbf{e}_{N_i}{}^\dagger\mathcal{Y} \tag{A.4a}$$

$$Y_{N_i} = \frac{1}{2}\begin{pmatrix} \text{Re}\{\mathbf{Y}_{N_i} + \mathbf{Y}^\dagger_{N_i}\} & \text{Im}\{\mathbf{Y}^\dagger_{N_i} - \mathbf{Y}_{N_i}\} \\ \text{Im}\{\mathbf{Y}_{N_i} - \mathbf{Y}^\dagger_{N_i}\} & \text{Re}\{\mathbf{Y}_{N_i} + \mathbf{Y}^\dagger_{N_i}\} \end{pmatrix} \tag{A.4b}$$

$$\overline{Y}_{N_i} = \frac{1}{2}\begin{pmatrix} \text{Im}\{\mathbf{Y}_{N_i} + \mathbf{Y}^\dagger_{N_i}\} & \text{Re}\{\mathbf{Y}_{N_i} - \mathbf{Y}^\dagger_{N_i}\} \\ \text{Re}\{\mathbf{Y}^\dagger_{N_i} - \mathbf{Y}_{N_i}\} & \text{Im}\{\mathbf{Y}_{N_i} + \mathbf{Y}^\dagger_{N_i}\} \end{pmatrix} \tag{A.4c}$$

$$\mathbf{Y}_{T_{r_{N_iN_j}}} = \left(\frac{y'_{T_{r_{N_iN_j}}}}{2} + y_{T_{r_{N_iN_j}}} \right)\mathbf{e}_{N_i}\mathbf{e}_{N_i}{}^\dagger - (y_{T_{r_{N_iN_j}}})\mathbf{e}_{N_i}\mathbf{e}_{N_j}{}^\dagger \tag{A.4d}$$

$$Y_{T_{r_{N_iN_j}}} = \frac{1}{2}\begin{pmatrix} \text{Re}\{\mathbf{Y}_{T_{r_{N_iN_j}}} + \mathbf{Y}^\dagger_{T_{r_{N_iN_j}}}\} & \text{Im}\{\mathbf{Y}^\dagger_{T_{r_{N_iN_j}}} - \mathbf{Y}_{T_{r_{N_iN_j}}}\} \\ \text{Im}\{\mathbf{Y}_{T_{r_{N_iN_j}}} - \mathbf{Y}^\dagger_{T_{r_{N_iN_j}}}\} & \text{Re}\{\mathbf{Y}_{T_{r_{N_iN_j}}} + \mathbf{Y}^\dagger_{T_{r_{N_iN_j}}}\} \end{pmatrix} \tag{A.4e}$$

$$\overline{Y}_{T_{r_{N_iN_j}}} = \frac{1}{2}\begin{pmatrix} \mathrm{Im}\{\mathbf{Y}_{T_{r_{N_iN_j}}} + \mathbf{Y}^{\dagger}_{T_{r_{N_iN_j}}}\} & \mathrm{Re}\{\mathbf{Y}_{T_{r_{N_iN_j}}} - \mathbf{Y}^{\dagger}_{T_{r_{N_iN_j}}}\} \\ \mathrm{Re}\{\mathbf{Y}^{\dagger}_{T_{r_{N_iN_j}}} - \mathbf{Y}_{T_{r_{N_iN_j}}}\} & \mathrm{Im}\{\mathbf{Y}_{T_{r_{N_iN_j}}} + \mathbf{Y}^{\dagger}_{T_{r_{N_iN_j}}}\} \end{pmatrix} \tag{A.5a}$$

$$K_{T_{r_{N_iN_j}}} = \begin{pmatrix} \mathrm{Re}\{\mathbf{Y}^{\dagger}_{T_{r_{N_iN_j}}}\} & 0 \\ 0 & \mathrm{Im}\{-\mathbf{Y}^{\dagger}_{T_{r_{N_iN_j}}}\} \end{pmatrix} \tag{A.5b}$$

$$\overline{K}_{T_{r_{N_iN_j}}} = \begin{pmatrix} \mathrm{Im}\{\mathbf{Y}^{\dagger}_{T_{r_{N_iN_j}}}\} & 0 \\ 0 & \mathrm{Re}\{\mathbf{Y}^{\dagger}_{T_{r_{N_iN_j}}}\} \end{pmatrix} \tag{A.5c}$$

$$M_{N_i} = \begin{pmatrix} \mathbf{e}_{N_i}{\mathbf{e}_{N_i}}^{\dagger} & 0 \\ 0 & \mathbf{e}_{N_i}{\mathbf{e}_{N_i}}^{\dagger} \end{pmatrix} \tag{A.6a}$$

$$M_{N_iN_j} = \begin{pmatrix} (\mathbf{e}_{N_i} - \mathbf{e}_{N_j})(\mathbf{e}_{N_i} - \mathbf{e}_{N_j})^{\dagger} & 0 \\ 0 & (\mathbf{e}_{N_i} - \mathbf{e}_{N_j})(\mathbf{e}_{N_i} - \mathbf{e}_{N_j})^{\dagger} \end{pmatrix} \tag{A.6b}$$

$$\Theta_{T_{r_{N_iN_j}}} = K_{T_{r_{N_iN_j}}} \mathcal{E}_{N_i}{\mathcal{E}_{N_i}}^{\dagger} {K_{T_{r_{N_iN_j}}}}^{\dagger} + \overline{K}_{T_{r_{N_iN_j}}} \mathcal{E}_{N_i}{\mathcal{E}_{N_i}}^{\dagger} \overline{K}^{\dagger}_{T_{r_{N_iN_j}}} \tag{A.6c}$$

$$X = [\mathrm{Re}\{V\}^{\dagger}\ \mathrm{Im}\{V\}^{\dagger}]^{\dagger} \tag{A.6d}$$

$$W = XX^{\dagger} \tag{A.6e}$$

A.3.3 OPF Reformulation

A.3.3.1 OPF Reformulation

With the above, we will now present a slightly reformulated version of the OPF from (A.2). But before we do that, we need to define a few other quantities as follows:

$$\begin{aligned} P_{N_i,inj} &= \sum_{g_q \in N_i} P_{g_{q_{N_i}}} - P_{D_{d_{N_i}}},\ \forall N_i \in \mathcal{N} \\ &= \mathrm{Re}\{V_{N_i}{I_{N_i}}^{H}\} = \mathrm{Re}\{V^{H}\mathbf{e}_{N_i}{\mathbf{e}_{N_i}}^{H}I\} = \mathrm{Re}\{V^{H}\mathbf{Y}_{N_i}V\} \end{aligned}$$

$$
\begin{aligned}
&= X^{\dagger}\begin{pmatrix}\mathrm{Re}\{\mathbf{Y}_{N_i}\} & -\mathrm{Im}\{\mathbf{Y}_{N_i}\}\\ \mathrm{Im}\{\mathbf{Y}_{N_i}\} & \mathrm{Re}\{\mathbf{Y}_{N_i}\}\end{pmatrix}X\\
&= X^{\dagger}\frac{1}{2}\begin{pmatrix}\mathrm{Re}\{\mathbf{Y}_{N_i}+\mathbf{Y}_{N_i}{}^{\dagger}\} & \mathrm{Im}\{\mathbf{Y}_{N_i}{}^{\dagger}-\mathbf{Y}_{N_i}\}\\ \mathrm{Im}\{\mathbf{Y}_{N_i}-\mathbf{Y}_{N_i}{}^{\dagger}\} & \mathrm{Re}\{\mathbf{Y}_{N_i}+\mathbf{Y}_{N_i}{}^{\dagger}\}\end{pmatrix}X = X^{\dagger}Y_{N_i}X\\
&= Tr\{Y_{N_i}XX^{\dagger}\} = Tr\{Y_{N_i}W\} \qquad \text{(A.7a)}
\end{aligned}
$$

$$
\begin{aligned}
Similarly,\ Q_{N_i,inj} &= \sum_{g_q\in N_i} Q_{g_{q_{N_i}}} - Q_{D_{d_{N_i}}},\ \forall N_i \in \mathcal{N}\\
&= Tr\{\overline{Y}_{N_i}W\} \qquad \text{(A.8a)}
\end{aligned}
$$

$$
|V_{N_i}|^2 = Tr\{M_{N_i}W\},:\forall N_i \in \mathcal{N} \qquad \text{(A.8b)}
$$

$$
P_{N_i,inj} = -P_{D_{d_{N_i}}},\ \forall N_i \in \mathcal{N} \backsim G \qquad \text{(A.8c)}
$$

$$
Q_{N_i,inj} = -Q_{D_{d_{N_i}}},\ \forall N_i \in \mathcal{N} \backsim G \qquad \text{(A.8d)}
$$

$$
\overline{P}_{N_i} = \underline{P}_{N_i} = \overline{Q}_{N_i} = \underline{Q}_{N_i} = 0, \forall N_i \in \mathcal{N} \backsim G \qquad \text{(A.8e)}
$$

$$
\begin{aligned}
I_{T_{r_{N_iN_j}}} &= (V_{N_i}-V_{N_j})y_{T_{r_{N_iN_j}}} + V_{N_i}\frac{y'_{T_{r_{N_iN_j}}}}{2} = X^{\dagger}K_{T_{r_{N_iN_j}}}\mathcal{E}_{N_i} + \sqrt{-1}(X^{\dagger}\overline{K}_{T_{r_{N_iN_j}}}\mathcal{E}_{N_i}),\ \forall T_r \in T\\
|I_{T_{r_{N_iN_j}}}|^2 &= X^{\dagger}(K_{T_{r_{N_iN_j}}}\mathcal{E}_{N_i}\mathcal{E}_{N_i}{}^{\dagger}K_{T_{r_{N_iN_j}}}{}^{\dagger} + \overline{K}_{T_{r_{N_iN_j}}}\mathcal{E}_{N_i}\mathcal{E}_{N_i}{}^{\dagger}\overline{K}^{\dagger}_{T_{r_{N_iN_j}}})X\\
&= Tr\{\Theta_{T_{r_{N_iN_j}}}XX^{\dagger}\} = Tr\{\Theta_{T_{r_{N_iN_j}}}W\} \qquad \text{(A.8f)}
\end{aligned}
$$

$$
\begin{aligned}
S_{T_{r_{N_iN_j}}}{}^{H} &= V_{N_i}{}^{H}\left(V_{N_i}\frac{y_{T_{r_{N_iN_j}}}}{2}'\right) + V_{N_i}{}^{H}(V_{N_i}-V_{N_j})y_{T_{r_{N_iN_j}}} = V^{H}\mathbf{Y}_{T_{r_{N_iN_j}}}V,\ \forall T_r \in T\\
&= Tr\{Y_{T_{r_{N_iN_j}}}W\} - \sqrt{-1}Tr\{\overline{Y}_{T_{r_{N_iN_j}}}W\} \qquad \text{(A.9a)}
\end{aligned}
$$

$$
P_{T_{r_{N_iN_j}}} = Tr\{Y_{T_{r_{N_iN_j}}}W\},\ \forall T_r \in T \qquad \text{(A.9b)}
$$

$$
|V_{N_iN_j}|^2 = Tr\{M_{N_iN_j}W\},:\forall T_r \in T \qquad \text{(A.9c)}
$$

Hence, combining all the information from above, the reformulated OPF is (with corresponding Lagrange Multipliers shown to the right of $\leftrightarrow$ symbol):

$$\min_{W} \sum_{N_i \in \mathcal{N}} \alpha_{g_q}(Tr\{Y_{N_i}W\} + P_{D_{d_{N_i}}})^2 + \beta_{g_q}(Tr\{Y_{N_i}W\} + P_{D_{d_{N_i}}}) + \gamma_{g_q} \tag{A.10a}$$

$$Subject\ to:$$

$$\forall N_i \in \mathcal{N}$$

$$\underline{P}_{N_i} - P_{D_{d_{N_i}}} \leqslant Tr\{Y_{N_i}W\} \leqslant \overline{P}_{N_i} - P_{D_{d_{N_i}}} \leftrightarrow \underline{\sigma}^P_{N_i}, \overline{\sigma}^P_{N_i} \tag{A.10b}$$

$$\underline{Q}_{N_i} - Q_{D_{d_{N_i}}} \leqslant Tr\{\overline{Y}_{N_i}W\} \leqslant \overline{Q}_{N_i} - Q_{D_{d_{N_i}}} \leftrightarrow \underline{\sigma}^Q_{N_i}, \overline{\sigma}^Q_{N_i} \tag{A.10c}$$

$$(\underline{V}_{N_i})^2 \leqslant Tr\{M_{N_i}W\} \leqslant (\overline{V}_{N_i})^2 \leftrightarrow \underline{\gamma}^V_{N_i}, \overline{\gamma}^V_{N_i} \tag{A.10d}$$

$$\forall T_r \in T$$

$$Tr\{Y_{T_r}W\} \leqslant \overline{P}^{(0)}_{T_r} \leftrightarrow \lambda^P_{T_r} \tag{A.11a}$$

$$Tr\{\Theta_{T_r}W\} \leqslant (\overline{I}^{(0)}_{T_r})^2 \leftrightarrow \gamma^I_{T_r} \tag{A.11b}$$

$$Tr\{M_{T_r}W\} \leqslant (\overline{\Delta V}_{T_r})^2 \leftrightarrow \mu^{\Delta V}_{T_r} \tag{A.11c}$$

$$\begin{pmatrix} (\overline{S}^{(0)}_{T_r})^2 & -Tr\{Y_{T_r}W\} & -Tr\{\overline{Y}_{T_r}W\} \\ -Tr\{Y_{T_r}W\} & 1 & 0 \\ -Tr\{\overline{Y}_{T_r}W\} & 0 & 1 \end{pmatrix} \succeq 0 \leftrightarrow \begin{pmatrix} r^1_{T_r} & r^2_{T_r} & r^3_{T_r} \\ r^2_{T_r} & r^4_{T_r} & r^6_{T_r} \\ r^3_{T_r} & r^6_{T_r} & r^5_{T_r} \end{pmatrix} \tag{A.11d}$$

$$W \succeq 0 \tag{A.11e}$$

$$rank(W) = 1 \tag{A.11f}$$

A.3.3.2 Schur's Complements

Consider a requirement $C_{g_q}(P_{g_q}) \leqslant \mathbb{A}_{g_q}$, with C_{g_q} defined in (A.2a) which is equivalent to

$$\begin{pmatrix} \beta_{g_q} Tr\{Y_{g_{q_{N_i}}} W\} - \mathbb{A}_{g_q} + a_{g_{q_{N_i}}} & \sqrt{\alpha_{g_q}} Tr\{Y_{g_{q_{N_i}}} W\} + b_{g_{q_{N_i}}} \\ \sqrt{\alpha_{g_q}} Tr\{Y_{g_{q_{N_i}}} W\} + b_{g_{q_{N_i}}} & -1 \end{pmatrix} \preceq 0 \tag{A.12a}$$

$$a_{g_{q_{N_i}}} = \gamma_{g_q} + \beta_{g_q} P_{D_{d_{N_i}}}, b_{g_{q_{N_i}}} = \sqrt{\alpha_{g_q}} P_{D_{d_{N_i}}} \tag{A.12b}$$

Hence, the Reformulated OPF from above can further be slightly modified into the four different forms presented next

A.3.3.3 Optimization 1

$$\min_{W} \sum_{g_q \in G} \mathbb{A}_{g_q} \tag{A.13a}$$

$$Subject\ to:$$

$$\underline{P}_{N_i} - P_{D_{d_{N_i}}} \leqslant Tr\{Y_{N_i} W\} \leqslant \overline{P}_{N_i} - P_{D_{d_{N_i}}} \leftrightarrow \underline{\sigma}^P_{N_i}, \overline{\sigma}^P_{N_i} \tag{A.13b}$$

$$\underline{Q}_{N_i} - Q_{D_{d_{N_i}}} \leqslant Tr\{\overline{Y}_{N_i} W\} \leqslant \overline{Q}_{N_i} - Q_{D_{d_{N_i}}} \leftrightarrow \underline{\sigma}^Q_{N_i}, \overline{\sigma}^Q_{N_i} \tag{A.13c}$$

$$(\underline{V}_{N_i})^2 \leqslant Tr\{M_{N_i} W\} \leqslant (\overline{V}_{N_i})^2 \leftrightarrow \underline{\gamma}^V_{N_i}, \overline{\gamma}^V_{N_i} \tag{A.13d}$$

$$\begin{pmatrix} \beta_{g_q} Tr\{Y_{g_{q_{N_i}}} W\} - \mathbb{A}_{g_q} + a_{g_{q_{N_i}}} & \sqrt{\alpha_{g_q}} Tr\{Y_{g_{q_{N_i}}} W\} + b_{g_{q_{N_i}}} \\ \sqrt{\alpha_{g_q}} Tr\{Y_{g_{q_{N_i}}} W\} + b_{g_{q_{N_i}}} & -1 \end{pmatrix} \preceq 0 \leftrightarrow \begin{pmatrix} 1 & r^1_{g_{q_{N_i}}} \\ r^1_{g_{q_{N_i}}} & r^2_{g_{q_{N_i}}} \end{pmatrix} \tag{A.13e}$$

$$\forall T_r \in T$$

$$Tr\{Y_{T_r}W\} \leqslant \overline{P}_{T_r}^{(0)} \leftrightarrow \lambda_{T_r}^{P} \tag{A.14a}$$

$$Tr\{\Theta_{T_r}W\} \leqslant (\overline{I}_{T_r}^{(0)})^2 \leftrightarrow \gamma_{T_r}^{I} \tag{A.14b}$$

$$Tr\{M_{T_r}W\} \leqslant (\overline{\Delta V}_{T_r})^2 \leftrightarrow \mu_{T_r}^{\Delta V} \tag{A.14c}$$

$$\begin{pmatrix} (\overline{S}_{T_r}^{(0)})^2 & -Tr\{Y_{T_r}W\} & -Tr\{\overline{Y}_{T_r}W\} \\ -Tr\{Y_{T_r}W\} & 1 & 0 \\ -Tr\{\overline{Y}_{T_r}W\} & 0 & 1 \end{pmatrix} \succeq 0 \leftrightarrow \begin{pmatrix} r_{T_r}^1 & r_{T_r}^2 & r_{T_r}^3 \\ r_{T_r}^2 & r_{T_r}^4 & r_{T_r}^6 \\ r_{T_r}^3 & r_{T_r}^6 & r_{T_r}^5 \end{pmatrix} \tag{A.14d}$$

$$W = XX^{\dagger} \tag{A.14e}$$

A.3.3.4 Optimization 2

$$\min_{W} \sum_{g_q \in G} \mathbb{A}_{g_q} \tag{A.15a}$$

$$Subject\ to:$$

$$\underline{P}_{N_i} - P_{D_{d_{N_i}}} \leqslant Tr\{Y_{N_i}W\} \leqslant \overline{P}_{N_i} - P_{D_{d_{N_i}}} \leftrightarrow \underline{\sigma}_{N_i}^{P}, \overline{\sigma}_{N_i}^{P} \tag{A.15b}$$

$$\underline{Q}_{N_i} - Q_{D_{d_{N_i}}} \leqslant Tr\{\overline{Y}_{N_i}W\} \leqslant \overline{Q}_{N_i} - Q_{D_{d_{N_i}}} \leftrightarrow \underline{\sigma}_{N_i}^{Q}, \overline{\sigma}_{N_i}^{Q} \tag{A.15c}$$

$$(\underline{V}_{N_i})^2 \leqslant Tr\{M_{N_i}W\} \leqslant (\overline{V}_{N_i})^2 \leftrightarrow \underline{\gamma}_{N_i}^{V}, \overline{\gamma}_{N_i}^{V} \tag{A.15d}$$

$$\begin{pmatrix} \beta_{g_q} Tr\{Y_{g_{q_{N_i}}}W\} - \mathbb{A}_{g_q} + a_{g_{q_{N_i}}} & \sqrt{\alpha_{g_q}} Tr\{Y_{g_{q_{N_i}}}W\} + b_{g_{q_{N_i}}} \\ \sqrt{\alpha_{g_q}} Tr\{Y_{g_{q_{N_i}}}W\} + b_{g_{q_{N_i}}} & -1 \end{pmatrix} \preceq 0 \leftrightarrow \begin{pmatrix} 1 & r_{g_{q_{N_i}}}^1 \\ r_{g_{q_{N_i}}}^1 & r_{g_{q_{N_i}}}^2 \end{pmatrix} \tag{A.15e}$$

$$\forall T_r \in T$$

$$Tr\{Y_{T_r}W\} \leqslant \overline{P}_{T_r}^{(0)} \leftrightarrow \lambda_{T_r}^{P} \tag{A.16a}$$

$$Tr\{\Theta_{T_r}W\} \leqslant (\overline{I}_{T_r}^{(0)})^2 \leftrightarrow \gamma_{T_r}^{I} \tag{A.16b}$$

$$Tr\{M_{T_r}W\} \leqslant (\overline{\Delta V}_{T_r})^2 \leftrightarrow \mu_{T_r}^{\Delta V} \tag{A.16c}$$

$$\begin{pmatrix} (\overline{S}_{T_r}^{(0)})^2 & -Tr\{Y_{T_r}W\} & -Tr\{\overline{Y}_{T_r}W\} \\ -Tr\{Y_{T_r}W\} & 1 & 0 \\ -Tr\{\overline{Y}_{T_r}W\} & 0 & 1 \end{pmatrix} \succeq 0 \leftrightarrow \begin{pmatrix} r_{T_r}^1 & r_{T_r}^2 & r_{T_r}^3 \\ r_{T_r}^2 & r_{T_r}^4 & r_{T_r}^6 \\ r_{T_r}^3 & r_{T_r}^6 & r_{T_r}^5 \end{pmatrix} \tag{A.16d}$$

$$W \succeq 0 \tag{A.16e}$$

$$rank(W) = 1 \tag{A.16f}$$

A.3.3.5 Optimization 3

$$\min_{W} \sum_{g_q \in G} \mathbb{A}_{g_q} \tag{A.17a}$$

$$Subject\ to:$$

$$\underline{P}_{N_i} - P_{D_{d_{N_i}}} \leqslant Tr\{Y_{N_i}W\} \leqslant \overline{P}_{N_i} - P_{D_{d_{N_i}}} \leftrightarrow \underline{\sigma}_{N_i}^P, \overline{\sigma}_{N_i}^P \tag{A.17b}$$

$$\underline{Q}_{N_i} - Q_{D_{d_{N_i}}} \leqslant Tr\{\overline{Y}_{N_i}W\} \leqslant \overline{Q}_{N_i} - Q_{D_{d_{N_i}}} \leftrightarrow \underline{\sigma}_{N_i}^Q, \overline{\sigma}_{N_i}^Q \tag{A.17c}$$

$$(\underline{V}_{N_i})^2 \leqslant Tr\{M_{N_i}W\} \leqslant (\overline{V}_{N_i})^2 \leftrightarrow \underline{\gamma}_{N_i}^V, \overline{\gamma}_{N_i}^V \tag{A.17d}$$

$$\begin{pmatrix} \beta_{g_q} Tr\{Y_{g_{q_{N_i}}}W\} - \mathbb{A}_{g_q} + a_{g_{q_{N_i}}} & \sqrt{\alpha_{g_q}} Tr\{Y_{g_{q_{N_i}}}W\} + b_{g_{q_{N_i}}} \\ \sqrt{\alpha_{g_q}} Tr\{Y_{g_{q_{N_i}}}W\} + b_{g_{q_{N_i}}} & -1 \end{pmatrix} \preceq 0 \leftrightarrow \begin{pmatrix} 1 & r_{g_{q_{N_i}}}^1 \\ r_{g_{q_{N_i}}}^1 & r_{g_{q_{N_i}}}^2 \end{pmatrix} \tag{A.17e}$$

$$\forall T_r \in T$$

$$Tr\{Y_{T_r}W\} \leqslant \overline{P}_{T_r}^{(0)} \leftrightarrow \lambda_{T_r}^P \tag{A.18a}$$

$$Tr\{\Theta_{T_r}W\} \leqslant (\overline{I}_{T_r}^{(0)})^2 \leftrightarrow \gamma_{T_r}^I \tag{A.18b}$$

$$Tr\{M_{T_r}W\} \leqslant (\overline{\Delta V}_{T_r})^2 \leftrightarrow \mu_{T_r}^{\Delta V} \tag{A.18c}$$

$$\begin{pmatrix} (\overline{S}_{T_r}^{(0)})^2 & -Tr\{Y_{T_r}W\} & -Tr\{\overline{Y}_{T_r}W\} \\ -Tr\{Y_{T_r}W\} & 1 & 0 \\ -Tr\{\overline{Y}_{T_r}W\} & 0 & 1 \end{pmatrix} \succeq 0 \leftrightarrow \begin{pmatrix} r_{T_r}^1 & r_{T_r}^2 & r_{T_r}^3 \\ r_{T_r}^2 & r_{T_r}^4 & r_{T_r}^6 \\ r_{T_r}^3 & r_{T_r}^6 & r_{T_r}^5 \end{pmatrix} \tag{A.18d}$$

$$W \succeq 0 \tag{A.18e}$$

A.3.3.6 Lagrange Multipliers

Here we consider some relations amongst Lagrange multipliers.

$$\sigma^P_{N_i} = -\underline{\sigma}^P_{N_i} + \overline{\sigma}^P_{N_i} + \beta_{g_q} + 2\sqrt{\alpha_{g_q}} r^1_{g_{q_{N_i}}} \forall \text{Generator Nodes} \tag{A.19a}$$

$$\sigma^P_{N_i} = -\underline{\sigma}^P_{N_i} + \overline{\sigma}^P_{N_i}, else \tag{A.19b}$$

$$\sigma^Q_{N_i} = -\underline{\sigma}^Q_{N_i} + \overline{\sigma}^Q_{N_i} \forall N_i \in \mathcal{N} \tag{A.19c}$$

$$\gamma^V_{N_i} = -\underline{\gamma}^V_{N_i} + \overline{\gamma}^V_{N_i} \forall N_i \in \mathcal{N} \tag{A.19d}$$

$$\mathcal{X} = \left[\underline{\sigma}^P_{N_i}, \underline{\sigma}^Q_{N_i}, \underline{\gamma}^V_{N_i}, \overline{\sigma}^P_{N_i}, \overline{\sigma}^Q_{N_i}, \overline{\gamma}^V_{N_i}, \lambda^P_{T_r}, \gamma^I_{T_r}, \mu^{\Delta V}_{T_r}\right] \tag{A.19e}$$

$$\mathcal{R} = \left[\begin{pmatrix} r^1_{T_r} & r^2_{T_r} & r^3_{T_r} \\ r^2_{T_r} & r^4_{T_r} & r^6_{T_r} \\ r^3_{T_r} & r^6_{T_r} & r^5_{T_r} \end{pmatrix}, \begin{pmatrix} 1 & r^1_{g_{q_{N_i}}} \\ r^1_{g_{q_{N_i}}} & r^2_{g_{q_{N_i}}} \end{pmatrix} \right] \tag{A.19f}$$

A.3.3.7 Dual OPF Formulation

$$\begin{aligned} H(\mathcal{X}, \mathcal{R}) = \sum_{N_i \in \mathcal{N}} \Big\{ & \underline{\sigma}^P_{N_i} \underline{P}_{N_i} - \overline{\sigma}^P_{N_i} \overline{P}_{N_i} \\ & + \sigma^P_{N_i} P_{D_{d_{N_i}}} + \underline{\sigma}^Q_{N_i} \underline{Q}_{N_i} - \overline{\sigma}^Q_{N_i} \overline{Q}_{N_i} + \sigma^Q_{N_i} Q_{D_{d_{N_i}}} \\ & + \underline{\gamma}^V_{N_i} (\underline{V}_{N_i})^2 - \overline{\gamma}^V_{N_i} (\overline{V}_k)^2 \Big\} \\ & - \sum_{T_r \in T} \Big\{ \lambda^P_{T_r} \overline{P}^{(0)}_{T_r} + \mu^{\Delta V}_{T_r} \left(\overline{\Delta V}_{T_r}\right)^2 \\ & + \gamma^I_{T_r} \left(\overline{I}^{(0)}_{T_r}\right)^2 + r^1_{T_r} (\overline{S}^{(0)}_{T_r})^2 + r^4_{T_r} + r^6_{T_r} \Big\} \end{aligned} \tag{A.20a}$$

$$B(\mathcal{X},\mathcal{R}) = \sum_{N_i \in \mathcal{N}} \Big\{ \sigma^P_{N_i} Y_{N_i} + \sigma^Q_{N_i} \overline{Y}_{N_i} + \gamma^V_{N_i} M_{N_i} \Big\}$$
$$+ \sum_{T_r \in T} \Big\{ (2r^2_{T_r} + \lambda^P_{T_r}) Y_{T_r} + 2r^3_{T_r} \overline{Y}_{T_r} + \mu^{\Delta V}_{T_r} M_{T_r} + \gamma^I_{T_r} \Theta_{T_r} \Big\} \tag{A.20b}$$

A.3.3.8 The Lagrangian

$$H(\mathcal{X},\mathcal{R}) + Tr[B(\mathcal{X},\mathcal{R})XX^\dagger] + \textstyle\sum_{g_q \in G}(1 - r^0_{g_{q_{N_i}}})\{\mathbb{A}_{g_q} - \beta_{g_q}(1 + P_{D_{d_{N_i}}}) - \gamma_{g_q}\}$$

The Lagrange Dual Problem is

$\max_{\mathcal{X}\geq 0,\mathcal{R}\geq 0}\{\min_{X,\alpha}(H(\mathcal{X},\mathcal{R}) + Tr[B(\mathcal{X},\mathcal{R})XX^\dagger] + \sum_{g_q \in G}(1 - r^0_{g_{q_{N_i}}})\{\mathbb{A}_{g_q} - \beta_{g_q}(1 + P_{D_{d_{N_i}}}) - \gamma_{g_q}\})\}$

This gives rise to the Dual OPF

A.3.3.9 Optimization 4(Dual OPF)

$$\max_{\mathcal{X}\geq 0,\mathcal{R}\geq 0} H(\mathcal{X},\mathcal{R}) \tag{A.21a}$$
$$Subject\ to:$$
$$B(\mathcal{X},\mathcal{R}) \succeq 0 \tag{A.21b}$$
$$\begin{pmatrix} r^1_{T_r} & r^2_{T_r} & r^3_{T_r} \\ r^2_{T_r} & r^4_{T_r} & r^6_{T_r} \\ r^3_{T_r} & r^6_{T_r} & r^5_{T_r} \end{pmatrix} \succeq 0 \forall T_r \in T \tag{A.21c}$$
$$\begin{pmatrix} 1 & r^1_{g_{q_{N_i}}} \\ r^1_{g_{q_{N_i}}} & r^2_{g_{q_{N_i}}} \end{pmatrix} \succeq 0 \forall g_q \in G \tag{A.21d}$$

A.3.4 Bridging the Duality Gap

A.3.4.1 Duality Gap (Slater's Constraint Qualification)

- Existence of finite optimum of Reformulated Primal OPF (Optimization 1) guarantees finite optimum of Dual OPF (Optimization 4).

- Non-empty interior of the feasible region of Optimization 4.

The above conditions are necessary to establish a zero duality gap between the primal and the dual OPF problems.

A.4 SDP-SCOPF Semidefinite Programming Security Constrained OPF

A.4.1 Classical Primal AC SCOPF Formulation (Without Generation Reserves)

In this formulation of the AC-SCOPF, the generation reserve capacities are not represented explicitly, but the formulation takes into account decision variables, $\tilde{\Delta}_P^{(c)}$, $\tilde{\Delta}_Q^{(c)}$, $\tilde{\tilde{\Delta}}_P^{(c)}$, and $\tilde{\tilde{\Delta}}_Q^{(c)}$ corresponding to each contingency scenario, c for real and reactive power generation, respectively, such that the change in real and reactive power output from the base case operation to the scenario, c is limited by a fraction (called the participation factor) of $\tilde{\Delta}_P^{(c)}$, $\tilde{\Delta}_Q^{(c)}$, $\tilde{\tilde{\Delta}}_P^{(c)}$, and $\tilde{\tilde{\Delta}}_Q^{(c)}$. Implicitly, $\tilde{\Delta}_P^{(c)}$, $\tilde{\Delta}_Q^{(c)}$, $\tilde{\tilde{\Delta}}_P^{(c)}$, and $\tilde{\tilde{\Delta}}_Q^{(c)}$ represent the changes in real and reactive power losses in case of a change of network topology due to line contingency and a loss of generation, for generator contingency. The role of $\tilde{\Delta}_P^{(c)}$, $\tilde{\Delta}_Q^{(c)}$, $\tilde{\tilde{\Delta}}_P^{(c)}$, and $\tilde{\tilde{\Delta}}_Q^{(c)}$ is also to make sure that the generating units that are operating close to their upper limits need to ramp up less, that the others, as well as those with lower values of ramp-rate limits also have to change their productions less than those that have higher values of ramp-rate limits.

A.4.1.1 Classical AC SCOPF Formulation: The Primal Problem (Base-Case Constraints)

$$\textbf{Objective Function}: \min_{V,P_g,Q_g,\tilde{\Delta}_P^{(c)},\tilde{\Delta}_Q^{(c)},\Delta P^{(c)},\Delta Q^{(c)},\tilde{\tilde{\Delta}}_P^{(c)},\tilde{\tilde{\Delta}}_Q^{(c)}}$$

$$\sum_{g_q \in G} \Big(C_{g_q}(P_{g_q}^{(0)}) = \alpha_{g_q}(P_{g_q}^{(0)})^2 + \beta_{g_q} P_{g_q}^{(0)} + \gamma_{g_q} \Big)$$

$$+ \sum_{c \in \mathcal{L}} \Big(C^{(c)} \{ \tilde{\Delta}_P^{(c)} + \tilde{\Delta}_Q^{(c)} \} \Big) \tag{A.22a}$$

$$\textbf{Subject to}: \underline{P}_{g_q} \leqslant P_{g_q}^{(0)} \leqslant \overline{P}_{g_q}, \ \forall g_q \in G \tag{A.22b}$$

$$\underline{Q}_{g_q} \leqslant Q_{g_q}^{(0)} \leqslant \overline{Q}_{g_q}, \ \forall g_q \in G \tag{A.22c}$$

$$\sum_{g_q \in N_i} P_{g_{q_{N_i}}}^{(0)} - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \sum_{T_r \in N_i N_{\bar{i}}} Re\{ V_{N_i}^{(0)} (V_{N_i}^{(0)} - V_{N_{\bar{i}}}^{(0)})^H y_{T_{r_{N_i N_{\bar{i}}}}}^{(0)H} \} \tag{A.22d}$$

$$\sum_{g_q \in N_i} Q_{g_{q_{N_i}}}^{(0)} - Q_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \sum_{T_r \in N_i N_{\bar{i}}} Im\{ V_{N_i}^{(0)} (V_{N_i}^{(0)} - V_{N_{\bar{i}}}^{(0)})^H y_{T_{r_{N_i N_{\bar{i}}}}}^{(0)H} \} \tag{A.22e}$$

$$\underline{V}_{N_i} \leqslant |V_{N_i}^{(0)}| \leqslant \overline{V}_{N_i}, \ \forall N_i \in \mathcal{N} \tag{A.22f}$$

$$|S_{T_r}^{(0)}| \leqslant \overline{S}_{T_r}^{(0)}, \ \forall T_r \in T \tag{A.22g}$$

$$\tag{A.22h}$$

A.4.1.2 Classical AC SCOPF Formulation: The Primal Problem (Base-Case Constraints)

$$|P_{T_r}^{(0)}| \leqslant \overline{P}_{T_r}^{(0)},\ \forall T_r \in T \tag{A.23a}$$

$$(|I_{T_r}^{(0)}|)^2 \leqslant (\overline{I}_{T_r}^{(0)})^2,\ \forall T_r \in T \tag{A.23b}$$

$$|V_{N_i}^{(0)} - V_{N_{\bar{i}}}^{(0)}| \leqslant \overline{\Delta V_{T_{r_{N_i N_{\bar{i}}}}}},\ \forall T_r \in T \tag{A.23c}$$

A.4.1.3 Classical AC SCOPF Formulation: The Primal Problem (Contingency Scenarios)

$$\underline{P}_{g_q} \leqslant (P_{g_q}^{(0)} + \Delta P_{g_q}^{(c)}) \leqslant \overline{P}_{g_q},\ \forall c \in \mathcal{L}\ \forall g_q \in G \tag{A.24a}$$

$$\underline{Q}_{g_q} \leqslant (Q_{g_q}^{(0)} + \Delta Q_{g_q}^{(c)}) \leqslant \overline{Q}_{g_q},\ \forall c \in \mathcal{L}\ \forall g_q \in G \tag{A.24b}$$

$$(P_{g_q}^{(0)} + \Delta P_{g_q}^{(c)}) - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \sum_{T_r \in N_i N_{\bar{i}}} Re\{V_{N_i}^{(c)} (V_{N_i}^{(c)} - V_{N_{\bar{i}}}^{(c)})^H y_{T_{r_{N_i N_{\bar{i}}}}}^{(c)H}\} \tag{A.24c}$$

$$(Q_{g_q}^{(0)} + \Delta Q_{g_q}^{(c)}) - Q_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \sum_{T_r \in N_i N_{\bar{i}}} Im\{V_{N_i}^{(c)} (V_{N_i}^{(c)} - V_{N_{\bar{i}}}^{(c)})^H y_{T_{r_{N_i N_{\bar{i}}}}}^{(c)H}\} \tag{A.24d}$$

$$\underline{V}_{N_i} \leqslant |V_{N_i}^{(c)}| \leqslant \overline{V}_{N_i},\ \forall N_i \in \mathcal{N} \tag{A.24e}$$

$$|S_{T_r}^{(c)}| \leqslant \overline{S}_{T_r}^{(c)},\ \forall T_r \in T \tag{A.24f}$$

$$|P_{T_r}^{(c)}| \leqslant \overline{P}_{T_r}^{(c)},\ \forall T_r \in T \tag{A.24g}$$

$$(|I_{T_r}^{(c)}|)^2 \leqslant (\overline{I}_{T_r}^{(c)})^2,\ \forall T_r \in T \tag{A.24h}$$

$$|V_{N_i}^{(c)} - V_{N_{\bar{i}}}^{(c)}| \leqslant \Delta\overline{V}_{T_{r_{N_i N_{\bar{i}}}}},\ \forall T_r \in T \tag{A.24i}$$

A.4.1.4 Classical AC SCOPF Formulation: The Primal Problem (Ramping Constraints)

$$-\delta^{(c)}\tilde{\tilde{\Delta}}_P^{(c)} \leqslant \Delta P_{g_q}^{(c)} \leqslant \delta_{g_q}^{(c)}\tilde{\Delta}_P^{(c)} \tag{A.25a}$$

$$-\delta^{(c)}\tilde{\tilde{\Delta}}_Q^{(c)} \leqslant \Delta Q_{g_q}^{(c)} \leqslant \delta_{g_q}^{(c)}\tilde{\Delta}_Q^{(c)} \tag{A.25b}$$

$$-\overline{R}_{Pg_q} \leqslant \Delta P_{g_q}^{(c)} \leqslant \overline{R}_{Pg_q} \tag{A.25c}$$

$$-\overline{R}_{Qg_q} \leqslant \Delta Q_{g_q}^{(c)} \leqslant \overline{R}_{Qg_q} \tag{A.25d}$$

$$\tilde{\tilde{\Delta}}_P^{(c)} \geqslant 0 \tag{A.25e}$$

$$\tilde{\tilde{\Delta}}_Q^{(c)} \geqslant 0 \tag{A.25f}$$

A.4.2 Classical Primal AC SCOPF Formulation (With Generation Reserves)

In this case, the maximum amounts of generation reserves for real and reactive powers committed by a generator g_q, which we indicate as $AS_{Pg_q}^{max}$ and $AS_{Qg_q}^{max}$ are the decision variables, and the change in generation, which is equal to the participation factor of each generator, multiplied by the shortfall (or excess) in generation (indicated as $\delta_{g_q}^{(c)}\tilde{\Delta}_P^{(c)}$ and $\delta_{g_q}^{(c)}\tilde{\Delta}_Q^{(c)}$, as before) are limited by the respective decision variables corresponding to reserves commitment. The rest of the formulation is the same as before.

A.4.2.1 Classical AC SCOPF Formulation: The Primal Problem (Base-Case Constraints)

$$\textbf{Objective Function}: \min_{V,P_g,Q_g,AS^{max}_{Pg_q},AS^{max}_{Qg_q}} \sum_{g_q \in G} \Big(C_{g_q}(P^{(0)}_{g_q}) = \alpha_{g_q}(P^{(0)}_{g_q})^2 + \beta_{g_q} P^{(0)}_{g_q} + \gamma_{g_q} \Big)$$

$$+ \sum_{g_q \in G} \Big(C^{(AS)}_{Pg_q} AS^{max}_{Pg_q} + C^{(AS)}_{Qg_q} AS^{max}_{Qg_q} \Big) \tag{A.26a}$$

$$\textbf{Subject to}: \underline{P}_{g_q} \leqslant P^{(0)}_{g_q} + AS^{max}_{Pg_q} \leqslant \overline{P}_{g_q},\ \forall g_q \in G \tag{A.26b}$$

$$\underline{Q}_{g_q} \leqslant Q^{(0)}_{g_q} + AS^{max}_{Qg_q} \leqslant \overline{Q}_{g_q},\ \forall g_q \in G \tag{A.26c}$$

$$\sum_{g_q \in N_i} P^{(0)}_{g_{q_{N_i}}} - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \sum_{T_r \in N_i N_{\bar{i}}} Re\{V^{(0)}_{N_i}(V^{(0)}_{N_i} - V^{(0)}_{N_{\bar{i}}})^H y^{(0)H}_{T_{r_{N_i N_{\bar{i}}}}}\} \tag{A.26d}$$

$$\sum_{g_q \in N_i} Q^{(0)}_{g_{q_{N_i}}} - Q_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \sum_{T_r \in N_i N_{\bar{i}}} Im\{V^{(0)}_{N_i}(V^{(0)}_{N_i} - V^{(0)}_{N_{\bar{i}}})^H y^{(0)H}_{T_{r_{N_i N_{\bar{i}}}}}\} \tag{A.26e}$$

$$\underline{V}_{N_i} \leqslant |V^{(0)}_{N_i}| \leqslant \overline{V}_{N_i},\ \forall N_i \in \mathcal{N} \tag{A.26f}$$

$$|S^{(0)}_{T_r}| \leqslant \overline{S}^{(0)}_{T_r},\ \forall T_r \in T \tag{A.26g}$$

$$\tag{A.26h}$$

A.4.2.2 Classical AC SCOPF Formulation: The Primal Problem (Base-Case Constraints)

$$|P^{(0)}_{T_r}| \leqslant \overline{P}^{(0)}_{T_r},\ \forall T_r \in T \tag{A.27a}$$

$$(|I^{(0)}_{T_r}|)^2 \leqslant (\overline{I}^{(0)}_{T_r})^2,\ \forall T_r \in T \tag{A.27b}$$

$$|V^{(0)}_{N_i} - V^{(0)}_{N_{\bar{i}}}| \leqslant \overline{\Delta V_{T_{r_{N_i N_{\bar{i}}}}}},\ \forall T_r \in T \tag{A.27c}$$

A.4.2.3 Classical AC SCOPF Formulation: The Primal Problem (Contingency Scenarios)

$$\underline{P}_{g_q} \leqslant (P_{g_q}^{(0)} + \delta_{g_q}^{(c)} \tilde{\Delta}_P^{(c)}) \leqslant \overline{P}_{g_q}, \ \forall c \in \mathcal{L} \ \forall g_q \in G \tag{A.28a}$$

$$\underline{Q}_{g_q} \leqslant (Q_{g_q}^{(0)} + \delta_{g_q}^{(c)} \tilde{\Delta}_Q^{(c)}) \leqslant \overline{Q}_{g_q}, \ \forall c \in \mathcal{L} \ \forall g_q \in G \tag{A.28b}$$

$$(P_{g_q}^{(0)} + \delta_{g_q}^{(c)} \tilde{\Delta}_P^{(c)}) - P_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \sum_{T_r \in N_i N_{\bar{i}}} Re\{V_{N_i}^{(c)} (V_{N_i}^{(c)} - V_{N_{\bar{i}}}^{(c)})^H y_{T_{r_{N_i N_{\bar{i}}}}}^{(c)H}\} \tag{A.28c}$$

$$(Q_{g_q}^{(0)} + \delta_{g_q}^{(c)} \tilde{\Delta}_Q^{(c)}) - Q_{D_{d_{N_i}}} = \sum_{N_{\bar{i}} \in J(N_i)} \sum_{T_r \in N_i N_{\bar{i}}} Im\{V_{N_i}^{(c)} (V_{N_i}^{(c)} - V_{N_{\bar{i}}}^{(c)})^H y_{T_{r_{N_i N_{\bar{i}}}}}^{(c)H}\} \tag{A.28d}$$

$$\underline{V}_{N_i} \leqslant |V_{N_i}^{(c)}| \leqslant \overline{V}_{N_i}, \ \forall N_i \in \mathcal{N} \tag{A.28e}$$

$$|S_{T_r}^{(c)}| \leqslant \overline{S}_{T_r}^{(c)}, \ \forall T_r \in T \tag{A.28f}$$

$$|P_{T_r}^{(c)}| \leqslant \overline{P}_{T_r}^{(c)}, \ \forall T_r \in T \tag{A.28g}$$

$$(|I_{T_r}^{(c)}|)^2 \leqslant (\overline{I}_{T_r}^{(c)})^2, \ \forall T_r \in T \tag{A.28h}$$

$$|V_{N_i}^{(c)} - V_{N_{\bar{i}}}^{(c)}| \leqslant \overline{\Delta V_{T_{r_{N_i N_{\bar{i}}}}}}, \ \forall T_r \in T \tag{A.28i}$$

A.4.2.4 Classical AC SCOPF Formulation: The Primal Problem (Ramping Constraints)

$$\delta_{g_q}^{(c)} \tilde{\Delta}_P^{(c)} \leqslant AS_{Pg_q}^{max} \tag{A.29a}$$

$$\delta_{g_q}^{(c)} \tilde{\Delta}_Q^{(c)} \leqslant AS_{Pg_q}^{max} \tag{A.29b}$$

$$\sum_{g_q \in G} AS_{Pg_q}^{max} \geqslant \overline{P}_B \tag{A.29c}$$

$$\sum_{g_q \in G} AS_{Qg_q}^{max} \geqslant \overline{Q}_B \tag{A.29d}$$

$$-\overline{R}_{Pg_q} \leqslant \delta_{g_q}^{(c)} \tilde{\Delta}_P^{(c)} \leqslant \overline{R}_{Pg_q} \tag{A.29e}$$

$$-\overline{R}_{Qg_q} \leqslant \delta_{g_q}^{(c)} \tilde{\Delta}_Q^{(c)} \leqslant \overline{R}_{Qg_q} \tag{A.29f}$$

Theabove formulations can be extended to multiple time interval post-contingency restoration models, which can then be solved using SDP relaxation, combining it with a tree decomposition and ADMM and APP decompositions, as before.

Appendix B

Future Work: Unit Commitment and Combined Cycle Power Plant Scheduling

B.0.1 Introduction and Problem Description

The Unit Commitment and Economic Dispatch problems deal with the optimal allocation of load demands to the different generators in the system so that the cost of generating power is minimized, while satisfying different system costraints like maximum and minimum generating limits, minimum up and down times, ramping constraints, transmission limits etc. As the name implies, the entire problem consisits of two parts, namely:

1) **The Unit Commitment Problem**, which which solves the problem of deciding which generators to switch on and which all to keep off during any particular time, and

2) **The Economic Dispatch or Optimal Power Flow Problem**, which decides how much power to generate from the units that are switched on (Economic Dispatch, when the transmission line limits are not considered and Optimal Power Flow otherwise). Although the second problem is continuous (although not convex in it's actual form. Nevertheless it can be convexified with minor loss of accuracy), the first part consists of binary integer variables representing switching states and hence is a non-convex integer optimization problem. In

this future work, we will be looking at an approximate model for implementing the same in a *Mixed Integer Linear Programming (MILP)* framework.

B.1 Problem Formulation

B.1.1 Assumptions of the Model

In order to simplify the mathematical modeling and also to expedite the computations, the following assumptions are made:

- The cost curve for all the dispatchable generators are linear and increasing. This means that the incremental cost is a constant (which is the product of average heat rate and fuel cost for the particular generating unit) with or without a non-zero positive no load or minimum power output cost. In a later version we might be including a piecewise linear increasing or convex polynomial cost function as well.

- Instead of representing the actual transmission lines, we represent a reduced set of constraints, whereby, we only represent the most significant transmission constraints (four for ERCOT in the current version, but can be changed and specified by the user).

- Instead of having to represent the actual values of line reactances in the formulation (and consequently take care of presence of transformers), we have made use of generation shift factor matrix that simplifies calculation.

- In the present model, we haven't taken into account the contribution of load demands to congest or relieve the line loading.

- It is assumed that the solar and wind power plants have a predecided hourly generation profile that has already been provided as part of the input data and hence the switching state of those units as well as their power generation are not included in the decision variable vector of the problem.

- Also, the base load units' (coal, nuclear etc) power outputs are constant throughout the entire dispatch horizon and are predecided as well. These are the must-run units and hence their switching state as well as power outputs are not parts of the decision variables.

With all the above assumptions we will now look at the different sets, indices, elements, describe the variables and parameters of the problem, describe and explain the different symbols, and write down the mathematical model. The formulation presented here is a somewhat modified version of the ones presented in [216] and [389] by Jiang, Zhang et al. However, the authors of the above two papers didn't include ramp-rate constraints. A good reference for how to include ramp-rate constraints is by Frangioni, Gentile, and Lacalandra in [156]. For a detailed derivation and more in-depth study of the formulations, we refer the readers to the above-mentioned references. In our formulation, we have taken into consideration, the possibility that some units are already switched on/off and have stayed in that state for a time duration less than the minimum up/down times allowed respectively, before the simulation starts. Also, our formulation of the ramping and transmission constraints are slightly different than the references and more in tune with the requirements of the present model.

B.1.2 Sets

: Here we will describe the different sets for the problem.

τ : Set of dispatch time intervals (in hours),

$\mathcal{L}$: Set of Transmission Lines/Commercially significant constraints,

$\mathcal{G}$: Set of Dispatchable Generators,

$\mathcal{G}_{nd}$: Set of Non-Dispatchable Generators (typically solar, wind, and/or base-load units),

$\mathcal{Z}$: Set of load-zones

B.1.3 Indices

i : Index of Disatchable Generators,

j : Index of Loads,

t : Index of Time (in hours),

l : Index of Transmission lines,

q : Index of non-dispatchable Generators,

z : Index of load-zones

B.1.4 Cardinality

G : Total number of Dispatchable Generators,

ND : Total number of non-dispatchable Generators,

NZ : Total number of load zones,

$\mathcal{D}$: Total number of Loads (Same as the number of hours the simulation is run for),

T : Total number of hours for which the simulation is run,

$|\mathcal{L}|$: Total number of Transmission lines/CSCs

B.1.5 Variables and Parameters

$P_{g_i}^{(t)}$: Power output in MW of i^{th} dispatchable generator at hour, t (Decision Variables),

$X_i^{(t)}$: Switching state of i^{th} dispatchable generator at hour, t (Decision Variables; 0 for "OFF" and 1 for "ON"),

$P_{g_q}^{(t)}$: Predecided hourly generation in MW of q^{th} non-dispatchable generator (Solar, Wind, or Base-Load unit) at hour, t,

$u_i^{(t)}$: Variable for switching on transition for the i^{th} dispatchable generator at hour, t; If $X_i^{(t-1)} = 0$ and $X_i^{(t)} = 1$, then $u_i^{(t)} = 1$, otherwise 0 (Decision Variables),

$d_i^{(t)}$: Variable for switching off transition for the i^{th} dispatchable generator at hour, t; If $X_i^{(t-1)} = 1$ and $X_i^{(t)} = 0$, then $d_i^{(t)} = 1$, otherwise 0 (Decision Variables),

SU_i, SD_i : Start-Up and Shut-Down costs respectively (in \$) for the i^{th} dispatchable generator,

UT_i, DT_i : Minimum Up and Minimum Down times respectively (in hours) for the i^{th} dispatchable generator,

U_i, $\mathcal{D}_i$: Time for which a previously switched on Generator was on and time for which a previously switched off Generator was off respectively (in hours) for the i^{th} dispatchable generator before the start of the simulation,

$R_{g_i}^{max}$, $R_{g_i}^{min}$: Maximum Ramp-up and maximum Ramp-down limits (MW/hour) of the i^{th} dispatchable generator,

$D_z^{(t)}$: Total load demand in MW at hour t in zone z,

SF_{index}^{l} : Shift Factor from the $index^{th}$ element to the l^{th} transmission line/constraint,

$P_{g_i}^{max}$, $P_{g_i}^{min}$: Maximum and minimum power generation limits (MW) of the i^{th} dispatchable generator,

L_l : Transmission limit (in MW) of the l^{th} transmission line/CSC,

c_{i1}, c_{i0} : Linear Cost Coeffcicient (product of average heat-rate and fuel cost) and no-load/minimum output cost for the i^{th} dispatchable generator

B.1.6 Optimization Model for the Problem

With the above mentioned sets, indices, cardinalities, variables, and parameters, we can now write down the Unit Commitment and Economic Dispatch/Optimal Power Flow Problem as the following *Mixed Integer Linear Programming Problem (MILP)*:

$$\min_{P_{g_i}^{(t)} \in \mathbb{R}, \mathrm{X_i^{(t)}} \in \{0,1\}} \sum_{t=1}^{T} \sum_{i=1}^{G} (c_{i1} P_{g_i}^{(t)} + c_{i0} X_i^{(t)} + u_i^{(t)} SU_i + d_i^{(t)} SD_i) \tag{B.1a}$$

$$Subject\ to:$$

$$Max/Min\ Generation\ Limits \rightarrow X_i^{(t)} P_{g_i}^{min} \leq P_{g_i}^{(t)} \leq X_i^{(t)} P_{g_i}^{max}, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.1b}$$

$$Supply - Demand\ Balance \rightarrow \sum_{i=1}^{G} P_{g_i}^{(t)} + \sum_{q=1}^{ND} P_{g_q}^{(t)} = \sum_{z=1}^{NZ} D_z^{(t)}, \forall t \in \tau \tag{B.1c}$$

$$Min\ Up - time \rightarrow -X_i^{(t-1)} + X_i^{(t)} - X_i^{(k)} \leq 0, \forall t \in \tau, \forall i \in \mathcal{G}, 1 \leq k - (t-1) \leq UT_i$$

$$If\ X_i^{(0)} = 1\ and\ UT_i - U_i > 0,$$

$$-X_i^{(0)} - X_i^{(1)} - X_i^{(k)} \leq -3, \forall i \in \mathcal{G}, 1 \leq k \leq UT_i - U_i \tag{B.1d}$$

$$Min\ Down - time \rightarrow X_i^{(t-1)} - X_i^{(t)} + X_i^{(k)} \leq 1, \forall t \in \tau, \forall i \in \mathcal{G}, 1 \leq k - (t-1) \leq DT_i$$

$$If\ X_i^{(0)} = 0\ and\ DT_i - \mathcal{D}_i > 0,$$

$$X_i^{(0)} + X_i^{(1)} + X_i^{(k)} \leq 0, \forall i \in \mathcal{G}, 1 \leq k \leq DT_i - \mathcal{D}_i \tag{B.1e}$$

$$Start - Up \rightarrow -X_i^{(t-1)} + X_i^{(t)} - u_i^{(t)} \leq 0, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.1f}$$

$$Shut - Down \rightarrow X_i^{(t-1)} - X_i^{(t)} - d_i^{(t)} \leq 0, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.1g}$$

$$Ramp\ Limits \rightarrow X_i^{(t-1)} R_{g_i}^{min} \leq P_{g_i}^{(t)} - P_{g_i}^{(t-1)} \leq X_i^{(t)} R_{g_i}^{max}, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.1h}$$

$$Flow\ Limits \rightarrow -L_l \leq \sum_{i=1}^{G} SF_i^l P_{g_i}^{(t)} + \sum_{q=1}^{ND} SF_q^l P_{g_q}^{(t)} - \sum_{z=1}^{NZ} SF_z^l D_z^{(t)} \leq L_l, \forall t \in \tau, \forall l \in \mathcal{L} \tag{B.1i}$$

The above set of equations is very self-explanatory as the names suggest. As it can be observed, the double sigma notation in the objective function corresponds to summation of costs across all the generators and all the hours in the entire dispatch simulation horizon. We haven't showed the production cost for the predecided output generators, which will just be a constant and same as the first two terms in the objective (which respectively represent the running cost for producing power and the no-load cost and hence is multiplied by the switching state). The third and fourth terms in the objective represents the start-up and shut-down costs for the dispatchable generators.

The next set of constraints represents the maximum and minimum generating ranges for the generators. As can be seen from the equations, the limits are multiplied by the switching variable so that when it is 0, representing an "OFF" state, the power generation value is forced to be 0 as well.

In the supply-demand balance constraint, both the dispatchable generators' outputs as well as the non-dispatchable MW values are taken into account at the left side and equated to hourly demand values at the right side.

Let us now describe the next two set of constraints corresponding to the minimum up and down-times respectively. In the case when the generator is previously switched on and has run for a time duration less than the minimum up-time, then from the equations, the first term $X_i^{(0)}$ will be equal to 1 and hence, $X_i^{(1)}$ and X_i^k has to be equal to 1 for the remaining duration of minimum up-time in order for the inequality to hold, thereby guaranteeing the minimum up-time on condition of the units. A similar explanation can be given for the other equations as well. As mentioned in the assumptions, since we are not considering the impact of the load demand on the line loading, we have excluded the term representing that in the last equation.

We will now give a brief description of the final form of the equations that has been modified slightly from the previous ones, which is suitable for the purposes of writing a code, that calles an MILP solver (for instance, the GLPK solver) to solve it. In the description that follows, the coefficients and/or right hand sides whose values need to be told to the solver are written in bold-face (The other ones being the decision variables).

B.1.7 Constraints and coefficients corresponding to supply-demand balance

$$\mathbf{1}.P_{g_1}^{(t)} + \mathbf{1}.P_{g_2}^{(t)} + ... + \mathbf{1}.P_{g_G}^{(t)} = \sum_{\mathbf{z=1}}^{\mathbf{NZ}} \mathbf{D_z^{(t)}} - \sum_{\mathbf{q=1}}^{\mathbf{ND}} \mathbf{P_{g_q}^{(t)}}, \forall t \in \tau \tag{B.2a}$$

B.1.8 Constraints and coefficients corresponding to power generation lower bounds

$$\mathbf{1}.P_{g_i}^{(t)} + X_i^{(t)}.(-\mathbf{P_{g_i}^{min}}) \geq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.3a}$$

B.1.9 Constraints and coefficients corresponding to power generation upper bounds

$$\mathbf{1}.P_{g_i}^{(t)} + X_i^{(t)}.(-\mathbf{P_{g_i}^{max}}) \leq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.4a}$$

B.1.10 Constraints and coefficients corresponding to power generation Ramp up limit

$$\mathbf{1}.P_{g_i}^{(1)} + X_i^{(1)}(-\mathbf{R_{g_i}^{max}}) \leq \mathbf{P_{g_i}^{(0)}}, t = 1, \forall i \in \mathcal{G} \tag{B.5a}$$

$$\mathbf{1}.P_{g_i}^{(t)} + (-\mathbf{1}).P_{g_i}^{(t-1)} + X_i^{(t)}(-\mathbf{R_{g_i}^{max}}) \leq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.5b}$$

B.1.11 Constraints and coefficients corresponding to power generation Ramp down limit

$$\mathbf{1}.P_{g_i}^{(1)} \geq \mathbf{P_{g_i}^{(0)}} + \mathbf{X_i^{(0)}}\mathbf{R_{g_i}^{max}}, t = 1, \forall i \in \mathcal{G} \tag{B.6a}$$

$$\mathbf{1}.P_{g_i}^{(t)} + (-\mathbf{1}).P_{g_i}^{(t-1)} + X_i^{(t-1)}(-\mathbf{R_{g_i}^{min}}) \geq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.6b}$$

B.1.12 Constraints and coefficients corresponding to power generator Start-Up

$$\mathbf{1}.X_i^{(1)} + (-\mathbf{1}).u_i^{(1)} \leq \mathbf{X_i^{(0)}}, t = 1, \forall i \in \mathcal{G} \tag{B.7a}$$

$$(-\mathbf{1}).X_i^{(t-1)} + \mathbf{1}.X_i^{(t)} + (-\mathbf{1}).u_i^{(t)} \leq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.7b}$$

B.1.13 Constraints and coefficients corresponding to power generator Shut-Down

$$(-\mathbf{1}).X_i^{(1)} + (-\mathbf{1}).d_i^{(1)} \leq -\mathbf{X_i^{(0)}}, t = 1, \forall i \in \mathcal{G} \tag{B.8a}$$

$$\mathbf{1}.X_i^{(t-1)} + (-\mathbf{1}).X_i^{(t)} + (-\mathbf{1}).d_i^{(t)} \leq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{G} \tag{B.8b}$$

B.1.14 Constraints and coefficients corresponding to power generator Minimum Up-time

$$(-\mathbf{1}).X_i^{(1)} + (-\mathbf{1}).X_i^{(k)} \leq \mathbf{X_i^{(0)}} - \mathbf{3}, t = 1, \forall i \in \mathcal{G}, 1 \leq k \leq UT_i - U_i \; if \; UT_i - U_i > 0, X_i^{(0)} = 1 \tag{B.9a}$$

$$(-\mathbf{1}).X_i^{(t-1)} + \mathbf{1}.X_i^{(t)} + (-\mathbf{1}).X_i^{(k)} \leq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{G}, 1 \leq k - (t-1) \leq UT_i, \; otherwise \tag{B.9b}$$

B.1.15 Constraints and coefficients corresponding to power generator Minimum Down-time

$$\mathbf{1}.X_i^{(1)} + \mathbf{1}.X_i^{(k)} \leq -\mathbf{X_i^{(0)}}, t = 1, \forall i \in \mathcal{G}, 1 \leq k \leq DT_i - \mathcal{D}_i \; if \; DT_i - \mathcal{D}_i > 0, X_i^{(0)} = 0 \tag{B.10a}$$

$$\mathbf{1}.X_i^{(t-1)} + (-\mathbf{1}).X_i^{(t)} + \mathbf{1}.X_i^{(k)} \leq \mathbf{1}, \forall t \in \tau, \forall i \in \mathcal{G}, 1 \leq k - (t-1) \leq UT_i, \; otherwise \tag{B.10b}$$

B.1.16 Constraints and coefficients corresponding to Forward Transmission Limits

$$\mathbf{SF_1^l} P_{g_1}^{(t)} + \mathbf{SF_2^l} P_{g_2}^{(t)} + ... + \mathbf{SF_G^l} P_{g_G}^{(t)} \leq \mathbf{L_l} - \sum_{\mathbf{q=1}}^{\mathbf{ND}} \mathbf{SF_q^l P_{g_q}^{(t)}} + \sum_{\mathbf{z=1}}^{\mathbf{NZ}} \mathbf{SF_z^l D_z^{(t)}}, \forall t \in \tau, \forall l \in \mathcal{L} \tag{B.11a}$$

B.1.17 Constraints and coefficients corresponding to Reverse Transmission Limits

$$\mathbf{SF^l_1} P^{(t)}_{g_1} + \mathbf{SF^l_2} P^{(t)}_{g_2} + ... + \mathbf{SF^l_G} P^{(t)}_{g_G} \geq -\mathbf{L_l} - \sum_{\mathbf{q=1}}^{\mathbf{ND}} \mathbf{SF^l_q P^{(t)}_{g_q}} + \sum_{\mathbf{z=1}}^{\mathbf{NZ}} \mathbf{SF^l_z D^{(t)}_z}, \forall t \in \tau, \forall l \in \mathcal{L} \tag{B.12a}$$

B.2 Description of the Software

In this section we will give a broad overview of the way the software for the above optimization model would work and the underlying ideas and philosophy we would like to adopt while writing the program. The structure of the program is as follows. The program execution starts at the main method, whereby an object of the "Nettran" class is initialized. Nettran class further has, as its data members, objects of the "Powergenerator" class (whose objects represent the dispatchable generators) and "preDecGen" class (which inherits from the Powergenerator class and whose objects represent predecided output generators), objects of "Tranline" class representing the transmission elements. The class diagram for the program is as shown below.

The figure B.2 shows the creation of the different objects of the different classes.

The figure B.3 shows the overall execution of a software, that can be written for this application.

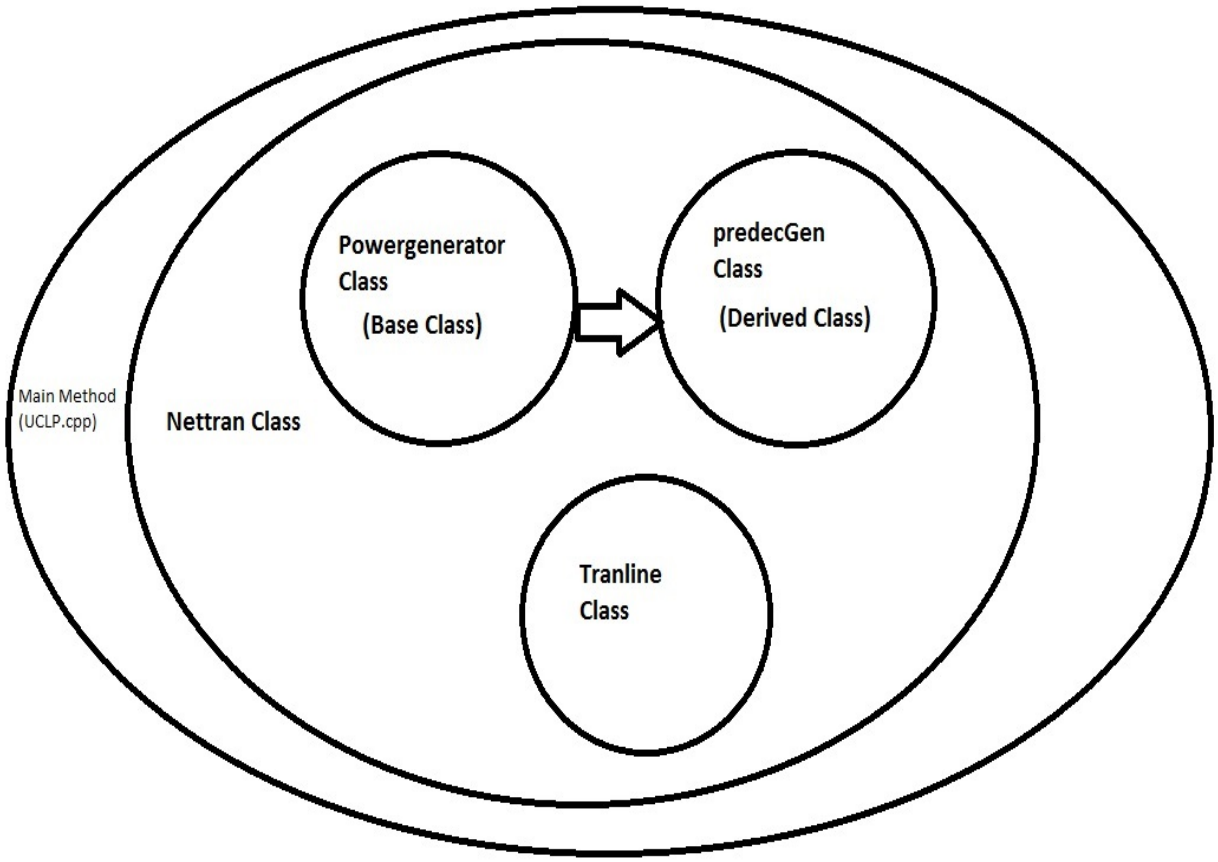


Figure B.1: The Class Hierarchy of the Program

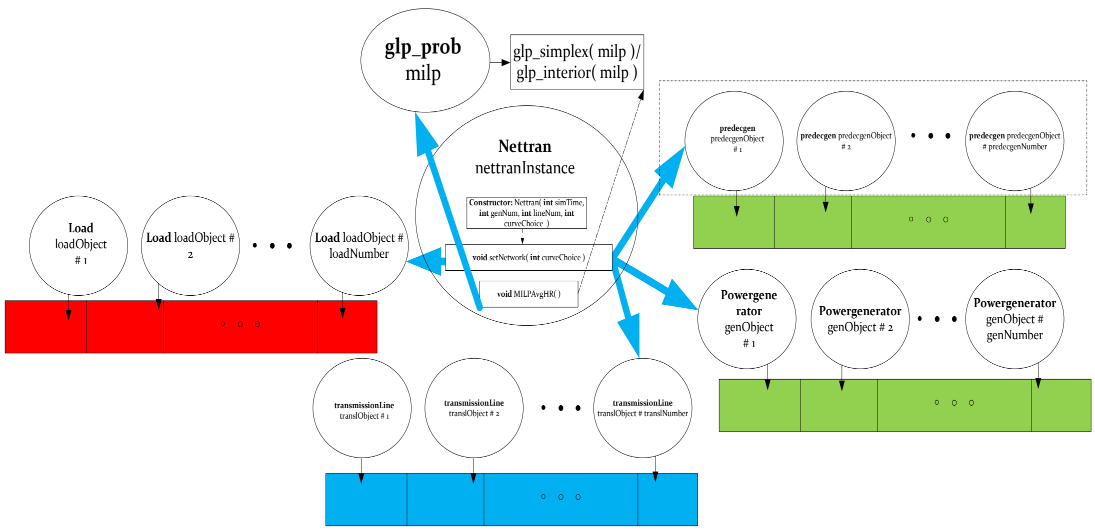


Figure B.2: Unit Commoitment MILP: Objects

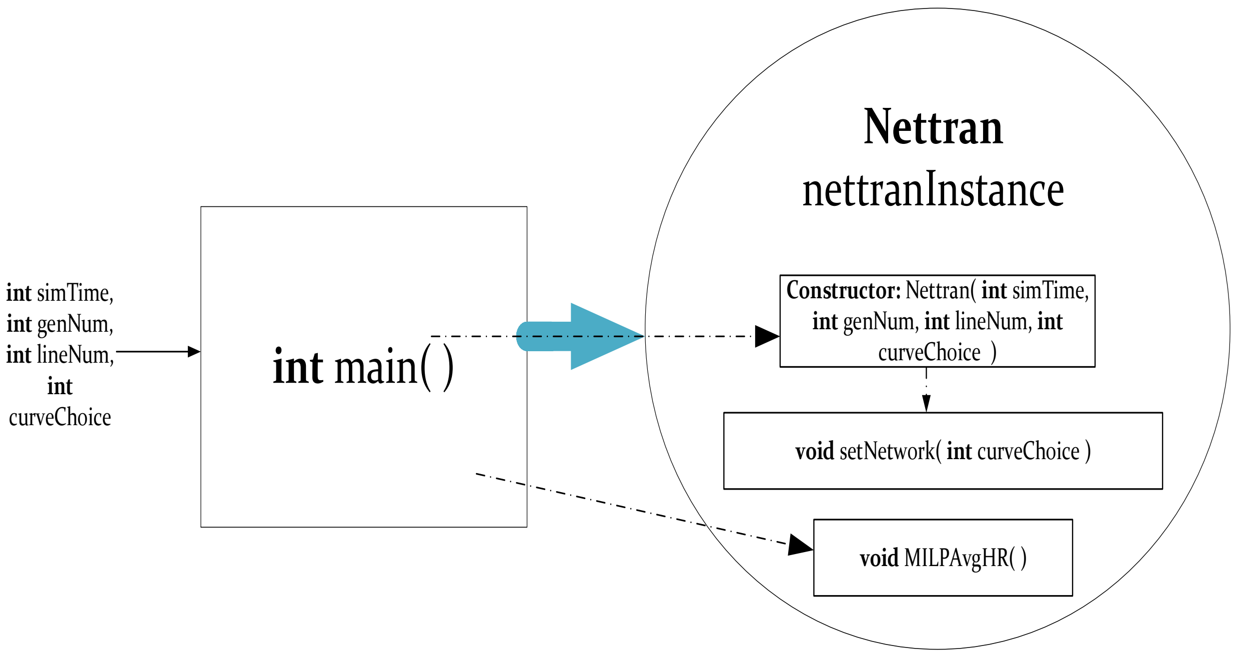


Figure B.3: Unit Commitment MILP: Software

The **Nettran** class object represent the physical network and has two main functionalities, namely, instantiation of the data members, which include the objects of the other three classes and running the GLPK MILP algorithm. The first function is achieved with the constructor method, which in its turn calls the “setNetwork()”, “setLoadVal()”, “setShiftVal()”, “setGenVal()”, “setTranVal()” methods, which respectively set the simulation mode (type of the objective function; average heat rate, piecewise linear, polynomial etc.), populates the vector of hourly load demand values, populates the matrix of generator shift factor values (rows representing the generator serial numbers and columns representing the transmission lines), creates the generator objects (both dispatchable and predecided hourly generators) and stores them in a vector (since the predecided output generators belong to the subclass of that of the dispatchable generators, both types of objects can be stored in the same vector. Storing on the same vector, both these types of objects makes it easier to refer to the associated shift factors), creates the transmission line/CSC objects and stores them in a vector. Depending on what type of objective function has been chosen, the second functionality is implemented with either the “MILPAvgHR()” (for average heat rate), “MILPPiecewiseLin()” (for piecewise linear objective), or “MILPPolynomial()” (for polynomial objective). Currently, only the one for the average heat rate has been implemented. Within the “MILPAvgHR()” method, before calling the Simplex and MIP GLPK solvers to solve the problem instance, the objective coefficients (also called the columns) along with the type and range of decision variables (continuous, discrete, integer, binary etc.) are specified. Also, the non-zero entires of the left had side of the constraint coefficient matrix has been specified corresponding to each group of constraints listed previously along with the relevant row number (which represents the serial number of the constraint equation/inequality) and the column number (which represents the serial position of the particular decision variable in the decision vector; here, we have put all the P_{g_i}s of the different hours, in the increasing order of hours, and in increasing order of generators' serial numbers within a particular hour, as the first group of variables, followed by X_is, u_is, and d_is in an exactly similar way).

B.3 Combined Cycle Dispatch Allocation Model

B.3.1 Introduction and Problem Description

In this future work the aim is to develop an optimization model that will help us in deciding which particular configuration of a Combined Cycle power plant to switch on to (ie a specific combination of a set of particular combustion turbines and steam turbines), how much MW to sell in the energy market and capacity to reserve, or, whether to switch off the plant, given a temporal variation of the Locational Marginal Price (LMP) and the Market Clearing Price for Capacity (MCPC)/ Ancillary Services Price for the particular node to which the plant in connected. The other constraints are highlighted in the material that follows. This problem is just a variant of the unit commitment problem, that we just described.

B.4 Problem Formulation

B.4.1 Sets

: Here we will describe the different sets for the problem.

τ : Set of dispatch time intervals (in hours),

$\mathcal{C}$: Set of Configurations

B.4.2 Indices

i : Index of configurations,

t : Index of Time (in hours)

B.4.3 Cardinality

N : Total number of configurations,

T : Total number of hours for which the simulation is run

B.4.4 Variables and Parameters

$P_{g_i}^{(t)}$: Power output in MW of i^{th} configuration at hour, t (Decision Variables),

$X_i^{(t)}$: Switching state of i^{th} configuration at hour, t (Decision Variables; 0 for"OFF" and 1 for"ON"),

$C_i^{(t)}$: Reserved capacity for Ancillary Services in MW of i^{th} configuration at hour, t (Decision Variables),

$u_{ij}^{(t)}$: Variable for switching on transition from the j^{th} configuration to the i^{th} configuration at hour, t; If $X_j^{(t-1)} = 1$ and $X_i^{(t)} = 1$, then $u_{ij}^{(t)} = 1$, otherwise 0 (Decision Variables),

SU_{ij} : Transition cost (in \$) for transition from the j^{th} configuration to the i^{th} configuration,

UT, DT : Minimum Up and Minimum Down times respectively (in hours) for the Combined Cycle plant

U, $\mathcal{D}$: Time for which the previously switched on plant was on and time for which the previously switched off plant was off respectively (in hours) before the start of the simulation,

$R_{g_i}^{max}$, $R_{g_i}^{min}$: Maximum Ramp-up and maximum Ramp-down limits (MW/hour) of the i^{th} configuration,

$P_{g_i}^{max}$, $P_{g_i}^{min}$: Maximum and minimum power generation limits (MW) of the i^{th} configuration,

c_{i3},c_{i2},c_{i1}, c_{i0} : Cubic Cost Coefficient, Quadratic Cost Coefficient, Linear Cost Coeffcicient (product of average heat-rate and fuel cost), and no-load/minimum output cost for the i^{th} configuration,

$p(t)$: Probability that the reserved capacity for the i^{th} configuration will be deployed in the energy market at t^{th} hour,

$F(t, i)$: Hourly fuel cost for the i^{th} configuration (Adjusted by the configuration specific fuel adder),

A, B, C, D, VOM : Cubic, Quadratic, Linear, Minimum/No load cost coefficients and Variable O & M costs respectively of the configurations,

$ASPercent$: Maximum limit on the reserve capacity for a particular configuration expressed as a percentage of the maximum generating limit of that configuration,

$LMP_{node}^{(t)}$: Locational Marginal Price of the $node^{th}$ node at hour t,

$MCPC^{(t)}$: Ancillary Service price at hour, t

Here, $c_{i3} = A.F(t, i)$, $c_{i2} = B.F(t, i)$, $c_{i1} = C.F(t, i) + VOM$, $c_{i0} = D.F(t, i)$

B.4.5 Optimization Model for the Problem

With the above mentioned sets, indices, cardinalities, variables, and parameters, we can now write down the Combined Cycle dispatch configuration allocation Problem as the following Mixed Integer Linear Programming Problem (MILP):

$$\max_{P_{g_i}^{(t)} \in \mathbb{R}_+, \mathrm{X_i^{(t)}} \in \{0,1\}, \mathrm{u_{ij}^{(t)}} \in \{0,1\}} \sum_{t=1}^{T} \sum_{i=1}^{N} \{LMP_{node}^{(t)}[P_{g_i}^{(t)} + pC_i^{(t)}] + MCPC^{(t)}[C_i^{(t)}]$$
$$-(p\{c_{i3}[P_{g_i}^{(t)} + C_i^{(t)}]^3 + c_{i2}[P_{g_i}^{(t)} + C_i^{(t)}]^2 + c_{i1}[P_{g_i}^{(t)} + C_i^{(t)}] + c_{i0}X_i^{(t)}\}$$
$$+(1-p)\{c_{i3}[P_{g_i}^{(t)}]^3 + c_{i2}[P_{g_i}^{(t)}]^2 + c_{i1}P_{g_i}^{(t)} + c_{i0}X_i^{(t)}\}$$
$$+ \sum_{j \in \mathcal{C}/\{i\}} u_{ij}^{(t)} SU_{ij})\} \tag{B.13a}$$
$$Subject\ to:$$

$$Max/Min\ Generation\ Limits \rightarrow X_i^{(t)} P_{g_i}^{min} \leq P_{g_i}^{(t)} + C_i^{(t)} \leq X_i^{(t)} P_{g_i}^{max}, \forall t \in \tau, \forall i \in \mathcal{C} \tag{B.13b}$$

$$Exclusion\ Constraint \rightarrow \sum_{i=1}^{N} X_i^{(t)} \leq 1, \forall t \in \tau \tag{B.13c}$$

$$Plant\ Min\ Up-time \rightarrow \sum_{i=1}^{N}(-X_i^{(t-1)} + X_i^{(t)} - X_i^{(k)}) \leq 0, \forall t \in \tau, 1 \leq k-(t-1) \leq UT$$
$$If\ \sum_{i=1}^{N} X_i^{(0)} = 1\ and\ UT - U > 0,$$
$$\sum_{i=1}^{N}(-X_i^{(0)} - X_i^{(1)} - X_i^{(k)}) \leq -3, \forall 1 \leq k \leq UT - U \tag{B.13d}$$

$$Plant\ Min\ Down-time \rightarrow \sum_{i=1}^{N}(X_i^{(t-1)} - X_i^{(t)} + X_i^{(k)}) \leq 1, \forall t \in \tau, 1 \leq k-(t-1) \leq DT$$
$$If\ \sum_{i=1}^{N} X_i^{(0)} = 0\ and\ DT - \mathcal{D} > 0,$$

$$\sum_{i=1}^{N}(X_i^{(0)} + X_i^{(1)} + X_i^{(k)}) \leq 0, \forall 1 \leq k \leq DT - \mathcal{D} \tag{B.13e}$$

$$Configuration\ Switching \rightarrow 1 - X_j^{(t-1)} - X_i^{(t)} + 2u_{ij}^{(t)} \geq 0, \forall t \in \tau, \forall i, j \in \mathcal{C} \cup \{0\} \tag{B.13f}$$

$$Ramp\ Limits \rightarrow X_i^{(t-1)} R_{g_i}^{min} \leq P_{g_i}^{(t)} + C_i^{(t)} - P_{g_i}^{(t-1)} \leq X_i^{(t)} R_{g_i}^{max}, \forall t \in \tau, \forall i \in \mathcal{C} \tag{B.13g}$$

From the above model, the objective is the profit maximization, and hence the first couple terms represent the expected revenue from sale of energy and ancillary services commodities, the next couple terms represent the expectedoperating cost to operate the plant in a particular configuration, and the last term represents the cost incurred due to configuration switching.

The first set of constraints are for enforching the condition that the generation and capacity reserved for any configuration cannot exceed the bounds allowed if that particular configuration in run. Otherwise, it's zero. The second set of constraints state that at a time, only one configuration can be operated. The next two sets of constraints are for the minimum up and down times. Finally, the last two sets of constraints tell us about when it is profitable to switch to what configuration depending upon both the operating costs and the transition costs and the ramping limits within a configuration. The reserve term in the future time here implies that even in the worst possible case when the plant is called to deploy all its reserves in the energy market, it can still ramp up or down safely. We will now slightly reformulate the problem to convert it into a piecewise linear program.

In order to carry out the piecewise linear equivalent formulation of the problem, we will do the following substitutions:

$$(P+C)_i^{(t)} = \sum_{m=1}^{r+1} y_{im}^{(t)}, 0 \leq y_{im}^{(t)} \leq Y_{i(m+1)}^{(t)} - Y_{im}^{(t)},\ m = 1,2,...,r,\ \forall i \in \mathcal{C}, \forall t \in \tau \tag{B.14}$$

$$P_i^{(t)} = \sum_{m=1}^{r+1} z_{im}^{(t)}, 0 \leq z_{im}^{(t)} \leq Z_{i(m+1)}^{(t)} - Z_{im}^{(t)},\ m = 1,2,...,r,\ \forall i \in \mathcal{C}, \forall t \in \tau \tag{B.15}$$

and

$$ASPercent.P_{g_i}^{max} \geq \sum_{m=1}^{r+1} y_{im}^{(t)} - \sum_{m=1}^{r+1} z_{im}^{(t)} > 0, \forall i \in \mathcal{C}, \forall t \in \tau \tag{B.16}$$

where $Z_{i(r+1)}^{(t)} = P_i^{max}$

$Z_{i1}^{(t)} = P_i^{min}$

$Y_{i(r+1)}^{(t)} = P_i^{max}$

$Y_{i1}^{(t)} = P_i^{min}$ Then,

$A[P_{g_i}^{(t)} + C_i^{(t)}]^3 + B[P_{g_i}^{(t)} + C_i^{(t)}]^2 + C[P_{g_i}^{(t)} + C_i^{(t)}] = \sum_{m=1}^{r+1} S_{im} y_{im}^{(t)}$,

$S_{im} = \frac{D_{i(m+1)} - D_{im}}{Y_{i(m+1)} - Y_{im}}$

$A[P_{g_i}^{(t)}]^3 + B[P_{g_i}^{(t)}]^2 + B[P_{g_i}^{(t)}]^2 + C[P_{g_i}^{(t)}] = \sum_{m=1}^{r+1} F_{im} z_{im}^{(t)}$

$F_{im} = \frac{E_{i(m+1)} - E_{im}}{Z_{i(m+1)} - Z_{im}}$

Z, Y s are the break points for energy and energy+reserves, S, F s are the slopes of the linearized segments of the respective cost functions (without the no load cost), and D, K are the y-axis intercepts of the quadratic cost curves corresponding to the break-points. Let us now look at the reformulation of the objective function first in terms of the new piecewise linear variables and then the other constraints in terms of the same set of variables. It will eventually lead us to the way the problem is posed to an MILP solver (for instance, the GLPK solver) in terms of the coefficients.

$$
\begin{aligned}
&\sum_{t=1}^{T}\sum_{i=1}^{N}\{LMP^{(t)}_{node}[P^{(t)}_{g_i}+pC^{(t)}_i]+MCPC^{(t)}[C^{(t)}_i] \\
&-(p\{c_{i3}[P^{(t)}_{g_i}+C^{(t)}_i]^3+c_{i2}[P^{(t)}_{g_i}+C^{(t)}_i]^2+c_{i1}[P^{(t)}_{g_i}+C^{(t)}_i]+c_{i0}X^{(t)}_i\} \\
&+(1-p)\{c_{i3}[P^{(t)}_{g_i}]^3+c_{i2}[P^{(t)}_{g_i}]^2+c_{i1}P^{(t)}_{g_i}+c_{i0}X^{(t)}_i\} \\
&+\sum_{j\in\mathcal{C}/\{i\}}u^{(t)}_{ij}SU_{ij})\} \\
&=\sum_{t=1}^{T}\sum_{i=1}^{N}\{LMP^{(t)}_{node}p[P^{(t)}_{g_i}+C^{(t)}_i]+LMP^{(t)}_{node}(1-p)[P^{(t)}_{g_i}]+MCPC^{(t)}[C^{(t)}_i] \\
&-(p\{c_{i3}[P^{(t)}_{g_i}+C^{(t)}_i]^3+c_{i2}[P^{(t)}_{g_i}+C^{(t)}_i]^2+c_{i1}[P^{(t)}_{g_i}+C^{(t)}_i]\} \\
&+(1-p)\{c_{i3}[P^{(t)}_{g_i}]^3+c_{i2}[P^{(t)}_{g_i}]^2+c_{i1}P^{(t)}_{g_i}\}+c_{i0}X^{(t)}_i \\
&+\sum_{j\in\mathcal{C}/\{i\}}u^{(t)}_{ij}SU_{ij})\} \\
&=\sum_{t=1}^{T}\sum_{i=1}^{N}\{(LMP^{(t)}_{node}-VOM)p[\sum_{m=1}^{r+1}y^{(t)}_{im}]+(LMP^{(t)}_{node}-VOM)(1-p)[\sum_{m=1}^{r+1}z^{(t)}_{im}] \\
&+MCPC^{(t)}[\sum_{m=1}^{r+1}y^{(t)}_{im}-\sum_{m=1}^{r+1}z^{(t)}_{im}] \\
&-pF(t,i)[\sum_{m=1}^{r+1}S_{im}y^{(t)}_{im}]-(1-p)F(t,i)[\sum_{m=1}^{r+1}F_{im}z^{(t)}_{im}]-c_{i0}X^{(t)}_i \\
&-\sum_{j\in\mathcal{C}/\{i\}}u^{(t)}_{ij}SU_{ij}\} \\
&=\sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{\mathbf{i=1}}^{\mathbf{N}}\sum_{\mathbf{m=1}}^{\mathbf{r+1}}((\mathbf{LMP^{(t)}_{node}}-\mathbf{VOM})\mathbf{p}+\mathbf{MCPC^{(t)}}-\mathbf{pF(t,i)S_{im}})y^{(t)}_{im} \\
&+\sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{\mathbf{i=1}}^{\mathbf{N}}\sum_{\mathbf{m=1}}^{\mathbf{r+1}}((\mathbf{LMP^{(t)}_{node}}-\mathbf{VOM})(\mathbf{1-p})-\mathbf{MCPC^{(t)}}-(\mathbf{1-p})\mathbf{F(t,i)F_{im}})z^{(t)}_{im}
\end{aligned}
$$

$$-\sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{\mathbf{i=1}}^{\mathbf{N}}\mathbf{c_{i0}}X_i^{(t)} - \sum_{\mathbf{t=1}}^{\mathbf{T}}\sum_{\mathbf{i=1}}^{\mathbf{N}}\sum_{\mathbf{j}\in\mathcal{C}/\{\mathbf{i}\}}\mathbf{SU_{ij}}u_{ij}^{(t)} \tag{B.17a}$$

$$\tag{B.17b}$$

The boldface coefficients above are the ones that are needed by the MILP solver while stating the objective coefficients or the columns.

B.4.6 Constraints and coefficients corresponding to Exclusion Condition

$$\mathbf{1}.X_1^{(t)} + \mathbf{1}.X_2^{(t)} + ... + \mathbf{1}.X_N^{(t)} \leq \mathbf{1}, \forall t \in \tau \tag{B.18a}$$

B.4.7 Constraints and coefficients corresponding to power generation lower bounds

$$\mathbf{1}.\sum_{m=1}^{r+1} y_{im}^{(t)} + X_i^{(t)}.(-\mathbf{P_{g_i}^{min}}) \geq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{C} \tag{B.19a}$$

B.4.8 Constraints and coefficients corresponding to power generation upper bounds

$$\mathbf{1}.\sum_{m=1}^{r+1} y_{im}^{(t)} + X_i^{(t)}.(-\mathbf{P_{g_i}^{max}}) \leq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{C} \tag{B.20a}$$

B.4.9 Constraints and coefficients corresponding to Configuration Ramp up limit

$$\mathbf{1}.\sum_{m=1}^{r+1} y_{im}^{(1)} + X_i^{(1)}(-\mathbf{R_{g_i}^{max}}) \leq \mathbf{P_{g_i}^{(0)}}, t=1, \forall i \in \mathcal{C} \tag{B.21a}$$

$$\mathbf{1}.\sum_{m=1}^{r+1} y_{im}^{(t)} + (-\mathbf{1}).\sum_{m=1}^{r+1} z_{im}^{(t-1)} + X_i^{(t)}(-\mathbf{R_{g_i}^{max}}) \leq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{C} \tag{B.21b}$$

B.4.10 Constraints and coefficients corresponding to Configuration Ramp down limit

$$\mathbf{1}.\sum_{m=1}^{r+1} y_{im}^{(1)} \geq \mathbf{P_{g_i}^{(0)}} + \mathbf{X_i^{(0)}}\mathbf{R_{g_i}^{max}}, t=1, \forall i \in \mathcal{C} \tag{B.22a}$$

$$\mathbf{1}.\sum_{m=1}^{r+1} y_{im}^{(t)} + (-\mathbf{1}).\sum_{m=1}^{r+1} z_{im}^{(t-1)} + X_i^{(t-1)}(-\mathbf{R_{g_i}^{min}}) \geq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{C} \tag{B.22b}$$

B.4.11 Constraints and coefficients corresponding to Configuration Switching from j^{th} to i^{th}

$$-\mathbf{1}.X_i^{(1)} + \mathbf{2}.u_{ij}^{(1)} \geq -\mathbf{1} + \mathbf{X_j^{(0)}}, t=1, \forall i,j \in \mathcal{C} \cup \{0\} \tag{B.23a}$$

$$-\mathbf{1}.X_i^{(t-1)} + -\mathbf{1}.X_i^{(t)} + \mathbf{2}.u_{ij}^{(t)} \geq -\mathbf{1}, \forall t \in \tau, \forall i,j \in \mathcal{C} \cup \{0\} \tag{B.23b}$$

B.4.12 Constraints and coefficients corresponding to Plant Minimum Up-time

$$(-\mathbf{1}).\sum_{i=1}^{N} X_i^{(1)} + (-\mathbf{1}).\sum_{i=1}^{N} X_i^{(k)} \leq \mathbf{X^{(0)}} - \mathbf{3}, t = 1, 1 \leq k \leq UT - U \ if \ UT - U > 0, X^{(0)} = 1 \tag{B.24a}$$

$$(-\mathbf{1}).\sum_{i=1}^{N} X_i^{(t-1)} + \mathbf{1}.\sum_{i=1}^{N} X_i^{(t)} + (-\mathbf{1}).\sum_{i=1}^{N} X_i^{(k)} \leq \mathbf{0}, \forall t \in \tau, 1 \leq k - (t-1) \leq UT, \ otherwise \tag{B.24b}$$

B.4.13 Constraints and coefficients corresponding to Plant Minimum Down-time

$$\mathbf{1}.\sum_{i=1}^{N} X_i^{(1)} + \mathbf{1}.\sum_{i=1}^{N} X_i^{(k)} \leq -\mathbf{X^{(0)}}, t = 1, 1 \leq k \leq DT - \mathcal{D} \ if \ DT - \mathcal{D} > 0, X^{(0)} = 0 \tag{B.25a}$$

$$\mathbf{1}.\sum_{i=1}^{N} X_i^{(t-1)} + (-\mathbf{1}).\sum_{i=1}^{N} X_i^{(t)} + \mathbf{1}.\sum_{i=1}^{N} X_i^{(k)} \leq \mathbf{1}, \forall t \in \tau, 1 \leq k - (t-1) \leq UT, \ otherwise \tag{B.25b}$$

B.4.14 Constraints and coefficients corresponding to Ancillary Services upper limit

$$\mathbf{1}.\sum_{m=1}^{r+1} y_{im}^{(t)} + (-\mathbf{1}).\sum_{m=1}^{r+1} z_{im}^{(t)} \leq \mathbf{ASPercent}.\mathbf{P_{g_i}^{max}}, \forall t \in \tau, \forall i \in \mathcal{C} \tag{B.26a}$$

B.4.15 Constraints and coefficients corresponding to Ancillary Services lower limit

$$\mathbf{1}.\sum_{m=1}^{r+1} y_{im}^{(t)} + (-\mathbf{1}).\sum_{m=1}^{r+1} z_{im}^{(t)} \geq \mathbf{0}, \forall t \in \tau, \forall i \in \mathcal{C} \tag{B.27a}$$

B.5 Description of the Software

A program, written to implement the above model, will begin its execution at the main method. In the main method, an object of the "Plant" class is instantiated, which in trun instantiates several objects of the "Configuration" class dependent on the number of configurations to be considered. The object of the Plant class has the "MILPPiece-WiseDisp()" function, which calls the GLPK Solver and solves the optimization problem. The piecewise linear approximation calculation is carried out within the member function "setConfiguration()" of the Plant class. The processcan be pictorially shown in figures B.5 and B.4.

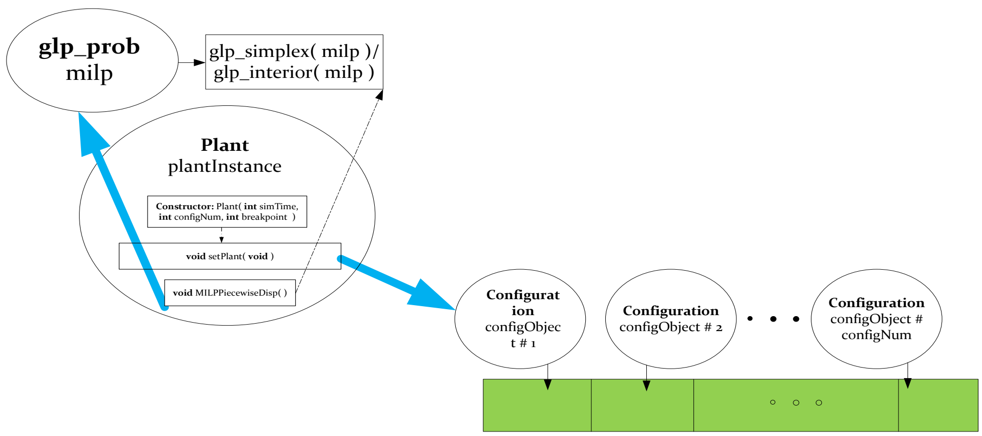


Figure B.4: Combined Cycle Configuration: Object Creation

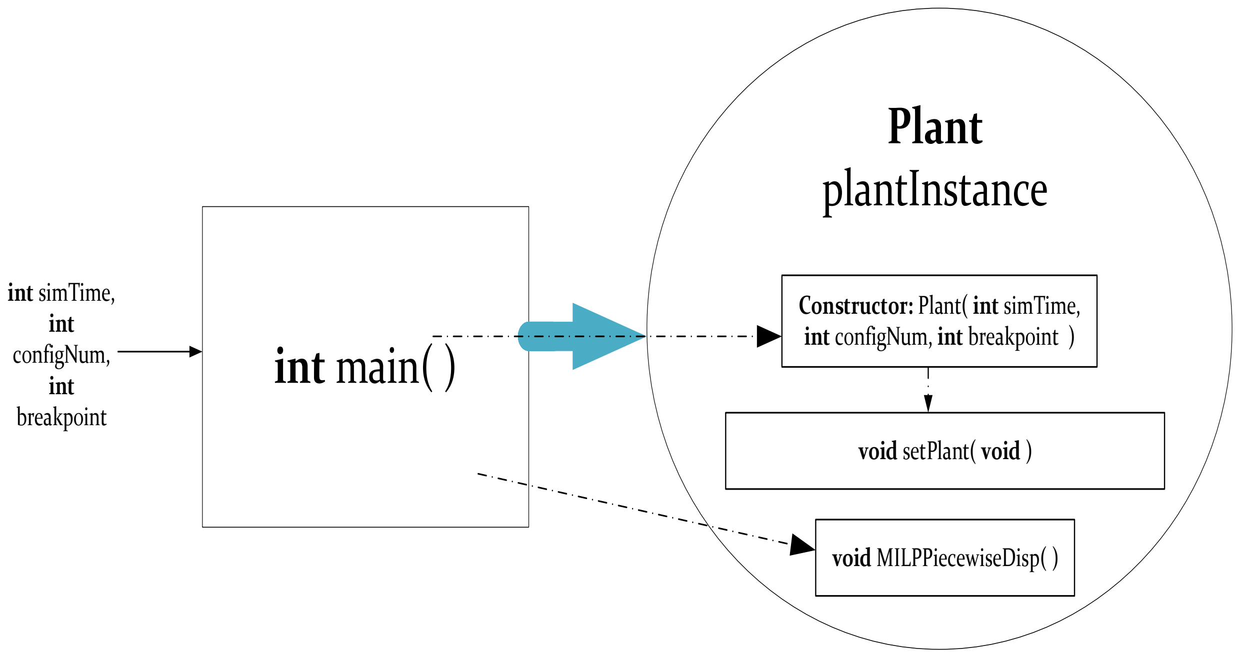


Figure B.5: Combined Cycle Configuration: Execution

Appendix C

Future Work: Advanced Mathematics for Grid Operation Enhancing Demand Response and Renewables

C.1 Market Sector, Background, Objectives, and Goals

The traditional view of the electrical power system, of controllable generation and probabilistic, but inflexible demand is changing due to inclusion of intermittent and probabilistic renewable generation and price-responsive smart loads and new energy resources such as plug-in electric vehicles, battery packs, consumer end solar PV etc. [202, 255, 155, 329, 333, 192, 222, 6]. This means, that the generation and demand now share a lot of common features. Closely aligned to this is the problem of contingency modeling enhancements and need for developing a preventive-corrective dispatch scheme as emphasized by the CAISO in their straw proposals. [203, 204, 210, 206, 208, 212, 209, 205, 207, 211]. These are multi-dispatch interval OPF problems, for which the ISOs (especially CAISO) doesn't currently have a scheme, which represents not only the post-contingency restoration, while abiding by the thermal line limits, but also generates appropriate pricing signals to incentivize the different agents mentioned above to provide energy and reserves. In this future work, we will explicitly solve the multi-stage dispatch problem, taking into account the "dynamic line rating", and producing the market mechanisms for implementing demand response (DR), enhancing participation of renewable and smart loads etc (Solving this new SCOPF will automatically generate prices, which reflect need for reserves or additional resources to ensure system reliability). The main goals of this work will be as follows:

- To maximize the penetration of renewables into the grid, like solar, wind, hydro etc.
- To make use of flexible demands like EVs/PHEVs, battery charging, DRs, and load acting as reserves to attain system restoration and reliability, in case of unforeseen events as outages. [47, 309, 178, 300, 54].
- To explore the possibilities of incentivizing the consumers to participate directly in the wholesale market layer through pricing schemes.

C.2 Technical Approach

In order to address each of the above-mentioned objectives, we will formulate the problem, in several stages, as mathematical optimization problems over multiple dispatch intervals, building in look-ahead capabilites. We will make use of the model predictive control paradigm, [379, 272, 370, 29, 307] since we want to build resiliency against suspected conditions in our formulation. Since the demand variation and renewables availability for the future are inherently probabilistic, we will adpot a dynamic programming approach to solve our optimization problem. Such problems become computationally expensive to solve. Hence we will make use of several model reductions and specifically of the sensitivity analysis tools [17, 50] and also the recent advances in distributed computations for power problems [233, 301, 71, 49] to solve our problem fast. Given below is a broad brush of the how the problem formulation is going to look like:

$$V(\hat{S^{(t)}}, \mathbf{P}^{(t-1)}, \pi^{(t-1)}) = \max_{\mathbf{P}^{(t)} \in \mathcal{F}, \pi^{(t)}} \sum_{t=0}^{T} [NU(\mathbf{P}^{(t)} | \hat{S^{(t)}}) + \sum_{S^{(t+1)} \in \mathcal{S}} p(S^{(t+1)} | \hat{S^{(t)}}) V(S^{(t+1)}, \mathbf{P}^{(t)}, \pi^{(t)})] \tag{C.1}$$

where, $V(.,.,.)$ = Value function associated with the state transitions, $p(.|.)$ = Transition probabilities, $\mathcal{S}$ = State space of the system, $NU(.)$ = Net utility function (Utility-Cost), $\pi^{(t)}$ = Vector of nodal prices for power at time slot t, $S^{(t+1)}$ = Future state at time $t+1$, $\hat{S^{(t)}}$ = Realization of current state, $\mathcal{F}$ = Feasible set of power injections and other state variables, which obeys the constraints imposed by the Look-Ahead SCOPF problem, $\mathbf{P}^{(t)}$ = Vector of power generation and consumption, which is composed of sub-vectors of conventional generation, wind, solar PV, solar concentrated, other renewables, conventional load, rechargeable batteries, PEVs/PHEVs, residential solar etc.

Observe, that (C.1) encapsulates the pricing incentives for motivating the loads to act as reserves and/or participate in the energy market by actively supplying power to the grid, in response to the pricing signals through the value function and an appropriate design of the utility function, based on which, the above model can even enhance the participation of smart/price responsive loads as well as renewable penetration. Figure C.1 shows a schematic of the layers of the market, where we would also like to explore the possible underpinnings with and without the presence of the aggregator (and hence, it's dashed).

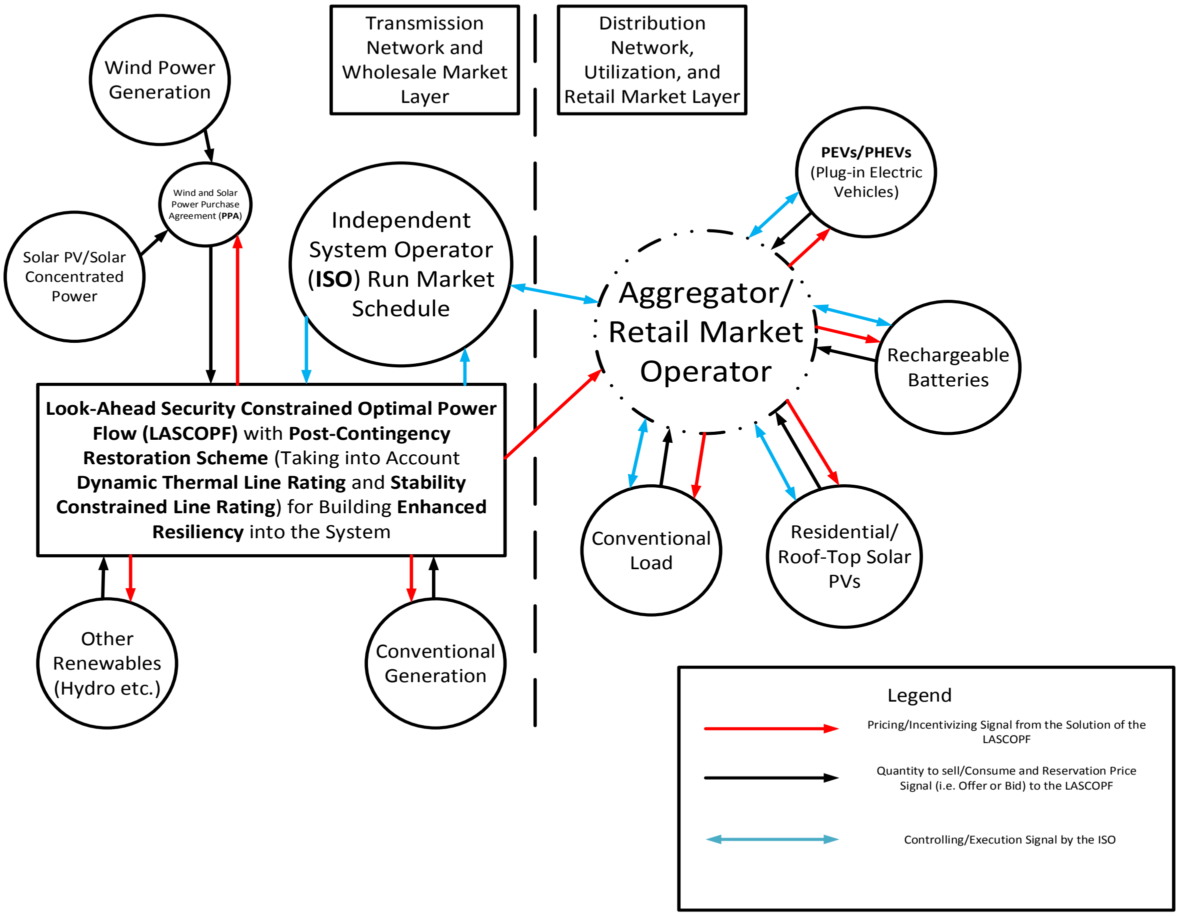


Figure C.1: Schematic Diagram of the Proposed Scheme of Enhancing Renewable Penetration in the Dynamic Operation

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Vita

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This dissertation was typeset with LaTeX[†] by the author.

[†] LaTeX is a document preparation system developed by Leslie Lamport as a special version of Donald Knuth's TeX Program.