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Conics and Geometry

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Conics and Geometry

by

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Report

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Acknowledgements

I would like to acknowledge Doctor Mark Daniels for believing in me even in times that I did not believe that I could make it.

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Abstract

Conics and Geometry

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The University of Texas at Austin, 2010

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Conics and Geometry is a report that focuses on the development of new approaches in mathematics by breaking from the accepted norm of the time. The conics themselves have their beginning in this manner. The author uses three ancient problems in geometry to illustrate this trend. Doubling the cube, squaring the circle, and trisecting an angle have intrigued mathematicians for centuries. The author shows various approaches at solving these three problems: Hippias' Quadratrix to trisect an angle and square the circle, Pappus' hyperbola to trisect an angle, and Little and Harris' simultaneous solution to all three problems. After presenting these approaches, the focus turns to the conic sections in the non-Euclidean geometry known as Taxicab geometry.

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INTRODUCTION:

Though geometry and the conic sections are intertwined both historically and mathematically, their development became the platform for the conflict between the traditional and newer approaches to mathematics. Boyer describes the conflict well

> "The three famous problems of antiquity were to be solved through the use of lines and circles only; but, fortunately for the development of mathematics, all three of the classical problems are unsolvable under this restriction. Here one sees a remarkable instance of the fruitfulness of failures, for, defeated in all frontal assaults on the problems, geometers sought partial satisfaction in a disingenuous infraction of the rules. They invented new curves to aid them" [**1**, p. 294-295].

If one considers the ad hoc manner in which secondary mathematics topics are presented in textbooks, then this conflict shows its influence still today. Though some may view the conflict between traditional and modern approaches to mathematics as negative, the author believes that it has always led to growth and interesting discoveries in mathematics. This fact is definitely true of the conics whose origin is believed by most to be the result of Menaechmus breaking one of the oldest traditions in mathematics – Plato's Restriction. Buchanan describes the restriction as "his dictum that only the ruler and compass could be used in the constructions of elementary geometry. The ruler must not be graduated, i. e., distances could be transferred from one part of the figure to another only by use of compass and not by ruler" [**2**, p. 14]. Menaechmus is thought to have been working on one of the three classic problems of geometry, the Delian Problem or duplicating the cube. Instead of following Plato's Restriction, he was as Boyer put it "following the path which Archytas had suggested – that is, he sought to solve the Delian problem by a consideration of sections of geometrical solids" [**1**, p. 297]. By taking this approach, Menaechmus discovered the ellipse, hyperbola, and parabola [**1**, p. 296]. His approach added a new category of curves to the existing line and circle. Lloyd describes the latter as being known as plane loci and the former as solid loci [**6**, p. 293]. Boyer adds a third category known as linear loci that represented all other curves [**1**, p. 297]. The three classic problems; the Delian problem, trisection of any angle, and squaring a circle, have fascinated mathematicians for centuries. Though the impossibility of solving them under Plato's restriction was probably known to the Greeks, it was not formally proved until as late as the nineteenth century in the case of trisecting an angle. The absence of formal proof can be attributed to the lack of the analytical method until the seventeenth century. This new invention is a further departure from the strictly traditional synthetic method known to mathematicians of that day. It highlights another example of the conflict between the traditional and newer approaches. Some developments take centuries to establish themselves. Even as they are established, the traditional methods linger. This is not necessarily a bad thing. It is the goal of this report to explore various approaches, both traditional and modern, to the classic three problems and the conics.

THE THREE ANCIENT PROBLEMS:

The first approach the author will consider was invented by Hippias in 420 B.C. While attempting to solve the trisection and squaring the circle problems, he invented the Quadratrix. Yates describes the process he used [**7**, p.191-192]. This new curve is formed by drawing a quadrant of the unit circle as in Fig. 1. As the point D moves along the axis $\overline{OC}$ at a constant rate, the point E moves at a constant rate along $\overset{\frown}{BC}$. The intersection of $\overline{OE}$ and the segment through point D and perpendicular to that axis is the point P. The path P makes as E and D move at a constant rate is the curve known as the quadratrix. Under the construction, the following ratios are equivalent:

$$\frac{\overline{OD}}{\overline{OF}} = \frac{\overset{\frown}{BE}}{\overset{\frown}{BA}} = \frac{\theta}{\phi} \tag{1}$$

The construction of the curve makes trisection fairly simple due to (1). If $\angle AOB = \phi$ is the angle to be trisected, then we construct $\overline{OF}$ along $\overline{OC}$ as shown. It is then possible to trisect $\overline{OF}$ with a compass and straight edge. Therefore $\overline{OD} = \frac{1}{3}\overline{OF}$ as can be seen in Figure 1. By substitution in (1) we have:

$$\frac{\frac{1}{3}\overline{OF}}{\overline{OF}} = \frac{\theta}{\phi},$$
$$\theta = \frac{1}{3}\phi.$$

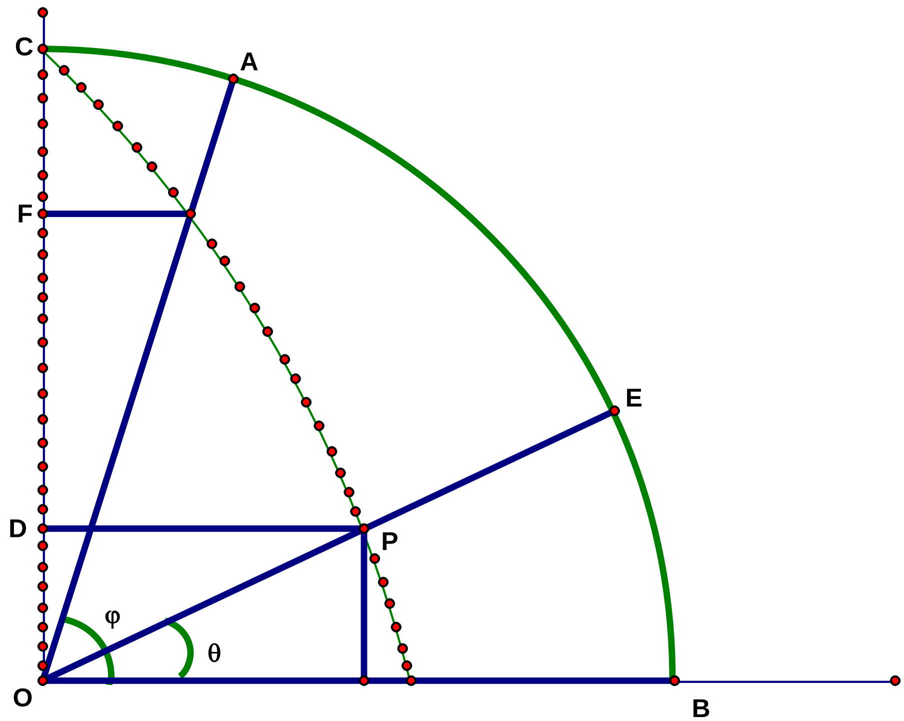

Figure 1: Construction of the Quadratrix

This result would satisfy as a solution to the trisection problem synthetically. Yates takes it one step further in order to find the rectangular equation of the curve by applying analytical methods. He takes $\overline{OB}$ and $\overline{OC}$ to be the positive x and y axes respectively and the coordinates of P to be (x,y). Under those conditions, the following is true:

$$\frac{\overline{OD}}{\overline{OC}} = \frac{\theta}{\frac{\pi}{2}}, \tag{1.2}$$

$$x = \left(\overline{OP}\right)\cos\theta\,, \tag{1.3}$$

$$\overline{OC} = 1\,, \tag{1.4}$$

$$\overline{OD} = y\,. \tag{1.5}$$

Using basic trigonometric relationships yields

$$\cot\theta = \frac{x}{y}$$

Then by multiplying both sides by *y*,

$$x = y\cot\theta$$

And by substituting (1.4) and (1.5) in (1.2),

$$y = \frac{2\theta}{\pi} \tag{1.6}$$

These two equations form the parametric of the Quadratrix. Yates leaves it to the reader to eliminate θ to find the rectangular equation. By using the definition of cotangent and solving for θ

$$\theta = \tan^{-1}\left(\frac{y}{x}\right).$$

Now substitute this result in (1.6)

$$y = \frac{2\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}{\pi}$$

$$\frac{\pi y}{2} = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan\left(\frac{\pi y}{2}\right) = \frac{y}{x}$$

$$x = \frac{y}{\tan\left(\frac{\pi y}{2}\right)}. \tag{1.7}$$

The equation in 1.7 is important for solving the second problem of squaring the circle. In order to do this, a length is needed in terms of π. This can be achieved by taking the limit as y approaches zero of (1.7) in order to obtain the value of x where the curve intersects $\overline{OB}$. Since taking this limit leads to an indeterminate, one must use L'Hôpital's Rule:

$$x = \frac{\frac{d}{dy} y}{\frac{d}{dy}\tan(\frac{\pi y}{2})} = \frac{1}{\frac{\pi}{2}(\sec^2(\frac{\pi y}{2}))} = \frac{\frac{2}{\pi}}{\sec^2(\frac{\pi y}{2})} \tag{1.8}$$

Now take the limit of the result from (1.8):

$$x = \frac{\lim\limits_{y\to 0}(\frac{2}{\pi})}{\lim\limits_{y\to 0}(\sec^2(\frac{\pi y}{2}))} = \frac{2}{\pi}$$

Thus two of the ancient problems can be solved using this creative curve. The problem of trisecting an angle was also solved by Pappus approximately 120 years later. He chose to make use of some properties of conic sections. Yates describes his method as follows [**7**, p. 194-195]: A unit circle is constructed such that the center is located at the vertex of the angle one wishes to trisect (see Figure 2). $\angle AOB$ is bisected by $\overline{OC}$. The point P is allowed to move keeping the distance from it to point B twice the distance as point P to $\overline{OC}$. In such a manner the path of P traces one branch of a hyperbola with $\overline{OC}$ as its directrix and point B its focus. Then the branch of the hyperbola is reflected across $\overline{OC}$ to have point P' correspond with point P.

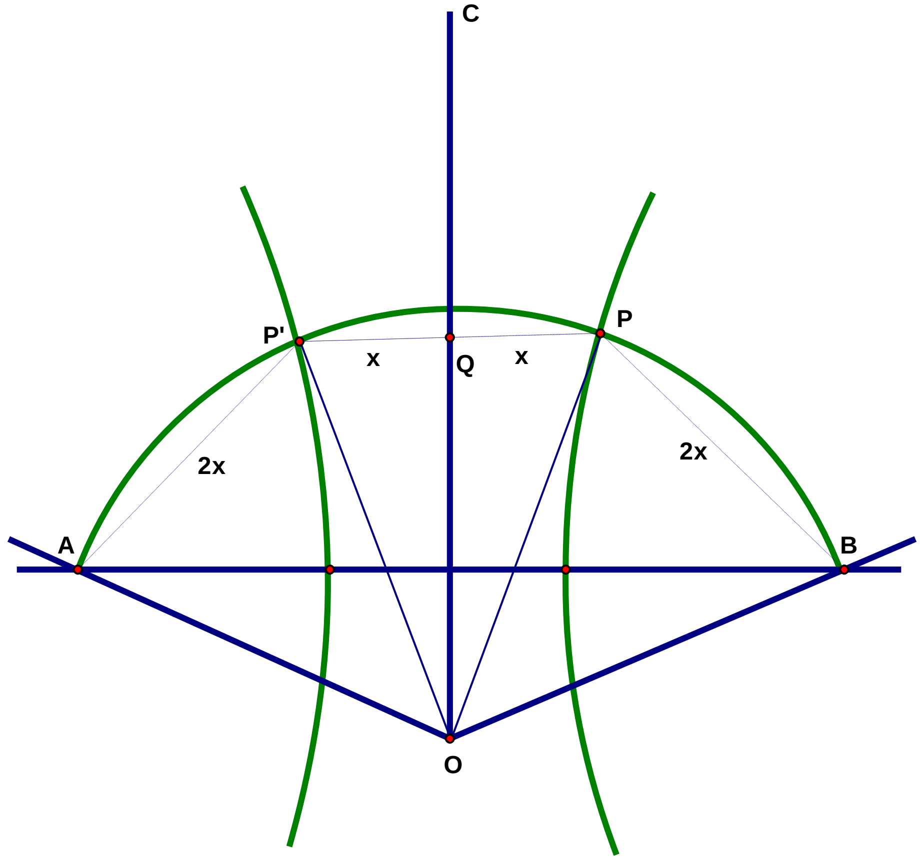

Figure 2: Hyperbola of Pappus

The idea to use the hyperbola in this manner was genius. By construction and the properties of reflection we have:

$$\overline{PB} \cong \overline{PP'} \cong \overline{P'A}$$

Since congruent chords have congruent arcs and congruent arcs have congruent central angles, the trisection is complete synthetically. Yates' analytical approach was to let $\overline{OC}$ be the *y* axis and $\overline{AB}$ be the *x* axis. He then designated *2c* as the distance for $\overline{AB}$ and the coordinates for *P* to be (*x,y*). Then using the distance formula and the fact that the distance from *P* to *B* is always twice that from *P* to *Q* the following equation is formed:

$$\sqrt{(x-c)^2+y^2}=2x$$

$$y^2-3x^2-2cx+c^2=0\,. \tag{2.1}$$

To find the coordinates of P and P', one would have to solve (2.1) and the equation of the circle simultaneously. This lengthy process would provide those points and the two other intersections of the circle and hyperbola. It is interesting to the author that up until the time that analytic geometry was invented, new curves were constructed with the purpose to help solve problems. With analytical geometry however, the mathematics led to an infinite number of curves. The synthetic approach has an appeal because of its elegance and beauty. The analytical approach has more flexibility and far-reaching consequences.

Boyer points out that Fermat was the first to direct his "attention to an infinite number of plane curves known as the parabolas and hyperbolas of higher degree given by the equation $y=kx^n$, where n is a positive or negative rational number. This represents the first systematic use of what is without doubt the most important of all methods of curve definition – analytically through equations" [**1**, p. 302]. It is yet another instance in history where a new approach opened numerous doors. After viewing some traditional approaches to the three ancient problems, the author turns to a modern example, however traditional in nature it may be.

Uniquely, Little and Harris found a simultaneous solution to all three of the ancient problems [**5**, p.310-311]. They achieved this by carefully defining three curves. The curves are

$$f_1(\theta)=3\sin\theta \tag{3.1}$$

$$f_2(\theta)=2+\sin 3\theta \tag{3.2}$$

$$f_3(\theta) = -6\cos\theta \tag{3.3}$$

Each is shown in Figure 3. Note that the circle has its center at (-3,0) and a radius of 3. It is also important to note that the circle is generated with θ values from 0 to 2π and (3.1) and (3.2) are generated with θ values from 0 to π.

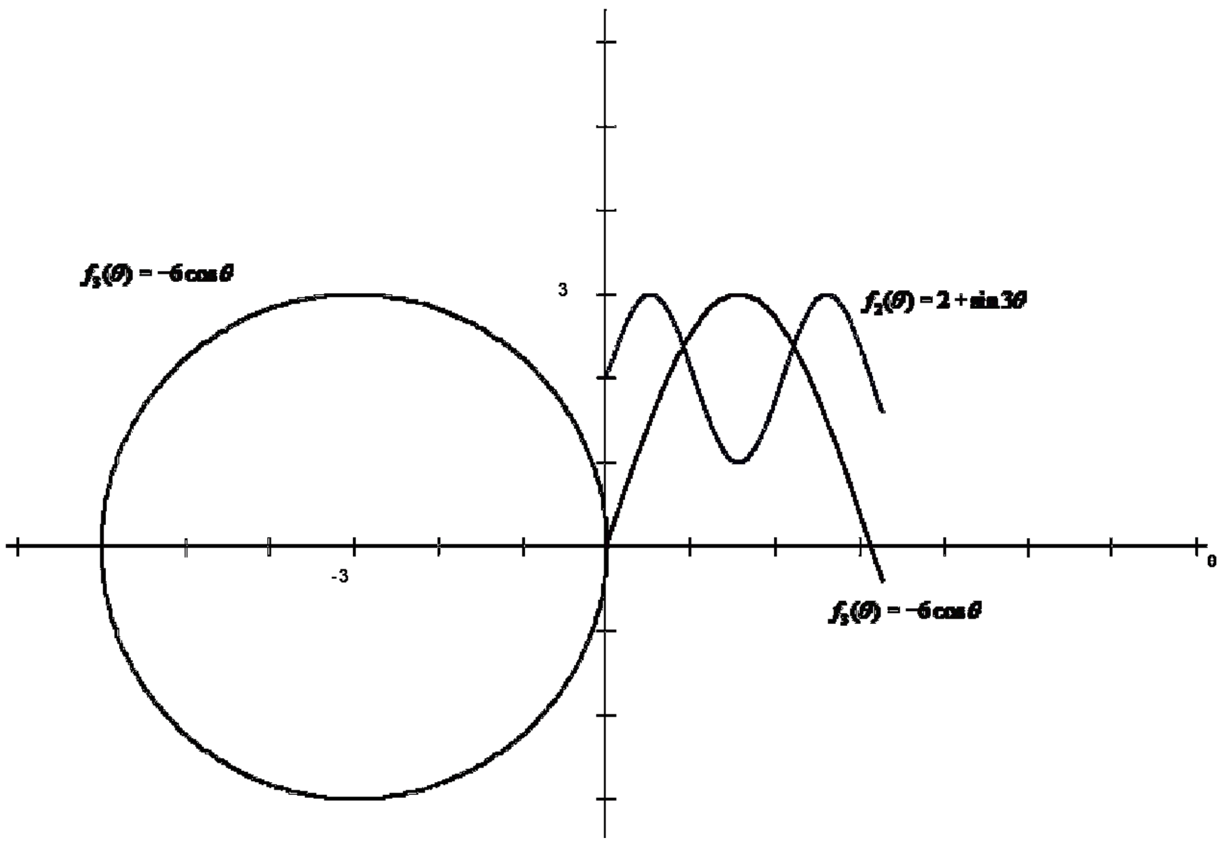

Figure 3: Little and Harris' curves

In order to solve the trisection problem, the (3.1) and (3.3) will be used. The angle to be trisected is constructed such that the vertex is at the center of (3.3) and the initial side coinciding with the θ axis as is shown in figure 4. The terminal side of the angle intersects (3.3) at point A and let α be the measure of the angle. Now by constructing a line parallel to the θ axis and through point A, the intersection with (3.1) is constructed and labeled R. Then by constructing a line perpendicular to the θ axis and through the point R, the length $\overline{OB}$ is achieved. By this clever construction, the length of $\overline{OB}$ is α, the same as the angle to be trisected. The segment $\overline{OB}$ is trisected and point Q is a third of the length of $\overline{OB}$. The process is reversed by constructing the perpendicular through Q and the parallel through R' to get the intersection with (3.3) at A'. In doing so, the angle formed by connecting O' with A' is the solution. The solution to squaring a circle is also apparent. Due to the period in which (3.1) is generated, the length of $\overline{OP}$ is π. Thus making it possible to construct a length of $\sqrt{\pi}$ which would be the side of the square needed to solve the problem.

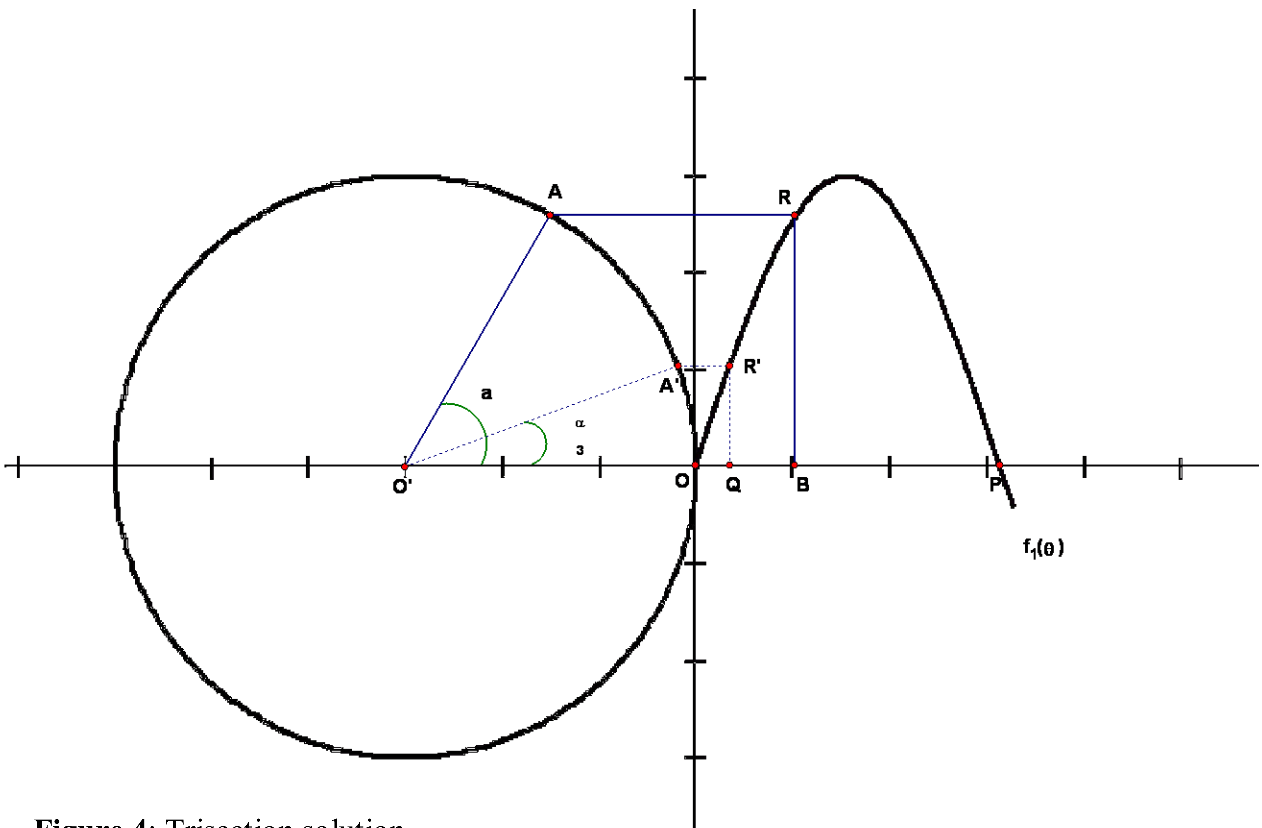

Figure 4: Trisection solution

Where these first two solutions could be easily classified as synthetic in nature, the solution to the doubling of a cube is not possible without the equations themselves. The solution is achieved by finding the simultaneous solution to (3.1) and (3.2) as follows:

$$3\sin\theta = \sin(3\theta) + 2 \tag{3.4}$$

It is necessary to manipulate $\sin 3\theta$ using various trigonometric properties:

$$\begin{aligned}
\sin 3\theta &= \sin(2\theta + \theta) \\
&= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\
&= 2\sin\theta\cos^2\theta + 2\sin\theta\cos^2\theta - \sin\theta \\
&= 4\sin\theta\cos^2\theta - \sin\theta \\
&= \sin\theta\left(4\cos^2\theta - 1\right) \\
&= \sin\theta\left(4 - 4\sin^2\theta - 1\right) \\
&= 3\sin\theta - 4\sin^3\theta
\end{aligned}$$

By substituting this result in (3.4):

$$3\sin\theta = 3\sin\theta - 4\sin^3\theta + 2$$
$$\sin^3\theta = \tfrac{1}{2}$$

This result gives us a value for the intersection and therefore corresponding length along the θ axis of $\frac{3}{\sqrt[3]{2}}$. It is then possible to construct a length of $\sqrt[3]{2}$ which is the necessary length for the side of the cube to be doubled. Therefore Little and Harris were able to solve all three ancient problems using three curves. An interesting endeavor would be to find a solution to all three using only two curves.

CONIC SECTIONS IN NON-EUCLIDEAN GEOMETRY

The same departure from the norm that brought about the discovery of the conic sections also led mathematicians to venture into other geometries. As with all new mathematics, these were met with much resistance. Johnson and Libeskind explain that it wasn't "until 1868 that an Italian mathematician named Eugenio Beltrami (1864-1900) proved beyond a doubt that these new geometries were every bit as valid as Euclid's own." [**3**, p. 6].

The author will now explore the conic sections in the non-Euclidean geometry known as Taxicab Geometry. Johnson and Libeskind describe this geometry as still working with the Euclidean coordinate plane, where lines, angles, and points are all the same, but the manner in which distance is measured has changed [**3**, p. 39]. The normal distance formula in Euclidean geometry derived from the Pythagorean Theorem is replaced by the *taxicab metric*:

$$d_T(P(x_1, y_1), Q(x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

where d_T denotes the taxicab distance between the points P and Q. Laatsch approaches the analog to conic sections in taxicab geometry using the traditional focus/directrix definition of the ellipse, parabola, and hyperbola [**4**, p. 205]. The definition of the distance from a point to a line is necessary. Let L be a line in the plane P and P be a point not on L. The taxicab distance from P to L is defined as:

$$d_T(P, L) = \min\{d_T(P, Q) : Q \in L\} \tag{4}$$

It soon becomes apparent that this distance depends on the slope of L. This fact is crucial when using the focus/directrix definition. Laatsch describes it well as "if L is a vertical line or if its slope m satisfies $|m| \geq 1$, then $d_T(P, L)$ is the horizontal segment joining P to L; however, if $|m| \leq 1$, then $d_T(P, L)$ is the vertical segment joining P to L." [**4**, p. 205].

For the purposes of this paper, unless otherwise noted, the author will take the same premise as Laatsch by fixing all lines and points in the coordinate plane *P*. Let F be the focus and L be the directrix, which does not contain F. Laatsch defines the set of all points P in the plane *P* such that

$$\frac{d_T(P, F)}{d_T(P, L)} = e, \tag{4.1}$$

where e is the eccentricity, as a projected pyramidal section, or p-section [**4**, p.206]. The resulting plane figure will be called an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$.

As previously stated, the taxicab distance from a point to a line depends on the slope of the line. For this reason, the *p*-sections are broken into cases. Laatsch describes three cases, but the author will add a sub case to one of these cases. The parallel case is when the directrix is a horizontal or vertical line. The diagonal case is when the directrix has a slope m such that $|m| = 1$. The general case is for all other slopes [**4**, p. 206]. It is in the general case that the author distinguishes between $0 < |m| < 1$ and $|m| > 1$ because these force the denominator in 4.1 to differ and thus affecting the graphs. The graph of each case of the parabola and one case for the ellipse and hyperbola are shown in this report (the graphs of the other cases are included in Appendix A). Though Laatsch changed the focus for some of his graphs, the author chose to keep the focus constant at (3,0) throughout the paper, unless otherwise noted. The first case is a general case parabola with $|m| > 1$ and L: $y = 2x$ (see figure 5). The equation is:

$$|x-3|+|y|=\left|x-\tfrac{y}{2}\right| \tag{4.2}$$

By changing the slope to $\frac{1}{2}$, we get the general case with $0<|m|<1$ and L: $y=\frac{1}{2}x$ (see Figure 6) with the equation:

$$|x-3|+|y|=\left|y-\frac{x}{2}\right| \tag{4.3}$$

Notice how the taxicab distance changes from the point to the line as the slope changes. The diagonal case with $|m|=1$ and L: $y=x$ (Figure 7) yields the equation:

$$|x-3|+|y|=|x-y| \tag{4.4}$$

The parallel case with $|m|$ undefined and L: $x=-2$ (Figure 8) yields the equation:

$$|x-3|+|y|=|x+2| \tag{4.5}$$

Following are the graphs for the 4 cases of parabolas:

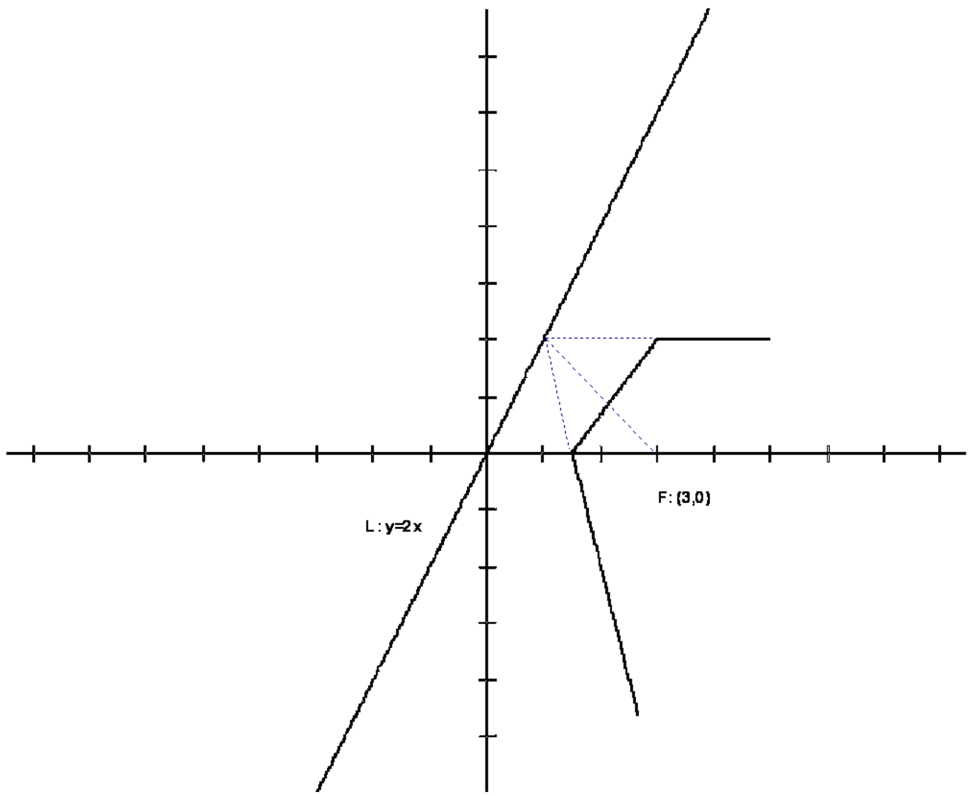

Figure 5: Taxicab parabola, general case, L: $y = 2x$, F: (3,0); its equation is 4.2

It is important to note that, as a leg of the graph reaches the vertical or horizontal line that intersects the focus, the graph has a bend in it. The graphing software used by the author would not show the lower bend.

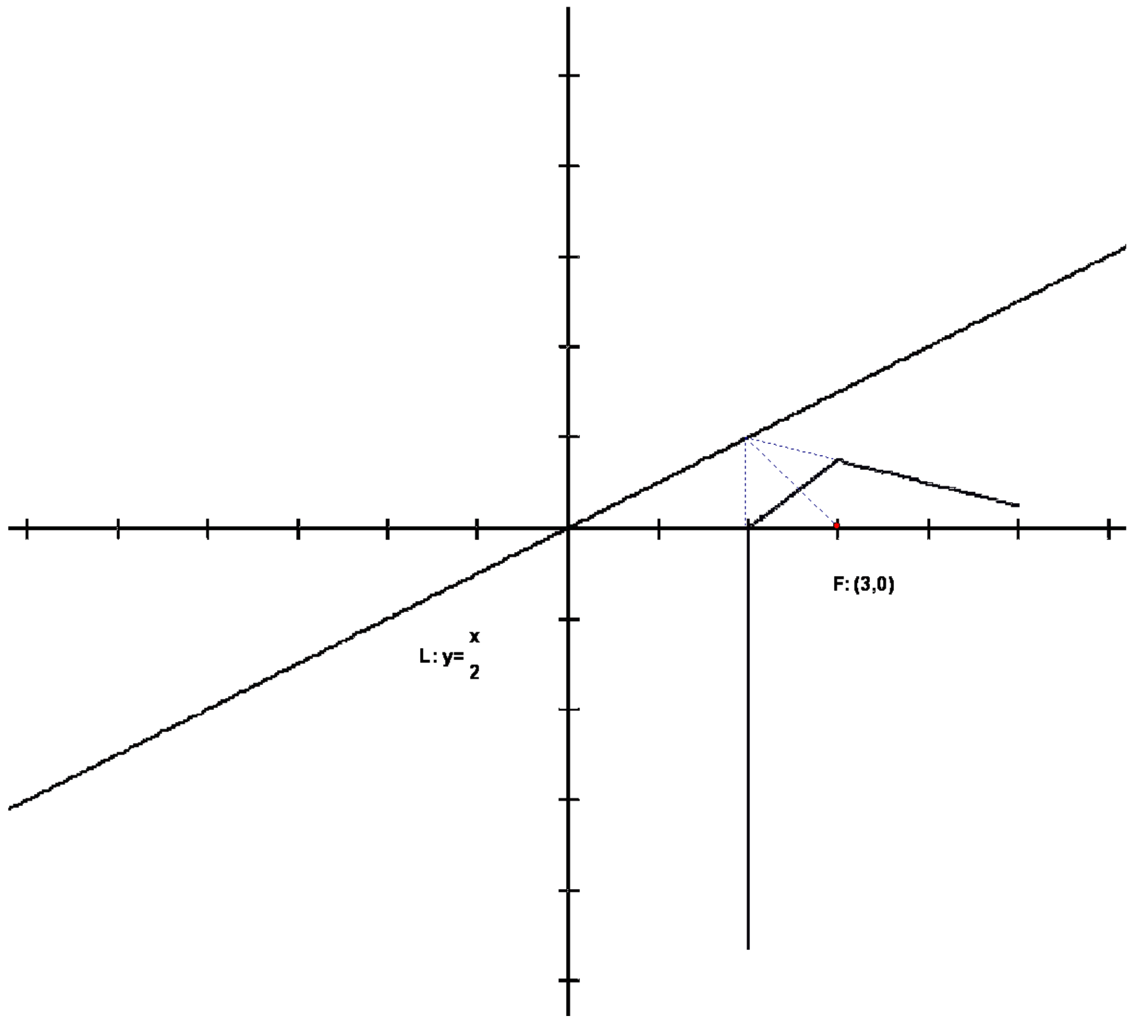

Figure 6: Taxicab parabola, general case, L: $y = \frac{x}{2}$, F: (3,0); its equation is 4.3

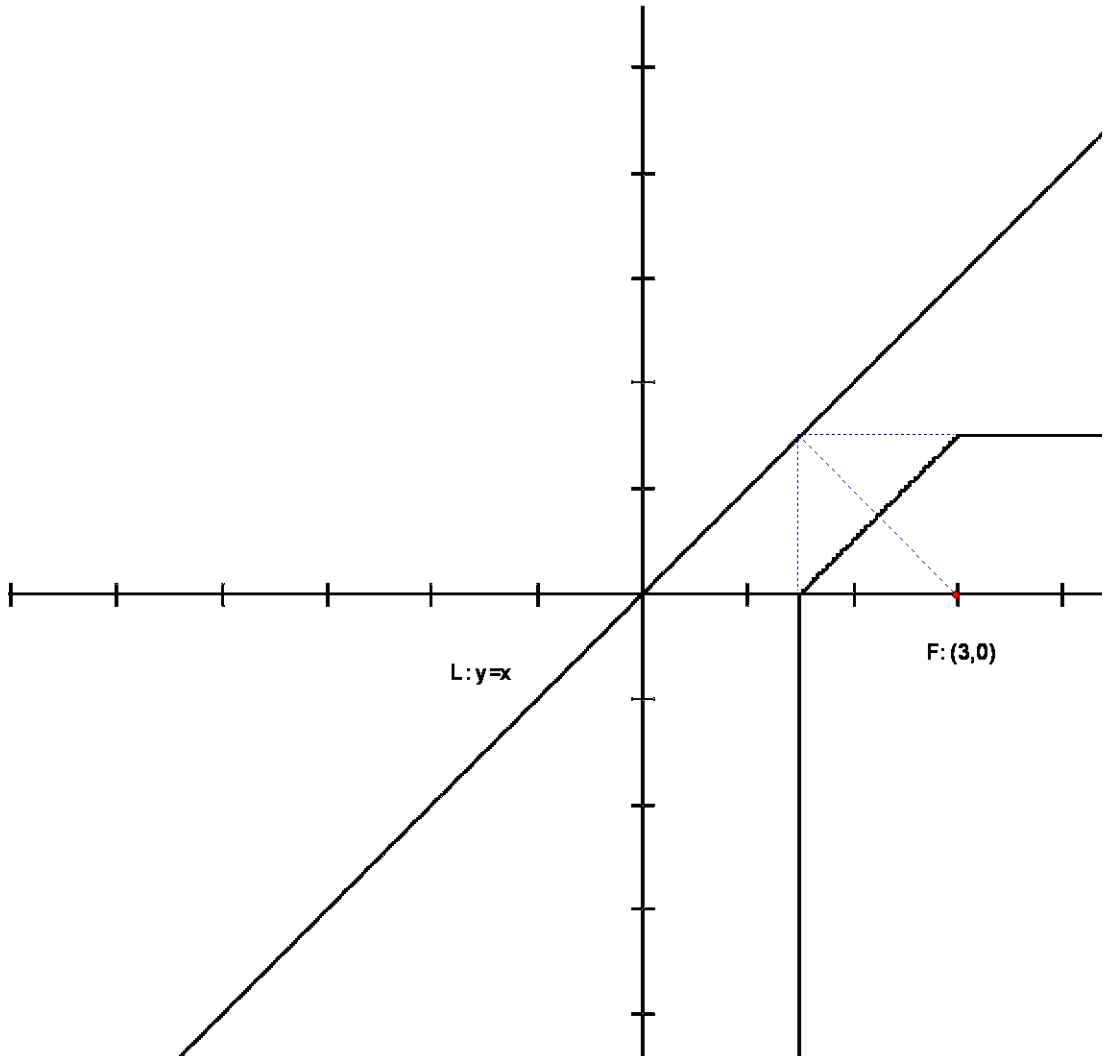

Figure 7: Taxicab parabola, diagonal case, L: $y = x$, F: (3,0); its equation is 4.4

Laatsch points out that the symmetry in Figure 7 is typical of the diagonal and parallel cases. The graph will be symmetric to a line perpendicular to the directrix [**4**, p.206].

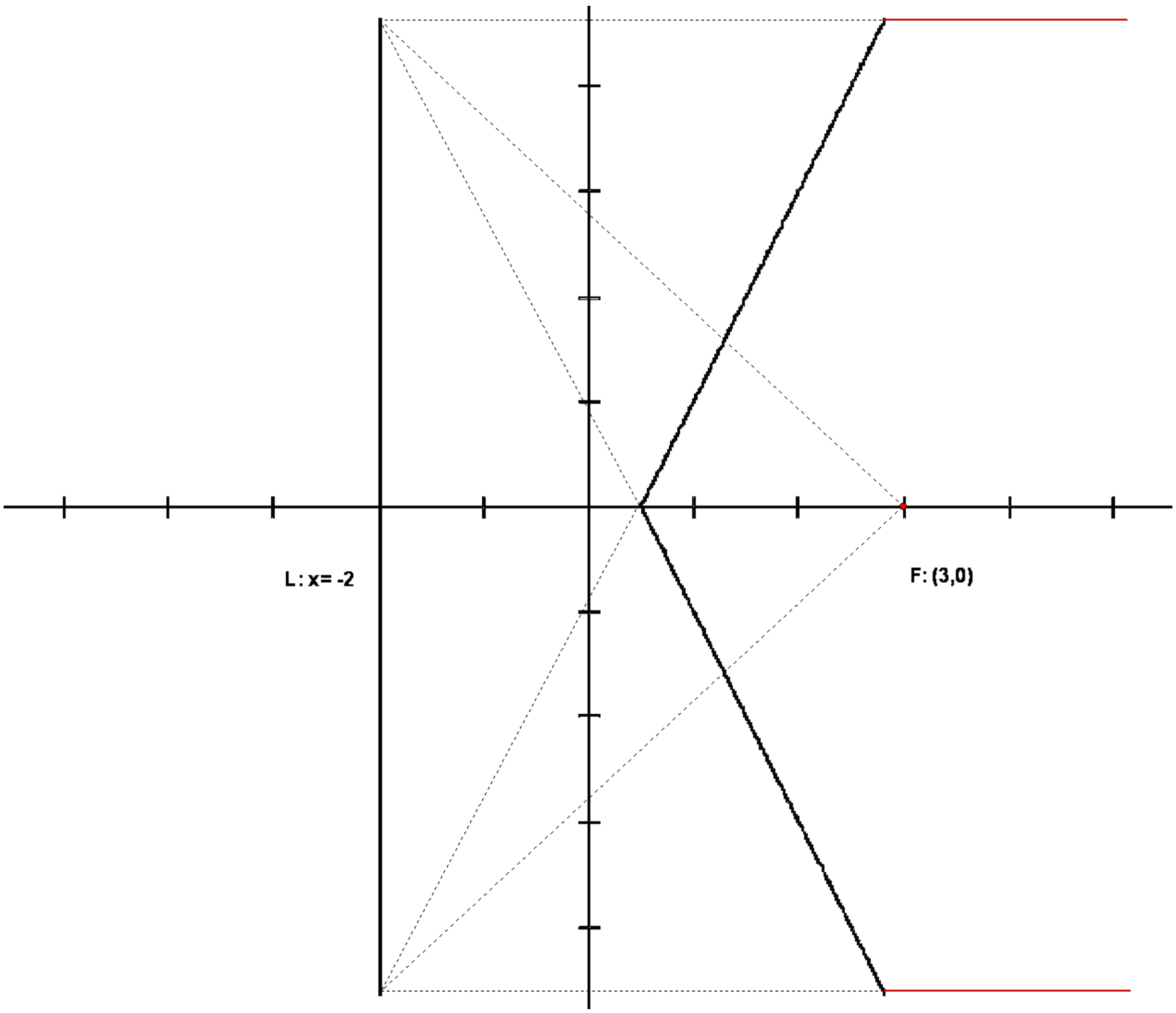

Figure 8: Taxicab parabola, parallel case, L: $x = -2$, F: (3,0); its equation is 4.5

The taxicab ellipse is a quadrilateral. The parallel case is included here (see Figure 9) and will be used to show that the vertices of the taxicab ellipse are points on the Euclidean ellipse with the same directrix, focus, and eccentricity. Again using the proportion is 4.1, the equation for the ellipse is:

$$|x-3|+|y|=|x| \tag{5}$$

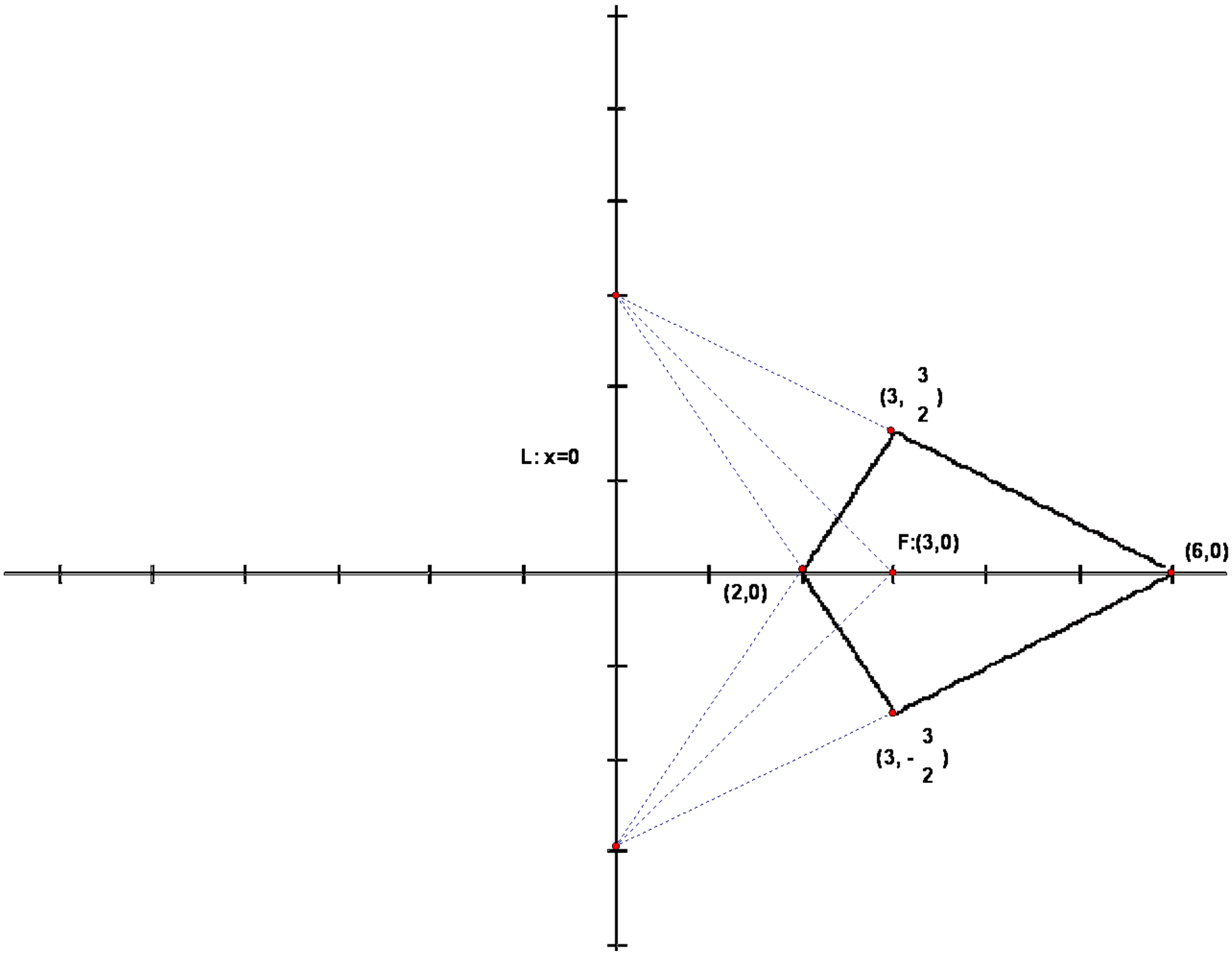

Figure 9: Taxicab ellipse, parallel case, L: $x = 0$, F: (3,0); its equation is (5)

To show that the vertices shown in Figure 9 are on the Euclidean ellipse (denoted E_E) with directrix $L: x = 0$, focus F: (3,0), and eccentricity $\frac{1}{2}$, we consider the locus of the point $P(x,y)$ satisfying the condition:

$$|d_1| = e|d_2| \tag{5.1}$$

where d_1 is the Euclidean distance from P to F and d_2 is the Euclidean distance from P to L. By use of the distance formula in (5.1) we get:

$$E_E : \sqrt{(x-3)^2 + y^2} = \tfrac{1}{2}|x| \tag{5.2}$$

Substituting each vertex point into 5.2 will show that they are indeed on E_E.

First the point (2,0):

$$\sqrt{(2-3)^2 + (0)^2} = \tfrac{1}{2}|2|$$

$$\sqrt{1} = 1$$

$$1=1$$

Now $(3, \tfrac{3}{2})$:

$$\sqrt{(3-3)^2 + (\tfrac{3}{2})^2} = \tfrac{1}{2}|3|$$

$$\sqrt{(\tfrac{3}{2})^2} = \tfrac{3}{2}$$

$$\frac{3}{2} = \frac{3}{2}$$

In a similar way, it can be shown that the vertices (6,0) and $(3, -\tfrac{3}{2})$ also lie on E_E. The hyperbola presents a wrinkle in the general case; which will be illuminated below. In the parallel and diagonal cases (which can be found in Appendix A) the hyperbola has one axis of symmetry. In the general case, the shape of the graph depends not only on the slope of the directrix, but also on the eccentricity [**4**, p.207]. Therefore the general case will be shown in three subcases: $|m| < e$, $|m| = e$, and $|m| > e$. The first subcase (see Figure 10) renders the following equation (note e=3):

$$|x-3|+|y|=3\left|x-\tfrac{y}{2}\right| \tag{6}$$

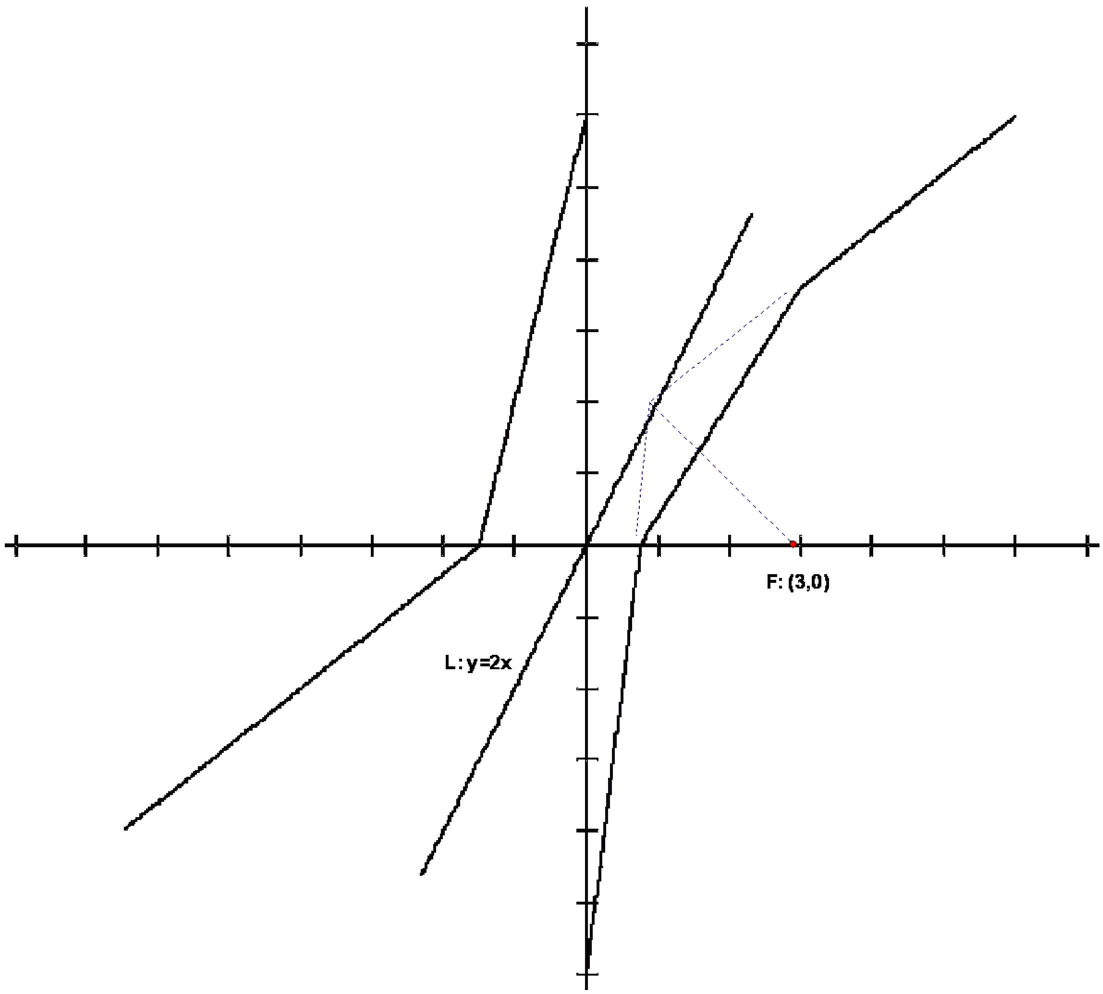

Figure 10: Taxicab hyperbola, general case, L: $y = 2x$, F: (3,0); its equation is (6)

The next subcase with $|m| = e$ (see Figure 11) is as follows:

$$|x-3|+|y|=3\left|x-\tfrac{y}{3}\right| \tag{6.1}$$

The last subcase with $|m| > e$ (see figure 12) renders the equation:

$$|x-3|+|y|=3\left|x-\tfrac{y}{10}\right| \tag{6.2}$$

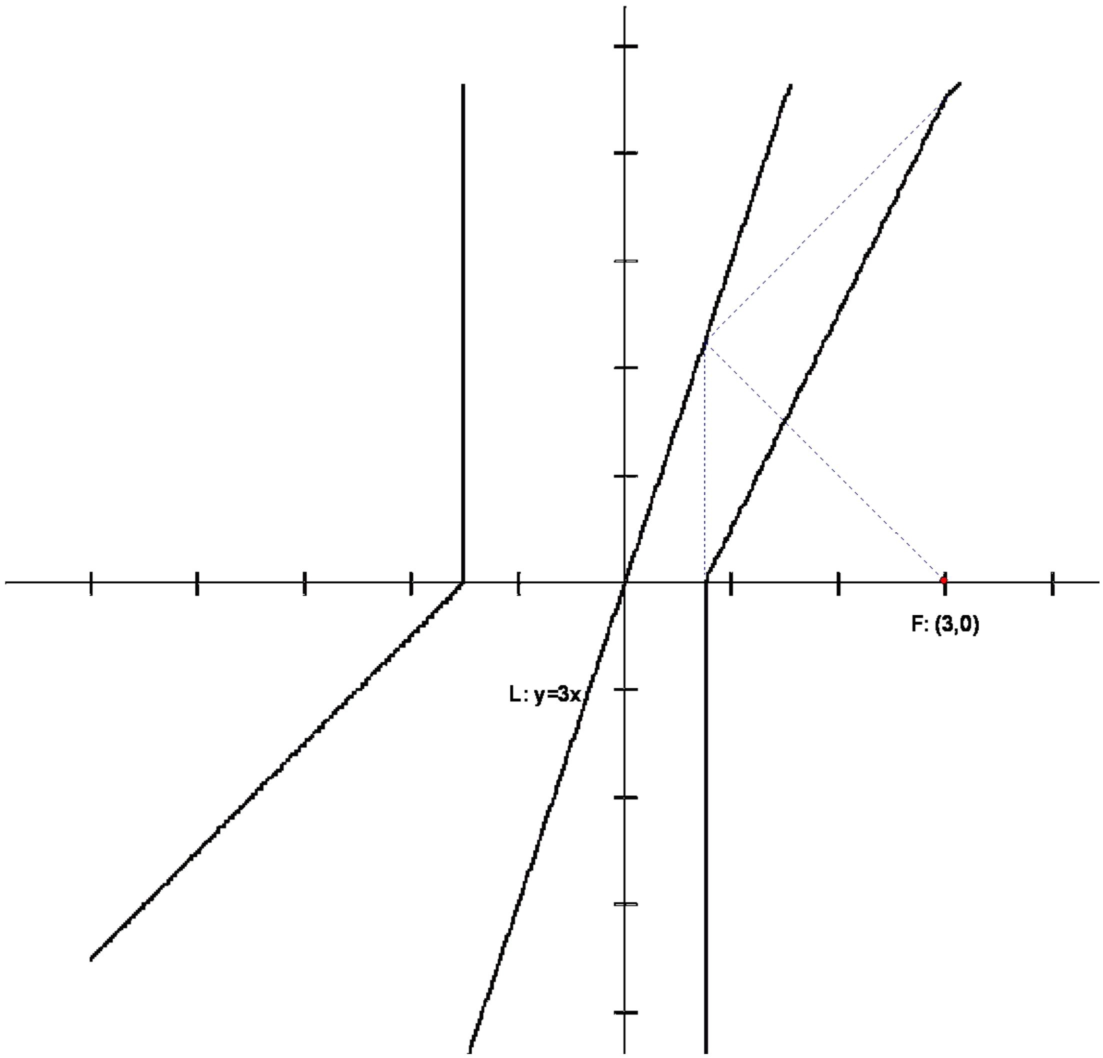

Figure 11: Taxicab hyperbola, general case, L: $y = 3x$, F: (3,0); its equation is (6.1)

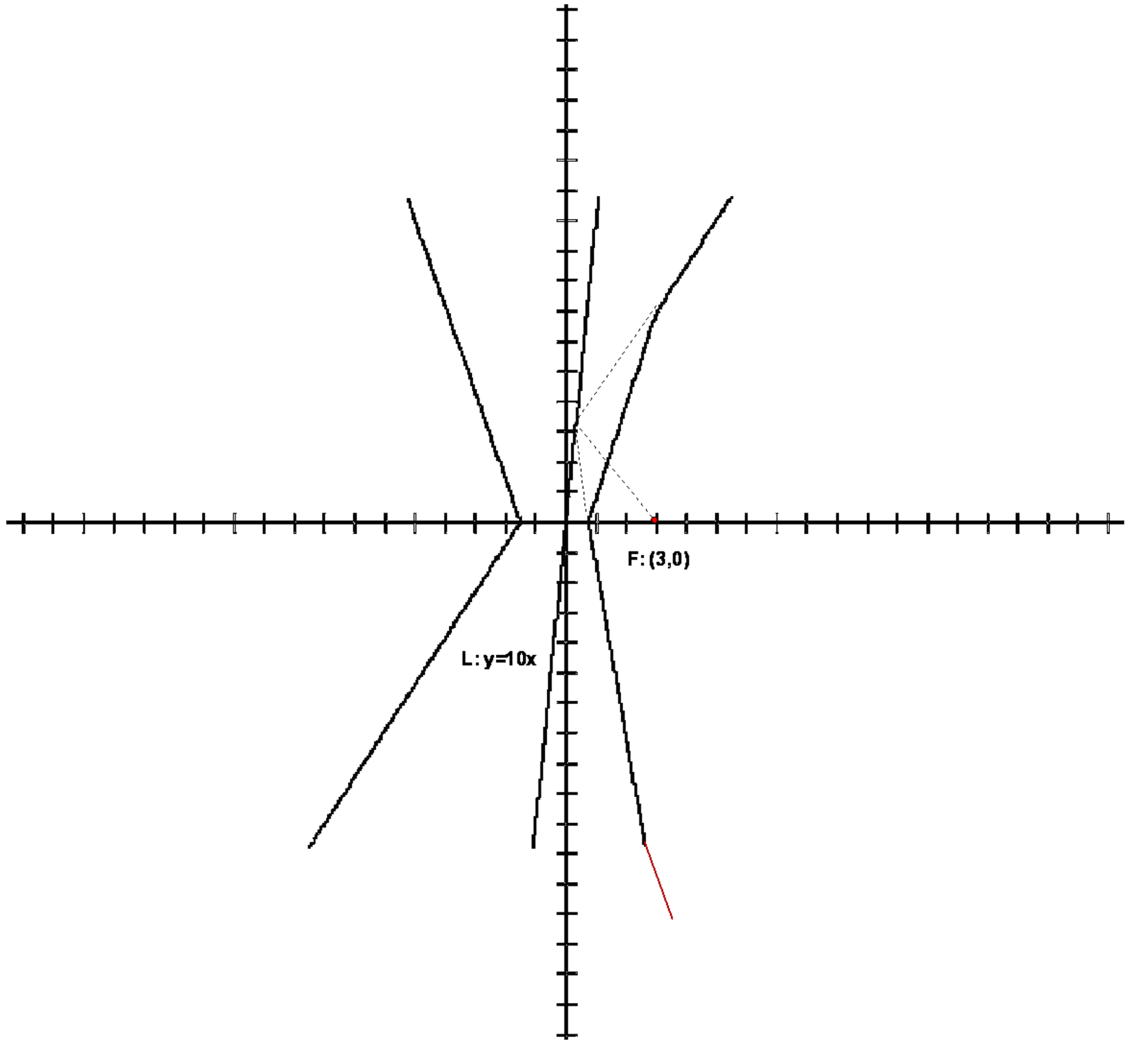

Figure 12: Taxicab hyperbola, general case, $L{:}y = 10x$, F: (3,0); its equation is (6.2)

Laatsch observes that as $|m|$ increases until it $|m|=e$, the lower ray of the right hand branch increases in slope until it becomes vertical. At that moment, the upper ray and segment of the left hand branch also form a vertical ray. If $|m|$ is increased so that $|m|$ $>e$, then the upper ray on the left hand branch assumes a negative slope as does the lower portion of the right hand branch. There is an apparent symmetry with the opposite segments that leads Laatsch to the conclusion that “each *p*-section is the union of segments or rays from exactly four distinct lines (three for the diagonal parabola)” [**4**, p. 208]. The dashed lines in each graph are included to illustrate Laatsch’s first theorem:

Theorem 1. *Two alternate sides of a p-section, if extended, will intersect at a point Q on the directrix of the section. A line of slope* ± 1 *through the focus of the section also passes through Q.*

As it turns out, this property is the consequence of slicing the square pyramid. Therefore, taking the directrix/focus approach in Taxicab geometry leads to the sectioning of the square pyramid with the same consistency as sectioning of the right cone in Euclidean geometry.

CONCLUSION:

Not only are geometry and conic sections intertwined throughout the history of mathematics, but they are the source of many new mathematical endeavors. Interestingly, the practical application of these endeavors take centuries to develop. This is certainly true of the conic sections. Buchanan put it well when he spoke of the Greeks and said “They developed geometry not for its utility but for its beauty and its adaptability to a perfect system of logic. A notable example of this is in their study of the Conic Sections. They knew the ellipse, parabola and hyperbola and most of their properties. They obtained these as exercises in solid geometry without the remotest idea so far we can tell that there ever would be any use for them. Now they are in use in Astronomy, Physics, Engineering, Ballistics and in almost every field of applied science” [**2**, p.15]. The exploration of these examples has been rewarding and enlightening. It has been rewarding in learning about the mathematics involved. It has been enlightening to view teaching in such a way as to incorporate openness to new methods of mathematics.

Appendix A: Graphs of *P*-Sections

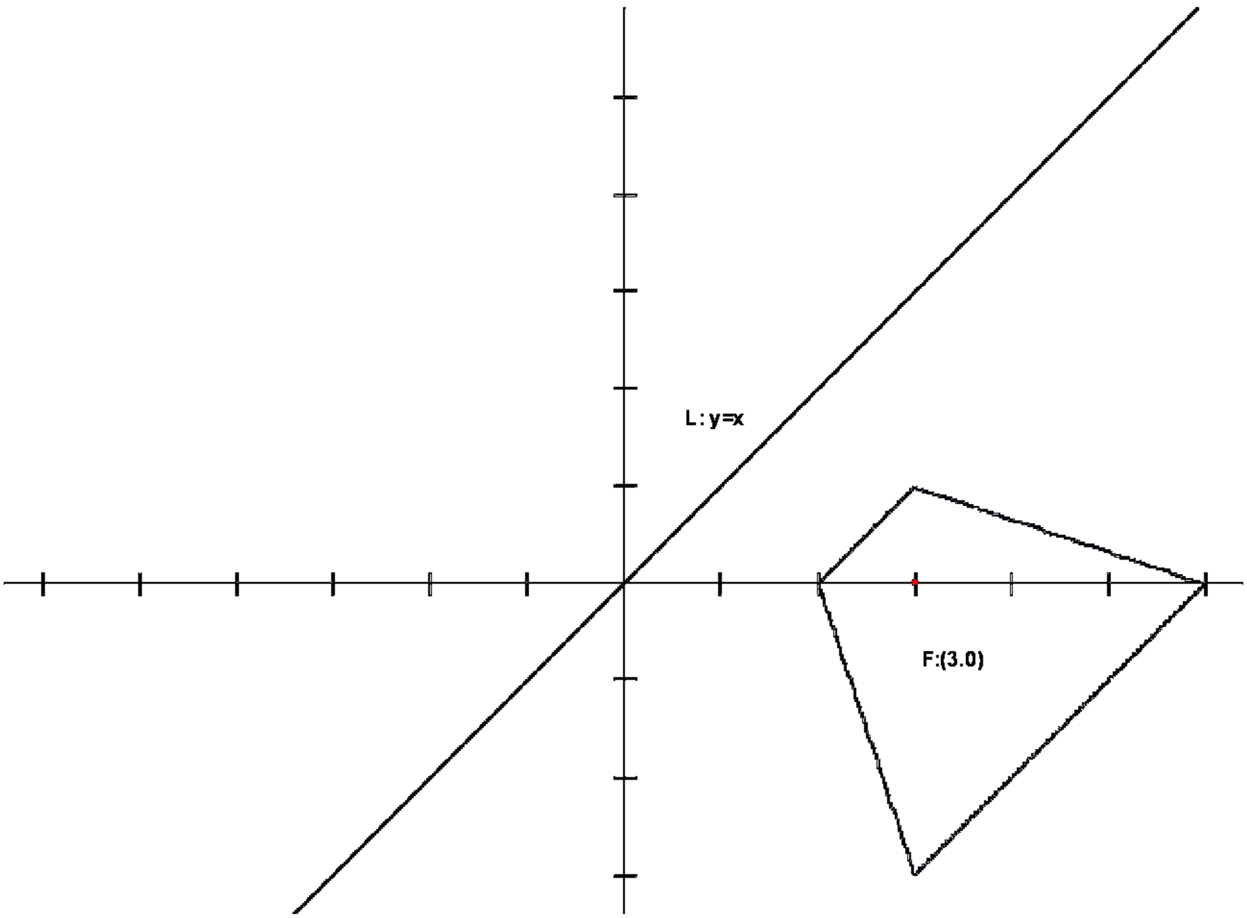

Figure A1: Taxicab ellipse, diagonal case, L: $y = x$, F: (3,0)

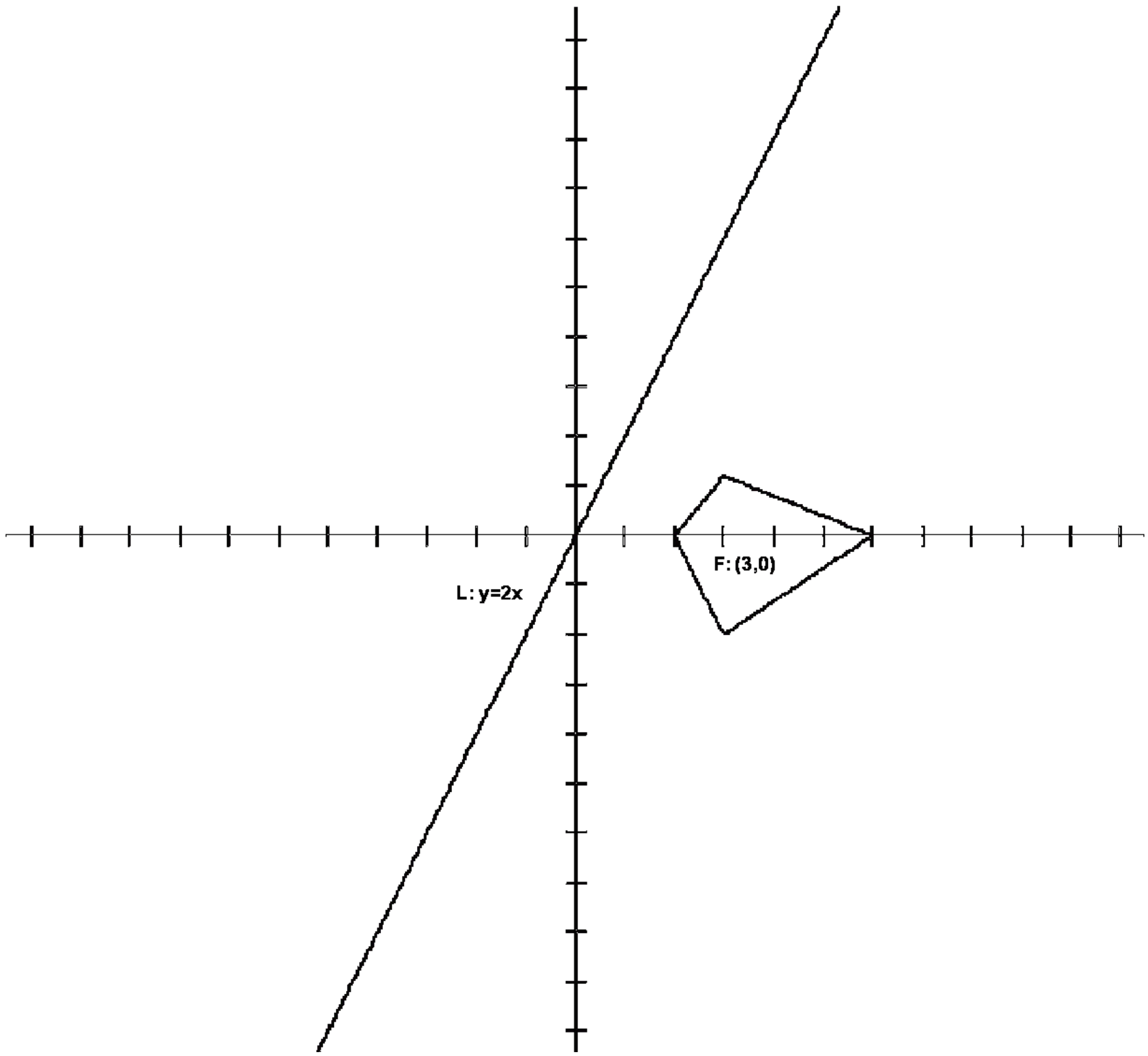

Figure A2: Taxicab ellipse, general case, L: $y = 2x$, F: (3,0)

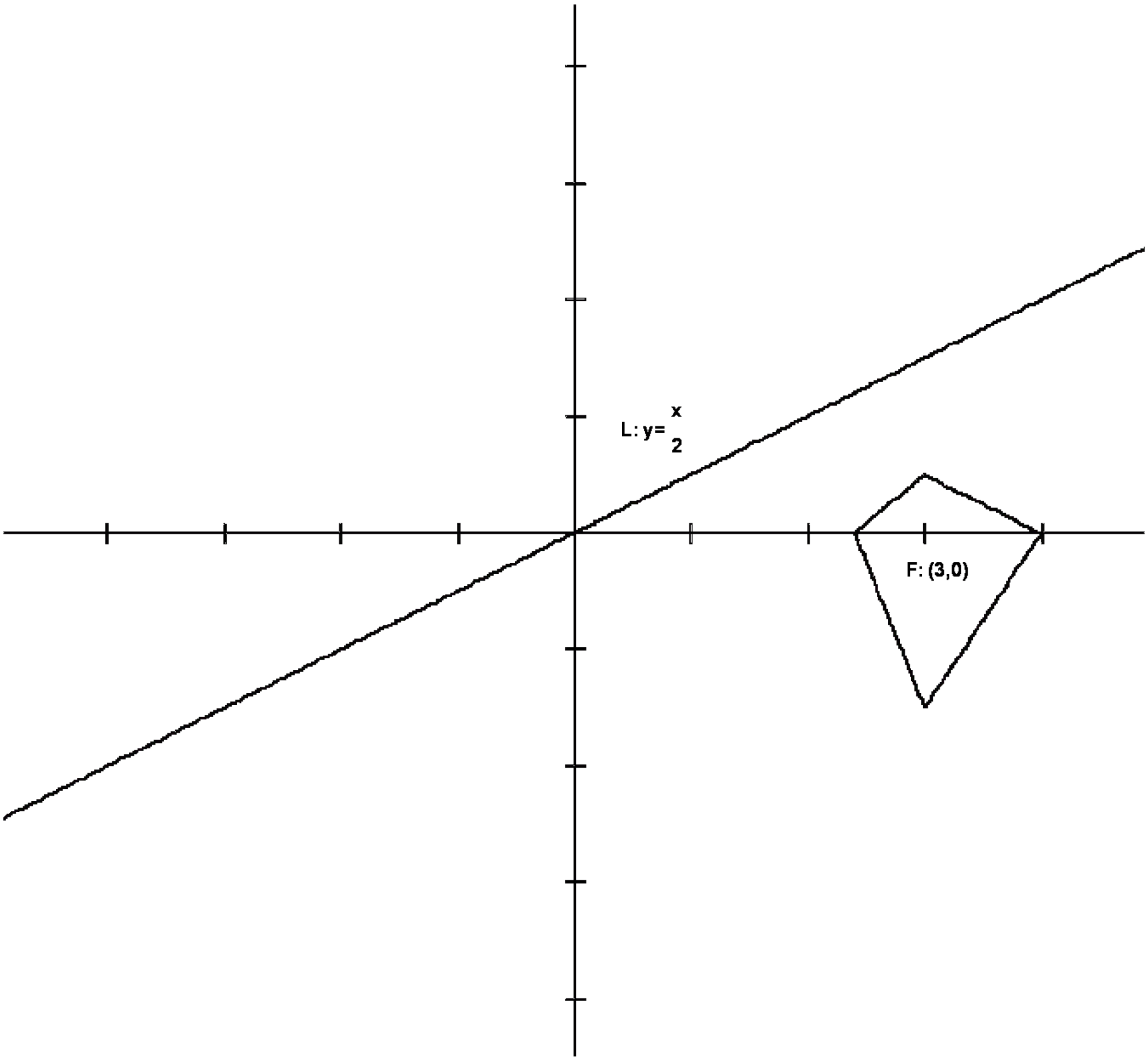

Figure A3: Taxicab ellipse, general case, L: $y = \frac{x}{2}$, F: (3,0)

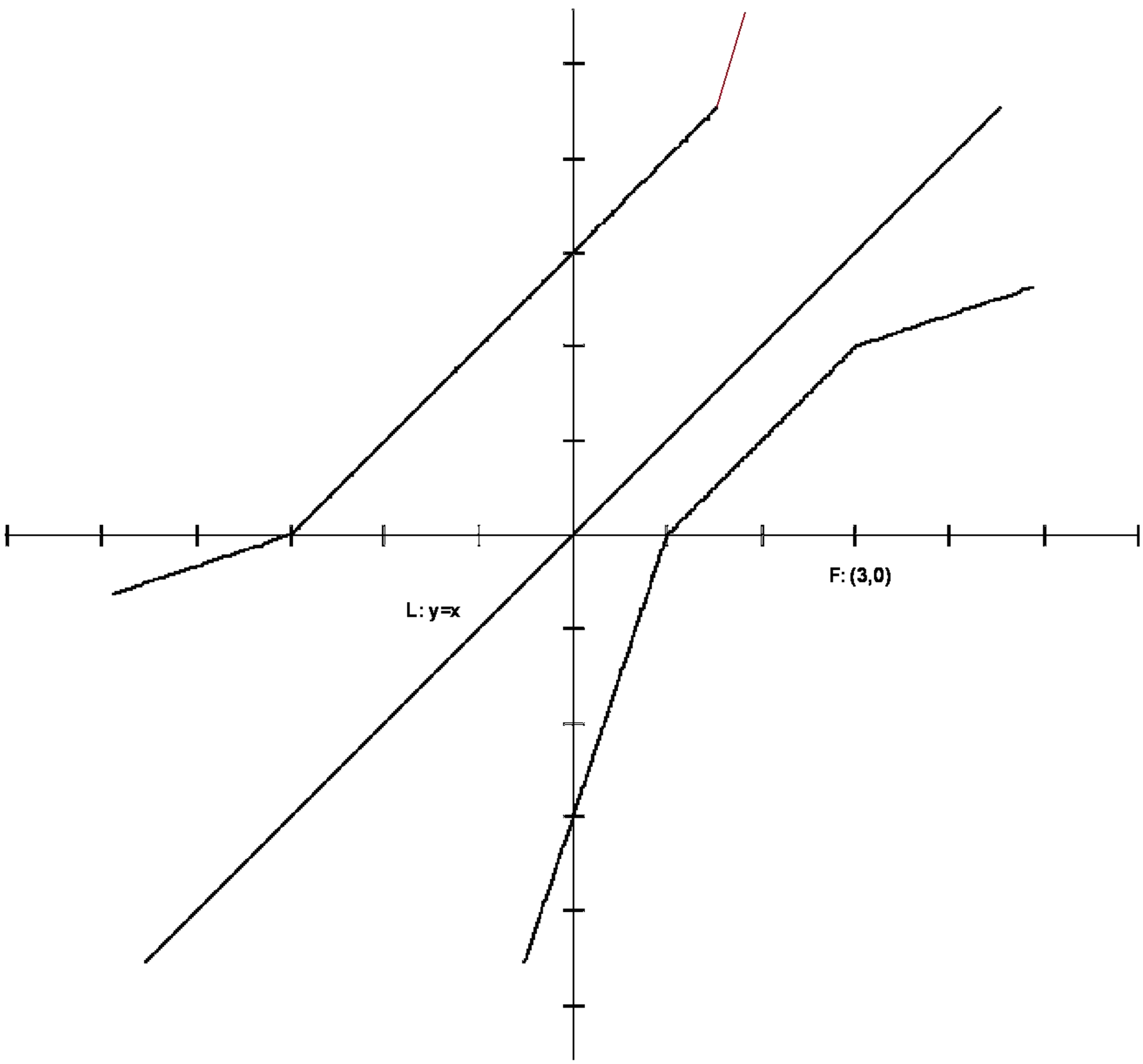

Figure A4: Taxicab hyperbola, diagonal case, $L{:}y = x$, F: (3,0)

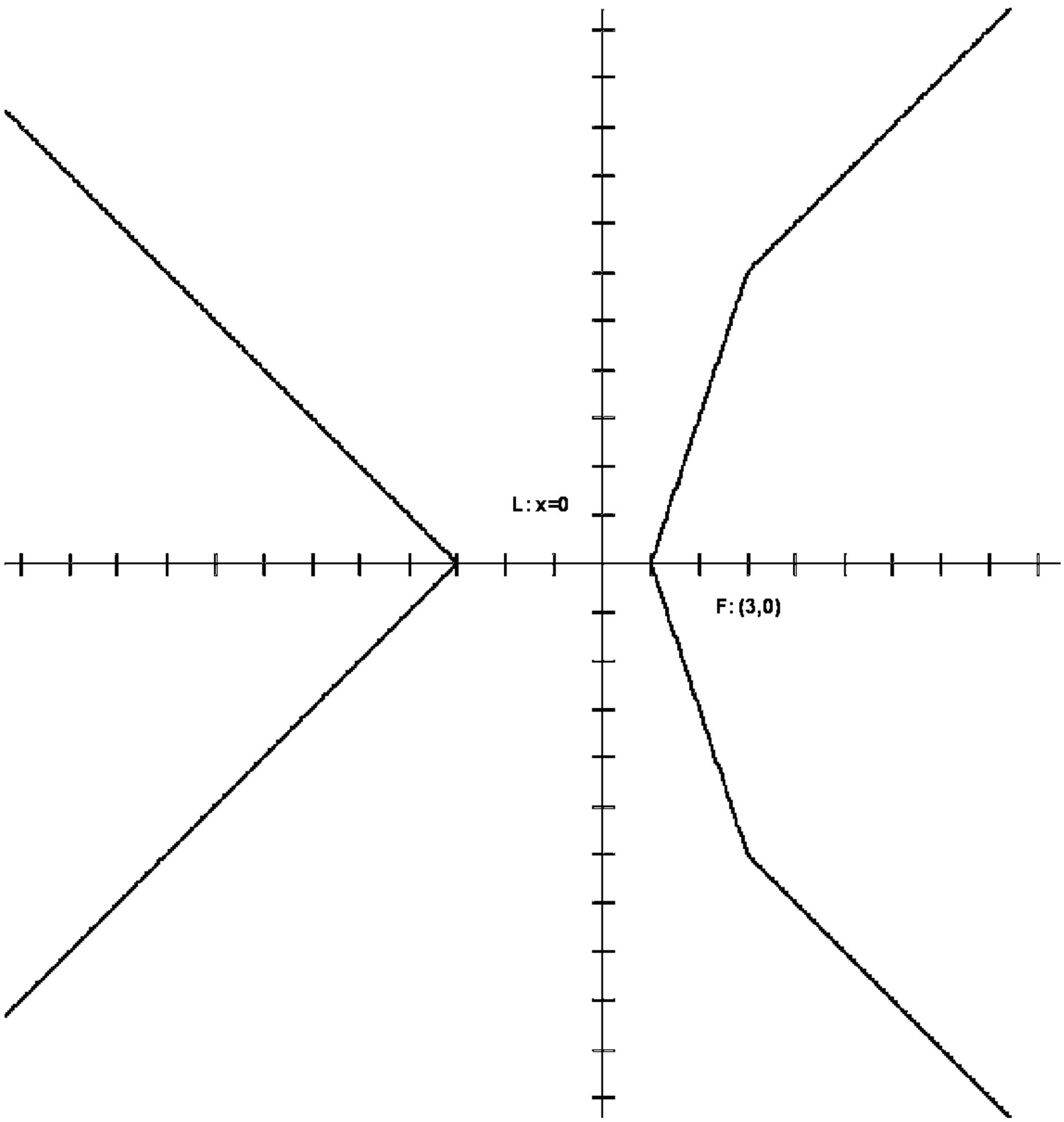

Figure A5: Taxicab hyperbola, parallel case, $L\colon x=0$, F: (3,0)

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Vita

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